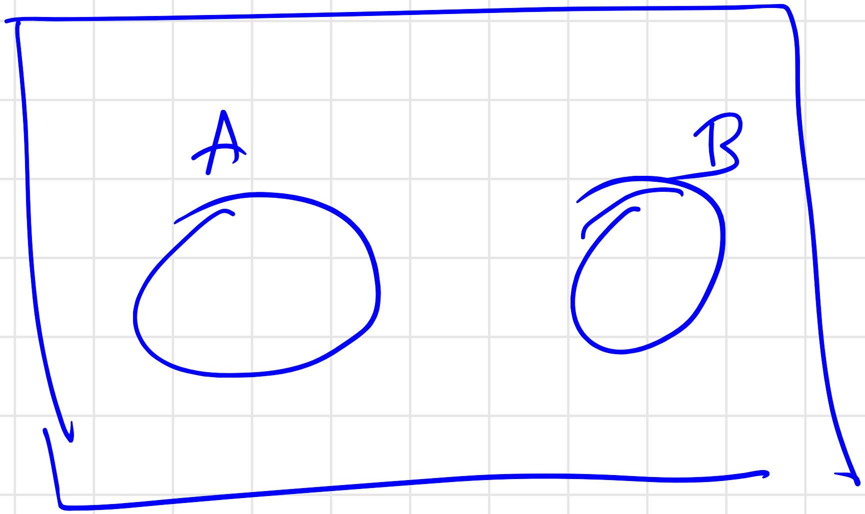
Exercise 1.6

If A and B are mutually exclusive events with P(A) = 0.45 and P(B) = 0.30, find $p = P(A \cap B^c)$ and $q = P(A^c \cap B^c)$.

A



$(A \cup B)^c$

The classical probability concept

If the sample space is finite and if each sample point is equally likely to occur, that is if $S = \{x_1, x_2, \cdot, x_N\}$ and $P(\{x_i\}) = p, \forall i$, then by the axioms of probability, we get that $P(\{x_i\}) = \frac{1}{N}, \forall i$. In this scenario, any event E has the probability,

$$P(E) = \frac{\text{Number of sample points in } E}{N}$$

Exercise 1.7

A car rental agency has 5 compact cars and 3 intermediate-size cars. If four of the cars are randomly selected for a safety check, what is the probability of getting two of each kind

$$\frac{{}^5C_2 \times {}^3C_2}{{}^8C_4}$$

Conditional Probability

Conditional Probability

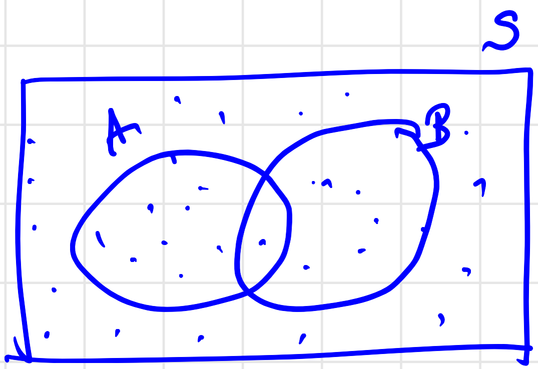
Definition 1.5

If A and B are any events in $\mathcal{S}$ and $P(B) \neq 0$, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Remark

Note that the conditional probability with respect to B satisfies all the axioms of probability, if the sample space is taken as B.



- $P(A \mid B) \geq 0$
- $P(B \mid B) = 1$
- $P(\cup A_j \mid B) = \sum_j P(A_j \mid B)$ if A_j's are mutually exclusive

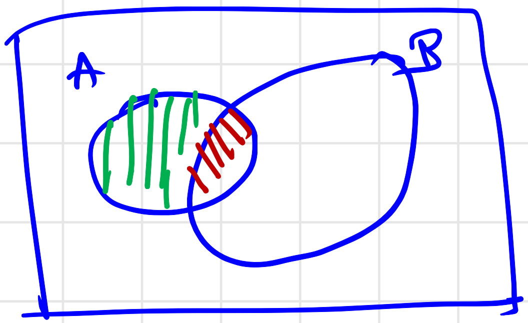
Conditional Probability

Theorem 1.3

If A and B are any events in $\mathcal{S}$, then

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B \mid A) \\ &= P(B) \cdot P(A \mid B) \\ \Rightarrow P(A) &= P(B) \cdot P(A \mid B) + P(B^c) \cdot P(A \mid B^c) \end{aligned}$$

$P(A \cap B)$ $\quad$ $P(A \cap B^c)$



Independent Events

Definition 1.6

If A and B are any two events in a sample space $\mathcal{S}$, we say that A is independent of B if $P(A \mid B) = P(A)$.

Remark

If A is independent of B, $P(A \cap B) = P(A) \cdot P(B)$. Therefore B is also independent of A. Hence we simply that A and B are independent events. A collection of events $A_1, A_2, \ldots, A_n$ is said to be (mutually) independent if and only if the probability of the intersections of any $2, 3, \ldots,$ or k of these events equals the product of their respective probabilities.

Example 1.4

Two fair dice are rolled. Show that the event, 'the sum is 7' is independent of the score shown by the first die.

$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$. $P(A) = \frac{6}{36} = \frac{1}{6}$

$B = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$ $P(B) = 1/6$.

$A \cap B = \{(1,6)\}$. $P(A \cap B) = 1/36$

$P(A|B) = P(A \cap B)/P(B) = \frac{1/36}{1/6} = 1/6 = P(A)$.

A paradox

Galton's Paradox: Suppose you flip three fair coins. At least two are always alike. Then the third is a head or a tail with probability $1/2$. So the chance that all three are the same is $1/2$.

Random variable

The sample space of a random experiment could be any set and only set theoretic operations can be defined on them. To analyse the experiment by bringing in a numerical aspect to events, we introduce the notion of random variable.

Definition 1.7

A random variable X is a function from the sample space to the real numbers. That is, $X : S \to \mathbb{R}$.

Example 1.5 (Bernoulli trial)

A series of trials with exactly two possible outcomes for each trial (called "Success" and "Failure"), in which the probability of Success is the same every time the experiment is conducted. Then the function X defined on S as

$$X(\text{Success}) = 0; X(\text{Failure}) = 1$$

is a random variable.

Distribution function

Definition 1.8

If $X : S \to \mathbb{R}$ is a random variable, for any $a \in \mathbb{R}$, $E_a = \{s \in S | X(s) \leq a\}$ is an event. By $P(X \leq a)$ we mean the probability of this event $P(E_a)$. The cumulative distribution function, or the distribution function, F_X of X is defined by a by $F_X(a) = P(X \leq a)$.

$\mathbb{R} \to [0,1]$

Remark

1. $F_x(a)$ is defined for all $a \in \mathbb{R}$. $F_X(a) \in [0,1]$
2. $a \leq b \Rightarrow F_x(a) \leq F_x(b)$. Why? $a \leq b \Rightarrow E_a \subseteq E_b$
3. $\lim_{t\to\infty} F_x(t) = 1$; $\lim_{t\to-\infty} F_x(t) = 0$