Beta distribution

Definition 1.22

A random variable X is said to follow the beta distribution with parameters $m > 0$ and $n > 0$ if its p.d.f is given by

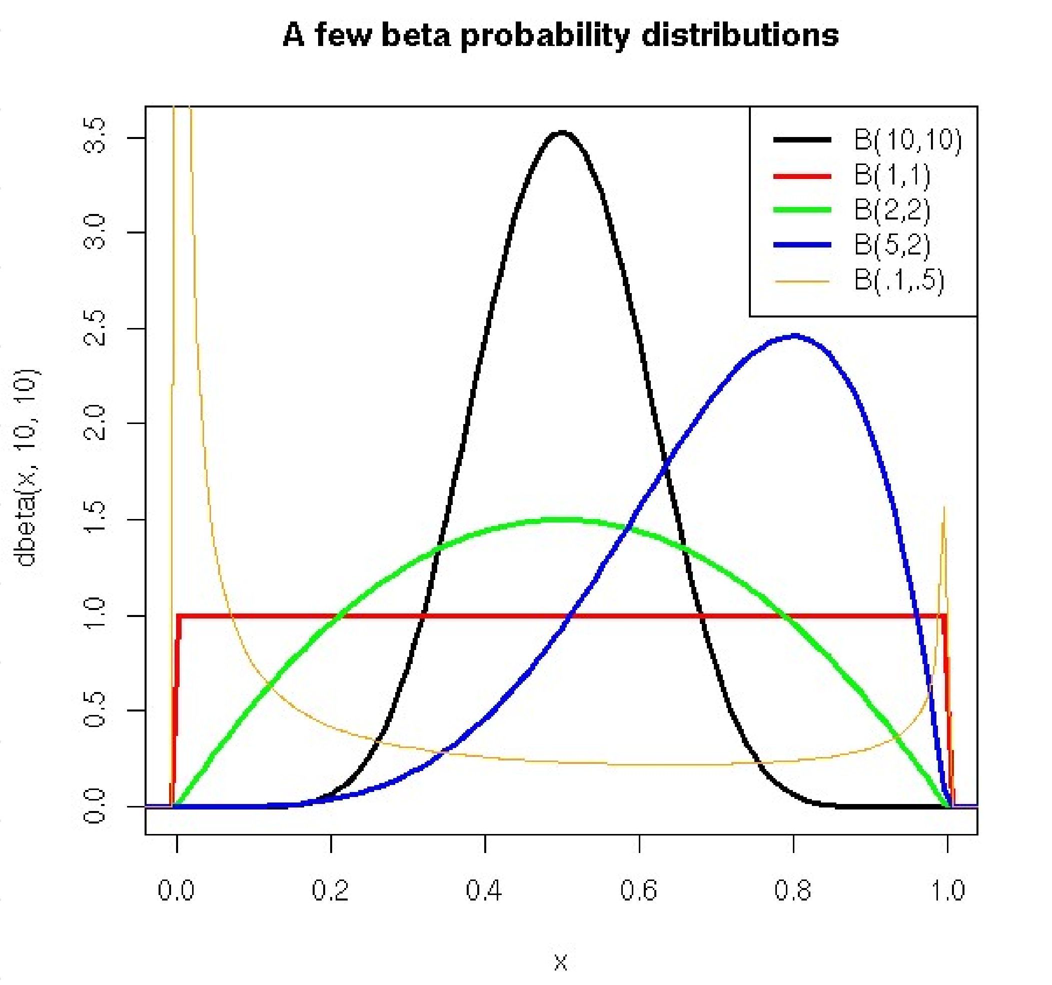
$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Theorem 1.24

The mean and the variance of beta distribution are given by

$$\mu = \frac{m}{m+n} \quad \text{and} \quad \sigma^2 = \frac{mn}{(m+n)^2(m+n+1)}$$

Beta distribution



Credit: *Vincent Zoonekynd / CC BY-SA*

Weibull distribution

Definition 1.23

A random variable X is said to follow the Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ if its p.d.f is given by

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Theorem 1.25

The mean and the variance of Weibull distribution are given b

$$\mu = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right)$$

$$\sigma^2 = \alpha^{-2/\beta}\left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\}$$

Normal Distribution

Definition 1.24

A random variable is said to be normally distributed with parameters μ and σ^2, and we write $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$, if its density is

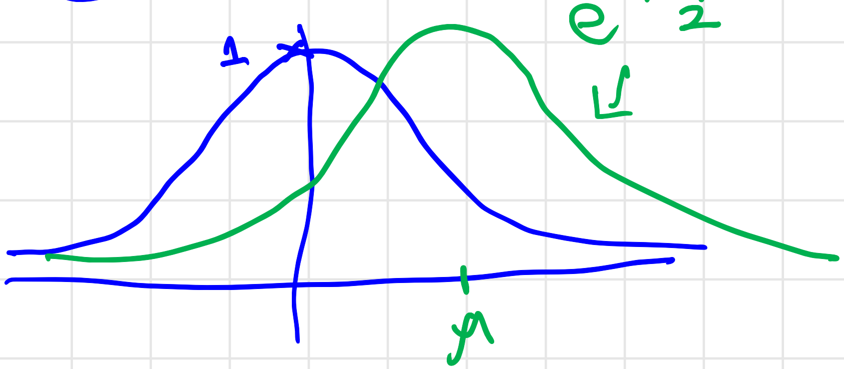
$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ when $\mu = 0$, $\sigma^2 = 1$

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\, dx &= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, dy \\
&= \frac{2}{\sqrt{2\pi\sigma}} \int_{0}^{\infty} e^{-y^2/2\sigma^2}\, dy \\
&= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2}\, dz \\
&= \frac{1}{\sqrt{\pi}} \Gamma(1/2) = 1
\end{aligned}$$

e^{-x^2}

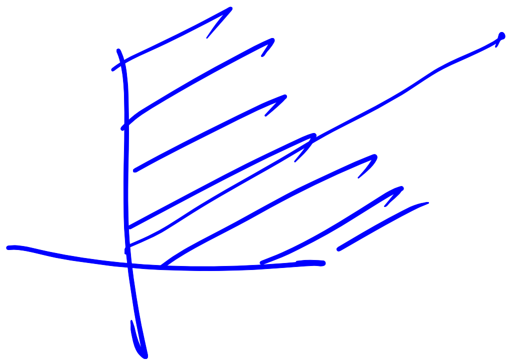
$e^{-\frac{(x-\mu)^2}{2}}$



Ex

$$\int_0^\infty e^{-x^2}\,dx = ?\ A$$

Then
$$A^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$



$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2}\, r\, dr\, d\theta$$

$$=$$

Normal Distribution

Theorem 1.26

If $X \sim \mathcal{N}(\mu, \sigma^2)$ then for any constants a and b, $b \neq 0$, the random variable $Y = a + bX$ is also a normal random variable with parameters $a + b\mu$ and $b^2\sigma^2$. That is $Y \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$

Proof

let $F_Y(y)$ be the distribution function of Y. Then, for $b > 0$

$$F_Y(y) = P(Y \leq y) = P(a + bX \leq y)$$
$$= P\left(X \leq \frac{y-a}{b}\right)$$
$$= Fx\left(\frac{y-a}{b}\right) \downarrow \frac{d}{dy}$$

f_Y

where F_X is the distribution function of X. $f_X\left(\frac{y-a}{b}\right) \cdot \frac{1}{b} =$

(Proof continued)

Similarly, if $b < 0$, then

$$
\begin{aligned}
F_Y(y) &= P(a + bX \leq y) \\
&= P\left(X \geq \frac{y-a}{b}\right) \\
&= 1 - F_X\left(\frac{y-a}{b}\right)
\end{aligned}
$$

Differentiation yields that the density function of Y is

$$
f_Y(y) = \begin{cases} \frac{1}{b} f_X\left(\frac{y-a}{b}\right), & \text{if} \quad b > 0 \\ -\frac{1}{b} f_X\left(\frac{y-a}{b}\right), & \text{if} \quad b < 0 \end{cases}
$$

Normal Distribution

(Proof continued)

That is,

$$
\begin{aligned}
f_Y(y) &= \frac{1}{|b|} f_X\left(\frac{y-a}{b}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma|b|} e^{-\left(\frac{y-a}{b}-\mu\right)^2/2\sigma^2} \\
&= \frac{1}{\sqrt{2\pi}\sigma|b|} e^{-(y-a-b\mu)^2/2b^2\sigma^2}
\end{aligned}
$$

$\Rightarrow Y \sim N(a+b\mu, b^2\sigma^2)$

Remark

If $X \sim \mathcal{N}(\mu, \sigma^2)$ then, $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$, the standard normal distribution.

Mean and variance of normal distribution

Theorem 1.27

If $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ then $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

Proof

Since $E\left(\frac{X-\mu}{\sigma}\right) = \frac{E(X)-\mu}{\sigma}$ and $\mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{\mathrm{Var}(X)}{\sigma^2}$, it is enough to consider $\mathcal{N}(0,1)$. For the standard normal distribution X,

$$E(X) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}}\left[- e^{-y^2/2}\Big|_{-\infty}^{\infty}\right] = 0$$

$$\mathrm{Var}(X) = E(X^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy$$

$$= \frac{1}{\sqrt{2\pi}}\left(- y e^{-y^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-y^2/2} dy\right) = 1.$$