**A short note on finding roots of nonlinear equations**

Consider solving a nonlinear equation in one variable $x$

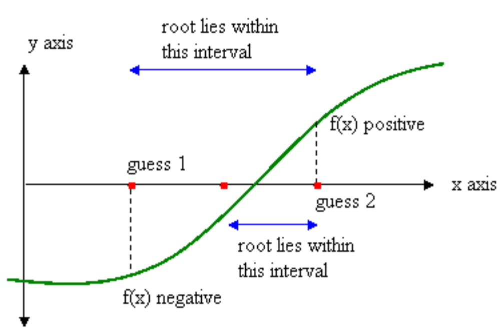
$$f(x) = 0 \tag{1}$$

In this course we will assume $f(x)$ to be a smoothly varying function that can have extrema, minima and/or maxima, over the range we are interested in. A few typical examples of $f(x)$ are

$$f(x) = \cos x - x^3, \quad 3x + \sin x - \exp x, \quad x \exp x - 2, \quad x^3 + 3x - 5 = 0 \tag{2}$$

If a value $x_0$ in the interval $(a_0, b_0)$ satisfies the equation (1) *i.e.* $f(x_0) = 0$, then $x_0$ is a *root* or *zero* of the function $f(x)$ and is **one** solution in that interval. Since $f(x)$ is a continuous function and there exist two points $a_0$ and $b_0$ such that $f(a_0)$ and $f(b_0)$ are of opposite signs, then according to *intermediate value theorem*, the function $f(x)$ has a root in the interval $(a_0, b_0)$.

Finding root numerically always start with guesses, two $x$-values $a_0$, $b_0$ either found by trial-and-error or educated guess, at which $f(x)$ has opposite signs. Since $f(x)$ is continuous and one root of it is guaranteed to lie between these two values, we say these $a_0$ and $b_0$ *bracket* the root. Then we proceed by iterations to produce a sequence of shrinking intervals $(a_0, b_0) \to (a_i, b_i)$ such that the shrunk intervals always contain one root of $f(x)$.




For convergence, it is necessary to a *good* initial guess. This might be achieved by plotting $f(x)$ vs. $x$ to get some idea of the root. In this course we will learn 4 methods for finding roots of nonlinear equations including the one specialised for finding the roots of polynomials.

1. Bisection method

2. False position (Regula falsi) method

3. Newton-Raphson method

4. Laguerre's method

***Bisection method***

The bisection method is the simplest but relatively slow method of finding root of nonlinear equations. The method is guaranteed to converge to a root of $f(x)$ if the function is continuous in the interval $[a, b]$ where $f(a)$ and $f(b)$ have opposite signs. But bracketing can go wrong if $f(x)$ has double roots or $f(x) = 0$ is an extrema or $f(x)$ has many roots over the interval choosen. The steps involve in bracketing are,

1. Choose $a$ and $b$, where $a < b$, and calculate $f(a)$ and $f(b)$.

2. If $f(a) * f(b) < 0$ then bracketing done. Proceed to execute bisection method.

3. If $f(a) * f(b) > 0$ *i.e.* same sign, then check whether $|f(a)| \lesseqgtr |f(b)|$.

4. If $|f(a)| < |f(b)|$, shift $a$ further to the left by using, say, $a = a - \beta * (b - a)$ and then go back to second step. Choose your own $\beta$, say 1.5.

5. If $|f(a)| > |f(b)|$, shift $b$ further to the right by using, say, $b = b + \beta * (b - a)$ and then go back to second step. Choose your own $\beta$, say 1.5.

6. Give up if you can't satisfy the condition $f(a) * f(b) < 0$ in $10 - 12$ iterations. Start with a new pair $[a', b']$ and do the thing all over again.

Now the bisection method proceeds as

1. Choose appropriate $[a, b]$, where $a < b$, to bracket the root *i.e.* $f(a) * f(b) < 0$.

2. Bisect the interval, the midpoint of the interval is taken as first apprximation with $a_1 = a$ and $b_1 = b$
$$c_1 = \frac{b_1 + a_1}{2} \tag{3}$$
The maximum absolute error of this approximation is
$$|c_1 - \bar{x}| \leq \frac{b_1 - a_1}{2} = \frac{b_a}{2} \tag{4}$$

3. If the error in (4) is considered too large, repeat the above step with new interval either $[a_2, b_2] = [a_1, c_1]$ or $[c_1, b_1]$ depending on the sign of $f(c_1)$. The new bisection or midpoint is $c_2 = (b_2 + a_2)/2$ and maximum abslute error is
$$|c_2 - \bar{x}| \leq \frac{b_2 - a_2}{2} = \frac{b - a}{4}$$

4. If in the $n$-th step the corresponding values are $a_n, b_n, c_n$ then

$$c_n = \frac{b_n + a_n}{2} \quad \rightarrow \quad |c_n - \bar{x}| \leq \frac{b_n - a_n}{2} = \frac{b-a}{2^n} \tag{5}$$

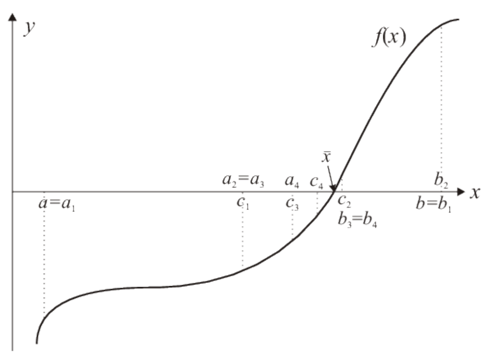
So the method converges since in the limit $n \to \infty$ the factor $2^{-n} \to 0$, but as we see it converges rather slowly.

5. If our desired maximum error $\epsilon$, say $\epsilon = 10^{-4}$, is reached, we stop

$$|c_n - \bar{x}| \leq \epsilon \Rightarrow \frac{b-a}{2^n} \leq \epsilon \tag{6}$$

Along with the above convergence criteria, we can also test if $|f(c)| < \epsilon$ since at root $x_0$ implies $f(x_0) = 0$.

The bisection steps are schematically shown in the figure below.




This method has slowest convergence of all other root finding methods but it is a sure shot to root provided you can bracket properly. But still one cannot get root beyond certain precision because the difference between $b$ and $a$ is limited by floating point precision *i.e.* as the difference $(b_n - a_n)$ decreases. Therefore, the accuracy can never reach machine precision.