The Algorithm for Doolittle's Method for LU Decompositions

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Recall from the Doolittle's Method for LU Decompositions page that we can factor a square $n \times n$ matrix into an LU decomposition, $A = LU$ (where L is an $n \times n$ lower triangular matrix whose main diagonal consists of 1's and where U is an $n \times n$ upper triangular matrix) using Doolittle's method. Doolittle's method provides an alternative way to factor A into an LU decomposition without going through the hassle of Gaussian Elimination.

Recall that for a general $n \times n$ matrix A, we assume that an LU decomposition exists, and write the form of L and U explicitly. We then systematically solve for the entries in in L and U from the equations that result from the multiplications necessary for $A = LU$.

We will now give the general algorithm for Doolittle's method.

Doolittle's Method Algorithm

For each $k = 1, 2, \ldots, n$:

- $u_{k,m} = a_{k,m} - \sum_{j=1}^{k-1} l_{k,j} u_{j,m}$ for $m = k, k+1, \ldots, n$ produces the k^{th} row of U.
- $l_{i,k} = \frac{(a_{i,k} - \sum_{j=1}^{k-1} l_{i,j} u_{j,k})}{u_{kk}}$, for $i = k+1, k+2, \ldots, n$ and $l_{i,i} = 1$ produces the k^{th} column of L.

Example 1

For $n = 3$, verify the algorithm given above for Doolittle's method.

Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$. We construct U and L systematically.

For $k = 1$ we have that:

$$\begin{aligned} u_{1,1} &= a_{1,1} - \sum_{j=1}^{0} l_{1,j} u_{j,1} = a_{1,1} \quad (m = 1) \\ u_{1,2} &= a_{1,2} - \sum_{j=1}^{0} l_{1,j} u_{j,2} = a_{1,2} \quad (m = 2) \\ u_{1,3} &= a_{1,3} - \sum_{j=1}^{0} l_{1,j} u_{j,3} = a_{1,3} \quad (m = 3) \end{aligned} \tag{1}$$

$$\begin{aligned} l_{1,1} &= 1 \\ l_{2,1} &= \frac{a_{2,1} - \sum_{j=1}^{0} l_{2,j} u_{j,1}}{u_{1,1}} = \frac{a_{2,1}}{u_{1,1}} \quad (i = 2) \\ l_{3,1} &= \frac{a_{3,1} - \sum_{j=1}^{0} l_{3,j} u_{j,1}}{u_{1,1}} = \frac{a_{3,1}}{u_{1,1}} \quad (i = 3) \end{aligned} \tag{2}$$

For $k = 2$ we have that:

(3)

$$u_{2,2} = a_{2,2} - \sum_{j=1}^{1} l_{2,j} u_{j,2} = a_{2,2} - l_{2,1} u_{1,2} \quad (m = 2)$$
$$u_{2,3} = a_{2,3} - \sum_{j=1}^{1} l_{2,j} u_{j,3} = a_{2,3} - l_{2,1} u_{1,3} \quad (m = 3)$$

$$\begin{aligned} l_{2,2} &= 1 \\ l_{3,2} &= \frac{a_{3,2} - \sum_{j=1}^{1} l_{3,j} u_{j,2}}{u_{2,2}} = \frac{a_{3,2} - l_{3,1} u_{1,2}}{u_{2,2}} \end{aligned} \tag{4}$$

For $k = 3$ we have that:

$$u_{3,3} = a_{3,3} - \sum_{j=1}^{2} l_{3,j} u_{j,3} = a_{3,3} - l_{3,1} u_{1,3} - l_{3,2} u_{2,3} \quad (m = 3) \tag{5}$$

$$l_{3,3} = 1 \tag{6}$$

The algorithm terminates at this step.

We get the following equations for the entries in U:

$$\begin{aligned} u_{1,1} &= a_{1,1} \\ u_{1,2} &= a_{1,2} \\ u_{1,3} &= a_{1,3} \\ u_{2,2} &= a_{2,2} - l_{2,1} u_{1,2} \\ u_{2,3} &= a_{2,3} - l_{2,1} u_{1,3} \\ u_{3,3} &= a_{3,3} - l_{3,1} u_{1,3} - l_{3,2} u_{2,3} \end{aligned} \tag{7}$$

And we get the following equations for the entries in L:

$$\begin{aligned} l_{2,1} &= \frac{a_{2,1}}{u_{1,1}} \\ l_{3,1} &= \frac{a_{3,1}}{u_{1,1}} \\ l_{3,2} &= \frac{a_{3,2} - l_{3,1} u_{1,2}}{u_{2,2}} \end{aligned} \tag{8}$$

Furthermore, $l_{1,1} = l_{2,2} = l_{3,3} = 1$.

If you compare these equations with those on the Doolittle's Method for LU Decompositions page, you'll see they're equivalent.