Example 9

- *Find the eigenvalues and eigenvectors of the matrix* $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$.

Solution

Let
$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 4 - 3 = 2$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}$$

$$= (-12 + 12) + (-3 + 0) + (4 + 0)$$

$$= 1$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{vmatrix}$$

$$= 1(-12 + 12) - 6(0 - 0) - 4(0 - 0)$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda = 0, 1, 1$$

(a) For $\lambda = 0$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 6y - 4z = 0$$

$$0x + 4y + 2z = 0$$

$$0x - 6y - 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & -4 \\ 4 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}} = t$$

$$\frac{x}{4} = \frac{y}{-2} = \frac{z}{4} = t$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ 2t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = t\mathbf{x}_1, \text{ where } \mathbf{x}_1 \text{ is an eigenvector corresponding to } \lambda = 0.$$

(b) For $\lambda = 1$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + 3y + 2z = 0$$

Let

$$x = t_1 \text{ and } z = t_2$$

$$y = -\frac{2}{3}t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t_1 \\ -\frac{2}{3}t_2 \\ t_2 \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{2}{3}t_2 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 1$.

Example 10

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

[Winter 2018]

Solution

Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

e S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 2 = 5$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= (4-2) + (2+2) + (2-0)$$

$$= 2 + 4 + 2$$

$$= 8$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= 1(4-2) - 2(0+1) + 2(0+2)$$

$$= 2 - 2 + 4$$

$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

(a) For $\lambda = 1$,

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + \ y + z = 0$$

$$-x + 2y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}} = t$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

(b) For $\lambda = 2$,

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 2y + 2z = 0$$
$$0x + 0y + \ \ z = 0$$
$$-x + 2y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix}} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{0} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = t\mathbf{x}_2 \text{ where } \mathbf{x}_2 \text{ is an eigenvector corresponding to } \lambda = 2.$$

Hence, there is only one eigenvector corresponding to repeated root $\lambda = 2$.

Example 11

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 0 + 0 + 3 = 3$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= (0+3) + (0) + (0)$$

$$= 3$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{vmatrix}$$

$$= 0 - 1(0-1) + 0$$

$$= 1$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\lambda = 1, 1, 1$$

For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + y + 0z = 0$$

$$0x - y + z = 0$$

$$x - 3y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

Hence, there is only one eigenvector corresponding to repeated root $\lambda = 1$.

Example 12

Find the values of μ which satisfy the equation $A^{100}\mathbf{x} = \mu\mathbf{x}$

where
$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 - 2 + 0 = 0$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}$$

$$= (0+2) + (0+1) + (-4-0)$$

$$= -1$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 2(0+2) - 1(0+2) - 1(0+2)$$

$$= 4 - 2 - 2$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda = 0$$

$$\lambda = 0, 1, -1$$

If λ is an eigen value of A, it satisfies the equation $A\mathbf{x} = \lambda\mathbf{x}$.

For equation $A^{100}\mathbf{x} = \mu\mathbf{x}$, μ represents eigen values of A^{100}. Eigenvalues of $A^{100} = \lambda^{100}$, i.e., 0, 1, 1.

Hence, values of μ are 0, 1, 1.

Example 13

Find the characteristic roots and characteristic vectors of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *and verify that characteristic roots are of unit modulus.*

Solution

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}\cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda\end{vmatrix}=0$$

$$(\cos\theta-\lambda)^2+\sin^2\theta=0$$

$$(\cos\theta-\lambda)^2=-\sin^2\theta$$

$$\cos\theta-\lambda=\pm i\sin\theta$$

$$\lambda=\cos\theta\pm i\sin\theta$$

$$|\lambda|=\sqrt{\cos^2\theta+\sin^2\theta}=1$$

Hence, the characteristic roots are of unit modules.

(a) For $\lambda=\cos\theta+i\sin\theta$, $\quad [A-\lambda I]\mathbf{x}=\mathbf{0}$

$$\begin{bmatrix}-i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$-i\sin\theta x-\sin\theta y=0$$

Let

$$y=t$$
$$x=it$$

$$\mathbf{x}=\begin{bmatrix}x \\ y\end{bmatrix}=\begin{bmatrix}it \\ t\end{bmatrix}=t\begin{bmatrix}i \\ 1\end{bmatrix}=t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda=\cos\theta+i\sin\theta$.

(b) For $\lambda=\cos\theta-i\sin\theta$, $\quad [A-\lambda I]\mathbf{x}=\mathbf{0}$

$$\begin{bmatrix}i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$i\sin\theta x-\sin\theta y=0$$

Let

$$y=t$$
$$x=-it$$

$$\mathbf{x}=\begin{bmatrix}x \\ y\end{bmatrix}=\begin{bmatrix}-it \\ t\end{bmatrix}=t\begin{bmatrix}-i \\ 1\end{bmatrix}=t\mathbf{x}_2$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda=\cos\theta-i\sin\theta$.

Example 14

Find orthogonal eigenvectors for the matrix $\begin{bmatrix}1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9\end{bmatrix}$.

Solution

Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 4 + 9 = 14$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= (36 - 36) + (9 - 9) + (4 - 4)$$

$$= 0$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix}$$

$$= 1(36 - 36) - 2(18 - 18) + 3(12 - 12)$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 14\lambda^2 = 0$$

$$\lambda = 0, 0, 14$$

(a) For $\lambda = 14$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-13x + 2y + 3z = 0$$

$$2x - 10y + 6z = 0$$

$$3x + 6y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 3 \\ -10 & 6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -13 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -13 & 2 \\ 2 & -10 \end{vmatrix}} = t$$

$$\frac{x}{42} = \frac{y}{84} = \frac{z}{126} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t$$

$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 14$.

(b) For $\lambda = 0$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + 3z = 0$$

Let

$$y = t_1 \text{ and } z = t_2$$

$$x = -2t_1 - 3t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 - 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 0$.

Since $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal, we must choose $\mathbf{x}_3$ such that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are orthogonal.

Let

$$\mathbf{x}_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

For $\mathbf{x}_1$ and $\mathbf{x}_3$ to be orthogonal, $\mathbf{x}_1^T\mathbf{x}_3 = 0$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$l + 2m + 3n = 0 \qquad \text{...(1)}$$

For $\mathbf{x}_2$ and $\mathbf{x}_3$ to be orthogonal, $\quad \mathbf{x}_2^T\mathbf{x}_3 = 0$

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$-2l + m = 0 \qquad \ldots(2)$$

Solving Eqs (1) and (2) by Cramer's rule,

$$\frac{l}{\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}} = t$$

$$\frac{l}{-3} = \frac{m}{-6} = \frac{n}{5} = t$$

$$\frac{l}{3} = \frac{m}{6} = \frac{n}{5} = t$$

$$\mathbf{x} = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 3t \\ 6t \\ 5t \end{bmatrix} = t\begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 14$.

EXERCISE 10.5

1. Find the sum and product of the eigenvalues of the following matrices:

(i) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

[**Ans.:** (i) –3, 4 (ii) –1, 45]

2. The matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & 2 & 3 \end{bmatrix}$ is singular. One of its eigenvalues is 2. Find the other two eigenvalues.

[**Ans.:** $1+\sqrt{5}, 1-\sqrt{5}$]

3. If two of the eigenvalues of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are 2 and 8, find the third eigenvalue.

[**Ans.:** 2]

10.14 CAYLEY–HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation.

Proof: Let A be an n-rowed square matrix. Its characteristic equation is

$$|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + p + a_n)$$

$$(A-\lambda I)\ adj\ (A-\lambda I) = |A-\lambda I|\, I \qquad \text{...(10.16)}$$

$$[\{A\ adj(A) = |A|\, I]$$

Since adj $(A - \lambda I)$ has element as cofactors of elements of $|A - \lambda I|$, the elements of adj $(A - \lambda I)$ are polynomials in λ of degree n – 1 or less. Hence, adj $(A - \lambda I)$ can be written as a matrix polynomial in λ.

adj $(A - \lambda \mathrm{I}) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}$

where $B_0, B_1, \ldots, B_{n-1}$ are matrices of order n.

$$(A-\lambda I)\ adj\ (A-\lambda I) = (A-\lambda I)[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}]$$

$$|A-\lambda I|\, I = (A-\lambda I)[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}]$$

$$(-1)^n[I\lambda^n + a_1 I\lambda^{n-1} + a_2 I\lambda^{n-2} + \cdots + a_{n-1}I\lambda + a_n I]$$

$$= (-IB_0)\lambda^n + (AB_0 - IB_1)\lambda^{n-1} + (AB_1 - IB_2)\lambda^{n-2} + \cdots + (AB_{n-2} - IB_{n-1})\lambda + AB_{n-1}$$

Equating corresponding coefficients,

$$-IB_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n a_2 I$$

$$\vdots \qquad \vdots$$

$$AB_{n-2} - IB_{n-1} = (-1)^n a_{n-1} I$$

$$AB_{n-1} = (-1)^n a_n I$$

Premultiplying the above equations successively by $A^n, A^{n-1}, A^{n-2}, \ldots I$ and adding,

$$(-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I] = 0$$

Hence,

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0 \qquad \text{...(10.17)}$$

Corollary: If A is a non-singular matrix, i.e. det $(A) \neq 0$ then premultiplying equation (1) by A^{-1}, we get

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_n A^{-1} = 0$$

$$A^{-1} = -\frac{1}{a^n}[A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I]$$

Example 1

Apply Cayley–Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ *and deduce that* $A^8 = 625I$.

Solution

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 - 1 = 0$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= -1 - 4$$

$$= -5$$

Hence, the characteristic equation is

$$\lambda^2 - 5 = 0$$

By Cayley–Hamilton theorem, the matrix A satisfies its own characteristic equation.

$$A^2 - 5I = \mathbf{0}$$

$$A^2 = 5I$$

$$A^4 = 25I$$

$$A^8 = 625I$$

Example 2

Verify Cayley–Hamilton theorem for the following matrix and hence, find A^{-1} *and* A^4.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2+2+2=6$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$=(4-1)+(4-1)+(4-1)$$

$$=9$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$=2(4-1)+1(-2+1)+1(1-2)$$

$$=6-1-1$$

$$=4$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0} \qquad \text{...(1)}$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^3-6A^2+9A-4I)=\mathbf{0}$$

$$A^2-6A+9I-4A^{-1}=\mathbf{0}$$

$$4A^{-1}=(A^2-6A+9I)$$

$$=\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}-\begin{bmatrix}12&-6&6\\-6&12&-6\\6&-6&12\end{bmatrix}+\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}$$

$$=\begin{bmatrix}3&1&-1\\1&3&1\\-1&1&3\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3&1&-1\\1&3&1\\-1&1&3\end{bmatrix}$$

Multiplying Eq. (1) by A,

$$A(A^3-6A^2+9A-4I)=\mathbf{0}$$

$$A^4-6A^3+9A^2-4A=\mathbf{0}$$

$$A^4=6A^3-9A^2+4A$$

$$=\begin{bmatrix}132&-126&126\\-126&132&-126\\126&-126&132\end{bmatrix}-\begin{bmatrix}54&-45&45\\-45&54&-45\\45&-45&54\end{bmatrix}+\begin{bmatrix}8&-4&4\\-4&8&-4\\4&-4&8\end{bmatrix}$$

$$=\begin{bmatrix}86&-85&85\\-85&86&-85\\85&-85&86\end{bmatrix}$$

Example 3

Show that matrix $A=\begin{bmatrix}0&c&-b\\-c&0&a\\b&-a&0\end{bmatrix}$ *satisfies Cayley–Hamilton theorem and hence find* A^{-1}*, if it exists.*

Solution

$$A=\begin{bmatrix}0&c&-b\\-c&0&a\\b&-a&0\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where S_1 = Sum of the principal diagonal elements of $A=0$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}+\begin{vmatrix} 0 & -b \\ b & 0 \end{vmatrix}+\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$$

$$=(0+a^2)+(0+b^2)+(0+c^2)$$

$$=a^2+b^2+c^2$$

$$S_3=\det(A)=\begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}$$

$$=0-c(0-ab)-b(ac-0)$$

$$=abc-abc$$

$$=0$$

Hence, the characteristic equation is

$$\lambda^3+(a^2+b^2+c^2)\lambda=0$$

$$A^2=\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}=\begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$$A^3=\begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -c^3-cb^2-ca^2 & b^3+bc^2+ba^2 \\ c^3+ca^2+cb^2 & 0 & -ab^2-ac^2-a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+a^3 & 0 \end{bmatrix}$$

$$=-(a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$=-(a^2+b^2+c^2)A$$

$$A^3+(a^2+b^2+c^2)A=\mathbf{0}$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

$$\det(A) = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = -c(0-ab) - b(ac-0)$$

$$= abc - abc = 0$$

Hence, A^{-1} does not exist.

Example 4

Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley–Hamilton theorem for this matrix. Find A^{-1} and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

Solution

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 3 = 4$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 3 - 8$$

$$= -5$$

Hence, the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = -1, 5$$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \quad \text{... (1)}$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^2-4A-5)=0$$
$$A-4I-5A^{-1}=0$$
$$4A^{-1}=\frac{1}{5}(A-4I)$$
$$=\frac{1}{5}\begin{bmatrix}-3 & 4\\ 2 & -1\end{bmatrix}$$

Now, $A^5-4A^4-7A^3+11A^2-A-10I = A^3(A^2-4A-5I)-2A(A^2-4A-5I)$
$+3(A^2-4A-5I)+A+5I$
$=(A^2-4A-5I)(A^3-2A+3I)+(A+5I)$
$=0+(A+5I)$ [Using Eq. (1)]
$=A+5I$

which is a linear polynomial in A.

Example 5

Find the characteristic equation of the matrix $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ *and hence find the matrix represented by* $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

Solution

$$A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 2-\lambda\end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where S_1 = Sum of the principal diagonal elements of $A=2+1+2=5$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}$$
$$=(2-0)+(4-1)+(2-0)$$
$$=7$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1)$$
$$= 4 - 0 - 1$$
$$= 3$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley–Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(1)$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$
$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$
$$= 0 + (A^2 + A + I) \qquad \text{[Using Eq.(1)]}$$
$$= A^2 + A + I$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 6

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *prove by induction that for every integer* $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. *Hence, find* A^{50}.

Solution

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where S_1 = Sum of the principal diagonal elements of $A = 1+0+0=1$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}+\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}+\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$=(0-1)+(0-0)+(0-0)$$

$$=-1$$

$$S_3=\det(A)=\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$=1(0-1)+0+0$$

$$=-1$$

Hence, the characteristic equation is

$$\lambda^3-\lambda^2-\lambda+1=0$$

By Cayley–Hamilton theorem,

$$A^3-A^2-A+I=0$$

$$A^3=A^2+A-I$$

$$=A^{3-2}+A^2-I \quad \text{...(1)}$$

Hence, $A^n=A^{n-2}+A^2-I$ is true for $n=3$.

Assuming that Eq. (1) is true for $n=k$,

$$A^k=A^{k-2}+A^2-I$$

Multiplying both the sides by A

$$A^{k+1}=A^{k+1}+A^3-A$$

Substituting the value of A^3

$$A^{k+1}=A^{k+1}+(A^2+A-I)-A$$

$$A^{(k+1)-2}+A^2-I$$

Hence, $A^n=A^{n-2}+A^2-I$ is true for $n=k+1$

Thus, by mathematical induction, it is true for $n\geq 3$.

We have, $\quad A^n=A^{n-2}+A^2-I$

$$= (A^{n-4} + A^2 - I) + A^2 - I$$
$$= A^{n-4} + 2(A^2 - I)$$
$$= (A^{n-6} + A^2 - I) + 2(A^2 - I)$$
$$= A^{n-6} + 3(A^2 - I)$$

....

$$A^n = A^{n-2r} + r(A^2 - I)$$

Putting $n = 50$ and $r = 24$,

$$A^{50} = A^{50-2(24)} + 24(A^2 - I)$$
$$= A^2 + 24A^2 - 24I$$
$$= 25A^2 - 24I$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

EXERCISE 10.6

1. Verify Cayley–Hamilton theorem for the matrix A and hence, find A^{-1} and A^4.

(i) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

7. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ and evaluate $2A^4 - 5A^3 - 7A - 6I$.

$$\left[\text{Ans.: } \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}\right]$$

10.15 SIMILARITY TRANSFORMATION

If A and B are two square matrices of order n then B is said to be similar to A, if there exists a nonsingular matrix P such that

$$B = P^{-1}AP$$

Note: (1) Similarity of matrices is an equivalence relation.

(2) Similar matrices have the same determinant.

(3) Similar matrices have the same characteristic polynomial and hence the same eigenvalues. If $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to the eigenvalue λ where $B = P^{-1}AP$.

10.16 DIAGONALIZATION OF A MATRIX

A matrix A is said to be *diagonalizable* if it is similar to a diagonal matrix.

A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix, also known as *spectral matrix*. The matrix P is then said to diagonalize A or transform A to a diagonal form. P is known as the *modal matrix*.

Note (1) An $n \times n$ matrix is diagonalizable if and only if it possesses n linearly independent eigenvectors.

(2) If the eigenvalues of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

(3) If A is similar to a diagonal matrix D, the diagonal elements of D are the eigenvalues of A..

10.16.1 Orthogonally Similar Matrices

If A and B are two square matrices of order n then B is said to be orthogonally similar to A if there exists an orthogonal matrix P such that

$$B = P^{-1}AP$$

Since P is orthogonal, $P^{-1} = P^T$

$$B = P^{-1}AP = P^TAP$$

Note: (1) Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

(2) A real symmetric matrix of order n has n mutually orthogonal real eigenvectors.

(3) Any two eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.

Note: To find the orthogonal matrix P, each element of the eigenvector is divided by its norm (length).

Working Rule for Diagonalization of Square Matrix A

(i) Find the eigenvalues of the square matrix A.
(ii) Find the eigenvectors corresponding to each eigenvalue.
(iii) Find the modal matrix P having the normalized eigenvectors as its column vectors.
(iv) Find the diagonal matrix $D = P^T AP$. The diagonal matrix D has eigenvalues as its diagonal elements.

Example 1

Show that the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ *is not diagonalizable.*

Solution

Let
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic equation is
$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 + 2 + 2 = 6$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$$
$$= (4-0) + (4-0) + (4-0)$$
$$= 12$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$
$$= 2(4-0) - 1(0-0) + 0$$
$$= 8 - 0 + 0$$
$$= 8$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda = 2, 2, 2$$

For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + y + 0z = 0$$

$$0x + 0y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}} = t$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

Since the matrix A has only one linearly independent eigenvector which is less than its order 3, matrix A is not diagonalizable.

Example 2

Show that the matrix $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ *is not diagonalizable.*

Solution

Let

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ 1 & 2-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 3 = 6$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= (6-4)+(3-0)+(2+2)$$

$$= 9$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 1(6-4)+2(3-2)+0$$

$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\lambda = 1, 1, 4$$

(a) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x - 2y + 0z = 0$$

$$x + y + 2z = 0$$

$$x + 2y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}} = t$$

$$\frac{x}{-4} = \frac{y}{0} = \frac{z}{2} = t$$

$$\frac{x}{2} = \frac{y}{0} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 4$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -3 & -2 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x - 2y + 0z = 0$$

$$x - 2y + 2z = 0$$

$$x + 2y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-2 & 0\\ -2 & 2\end{vmatrix}} = -\frac{y}{\begin{vmatrix}-3 & 0\\ 1 & 2\end{vmatrix}} = \frac{z}{\begin{vmatrix}-3 & -2\\ 1 & -2\end{vmatrix}} = t$$

$$\frac{x}{-4} = \frac{y}{6} = \frac{z}{8} = t$$

$$\frac{x}{-2} = \frac{y}{3} = \frac{z}{4} = t$$

$\mathbf{x} = \begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} -2t\\ 3t\\ 4t\end{bmatrix} = t\begin{bmatrix} -2\\ 3\\ 4\end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 1$.

Since the matrix A has two linearly independent eigenvectors which is less than its order 3, the matrix A is not diagonalizable.

Example 3

Determine a diagonal matrix orthogonally similar to the real symmetric matrix $\begin{bmatrix} 3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$. *Also find the modal matrix.*

Solution

Let

$$A = \begin{bmatrix} 3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 3 + 5 + 3 = 11$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 5 & -1\\ -1 & 3\end{vmatrix} + \begin{vmatrix} 3 & 1\\ 1 & 3\end{vmatrix} + \begin{vmatrix} 3 & -1\\ -1 & 5\end{vmatrix}$$

$$= (15-1) + (9-1) + (15-1)$$

$$= 36$$

$$S_3 = \det(A) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4$$

$$= 36$$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(a) For $\lambda = 2$, $$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y + z = 0$$

$$-x + 3y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}} = t$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2} = t$$

$$\frac{x}{-1} = \frac{y}{0} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 2$.

(b) For $\lambda = 3$, $$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x - y + z = 0$$

$$-x + 2y - z = 0$$

$$x - y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}} = t$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_2$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 6$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x - y + z = 0$$

$$x + y - z = 0$$

$$x - y - 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}} = t$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2} = t$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 6$.

Since matrix A has three linearly independent eigenvectors which is same as its order, the matrix A is diagonalizable.

Length of eigenvector $\mathbf{x}_1 = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

Length of eigenvector $\mathbf{x}_2 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Length of eigenvector $\mathbf{x}_3 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

The modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.

Example 4

Determine a diagonal matrix orthogonally similar to the real symmetric matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. *Also find the modal matrix.*