Binomial Distribution

Definition 1.14

For some $n \in \mathbb{N}$ and $0 < p < 1$

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \ldots, n$$

.

Remark

$b(x; n, p) = b(\cancel{x}; n, 1-p)$

$n-x$

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1$$

Expectation of binomial distribution

Theorem 1.13

The expectation of binomial distribution $b(x; n, p)$ is np.

Proof

$$E(x) = \mu = \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$(n-1)-(x-1)$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

$((n-1)-(x-1))!$

$m = n-1$

$y = x-1$

$$= np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y}$$

$$= np$$

Exercise

At least one-half of an airplane's engines are required to function in order for it to operate. It is known that each engine independently functions with probability 0.4.

Exercise 1.14

Find the probability for a successful flight of a two engine plane.

$x = 0, 1, 2$ # number of engines functioning.

$P(X=x) = \binom{2}{x} (0.4)^x (0.6)^{2-x}$

$P(X \geq 1) = P(X=1) + P(X=2) =$

$= 1 - P(X=0)$

$= 1 - (0.6)^2 = \underline{\underline{0.64}}$

Exercise

At least one-half of an airplane's engines are required to function in order for it to operate. It is known that each engine independently functions with probability 0.4.

Exercise 1.14

Find the probability for a successful flight of a two engine plane.

Exercise 1.15

Find the probability for a successful flight of a four engine plane.

$P(X=x) = \binom{4}{x}(0.4)^x(0.6)^{4-x}, \quad x = 0, 1, 2, 3, 4.$

$P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - (0.6)^4 - 4(0.4)(0.6)^3$

$=$

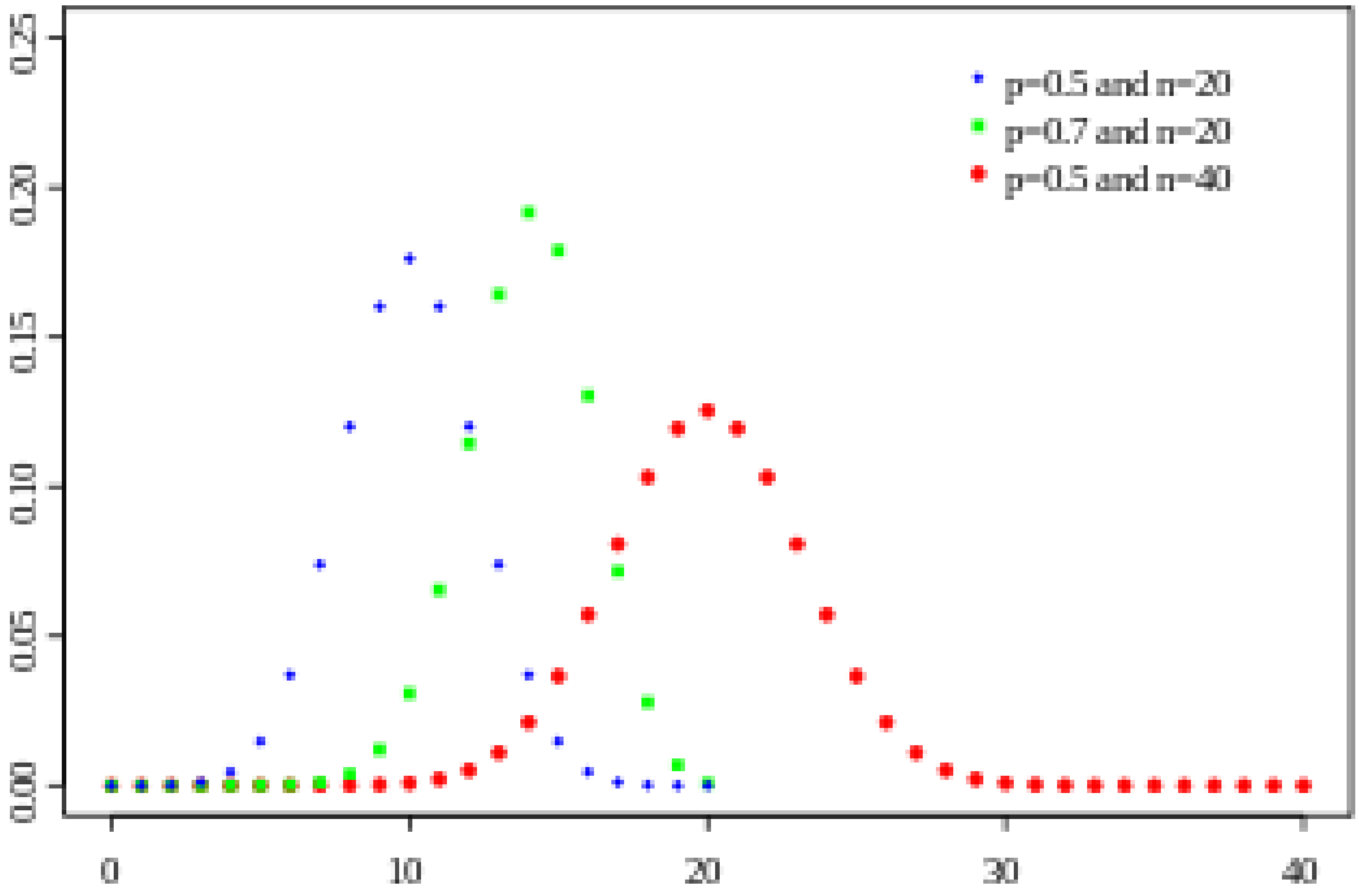
Variance of binomial distribution

$$
\begin{aligned}
E(X(X-1)) &= \sum_{x=0}^{n} x(x-1)\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} \\
&= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!}p^x(1-p)^{n-x} \\
&= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!}p^{x-2}(1-p)^{n-x} \\
&= n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!(m-y)!}p^{y}(1-p)^{m-y} \\
&= n(n-1)p^2(p+(1-p))^m = n(n-1)p^2
\end{aligned}
$$

$E(X) = E(X(X-1)) + E(X)$

$$
\begin{aligned}
\text{Hence } \sigma^2(X) &= E\left(X^2\right) - E(X)^2 = E(X(X-1)) + E(X) - E(X)^2 \\
&= n(n-1)p^2 + np - (np)^2 = [np(1-p)]
\end{aligned}
$$

The shape of the Binomial distribution



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Hypergeometric distribution

Definition 1.15

For $a, n, N \in \mathbb{N}$ such that $a \leq N, n \leq N$,

$$h(x; n, a, N) = \frac{\begin{pmatrix} a \\ x \end{pmatrix} \begin{pmatrix} N - a \\ n - x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}} \quad \text{for } x = 0, 1, \ldots, n.$$

Hypergeometric distribution

Definition 1.15

For $a, n, N \in \mathbb{N}$ such that $a \leq N, n \leq N$,

$$h(x; n, a, N) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}} \quad \text{for } x = 0, 1, \ldots, n.$$

Remark

$h(x; n, a, N) = h(x; a, n, N)$

Hypergeometric distribution

$h(x; n, a, N)$ represents the probability of obtaining x items of Type I while choosing n items, without replacement, from a collection of N items of which a are of Type I (Success) and the remaining $N - a$ are of type II (Failure). Since items are picked without replacement, this distribution is different from Binomial distribution. But if n is small compared to N, the difference (between picking with replacement and without replacement) is not significant and so the binomial distribution with the parameters n and $p = \frac{a}{N}$ will be a good approximation to $h(x; n, a, N)$.