1. Indeterminate Forms

* Indeterminate Forms of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{\ln 1}{1-1} = \frac{0}{0} \text{ (undefined)}$$

$$\lim_{x\to 1} \frac{x^2-x}{x^2-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{x\to \infty} \frac{\ln x}{x-1} = \frac{\ln \infty}{\infty - 1} = \frac{\infty}{\infty}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \left(\frac{0}{0} \text{ Form}\right)$$

⟹ **L' Hospital's Rule :-** Suppose f and g are differentable and $g'(x) = 0$ on an open interval I that contains a (except possibly a) suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

or

that $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

if the limit on the right side exists (or is ∞ or $-\infty$)

:- Examples :-

(1) $\lim_{x \to 0} \frac{3x - \sin x}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{d/dx\,(3x - \sin x)}{d/dx\,(x)}$ (By L' Hospital Rule)

$= \lim_{x \to 0} \frac{3 - \cos x}{1}$

$= \frac{3 - \cos 0}{1}$

$= \frac{3 - 1}{1}$

$= 2$

(2) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{d/dx\,(\sqrt{1+x} - 1)}{d/dx\,(x)}$ (By L' Hospital Rule)

$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}} - 0}{1}$

$= \lim_{x \to 0} \frac{1}{2\sqrt{1+x}}$

$= \frac{1}{2\sqrt{1+0}} = \frac{1}{2}$

(3) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2}}{2x}$ (By L' Hospital Rule)

$= \frac{1}{4} \lim_{x \to 0} \frac{(1+x)^{-\frac{1}{2}} - 1}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \frac{1}{4} \lim_{x \to 0} \frac{(-\frac{1}{2})(1+x)^{-\frac{1}{2}-1} - 0}{1}$

(By L' Hospital Rule)

$= -\frac{1}{8} \lim_{x \to 0} (1+x)^{-\frac{3}{2}}$

$= -\frac{1}{8} (1+0)^{-\frac{3}{2}}$

$= -\frac{1}{8}$

(4) $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\left(\frac{0}{0} \text{ Form}\right)$

$= \lim_{x \to 1} \frac{1/x}{1-0}$ (By L' Hospital Rule)

$= \lim_{x \to 1} \frac{1}{x}$

$= \frac{1}{1} = 1$

(5) $\lim_{x \to \infty} \frac{e^x}{x^2}$ $\left(\frac{\infty}{\infty} \text{ Form}\right)$

$= \lim_{x \to \infty} \frac{e^x}{2x}$ (By L' Hospital Rule) $\left(\frac{\infty}{\infty} \text{ Form}\right)$

$= \lim_{x \to \infty} \frac{e^x}{2}$ (By L' Hospital Rule)

$= \frac{e^\infty}{2} = \infty$

(6) $\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$

$= \lim_{x \to \infty} \frac{\ln x}{x^{1/3}}$ $\left(\frac{\infty}{\infty} \text{ Form}\right)$

$= \lim_{x \to \infty} \frac{1/x}{1/3 (x)^{-2/3}}$ (By L' Hospital Rule)

$= 3 \lim_{x \to \infty} \frac{x^{2/3}}{x}$

$= 3 \lim_{x \to \infty} \frac{1}{x^{1-2/3}}$

$= 3 \lim_{x \to \infty} \frac{1}{x^{1/3}}$

$= 3 \frac{1}{(\infty)^{1/3}} = 3\left(\frac{1}{\infty}\right) = 3(0) = 0$

(7) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$ (By L' Hospital Rule) $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0} \frac{\cancel{2}\sec x \cdot \sec x \cdot \tan x - 0}{3(\cancel{2}x)}$

$= \lim_{x\to 0} \frac{\sec^2 x \cdot \tan x}{3x}$

$= \frac{1}{3} \lim_{x\to 0} \sec^2 x \cdot \frac{\tan x}{x}$

$= \frac{1}{3}(1)(1) = \frac{1}{3}$

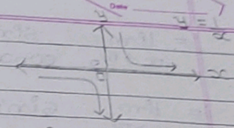
(8) $\lim_{x\to 0^+} \frac{\sin x}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0^+} \frac{\cos x}{2x}$ (By L' Hospital Rule)

$= \frac{1}{2} \lim_{x\to 0^+} \cos x \lim_{x\to 0^+} \frac{1}{x}$

$= \frac{1}{2}(1)(\infty)$

$= \infty$



Graph of $1/x$

(9) $\lim_{x\to 0^-} \frac{\sin x}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0^-} \frac{\cos x}{2x}$

$= \frac{1}{2} \lim_{x\to 0^-} \frac{1}{x}$

$= -\infty$

(10) $\lim_{x\to \pi/2} \frac{\sec x}{1 + \tan x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$= \lim_{x\to \pi/2} \frac{\sec x \cdot \tan x}{\sec^2 x}$ (By L' Hospital Rule)

$= \lim_{x\to \pi/2} \frac{\tan x}{\sec x}$

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$$= \lim_{x \to \pi/2} \frac{\sin x}{\cos x \cdot \sec x}$$

$$= \lim_{x \to \pi/2} \sin x$$

$$= 1$$

(11) $$\lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5x}$$

$$= \lim_{x \to \infty} \frac{x(1-2x)}{x(3x+5)}$$

$$= \lim_{x \to \infty} \frac{1-2x}{3x+5} \quad \left(\frac{-\infty}{\infty} \text{ Form}\right)$$

$$= \lim_{x \to \infty} \frac{(-2)}{3} \quad \left(\text{By L' Hospital Rule}\right)$$

$$= \frac{-2}{3}$$

(12) $$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{d/dx\,(e^x - 1 - x)}{d/dx\,(x^2)} \quad \left(\text{By L' Hospital Rule}\right)$$

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$$= \lim_{x \to 0} \frac{e^x - 1}{2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{d/dx\,(e^x - 1)}{d/dx\,(2x)}$$

$$= \lim_{x \to 0} \frac{e^x}{2}$$

$$= \frac{e^0}{2} = \frac{1}{2}$$

(13) $$\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^3 \cdot \frac{\tan x}{x}}$$

$$= \lim_{x \to 0} \left\{ \frac{\tan x - x}{x^3} \cdot \frac{x}{\tan x} \right\}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^3} \cdot \lim_{x \to 0} \frac{x}{\tan x}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$ (By L' Hospital Rule)

$= \lim_{x\to 0} \frac{\cancel{2}\sec x \cdot \sec x \cdot \tan x}{3(\cancel{2}x)}$

$= \frac{1}{3} \lim_{x\to 0} \left\{ \sec^2 x \cdot \frac{\tan x}{x} \right\}$

$= \frac{1}{3} \lim_{x\to 0} \sec^2 x \cdot \lim_{x\to 0} \frac{\tan x}{x}$

$= \frac{1}{3} \sec^2 0 = \frac{1}{3}$

(14) $\lim_{x\to \pi/2} \frac{3\sec x}{1+\tan x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$ $\frac{3\sec \pi/2}{1+\tan \pi/2} = \frac{\infty}{\infty}$

$= \lim_{x\to \pi/2} \frac{3\sec x \cdot \tan x}{\sec^2 x}$ (By L' Hospital Rule)

$= 3 \lim_{x\to \pi/2} \frac{\tan x}{\sec x}$

$= 3 \lim_{x\to \pi/2} \frac{\sin x}{\sec x \cdot \cos x}$

$= 3 \lim_{x\to \pi/2} \sin x = 3 \sin\frac{\pi}{2} = 3\cdot 1 = 3$

* Indeterminate Form of the type $0\cdot\infty$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$)

then $\lim_{x\to a} f(x)g(x)$ is called an indeterminate form of type $(0\cdot\infty)$ $(0\cdot -\infty)$

- NOTE :- We can write the product fg as $fg = \frac{f}{(1/g)}$ or $fg = \frac{g}{(1/f)}$ to deal with indeterminate form $0\cdot\infty$

(1) $\lim_{x\to 0^+} x \ln x$

$= \lim_{x\to 0^+} \frac{\ln x}{1/x}$ $\left(\frac{-\infty}{\infty} \text{ form}\right)$

$= \lim_{x\to 0^+} \frac{1/x}{(-1/x^2)}$ (By L' Hospital Rule)

$= \lim_{x\to 0^+} (-x)$

$= 0$

(2) $\lim\limits_{x\to\infty} x\sin\frac{1}{x}$ ($\infty \cdot 0$ Form)

$= \lim\limits_{x\to\infty} \frac{\sin 1/x}{1/x}$

Let $\frac{1}{x} = h$ Then as $x \to \infty$, $h \to 0^+$

Hence $\lim\limits_{x\to\infty} x\sin\frac{1}{x} = \lim\limits_{h\to 0^+} \frac{\sin h}{h}$.

$= \lim\limits_{h\to 0^+} \frac{\cosh}{1}$ (By L' Hospital Rule)

$= \frac{\cos 0}{1}$

$= 1$

(3) $\lim\limits_{x\to\infty} (a^{\frac{1}{x}} - 1)x$ ($0 \cdot \infty$ Form)

$= \lim\limits_{x\to\infty} \frac{a^{\frac{1}{x}} - 1}{1/x}$

$\frac{1}{x} = h$, $x \to 0$, $h \to 0^+$

$= \lim\limits_{h\to 0^+} \frac{a^h - 1}{h}$ $\left(\frac{0}{0} \text{ Form}\right)$

$= \lim\limits_{h\to 0^+} \frac{a^h \cdot \log a}{1}$ (By L' Hospital Rule)

$= a^0 \log a$

$= \log a$

(4) $\lim\limits_{x\to 1} \tan^2\left(\frac{\pi x}{2}\right)(1 + \sec\pi x)$ ($\infty \cdot 0$ Form)

$= \lim\limits_{x\to 1} \frac{(1 + \sec\pi x)}{\cot^2(\pi x/2)}$ $\left(\frac{0}{0} \text{ Form}\right)$

$= \lim\limits_{x\to 1} \frac{\pi \sec\pi x \cdot \tan\pi x}{2\cot(\pi x/2)(-\operatorname{cosec}^2 \pi x/2) \cdot \pi/2}$ (By L' Hospital Rule)

$= -\lim\limits_{x\to 1} \frac{\sec\pi x}{\operatorname{cosec}^2 \pi x/2} \cdot \lim\limits_{x\to 1} \frac{\tan\pi x}{\cot(\pi x/2)}$

$= -\frac{\sec\pi}{\operatorname{cosec}^2 \pi/2} \cdot \lim\limits_{x\to 1} \frac{\tan\pi x}{\cot(\pi x/2)}$

$= \lim\limits_{x\to 1} \frac{\tan\pi x}{\cot(\pi x/2)}$ $\left(\frac{0}{0} \text{ Form}\right)$

$= \lim\limits_{x\to 1} \frac{\pi \sec^2 \pi x}{-\pi/2 \operatorname{cosec}^2(\pi x/2)}$ (By L' Hospital Rule)

$$= -2 \cdot \frac{\sec^2 \pi}{\operatorname{cosec}^2 \pi/2}$$

$$= -2$$

(5) $\lim_{x \to a} \log\left(2 - \frac{x}{a}\right) \cot(x-a)$ $(0 \cdot \infty$ form$)$

$$= \lim_{x \to a} \frac{\log(2 - x/a)}{\tan(x-a)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to a} \frac{\frac{1}{2 - x/a}\,(-1/a)}{\sec^2(x-a)} \quad (\text{By L'Hospital Rule})$$

$$= -\frac{1}{a}$$

(6) $\lim_{x \to 1/2} \left(x^2 - \frac{1}{4}\right) \cdot \tan \pi x$ $(0 \cdot \infty$ form$)$

$$= \lim_{x \to 1/2} \frac{x^2 - 1/4}{\cot \pi x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1/2} \frac{2x}{(-\operatorname{cosec}^2 \pi x) \cdot \pi}$$

$$= -\frac{2}{\pi} \cdot \frac{1/2}{\operatorname{cosec}^2 \pi/2} = -\frac{1}{\pi}$$

(7) $\lim_{x \to a} \ln\left(2 - \frac{x}{a}\right) \cot(x-a)$ $(0 \cdot \infty$ form$)$

$$= \lim_{x \to a} \frac{\ln(2 - x/a)}{\tan(x-a)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to a} \frac{\frac{1}{2 - x/a} \cdot (-1/a)}{\sec^2(x-a)} \quad (\text{By L'Hospital Rule})$$

$$= \frac{-1}{a \sec^2 0} = -\frac{1}{a}$$

$$0 \cdot \infty \longrightarrow \frac{0}{\left(\frac{1}{\infty}\right)} \longrightarrow \frac{0}{0}$$

$$\frac{\infty}{\left(\frac{1}{0}\right)} \longrightarrow \frac{\infty}{\infty}$$

* Indeterminate Form of type $\infty - \infty$

$\Rightarrow$ If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit $\lim_{x\to a} \{f(x) - g(x)\}$ is called an Indeterminate form of type $\infty - \infty$.

- **NOTE** :- To find above limit we try to convert the difference in a quotient (for instance, by using a common denominator, rationalization or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

(1) $\lim_{x\to \pi/2^-} (\sec x - \tan x)$ $(\infty - \infty$ form$)$

$$= \lim_{x\to \pi/2^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to \pi/2^-} \frac{1 - \sin x}{\cos x} \quad \left(\frac{0}{0} \text{ Form}\right)$$

$$= \lim_{x\to \pi/2^-} \frac{0 - \cos x}{(-\sin x)} \quad (\text{By L' Hospital Rule})$$

$$= \lim_{x\to \pi/2^-} \frac{\cos x}{\sin x}$$

$$= \frac{\cos \pi/2}{\sin \pi/2} = \frac{0}{1} = 0$$

(2) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $(\infty - \infty$ form$)$

$$= \lim_{x\to 0} \left(\frac{x - \sin x}{x \sin x}\right) \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{x\cos x + \sin x} \quad (\text{By L' Hospital Rule}) \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{0 - (-\sin x)}{x(-\sin x) + \cos x + \cos x} \quad (\text{By L' Hospital Rule})$$

$$= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x}$$

$$= \frac{0}{2(1) - 0}$$

$$= 0$$

(3) $\lim\limits_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)$ ($\infty-\infty$ form)

$= \lim\limits_{x\to 1} \frac{x\log x - x + 1}{(x-1)\log x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 1} \frac{x\cdot 1/x + \log x - 1}{(x-1)\,1/x + \log x}$ (By L' Hospital Rule)

$= \lim\limits_{x\to 1} \frac{1+\log x - 1}{\frac{x-1}{x} + \log x}$

$= \lim\limits_{x\to 1} \frac{\log x}{1-\frac{1}{x}+\log x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 1} \frac{1/x}{0-\left(\frac{-1}{x^2}\right)+\frac{1}{x}}$ (By L' Hospital Rule)

$= \lim\limits_{x\to 1} \frac{1/x}{\frac{1}{x^2}+\frac{1}{x}}$

$= \frac{1}{1+1}$

$= \frac{1}{2}$

(4) $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ ($\infty-\infty$ form)

$= \lim\limits_{x\to 0} \frac{\sin^2 x - x^2}{x^2\sin^2 x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \frac{\sin^2 x - x^2}{x^4\,\frac{\sin^2 x}{x^2}}$

$= \lim\limits_{x\to 0} \left\{\frac{\sin^2 x - x^2}{x^4}\cdot\frac{x^2}{\sin^2 x}\right\}$

$= \lim\limits_{x\to 0} \frac{\sin^2 x - x^2}{x^4}\cdot\lim\limits_{x\to 0}\left(\frac{x}{\sin x}\right)^2$

$= \lim\limits_{x\to 0} \frac{\sin^2 x - x^2}{x^4}$ $\left(\frac{0}{0}\text{ form}\right)$ $\left(\because \lim\limits_{x\to 0}\left(\frac{x}{\sin x}\right)^2 = 1\right)$

$= \lim\limits_{x\to 0} \frac{2\sin x\cdot\cos x - 2x}{4x^3}$ (By L' Hospital Rule)

$= \lim\limits_{x\to 0} \frac{\cancel{2}\sin 2x - 2x}{4x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \frac{2\cos 2x - 2}{12x^2}$ (By L' Hospital Rule)

$$= \lim_{x\to 0} \frac{\cos 2x - 1}{6x^2} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2\sin 2x}{12x} \left(\text{By L' Hospital Rule}\right)$$

$$= \frac{-1}{6} \lim_{x\to 0} \frac{\sin 2x}{x} \left(\frac{0}{0} \text{ Form}\right)$$

$$= \frac{-1}{6} \lim_{x\to 0} \frac{2\cos 2x}{1} \left(\because \text{By L' Hospital Rule}\right)$$

$$= \frac{-1}{6}(2)$$

$$= \frac{-1}{3}$$

* Indeterminate forms of the type 0^0, ∞^0 and 1^∞

⇒ Some indeterminate forms arise from the limit $\lim_{x\to a} f(x)^{g(x)}$

(1) If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ is called an indeterminate form of type 0^0

(2) If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ is called an indeterminate form of type ∞^0

(3) If $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\pm\infty$ is called an indeterminate form of type 1^∞

⇒ Each of the above three cases can be treated by taking the natural logarithm:

let $y = \{f(x)\}^{g(x)}$, then $\ln y = g(x) \ln f(x)$

(1) $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$ (type 1^∞)

$\Rightarrow \lim_{x\to 0^+} (1+\sin 4x) = 1+\sin 0 = 1$

and $\lim_{x\to 0^+} \cot x = \cot 0 = \frac{\cos 0}{\sin 0} = \frac{1}{0} = \infty$

Given limit is indeterminate form of type 1^∞

Let $y = (1+\sin 4x)^{\cot x}$

Then $\ln y = \cot x \cdot \ln(1+\sin 4x)$

$\therefore \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \cot x \cdot \ln(1+\sin 4x)$ ($\infty \cdot 0$ form)

$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x}$ (By L'Hospitale Rule)

$\therefore \lim_{x\to 0^+} \ln y = 4$

$\therefore \ln \lim_{x\to 0^+} y = 4$ ($\because \ln x$ is continuous function)

$\ln \alpha = \beta \Leftrightarrow \alpha = e^\beta$

$\ln 0 \to -\infty$

$\therefore \lim_{x\to 0^+} y = e^4$

$\therefore \lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = e^4$

(2) $\lim_{x\to 0^+} x^x$ (0^0 form)

Let $y = x^x$

Then $\ln y = \ln x^x = x \ln x$

$\therefore \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln x$ ($0 \cdot \infty$ form)

$= \lim_{x\to 0^+} \frac{\ln x}{(1/x)}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0^+} \frac{1/x}{(-1/x^2)}$ (By L'Hospital Rule)

$= -\lim_{x\to 0^+} x$

$\therefore \lim_{x\to 0^+} \ln y = 0$

$\therefore \ln \lim_{x\to 0^+} \ln y = 0$ ($\because \ln x$ is continuous)

$$\therefore \lim_{x\to 0^+} y = e^0 = 1$$

$$\therefore \lim_{x\to 0^+} x^x = 1$$

(3) $\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\tan x}$

Let $y = \left(\frac{1}{x}\right)^{\tan x}$

Then $\ln y = \ln\left(\frac{1}{x}\right)^{\tan x} = \tan x \ln\left(\frac{1}{x}\right)$

Hence $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \left\{\tan x \cdot \ln\left(\frac{1}{x}\right)\right\}$ $(0\cdot\infty \text{ Form})$

$$= \lim_{x\to 0^+} \frac{\ln(1/x)}{\cot x} \quad \left(\frac{\infty}{\infty} \text{ Form}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{(1/x)}\cdot\left(-1/x^2\right)}{-\operatorname{cosec}^2 x} \quad \left(\begin{matrix}\text{By L'}\\ \text{Hospital}\\ \text{Rule}\end{matrix}\right)$$

$$= \lim_{x\to 0^+} \frac{1/x}{1/\sin^2 x}$$

$$= \lim_{x\to 0^+} \frac{\sin^2 x}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0^+} \frac{2\sin x\cos x}{1}$$

$$= \lim_{x\to 0^+} \sin 2x$$

$$\lim_{x\to 0^+} \ln y = 0$$

$$\therefore \ln \lim_{x\to 0^+} y = 0 \quad (\because \ln x \text{ is continuous})$$

$$\therefore \lim_{x\to 0^+} y = e^0 = 1$$

$$\therefore \lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\tan x} = 1$$

(4) $\lim_{x\to 0^+} \left[\frac{a^x+b^x+c^x}{3}\right]^{\frac{1}{3x}}$ (1^∞ form)

Let $y = \left[\frac{a^x+b^x+c^x}{3}\right]^{\frac{1}{3x}}$

Then $\ln y = \ln\left[\frac{a^x+b^x+c^x}{3}\right]^{\frac{1}{3x}}$

$$\ln y = \frac{1}{3x} \ln\left[\frac{a^x+b^x+c^x}{3}\right]$$

$\therefore \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{1}{3x} \ln\left[\frac{a^x+b^x+c^x}{3}\right]$ $(\infty\cdot 0 \text{ form})$

$$= \frac{1}{3} \lim_{x\to 0^+} \frac{\ln\left[\frac{a^x+b^x+c^x}{3}\right]}{x} \quad \left(\frac{0}{0}\right)$$

$$= \frac{1}{3} \lim_{x\to 0^+} \frac{1}{\frac{a^x+b^x+c^x}{3}} \cdot \frac{d}{dx}\left[\frac{a^x+b^x+c^x}{3}\right]$$

(By L' Hospital Rule)

$$= \frac{1}{3} \lim_{x\to 0^+} \frac{3}{a^x+b^x+c^x} \cdot \frac{a^x\ln a+b^x\ln b+c^x\ln c}{3}$$

$$= \frac{1}{3} \lim_{x\to 0^+} \frac{a^x\ln a+b^x\ln b+c^x\ln c}{a^x+b^x+c^x}$$

$$= \frac{1}{3} \frac{\ln a+\ln b+\ln c}{1+1+1}$$

$$= \frac{1}{3} \frac{\ln(abc)}{3}$$

$$= \frac{1}{9} \ln(abc)$$

$$\lim_{x\to 0^+} \ln y = \ln(abc)^{\frac{1}{9}}$$

$$\therefore \ln \lim_{x\to 0^+} y = \ln(abc)^{\frac{1}{9}}$$

$$\lim_{x\to 0^+} y = (abc)^{\frac{1}{9}}$$

$$\therefore \lim_{x\to 0^+} \left[\frac{a^x+b^x+c^x}{3}\right]^{\frac{1}{3x}} = (abc)^{\frac{1}{9}}$$

(5) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ (1^∞ Form)

Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Then $\ln y = \ln\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

$$= \frac{1}{x^2} \ln\left(\frac{\tan x}{x}\right)$$

$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{1}{x^2} \ln\left(\frac{\tan x}{x}\right)$ $(\infty\cdot 0)$

$$= \lim_{x\to 0} \frac{\ln\left(\frac{\tan x}{x}\right)}{x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{x}{\tan x} \cdot \frac{x\sec^2 x - \tan x}{x^2}}{2x}$$

(By L' Hospital Rule)

$$= \frac{1}{2} \lim_{x\to 0} \frac{x}{\tan x} \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^3}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{x \cdot 2\sec^2 x \cdot \tan x + \sec^2 x - \sec^2 x}{3x^2}$$

$$= \frac{1}{6} \lim_{x\to 0} \frac{2x\sec^2 x \cdot \tan x}{x^2}$$

$$= \frac{2}{6} \lim_{x\to 0} \frac{\sec^2 x \cdot \tan x}{x}$$

$$= \frac{1}{3} \lim_{x\to 0} \sec^2 x \cdot \lim_{x\to 0} \frac{\tan x}{x}$$

$$= \frac{1}{3} (\sec^2 0)(1)$$

$$\lim_{x\to 0^+} \ln y = \frac{1}{3}$$

$$\therefore \ln \lim_{x\to 0} y = \frac{1}{3}$$

$$\therefore \lim_{x\to 0} y = e^{1/3}$$

$$\therefore \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$$

(6) $\lim_{x\to 0} \left(\frac{1}{x}\right)^{1-\cos x}$ $(\infty^0 \text{ form})$

Let $y = \left(\frac{1}{x}\right)^{1-\cos x}$

$$\therefore \ln y = \ln \left(\frac{1}{x}\right)^{1-\cos x}$$

$$\ln y = (1-\cos x) \ln \left(\frac{1}{x}\right)$$

$$\therefore \ln y = 2\sin^2 \frac{x}{2} \ln \left(\frac{1}{x}\right)$$

$$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} 2\sin^2 \frac{x}{2} \cdot \ln\left(\frac{1}{x}\right) \quad (0 \cdot \infty)$$

$$= \lim_{x\to 0} \frac{2\ln(1/x)}{\operatorname{cosec}^2 x/2}$$

$$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{2x(-1/x^2)}{2\operatorname{cosec}\frac{x}{2}\left(-\operatorname{cosec}\frac{x}{2} \cdot \cot\frac{x}{2}\right)\frac{1}{2}}$$

$$= \lim_{x\to 0} \frac{(+1/x)}{\operatorname{cosec}^2 \frac{x}{2} \cot \frac{x}{2}}$$

$$= \lim_{x\to 0} \frac{2}{x} \sin^2 \frac{x}{2} \cdot \frac{\sin x/2}{\cos x/2}$$

$$= \lim_{x\to 0} \frac{\sin x/2}{x/2} \lim_{x\to 0} \frac{\sin^2 x/2}{\cos x/2}$$

$$= \frac{0}{1}$$

$$\therefore \lim_{x\to0} \ln y = 0$$

$$\therefore \ln \lim_{x\to0} y = 0$$

$$\therefore \lim_{x\to0} y = e^0 = 1$$

$$\therefore \lim_{x\to0} \left(\frac{1}{x}\right)^{1-\cos x} = 1$$

(7) $\lim_{x\to1} (2-x)^{\tan\frac{\pi x}{2}}$ $(1^\infty$ Form$)$

Let $y = (2-x)^{\tan\frac{\pi x}{2}}$

$$\therefore \ln y = \tan\frac{\pi x}{2} \ln(2-x)$$

$$\therefore \lim_{x\to1} \ln y = \lim_{x\to1} \tan\frac{\pi x}{2} \ln(2-x)$$

$(\infty \cdot 0$ form$)$

$$\therefore \lim_{x\to1} \ln y = \lim_{x\to1} \frac{\ln(2-x)}{\cot\frac{\pi x}{2}} \left(\frac{0}{0} \text{ Form}\right)$$

$$= \lim_{x\to1} \frac{\left(-\frac{1}{2-x}\right)}{\left(-\operatorname{cosec}^2\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}$$

$$= \frac{-1}{(-\pi/2)(1)^2}$$

$$\therefore \lim_{x\to1} \ln y = \frac{2}{\pi}$$

$$\therefore \ln \lim_{x\to1} y = \frac{2}{\pi}$$

$$\therefore \lim_{x\to1} y = e^{\frac{2}{\pi}}$$

$$\therefore \lim_{x\to1} (2-x)^{\tan\frac{\pi x}{2}} = e^{\frac{2}{\pi}}$$

(8) $\lim_{x\to1} (x-1)^{(x-1)}$ $(0^0$ form$)$

Let $y = \cancel{\lim_{x\to1}} (x-1)^{(x-1)}$

$$\therefore \ln y = \ln (x-1)^{(x-1)}$$

$$= (x-1) \ln(x-1)$$

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$$\therefore \lim_{x\to 1} \ln y = \lim_{x\to 1} (x-1) \ln (x-1)$$
$(0 \cdot \infty \text{ form})$

$$= \lim_{x\to 1} \frac{\ln (x-1)}{1/x-1} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 1} \frac{1/x-1}{(-1/x-1)^2}$$

$$= \lim_{x\to 1} (-1)(x-1)$$

$$\lim_{x\to 1} \ln y = 0$$

$$\therefore \ln \lim_{x\to 1} y = 0$$

$$\therefore \lim_{x\to 1} y = e^0 = 1$$

$$\therefore \lim_{x\to 1} (x-1)^{(x-1)} = 1$$