# Distribution function

## Definition 1.8

If $X : S \to \mathbb{R}$ is a random variable, for any $a \in \mathbb{R}$, $E_a = \{s \in S | X(s) \leq a\}$ is an event. By $P(X \leq a)$ we mean the probability of this event $P(E_a)$. The cumulative distribution function, or the distribution function, $F_X$ of $X$ is defined by $a$ by $F_X(a) = P(X \leq a)$.

## Remark

1. $F_x(a)$ is defined for all $a \in \mathbb{R}$.
2. $a \leq b \Rightarrow F_x(a) \leq F_x(b)$.
3. $\lim\limits_{t\to\infty} F_x(t) = 1; \lim\limits_{t\to-\infty} F_x(t) = 0$

a.k.a the cumulative distribution function.

# Discrete Random variable

## Definition 1.9

If the range of a random variable is a discrete set, then it is called a discrete random variable.

## Remark

1. If the sample space is discrete, any random variable defined on that sample space will be discrete. $X : S \to \mathbb{R}$
2. For any $a \in \mathbb{R}$, $E_a = \{s \in S | X(s) = a\}$ is an event. By $P(X = a)$ we mean the probability of this event $P(E_a)$.

we are considering discrete random variable from now onwards

# Probability distribution function

## Definition 1.10

If $X$ is a discrete random variable, the function $f_X : \mathbb{R} \to \mathbb{R}$ defined by $f_X(a) = P(X = a)$ is called the probability distribution function or the probability mass function of $X$.

p.d.f.

## Remark

1. $f_X(a) \geq 0, \forall a \in \mathbb{R}$
2. $\sum_{a \in \text{Range of } X} f_X(a) = 1$

$f_X : \mathbb{R} \to [0,1]$

# Probability distribution function

## Exercise 1.8 (Source https://www.probabilitycourse.com)

Let $X$ be the number of the cars being repaired at a repair shop. We have the following information: At any time, there are at most 3 cars being repaired. The probability of having 2 cars at the shop is the same as the probability of having one car. The probability of having no car at the shop is the same as the probability of having 3 cars. The probability of having 1 or 2 cars is half of the probability of having 0 or 3 cars. If $f_X$ is the p.d.f. of $X$, find $f_X(0)$.

$P(X=2) = P(X=1) \Rightarrow f_X(2) = f_X(1)$

$P(X=0) = P(X=3) \Rightarrow f_X(0) = f_X(3)$

$P(X=1 \text{ or } 2) = \frac{1}{2} P(X=0 \text{ or } 3) \Rightarrow f_X(1) + f_X(2) = \frac{1}{2}(f_X(0) + f_X(3))$

$\underbrace{f_X(0)}_{a} + \underbrace{f_X(1)}_{b} + \underbrace{f_X(2)}_{b} + \underbrace{f_X(3)}_{a} = 1.$

$2b = \frac{1}{2}(2a)$

$a + b = \frac{1}{2}$

$a = \frac{1}{3}$

$b = \frac{1}{6}$

# Examples of Probability distribution function

Check whether the following can serve as probability distributions

## Example 1.6

1. $f(x) = \frac{x-2}{2}; x = 1, 2, 3, 4.$ ✗ $f_x(1) < 0$
2. $f(x) = \frac{(x-1)^2}{4}; x = 0, 1, 2, 3.$ ✗ $\sum f(x) \neq 1$
3. $f(x) = \frac{1}{n}; x = \cancel{0}, 1, \ldots, n$ for some $n \in \mathbb{N}$.

$n=1$ $f(1)=1$

$n=2$ $f(1)=1/2$, $f(2)=1/2$

$n=3$ $f(1) = 1/3$, $f(2)=1/3$, $f(3)=1/3$

✓ possible

$f(x) = \frac{1}{n+1}$, $x = 0, 1, \ldots n$. ✓

(for any $n \in \mathbb{N}$ fixed)

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3. $f(x) = \frac{1}{n}; x = 0, 1, \ldots, n$ for some $n \in \mathbb{N}$.     (Uniform Distribution)
4. Given that $f(x) = \frac{k}{2^x}$ is ~~a~~ the probability distribution for a random variable that can take on the values $x = 0, 1, 2,$and 3. Find $k$.

$$k\left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3}\right) = 1 \implies k =$$

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4. Given that $f(x) = \frac{k}{2^x}$ is a probability distribution for a random variable that can take on the values $x = 0, 1, 2,$and 3. Find $k$.
5. $\binom{n}{x}p^x(1-p)^{n-x}, x = 0, 1, \dots, n$ for some $n \in \mathbb{N}$ and $0 < p < 1$.

$f(x) =$

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1$$

# Expectation

## Definition 1.11

Let $X$ be a random variable with probability mass function $f(x)$ such that $\sum_x |x|f(x) < \infty$. The expected value of $X$, denoted by $\mathbf{E}(X)$, is defined as

$$\mathbf{E}(X) = \sum_x xf(x)$$

This is also known as the mean, or average or first moment of $X$ and is usually denoted by $\mu$.

## Remark

Note that $X$ can take (countable) infinite number of values and so the expression for $\mathbf{E}(X)$ could be an infinite series. $\mathbf{E}(X)$ is defined only when this series is absolutely convergent.

# Expectation of a function of a random variable

## Remark

Let $X$ be a random variable defined on $S$ with mass function $f(x)$, and let $g(.)$ be a real valued function defined on $\mathbb{R}$. Then $Y = g(X)$ is also a random variable on $S$ with the p.d.f.

$$f_Y(y) = \mathbf{P}(g(X) = y) = \sum_{x:g(x)=y} \mathbf{P}(X = x) = \sum_{x:g(x)=y} f_X(x)$$

## Theorem 1.7

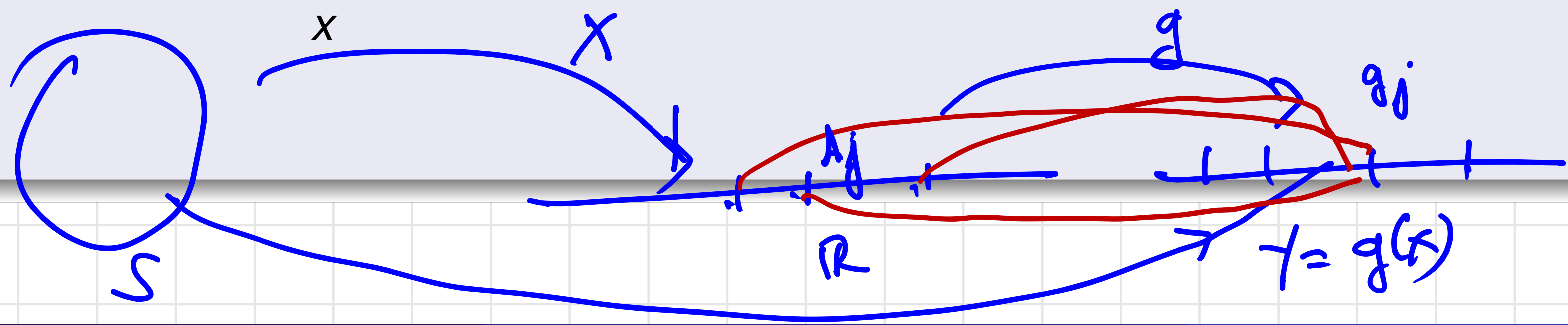
$$\mathbf{E}(g(X)) = \sum_x g(x)f(x)$$

# Expectation of a function of a random variable

## Proof.

Let $(g_j)$ denote the possible values of $g(X)$, and for each $j$ define the set $A_j = \{x : g(x) = g_j\}$. Then $\mathbf{P}\left(g(X) = g_j\right) = \mathbf{P}\left(X \in A_j\right)$, and therefore, provided all the following summations converge absolutely, we have

$$\begin{aligned}
\mathbf{E}(g(X)) &= \sum_j g_j \mathbf{P}\left(g(X) = g_j\right) = \sum_j g_j \sum_{x \in A_j} f(x) \\
&= \sum_j \sum_{x \in A_j} g(x) f(x), \quad \text{because } g(x) = g_j \quad \text{for } x \in A_j \\
&= \sum_x g(x) f(x), \quad \text{because } A_j \cap A_k = \phi \quad \text{for } j \neq k
\end{aligned}$$

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