Expectation of a function of a random variable

Theorem 1.8

Let X be a random variable and let a and b be constants. Then

1. $\mathbf{E}(aX + b) = a\mathbf{E}(X) + b$ | Note $a = 0 \Rightarrow E(b) = b$
2. *If* $\mathbf{P}(a < X \leq b) = 1$, *then* $a < \mathbf{E}(X) \leq b$
3. *If* $g(X)$ *and* $h(X)$ *have (finite) means, then*

$$\mathbf{E}(g(X) + h(X)) = \mathbf{E}(g(X)) + \mathbf{E}(h(X))$$

① $E(aX+b) = \sum_x (ax+b) f(x) = a \sum_x x \cdot f(x) + b \sum_x f(x)$

$= a \cdot E(X) + b \cdot 1 = a\,E(X) + b.$

② $E(X) = \sum_x x \cdot f_X(x) = \sum_{x<a} x \cdot f_X(x) + \sum_{a \leq x \leq b} x \cdot f_X(x) + \sum_{x>b} x \cdot f_X(x).$

$a = a \sum_x f_X(x) \leq = \sum_{a \leq x \leq b} x \cdot f_X(x) \leq b \cdot \sum_x f_X(x) = b.$

Expectation of the indicator random variable

Example 1.9

Suppose that X is the indicator random variable of an event A. That is

$$X_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

Then $\mathbf{E}(X) = 0.f_X(0) + 1.f_X(1) = \mathbf{P}(A)$

$$\left(P(X=1) = P(\{a \in S : X(a) = 1\}) = P(A) \right)$$

Exercise

Exercise 1.9

Let X have the p.d.f

$$f_X(x) = a\frac{\cancel{4}\,1}{x(x+1)(x+2)}, \quad x = 1, 2, \ldots$$

$$\sum_{1}^{\infty} \frac{1}{x(x+1)(x+2)} = S_1$$

$$a = 1/S_1$$

and let Y have the p.d.f

$$f_Y(x) = b\frac{1}{x(x+1)}, \quad x = 1, 2, \ldots$$

$$\sum_{x=1}^{\infty} \frac{1}{x(x+1)} = S_2$$

$$b = \frac{1}{S_2}$$

for some constants a and b. Which of these have an expected value.

$$\sum_{x=1}^{\infty} \frac{1}{(x+1)(x+2)} < \infty$$

but $\sum_{x=1}^{\infty} \frac{1}{(x+1)} \nless \infty.$

Variance

Theorem 1.10

If $\mathbf{E}(X)$ *and* $\mathbf{E}(X^2)$ *exist, then* $(\mathbf{E}(X))^2 \leq (\mathbf{E}(|X|))^2 \leq \mathbf{E}\left(X^2\right)$

Proof

First, note that $(|X| - \mathbf{E}(|X|))^2 \geq 0$. Hence, Theorem 1.8

$0 \leq \mathbf{E}((|X| - \mathbf{E}(|X|)^2) = \mathbf{E}\left(|X|^2 + \mathbf{E}(|X|)^2 - 2\mathbf{E}(|X|)\cdot\mathbf{E}(|X|)\right)$

$$= \mathbf{E}\left(X^2\right) - (\mathbf{E}(|X|))^2$$

which proves the second inequality. Also, $|X| - X \geq 0$, hence, $\mathbf{E}(X) \leq \mathbf{E}(|X|)$

$E(|X| - X) \geq 0$

$E(|X|) - E(X) \geq 0$

$E(|X|)^2 \geq E(X)^2 \Leftarrow E(|X|) \geq E(X)$.

Variance

$E(f) < \infty$

Definition 1.12

If X is a random variable with $\mathbf{E}\left(X^2\right) < \infty$, its variance is defined as

$$\sigma^2(X) = \mathbf{E}((X - \overset{\mu}{\mathbf{E}}(X))^2) = \mathbf{E}\left(X^2\right) - (\mathbf{E}(X))^2.$$

Its square root $\sigma(X)$ is called the standard deviation of X.

$$\sigma^2(x) = E((x-\mu)^2) = E(x^2+\mu^2-2\mu x)$$

$$= E(x^2)+\mu^2-2\mu E(x) = E(x^2)+\mu^2-2\mu^2$$

$$= E(x^2)-\mu^2 = E(x^2) - E(x)^2$$

Variance

Theorem 1.11

If $\mathbf{E}\left(X^2\right) < \infty$, *and a and b are constants then*

$$\text{var}(aX + b) = a^2 \text{var}(X)$$

Proof

Using Theorem 1.8 we get,

$$\begin{aligned}\text{var}(aX + b) &= \mathbf{E}\left((aX + b - \mathbf{E}(aX + b))^2\right) = \mathbf{E}\left((a(X - \mathbf{E}(X)) + b - b)^2\right) \\ &= \mathbf{E}\left(a^2(X - \mathbf{E}(X))^2\right) = a^2 \text{var}(X)\end{aligned}$$

Geometric distribution

Definition 1.13

The number X of Bernoulli trials needed to get one success, follows the geometric distribution with p.d.f. $0 < p < 1$

$$f(x) = (1-p)^{x-1}p; \quad x = 1, 2, \ldots.$$

Theorem 1.12

If X follows the geometric distribution with parameter p ,

$$E(X) = \frac{1}{p}, \qquad \operatorname{var}(X) = \frac{1-p}{p^2}$$

Proof Ex!

Ex $\sum_{x=1}^{\infty}(1-p)^{x-1}p = 1$

$$\sum_{x=1}^{\infty} x \cdot p(1-p)^{x-1} = \frac{1}{p}$$

$$E(x^2) = \sum_{x=1}^{\infty} x^2 p(1-p)^{x-1} = ?$$