Monsoon Semester 2022-23: Tutorial 1

MA 2001D: Mathematics 3

Department of Mathematics
National Institute of Technology
Calicut

Question 1

Let X be a discrete random variable taking the values from $\{1, 2, 3, 4, 5, \cdots\}$ and let $f(x) = \frac{1}{2^x}$ be the probability mass function. Find,

(a) $P(X \geq 75)$

(b) P(X is even)

(c) P(X is divisible by 3)

Answer

(a)

$$P(X \geq 75) = \sum_{x=75}^{\infty} \frac{1}{2^x} = \frac{1}{2^{75}} + \frac{1}{2^{76}} + \cdots = \frac{1}{2^{75}}(1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots)$$

$$= \frac{1}{2^{75}}(\frac{1}{(1-\frac{1}{2})}) = \frac{2}{2^{75}} = \frac{1}{2^{74}}.$$

(b)

$$P(X = 2x) = \sum_{x=1}^{\infty} \frac{1}{2^{2x}} = \sum_{x=1}^{\infty} \frac{1}{4^x} = \frac{1}{4^1} + \frac{1}{4^2} + \cdots = \frac{1}{4^1}(1 + \frac{1}{4^1} + \frac{1}{4^2} + \cdots)$$

$$= \frac{1}{4}(\frac{1}{(1-\frac{1}{4})}) = \frac{1}{4}.\frac{4}{3} = \frac{1}{3}.$$

(c)

$$P(3 \mid X) = \sum_{x=1}^{\infty} \frac{1}{2^{3x}} = \sum_{x=1}^{\infty} \frac{1}{8^x} = \frac{1}{8^1}(1 + \frac{1}{8^1} + \frac{1}{8^2} + \cdots) = \frac{1}{8}.\frac{8}{7} = \frac{1}{7}.$$

Question 2

The probability distribution of a discrete random variable X is given by

$$\begin{array}{lcrrrrrr} X & : & -2 & -1 & 0 & 1 & 2 & 3 \\ P(X=x) & : & 0.1 & c & 0.2 & 2c & 0.3 & c \end{array}$$

Find (a) value of c (b) mean (c) variance (d) $P(X \geq 1)$ (e) $P(X < 3)$ (f) $P(-1 < X < 2)$ (g) Distribution function of X

Answer

a) Sum of all probabilities equal to 1 $\implies c = 0.1$
b) Mean $= \mu = \sum xP(x) = 0.8$
c) Variance $= \sum (x-\mu)^2 P(x) = 2.16$
d) $P(X \geq 1) = 0.2 + 0.3 + 0.1 = 0.6$
(e) $P(X < 3) = 0.9$
(f) $P(-1 < X < 2) = 0.2 + 0.2 = 0.4$
(g) Distribution function of X

X	:	-2	-1	0	1	2	3
$P(X = x)$	:	0.1	0.2	0.4	0.6	0.9	1

Question 3

Let X be a discrete random variable taking the values from $S = \{0, 1, 2, 3, 4\}$ and let the probability mass function be $f(x) = \frac{4!}{x!(4-x)!}(1/2)^4$. Find $P(A)$ if $A = \{0, 1, 2\}$.

Answer

$$P(A) = \sum_{x_i \in A} f(x_i) \quad \text{where} \quad f(x) = \frac{4!}{x!(4-x)!}\left(\frac{1}{2}\right)^4 \tag{1}$$

$$= f(0) + f(1) + f(2) \tag{2}$$

$$= \frac{1}{2^4} + \frac{4}{2^4} + \frac{6}{2^4} = \frac{11}{16} \tag{3}$$

$$\tag{4}$$

Question 4

The p.d.f of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x+1}{k}, & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find k.

(b) Obtain the distribution function $F(X)$

(c) Compute $P(X \geq 2)$

(d) Find Mean and Variance of X

(e) Find $P(2 < X < 3.5)$

Answer I

(a)

$$\int_x f(x)dx = 1 \implies \int_2^4 \frac{x+1}{k} = 1 \implies \frac{1}{k}[\frac{x^2}{2} + x]_2^4 = 1 \implies k = 8.$$

(b)

$$F(x) = \int_2^x f(x)dx = \int_2^x \frac{x+1}{8}dx = \frac{1}{8}[\frac{x^2}{2} + x]_2^x = \frac{x^2+2x-8}{16}.$$

(c) $p(x \geq 2) = 1 - p(x < 2) = 1 - F(2) = 1 - 0 = 1.$

(d)

$$E(x) = \int xf(x)dx = \int_2^4 x(\frac{x+1}{8})dx = \frac{37}{12}.$$

$$E(X^2) = \int x^2 f(x)dx = \int_2^4 x^2(\frac{x+1}{8})dx = \frac{59}{6}.$$

$v(x) = E(X^2) - (E(X))^2 = \frac{47}{144}.$

Answer II

(e)

$$p(2 < x < 3.5) = F(3.5) - F(2) = \frac{(3.5)^2 + 2(3.5) - 8}{16} = \frac{11.25}{16}.$$

Question 5

If the density function of a random variable is

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x \le c \\ 0, & \text{otherwise} \end{cases}, \text{ find a) } c, \text{ b) } F(x), \text{ c) } P(0.5 < X < 0.6c).$$

Solution.

(a) Using the fact that $\int_0^1 x\,dx0 + \int_1^c (2 - x)\,dx = 1$, we get $c^2 - 4c + 4 = 0$. Therefore, $c = 2$.

(b) $F(x) = \int_0^x f(t)\,dt$. Therefore, $F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{2}, & 0 < x < 1 \\ \frac{-x^2+4x-2}{2}, & 1 \le x \le 2 \\ 1, & x > 2 \end{cases}$.

(c) $P(0.5 < X < 1.2) = \int_0^1 x\,dx + \int_1^{1.2} (2 - x)\,dx = 0.555.$

Question 6

If the probability density function of X is given by $f(x) = a + bx^2$, $0 \le x \le 1$ and $f(x) = 0$, otherwise with $E(X) = \frac{3}{5}$, then find a and b.

Answer

$\int\limits_x f(x)dx = 1 \implies \int\limits_0^1 a + bx^2 dx = 1 \implies (ax + \frac{bx^3}{3})_0^1 = 1 \implies 3a + b = 3$

Now, $\int\limits_x xf(x)dx = 3/5 \implies \int\limits_0^1 x(a + bx^2)dx = 3/5 \implies (a\frac{ax^2}{2} + \frac{bx^4}{4})_0^1 = 3/5 \implies 10a + 5b = 12$

Thus, $a = 3/5$ and $b = 6/5$.

Question 7

Let,

$$f(x) = \begin{cases} \frac{1}{x^2}, & 1 < x < \infty \\ 0, & otherwise. \end{cases}$$

Be the pdf of X.
If A=$\{1 < X < 2\}$ B=$\{\frac{1}{4} < X < 5\}$, then
find $(a)P(A), (b)P(B), (c)P(AUB), (d)P(AnB)$

Answer

(a) $\int_1^2 f(x)dx$.
(b) $\int_{\frac{1}{4}}^5 f(x)dx. = \int_1^5 f(x)dx$.
(c) $\int_{\frac{1}{4}}^5 f(x)dx. = \int_1^5 f(x)dx$.
(d) $\int_1^2 f(x)dx$.

Question 8

Let the continuous random variable X have probability density function

$$f(x) = \begin{cases} \frac{2}{x^3}, & 1 < x < \infty \\ 0, & \textit{otherwise}. \end{cases}$$

Find the distribution function $F(x)$.

Answer

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

When $x \leq 1$, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0dt = 0.$

When $1 < x < \infty$,

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{1} f(t)dt + \int_{1}^{x} f(t)dt = 0 + \int_{1}^{x} \frac{2}{t^3} = 1 - \frac{1}{x^2}.$$

Thus, $F(x) = \begin{cases} 0 & x \leq 1, \\ 1 - \dfrac{1}{x^2} & 1 \leq x \leq \infty. \end{cases}$

Question 9

The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - (1 + x)e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the corresponding density function of X.

Answer

$$
\begin{aligned}
f(x) &= \frac{d}{dx}F(x) \\
&= e^{-x} + xe^{-x} - e^{-x} \\
&= xe^{-x}
\end{aligned}
$$

Question 10

A continuous random variable has the probability density function given by

$$f(x) = \begin{cases} kxe^{-kx} & , \quad x \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Determine (i) k (ii) Mean and (iii) Variance.

Answer I

1 We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\implies \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\implies k\int_{0}^{\infty} xe^{-kx}\,dx = 1$$

Integrating we get $k = 1$

$$2 \qquad \text{Mean}, \mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \int_{0}^{\infty} x^2 e^{-x}\,dx = 2$$

$$3 \qquad \text{Variance}, \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= \int_{-\infty}^{\infty} x^3 e^{-x}\,dx - 4 = 2$$

Question 11

Obtain the moment generating function of the random variable X having probability density function given by

$$f(x) = \begin{cases} x, \text{ if } 0 \leq x \leq 1 \\ 2 - x, \text{ if } 1 < x \leq 2 \\ 0, \text{ otherwise} \end{cases}$$

Solution

The moment generating function of X is given by

$$\begin{aligned}
M_x(t) &= \mathbb{E}[e^{tx}] \\
&= \int_0^{\infty} e^{tx} f(x) dx \\
&= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx}(2-x) dx + 0 \int_2^{\infty} e^{tx} dx \\
&= \left[x \frac{e^{tx}}{t} \right]_0^1 - \int_0^1 \frac{e^{tx}}{t} dx + \left[(2-x) \frac{e^{tx}}{t} \right]_2^1 + \int_1^2 \frac{e^{tx}}{t} dx \\
&= \frac{e^t}{t} - 0 - \left[\frac{e^{tx}}{t^2} \right]_0^1 + 0 - \frac{e^t}{t} + \left[\frac{e^{tx}}{t^2} \right]_1^2 \\
&= -\frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t^2} \\
&= \frac{e^{2t} - 2e^t + 1}{t^2}
\end{aligned}$$

Question 12

Let X be a discrete random variabe and $f(x) = ab^x$ where a and b are positive numbers such that $a + b = 1$ and X takes the values $0, 1, 2...$ .Find the moment generating function of X.

Solution

$$
\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \sum_{x=0}^{\infty} e^{tx} P(x) \\
&= a + abe^t + a(be^t)^2 + \dots \\
&= \frac{a}{(1 - e^t b)} \text{ provided } \ e^t b < 1
\end{aligned}
$$

Question 13

The random variable x denote the number obtained when a fair die is thrown. Prove that the Chebyshev's inequality gives $P(|x - 3.5| \geq 2.5) \leq 0.47$. Compare this with the actual probability.

Solution

Here x is a random variable which takes the values $1, 2, \ldots, 6$, each with probability $1/6$. Hence

$$E(x) = \frac{1}{6}(1 + 2 + \ldots + 6) = \frac{7}{2}.$$
$$E(x^2) = \frac{1}{6}(1^2 + 2^2 + \ldots + 6^2) = \frac{91}{6}.$$

$\therefore Var(x) = E(x^2) - (E(x))^2 = 2.9167$. For $k > 0$, Chebychev's inequality gives

$$P(|x - E(x)| > k) < \frac{Varx}{k^2}$$

Choosing $k = 2.5$, we get

$$P(|x - \mu| > 2.5) < 0.47$$

The actual probability is given by

$$p = P(|x - 3.5| > 2.5)$$

$$= P\,(\text{x lies outside the limits}(3.5 - 2.5, 3.5 + 2.5), i.e., (1, 6))$$

But since x is the number on the dice when thrown, it cannot lie outside the limits of 1 and 6. $\therefore p = p(\phi) = 0$.

Question 14

Let X be a continuous random variable having probability density function $f(x) = \frac{1}{4}, -1 < x < 3$ and $f(x) = 0$, otherwise. Establish the Chebyshev's inequality and compare the result with actual probability.

Solution. It is easy to calculate that $\mu = \mathbb{E}(X) = 1$ and $\mathbb{E}(X^2) = \frac{7}{3}$.
Thus, $Var(X) = \sigma^2 = \frac{4}{3}$ and hence $\sigma = 1.1547$.
Hence by Chebyshev's inequality, we have $P(|X - 1| \geq 1.1547k) \leq \frac{1}{k^2}$.
For $k = 2$, we get $P(|X - 1| \geq 2.3094) \leq \frac{1}{4}$. But, $|X - 1| \geq 2.3094$ implies $X \in (-\infty, -1.30494) \cup (3.30494, \infty)$. Clearly $P(X) = 0$ from the definition.

Question 15

The random variable X has the density function

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

Using Chebyhev's inequality find an approximate value of $P\{|X - 0.5| < 2\}$. Also find the actual probability.

Solution

Chebyshev's inequality is given by
$P\{|X-\mu| \leq k\sigma\} \geq 1-\frac{1}{k^2}$
Since the random variable follows exponential distribution
$E(X)=\frac{1}{\lambda}=\frac{1}{2}$
Similarly, $\sigma=\frac{1}{\lambda}=\frac{1}{2}$
Hence $k=4$. Therefore, $P\{|X-0.5|<2\} \geq 1-\frac{1}{16}=0.9375$.
Actual probability is calculated as follows:

$$
\begin{aligned}
P\{|X-0.5|<2\} &= P\{-1.5<X<2.5\} \\
&= \int_{-1.5}^{2.5} f(x)dx \\
&= \int_{0}^{2.5} f(x)dx \\
&= \int_{0}^{2.5} 2e^{-2x}dx \\
&= 0.9932
\end{aligned}
$$

Question 16

Boxes of bolts have an average of 100 bolts with a S.D of 3 . Find an approximate probability that the number of bolts in a box is between 95 and 105 , using Chebyshev's inequality.

Solution

Using Chebychev's inequality, we get $P[|S - E(S)| < k\sigma] \geq 1 - \frac{1}{k^2}$
$\Rightarrow \quad P||S - 100 \;|< 3k| \geq 1 - \frac{1}{k^2}$
$\Rightarrow \quad P \mid 100 - 3k < S < 100 + 3k| \geq 1 - \frac{1}{k^2}$ Taking $\quad k = \frac{5}{3}$, we get
$P(95 \leq S \leq 105) \geq 1 - \frac{1}{25/9} = \frac{16}{9}$

Question 17

With usual notations, Find p of the Binomial distribution, If $n = 6$ and $P(X = 4) = P(X = 2)$.

Solution

Given : $n = 6$ and $P(X = 4) = P(X = 2)$.
To find : p

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(X = 4) = P(X = 2)$$

$$\implies \binom{6}{4} p^4 (1-p)^2 = \binom{6}{2} p^2 (1-p)^4$$

$$\implies \qquad p = \frac{1}{2}.$$

Question 18

The mean and variance of a Binomial distribution are respectively given by 4 and 3. Find (a) Obtain the binomial distribution (b) Find $P(X \geq 2)$ (c) $P(1 \leq X \leq 3)$ (d)$P(X \leq 1)$.

Answer

Given: $np = 4$ and $npq = 3$
Hence $q = \frac{3}{4}$, $p = \frac{1}{4}$ and $n = 16$
(a) The Binomial distribution is
$f(x) = P(X = x) = ^{16} C_x (\frac{1}{4})^x (\frac{3}{4})^{n-x}; x = 0, 1, 2, ...16.$
(b)$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$
(c) $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$
(d) $P(X \leq 1) = P(X = 0) + P(X = 1)$

Question 19

A coin is tossed until tail appears. What is the expected value of number of tosses?

Solution

Let X_i denote the random variable that a tail appears at the i^{th} toss where $i = 1, 2, 3, ...$ The distribution function will be:

$$\begin{array}{ccccc} X_i : & 1 & 2 & 3 & 4 \\ P(X_i) : & \frac{1}{2} & \left(\frac{1}{2}\right)^2 & \left(\frac{1}{2}\right)^3 & \left(\frac{1}{2}\right)^4 \end{array}$$

$$\begin{aligned} E(X) &= \sum x_i P(X = x_i) \\ &= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + ... \\ &= \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} \\ &= 2 \end{aligned}$$

Question 20

If the probability of success is 0.04, how many trials are necessary in order that the probability of at least one success is greater than 0.5?

Answer I

Let X = number of successes.
p = probability of success in a single trial
$\therefore p = 0.04$ and $q = 1 - p = 1 - 0.04 = 0.96$
$\therefore X \sim B(n, 0.04)$
The p.m.f of X is given by
$P(X = x) = \binom{n}{x} p^x q^{n-x}$
i.e ${}^nC_0 (0.04)^x (0.96)^{n-x}$

$$\begin{aligned} P(\text{at least one success}) &= P(X \geq 1) = 1 - P(X < 1) \\ &= 1 - P(X = 0) = 1 - p(0) \\ &= 1 -^n C_0 (0.04)^0 (0.96)^{n-0} \\ &= 1 - 0.96^n \end{aligned}$$

Answer II

Given: $P(X \geq 1) > 0.5$

$$
\begin{aligned}
i.e \quad & 1 - 0.96^n > 0.5 \\
i.e \quad & 1 - 0.5 > 0.96^n \\
i.e \quad & 0.5 > 0.96^n \\
i.e \quad & nlog(0.96) < log(0.5) \\
i.e \quad & n < \frac{log 0.5}{log 0.96} \\
i.e \quad & n < 16.97 \\
& n = 16
\end{aligned}
$$

Question 21

Six dice are thrown 729 times. How many times do you expect at least 3 dice show a 5 or 6?

Answer

The probability of showing 5 or 6 in a single throw of a die is $\frac{1}{3}$. When six dice are thrown, the probability that at least 3 dice will show 5 or 6 is

$$1 - \left[{}^6C_0\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0 + {}^6C_1\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1 + {}^6C_2\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2\right]$$

$$= 1 - \left(\frac{2}{3}\right)^4\left[\frac{4}{9} + \frac{12}{9} + \frac{15}{9}\right] = \frac{233}{729}$$

Treating this (when six dice are thrown, at least 3 dice will show 5 or 6) as Success in a Bernoulli trail, with $p = \frac{233}{729}$, the expected number of times at least 3 dice show a 5 or 6 if the trial is repeated 729 times is

$$E(X) = np = 729 \times \frac{233}{729} = 233.$$

Question 22

In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 2000 such samples how many would be expected to contain at least 3 defectives?

Answer

X = random variable of number of defectives
$p = \frac{2}{20} = 0.1, q = 0.9$

$$
\begin{aligned}
P(\text{at least 3 defectives}) &= P(X \geq 3) \\
&= 1 - P(X < 3) \\
&= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\
&= 1 - \binom{20}{0}(0.1)^0(0.9)^{20} - \binom{20}{1}(0.1)^1(0.9)^{19} \\
&- \binom{20}{2}(0.1)^2(0.9)^{18} \\
&= 1 - \{0.1216 + 0.2702 + 0.2852\} \\
&= 0.323
\end{aligned}
$$

Out of 2000 samples, $2000 \times 0.323 = 646$ samples are expected to contain at least 3 defectives.

Question 23

A production process is known to produce 20% defectives. The manager decides to stop the process for adjustment if a random sample of 6 units selected with replacement yields 2 or more defectives. What is the probability that the process will be stopped?

Answer

Let X be the random variable of the number of defectives.
$X \sim \mathrm{B}(n, p)$. Here probability of success, $p = \frac{20}{100} = 0.2$ and probability of failure, $q = 1 - p = 0.8$. Also, sample size, $n = 6$. Then,

$$
\begin{aligned}
P(\text{process will be stopped}) &= P(X \geq 2) \\
&= 1 - P(X < 2) \\
&= 1 - [P(X = 0) + P(X = 1)] \\
&= 1 - \left[\binom{6}{0}(0.2)^0(0.8)^6 + \binom{6}{1}(0.2)^1(0.8)^5\right] \\
&= 1 - [0.2621 + 0.3932] \\
&= 0.3447
\end{aligned}
$$

Question 24

The probability that a man hits the target is $1/5$. How many times must he fire, so that the probability of hitting the target at least once is more that 90percentage?

Answer

Probability of man hitting a target $= 1/5$
Here $p = 1/5$
The probability of not hitting the target in n trails is q^n
Therefore to find the smallest n of which the probability of hitting at once in more than 90 percentage

$$1 - q^n > 0.9$$

$$1 - (4/5)^n > 0.9$$

$$(5/4)^n > 10$$

$$n > 7$$

Therefore smallest $n = 8$

Question 25

If $M_x(t) = e^{3(e^t-1)}$, identify the distribution of the random variable X and hence find $P(X = 2)$ and $P(X \geq 2)$.

Solution

We can match this MGF to a known MGF of one of the distributions we considered. Observe that $m(t) = e^{3(e^t-1)} = e^{\lambda(e^t-1)}$, where $\lambda = 3$. Thus $X \sim \text{Poisson}(3)$, and therefore

$$P(X = 2) = e^{-\lambda}\frac{\lambda^2}{2!} = e^{-3}\frac{3^2}{2!} \approx 0.224.$$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - e^{-3}\frac{3^0}{0!} - e^{-3}\frac{3^1}{1!} \approx 0.8$$

Question 26

The monthly breakdowns of a computer system is a random variable X having Poisson distribution with $E\left(X^2\right) = 15$. Find the probability that the computer will function for a month (i) without breakdown. (ii) with exactly one breakdown.

Answer

Let X denote the monthly breakdowns of a computer system.
$X \sim \text{Poisson}(\lambda)$
$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, ...$
$E(X^2) = 15$
$E(X^2) = Var(X) + [E(X)]^2$
For Poisson distribution, $E(X) = V(X) = \lambda$
$\implies \qquad E(X^2) = \lambda + \lambda^2 = 15$
i.e $\qquad \lambda^2 + \lambda - 15 = 0$
$\implies \lambda = \frac{-1 \pm \sqrt{61}}{2}$
$E(X) = \frac{-1+\sqrt{61}}{2}$is $+$ve, since X denotes the number of monthly breakdowns.Therefore

1. $P(\text{Without breakdown}) = P(X = 0) = \frac{e^{-\frac{-1+\sqrt{61}}{2}}(\frac{-1+\sqrt{61}}{2})^0}{0!} = e^{\frac{1-\sqrt{61}}{2}}$
2. $P(\text{Exactly one breakdown}) = P(X = 1) = \frac{e^{-\frac{-1+\sqrt{61}}{2}}(\frac{-1+\sqrt{61}}{2})^1}{1!} = (\frac{-1+\sqrt{61}}{2})e^{\frac{1-\sqrt{61}}{2}}$.

Question 27

A company owns 400 laptops. Each laptop has an 9% probability of not working. If you randomly select 20 laptops for your sales people, then
(a) What is the likelihood that 5 will be broken?
(b) What is the likelihood that they will all work?
(c) What is the likelihood that they will all broken?

Solution

Binomial Problem with n = 20 and p = 0.08
(a) $20C_5(0.08)^5(0.92)^{15} = .0145$
(b) $20C_0(.08)^0(.92)^{20} = .1887$
(c) $20C_{20}(.08)^{20}(.92)^0 = .000000000000000000000001$

Question 28

If X and Y are binomial random variables with respective parameters (n, p) and $(n, (1-p))$ verify the following

i) $P(X \leq i) = P(Y \geq n - i)$

ii) $P(X = k) = P(Y = n - k)$

Answer

(ii)

$$\begin{aligned} P(X=k) &= {}^{n}C_{k}\,(p)^{k}\,(1-p)^{n-k} \\ &= {}^{n}C_{n-k}\,(p)^{n-(n-k)}\,(1-p)^{n-k} \\ &= {}^{n}C_{n-k}\,(1-p)^{n-k}\,(p)^{n-(n-k)} \\ &= P(Y=n-k). \end{aligned}$$

(i) $P(X \leq i) = P(X=0) + P(X=1) + ... + P(X=i)$.
Now

$$\begin{aligned} P(Y \geq n-i) &= P(Y=n-i) + P(Y=n-(i-1)) + ... + P(Y=n) \\ &= P(X=i) + P(X=i-1) + ... + P(X=0) \quad from(ii) \\ &= P(X \leq i). \end{aligned}$$

question 29

At least one half of the aeroplane engines are required to function in order for it to operate. If each engine functions independently with a probability p, for what values of p, is a 4 engine plane more likely to operate than a 2 engine plane?

Answer

$p =$ P(Engine function), $p - 1 =$ P(Engine doesn't function).
Given that at least one half of the aeroplane engines are required to function in order for it to operate,
Let F- Engine function and N -Engine doesn't function, then.
P(4 engines planes works)= P{ FFFF, FFFN, FFNF, FNFF, NFFF FFNN, FNNF, NNFF, NFNF, FNFN, NFFN}

$$= p^4 + 4p^3(1-p) + 6p^2(1-p)^2$$

P(2 engines planes works)= P{ FF, FN, NF}

$$= p^2 + 2p(1-p)$$

4 engine plane more likely to operate than a 2 engine plane, therefore

$$p^4 + 4p^3(1-p) + 6p^2(1-p)^2 > p^2 + 2p(1-p)$$
$$p^4 + 4p^3 - 4p^4 + 6p^2 - 12p^3 + 6p^4 - p^2 - 2p + 2p^2 > 0$$
$$3p^4 - 8p^3 + 7p^2 - 2p > 0 \implies 3p^3 - 8p^2 + 7p - 2 > 0$$

No $p \in (0,1)$ satisfies this condition. Thus, always 2 engine plane works better than 4 engine plane.

Question 30

In a given city, 5% of all drivers get at least one parking ticket per year. Use the Poisson approximation to the binomial distribution to determine the probabilities that among 80 randomly chosen drivers in this city,
a) 4 will get at least one parking ticket in any given year;
b) at least 4 will get at least one parking ticket in any given year;
c) anywhere from 3 to 7 inclusive, will get at least one parking ticket in any given year.

Answer

Probability that a driver get at least one parking ticket per year, $p = 5\% = 0.05$. Sample Size, $n = 80$.

Let X denote the number of drivers that get at least one parking ticket per year, then X follows binomial distribution with parameters n=80 and p=0.05.

Now, X can be approximated by Poisson with mean of $X = np = 80x0.05 = 4$

(a) Probability that 5 will get at least one parking ticket per year $= P(X = 5)$

Using Poisson approximation to Binomial, $P(X = 5) = \frac{e^{-\lambda}\lambda^5}{5!} = \frac{e^{-4}4^5}{5!} = 0.1563$

(b) Probability that at least 4 will get at least one parking ticket per year $= P(X \geq 4)$

Using Poisson approximation to Binomial,

$P(X \geq 4) = 1 - P(X < 4)$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$

$= 1 - [\frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!}] = 0.5665$

(c) Probability that anywhere from 3 to 6 inclusive,will get at least one parking ticket per year $= P(3 \leq X \leq 6)$.
Using Poisson approximation to Binomial,
$P(3 \leq X \leq 6) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$
$= \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!} + \frac{e^{-4}4^5}{5!} + \frac{e^{-4}4^6}{6!} = 0.6512$

Question 31

In one out of 6 cases, material for bulletproof vests fails to meet puncture standards. If 405 specimens are tested, what does Chebyshev's theorem tell us about the probability of getting at most 30 or at least 105 cases that do not meet puncture standards?

Answer

$mean = 405\frac{1}{6} = 67.5$, SD $=[405\frac{1}{6}(1-\frac{1}{6})]^{0.5} = 7.5$
$z = \frac{(30-67.5)}{7.5} = -5$, $z = \frac{(105-67.5)}{7.5} = 5$
so k = 5(k must be greater than 1.) chebyshev theorem : The fraction of any set of numbers lying within k ,standard deviations of those numbers of the mean of those numbers is at least $= 1 - \frac{1}{5^2} = 0.96$.
$P(30 < x < 105) >= 0.96$ so :
$P(x <= 30 or x >= 105) = 1 - P(30 < x < 105) < 1 - 0.96 = 0.04$
so this tells that $P(x <= 30 or x >= 105) < 0.04$

Question 32

A bombing plane carrying three bombs flies directly above a railroad track. If a bomb falls within 40 meters of track, the track will be sufficiently damaged to disrupt the traffic. With a certain bomb site the point of impact of a bomb have the probability density function

$$f(x) = \begin{cases} \frac{100+x}{10000}, & -100 \leq x < 0 \\ \frac{100-x}{10000}, & 0 \leq x < 100 \\ 0, & \text{elsewhere} \end{cases}$$

where x represents the vertical deviation (in meter) from the aiming point, which is the track in this case. Find the cumulative distribution function. If all the tree bombs are used, what is the probability that the track will be damaged?

Answer

(a) Given that the bomb fall within 40m so it fall in the range of 0m to 100m Therefore probability density function (p.d.f) is $f(x) = \frac{100-x}{10000}$

A bombing plane flies directly above a railroad track. Assume that if a large(small) bomb falls within 40(15) feet of the track, the track will be sufficiently damaged so that traffic will be disrupted.

Let x denotes the vertical deviation (in meter) from the aiming point.

The probability that a large bomb will disrupt traffic is,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{40} \frac{100-x}{10000}dx = \frac{1}{10000}((100 \times 40) - \frac{(40)^2}{2}) = \frac{8}{25}$$

The probability that a small bomb will disrupt traffic is,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{15} \frac{100-x}{10000}dx = \frac{1}{10000}((100 \times 15) - \frac{(15)^2}{2}) = \frac{111}{800}$$

(b) If the plane can carry three large (eight small) bombs and uses all three (eight),

The probability that traffic will be disrupted is

Large bomb:

We use binomial distribution to find the probability

$X \sim B(n = 3, p = \frac{8}{25} = 0.32)$

$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} : x = 0, 1, 2, 3.$

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - 0.314432 = 0.685568$

small bomb:

We use binomial distribution to find the probability

$X \sim B(n = 8, p = \frac{111}{800} = 0.13875)$

$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} : x = 0, 1, 2, 3.$

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - 0.302715 = 0.67285$

Question 33

A multiple choice test consists of 8 questions with 3 answers to each question (of which only one is correct). A student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 and the third answer if he gets 5 or 6. To get a distinction, the student must secure at least 75% correct answers. If there is no negative marking, what is the probability that the student secures a distinction?

Solution.

Probability of getting correct answer, $p = 2/6 = 1/3$ and probability of getting wrong answer, $q = 4/6 = 2/3$. Therefore, probability of getting x correct answers is $P(X = x) =^n C_x p^x q^{n-x}$.

To get distinction, 75% correct answers should be there. Hence, we need to find $P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8)$. By calculation, we get $P(X \geq 6) = \frac{43}{2187} = 0.01966$.

Question 34

A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of each day on which

(a) Neither car is used

(b) Some demand is refused.

Solution

Let X denote number of cars hired out per day.
Poisson distribution mean = m = 1.5
$P(X = x) = \frac{e^{-m}m^x}{x!} = \frac{e^{-1.5}(1.5^x)}{x!}$
(a) P(neither car is used): $P(X = 0) = \frac{e^{-1.5}1.5^0}{0!} = 0.2231$
(b) P(Some demand is refused) = P(Demand is more than 2 cars per days):

$$\begin{aligned} P(x > 2) &= 1 - P(x \leq 2) \\ &= 1 - [P(x = 0) + P(x = 1) + P(x = 2)] \\ &= 1 - \left[\frac{(e^{-1.5})(1.5^0)}{0!} + \frac{(e^{-1.5})(1.5^1)}{1!} + \frac{(e^{-1.5})(1.5^2)}{2!}\right] \\ &= 1 - e^{-1.5}\left[1 + 1.5 + \left(\frac{2.25}{2}\right)\right] = 0.1912 \end{aligned}$$

Proportion of days on which neither car is used= 0.2231= 22.31 %
Proportion of days on which some demand is refused =0.1912=19.12 %

Question 35

A boy is throwing stones at a target what is the probability that the 4 th attempt is the first hit, if the probability of hitting target is .3 ?

Question 36

The mean yield for one-acre plot is 662 kilos with a standard deviation 32 kilos. Assuming normal distribution, how many one acre plot in a batch of 1000 plots would you expect to have yield

a. Over 705 kilos

b. Below 640 kilos.

Answer

Question 37

Find the m.g.f of exponential distribution and hence find mean and variance.

The moment-generating function (mgf) of a random variable X is given by

$$M_X(t) = E[e^{tX}], \quad \text{for } t \in \mathbb{R}.$$

PDF of Exponential distriution is

$$p_X(x) = \lambda \cdot e^{-\lambda x}$$

for $x > 0$, and 0 for $x \leq 0$. Deriving the MGF:

$$\begin{aligned}
M_X(t) &= \mathbb{E}\left[e^{tX}\right] \\
&= \int_{-\infty}^{\infty} x \cdot p_X(x) dx \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx \\
&= \int_{0}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx \\
&= \lambda \int_{0}^{\infty} e^{tx} \cdot e^{-\lambda x} dx \\
&= \lambda \int_{0}^{\infty} e^{tx - \lambda x} dx
\end{aligned}$$

$$= \lambda \int_0^\infty e^{x(t-\lambda)} dx$$
$$= \lambda \cdot \frac{1}{\lambda - t}$$
$$= \frac{\lambda}{\lambda - t}$$

Getting moments of exponential distributions by derivating MGF

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

First moment (expectation)

$$M_X^{(1)}(t) = \frac{\partial}{\partial t}\left(\frac{\lambda}{\lambda - t}\right) = \frac{\lambda}{(\lambda - t)^2}$$

And evaluate at $t = 0 : \left.\frac{\lambda}{(\lambda-t)^2}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

Second moment

$$M_X^{(2)}(t) = \frac{\partial^2}{\partial^2 t}\left(\frac{\lambda}{\lambda - t}\right) = \frac{2\lambda}{(\lambda - t)^3}$$

$\frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = \frac{2}{\lambda^2}$

$\text{Var}(X) = E\left(X^2\right) - E(X)^2 = M_X^{(2)}(0) - \left[M_X^{(1)}(0)\right]^2 = \frac{1}{\lambda^2}$

Question 39

If two random variables X_1 and X_2 have the joint density function given by

$$f(x_1, x_2) = \begin{cases} x_1 x_2, \ \ 0 < x_1 < 1, \ \ 0 < x_2 < 2 \\ 0, \ \text{otherwise} \end{cases}$$

Find the probability that
(a) Both random variables will take on values less than 1
(b) The sum of the values taken on by the two random variables will be less than 1.

Solution