Exercise

Exercise 1.21

Suppose the p.d.f. of a random variable is given by

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}.$$

a. Find k

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a. Find k
b. Find $P(-1 < x < 1)$.

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

a) $\Rightarrow k\int_0^{\infty} x e^{-4x^2}\,dx = 1 \Rightarrow k = \boxed{}$

b) $\int_{-1}^{1} f(x)\,dx = k\int_0^{1} x e^{-4x^2}\,dx = \boxed{}$

Mean and variance of a continuous random variable

Definition 1.19

The mean of a continuous random variable X with p.d.f is given by

$$E(X) = \int_{-\infty}^{\infty} xf(x)\, dx.$$

provided $\int_{-\infty}^{\infty} |x|f(x)\, dx < \infty$.

Theorem 1.22

If $g : \mathbb{R} \to \mathbb{R}$ and if $g(X)$ has a mean, then it is given by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\, dx.$$

$E(X^2) - E(X)^2$

Remark

The variance of X is given by $\sigma^2(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (E(X))^2$.

Mean variance of uniform random variable

If X is a continuous random variable with uniform distribution over $[\alpha, \beta]$

$$f(x) = \begin{cases} 0, & x < \alpha \\ 1/\beta-\alpha, & \alpha \le x \le \beta \\ 0, & x > \beta \end{cases}$$

$$E[X] = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx$$
$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{\alpha + \beta}{2}$$

$$E\left[X^2\right] = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} x^2 dx$$
$$= \frac{\beta^3 - \alpha^3}{3(\beta - \alpha)} = \frac{\beta^2 + \alpha\beta + \alpha^2}{3}$$

and so

$$\text{Var}(X) = \frac{\beta^2 + \alpha\beta + \alpha^2}{3} - \left(\frac{\alpha + \beta}{2}\right)^2 = \frac{(\beta - \alpha)^2}{12}$$

Cauchy distribution

Exercise 1.23

Let $f(x) = c/\left(1 + x^2\right), -\infty < x < \infty$ be the p.d.f. of a continuous random variable X. Find c.

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Let $f(x) = c/\left(1 + x^2\right), -\infty < x < \infty$ be the p.d.f. of a continuous random variable X. Find c.

$\int_{-\infty}^{\infty} \frac{c}{1+x^2} dx = 1 \Rightarrow$

$$\frac{1}{c} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan x|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

This density is known as the Cauchy density. The corresponding distribution function is given by

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x, \quad -\infty < x < \infty$$

Does this random variable have a mean and variance?

$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$

Gamma distribution

Definition 1.20

A continuous random variable X is said to follow the Gamma distribution with parameters $\alpha > 0, \beta > 0$ if its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0, \\ 0 & \text{elsewhere} \end{cases}$$

Recall that

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \alpha > 0.$$

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \quad \alpha > 1.$$

Hence, that $\Gamma(\alpha) = (\alpha-1)!$ when α is a positive integer. The Gamma random variable has mean $\mu = \alpha\beta$ and variance $\sigma^2 = \alpha\beta^2$.

$$\frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty x^{\alpha-1} e^{-x/\beta} dx \overset{x/\beta = y}{=} \frac{1}{} \int \qquad = \underline{1}.$$

Exponential distribution

Definition 1.21

An important spacial case of the Gamma distribution, that is when $\alpha = 1$, is called the exponential distribution. Its p.d.f. is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0, \\ 0 & \text{elsewhere} \end{cases}$$

$$P(X > a+b) = \int_{a+b}^{\infty} \lambda e^{-\lambda x} = \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_{a+b}^{\infty}$$

$$= e^{-\lambda(a+b)}$$

Lack of memory property of exponential distribution

An important property of exponentially distributed random variables is that if X is such a variable, then

$$\begin{aligned} P(X > a+b) &= e^{-\lambda(a+b)} \\ &= e^{-\lambda a}e^{-\lambda b} \\ &= P(X > a)P(X > b) \end{aligned}$$

That is, $\quad P(X > a+b \mid X > a) = P(X > b), \quad a \geq 0$ and $b \geq 0$.

$$\frac{P(X > a+b \ \& \ X > a)}{P(X > a)} = \frac{P(X > a+b)}{P(X > a)}$$