1 Combinatorics

Classical Problems			
HanoiTower(HT) min steps	$T_n = 2^n - 1$	Regions by n lines	$L_n = n(n+1)/2 + 1$
Regions by n Zig lines	$Z_n = 2n^2 - n + 1$	Joseph Problem (every m-th)	$F_1 = 0$, $F_i = (F_{i-1} + m)\%i$
Joseph Problem (every 2nd)	rotate n 1-bit to left	HanoiTower (no direct A to C)	$T_n = 3^n - 1$
Bounded regions by n lines	$(n^2 - 3n + 2)/2$	Joseph given pos j,find m.($\downarrow$con.)	$m \equiv 1 \pmod{\frac{L}{p}}$,
HT min steps A to C clockw.	$Q_n = 2R_{n-1} + 1$	$L(n) = lcm(1, ..., n)$, p prime $\in [\frac{n}{2}, n]$	$m \equiv j + 1 - n \pmod p$
HT min steps C to A clockw.	$R_n = 2R_{n-1} + Q_{n-1} + 2$	$\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$	$\sum_{i=1}^n i^3 = n^2(n+1)^2/4$
Egyptian Fraction	$\frac{m}{n} = \frac{1}{\lceil n/m \rceil} + (\frac{m}{n} - \frac{1}{\lceil n/m \rceil})$	Farey Seq given m/n, m'/n'	$m'' = \lfloor (n+N)/n' \rfloor m' - m$
Farey Seq given m/n, m''/n''	$m'/n' = \frac{m+m''}{n+n''}$	$m/n = 0/1$, $m'/n' = 1/N$	$n'' = \lfloor (n+N)/n' \rfloor n' - n$
#labeled rooted trees	n^{n-1}	#labeled unrooted trees	n^{n-2}
#SpanningTree of G (no SL)	$C(G) = D(G) - A(G)(\downarrow)$	Stirling's Formula	$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \frac{1}{12n})$
D : DegMat; A : AdjMat	$Ans = \lvert \det(C - 1r - 1c) \rvert$	Farey Seq	$mn' - m'n = -1$
		#ways $0 \to m$ in n steps (never < 0)	$\frac{m+1}{\frac{n+m}{2}+1}\binom{n}{\frac{n+m}{2}}$
$\#seq\langle a_0, ..., a_{mn}\rangle$ of 1's and $(1-m)$'s with sum $+1 = \binom{mn+1}{n}\frac{1}{mn+1} = \binom{mn}{n}\frac{1}{(m-1)n+1}$			

Binomial Coefficients		
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$, int $n \ge k \ge 0$	$\binom{n}{k} = \binom{n}{n-k}$, int $n \ge 0$, int k	$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$, int $k \ne 0$
$\binom{r}{k} = (-1)^k\binom{k-r-1}{k}$, int k	$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$, int m, k	$(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k}$, int $r \ge 0$ or $\lvert x/y \rvert < 1$
$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$, int k	$\sum_{k \le n}\binom{r+k}{k} = \binom{r+n+1}{n}$, int n	$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$, int $m, n \ge 0$
$\binom{r+s}{n} = \sum_k \binom{r}{k}\binom{s}{n-k}$, int n	$\sum_{k \le m}\binom{r}{k}(\frac{r}{2} - k) = \frac{m+1}{2}\binom{r}{m+1}$, int m	$\sum_{k \le m}\binom{r}{k}(-1)^k = (-1)^m\binom{r-1}{m}$, int m
$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$, int m, n	$\binom{\binom{k}{2}}{2} = 3\binom{k+1}{4}$ | $\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$	$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$ int $l \ge 0$, int m, n
$\sum_k \binom{n}{2k} = 2^{n-even(n)}$	$lcm_{i=0}^n\binom{n}{i} = \frac{L(n+1)}{n+1}$	$S(n,1) = S(n,n) = n \Rightarrow S(n,k) = \binom{n+1}{k} - \binom{n-1}{k-1}$
$\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}$, $F_n = n$-th Fib	$\sum_i \binom{n-i}{i} = F_{n+1}$	

Famous Numbers		
Catalan	$C_0 = 1$, $C_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \frac{4n-2}{n+1} C_{n-1}$	
Stirling 1st kind	$\left[{0 \atop 0}\right] = 1$, $\left[{n \atop 0}\right] = \left[{0 \atop n}\right] = 0$, $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$	#perms of n objs with exactly k cycles
Stirling 2nd kind	$\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$, $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$	#ways to partition n objs into k nonempty sets
Euler	$\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$	#perms of n objs with exactly k ascents
Euler 2nd Order	$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-k-1)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$	#perms of $1, 1, 2, 2, ..., n, n$ with exactly k ascents

Burnside's Lemma: $L = \frac{1}{\lvert G \rvert}\sum_{k=1}^n \lvert Z_k \rvert = \frac{1}{\lvert G \rvert}\sum_{a_i \in G} C_1(a_i)$. Z_k : the set of permutations in G under which k stays stable; $C_1(a_i)$: the number of cycles of order 1 in a_i. **Pólya's Theorem:** The number of colorings of n objects with m colors $L = \frac{1}{\lvert \overline{G} \rvert}\sum_{g_i \in \overline{G}} m^{c(g_i)}$. $\overline{G}$: the group over n objects; $c(g_i)$:the number of cycles in g_i.

Regular Polyhedron Coloring **with at most n colors (up to isomorph)**			
Description	*Formula*	Remarks	
vertices of octahedron or **faces of cube**	$(n^6 + 3n^4 + 12n^3 + 8n^2)/24$		(V, F, E)
vertices of cube or **faces of octahedron**	$(n^8 + 17n^4 + 6n^2)/24$	tetrahedron:	$(4, 4, 6)$
edges of cube or **edges of octahedron**	$(n^{12} + 6n^7 + 3n^6 + 8n^4 + 6n^3)/24$	cube:	$(8, 6, 12)$
vertices or **faces of tetrahedron**	$(n^4 + 11n^2)/12$	octahedron:	$(6, 8, 12)$
edges of tetrahedron	$(n^6 + 3n^4 + 8n^2)/12$	dodecahedron:	$(20, 12, 30)$
vertices of icosahedron or **faces of dodecahedron**	$(n^{12} + 15n^6 + 44n^4)/60$	icosahedron	$(12, 20, 30)$
vertices of dodecahedron or **faces of icosahedron**	$(n^{20} + 15n^{10} + 20n^8 + 24n^4)/60$		
edges of dodecahedron or **edges of icosahedron**	$(n^{30} + 15n^{16} + 20n^{10} + 24n^6)/60$	*This row may be wrong.*	

2 Number Theory

Classical Theorems

exp of p in $n!$ is $\sum_{i\geq 1}[\frac{n}{p^i}]$	$p_n \sim n\log n$; $\forall_{n>1}\exists_{n<p<2n} : p$ is prime	$\pi(x) \sim \frac{x}{\log x}$; $\mathrm{Norm}(\alpha\beta) = \mathrm{Norm}(\alpha)\cdot\mathrm{Norm}(\beta)$
$\mathrm{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$	$a \equiv b \pmod{x,y} \Rightarrow a \equiv b \pmod{\mathrm{lcm}(x,y)}$	All prime factors of $2^{2^n}+1$ have form $2^{n+2}k+1$
$(2^a-1, 2^b-1) = 2^{(a,b)}-1$	$ac \equiv bc \pmod m \Rightarrow a \equiv b (\bmod \frac{m}{\gcd(c,m)})$	n-plygn drawable $\Leftrightarrow n = 2^k \prod F_i$, F_i fermatNum
p_i is prime, $\prod_{p_i \leq n} p_i < 4^n$	$W \equiv d + [2.6m - 0.2] - 2C + Y + \left[\frac{Y}{4}\right] + \left[\frac{C}{4}\right] (\%7)$. $m = 11, 12, 1$ for Ja,Fe,Ma. J&F$\in$lastyear	

Classical Theorems

p prime $\Leftrightarrow (p-1)! \equiv -1(\%p)$	$a \perp m \Rightarrow a^{\phi(m)} = 1(\%m)$	Min general idx $\lambda(n)$: $\forall_a : a^{\lambda(n)} \equiv 1(\%n)$
$\sum_{d\mid n}\phi(d) = \sum_{d\mid n}\phi(n/d) = n$	$\sum_{m\perp n, m<n} m = \frac{n\phi(n)}{2}$	$\sum_{i=1}^{n}\sigma_0(i) = 2\sum_{i=1}^{[\sqrt{n}]}[n/j] - [\sqrt{n}]^2$
$(\sum_{d\mid n}\sigma_0(d))^2 = \sum_{d\mid n}\sigma_0(d)^3$	$\sum_{d\mid n} n\sigma_1(d)/d = \sum_{d\mid n} d\sigma_0(d)$	$[\sqrt{n}]$ Newton: $y = [\frac{x+[n/x]}{2}]$, $x_0 = 2^{[\frac{\log_2(n)+2}{2}]}$
$\sigma_0(p_1^{e_1}\cdots p_s^{e_s}) = \prod_{i=1}^{s}(e_i+1)$	$\sigma_1(p_1^{e_1}\cdots p_s^{e_s}) = \prod_{i=1}^{s}\frac{p_i^{e_i+1}-1}{p_i-1}$	$r_1 = 4$, $r_k \equiv r_{k-1}^2 - 2(\%M_p)$, M_p prime $\Leftrightarrow r_{p-1} \equiv 0(\%M_p)$
$\mu(p_1p_2\cdots p_s) = (-1)^s$, else 0	$\sum_{d\mid n}\mu(d) = 1$ if $n=1$, else 0	$F(n) = \sum_{d\mid n} f(d) \Leftrightarrow f(n) = \sum_{d\mid n}\mu(d)F(\frac{n}{d})$
$n = \sum_{d\mid n}\mu(\frac{n}{d})\sigma_1(d)$	$n = 2, 4, p^t, 2p^t \Leftrightarrow n$ has p_roots	$a \perp n$, then $a^i \equiv a^j(\%n) \Leftrightarrow i \equiv j(\%\mathrm{ord}_n(a))$
$1 = \sum_{d\mid n}\mu(\frac{n}{d})\sigma_0(d)$	$r = \mathrm{ord}_n(a)$, $\mathrm{ord}_n(a^u) = \frac{r}{\gcd(r,u)}$	r p_root of n, then r^u is p_root of $n \Leftrightarrow u \perp \phi(n)$
$\mathrm{ord}_n(a) = \mathrm{ord}_n(a^{-1})$	r p_root of $n \Leftrightarrow r^{-1}$ p_root of n	n has p_roots $\Leftrightarrow n$ has $\phi(\phi(n))$ p_roots
$a^n \equiv a^{\phi(m)+n\%\phi(m)}(\%m)$	$\lambda(2^t) = 2^{t-2}$, $\lambda(p^t) = \phi(p^t) = (p-1)p^{t-1}$, $\lambda(2^{t_0}p_1^{t_1}\cdots p_m^{t_m}) = lcm(\lambda(2^{t0}), \phi(p_1^{t_1}), \cdots, \phi(p_m^{t_m}))$	
$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2}(\%p)$	Legendre sym $\left(\frac{a}{p}\right) = 1$ if a is quad residue $\%p$; -1 if a is non-residue; 0 if $a = 0$	
$a \equiv b(\%p) \Rightarrow \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$	$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$; $\left(\frac{a^2}{p}\right) = 1$	$a \perp p$, s from $a, 2a, ..., \frac{p-1}{2}a(\%p)$ are $> \frac{p}{2} \Rightarrow \left(\frac{a}{p}\right) = (-1)^s$
$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$	Gauss Integer $\pi = a + bi$. $\mathrm{Norm}(\pi) = p$ prime $\Rightarrow \pi$ and $\overline{\pi}$ prime, p not prime	

3 Probability

Classical Formulae

Ballot.Always $\#A > k\#B$	$Pr = \frac{a-kb}{a+b}\binom{a+b}{a}$	Ballot.Always $\#B - \#A \leq k$	$Pr = 1 - \frac{n!m!}{(n+k+1)!(m-k-1)!}$
$E(X+Y) = EX + EY$	$E(\alpha X) = \alpha EX$	if X,Y indep. $\Leftrightarrow E(XY) = (EX)(EY)$	

A)Johnson重加权：添加源点S，它和所有的点连边，边权为0。从S点开始用bellmanford求其到其他所有点的最短路d[v]，若有负环则退出，否则原图所有边(u,v)的边权w'[u,v]改为d[u]+w[u,v]-d[v]。经此，所有边权变为非负且其最短路树不变，可以对其进行Dijkstra。

B)有容量上下限，无源汇的可行流：容量改为上界减去下界。新建源点S和汇点T，设M[v] = Sigma{v的入边下界和}-Sigma{v的出边下界和}。对于所有结点v，若M[v]>0则连边(S,v)容量为M,否则连边(v,T)容量为-M。从S到T求最大流，若从S出去的边全部满流则存在可行流，可行流为流量加下界。

C)有容量上下限，给定源汇s和t的最小流/最大流：t到s连边容量上界为无穷，二分下界，判断有无无源汇的可行流。

D)有容量上下限，给定源汇s和t的最小流，最大流方法2：按容量有上下限，无源汇的可行流方法构图后连边t到s，容量上界为无穷。从S到T求最大流，若从S出发的边全部满流，则存在可行流，每条边的实际流量为求得流量加下界，这样得到s到t的可行最小流。若要求最大流则再次从s到t再满足容量下界的前提下进行增广。

E)差分约束系统：把题目等价转换成一系列a-b<=c的限制条件，注意不要遗漏。每个a-b<=c的条件转换成一条b到a权为c的边。建立一个汇S到所有边的权为0，求S到其它点的最短路d[v],d[v]即满足约束条件的一组解。

F)最优比例成生树：z=Sigma(benifit[i]*x[i])-l*Sigma(cost[i]*x[i])=Sigma(d[i]*x[i])。z关于l严格单调递减，二分l，求生成树，使得z为0，则为最优比例。

G)最优比例成生圈：同上面，把每次求生成树改成判断正环存不存在。

H)二分图：最小覆盖数（最少的点覆盖所有边）= 最大匹配, 最大独立集 = 总点数-最大匹配

无向图的最小路径覆盖：构二分图，答案为无向图点数-最大匹配，最小支配集 = 最大匹配 + 孤立点数

I)弦图判定：首先从大到小编号，每次被编号的结点是当前没编号结点中具有最多相邻已编号结点的结点。再按编号的大小从小到大检查每一个结点，设minnod为比当前考察结点编号大且和它相邻的结点中编号最小的结点，若有另一个编号比当前考察结点大且和它相邻的结点不与minnod相邻，则此图不是弦图。无此情况此判为弦图。