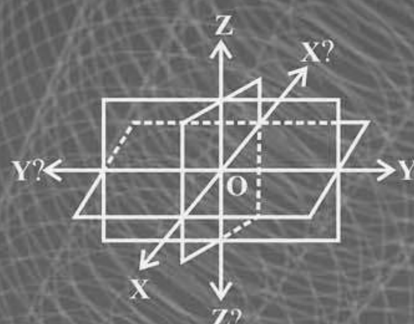
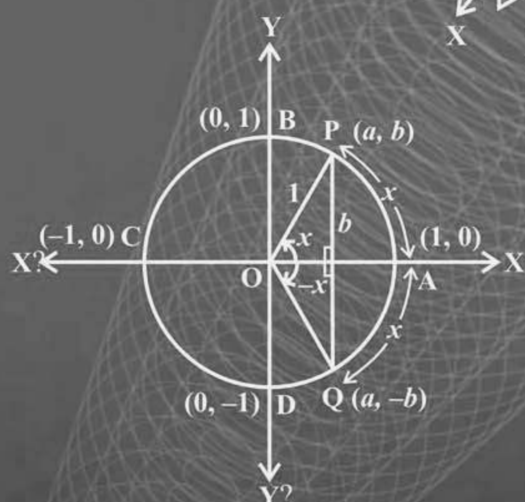
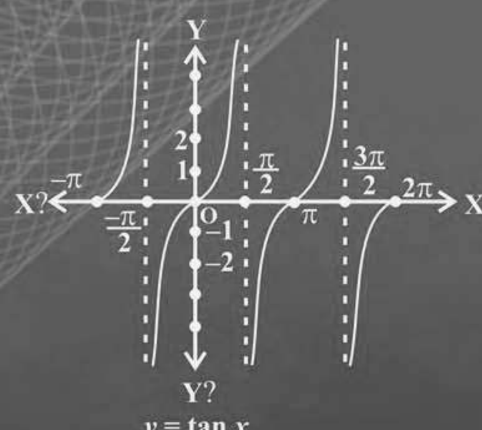
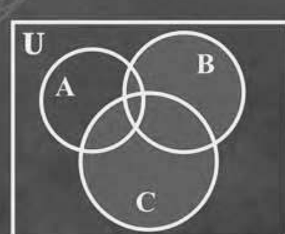
MATHEMATICS

Textbook for Class XI









MATHEMATICS

Class XI

Contents

	Foreword		*iii*
1.	**Sets**		**1**
	1.1	Introduction	1
	1.2	Sets and their Representations	1
	1.3	The Empty Set	5
	1.4	Finite and Infinite Sets	6
	1.5	Equal Sets	7
	1.6	Subsets	9
	1.7	Power Set	12
	1.8	Universal Set	12
	1.9	Venn Diagrams	13
	1.10	Operations on Sets	14
	1.11	Complement of a Set	18
	1.12	Practical Problems on Union and Intersection of Two Sets	21
2.	**Relations and Functions**		**30**
	2.1	Introduction	30
	2.2	Cartesian Product of Sets	30
	2.3	Relations	34
	2.4	Functions	36
3.	**Trigonometric Functions**		**49**
	3.1	Introduction	49
	3.2	Angles	49
	3.3	Trigonometric Functions	55
	3.4	Trigonometric Functions of Sum and Difference of Two Angles	63
	3.5	Trigonometric Equations	74
4.	**Principle of Mathematical Induction**		**86**
	4.1	Introduction	86
	4.2	Motivation	87
	4.3	The Principle of Mathematical Induction	88

5.	**Complex Numbers and Quadratic Equations**	**97**
5.1	Introduction	97
5.2	Complex Numbers	97
5.3	Algebra of Complex Numbers	98
5.4	The Modulus and the Conjugate of a Complex Number	102
5.5	Argand Plane and Polar Representation	104
5.6	Quadratic Equations	108
6.	**Linear Inequalities**	**116**
6.1	Introduction	116
6.2	Inequalities	116
6.3	Algebraic Solutions of Linear Inequalities in One Variable and their Graphical Representation	118
6.4	Graphical Solution of Linear Inequalities in Two Variables	123
6.5	Solution of System of Linear Inequalities in Two Variables	127
7.	**Permutations and Combinations**	**134**
7.1	Introduction	134
7.2	Fundamental Principle of Counting	134
7.3	Permutations	138
7.4	Combinations	148
8.	**Binomial Theorem**	**160**
8.1	Introduction	160
8.2	Binomial Theorem for Positive Integral Indices	160
8.3	General and Middle Terms	167
9.	**Sequences and Series**	**177**
9.1	Introduction	177
9.2	Sequences	177
9.3	Series	179
9.4	Arithmetic Progression (A.P.)	181
9.5	Geometric Progression (G.P.)	186
9.6	Relationship Between A.M. and G.M.	191
9.7	Sum to n terms of Special Series	194
10.	**Straight Lines**	**203**
10.1	Introduction	203
10.2	Slope of a Line	204
10.3	Various Forms of the Equation of a Line	212
10.4	General Equation of a Line	220
10.5	Distance of a Point From a Line	225

11. Conic Sections **236**
11.1 Introduction 236
11.2 Sections of a Cone 236
11.3 Circle 239
11.4 Parabola 242
11.5 Ellipse 247
11.6 Hyperbola 255

12. Introduction to Three Dimensional Geometry **268**
12.1 Introduction 268
12.2 Coordinate Axes and Coordinate Planes in Three Dimensional Space 269
12.3 Coordinates of a Point in Space 269
12.4 Distance between Two Points 271
12.5 Section Formula 273

13. Limits and Derivatives **281**
13.1 Introduction 281
13.2 Intuitive Idea of Derivatives 281
13.3 Limits 284
13.4 Limits of Trigonometric Functions 298
13.5 Derivatives 303

14. Mathematical Reasoning **321**
14.1 Introduction 321
14.2 Statements 321
14.3 New Statements from Old 324
14.4 Special Words/Phrases 329
14.5 Implications 335
14.6 Validating Statements 339

15. Statistics **347**
15.1 Introduction 347
15.2 Measures of Dispersion 349
15.3 Range 349
15.4 Mean Deviation 349
15.5 Variance and Standard Deviation 361
15.6 Analysis of Frequency Distributions 372

16. Probability **383**
16.1 Introduction 383
16.2 Random Experiments 384
16.3 Event 387
16.4 Axiomatic Approach to Probability 394

Appendix 1: Infinite Series **412**
A.1.1 Introduction 412
A.1.2 Binomial Theorem for any Index 412
A.1.3 Infinite Geometric Series 414
A.1.4 Exponential Series 416
A.1.5 Logarithmic Series 419

Appendix 2: Mathematical Modelling **421**
A.2.1 Introduction 421
A.2.2 Preliminaries 421
A.2.3 What is Mathematical Modelling 425

Answers **433**

Chapter 1

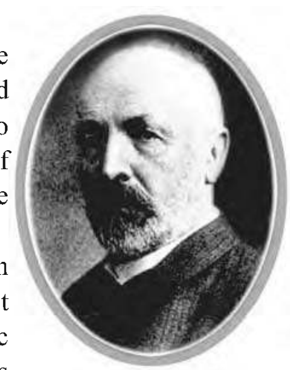
SETS

❖*In these days of conflict between ancient and modern studies; there must surely be something to be said for a study which did not begin with Pythagoras and will not end with Einstein; but is the oldest and the youngest.* — G.H. HARDY ❖

1.1 Introduction

The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.

The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He first encountered sets while working on "problems on trigonometric series". In this Chapter, we discuss some basic definitions and operations involving sets.



Georg Cantor (1845 1918)

1.2 Sets and their Representations

In everyday life, we often speak of collections of objects of a particular kind, such as, a pack of cards, a crowd of people, a cricket team, etc. In mathematics also, we come across collections, for example, of natural numbers, points, prime numbers, etc. More specially, we examine the following collections:

(i) Odd natural numbers less than 10, i.e., 1, 3, 5, 7, 9

(ii) The rivers of India

(iii) The vowels in the English alphabet, namely, *a, e, i, o, u*

(iv) Various kinds of triangles

(v) Prime factors of 210, namely, 2,3,5 and 7

(vi) The solution of the equation: $x^2 - 5x + 6 = 0$, viz, 2 and 3.

We note that each of the above example is a well-defined collection of objects in the sense that we can definitely decide whether a given particular object belongs to a given collection or not. For example, we can say that the river Nile does not belong to the collection of rivers of India. On the other hand, the river Ganga does belong to this colleciton.

We give below a few more examples of sets used particularly in mathematics, viz.

N : the set of all natural numbers
Z : the set of all integers
Q : the set of all rational numbers
R : the set of real numbers
$\mathbf{Z}^+$: the set of positive integers
$\mathbf{Q}^+$: the set of positive rational numbers, and
$\mathbf{R}^+$: the set of positive real numbers.

The symbols for the special sets given above will be referred to throughout this text.

Again the collection of five most renowned mathematicians of the world is not well-defined, because the criterion for determining a mathematician as most renowned may vary from person to person. Thus, it is not a well-defined collection.

We shall say that ***a set is a well-defined collection of objects.***

The following points may be noted :

(i) Objects, elements and members of a set are synonymous terms.
(ii) Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc.
(iii) The elements of a set are represented by small letters *a, b, c, x, y, z,* etc.

If *a* is an element of a set A, we say that " *a* belongs to A" the Greek symbol $\in$ (epsilon) is used to denote the phrase '*belongs to*'. Thus, we write $a \in$ A. If '*b*' is not an element of a set A, we write $b \notin$ A and read "*b* does not belong to A".

Thus, in the set V of vowels in the English alphabet, $a \in$ V but $b \notin$ V. In the set P of prime factors of 30, $3 \in$ P but $15 \notin$ P.

There are two methods of representing a set :

(i) Roster or tabular form
(ii) Set-builder form.

(i) In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. For example, the set of all even positive integers less than 7 is described in roster form as {2, 4, 6}. Some more examples of representing a set in roster form are given below :

(a) The set of all natural numbers which divide 42 is {1, 2, 3, 6, 7, 14, 21, 42}.

Note In roster form, the order in which the elements are listed is immaterial. Thus, the above set can also be represented as {1, 3, 7, 21, 2, 6, 14, 42}.

(b) The set of all vowels in the English alphabet is {*a, e, i, o, u*}.

(c) The set of odd natural numbers is represented by {1, 3, 5, . . .}. The dots tell us that the list of odd numbers continue indefinitely.

Note It may be noted that while writing the set in roster form an element is not generally repeated, i.e., all the elements are taken as distinct. For example, the set of letters forming the word 'SCHOOL' is { S, C, H, O, L} or {H, O, L, C, S}. Here, the order of listing elements has no relevance.

(ii) In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. For example, in the set {*a, e, i, o, u*}, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property. Denoting this set by V, we write

V = {x : x is a vowel in English alphabet}

It may be observed that we describe the element of the set by using a symbol x (any other symbol like the letters y, z, etc. could be used) which is followed by a colon " : ". After the sign of colon, we write the characteristic property possessed by the elements of the set and then enclose the whole description within braces. The above description of the set V is read as "the set of all x such that x is a vowel of the English alphabet". In this description the braces stand for "the set of all", the colon stands for "such that". For example, the set

A = {x : x is a natural number and $3 < x < 10$} is read as "the set of all x such that x is a natural number and x lies between 3 and 10. Hence, the numbers 4, 5, 6, 7, 8 and 9 are the elements of the set A.

If we denote the sets described in (*a*), (*b*) and (*c*) above in roster form by A, B, C, respectively, then A, B, C can also be represented in set-builder form as follows:

A= {x : x is a natural number which divides 42}

B= {y : y is a vowel in the English alphabet}

C= {z : z is an odd natural number}

Example 1 Write the solution set of the equation $x^2 + x - 2 = 0$ in roster form.

Solution The given equation can be written as

$$(x - 1)\ (x + 2) = 0, \text{ i. e., } \quad x = 1, -2$$

Therefore, the solution set of the given equation can be written in roster form as {1, – 2}.

Example 2 Write the set {x : x is a positive integer and $x^2 < 40$} in the roster form.

Solution The required numbers are 1, 2, 3, 4, 5, 6. So, the given set in the roster form is {1, 2, 3, 4, 5, 6}.

Example 3 Write the set A = {1, 4, 9, 16, 25, . . . }in set-builder form.

Solution We may write the set A as

A = {x : x is the square of a natural number}

Alternatively, we can write

$$A = \{x : x = n^2, \text{ where } n \in \mathbf{N}\}$$

Example 4 Write the set $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\}$ in the set-builder form.

Solution We see that each member in the given set has the numerator one less than the demominator. Also, the numerator begin from 1 and do not exceed 6. Hence, in the set-builder form the given set is

$$\left\{x : x = \frac{n}{n+1}, \text{ where } n \text{ is a natural number and } 1 \le n \le 6\right\}$$

Example 5 Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form :

(i)	{P, R, I, N, C, A, L}	(a) { x : x is a positive integer and is a divisor of 18}
(ii)	{ 0 }	(b) { x : x is an integer and $x^2 - 9 = 0$}
(iii)	{1, 2, 3, 6, 9, 18}	(c) {x : x is an integer and $x + 1 = 1$}
(iv)	{3, –3}	(d) {x : x is a letter of the word PRINCIPAL}

Solution Since in (d), there are 9 letters in the word PRINCIPAL and two letters P and I are repeated, so (i) matches (d). Similarly, (ii) matches (c) as $x + 1 = 1$ implies $x = 0$. Also, 1, 2 ,3, 6, 9, 18 are all divisors of 18 and so (iii) matches (a). Finally, $x^2 - 9 = 0$ implies $x = 3, -3$ and so (iv) matches (b).

EXERCISE 1.1

1. Which of the following are sets ? Justify your asnwer.
   (i) The collection of all the months of a year beginning with the letter J.
   (ii) The collection of ten most talented writers of India.
   (iii) A team of eleven best-cricket batsmen of the world.
   (iv) The collection of all boys in your class.
   (v) The collection of all natural numbers less than 100.
   (vi) A collection of novels written by the writer Munshi Prem Chand.
   (vii) The collection of all even integers.

(viii) The collection of questions in this Chapter.

(ix) A collection of most dangerous animals of the world.

2. Let A = {1, 2, 3, 4, 5, 6}. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces:

(i) 5. . .A	(ii) 8 . . . A	(iii) 0. . .A
(iv) 4. . . A	(v) 2. . .A	(vi) 10. . .A

3. Write the following sets in roster form:

(i) A = {x : x is an integer and $-3 < x < 7$}

(ii) B = {x : x is a natural number less than 6}

(iii) C = {x : x is a two-digit natural number such that the sum of its digits is 8}

(iv) D = {x : x is a prime number which is divisor of 60}

(v) E = The set of all letters in the word TRIGONOMETRY

(vi) F = The set of all letters in the word BETTER

4. Write the following sets in the set-builder form :

(i) (3, 6, 9, 12}	(ii) {2,4,8,16,32}	(iii) {5, 25, 125, 625}
(iv) {2, 4, 6, . . .}	(v) {1,4,9, . . .,100}	

5. List all the elements of the following sets :

(i) A = {x : x is an odd natural number}

(ii) B = {x : x is an integer, $-\frac{1}{2} < x < \frac{9}{2}$}

(iii) C = {x : x is an integer, $x^2 \leq 4$}

(iv) D = {x : x is a letter in the word "LOYAL"}

(v) E = {x : x is a month of a year not having 31 days}

(vi) F = {x : x is a consonant in the English alphabet which precedes k }.

6. Match each of the set on the left in the roster form with the same set on the right described in set-builder form:

(i) {1, 2, 3, 6}	(a) {x : x is a prime number and a divisor of 6}
(ii) {2, 3}	(b) {x : x is an odd natural number less than 10}
(iii) {M,A,T,H,E,I,C,S}	(c) {x : x is natural number and divisor of 6}
(iv) {1, 3, 5, 7, 9}	(d) {x : x is a letter of the word MATHEMATICS}.

1.3 The Empty Set

Consider the set

A = { x : x is a student of Class XI presently studying in a school }

We can go to the school and count the number of students presently studying in Class XI in the school. Thus, the set A contains a finite number of elements.

We now write another set B as follows:

B = { x : x is a student presently studying in both Classes X and XI }

We observe that a student cannot study simultaneously in both Classes X and XI. Thus, the set B contains no element at all.

Definition 1 A set which does not contain any element is called the *empty set* or the *null set* or the *void set*.

According to this definition, B is an empty set while A is not an empty set. The empty set is denoted by the symbol ϕ or { }.

We give below a few examples of empty sets.

(i) Let A = {x : 1 < x < 2, x is a natural number}. Then A is the empty set, because there is no natural number between 1 and 2.

(ii) B = {x : $x^2 - 2 = 0$ and x is rational number}. Then B is the empty set because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.

(iii) C = {x : x is an even prime number greater than 2}.Then C is the empty set, because 2 is the only even prime number.

(iv) D = { x : $x^2 = 4$, x is odd }. Then D is the empty set, because the equation $x^2 = 4$ is not satisfied by any odd value of x.

1.4 Finite and Infinite Sets

Let A = {1, 2, 3, 4, 5}, B = {a, b, c, d, e, g}

and C = { men living presently in different parts of the world}

We observe that A contains 5 elements and B contains 6 elements. How many elements does C contain? As it is, we do not know the number of elements in C, but it is some natural number which may be quite a big number. By number of elements of a set S, we mean the number of distinct elements of the set and we denote it by n (S). If n (S) is a natural number, then S is *non-empty finite* set.

Consider the set of natural numbers. We see that the number of elements of this set is not finite since there are infinite number of natural numbers. We say that the set of natural numbers is an infinite set. The sets A, B and C given above are finite sets and n(A) = 5, n(B) = 6 and n(C) = some finite number.

Definition 2 A set which is empty or consists of a definite number of elements is called *finite* otherwise, the set is called *infinite*.

Consider some examples :

(i) Let W be the set of the days of the week. Then W is finite.

(ii) Let S be the set of solutions of the equation $x^2 - 16 = 0$. Then S is finite.

(iii) Let G be the set of points on a line. Then G is infinite.

When we represent a set in the roster form, we write all the elements of the set within braces { }. It is not possible to write all the elements of an infinite set within braces { } because the numbers of elements of such a set is not finite. So, we represent some infinite set in the roster form by writing a few elements which clearly indicate the structure of the set followed (or preceded) by three dots.

For example, {1, 2, 3 . . .} is the set of natural numbers, {1, 3, 5, 7, . . .} is the set of odd natural numbers, {. . .,–3, –2, –1, 0,1, 2 ,3, . . .} is the set of integers. All these sets are infinite.

Note All infinite sets cannot be described in the roster form. For example, the set of real numbers cannot be described in this form, because the elements of this set do not follow any particular pattern.

Example 6 State which of the following sets are finite or infinite :

(i) $\{x : x \in \text{N and } (x - 1)(x - 2) = 0\}$

(ii) $\{x : x \in \text{N and } x^2 = 4\}$

(iii) $\{x : x \in \text{N and } 2x - 1 = 0\}$

(iv) $\{x : x \in \text{N and } x \text{ is prime}\}$

(v) $\{x : x \in \text{N and } x \text{ is odd}\}$

Solution (i) Given set = {1, 2}. Hence, it is finite.

(ii) Given set = {2}. Hence, it is finite.

(iii) Given set = ϕ. Hence, it is finite.

(iv) The given set is the set of all prime numbers and since set of prime numbers is infinite. Hence the given set is infinite

(v) Since there are infinite number of odd numbers, hence, the given set is infinite.

1.5 Equal Sets

Given two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal. Clearly, the two sets have exactly the same elements.

Definition 3 Two sets A and B are said to be *equal* if they have exactly the same elements and we write A = B. Otherwise, the sets are said to be *unequal* and we write A ≠ B.

We consider the following examples :

(i) Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B.

(ii) Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Then A and P are equal, since 2, 3 and 5 are the only prime factors of 30 and also these are less than 6.

Note A set does not change if one or more elements of the set are repeated. For example, the sets A = {1, 2, 3} and B = {2, 2, 1, 3, 3} are equal, since each element of A is in B and vice-versa. That is why we generally do not repeat any element in describing a set.

Example 7 Find the pairs of equal sets, if any, give reasons:

$A = \{0\}$, $B = \{x : x > 15 \text{ and } x < 5\}$,

$C = \{x : x - 5 = 0\}$, $D = \{x : x^2 = 25\}$,

$E = \{x : x \text{ is an integral positive root of the equation } x^2 - 2x - 15 = 0\}$.

Solution Since $0 \in A$ and 0 does not belong to any of the sets B, C, D and E, it follows that, $A \neq B$, $A \neq C$, $A \neq D$, $A \neq E$.

Since $B = \phi$ but none of the other sets are empty. Therefore $B \neq C$, $B \neq D$ and $B \neq E$. Also $C = \{5\}$ but $-5 \in D$, hence $C \neq D$.

Since $E = \{5\}$, $C = E$. Further, $D = \{-5, 5\}$ and $E = \{5\}$, we find that, $D \neq E$. Thus, the only pair of equal sets is C and E.

Example 8 Which of the following pairs of sets are equal? Justify your answer.

(i) X, the set of letters in "ALLOY" and B, the set of letters in "LOYAL".

(ii) $A = \{n : n \in Z \text{ and } n^2 \leq 4\}$ and $B = \{x : x \in R \text{ and } x^2 - 3x + 2 = 0\}$.

Solution (i) We have, X = {A, L, L, O, Y}, B = {L, O, Y, A, L}. Then X and B are equal sets as repetition of elements in a set do not change a set. Thus,

$$X = \{A, L, O, Y\} = B$$

(ii) $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2\}$. Since $0 \in A$ and $0 \notin B$, A and B are not equal sets.

EXERCISE 1.2

1. Which of the following are examples of the null set
   (i) Set of odd natural numbers divisible by 2
   (ii) Set of even prime numbers
   (iii) $\{x : x \text{ is a natural numbers}, x < 5 \text{ and } x > 7\}$
   (iv) $\{y : y \text{ is a point common to any two parallel lines}\}$
2. Which of the following sets are finite or infinite
   (i) The set of months of a year
   (ii) $\{1, 2, 3, \ldots\}$
   (iii) $\{1, 2, 3, \ldots 99, 100\}$
   (iv) The set of positive integers greater than 100
   (v) The set of prime numbers less than 99
3. State whether each of the following set is finite or infinite:
   (i) The set of lines which are parallel to the x-axis
   (ii) The set of letters in the English alphabet
   (iii) The set of numbers which are multiple of 5

(iv) The set of animals living on the earth

(v) The set of circles passing through the origin (0,0)

4. In the following, state whether A = B or not:

(i) A = $\{ a, b, c, d \}$ B = $\{ d, c, b, a \}$

(ii) A = $\{ 4, 8, 12, 16 \}$ B = $\{ 8, 4, 16, 18\}$

(iii) A = $\{2, 4, 6, 8, 10\}$ B = $\{ x : x$ is positive even integer and $x \leq 10\}$

(iv) A = $\{ x : x$ is a multiple of 10$\}$, B = $\{ 10, 15, 20, 25, 30, . . . \}$

5. Are the following pair of sets equal ? Give reasons.

(i) A = $\{2, 3\}$, B = $\{x : x$ is solution of $x^2 + 5x + 6 = 0\}$

(ii) A = $\{ x : x$ is a letter in the word FOLLOW$\}$
B = $\{ y : y$ is a letter in the word WOLF$\}$

6. From the sets given below, select equal sets :

A = $\{ 2, 4, 8, 12\}$, B = $\{ 1, 2, 3, 4\}$, C = $\{ 4, 8, 12, 14\}$, D = $\{ 3, 1, 4, 2\}$
E = $\{–1, 1\}$, F = $\{ 0, a\}$, G = $\{1, –1\}$, H = $\{ 0, 1\}$

1.6 Subsets

Consider the sets : X = set of all students in your school, Y = set of all students in your class.

We note that every element of Y is also an element of X; we say that Y is a subset of X. The fact that Y is subset of X is expressed in symbols as Y $\subset$ X. The symbol $\subset$ stands for 'is a subset of' or 'is contained in'.

Definition 4 A set A is said to be a subset of a set B if every element of A is also an element of B.

In other words, A $\subset$ B if whenever $a \in$ A, then $a \in$ B. It is often convenient to use the symbol "$\Rightarrow$" which means *implies*. Using this symbol, we can write the definiton of *subset* as follows:

$$A \subset B \text{ if } a \in A \Rightarrow a \in B$$

We read the above statement as **"***A is a subset of B if a is an element of A implies that a is also an element of B***"**. If A is not a subset of B, we write A $\not\subset$ B.

We may note that for A to be a subset of B, all that is needed is that every element of A is in B. It is possible that every element of B may or may not be in A. If it so happens that every element of B is also in A, then we shall also have B $\subset$ A. In this case, A and B are the same sets so that we have A $\subset$ B and B $\subset$ A $\Leftrightarrow$ A = B, where "$\Leftrightarrow$" is a symbol for two way implications, and is usually read as *if and only if* (briefly written as "iff").

It follows from the above definition that every set *A is a subset of itself,* i.e., A $\subset$ A. Since the empty set ϕ has no elements, we agree to say that ϕ *is a subset of every set***.** We now consider some examples :

(i) The set **Q** of rational numbers is a subset of the set **R** of real numbes, and we write $\mathbf{Q} \subset \mathrm{R}$.

(ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A and we write $\mathrm{B} \subset \mathrm{A}$.

(iii) Let $\mathrm{A} = \{1, 3, 5\}$ and $\mathrm{B} = \{x : x$ is an odd natural number less than $6\}$. Then $\mathrm{A} \subset \mathrm{B}$ and $\mathrm{B} \subset \mathrm{A}$ and hence $\mathrm{A} = \mathrm{B}$.

(iv) Let $\mathrm{A} = \{ a, e, i, o, u\}$ and $\mathrm{B} = \{ a, b, c, d\}$. Then A is not a subset of B, also B is not a subset of A.

Let A and B be two sets. If $\mathrm{A} \subset \mathrm{B}$ and $\mathrm{A} \neq \mathrm{B}$, then A is called a *proper subset of* B and B is called *superset* of A. For example,

$\mathrm{A} = \{1, 2, 3\}$ is a proper subset of $\mathrm{B} = \{1, 2, 3, 4\}$.

If a set A has only one element, we call it a *singleton set*. Thus, $\{ a \}$ is a singleton set.

Example 9 Consider the sets

$$\phi,\ \mathrm{A} = \{ 1, 3 \},\quad \mathrm{B} = \{1, 5, 9\},\quad \mathrm{C} = \{1, 3, 5, 7, 9\}.$$

Insert the symbol $\subset$ or $\not\subset$ between each of the following pair of sets:

(i) ϕ . . . B (ii) A . . . B (iii) A . . . C (iv) B . . . C

Solution (i) $\phi \subset \mathrm{B}$ as ϕ is a subset of every set.

(ii) $\mathrm{A} \not\subset \mathrm{B}$ as $3 \in \mathrm{A}$ and $3 \notin \mathrm{B}$

(iii) $\mathrm{A} \subset \mathrm{C}$ as $1, 3 \in \mathrm{A}$ also belongs to C

(iv) $\mathrm{B} \subset \mathrm{C}$ as each element of B is also an element of C.

Example 10 Let $\mathrm{A} = \{ a, e, i, o, u\}$ and $\mathrm{B} = \{ a, b, c, d\}$. Is A a subset of B ? No. (Why?). Is B a subset of A? No. (Why?)

Example 11 Let A, B and C be three sets. If $\mathrm{A} \in \mathrm{B}$ and $\mathrm{B} \subset \mathrm{C}$, is it true that $\mathrm{A} \subset \mathrm{C}$?. If not, give an example.

Solution No. Let $\mathrm{A} = \{1\}$, $\mathrm{B} = \{\{1\}, 2\}$ and $\mathrm{C} = \{\{1\}, 2, 3\}$. Here $\mathrm{A} \in \mathrm{B}$ as $\mathrm{A} = \{1\}$ and $\mathrm{B} \subset \mathrm{C}$. But $\mathrm{A} \not\subset \mathrm{C}$ as $1 \in \mathrm{A}$ and $1 \notin \mathrm{C}$.

Note that an element of a set can never be a subset of itself.

1.6.1 *Subsets of set of real numbers*

As noted in Section 1.6, there are many important subsets of **R**. We give below the names of some of these subsets.

The set of natural numbers $\mathbf{N} = \{1, 2, 3, 4, 5, \ldots\}$

The set of integers $\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of rational numbers $\mathbf{Q} = \{ x : x = \dfrac{p}{q}, p, q \in \mathbf{Z}$ and $q \neq 0\}$

which is read " **Q** is the set of all numbers x such that x equals the quotient $\frac{p}{q}$, where p and q are integers and q is not zero". Members of **Q** include -5 (which can be expressed as $-\frac{5}{1}$), $\frac{5}{7}$, $3\frac{1}{2}$ (which can be expressed as $\frac{7}{2}$) and $-\frac{11}{3}$.

The set of irrational numbers, denoted by **T**, is composed of all other real numbers. Thus $\mathbf{T} = \{x : x \in \mathbf{R} \text{ and } x \notin \mathbf{Q}\} = \mathbf{R} - \mathbf{Q}$., i.e., all real numbers that are not rational. Members of **T** include $\sqrt{2}$, $\sqrt{5}$ and π.

Some of the obvious relations among these subsets are:

$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}, \mathbf{Q} \subset \mathbf{R}, \mathbf{T} \subset \mathbf{R}, \mathbf{N} \not\subset \mathbf{T}$.

1.6.2 ***Intervals as subsets of R*** Let $a, b \in \mathbf{R}$ and $a < b$. Then the set of real numbers $\{ y : a < y < b\}$ is called an *open interval* and is denoted by (a, b). All the points between a and b belong to the open interval (a, b) but a, b themselves do not belong to this interval.

The interval which contains the end points also is called *closed interval* and is denoted by $[a, b]$. Thus

$[a, b] = \{x : a \le x \le b\}$

We can also have intervals closed at one end and open at the other, i.e.,

$[a, b) = \{x : a \le x < b\}$ is an *open interval* from a to b, including a but excluding b.

$(a, b] = \{ x : a < x \le b \}$ is an *open interval* from a to b including b but excluding a.

These notations provide an alternative way of designating the subsets of set of real numbers. For example , if A = (–3, 5) and B = [–7, 9], then A ⊂ B. The set $[0, \infty)$ defines the set of non-negative real numbers, while set $(-\infty, 0)$ defines the set of negative real numbers. The set $(-\infty, \infty)$ describes the set of real numbers in relation to a line extending from $-\infty$ to ∞.

On real number line, various types of intervals described above as subsets of **R**, are shown in the Fig 1.1.

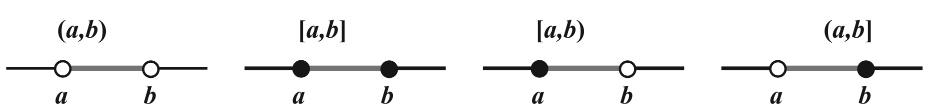


Fig 1.1

Here, we note that an interval contains infinitely many points.

For example, the set $\{x : x \in \mathbf{R}, -5 < x \le 7\}$, written in set-builder form, can be written in the form of interval as $(-5, 7]$ and the interval $[-3, 5)$ can be written in set-builder form as $\{x : -3 \le x < 5\}$.

The number $(b - a)$ is called the *length of any of the intervals* (a, b), $[a, b]$, $[a, b)$ or $(a, b]$.

1.7 Power Set

Consider the set {1, 2}. Let us write down all the subsets of the set {1, 2}. We know that ϕ is a subset of every set . So, ϕ is a subset of {1, 2}. We see that {1} and { 2 }are also subsets of {1, 2}. Also, we know that every set is a subset of itself. So, { 1, 2 } is a subset of {1, 2}. Thus, the set { 1, 2 } has, in all, four subsets, viz. ϕ, { 1 }, { 2 } and { 1, 2 }. The set of all these subsets is called the *power set* of { 1, 2 }.

Definition 5 The collection of all subsets of a set A is called the *power set* of A. It is denoted by P(A). In P(A), every element is a set.

Thus, as in above, if A = { 1, 2 }, then

$$P(A) = \{ \phi, \{ 1 \}, \{ 2 \}, \{ 1,2 \}\}$$

Also, note that $n [P (A)] = 4 = 2^2$

In general, if A is a set with $n(A) = m$, then it can be shown that $n [P(A)] = 2^m$.

1.8 Universal Set

Usually, in a particular context, we have to deal with the elements and subsets of a basic set which is relevant to that particular context. For example, while studying the system of numbers, we are interested in the set of natural numbers and its subsets such as the set of all prime numbers, the set of all even numbers, and so forth. This basic set is called the "*Universal Set*". The universal set is usually denoted by U, and all its subsets by the letters A, B, C, etc.

For example, for the set of all integers, the universal set can be the set of rational numbers or, for that matter, the set **R** of real numbers. For another example, in human population studies, the universal set consists of all the people in the world.

EXERCISE 1.3

1. Make correct statements by filling in the symbols $\subset$ or $\not\subset$ in the blank spaces :
   (i) { 2, 3, 4 } . . . { 1, 2, 3, 4,5 } (ii) { a, b, c } . . . { b, c, d }
   (iii) {x : x is a student of Class XI of your school}. . .{x : x student of your school}
   (iv) {x : x is a circle in the plane} . . .{x : x is a circle in the same plane with radius 1 unit}
   (v) {x : x is a triangle in a plane} . . . {x : x is a rectangle in the plane}
   (vi) {x : x is an equilateral triangle in a plane} . . . {x : x is a triangle in the same plane}
   (vii) {x : x is an even natural number} . . . {x : x is an integer}

2. Examine whether the following statements are true or false:
   (i) $\{ a, b \} \not\subset \{ b, c, a \}$
   (ii) $\{ a, e \} \subset \{ x : x$ is a vowel in the English alphabet$\}$
   (iii) $\{ 1, 2, 3 \} \subset \{ 1, 3, 5 \}$
   (iv) $\{ a \} \subset \{ a, b, c \}$
   (v) $\{ a \} \in \{ a, b, c \}$
   (vi) $\{ x : x$ is an even natural number less than $6\} \subset \{ x : x$ is a natural number which divides $36\}$
3. Let $A = \{ 1, 2, \{ 3, 4 \}, 5 \}$. Which of the following statements are incorrect and why?
   (i) $\{3, 4\} \subset A$ (ii) $\{3, 4\} \in A$ (iii) $\{\{3, 4\}\} \subset A$
   (iv) $1 \in A$ (v) $1 \subset A$ (vi) $\{1, 2, 5\} \subset A$
   (vii) $\{1, 2, 5\} \in A$ (viii) $\{1, 2, 3\} \subset A$ (ix) $\phi \in A$
   (x) $\phi \subset A$ (xi) $\{\phi\} \subset A$
4. Write down all the subsets of the following sets
   (i) $\{a\}$ (ii) $\{a, b\}$ (iii) $\{1, 2, 3\}$ (iv) ϕ
5. How many elements has P(A), if $A = \phi$?
6. Write the following as intervals :
   (i) $\{x : x \in R, -4 < x \leq 6\}$ (ii) $\{x : x \in R, -12 < x < -10\}$
   (iii) $\{x : x \in R, 0 \leq x < 7\}$ (iv) $\{x : x \in R, 3 \leq x \leq 4\}$
7. Write the following intervals in set-builder form :
   (i) $(-3, 0)$ (ii) $[6, 12]$ (iii) $(6, 12]$ (iv) $[-23, 5)$
8. What universal set(s) would you propose for each of the following :
   (i) The set of right triangles. (ii) The set of isosceles triangles.
9. Given the sets $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{0, 2, 4, 6, 8\}$, which of the following may be considered as universal set (s) for all the three sets A, B and C
   (i) $\{0, 1, 2, 3, 4, 5, 6\}$
   (ii) ϕ
   (iii) $\{0,1,2,3,4,5,6,7,8,9,10\}$
   (iv) $\{1,2,3,4,5,6,7,8\}$

1.9 Venn Diagrams

Most of the relationships between sets can be represented by means of diagrams which are known as *Venn diagrams*. Venn diagrams are named after the English logician, John Venn (1834-1883). These diagrams consist of rectangles and closed curves usually circles. The universal set is represented usually by a rectangle and its subsets by circles.

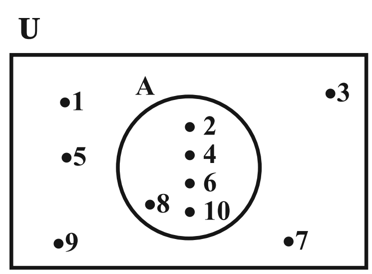


Fig 1.2

In Venn diagrams, the elements of the sets are written in their respective circles (Figs 1.2 and 1.3)

Illustration 1 In Fig 1.2, U = {1,2,3, ..., 10} is the universal set of which

A = {2,4,6,8,10} is a subset.

Illustration 2 In Fig 1.3, U = {1,2,3, ..., 10} is the universal set of which

A = {2,4,6,8,10} and B = {4, 6} are subsets, and also B ⊂ A.

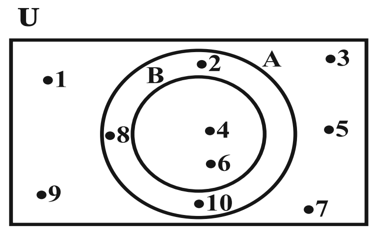


Fig 1.3

The reader will see an extensive use of the Venn diagrams when we discuss the union, intersection and difference of sets.

1.10 Operations on Sets

In earlier classes, we have learnt how to perform the operations of addition, subtraction, multiplication and division on numbers. Each one of these operations was performed on a pair of numbers to get another number. For example, when we perform the operation of addition on the pair of numbers 5 and 13, we get the number 18. Again, performing the operation of multiplication on the pair of numbers 5 and 13, we get 65. Similarly, there are some operations which when performed on two sets give rise to another set. We will now define certain operations on sets and examine their properties. Henceforth, we will refer all our sets as subsets of some universal set.

1.10.1 ***Union of sets*** Let A and B be any two sets. The union of A and B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. The symbol '∪' is used to denote the *union*. *Symbolically, we write* A ∪ B *and usually read as 'A union B'*.

Example 12 Let A = { 2, 4, 6, 8} and B = { 6, 8, 10, 12}. Find A ∪ B.

Solution We have A ∪ B = { 2, 4, 6, 8, 10, 12}

Note that the common elements 6 and 8 have been taken only once while writing A ∪ B.

Example 13 Let A = { *a, e, i, o, u* } and B = { *a, i, u* }. Show that A ∪ B = A

Solution We have, A ∪ B = { *a, e, i, o, u* } = A.

This example illustrates that union of sets A and its subset B is the set A itself, i.e., if B ⊂ A, then A ∪ B = A.

Example 14 Let X = {Ram, Geeta, Akbar} be the set of students of Class XI, who are in school hockey team. Let Y = {Geeta, David, Ashok} be the set of students from Class XI who are in the school football team. Find X ∪ Y and interpret the set.

Solution We have, X ∪ Y = {Ram, Geeta, Akbar, David, Ashok}. This is the set of students from Class XI who are in the hockey team or the football team or both.

Thus, we can define the union of two sets as follows:

Definition 6 The union of two sets A and B is the set C which consists of all those elements which are either in A or in B (including those which are in both). In symbols, we write. $A \cup B = \{ x : x \in A \text{ or } x \in B \}$

The union of two sets can be represented by a Venn diagram as shown in Fig 1.4.

The shaded portion in Fig 1.4 represents $A \cup B$.

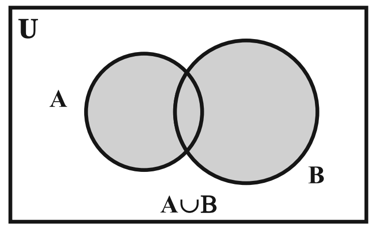


Fig 1.4

Some Properties of the Operation of Union

(i) $A \cup B = B \cup A$ (Commutative law)

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)

(iii) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of $\cup$)

(iv) $A \cup A = A$ (Idempotent law)

(v) $U \cup A = U$ (Law of U)

1.10.2 ***Intersection of sets*** The intersection of sets A and B is the set of all elements which are common to both A and B. The symbol '$\cap$' is used to denote the *intersection*. The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Example 15 Consider the sets A and B of Example 12. Find $A \cap B$.

Solution We see that 6, 8 are the only elements which are common to both A and B. Hence $A \cap B = \{ 6, 8 \}$.

Example 16 Consider the sets X and Y of Example 14. Find $X \cap Y$.

Solution We see that element 'Geeta' is the only element common to both. Hence, $X \cap Y = \{\text{Geeta}\}$.

Example 17 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{ 2, 3, 5, 7 \}$. Find $A \cap B$ and hence show that $A \cap B = B$.

Solution We have $A \cap B = \{ 2, 3, 5, 7 \} = B$. We note that $B \subset A$ and that $A \cap B = B$.

Definition 7 The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

The shaded portion in Fig 1.5 indicates the interseciton of A and B.

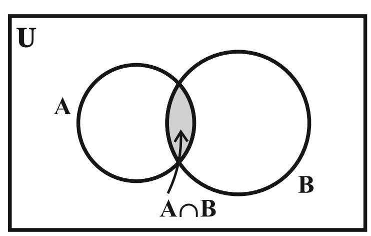


Fig 1.5

If A and B are two sets such that $A \cap B = \phi$, then A and B are called *disjoint sets.*

For example, let $A = \{ 2, 4, 6, 8 \}$ and $B = \{ 1, 3, 5, 7 \}$. Then A and B are disjoint sets, because there are no elements which are common to A and B. The disjoint sets can be represented by means of Venn diagram as shown in the Fig 1.6

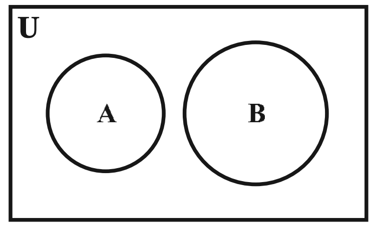


Fig 1.6

In the above diagram, A and B are disjoint sets.

Some Properties of Operation of Intersection

(i) $A \cap B = B \cap A$ (Commutative law).

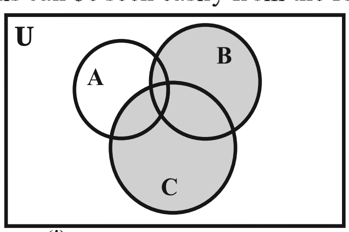
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law).

(iii) $\phi \cap A = \phi$, $U \cap A = A$ (Law of ϕ and U).

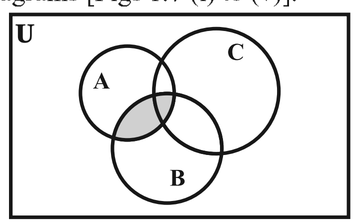
(iv) $A \cap A = A$ (Idempotent law)

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law) i. e., $\cap$ distributes over $\cup$

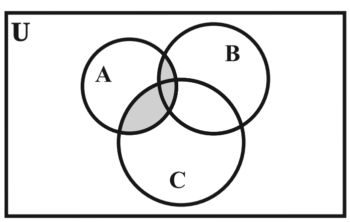
This can be seen easily from the following Venn diagrams [Figs 1.7 (i) to (v)].



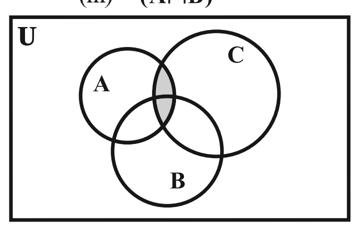
(i) **$(B \cup C)$**



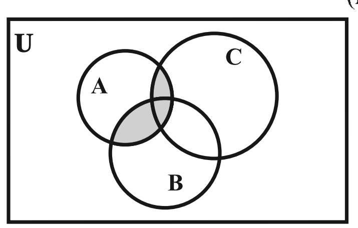
(iii) **$(A \cap B)$**



(ii) **$A \cap (B \cup C)$**



(iv) **$(A \cap C)$**



(v) $(A \cap B) \cup (A \cap C)$

Figs 1.7 (i) to (v)

1.10.3 ***Difference of sets*** The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write A – B and read as " A minus B".

Example 18 Let A = { 1, 2, 3, 4, 5, 6}, B = { 2, 4, 6, 8 }. Find A – B and B – A.

Solution We have, A – B = { 1, 3, 5 }, since the elements 1, 3, 5 belong to A but not to B and B – A = { 8 }, since the element 8 belongs to B and not to A.
We note that A – B ≠ B – A.

Example 19 Let V = { *a, e, i, o, u* } and B = { *a, i, k, u*}. Find V – B and B – V

Solution We have, V – B = { *e, o* }, since the elements *e, o* belong to V but not to B and B – V = { *k* }, since the element *k* belongs to B but not to V.

We note that V – B ≠ B – V. Using the set-builder notation, we can rewrite the definition of difference as

$$A – B = \{ x : x \in A \text{ and } x \notin B \}$$

The difference of two sets A and B can be represented by Venn diagram as shown in Fig 1.8. The shaded portion represents the difference of the two sets A and B.

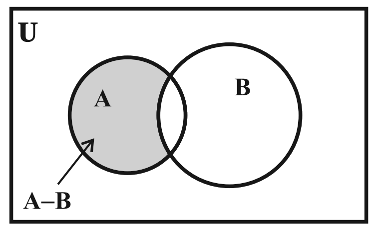


Fig 1.8

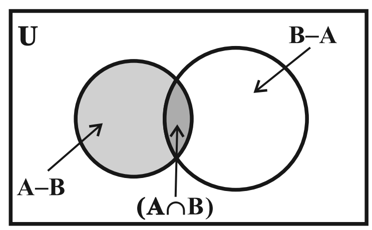


Fig 1.9

Remark The sets A – B, A ∩ B and B – A are mutually disjoint sets, i.e., the intersection of any of these two sets is the null set as shown in Fig 1.9.

EXERCISE 1.4

1. Find the union of each of the following pairs of sets :
   (i) X = {1, 3, 5} Y = {1, 2, 3}
   (ii) A = [*a, e, i, o, u*} B = {*a, b, c*}
   (iii) A = {x : x is a natural number and multiple of 3}
   B = {x : x is a natural number less than 6}
   (iv) A = {x : x is a natural number and $1 < x \leq 6$ }
   B = {x : x is a natural number and $6 < x < 10$ }
   (v) A = {1, 2, 3}, B = ϕ
2. Let A = { *a, b* }, B = {*a, b, c*}. Is A ⊂ B ? What is A ∪ B ?
3. If A and B are two sets such that A ⊂ B, then what is A ∪ B ?
4. If A = {1, 2, 3, 4}, B = {3, 4, 5, 6}, C = {5, 6, 7, 8 }and D = { 7, 8, 9, 10 }; find

(i) $A \cup B$ (ii) $A \cup C$ (iii) $B \cup C$ (iv) $B \cup D$
(v) $A \cup B \cup C$ (vi) $A \cup B \cup D$ (vii) $B \cup C \cup D$

5. Find the intersection of each pair of sets of question 1 above.
6. If A = { 3, 5, 7, 9, 11 }, B = {7, 9, 11, 13}, C = {11, 13, 15}and D = {15, 17}; find
(i) $A \cap B$ (ii) $B \cap C$ (iii) $A \cap C \cap D$
(iv) $A \cap C$ (v) $B \cap D$ (vi) $A \cap (B \cup C)$
(vii) $A \cap D$ (viii) $A \cap (B \cup D)$ (ix) $(A \cap B) \cap (B \cup C)$
(x) $(A \cup D) \cap (B \cup C)$
7. If A = {*x* : *x* is a natural number }, B = {*x* : *x* is an even natural number}
C = {*x* : *x* is an odd natural number}andD = {*x* : *x* is a prime number }, find
(i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$
(iv) $B \cap C$ (v) $B \cap D$ (vi) $C \cap D$
8. Which of the following pairs of sets are disjoint
(i) {1, 2, 3, 4} and {*x* : *x* is a natural number and $4 \le x \le 6$ }
(ii) { *a, e, i, o, u* } and { *c, d, e, f* }
(iii) {*x* : *x* is an even integer } and {*x* : *x* is an odd integer}
9. If A = {3, 6, 9, 12, 15, 18, 21}, B = { 4, 8, 12, 16, 20 },
C = { 2, 4, 6, 8, 10, 12, 14, 16 }, D = {5, 10, 15, 20 }; find
(i) A – B (ii) A – C (iii) A – D (iv) B – A
(v) C – A (vi) D – A (vii) B – C (viii) B – D
(ix) C – B (x) D – B (xi) C – D (xii) D – C
10. If X= { *a, b, c, d* } and Y = { *f, b, d, g*}, find
(i) X – Y (ii) Y – X (iii) $X \cap Y$
11. If **R** is the set of real numbers and **Q** is the set of rational numbers, then what is **R** – **Q**?
12. State whether each of the following statement is true or false. Justify your answer.
(i) { 2, 3, 4, 5 } and { 3, 6} are disjoint sets.
(ii) { *a, e, i, o, u* } and { *a, b, c, d* }are disjoint sets.
(iii) { 2, 6, 10, 14 } and { 3, 7, 11, 15} are disjoint sets.
(iv) { 2, 6, 10 } and { 3, 7, 11} are disjoint sets.

1.11 Complement of a Set

Let U be the universal set which consists of all prime numbers and A be the subset of U which consists of all those prime numbers that are not divisors of 42. Thus, A = {*x* : $x \in$ U and *x* is not a divisor of 42 }. We see that $2 \in$ U but $2 \notin$ A, because 2 is divisor of 42. Similarly, $3 \in$ U but $3 \notin$ A, and $7 \in$ U but $7 \notin$ A. Now 2, 3 and 7 are the only elements of U which do not belong to A. The set of these three prime numbers, i.e., the set {2, 3, 7} is called the *Complement* of A with respect to U, and is denoted by

A'. So we have $A' = \{2, 3, 7\}$. Thus, we see that

$$A' = \{x : x \in U \text{ and } x \notin A\}.$$ This leads to the following definition.

Definition 8 Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A' to denote the complement of A with respect to U. Thus,

$$A' = \{x : x \in U \text{ and } x \notin A\}. \text{ Obviously } A' = U - A$$

We note that the complement of a set A can be looked upon, alternatively, as the difference between a universal set U and the set A.

Example 20 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$. Find A'.

Solution We note that 2, 4, 6, 8, 10 are the only elements of U which do not belong to A. Hence $A' = \{2, 4, 6, 8, 10\}$.

Example 21 Let U be universal set of all the students of Class XI of a coeducational school and A be the set of all girls in Class XI. Find A'.

Solution Since A is the set of all girls, A' is clearly the set of all boys in the class.

> **Note** If A is a subset of the universal set U, then its complement A' is also a subset of U.
>
> Again in Example 20 above, we have $A' = \{2, 4, 6, 8, 10\}$
>
> Hence $(A')' = \{x : x \in U \text{ and } x \notin A'\}$
>
> $= \{1, 3, 5, 7, 9\} = A$
>
> It is clear from the definition of the complement that for any subset of the universal set U, we have $(A')' = A$

Now, we want to find the results for $(A \cup B)'$ and $A' \cap B'$ in the followng example.

Example 22 Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3\}$ and $B = \{3, 4, 5\}$.

Find A', B', $A' \cap B'$, $A \cup B$ and hence show that $(A \cup B)' = A' \cap B'$.

Solution Clearly $A' = \{1, 4, 5, 6\}$, $B' = \{1, 2, 6\}$. Hence $A' \cap B' = \{1, 6\}$

Also $A \cup B = \{2, 3, 4, 5\}$, so that $(A \cup B)' = \{1, 6\}$

$$(A \cup B)' = \{1, 6\} = A' \cap B'$$

It can be shown that the above result is true in general. If A and B are any two subsets of the universal set U, then

$(A \cup B)' = A' \cap B'$. Similarly, $(A \cap B)' = A' \cup B'$. These two results are stated in words as follows :

The complement of the union of two sets is the intersection of their complements and the complement of the intersection of two sets is the union of their complements. These are called *De Morgan's laws*. These are named after the mathematician De Morgan.

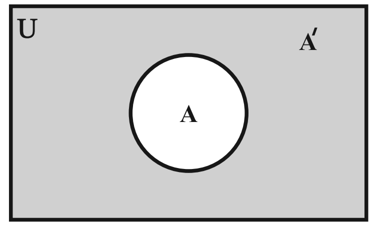


Fig 1.10

The complement A′ of a set A can be represented by a Venn diagram as shown in Fig 1.10.

The shaded portion represents the complement of the set A.

Some Properties of Complement Sets

1. Complement laws: (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$

2. De Morgan's law: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

3. Law of double complementation : $(A')' = A$

4. Laws of empty set and universal set $\phi' = U$ and $U' = \phi$.

These laws can be verified by using Venn diagrams.

EXERCISE 1.5

1. Let U = { 1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = { 1, 2, 3, 4}, B = { 2, 4, 6, 8 } and C = { 3, 4, 5, 6 }. Find (i) A' (ii) B' (iii) $(A \cup C)'$ (iv) $(A \cup B)'$ (v) $(A')'$ (vi) $(B - C)'$
2. If U = { *a, b, c, d, e, f, g, h*}, find the complements of the following sets :
   (i) A = {*a, b, c*} (ii) B = {*d, e, f, g*}
   (iii) C = {*a, c, e, g*} (iv) D = { *f, g, h, a*}
3. Taking the set of natural numbers as the universal set, write down the complements of the following sets:
   (i) {x : x is an even natural number} (ii) { x : x is an odd natural number }
   (iii) {x : x is a positive multiple of 3} (iv) { x : x is a prime number }
   (v) {x : x is a natural number divisible by 3 and 5}
   (vi) { x : x is a perfect square } (vii) { x : x is a perfect cube}
   (viii) { $x : x + 5 = 8$ } (ix) { $x : 2x + 5 = 9$}
   (x) { $x : x \geq 7$ } (xi) { $x : x \in N$ and $2x + 1 > 10$ }
4. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = {2, 4, 6, 8} and B = { 2, 3, 5, 7}. Verify that
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
5. Draw appropriate Venn diagram for each of the following :
   (i) $(A \cup B)'$, (ii) $A' \cap B'$, (iii) $(A \cap B)'$, (iv) $A' \cup B'$
6. Let U be the set of all triangles in a plane. If A is the set of all triangles with at least one angle different from 60°, what is A′?

7. Fill in the blanks to make each of the following a true statement :

(i) $A \cup A' = \ldots$ (ii) $\phi' \cap A = \ldots$

(iii) $A \cap A' = \ldots$ (iv) $U' \cap A = \ldots$

1.12 Practical Problems on Union and Intersection of Two Sets

In earlier Section, we have learnt union, intersection and difference of two sets. In this Section, we will go through some practical problems related to our daily life.The formulae derived in this Section will also be used in subsequent Chapter on Probability (Chapter 16).

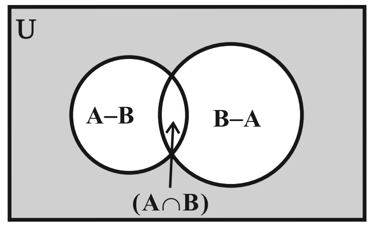


Fig 1.11

Let A and B be finite sets. If $A \cap B = \phi$, then

(i) $n(A \cup B) = n(A) + n(B)$ ... (1)

The elements in $A \cup B$ are either in A or in B but not in both as $A \cap B = \phi$. So, (1) follows immediately.

In general, if A and B are finite sets, then

(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ... (2)

Note that the sets A – B, $A \cap B$ and B – A are disjoint and their union is $A \cup B$ (Fig 1.11). Therefore

$$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$$
$$= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B)$$
$$= n(A) + n(B) - n(A \cap B), \text{ which verifies (2)}$$

(iii) If A, B and C are finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) \quad \ldots (3)$$

In fact, we have

$$n(A \cup B \cup C) = n(A) + n(B \cup C) - n[A \cap (B \cup C)] \quad [\text{ by (2) }]$$
$$= n(A) + n(B) + n(C) - n(B \cap C) - n[A \cap (B \cup C)] \quad [\text{ by (2) }]$$

Since $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we get

$$n[A \cap (B \cup C)] = n(A \cap B) + n(A \cap C) - n[(A \cap B) \cap (A \cap C)]$$
$$= n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)$$

Therefore

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

This proves (3).

Example 23 If X and Y are two sets such that $X \cup Y$ has 50 elements, X has 28 elements and Y has 32 elements, how many elements does $X \cap Y$ have ?

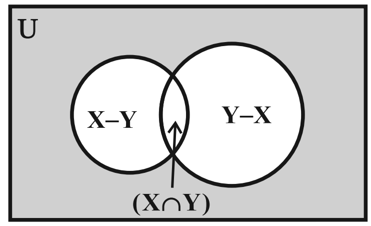


Fig 1.12

Solution Given that

$n (X \cup Y) = 50$, $n (X) = 28$, $n (Y) = 32$,
$n (X \cap Y) = ?$

By using the formula

$$n (X \cup Y) = n (X) + n (Y) - n (X \cap Y),$$

we find that

$$n (X \cap Y) = n (X) + n (Y) - n (X \cup Y)$$
$$= 28 + 32 - 50 = 10$$

Alternatively, suppose $n (X \cap Y) = k$, then

$n (X - Y) = 28 - k$, $n (Y - X) = 32 - k$ (by Venn diagram in Fig 1.12)

This gives $50 = n (X \cup Y) = n (X - Y) + n (X \cap Y) + n (Y - X)$
$= (28 - k) + k + (32 - k)$

Hence $k = 10$.

Example 24 In a school there are 20 teachers who teach mathematics or physics. Of these, 12 teach mathematics and 4 teach both physics and mathematics. How many teach physics ?

Solution Let M denote the set of teachers who teach mathematics and P denote the set of teachers who teach physics. In the statement of the problem, the word 'or' gives us a clue of union and the word 'and' gives us a clue of intersection. We, therefore, have

$$n (M \cup P) = 20 \text{ , } n (M) = 12 \text{ and } n (M \cap P) = 4$$

We wish to determine $n (P)$.

Using the result

$$n (M \cup P) = n (M) + n (P) - n (M \cap P),$$

we obtain

$$20 = 12 + n (P) - 4$$

Thus $n (P) = 12$

Hence 12 teachers teach physics.

Example 25 In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football ?

Solution Let X be the set of students who like to play cricket and Y be the set of students who like to play football. Then $X \cup Y$ is the set of students who like to play at least one game, and $X \cap Y$ is the set of students who like to play both games.

Given $n (X) = 24$, $n (Y) = 16$, $n (X \cup Y) = 35$, $n (X \cap Y) = ?$

Using the formula $n (X \cup Y) = n (X) + n (Y) - n (X \cap Y)$, we get

$$35 = 24 + 16 - n (X \cap Y)$$

Thus, $n(X \cap Y) = 5$
i.e., 5 students like to play both games.

Example 26 In a survey of 400 students in a school, 100 were listed as taking apple juice, 150 as taking orange juice and 75 were listed as taking both apple as well as orange juice. Find how many students were taking neither apple juice nor orange juice.

Solution Let U denote the set of surveyed students and A denote the set of students taking apple juice and B denote the set of students taking orange juice. Then

$$n(U) = 400,\ n(A) = 100,\ n(B) = 150 \text{ and } n(A \cap B) = 75.$$

Now $n(A' \cap B') = n(A \cup B)'$

$$= n(U) - n(A \cup B)$$
$$= n(U) - n(A) - n(B) + n(A \cap B)$$
$$= 400 - 100 - 150 + 75 = 225$$

Hence 225 students were taking neither apple juice nor orange juice.

Example 27 There are 200 individuals with a skin disorder, 120 had been exposed to the chemical C_1, 50 to chemical C_2, and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to

(i) Chemical C_1 but not chemical C_2 (ii) Chemical C_2 but not chemical C_1

(iii) Chemical C_1 or chemical C_2

Solution Let U denote the universal set consisting of individuals suffering from the skin disorder, A denote the set of individuals exposed to the chemical C_1 and B denote the set of individuals exposed to the chemical C_2.

Here $n(U) = 200,\ n(A) = 120,\ n(B) = 50$ and $n(A \cap B) = 30$

(i) From the Venn diagram given in Fig 1.13, we have

$$A = (A - B) \cup (A \cap B).$$

$n(A) = n(A - B) + n(A \cap B)$ (Since $A - B$) and $A \cap B$ are disjoint.)

or $n(A - B) = n(A) - n(A \cap B) = 120 - 30 = 90$

Hence, the number of individuals exposed to chemical C_1 but not to chemical C_2 is 90.

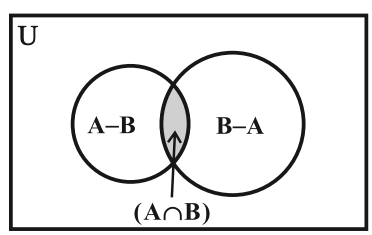


Fig 1.13

(ii) From the Fig 1.13, we have

$$B = (B - A) \cup (A \cap B).$$

and so, $n(B) = n(B - A) + n(A \cap B)$

(Since $B - A$ and $A \cap B$ are disjoint.)

or $n(B - A) = n(B) - n(A \cap B)$

$$= 50 - 30 = 20$$

Thus, the number of individuals exposed to chemical C_2 and not to chemical C_1 is 20.

(iii) The number of individuals exposed either to chemical C_1 or to chemical C_2, i.e.,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 120 + 50 - 30 = 140.$$

EXERCISE 1.6

1. If X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$, find $n(X \cap Y)$.
2. If X and Y are two sets such that $X \cup Y$ has 18 elements, X has 8 elements and Y has 15 elements ; how many elements does $X \cap Y$ have?
3. In a group of 400 people, 250 can speak Hindi and 200 can speak English. How many people can speak both Hindi and English?
4. If S and T are two sets such that S has 21 elements, T has 32 elements, and $S \cap T$ has 11 elements, how many elements does $S \cup T$ have?
5. If X and Y are two sets such that X has 40 elements, $X \cup Y$ has 60 elements and $X \cap Y$ has 10 elements, how many elements does Y have?
6. In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many people like both coffee and tea?
7. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket? How many like tennis?
8. In a committee, 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. How many speak at least one of these two languages?

Miscellaneous Examples

Example 28 Show that the set of letters needed to spell " CATARACT " and the set of letters needed to spell " TRACT" are equal.

Solution Let X be the set of letters in "CATARACT". Then

$$X = \{ C, A, T, R \}$$

Let Y be the set of letters in " TRACT". Then

$$Y = \{ T, R, A, C, T \} = \{ T, R, A, C \}$$

Since every element in X is in Y and every element in Y is in X. It follows that X = Y.

Example 29 List all the subsets of the set $\{ -1, 0, 1 \}$.

Solution Let $A = \{ -1, 0, 1 \}$. The subset of A having no element is the empty set ϕ. The subsets of A having one element are $\{ -1 \}$, $\{ 0 \}$, $\{ 1 \}$. The subsets of A having two elements are $\{-1, 0\}$, $\{-1, 1\}$, $\{0, 1\}$. The subset of A having three elements of A is A itself. So, all the subsets of A are ϕ, $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$, $\{-1, 1\}$, $\{0, 1\}$ and $\{-1, 0, 1\}$.

Example 30 Show that $A \cup B = A \cap B$ implies $A = B$

Solution Let $a \in A$. Then $a \in A \cup B$. Since $A \cup B = A \cap B$, $a \in A \cap B$. So $a \in B$. Therefore, $A \subset B$. Similarly, if $b \in B$, then $b \in A \cup B$. Since

$A \cup B = A \cap B$, $b \in A \cap B$. So, $b \in A$. Therefore, $B \subset A$. Thus, $A = B$

Example 31 For any sets A and B, show that

$P(A \cap B) = P(A) \cap P(B)$.

Solution Let $X \in P(A \cap B)$. Then $X \subset A \cap B$. So, $X \subset A$ and $X \subset B$. Therefore, $X \in P(A)$ and $X \in P(B)$ which implies $X \in P(A) \cap P(B)$. This gives $P(A \cap B) \subset P(A) \cap P(B)$. Let $Y \in P(A) \cap P(B)$. Then $Y \in P(A)$ and $Y \in P(B)$. So, $Y \subset A$ and $Y \subset B$. Therefore, $Y \subset A \cap B$, which implies $Y \in P(A \cap B)$. This gives $P(A) \cap P(B) \subset P(A \cap B)$

Hence $P(A \cap B) = P(A) \cap P(B)$.

Example 32 A market research group conducted a survey of 1000 consumers and reported that 720 consumers like product A and 450 consumers like product B, what is the least number that must have liked both products?

Solution Let U be the set of consumers questioned, S be the set of consumers who liked the product A and T be the set of consumers who like the product B. Given that

$n(U) = 1000$, $n(S) = 720$, $n(T) = 450$

So $\quad n(S \cup T) = n(S) + n(T) - n(S \cap T)$

$= 720 + 450 - n(S \cap T) = 1170 - n(S \cap T)$

Therefore, $n(S \cup T)$ is maximum when $n(S \cap T)$ is least. But $S \cup T \subset U$ implies $n(S \cup T) \leq n(U) = 1000$. So, maximum values of $n(S \cup T)$ is 1000. Thus, the least value of $n(S \cap T)$ is 170. Hence, the least number of consumers who liked both products is 170.

Example 33 Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

Solution Let U be the set of car owners investigated, M be the set of persons who owned car A and S be the set of persons who owned car B.

Given that $\quad n(U) = 500$, $n(M) = 400$, $n(S) = 200$ and $n(S \cap M) = 50$.

Then $\quad n(S \cup M) = n(S) + n(M) - n(S \cap M) = 200 + 400 - 50 = 550$

But $S \cup M \subset U$ implies $n(S \cup M) \leq n(U)$.

This is a contradiction. So, the given data is incorrect.

Example 34 A college warded 38 medals in football, 15 in basketball and 20 in cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports ?

Solution Let F, B and C denote the set of men who received medals in football, basketball and cricket, respectively.

Then $n(F) = 38$, $n(B) = 15$, $n(C) = 20$

$n(F \cup B \cup C) = 58$ and $n(F \cap B \cap C) = 3$

Therefore, $n(F \cup B \cup C) = n(F) + n(B) + n(C) - n(F \cap B) - n(F \cap C) - n(B \cap C) + n(F \cap B \cap C)$,

gives $n(F \cap B) + n(F \cap C) + n(B \cap C) = 18$

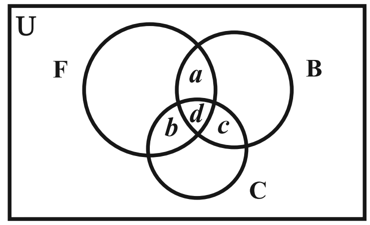


Fig 1.14

Consider the Venn diagram as given in Fig 1.14

Here, a denotes the number of men who got medals in football and basketball only, b denotes the number of men who got medals in football and cricket only, c denotes the number of men who got medals in basket ball and cricket only and d denotes the number of men who got medal in all the three. Thus, $d = n(F \cap B \cap C) = 3$ and $a + d + b + d + c + d = 18$

Therefore $a + b + c = 9$,

which is the number of people who got medals in exactly two of the three sports.

Miscellaneous Exercise on Chapter 1

1. Decide, among the following sets, which sets are subsets of one and another:
   $A = \{ x : x \in \mathbf{R}$ and x satisfy $x^2 - 8x + 12 = 0 \}$,
   $B = \{ 2, 4, 6 \}$, $C = \{ 2, 4, 6, 8, \ldots \}$, $D = \{ 6 \}$.
2. In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.
   (i) If $x \in A$ and $A \in B$, then $x \in B$
   (ii) If $A \subset B$ and $B \in C$, then $A \in C$
   (iii) If $A \subset B$ and $B \subset C$, then $A \subset C$
   (iv) If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$
   (v) If $x \in A$ and $A \not\subset B$, then $x \in B$
   (vi) If $A \subset B$ and $x \notin B$, then $x \notin A$
3. Let A, B, and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.
4. Show that the following four conditions are equivalent :
   (i) $A \subset B$ (ii) $A - B = \phi$ (iii) $A \cup B = B$ (iv) $A \cap B = A$
5. Show that if $A \subset B$, then $C - B \subset C - A$.
6. Assume that $P(A) = P(B)$. Show that $A = B$
7. Is it true that for any sets A and B, $P(A) \cup P(B) = P(A \cup B)$? Justify your answer.

8. Show that for any sets A and B,
$A = (A \cap B) \cup (A - B)$ and $A \cup (B - A) = (A \cup B)$
9. Using properties of sets, show that
(i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$.
10. Show that $A \cap B = A \cap C$ need not imply $B = C$.
11. Let A and B be sets. If $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, show that $A = B$.
(**Hints** $A = A \cap (A \cup X)$, $B = B \cap (B \cup X)$ and use Distributive law)
12. Find sets A, B and C such that $A \cap B$, $B \cap C$ and $A \cap C$ are non-empty sets and $A \cap B \cap C = \phi$.
13. In a survey of 600 students in a school, 150 students were found to be taking tea and 225 taking coffee, 100 were taking both tea and coffee. Find how many students were taking neither tea nor coffee?
14. In a group of students, 100 students know Hindi, 50 know English and 25 know both. Each of the students knows either Hindi or English. How many students are there in the group?
15. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find:
(i) the number of people who read at least one of the newspapers.
(ii) the number of people who read exactly one newspaper.
16. In a survey it was found that 21 people liked product A, 26 liked product B and 29 liked product C. If 14 people liked products A and B, 12 people liked products C and A, 14 people liked products B and C and 8 liked all the three products. Find how many liked product C only.

Summary

This chapter deals with some basic definitions and operations involving sets. These are summarised below:

- A set is a well-defined collection of objects.
- A set which does not contain any element is called *empty set*.
- A set which consists of a definite number of elements is called *finite set*, otherwise, the set is called *infinite set*.
- Two sets A and B are said to be equal if they have exactly the same elements.
- A set A is said to be subset of a set B, if every element of A is also an element of B. Intervals are subsets of **R**.
- A power set of a set A is collection of all subsets of A. It is denoted by P(A).

- The union of two sets A and B is the set of all those elements which are either in A or in B.
- The intersection of two sets A and B is the set of all elements which are common. The difference of two sets A and B in this order is the set of elements which belong to A but not to B.
- The complement of a subset A of universal set U is the set of all elements of U which are not the elements of A.
- For any two sets A and B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$
- If A and B are finite sets such that $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$.
  If $A \cap B \neq \phi$, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Historical Note

The modern theory of sets is considered to have been originated largely by the German mathematician Georg Cantor (1845-1918 A.D.). His papers on set theory appeared sometimes during 1874 A.D. to 1897 A.D. His study of set theory came when he was studying trigonometric series of the form $a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + ...$ He published in a paper in 1874 A.D. that the set of real numbers could not be put into one-to-one correspondence wih the integers. From 1879 onwards, he publishd several papers showing various properties of abstract sets.

Cantor's work was well received by another famous mathematician Richard Dedekind (1831-1916 A.D.). But Kronecker (1810-1893 A.D.) castigated him for regarding infinite set the same way as finite sets. Another German mathematician Gottlob Frege, at the turn of the century, presented the set theory as principles of logic. Till then the entire set theory was based on the assumption of the existence of the set of all sets. It was the famous Englih Philosopher Bertand Russell (1872-1970 A.D.) who showed in 1902 A.D. that the assumption of existence of a set of all sets leads to a contradiction. This led to the famous Russell's Paradox. Paul R.Halmos writes about it in his book 'Naïve Set Theory' that "nothing contains everything".

The Russell's Paradox was not the only one which arose in set theory. Many paradoxes were produced later by several mathematicians and logicians.

As a consequence of all these paradoxes, the first axiomatisation of set theory was published in 1908 A.D. by Ernst Zermelo. Another one was proposed by Abraham Fraenkel in 1922 A.D. John Von Neumann in 1925 A.D. introduced explicitly the axiom of regularity. Later in 1937 A.D. Paul Bernays gave a set of more satisfactory axiomatisation. A modification of these axioms was done by Kurt Gödel in his monograph in 1940 A.D. This was known as Von Neumann-Bernays (VNB) or Gödel-Bernays (GB) set theory.

Despite all these difficulties, Cantor's set theory is used in present day mathematics. In fact, these days most of the concepts and results in mathematics are expressed in the set theoretic language.

❖

Chapter 2

RELATIONS AND FUNCTIONS

❖*Mathematics is the indispensable instrument of all physical research. – BERTHELOT* ❖

2.1 Introduction

G. W. Leibnitz
(1646 1716)

Much of mathematics is about finding a pattern – a recognisable link between quantities that change. In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student. In mathematics also, we come across many relations such as number m is less than number n, line l is parallel to line m, set A is a subset of set B. In all these, we notice that a relation involves pairs of objects in certain order. In this Chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. Finally, we will learn about special relations which will qualify to be functions. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correspondence between one quantity with the other.

2.2 Cartesian Products of Sets

Suppose A is a set of 2 colours and B is a set of 3 objects, i.e.,

$$A = \{\text{red, blue}\} \text{ and } B = \{b, c, s\},$$

where b, c and s represent a particular bag, coat and shirt, respectively.

How many pairs of coloured objects can be made from these two sets? Proceeding in a very orderly manner, we can see that there will be 6 distinct pairs as given below:

(red, b), (red, c), (red, s), (blue, b), (blue, c), (blue, s).

Thus, we get 6 distinct objects (Fig 2.1).

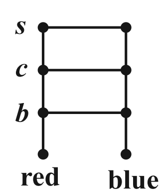


Fig 2.1

Let us recall from our earlier classes that an ordered pair of elements taken from any two sets P and Q is a pair of elements written in small brackets and grouped together in a particular order, i.e., (p,q), $p \in$ P and $q \in$ Q . This leads to the following definition:

Definition 1 Given two non-empty sets P and Q. The cartesian product P × Q is the set of all ordered pairs of elements from P and Q, i.e.,

$$\text{P} \times \text{Q} = \{ (p,q) : p \in \text{P}, q \in \text{Q} \}$$

If either P or Q is the null set, then P × Q will also be empty set, i.e., P × Q = ϕ

From the illustration given above we note that

A × B = {(red,b), (red,c), (red,s), (blue,b), (blue,c), (blue,s)}.

Again, consider the two sets:

A = {DL, MP, KA}, where DL, MP, KA represent Delhi, Madhya Pradesh and Karnataka, respectively and B = {01,02, 03}representing codes for the licence plates of vehicles issued by DL, MP and KA .

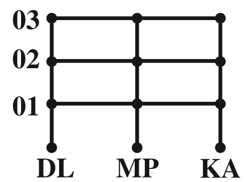


Fig 2.2

If the three states, Delhi, Madhya Pradesh and Karnataka were making codes for the licence plates of vehicles, with the restriction that the code begins with an element from set A, which are the pairs available from these sets and how many such pairs will there be (Fig 2.2)?

The available pairs are:(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03) and the product of set A and set B is given by

A × B = {(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03)}.

It can easily be seen that there will be 9 such pairs in the Cartesian product, since there are 3 elements in each of the sets A and B. This gives us 9 possible codes. Also note that the order in which these elements are paired is crucial. For example, the code (DL, 01) will not be the same as the code (01, DL).

As a final illustration, consider the two sets A= $\{a_1, a_2\}$ and B = $\{b_1, b_2, b_3, b_4\}$ (Fig 2.3).

A × B = $\{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, b_4)\}$.

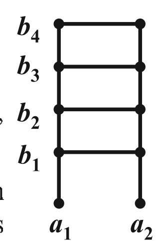


Fig 2.3

The 8 ordered pairs thus formed can represent the position of points in the plane if A and B are subsets of the set of real numbers and it is obvious that the point in the position (a_1, b_2) will be distinct from the point in the position (b_2, a_1).

Remarks

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.

(ii) If there are p elements in A and q elements in B, then there will be pq elements in $A \times B$, i.e., if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

(iii) If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.

(iv) $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is called an *ordered triplet*.

Example 1 If $(x + 1, y - 2) = (3,1)$, find the values of x and y.

Solution Since the ordered pairs are equal, the corresponding elements are equal.

Therefore $x + 1 = 3$ and $y - 2 = 1$.

Solving we get $x = 2$ and $y = 3$.

Example 2 If $P = \{a, b, c\}$ and $Q = \{r\}$, form the sets $P \times Q$ and $Q \times P$.
Are these two products equal?

Solution By the definition of the cartesian product,

$P \times Q = \{(a, r), (b, r), (c, r)\}$ and $Q \times P = \{(r, a), (r, b), (r, c)\}$

Since, by the definition of equality of ordered pairs, the pair (a, r) is not equal to the pair (r, a), we conclude that $P \times Q \neq Q \times P$.

However, the number of elements in each set will be the same.

Example 3 Let $A = \{1,2,3\}$, $B = \{3,4\}$ and $C = \{4,5,6\}$. Find

(i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$

(iii) $A \times (B \cup C)$ (iv) $(A \times B) \cup (A \times C)$

Solution (i) By the definition of the intersection of two sets, $(B \cap C) = \{4\}$.

Therefore, $A \times (B \cap C) = \{(1,4), (2,4), (3,4)\}$.

(ii) Now $(A \times B) = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

and $(A \times C) = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

Therefore, $(A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$.

(iii) Since, $(B \cup C) = \{3, 4, 5, 6\}$, we have

$A \times (B \cup C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

(iv) Using the sets $A \times B$ and $A \times C$ from part (ii) above, we obtain

$(A \times B) \cup (A \times C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

Example 4 If P = {1, 2}, form the set P × P × P.

Solution We have, P × P × P = {(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)}.

Example 5 If **R** is the set of all real numbers, what do the cartesian products **R** × **R** and **R** × **R** × **R** represent?

Solution The Cartesian product **R** × **R** represents the set **R** × **R**={(*x*, *y*) : *x*, *y* ∈ **R**} which represents the *coordinates of all the points in two dimensional space* and the cartisian product **R** × **R** × **R** represents the set **R** × **R** × **R** ={(*x*, *y*, *z*) : *x*, *y*, *z* ∈ **R**} which represents the *coordinates of all the points in three-dimensional space*.

Example 6 If A × B ={(*p*, *q*),(*p*, *r*), (*m*, *q*), (*m*, *r*)}, find A and B.

Solution A = set of first elements = {*p*, *m*}
B = set of second elements = {*q*, *r*}.

EXERCISE 2.1

1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of *x* and *y*.
2. If the set A has 3 elements and the set B = {3, 4, 5}, then find the number of elements in (A×B).
3. If G = {7, 8} and H = {5, 4, 2}, find G × H and H × G.
4. State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly.
   (i) If P = {*m*, *n*} and Q = { *n*, *m*}, then P × Q = {(*m*, *n*),(*n*, *m*)}.
   (ii) If A and B are non-empty sets, then A × B is a non-empty set of ordered pairs (*x*, *y*) such that *x* ∈ A and *y* ∈ B.
   (iii) If A = {1, 2}, B = {3, 4}, then A × (B ∩ φ) = φ.
5. If A = {–1, 1}, find A × A × A.
6. If A × B = {(*a*, *x*),(*a* , *y*), (*b*, *x*), (*b*, *y*)}. Find A and B.
7. Let A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}. Verify that
   (i) A × (B ∩ C) = (A × B) ∩ (A × C). (ii) A × C is a subset of B × D.
8. Let A = {1, 2} and B = {3, 4}. Write A × B. How many subsets will A × B have? List them.
9. Let A and B be two sets such that *n*(A) = 3 and *n*(B) = 2. If (*x*, 1), (*y*, 2), (*z*, 1) are in A × B, find A and B, where *x*, *y* and *z* are distinct elements.

10. The Cartesian product A × A has 9 elements among which are found (–1, 0) and (0,1). Find the set A and the remaining elements of A × A.

2.3 Relations

Consider the two sets P = {a, b, c} and Q = {Ali, Bhanu, Binoy, Chandra, Divya}. The cartesian product of P and Q has 15 ordered pairs which can be listed as P × Q = {(a, Ali), (a, Bhanu), (a, Binoy), ..., (c, Divya)}.

We can now obtain a subset of P × Q by introducing a relation R between the first element x and the second element y of each ordered pair (x, y) as

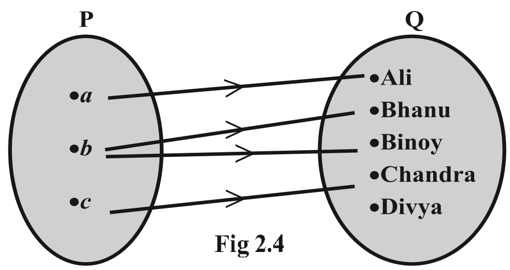


Fig 2.4

R= { (x,y): x is the first letter of the name y, $x \in$ P, $y \in$ Q}.

Then R = {(a, Ali), (b, Bhanu), (b, Binoy), (c, Chandra)}

A visual representation of this relation R (called an *arrow diagram*) is shown in Fig 2.4.

Definition 2 A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B. The second element is called the *image* of the first element.

Definition 3 The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

Definition 4 The set of all second elements in a relation R from a set A to a set B is called the *range* of the relation R. The whole set B is called the *codomain* of the relation R. Note that range $\subseteq$ codomain.

Remarks (i) A *relation* may be represented algebraically either by the *Roster method* or by the *Set-builder method.*

(ii) An arrow diagram is a visual representation of a relation.

Example 7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by

R = {(x, y) : $y = x + 1$ }

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

Solution (i) By the definition of the relation,

R = {(1,2), (2,3), (3,4), (4,5), (5,6)}.

The corresponding arrow diagram is shown in Fig 2.5.

(ii) We can see that the domain $=\{1, 2, 3, 4, 5,\}$

Similarly, the range = {2, 3, 4, 5, 6} and the codomain = {1, 2, 3, 4, 5, 6}.

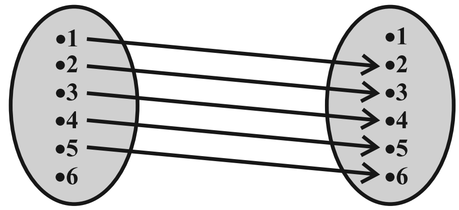


Fig 2.5

Example 8 The Fig 2.6 shows a relation between the sets P and Q. Write this relation (i) in set-builder form, (ii) in roster form. What is its domain and range?

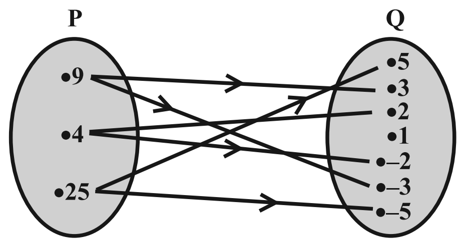


Fig 2.6

Solution It is obvious that the relation R is "x is the square of y".

(i) In set-builder form, R = $\{(x, y)$: x is the square of y, $x \in$ P, $y \in$ **Q**}

(ii) In roster form, R = {(9, 3), (9, –3), (4, 2), (4, –2), (25, 5), (25, –5)}

The domain of this relation is {4, 9, 25}.

The range of this relation is {– 2, 2, –3, 3, –5, 5}.

Note that the element 1 is not related to any element in set P.

The set Q is the codomain of this relation.

> **Note** The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B. If $n(\text{A}) = p$ and $n(\text{B}) = q$, then $n(\text{A} \times \text{B}) = pq$ and the total number of relations is 2^{pq}.

Example 9 Let A = {1, 2} and B = {3, 4}. Find the number of relations from A to B.

Solution We have,

A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}.

Since n (A×B) = 4, the number of subsets of A×B is 2^4. Therefore, the number of relations from A into B will be 2^4.

Remark A relation R from A to A is also stated as a relation on A.

EXERCISE 2.2

1. Let A = {1, 2, 3,...,14}. Define a relation R from A to A by R = $\{(x, y) : 3x - y = 0$, where $x, y \in$ A}. Write down its domain, codomain and range.

2. Define a relation R on the set **N** of natural numbers by R = $\{(x, y) : y = x + 5$, x is a natural number less than 4; $x, y \in \mathbf{N}\}$. Depict this relationship using roster form. Write down the domain and the range.
3. A = {1, 2, 3, 5} and B = {4, 6, 9}. Define a relation R from A to B by R = $\{(x, y)$: the difference between x and y is odd; $x \in$ A, $y \in$ B$\}$. Write R in roster form.
4. The Fig2.7 shows a relationship between the sets P and Q. Write this relation

   (i) in set-builder form (ii) roster form. What is its domain and range?

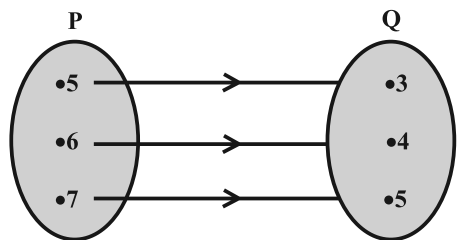


Fig 2.7

5. Let A = {1, 2, 3, 4, 6}. Let R be the relation on A defined by $\{(a, b): a, b \in A$, b is exactly divisible by $a\}$.
   (i) Write R in roster form
   (ii) Find the domain of R
   (iii) Find the range of R.
6. Determine the domain and range of the relation R defined by R = $\{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.
7. Write the relation R = $\{(x, x^3) : x$ is a prime number less than 10$\}$ in roster form.
8. Let A = $\{x, y, z\}$ and B = {1, 2}. Find the number of relations from A to B.
9. Let R be the relation on **Z** defined by R = $\{(a,b): a, b \in \mathbf{Z}, a - b$ is an integer$\}$. Find the domain and range of R.

2.4 Functions

In this Section, we study a special type of relation called *function.* It is one of the most important concepts in mathematics. We can, visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as 'map' or 'mapping' used to denote a function.

Definition 5 A relation f from a set A to a set B is said to be a *function* if every element of set A has one and only one image in set B.

In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the *image* of a under f and a is called the *preimage* of b under f.

The function f from A to B is denoted by f: A → B.

Looking at the previous examples, we can easily see that the relation in Example 7 is not a function because the element 6 has no image.

Again, the relation in Example 8 is not a function because the elements in the domain are connected to more than one images. Similarly, the relation in Example 9 is also not a function. (*Why*?) In the examples given below, we will see many more relations some of which are functions and others are not.

Example 10 Let **N** be the set of natural numbers and the relation R be defined on N such that R = $\{(x, y) : y = 2x, x, y \in \mathbf{N}\}$.

What is the domain, codomain and range of R? Is this relation a function?

Solution The domain of R is the set of natural numbers **N**. The codomain is also **N**. The range is the set of even natural numbers.

Since every natural number n has one and only one image, this relation is a function.

Example 11 Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?

(i) R = {(2,1),(3,1), (4,2)}, (ii) R = {(2,2),(2,4),(3,3), (4,4)}

(iii) R = {(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)}

Solution (i) Since 2, 3, 4 are the elements of domain of R having their unique images, this relation R is a function.

(ii) Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function.

(iii) Since every element has one and only one image, this relation is a function.

Definition 6 A function which has either R or one of its subsets as its range is called a *real valued function*. Further, if its domain is also either R or a subset of R, it is called a *real function*.

Example 12 Let **N** be the set of natural numbers. Define a real valued function

f : **N**→ **N** by $f(x) = 2x + 1$. Using this definition, complete the table given below.

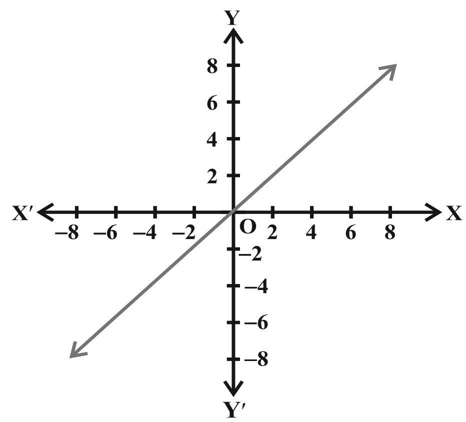
x	1	2	3	4	5	6	7
y	$f(1) = ...$	$f(2) = ...$	$f(3) = ...$	$f(4) = ...$	$f(5) = ...$	$f(6) = ...$	$f(7) = ...$

Solution The completed table is given by

x	1	2	3	4	5	6	7
y	$f(1) = 3$	$f(2) = 5$	$f(3) = 7$	$f(4) = 9$	$f(5) = 11$	$f(6) = 13$	$f(7) = 15$

2.4.1 *Some functions and their graphs*

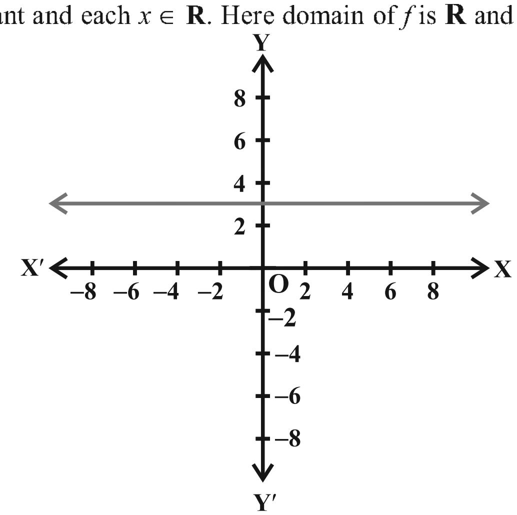
(i) **Identity function** Let **R** be the set of real numbers. Define the real valued function $f : \mathbf{R} \to \mathbf{R}$ by $y = f(x) = x$ for each $x \in \mathbf{R}$. Such a function is called the *identity function*. Here the domain and range of f are **R**. The graph is a straight line as shown in Fig 2.8. It passes through the origin.



$f(x) = x$

Fig 2.8

(ii) **Constant function** Define the function $f: \mathbf{R} \to \mathbf{R}$ by $y = f(x) = c, x \in \mathbf{R}$ where c is a constant and each $x \in \mathbf{R}$. Here domain of f is **R** and its range is $\{c\}$.



$f(x) = 3$

Fig 2.9

The graph is a line parallel to x-axis. For example, if $f(x)=3$ for each $x \in \mathbf{R}$, then its graph will be a line as shown in the Fig 2.9.

(iii) **Polynomial function** A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is said to be *polynomial function* if for each x in $\mathbf{R}$, $y = f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, ..., a_n \in \mathbf{R}$.

The functions defined by $f(x) = x^3 - x^2 + 2$, and $g(x) = x^4 + \sqrt{2}\,x$ are some examples of polynomial functions, whereas the function h defined by $h(x) = x^{\frac{2}{3}} + 2x$ is not a polynomial function.(*Why*?)

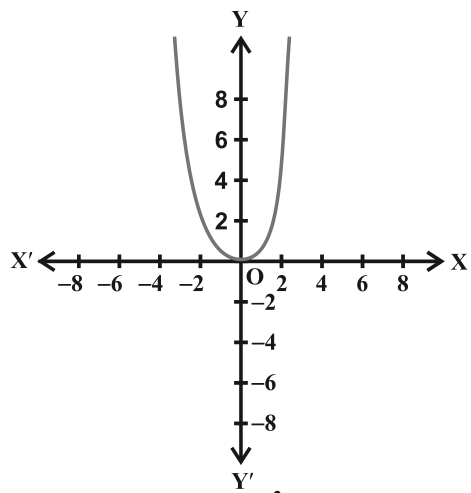
Example 13 Define the function $f: \mathbf{R} \rightarrow \mathbf{R}$ by $y = f(x) = x^2$, $x \in \mathbf{R}$. Complete the Table given below by using this definition. What is the domain and range of this function? Draw the graph of f.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

Solution The completed Table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

Domain of $f = \{x : x \in \mathbf{R}\}$. Range of $f = \{x : x \geq 0, x \in \mathbf{R}\}$. The graph of f is given by Fig 2.10



$f(x) = x^2$ **Fig 2.10**

Example 14 Draw the graph of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$, $x \in \mathbf{R}$.

Solution We have

$f(0) = 0, f(1) = 1, f(-1) = -1, f(2) = 8, f(-2) = -8, f(3) = 27; f(-3) = -27$, etc.

Therefore, $f = \{(x, x^3): x \in \mathbf{R}\}$.

The graph of f is given in Fig 2.11.

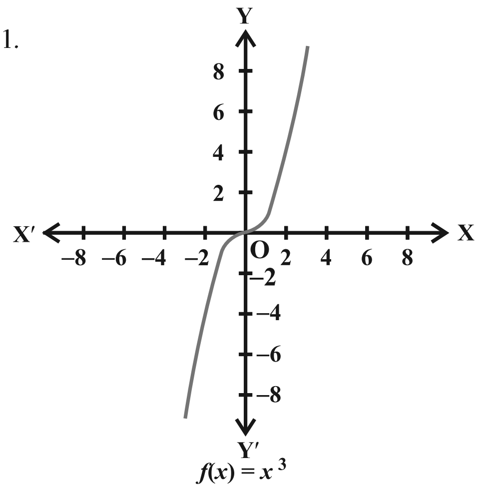


Fig 2.11

(iv) **Rational functions** are functions of the type $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.

Example 15 Define the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by $f(x) = \dfrac{1}{x}$, $x \in \mathbf{R} - \{0\}$. Complete the Table given below using this definition. What is the domain and range of this function?

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \dfrac{1}{x}$	...	...	...	...	...	...	...	...	...

Solution The completed Table is given by

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \dfrac{1}{x}$	– 0.5	– 0.67	–1	– 2	4	2	1	0.67	0.5

The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig 2.12.

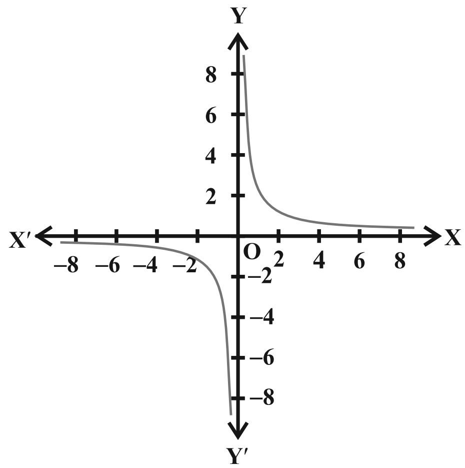
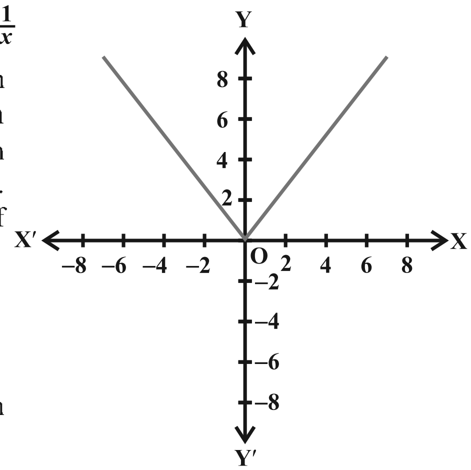


Fig 2.12 $f(x) = \frac{1}{x}$

(v) **The Modulus function** The function f: $\mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = |x|$ for each $x \in \mathbf{R}$ is called *modulus function*. For each non-negative value of x, $f(x)$ is equal to x. But for negative values of x, the value of $f(x)$ is the negative of the value of x, i.e.,

$$f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

The graph of the modulus function is given in Fig 2.13.



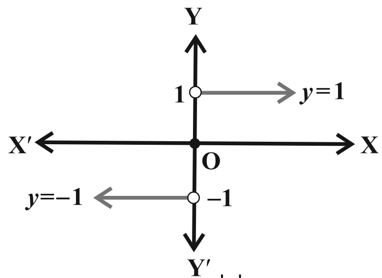
$f(x) = |x|$

Fig 2.13

(vi) **Signum function** The function f:$\mathbf{R} \rightarrow \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$$

is called the *signum function*. The domain of the signum function is $\mathbf{R}$ and the range is the set {–1, 0, 1}. The graph of the signum function is given by the Fig 2.14.



$f(x) = \dfrac{|x|}{x}$

Fig 2.14

(vii) **Greatest integer function** The function f: **R** → **R** defined by $f(x) = [x]$, $x \in$ **R** assumes the value of the greatest integer, less than or equal to x. Such a function is called the *greatest integer function*.

From the definition of $[x]$, we can see that

$[x] = -1$ for $-1 \le x < 0$

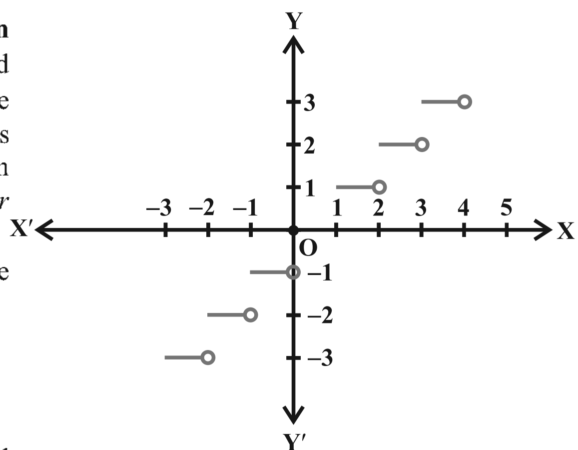
$[x] = 0$ for $0 \le x < 1$

$[x] = 1$ for $1 \le x < 2$

$[x] = 2$ for $2 \le x < 3$ and

so on.

The graph of the function is shown in Fig 2.15.



$f(x) = [x]$

Fig 2.15

2.4.2 ***Algebra of real functions*** In this Section, we shall learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real functions and divide one real function by another.

(i) **Addition of two real functions** Let f : X → **R** and g : X → **R** be any two real functions, where X ⊂ **R**. Then, we define $(f + g)$: X → **R** by

$(f + g)(x) = f(x) + g(x)$, for all $x \in$ X.

(ii) **Subtraction of a real function from another** Let f: X $\rightarrow$ **R** and g: X $\rightarrow$ **R** be any two real functions, where X $\subset$ **R**. Then, we define $(f - g)$: X$\rightarrow$**R** by $(f-g)(x) = f(x) - g(x)$, for all $x \in$ X.

(iii) **Multiplication by a scalar** Let f: X$\rightarrow$**R** be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product αf is a function from X to **R** defined by $(\alpha f)(x) = \alpha f(x)$, $x \in$ X.

(iv) **Multiplication of two real functions** The product (or multiplication) of two real functions f:X$\rightarrow$**R** and g:X$\rightarrow$**R** is a function fg:X$\rightarrow$**R** defined by $(fg)(x) = f(x)\, g(x)$, for all $x \in$ X.
This is also called *pointwise multiplication.*

(v) **Quotient of two real functions** Let f and g be two real functions defined from X$\rightarrow$**R** where X$\subset$**R**. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined by ,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0, x \in \text{X}$$

Example 16 Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions.Find

$$(f+g)(x), (f-g)(x), (fg)(x), \left(\frac{f}{g}\right)(x).$$

Solution We have,

$$(f+g)(x) = x^2 + 2x + 1,\ (f-g)(x) = x^2 - 2x - 1,$$

$$(fg)(x) = x^2(2x+1) = 2x^3 + x^2,\ \left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1}, x \neq -\frac{1}{2}$$

Example 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution We have

$$(f+g)(x) = \sqrt{x} + x,\ (f-g)(x) = \sqrt{x} - x,$$

$$(fg)\,x = \sqrt{x}(x) = x^{\frac{3}{2}} \text{ and } \left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}}, x \neq 0$$

EXERCISE 2.3

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.
   (i) $\{(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)\}$
   (ii) $\{(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)\}$
   (iii) $\{(1,3), (1,5), (2,5)\}$.
2. Find the domain and range of the following real functions:
   (i) $f(x) = -|x|$ (ii) $f(x) = \sqrt{9-x^2}$.
3. A function f is defined by $f(x) = 2x - 5$. Write down the values of
   (i) $f(0)$, (ii) $f(7)$, (iii) $f(-3)$.
4. The function 't' which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by $t(C) = \frac{9C}{5} + 32$.

Find (i) $t(0)$ (ii) $t(28)$ (iii) $t(-10)$ (iv) The value of C, when $t(C) = 212$.

5. Find the range of each of the following functions.
   (i) $f(x) = 2 - 3x$, $x \in \mathbf{R}$, $x > 0$.
   (ii) $f(x) = x^2 + 2$, x is a real number.
   (iii) $f(x) = x$, x is a real number.

Miscellaneous Examples

Example 18 Let $\mathbf{R}$ be the set of real numbers. Define the real function

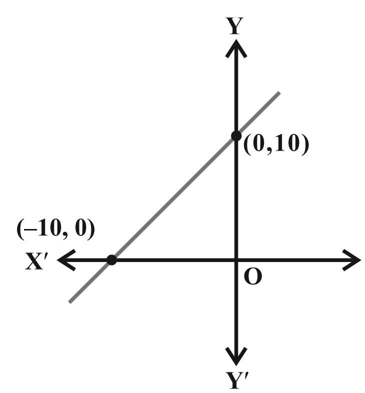
$$f: \mathbf{R} \to \mathbf{R} \text{ by } f(x) = x + 10$$

and sketch the graph of this function.

Solution Here $f(0) = 10$, $f(1) = 11$, $f(2) = 12$, ..., $f(10) = 20$, etc., and

$f(-1) = 9$, $f(-2) = 8$, ..., $f(-10) = 0$ and so on.

Therefore, shape of the graph of the given function assumes the form as shown in Fig 2.16.



$f(x) = x + 10$

Fig 2.16

Remark The function f defined by $f(x) = mx + c$, $x \in \mathbf{R}$, is called *linear function*, where m and c are constants. Above function is an example of a *linear function*.

Example 19 Let R be a relation from **Q** to **Q** defined by R = $\{(a,b): a,b \in$ **Q** and $a - b \in$ **Z**$\}$. Show that

(i) $(a,a) \in$ R for all $a \in$ **Q**

(ii) $(a,b) \in$ R implies that $(b, a) \in$ R

(iii) $(a,b) \in$ R and $(b,c) \in$ R implies that $(a,c) \in$ R

Solution (i) Since, $a - a = 0 \in$ **Z**, if follows that $(a, a) \in$ R.

(ii) $(a,b) \in$ R implies that $a - b \in$ **Z**. So, $b - a \in$ **Z**. Therefore, $(b, a) \in$ R

(iii) (a, b) and $(b, c) \in$ R implies that $a - b \in$ **Z**. $b - c \in$ **Z**. So, $a - c = (a - b) + (b - c) \in$ **Z**. Therefore, $(a,c) \in$ R

Example 20 Let $f = \{(1,1), (2,3), (0, -1), (-1, -3)\}$ be a linear function from **Z** into **Z**. Find $f(x)$.

Solution Since f is a linear function, $f(x) = mx + c$. Also, since $(1, 1), (0, -1) \in$ R, $f(1) = m + c = 1$ and $f(0) = c = -1$. This gives $m = 2$ and $f(x) = 2x - 1$.

Example 21 Find the domain of the function $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4}$

Solution Since $x^2 - 5x + 4 = (x - 4)(x - 1)$, the function $f(x)$ is defined for all real numbers except at $x = 4$ and $x = 1$. Hence the domain of f is **R** $- \{1, 4\}$.

Example 22 The function f is defined by

$$f(x) = \begin{cases} 1 - x, & x < 0 \\ 1 \quad , & x = 0 \\ x + 1, & x > 0 \end{cases}$$

Draw the graph of $f(x)$.

Solution Here, $f(x) = 1 - x$, $x < 0$, this gives

$f(-4) = 1 - (-4) = 5$;

$f(-3) = 1 - (-3) = 4$,

$f(-2) = 1 - (-2) = 3$

$f(-1) = 1 - (-1) = 2$; etc,

and $f(1) = 2, f(2) = 3, f(3) = 4$

$f(4) = 5$ and so on for $f(x) = x + 1, x > 0$.

Thus, the graph of f is as shown in Fig 2.17

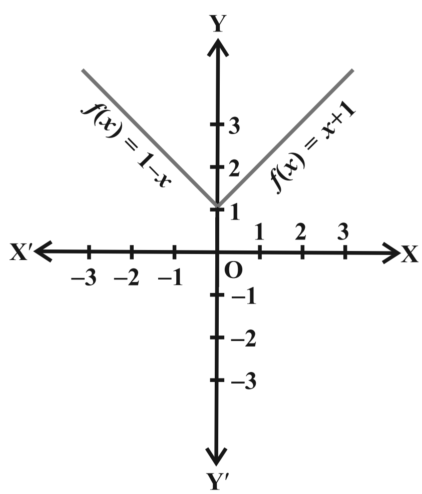


Fig 2.17

Miscellaneous Exercise on Chapter 2

1. The relation f is defined by $f(x) = \begin{cases} x^2, 0 \le x \le 3 \\ 3x, 3 \le x \le 10 \end{cases}$

   The relation g is defined by $g(x) = \begin{cases} x^2, 0 \le x \le 2 \\ 3x, 2 \le x \le 10 \end{cases}$

   Show that f is a function and g is not a function.

2. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1 - 1)}$.

3. Find the domain of the function $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

4. Find the domain and the range of the real function f defined by $f(x) = \sqrt{(x-1)}$.

5. Find the domain and the range of the real function f defined by $f(x) = |x - 1|$.

6. Let $f = \left\{\left(x, \dfrac{x^2}{1+x^2}\right) : x \in \mathbf{R}\right\}$ be a function from **R** into **R**. Determine the range of f.

7. Let $f, g : \mathbf{R} \to \mathbf{R}$ be defined, respectively by $f(x) = x + 1$, $g(x) = 2x - 3$. Find $f + g$, $f - g$ and $\dfrac{f}{g}$.

8. Let $f = \{(1,1), (2,3), (0,-1), (-1, -3)\}$ be a function from **Z** to **Z** defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.

9. Let R be a relation from **N** to **N** defined by R = $\{(a, b) : a, b \in \mathbf{N}$ and $a = b^2\}$. Are the following true?

   (i) $(a,a) \in$ R, for all $a \in$ **N** (ii) $(a,b) \in$ R, implies $(b,a) \in$ R

   (iii) $(a,b) \in$ R, $(b,c) \in$ R implies $(a,c) \in$ R.

   Justify your answer in each case.

10. Let A = {1,2,3,4}, B = {1,5,9,11,15,16} and f = {(1,5), (2,9), (3,1), (4,5), (2,11)} Are the following true?

    (i) f is a relation from A to B (ii) f is a function from A to B.

    Justify your answer in each case.

11. Let f be the subset of $\mathbf{Z} \times \mathbf{Z}$ *defined by* $f = \{(ab, a + b) : a, b \in \mathbf{Z}\}$. Is f a function from $\mathbf{Z}$ to $\mathbf{Z}$? Justify your answer.

12. Let A = {9,10,11,12,13} and let $f : \mathrm{A} \rightarrow \mathbf{N}$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f.

Summary

In this Chapter, we studied about relations and functions.The main features of this Chapter are as follows:

- ***Ordered pair*** A pair of elements grouped together in a particular order.
- ***Cartesian product*** A × B of two sets A and B is given by

  $\mathrm{A} \times \mathrm{B} = \{(a, b): a \in \mathrm{A}, b \in \mathrm{B}\}$

  In particular $\mathbf{R} \times \mathbf{R} = \{(x, y): x, y \in \mathbf{R}\}$

  and $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = (x, y, z): x, y, z \in \mathbf{R}\}$
- If $(a, b) = (x, y)$, then $a = x$ and $b = y$.
- If $n(\mathrm{A}) = p$ and $n(\mathrm{B}) = q$, then $n(\mathrm{A} \times \mathrm{B}) = pq$.
- $\mathrm{A} \times \phi = \phi$
- In general, $\mathrm{A} \times \mathrm{B} \neq \mathrm{B} \times \mathrm{A}$.
- ***Relation*** A relation R from a set A to a set B is a subset of the cartesian product A × B obtained by describing a relationship between the first element x and the second element y of the ordered pairs in A × B.
- The ***image*** of an element x under a relation R is given by y, where $(x, y) \in \mathrm{R}$,
- The ***domain*** of R is the set of all first elements of the ordered pairs in a relation R.
- The ***range*** of the relation R is the set of all second elements of the ordered pairs in a relation R.
- ***Function*** A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B.

  We write f: A→B, where $f(x) = y$.
- A is the domain and B is the codomain of f.
- The range of the function is the set of images.

- A real function has the set of real numbers or one of its subsets both as its domain and as its range.
- ***Algebra of functions*** For functions $f : X \rightarrow \mathbf{R}$ and $g : X \rightarrow \mathbf{R}$, we have

$$(f + g)(x) = f(x) + g(x),\ x \in X.$$

$$(f - g)(x) = f(x) - g(x),\ x \in X.$$

$$(f.g)(x) = f(x) . g(x),\ x \in X.$$

$$(kf)(x) = k\, f(x)\ ),\ x \in X.$$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)},\ x \in X,\ g(x) \neq 0.$$

Historical Note

The word FUNCTION first appears in a Latin manuscript "Methodus tangentium inversa, seu de fuctionibus" written by Gottfried Wilhelm Leibnitz (1646-1716) in 1673; Leibnitz used the word in the non-analytical sense. He considered a function in terms of "mathematical job" – the "employee" being just a curve.

On July 5, 1698, Johan Bernoulli, in a letter to Leibnitz, for the first time deliberately assigned a specialised use of the term *function* in the analytical sense. At the end of that month, Leibnitz replied showing his approval.

Function is found in English in 1779 in Chambers' Cylopaedia: "The term function is used in algebra, for an analytical expression any way compounded of a variable quantity, and of numbers, or constant quantities".

❖

Chapter 3

TRIGONOMETRIC FUNCTIONS

❖ *A mathematician knows how to solve a problem, he can not solve it. – MILNE* ❖

3.1 Introduction

Arya Bhatt (476 550 B.C.)

The word 'trigonometry' is derived from the Greek words '*trigon*' and '*metron*' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances. In this Chapter, we will generalise the concept of trigonometric ratios to trigonometric functions and study their properties.

3.2 Angles

Angle is a measure of rotation of a given ray about its initial point. The original ray is

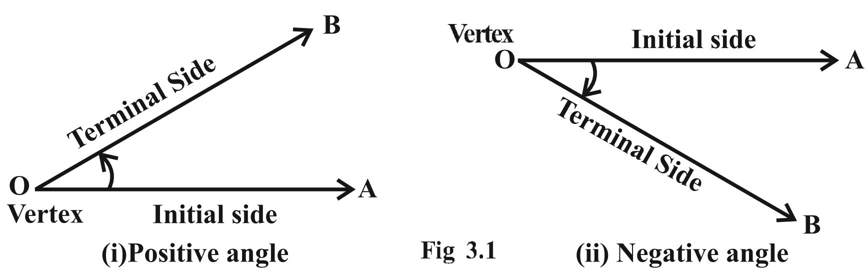


Fig 3.1

called the *initial side* and the final position of the ray after rotation is called the *terminal side* of the angle. The point of rotation is called the *vertex*. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is *negative* (Fig 3.1).

The measure of an angle is the amount of rotation performed to get the terminal side from the initial side. There are several units for measuring angles. The definition of an angle suggests a unit, viz. *one complete revolution* from the position of the initial side as indicated in Fig 3.2.

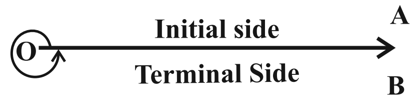


Fig 3.2

This is often convenient for large angles. For example, we can say that a rapidly spinning wheel is making an angle of say 15 revolution per second. We shall describe two other units of measurement of an angle which are most commonly used, viz. degree measure and radian measure.

3.2.1 ***Degree measure*** If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{\text{th}}$ of a revolution, the angle is said to have a measure of one *degree*, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds . One sixtieth of a degree is called a *minute*, written as 1′, and one sixtieth of a minute is called a *second*, written as 1″.

Thus, $1° = 60′$, $1′ = 60″$

Some of the angles whose measures are 360°,180°, 270°, 420°, – 30°, – 420° are shown in Fig 3.3.

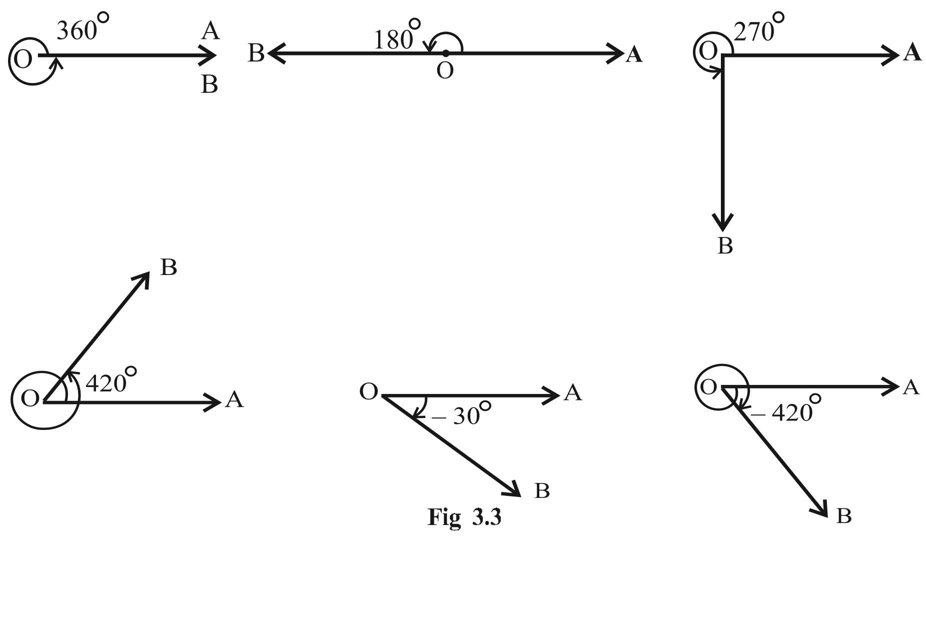


Fig 3.3

3.2.2 *Radian measure* There is another unit for measurement of an angle, called the *radian* measure. Angle subtended at the centre by an arc of length 1 unit in a unit circle (circle of radius 1 unit) is said to have a measure of 1 radian. In the Fig 3.4(i) to (iv), OA is the initial side and OB is the terminal side. The figures show the angles whose measures are 1 radian, –1 radian, $1\frac{1}{2}$ radian and $-1\frac{1}{2}$ radian.

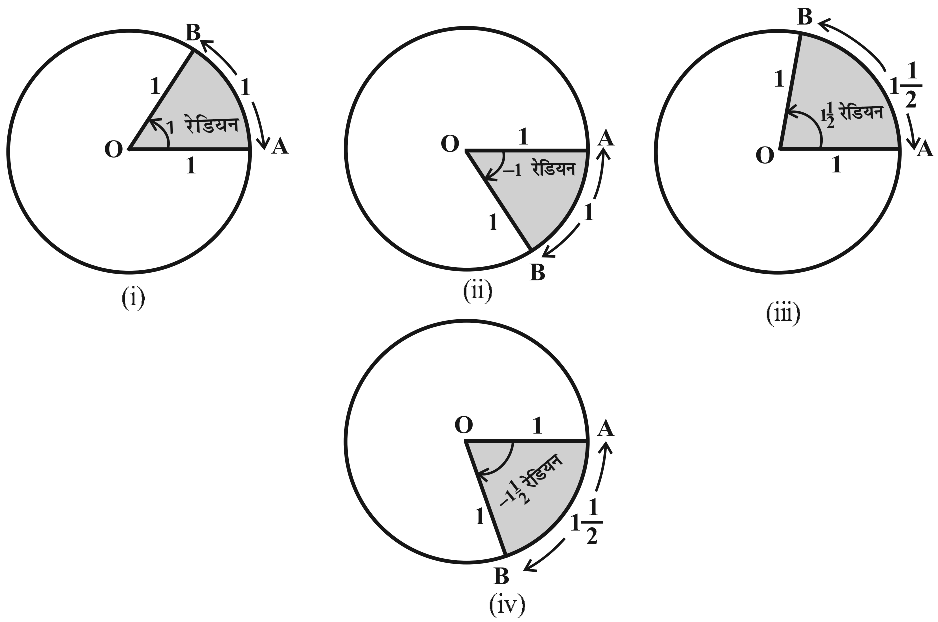


Fig 3.4 (i) to (iv)

We know that the circumference of a circle of radius 1 unit is 2π. Thus, one complete revolution of the initial side subtends an angle of 2π radian.

More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length l will subtend an angle whose measure is $\frac{l}{r}$ radian. Thus, if in a circle of radius r, an arc of length l subtends an angle θ radian at the centre, we have

$$\theta = \frac{l}{r} \text{ or } l = r\,\theta.$$

3.2.3 *Relation between radian and real numbers*

Consider the unit circle with centre O. Let A be any point on the circle. Consider OA as initial side of an angle. Then the length of an arc of the circle will give the radian measure of the angle which the arc will subtend at the centre of the circle. Consider the line PAQ which is tangent to the circle at A. Let the point A represent the real number zero, AP represents positive real number and AQ represents negative real numbers (Fig 3.5). If we rope the line AP in the anticlockwise direction along the circle, and AQ in the clockwise direction, then every real number will correspond to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

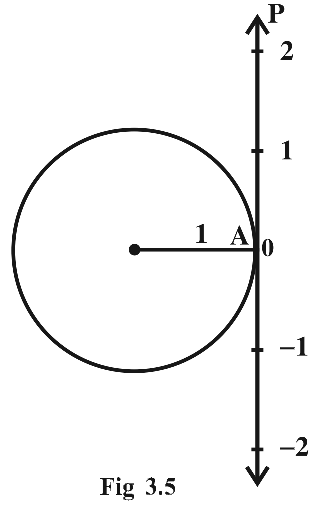


Fig 3.5

3.2.4 *Relation between degree and radian*

Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is 360°, it follows that

$$2\pi \text{ radian} = 360° \quad \text{or} \quad \pi \text{ radian} = 180°$$

The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. Using approximate value of π as $\frac{22}{7}$, we have

$$1 \text{ radian} = \frac{180°}{\pi} = 57°\ 16' \text{ approximately.}$$

Also $$1° = \frac{\pi}{180} \text{ radian} = 0.01746 \text{ radian approximately.}$$

The relation between degree measures and radian measure of some common angles are given in the following table:

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Notational Convention

Since angles are measured either in degrees or in radians, we adopt the convention that whenever we write angle $\theta°$, we mean the angle whose degree measure is θ and whenever we write angle β, we mean the angle whose radian measure is β.

Note that when an angle is expressed in radians, the word 'radian' is frequently omitted. Thus, $\pi = 180°$ and $\frac{\pi}{4} = 45°$ are written with the understanding that π and $\frac{\pi}{4}$ are radian measures. Thus, we can say that

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Example 1 Convert $40°\ 20'$ into radian measure.

Solution We know that $180° = \pi$ radian.

Hence $\quad 40°\ 20' = 40\frac{1}{3}$ degree $= \frac{\pi}{180} \times \frac{121}{3}$ radian $= \frac{121\pi}{540}$ radian.

Therefore $\quad 40°\ 20' = \frac{121\pi}{540}$ radian.

Example 2 Convert 6 radians into degree measure.

Solution We know that π radian = 180°.

Hence $\quad 6 \text{ radians} = \frac{180}{\pi} \times 6$ degree $= \frac{1080 \times 7}{22}$ degree

$= 343\frac{7}{11}$ degree $\quad = 343° + \frac{7 \times 60}{11}$ minute $\quad$ [as $1° = 60'$]

$= 343° + 38' + \frac{2}{11}$ minute $\quad$ [as $1' = 60''$]

$= 343° + 38' + 10.9'' \quad = 343° 38' 11''$ approximately.

Hence $\quad$ 6 radians = $343°\ 38'\ 11''$ approximately.

Example 3 Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).

Solution Here $l = 37.4$ cm and $\theta = 60° = \frac{60\pi}{180}$ radian $= \frac{\pi}{3}$

Hence, by $r = \frac{l}{\theta}$, we have

$$r = \frac{37.4 \times 3}{\pi} = \frac{37.4 \times 3 \times 7}{22} = 35.7 \text{ cm}$$

Example 4 The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use $\pi = 3.14$).

Solution In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 40 minutes, the minute hand turns through $\frac{2}{3}$ of a revolution. Therefore, $\theta = \frac{2}{3} \times 360°$ or $\frac{4\pi}{3}$ radian. Hence, the required distance travelled is given by

$$l = r\,\theta = 1.5 \times \frac{4\pi}{3} \text{ cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}.$$

Example 5 If the arcs of the same lengths in two circles subtend angles 65°and 110° at the centre, find the ratio of their radii.

Solution Let r_1 and r_2 be the radii of the two circles. Given that

$$\theta_1 = 65° = \frac{\pi}{180} \times 65 = \frac{13\pi}{36} \text{ radian}$$

and $$\theta_2 = 110° = \frac{\pi}{180} \times 110 = \frac{22\pi}{36} \text{ radian}$$

Let l be the length of each of the arc. Then $l = r_1\theta_1 = r_2\theta_2$, which gives

$$\frac{13\pi}{36} \times r_1 = \frac{22\pi}{36} \times r_2, \text{ i.e., } \frac{r_1}{r_2} = \frac{22}{13}$$

Hence $r_1 : r_2 = 22 : 13$.

EXERCISE 3.1

1. Find the radian measures corresponding to the following degree measures:

(i) 25° (ii) – 47°30′ (iii) 240° (iv) 520°

2. Find the degree measures corresponding to the following radian measurs (Use $\pi = \frac{22}{7}$).

(i) $\frac{11}{16}$ (ii) -4 (iii) $\frac{5\pi}{3}$ (iv) $\frac{7\pi}{6}$

3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?
4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = \frac{22}{7}$).
5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.
6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.
7. Find the angle in radian through which a pendulum swings if its length is 75 cm and th e tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

3.3 Trigonometric Functions

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of sides of a right angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

Consider a unit circle with centre at origin of the coordinate axes. Let P (a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x (Fig 3.6).

We define $\cos x = a$ and $\sin x = b$ Since ΔOMP is a right triangle, we have

$OM^2 + MP^2 = OP^2$ or $a^2 + b^2 = 1$

Thus, for every point on the unit circle, we have

$a^2 + b^2 = 1$ or $\cos^2 x + \sin^2 x = 1$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$,

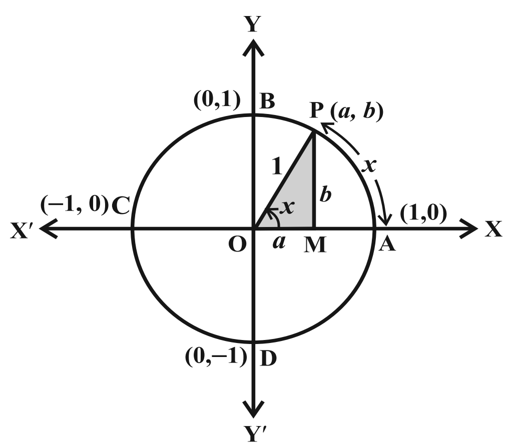


Fig 3.6

$\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called *quadrantal angles*. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$$\cos 0^\circ = 1 \qquad \sin 0^\circ = 0,$$

$$\cos \frac{\pi}{2} = 0 \qquad \sin \frac{\pi}{2} = 1$$

$$\cos\pi = -1 \qquad \sin\pi = 0$$

$$\cos \frac{3\pi}{2} = 0 \qquad \sin \frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1 \qquad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$$\sin (2n\pi + x) = \sin x,\ n \in \mathbf{Z},\quad \cos (2n\pi + x) = \cos x,\ n \in \mathbf{Z}$$

Further, $\sin x = 0$, if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, ...$, i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$ i.e., $\cos x$ vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

sin x = 0 implies $x = n\pi$, where n is any integer

cos x = 0 implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$$\text{cosec}\, x = \frac{1}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

$$\sec x = \frac{1}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\tan x = \frac{\sin x}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\cot x = \frac{\cos x}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that

$$1 + \tan^2 x = \sec^2 x \qquad \text{(why?)}$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x \qquad \text{(why?)}$$

In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table:

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

3.3.1 *Sign of trigonometric functions*

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be $(a, -b)$ (Fig 3.7). Therefore

$$\cos(-x) = \cos x$$

and $\quad \sin(-x) = -\sin x$

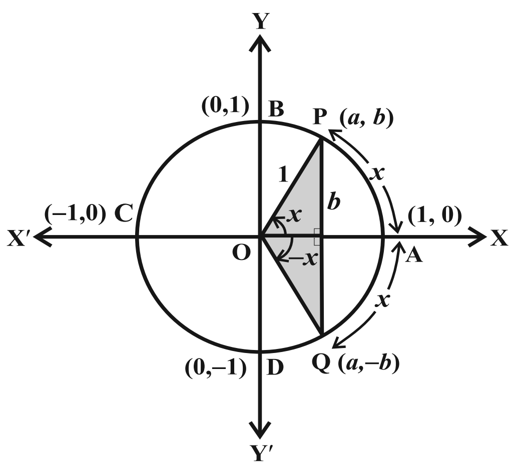


Fig 3.7

Since for every point P (a, b) on the unit circle, $-1 \leq a \leq 1$ and

$-1 \leq b \leq 1$, we have $-1 \leq \cos x \leq 1$ and $-1 \leq \sin x \leq 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive, in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, $\sin x$ is positive for $0 < x < \pi$, and negative for $\pi < x < 2\pi$. Similarly, $\cos x$ is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\operatorname{cosec} x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.3.2 ***Domain and range of trigonometric functions*** From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x,

$$-1 \leq \sin x \leq 1 \text{ and } -1 \leq \cos x \leq 1$$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval $[-1, 1]$, i.e., $-1 \leq y \leq 1$.

Since cosec $x = \dfrac{1}{\sin x}$, the domain of y = cosec x is the set $\{ x : x \in \mathbf{R}$ and $x \neq n\,\pi, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \geq 1$ or $y \leq -1\}$. Similarly, the domain of y = sec x is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\,\dfrac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \leq -1$ or $y \geq 1\}$. The domain of y = tan x is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\,\dfrac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set of all real numbers. The domain of y = cot x is the set $\{x : x \in \mathbf{R}$ and $x \neq n\,\pi, n \in \mathbf{Z}\}$ and the range is the set of all real numbers.

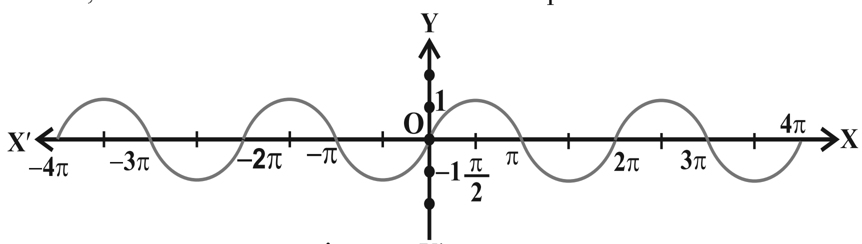
We further observe that in the first quadrant, as x increases from 0 to $\dfrac{\pi}{2}$, sin x increases from 0 to 1, as x increases from $\dfrac{\pi}{2}$ to π, sin x decreases from 1 to 0. In the third quadrant, as x increases from π to $\dfrac{3\pi}{2}$, sin x decreases from 0 to –1and finally, in the fourth quadrant, sin x increases from –1 to 0 as x increases from $\dfrac{3\pi}{2}$ to 2π.

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to 1	increases from 1 to 0
cos	decreases from 1 to 0	decreases from 0 to 1	increases from 1 to 0	increases from 0 to 1
tan	increases from 0 to ∞	increases from ∞to 0	increases from 0 to ∞	increases from ∞to 0
cot	decreases from ∞ to 0	decreases from 0 to ∞	decreases from ∞ to 0	decreases from 0to ∞
sec	increases from 1 to ∞	increases from ∞to 1	decreases from 1to ∞	decreases from ∞ to 1
cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from ∞to 1	decreases from 1to ∞

Similarly, we can discuss the behaviour of other trigonometric functions. In fact, we have the following table:

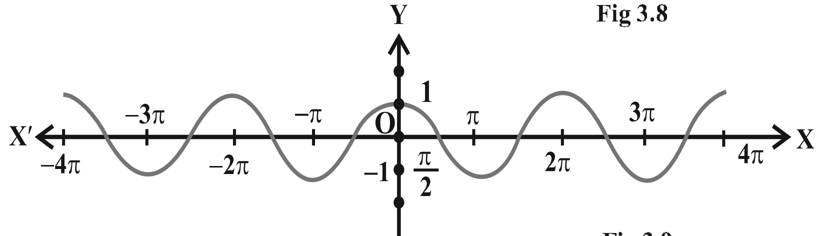
Remark In the above table, the statement tan x increases from 0 to ∞ (infinity) for $0 < x < \dfrac{\pi}{2}$ simply means that tan x increases as x increases for $0 < x < \dfrac{\pi}{2}$ and assumes arbitraily large positive values as x approaches to $\frac{\pi}{2}$. Similarly, to say that cosec x decreases from -1 to $-\infty$ (minus infinity) in the fourth quadrant means that cosec x decreases for $x \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as x approaches to 2π. The symbols ∞ and $-\infty$ simply specify certain types of behaviour of functions and variables.

We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We



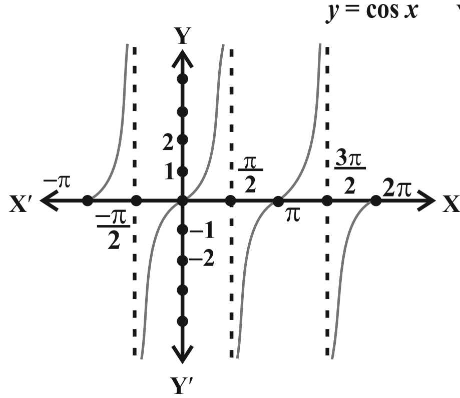
$y = \sin x$

Fig 3.8

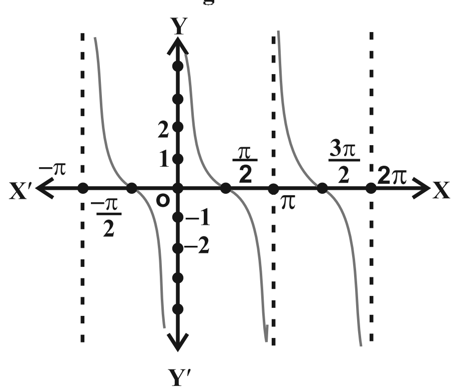


$y = \cos x$

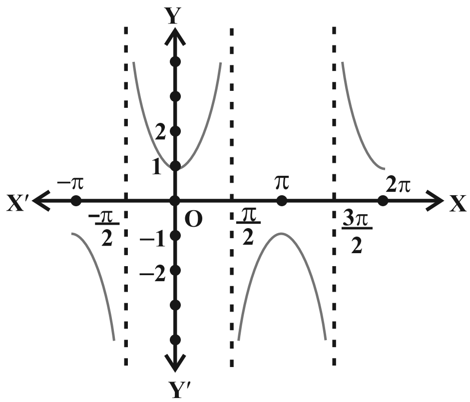
Fig 3.9



$y = \tan x$ **Fig 3.10**

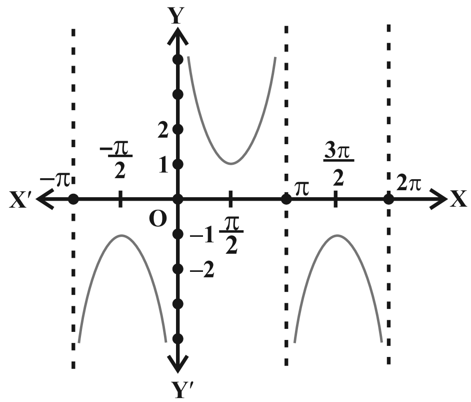


$y = \cot x$ **Fig 3.11**



y = **sec** x

Fig 3.12



y = **cosec** x

Fig 3.13

shall see in the next section that tan $(\pi + x) = \tan x$. Hence, values of tan x will repeat after an interval of π. Since cot x is reciprocal of tan x, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometic functions, we can sketch the graph of these functions. The graph of these functions are given above:

Example 6 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now $\sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, sin x is negative. Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\text{cosec } x = -\frac{5}{4}$$

Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \text{ and } \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}.$$

Example 7 If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence $\sec x = \pm \frac{13}{5}$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5},$$

which also gives

$$\cos x = -\frac{5}{13}$$

Further, we have

$$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right) \times \left(-\frac{5}{13}\right) = \frac{12}{13}$$

and $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin \left(10\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°.
Therefore, $\cos(-1710°) = \cos(-1710° + 5 \times 360°)$
$= \cos(-1710° + 1800°) = \cos 90° = 0.$

EXERCISE 3.2

Find the values of other five trigonometric functions in Exercises 1 to 5.

1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.
2. $\sin x = \frac{3}{5}$, x lies in second quadrant.
3. $\cot x = \frac{3}{4}$, x lies in third quadrant.
4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.
5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Find the values of the trigonometric functions in Exercises 6 to 10.

6. sin 765°
7. cosec (– 1410°)
8. $\tan \frac{19\pi}{3}$
9. $\sin(-\frac{11\pi}{3})$
10. $\cot(-\frac{15\pi}{4})$

3.4 Trigonometric Functions of Sum and Difference of Two Angles

In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called *trigonometric identities*. We have seen that

1. $\mathbf{\sin(-x) = -\sin x}$
2. $\mathbf{\cos(-x) = \cos x}$

We shall now prove some more results:

3. $\cos(x+y) = \cos x \cos y - \sin x \sin y$

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1 and y be the angle P_1OP_2. Then $(x + y)$ is the angle P_4OP_2. Also let $(-y)$ be the angle P_4OP_3. Therefore, P_1, P_2, P_3 and P_4 will have the coordinates $P_1(\cos x, \sin x)$, $P_2[\cos(x+y), \sin(x+y)]$, $P_3[\cos(-y), \sin(-y)]$ and $P_4(1, 0)$ (Fig 3.14).

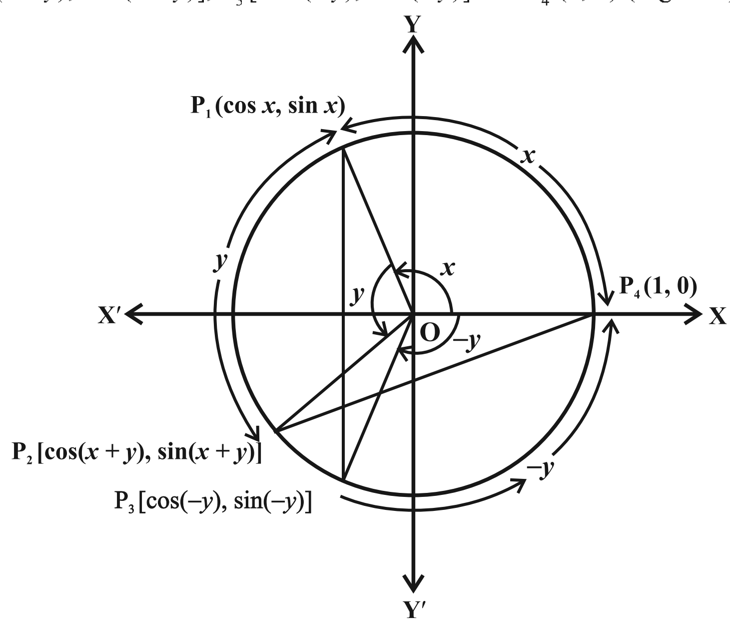


Fig 3.14

Consider the triangles P_1OP_3 and P_2OP_4. They are congruent (Why?). Therefore, P_1P_3 and P_2P_4 are equal. By using distance formula, we get

$$P_1P_3^2 = [\cos x - \cos(-y)]^2 + [\sin x - \sin(-y)]^2$$

$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$

$$= \cos^2 + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$

$$= 2 - 2(\cos x \cos y - \sin x \sin y) \qquad \text{(Why?)}$$

Also, $P_2P_4^2 = [1 - \cos(x+y)]^2 + [0 - \sin(x+y)]^2$

$$= 1 - 2\cos(x+y) + \cos^2(x+y) + \sin^2(x+y)$$

$$= 2 - 2\cos(x+y)$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$.
Therefore, $2 - 2(\cos x \cos y - \sin x \sin y) = 2 - 2\cos(x + y)$.

Hence $\cos(x + y) = \cos x \cos y - \sin x \sin y$

4. **$\cos(x - y) = \cos x \cos y + \sin x \sin y$**

Replacing y by $-y$ in identity 3, we get

$$\cos(x + (-y)) = \cos x \cos(-y) - \sin x \sin(-y)$$

or $$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

5. **$\cos\left(\frac{\pi}{2} - x\right) = \sin x$**

If we replace x by $\frac{\pi}{2}$ and y by x in Identity (4), we get

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = \sin x.$$

6. **$\sin\left(\frac{\pi}{2} - x\right) = \cos x$**

Using the Identity 5, we have

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x.$$

7. **$\sin(x + y) = \sin x \cos y + \cos x \sin y$**

We know that

$$\sin(x + y) = \cos\left(\frac{\pi}{2} - (x + y)\right) = \cos\left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y$$

$$= \sin x \cos y + \cos x \sin y$$

8. **$\sin(x - y) = \sin x \cos y - \cos x \sin y$**

If we replace y by $-y$, in the Identity 7, we get the result.

9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results:

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x \qquad \sin\left(\frac{\pi}{2} + x\right) = \cos x$$

$$\cos(\pi - x) = -\cos x \qquad \sin(\pi - x) = \sin x$$

$$\mathbf{\cos(\pi + x) = -\cos x} \qquad \mathbf{\sin(\pi + x) = -\sin x}$$
$$\mathbf{\cos(2\pi - x) = \cos x} \qquad \mathbf{\sin(2\pi - x) = -\sin x}$$

Similar results for tan x, cot x, sec x and cosec x can be obtianed from the results of sin x and cos x.

10. If none of the angles x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Since none of the x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, it follows that cos x, cos y and cos $(x + y)$ are non-zero. Now

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Dividing numerator and denominator by cos x cos y, we have

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

11. $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we replace y by $-y$ in Identity 10, we get

$$\tan(x - y) = \tan[x + (-y)]$$

$$= \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

12. If none of the angles x, y and $(x + y)$ is a multiple of π, then

$$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

Since, none of the x, y and $(x + y)$ is multiple of π, we find that $\sin x \sin y$ and $\sin (x + y)$ are non-zero. Now,

$$\cot (x + y) = \frac{\cos (x + y)}{\sin (x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

Dividing numerator and denominator by $\sin x \sin y$, we have

$$\cot (x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

13. $\cot (x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

If we replace y by $-y$ in identity 12, we get the result

14. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

We know that

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

Replacing y by x, we get

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \\ &= \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1 \end{aligned}$$

Again, $\quad \begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x. \end{aligned}$

We have $\quad \cos 2x = \cos^2 x - \sin^2 x = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$

Dividing each term by $\cos^2 x$, we get

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

15. $\sin 2x = 2 \sin x \cos x = \dfrac{2\tan x}{1 + \tan^2 x}$

We have

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

Replacing y by x, we get $\sin 2x = 2 \sin x \cos x$.

Again $\quad \sin 2x = \dfrac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$

Dividing each term by $\cos^2 x$, we get

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

16. $\mathbf{\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}}$

We know that

$$\tan (x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Replacing y by x, we get $\tan 2x = \dfrac{2 \tan x}{1-\tan^2 x}$

17. $\mathbf{\sin 3x = 3 \sin x - 4 \sin^3 x}$

We have,

$$\begin{aligned}\sin 3x &= \sin (2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= 2 \sin x \cos x \cos x + (1 - 2\sin^2 x) \sin x\\ &= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x\\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x\\ &= 3 \sin x - 4 \sin^3 x\end{aligned}$$

18. $\mathbf{\cos 3x = 4 \cos^3 x - 3 \cos x}$

We have,

$$\begin{aligned}\cos 3x &= \cos (2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= (2\cos^2 x - 1) \cos x - 2\sin x \cos x \sin x\\ &= (2\cos^2 x - 1) \cos x - 2\cos x (1 - \cos^2 x)\\ &= 2\cos^3 x - \cos x - 2\cos x + 2 \cos^3 x\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

19. $\mathbf{\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3\tan^2 x}}$

We have $\tan 3x = \tan (2x + x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \dfrac{2\tan x \,.\, \tan x}{1-\tan^2 x}}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

20. **(i)** $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

(ii) $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

(iii) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

(iv) $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

We know that

$\cos(x+y) = \cos x \cos y - \sin x \sin y$ ... (1)

and $\cos(x-y) = \cos x \cos y + \sin x \sin y$ ... (2)

Adding and subtracting (1) and (2), we get

$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$ ... (3)

and $\cos(x+y) - \cos(x-y) = -2\sin x \sin y$ ... (4)

Further $\sin(x+y) = \sin x \cos y + \cos x \sin y$ ... (5)

and $\sin(x-y) = \sin x \cos y - \cos x \sin y$ ... (6)

Adding and subtracting (5) and (6), we get

$\sin(x+y) + \sin(x-y) = 2\sin x \cos y$ ... (7)

$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$ ... (8)

Let $x + y = \theta$ and $x - y = \phi$. Therefore

$$x = \left(\frac{\theta+\phi}{2}\right) \text{and } y = \left(\frac{\theta-\phi}{2}\right)$$

Substituting the values of x and y in (3), (4), (7) and (8), we get

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Since θ and ϕ can take any real values, we can replace θ by x and ϕ by y.
Thus, we get

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2},$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

Remarks As a part of identities given in 20, we can prove the following results:

21. **(i)** $\mathbf{2\cos x\cos y = \cos(x+y) + \cos(x-y)}$
(ii) $\mathbf{-2\sin x\sin y = \cos(x+y) - \cos(x-y)}$
(iii) $\mathbf{2\sin x\cos y = \sin(x+y) + \sin(x-y)}$
(iv) $\mathbf{2\cos x\sin y = \sin(x+y) - \sin(x-y)}$.

Example 10 Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$$

Solution We have

$$\text{L.H.S.} = 3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

$$= 3\times\frac{1}{2}\times 2 - 4\sin\left(\pi - \frac{\pi}{6}\right)\times 1 = 3 - 4\sin\frac{\pi}{6}$$

$$= 3 - 4\times\frac{1}{2} = 1 = \text{R.H.S.}$$

Example 11 Find the value of sin 15°.

Solution We have

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45°\cos 30° - \cos 45°\sin 30°$$
$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 12 Find the value of $\tan\frac{13\pi}{12}$.

Solution We have

$$\tan\frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

Example 13 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Solution We have

$$\text{L.H.S.} \quad = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

Dividing the numerator and denominator by $\cos x \cos y$, we get

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Example 14 Show that

$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

Solution We know that $3x = 2x + x$

Therefore, $\tan 3x = \tan(2x + x)$

or $$\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

or $\tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

or $\tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

or $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$.

Example 15 Prove that

$$\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x$$

Solution Using the Identity 20(i), we have

$$\text{L.H.S.} = \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$

$$= 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-(\frac{\pi}{4}-x)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S.}$$

Example 16 Prove that $\dfrac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x} = \cot x$

Solution Using the Identities 20 (i) and 20 (iv), we get

$$\text{L.H.S.} = \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.}$$

Example 17 Prove that $=\dfrac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \tan x$

Solution We have

$$\text{L.H.S.} = \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \frac{\sin 5x+\sin x-2\sin 3x}{\cos 5x-\cos x}$$

$$= \frac{2\sin 3x\cos 2x-2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x(\cos 2x-1)}{\sin 3x\sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x = \text{R.H.S.}$$

EXERCISE 3.3

Prove that:

1. $\sin^2 \dfrac{\pi}{6} + \cos^2 \dfrac{\pi}{3} - \tan^2 \dfrac{\pi}{4} = -\dfrac{1}{2}$

2. $2\sin^2 \dfrac{\pi}{6} + \operatorname{cosec}^2 \dfrac{7\pi}{6} \cos^2 \dfrac{\pi}{3} = \dfrac{3}{2}$

3. $\cot^2 \dfrac{\pi}{6} + \operatorname{cosec} \dfrac{5\pi}{6} + 3\tan^2 \dfrac{\pi}{6} = 6$

4. $2\sin^2 \dfrac{3\pi}{4} + 2\cos^2 \dfrac{\pi}{4} + 2\sec^2 \dfrac{\pi}{3} = 10$

5. Find the value of:

   (i) $\sin 75°$ (ii) $\tan 15°$

6. Prove the following:

$$\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x+y)$$

7. $\dfrac{\tan\left(\dfrac{\pi}{4} + x\right)}{\tan\left(\dfrac{\pi}{4} - x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

8. $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\dfrac{\pi}{2} + x\right)} = \cot^2 x$

9. $\cos\left(\dfrac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\dfrac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1$

10. $\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x = \cos x$

11. $\cos\left(\dfrac{3\pi}{4} + x\right) - \cos\left(\dfrac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

12. $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

13. $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

14. $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

15. $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$

16. $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$

17. $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

18. $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan\dfrac{x-y}{2}$

19. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

20. $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$

21. $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

22. $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

23. $\tan 4x = \dfrac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$ **24.** $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

25. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

3.5 Trigonometric Equations

Equations involving trigonometric functions of a variable are called *trigonometric equations*. In this Section, we shall find the solutions of such equations. We have already learnt that the values of $\sin x$ and $\cos x$ repeat after an interval of 2π and the values of $\tan x$ repeat after an interval of π. The solutions of a trigonometric equation for which $0 \le x < 2\pi$ are called *principal solutions*. The expression involving integer 'n' which gives all solutions of a trigonometric equation is called the *general solution*. We shall use '**Z**' to denote the set of integers.

The following examples will be helpful in solving trigonometric equations:

Example 18 Find the principal solutions of the equation $\sin x = \dfrac{\sqrt{3}}{2}$.

Solution We know that, $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ and $\sin\dfrac{2\pi}{3} = \sin\left(\pi - \dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

Therefore, principal solutions are $x = \dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$.

Example 19 Find the principal solutions of the equation $\tan x = -\dfrac{1}{\sqrt{3}}$.

Solution We know that, $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$. Thus, $\tan\left(\pi - \dfrac{\pi}{6}\right) = -\tan\dfrac{\pi}{6} = -\dfrac{1}{\sqrt{3}}$

and $$\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

Thus $$\tan\frac{5\pi}{6} = \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}.$$

Therefore, principal solutions are $\dfrac{5\pi}{6}$ and $\dfrac{11\pi}{6}$.

We will now find the general solutions of trigonometric equations. We have already seen that:

$\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$

$\cos x = 0$ gives $x = (2n + 1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$.

We shall now prove the following results:

Theorem 1 For any real numbers x and y,

$$\sin x = \sin y \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}$$

Proof If $\sin x = \sin y$, then

$$\sin x - \sin y = 0 \text{ or } 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

which gives $\cos\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = (2n + 1)\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = (2n + 1)\pi + (-1)^{2n+1} y$ or $x = 2n\pi + (-1)^{2n} y$, where $n \in \mathbf{Z}$.

Combining these two results, we get

$$x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}.$$

Theorem 2 For any real numbers x and y, $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Proof If $\cos x = \cos y$, then

$$\cos x - \cos y = 0 \quad \text{i.e.,} \quad -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

Thus $\sin\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = n\pi$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = 2n\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Theorem 3 Prove that if x and y are not odd mulitple of $\frac{\pi}{2}$, then

$$\tan x = \tan y \text{ implies } x = n\pi + y, \text{ where } n \in \mathbf{Z}$$

Proof If $\tan x = \tan y$, then $\tan x - \tan y = 0$

or $$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

which gives $\sin(x - y) = 0$ (Why?)

Therefore $x - y = n\pi$, i.e., $x = n\pi + y$, where $n \in \mathbf{Z}$

Example 20 Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.

Solution We have $\sin x = -\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

Hence $\sin x = \sin\frac{4\pi}{3}$, which gives

$$x = n\pi + (-1)^n \frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$

> **Note** $\frac{4\pi}{3}$ is one such value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. One may take any other value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. The solutions obtained will be the same although these may apparently look different.

Example 21 Solve $\cos x = \frac{1}{2}$.

Solution We have, $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

Therefore $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$.

Example 22 Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$.

Solution We have, $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right)$

or $\tan 2x = \tan\left(x + \frac{5\pi}{6}\right)$

Therefore $2x = n\pi + x + \frac{5\pi}{6}$, where $n \in \mathbf{Z}$

or $x = n\pi + \frac{5\pi}{6}$, where $n \in \mathbf{Z}$.

Example 23 Solve $\sin 2x - \sin 4x + \sin 6x = 0$.

Solution The equation can be written as

$$\sin 6x + \sin 2x - \sin 4x = 0$$

or $2\sin 4x \cos 2x - \sin 4x = 0$

i.e. $\sin 4x(2\cos 2x - 1) = 0$

Therefore $\sin 4x = 0$ or $\cos 2x = \frac{1}{2}$

i.e. $\sin 4x = 0$ or $\cos 2x = \cos\frac{\pi}{3}$

Hence $4x = n\pi$ or $2x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$

i.e. $x = \frac{n\pi}{4}$ or $x = n\pi \pm \frac{\pi}{6}$, where $n \in \mathbf{Z}$.

Example 24 Solve $2\cos^2 x + 3\sin x = 0$

Solution The equation can be written as

$$2\left(1 - \sin^2 x\right) + 3\sin x = 0$$

or $2\sin^2 x - 3\sin x - 2 = 0$

or $(2\sin x + 1)(\sin x - 2) = 0$

Hence $\sin x = -\frac{1}{2}$ or $\sin x = 2$

But $\sin x = 2$ is not possible (Why?)

Therefore $\sin x = -\frac{1}{2} = \sin\frac{7\pi}{6}$.

Hence, the solution is given by

$$x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

EXERCISE 3.4

Find the principal and general solutions of the following equations:

1. $\tan x = \sqrt{3}$
2. $\sec x = 2$
3. $\cot x = -\sqrt{3}$
4. $\operatorname{cosec} x = -2$

Find the general solution for each of the following equations:

5. $\cos 4x = \cos 2x$
6. $\cos 3x + \cos x - \cos 2x = 0$
7. $\sin 2x + \cos x = 0$
8. $\sec^2 2x = 1 - \tan 2x$
9. $\sin x + \sin 3x + \sin 5x = 0$

Miscellaneous Examples

Example 25 If $\sin x = \frac{3}{5}$, $\cos y = -\frac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin (x + y)$.

Solution We know that

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \quad \ldots (1)$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Therefore $\cos x = \pm \frac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\frac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$

i.e. $\sin y = \pm \frac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \frac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get

$$\sin(x+y) = \frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13} \quad = -\frac{36}{65}-\frac{20}{65}=-\frac{56}{65}.$$

Example 26 Prove that

$$\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}.$$

Solution We have

$$\text{L.H.S.} = \frac{1}{2}\left[2\cos 2x \cos\frac{x}{2} - 2\cos\frac{9x}{2}\cos 3x\right]$$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right)+\cos\left(2x-\frac{x}{2}\right)-\cos\left(\frac{9x}{2}+3x\right)-\cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2}+\cos\frac{3x}{2}-\cos\frac{15x}{2}-\cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2}-\cos\frac{15x}{2}\right]$$

$$= \frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$= -\sin 5x \sin\left(-\frac{5x}{2}\right) = \sin 5x \sin\frac{5x}{2} = \text{R.H.S.}$$

Example 27 Find the value of $\tan\frac{\pi}{8}$.

Solution Let $x=\frac{\pi}{8}$. Then $2x=\frac{\pi}{4}$.

Now $\qquad \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

or $\qquad \tan\frac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

Let $y=\tan\frac{\pi}{8}$. Then $1=\dfrac{2y}{1-y^2}$

or $\quad y^2 + 2y - 1 = 0$

Therefore $\quad y = \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $\dfrac{\pi}{8}$ lies in the first quadrant, $y = \tan\dfrac{\pi}{8}$ is positve. Hence

$$\tan\frac{\pi}{8} = \sqrt{2} - 1.$$

Example 28 If $\tan x = \dfrac{3}{4}, \pi < x < \dfrac{3\pi}{2}$, find the value of $\sin\dfrac{x}{2}, \cos\dfrac{x}{2}$ and $\tan\dfrac{x}{2}$.

Solution Since $\pi < x < \dfrac{3\pi}{2}$, $\cos x$ is negative.

Also $\quad \dfrac{\pi}{2} < \dfrac{x}{2} < \dfrac{3\pi}{4}$.

Therefore, $\sin\dfrac{x}{2}$ is positive and $\cos\dfrac{x}{2}$ is negative.

Now $\quad \sec^2 x = 1 + \tan^2 x = 1 + \dfrac{9}{16} = \dfrac{25}{16}$

Therefore $\quad \cos^2 x = \dfrac{16}{25}$ or $\cos x = -\dfrac{4}{5}$ (Why?)

Now $\quad 2\sin^2\dfrac{x}{2} = 1 - \cos x = 1 + \dfrac{4}{5} = \dfrac{9}{5}$.

Therefore $\quad \sin^2\dfrac{x}{2} = \dfrac{9}{10}$

or $\quad \sin\dfrac{x}{2} = \dfrac{3}{\sqrt{10}}$ (Why?)

Again $\quad 2\cos^2\dfrac{x}{2} = 1 + \cos x = 1 - \dfrac{4}{5} = \dfrac{1}{5}$

Therefore $\quad \cos^2\dfrac{x}{2} = \dfrac{1}{10}$

or $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ (Why?)

Hence $\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3}{\sqrt{10}} \times \left(\frac{-\sqrt{10}}{1}\right) = -3.$

Example 29 Prove that $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$.

Solution We have

$$\text{L.H.S.} = \frac{1+\cos 2x}{2} + \frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2} + \frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}.$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-\cos 2x\right] = \frac{3}{2} = \text{R.H.S.}$$

Miscellaneous Exercise on Chapter 3

Prove that:

1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$
2. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

3. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2 \dfrac{x+y}{2}$

4. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \dfrac{x-y}{2}$

5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$

6. $\dfrac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

7. $\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \dfrac{x}{2} \cos \dfrac{3x}{2}$

Find $\sin \dfrac{x}{2}$, $\cos \dfrac{x}{2}$ and $\tan \dfrac{x}{2}$ in each of the following :

8. $\tan x = -\dfrac{4}{3}$, x in quadrant II

9. $\cos x = -\dfrac{1}{3}$, x in quadrant III

10. $\sin x = \dfrac{1}{4}$, x in quadrant II

Summary

- If in a circle of radius r, an arc of length l subtends and angle of θ radians, then $l = r\,\theta$
- Radian measure $= \dfrac{\pi}{180} \times$ Degree measure
- Degree measure $= \dfrac{180}{\pi} \times$ Radian measure
- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \operatorname{cosec}^2 x$
- $\cos (2n\pi + x) = \cos x$
- $\sin (2n\pi + x) = \sin x$
- $\sin (-x) = -\sin x$
- $\cos (-x) = \cos x$

- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$ $\quad$ $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

  $\cos(\pi - x) = -\cos x$ $\quad$ $\sin(\pi - x) = \sin x$

  $\cos(\pi + x) = -\cos x$ $\quad$ $\sin(\pi + x) = -\sin x$

  $\cos(2\pi - x) = \cos x$ $\quad$ $\sin(2\pi - x) = -\sin x$

- If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

  $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

- $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- If none of the angles x, y and $(x \pm y)$ is a multiple of π, then

  $$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

- $$\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

- $$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

◆ $\sin 2x = 2 \sin x \cos x = \dfrac{2 \tan x}{1 + \tan^2 x}$

◆ $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

◆ $\sin 3x = 3 \sin x - 4 \sin^3 x$

◆ $\cos 3x = 4 \cos^3 x - 3 \cos x$

◆ $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

◆ (i) $\cos x + \cos y = 2 \cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

(ii) $\cos x - \cos y = -2 \sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

(iii) $\sin x + \sin y = 2 \sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

(iv) $\sin x - \sin y = 2 \cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

◆ (i) $2 \cos x \cos y = \cos (x + y) + \cos (x - y)$

(ii) $- 2 \sin x \sin y = \cos (x + y) - \cos (x - y)$

(iii) $2 \sin x \cos y = \sin (x + y) + \sin (x - y)$

(iv) $2 \cos x \sin y = \sin (x + y) - \sin (x - y)$.

◆ $\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$.

◆ $\cos x = 0$ gives $x = (2n + 1) \dfrac{\pi}{2}$, where $n \in \mathbf{Z}$.

◆ $\sin x = \sin y$ implies $x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$.

◆ $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$.

◆ $\tan x = \tan y$ implies $x = n\pi + y$, where $n \in \mathbf{Z}$.

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476A.D.), Brahmagupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents the main contribution of the *siddhantas* (Sanskrit astronomical works) to the history of mathematics.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* (period) contains a proof for the expansion of sin (A + B). Exact expressin for sines or cosines of 18°, 36°, 54°, 72°, etc., are given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813 A.D.) The names of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\mathrm{H}}{\mathrm{S}} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

❖

Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

❖*Analysis and natural philosopy owe their most important discoveries to this fruitful means, which is called induction. Newton was indebted to it for his theorem of the binomial and the principle of universal gravity. – LAPLACE* ❖

4.1 Introduction

G. Peano (1858 1932)

One key basis for mathematical thinking is deductive reasoning. An informal, and example of deductive reasoning, borrowed from the study of logic, is an argument expressed in three statements:

(a) Socrates is a man.
(b) All men are mortal, therefore,
(c) Socrates is mortal.

If statements (a) and (b) are true, then the truth of (c) is established. To make this simple mathematical example, we could write:

(i) Eight is divisible by two.
(ii) Any number divisible by two is an even number, therefore,
(iii) Eight is an even number.

Thus, deduction in a nutshell is *given a statement to be proven, often called a conjecture or a theorem in mathematics, valid deductive steps are derived and a proof may or may not be established,* i.e., *deduction is the application of a general case to a particular case.*

In contrast to deduction, inductive reasoning depends on working with each case, and developing a conjecture by observing incidences till we have observed each and every case. It is frequently used in mathematics and is a key aspect of scientific reasoning, where collecting and analysing data is the norm. Thus, in simple language, we can say the word induction means the generalisation from particular cases or facts.

In algebra or in other discipline of mathematics, there are certain results or statements that are formulated in terms of n, where n is a positive integer. To prove such statements the well-suited principle that is used–based on the specific technique, is known as the *principle of mathematical induction*.

4.2 Motivation

In mathematics, we use a form of complete induction called mathematical induction. To understand the basic principles of mathematical induction, suppose a set of thin rectangular tiles are placed on one end, as shown in Fig 4.1.

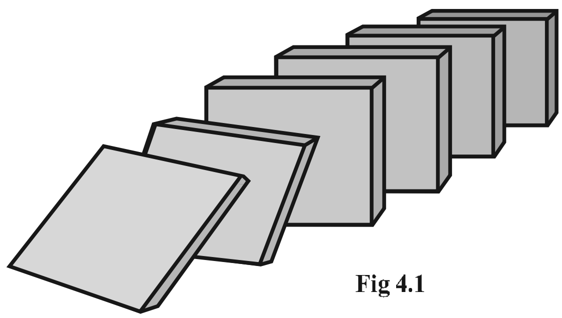


Fig 4.1

When the first tile is pushed in the indicated direction, all the tiles will fall. To be absolutely sure that all the tiles will fall, it is sufficient to know that

(a) The first tile falls, and

(b) In the event that any tile falls its successor necessarily falls.

This is the underlying principle of mathematical induction.

We know, the set of natural numbers **N** is a special ordered subset of the real numbers. In fact, **N** is the smallest subset of **R** with the following property:

A set S is said to be an inductive set if $1 \in$ S and $x + 1 \in$ S whenever $x \in$ S. Since **N** is the smallest subset of **R** which is an inductive set, it follows that any subset of **R** that is an inductive set must contain **N**.

Illustration

Suppose we wish to find the formula for the sum of positive integers 1, 2, 3,...,n, that is, a formula which will give the value of 1 + 2 + 3 when n = 3, the value 1 + 2 + 3 + 4, when n = 4 and so on and suppose that in some manner we are led to believe that the formula $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ is the correct one.

How can this formula actually be proved? We can, of course, verify the statement for as many positive integral values of n as we like, but this process will not prove the formula for all values of n. What is needed is some kind of chain reaction which will have the effect that once the formula is proved for a particular positive integer the formula will automatically follow for the next positive integer and the next indefinitely. Such a reaction may be considered as produced by the method of mathematical induction.

4.3 The Principle of Mathematical Induction

Suppose there is a given statement P(n) involving the natural number n such that

(i) *The statement is true for* $n = 1$, i.e., *P*(1) *is true, and*

(ii) *If the statement is true for* $n = k$ (*where k is some positive integer*), *then the statement is also true for* $n = k + 1$, i.e., *truth of P*(k) *implies the truth of P* ($k + 1$).

Then, P(n) is true for all natural numbers n.

Property (i) is simply a statement of fact. There may be situations when a statement is true for all $n \geq 4$. In this case, step 1 will start from $n = 4$ and we shall verify the result for $n = 4$, i.e., P(4).

Property (ii) is a conditional property. It does not assert that the given statement is true for $n = k$, but only that if it is true for $n = k$, then it is also true for $n = k + 1$. So, to prove that the property holds , only prove that conditional proposition:

If the statement is true for $n = k$, then it is also true for $n = k + 1$.

This is sometimes referred to as the inductive step. The assumption that the given statement is true for $n = k$ in this inductive step is called the *inductive hypothesis*.

For example, frequently in mathematics, a formula will be discovered that appears to fit a pattern like

$$1 = 1^2 = 1$$
$$4 = 2^2 = 1 + 3$$
$$9 = 3^2 = 1 + 3 + 5$$
$$16 = 4^2 = 1 + 3 + 5 + 7, \text{ etc.}$$

It is worth to be noted that the sum of the first two odd natural numbers is the square of second natural number, sum of the first three odd natural numbers is the square of third natural number and so on.Thus, from this pattern it appears that

$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$, i.e,

the sum of the first n odd natural numbers is the square of n.

Let us write

P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$.

We wish to prove that P(n) is true for all n.

The first step in a proof that uses mathematical induction is to prove that P (1) is true. This step is called the basic step. Obviously

$1 = 1^2$, i.e., P(1) is true.

The next step is called the *inductive step*. Here, we suppose that P (k) is true for some positive integer k and we need to prove that P $(k+1)$ is true. Since P (k) is true, we have

$$1 + 3 + 5 + 7 + ... + (2k-1) = k^2 \qquad ... (1)$$

Consider

$$1 + 3 + 5 + 7 + ... + (2k-1) + \{2(k+1) - 1\} \qquad ... (2)$$

$$= k^2 + (2k+1) = (k+1)^2 \qquad \text{[Using (1)]}$$

Therefore, P $(k+1)$ is true and the inductive proof is now completed.

Hence P(n) is true for all natural numbers n.

Example 1 For all $n \geq 1$, prove that

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution Let the given statement be P(n), i.e.,

$$P(n): \ 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$, $\quad$ P(1): $1 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{6} = 1$ which is true.

Assume that P(k) is true for some positive integers k, i.e.,

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad ... (1)$$

We shall now prove that P($k+1$) is also true. Now, we have

$$(1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{[Using (1)]}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$

Thus P($k+1$) is true, whenever P (k) is true.

Hence, from the principle of mathematical induction, the statement P(n) is true for all natural numbers **N**.

Example 2 Prove that $2^n > n$ for all positive integers n.

Solution Let P(n): $2^n > n$

When $n = 1$, $2^1 > 1$. Hence P(1) is true.

Assume that P(k) is true for any positive integers k, i.e.,

$$2^k > k \qquad \text{... (1)}$$

We shall now prove that P(k +1) is true whenever P(k) is true.

Multiplying both sides of (1) by 2, we get

$$2.\ 2^k > 2k$$

i.e., $2^{k+1} > 2k = k + k > k + 1$

Therefore, P(k + 1) is true when P(k) is true. Hence, by principle of mathematical induction, P(n) is true for every positive integer n.

Example 3 For all $n \geq 1$, prove that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

Solution We can write

$$P(n): \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

We note that P(1): $\frac{1}{1.2}=\frac{1}{2}=\frac{1}{1+1}$, which is true. Thus, P(n) is true for n = 1.

Assume that P(k) is true for some natural numbers k,

i.e., $$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}=\frac{k}{k+1} \qquad \text{... (1)}$$

We need to prove that P(k + 1) is true whenever P(k) is true. We have

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$= \left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \qquad \text{[Using (1)]}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k^2+2k+1)}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$

Thus P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all natural numbers.

Example 4 For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

Solution We can write

P(n) : $7^n - 3^n$ is divisible by 4.

We note that

P(1): $7^1 - 3^1 = 4$ which is divisile by 4. Thus P(n) is true for $n = 1$

Let P(k) be true for some natural number k,

i.e., P(k) : $7^k - 3^k$ is divisible by 4.

We can write $7^k - 3^k = 4d$, where $d \in \mathbf{N}$.

Now, we wish to prove that P(k + 1) is true whenever P(k) is true.

Now $\quad 7^{(k+1)} - 3^{(k+1)} = 7^{(k+1)} - 7.3^k + 7.3^k - 3^{(k+1)}$

$= 7(7^k - 3^k) + (7 - 3)3^k = 7(4d) + (7 - 3)3^k$

$= 7(4d) + 4.3^k = 4(7d + 3^k)$

From the last line, we see that $7^{(k+1)} - 3^{(k+1)}$ is divisible by 4. Thus, P(k + 1) is true when P(k) is true. Therefore, by princlple of mathematical induction the statement is true for every positive integer n.

Example 5 Prove that $(1 + x)^n \geq (1 + nx)$, for all natural number n, where $x > -1$.

Solution Let P(n) be the given statement,

i.e., P(n): $(1 + x)^n \geq (1 + nx)$, for $x > -1$.

We note that P(n) is true when $n = 1$, since $(1+x) \geq (1 + x)$ for $x > -1$

Assume that

P(k): $(1 + x)^k \geq (1 + kx)$, $x > -1$ is true. ... (1)

We want to prove that P(k + 1) is true for $x > -1$ whenever P(k) is true. ... (2)

Consider the identity

$(1 + x)^{k+1} = (1 + x)^k (1 + x)$

Given that $\quad x > -1$, so $(1+x) > 0$.

Therefore , by using $(1 + x)^k \geq (1 + kx)$, we have

$(1 + x)^{k+1} \geq (1 + kx)(1 + x)$

i.e. $\quad (1 + x)^{k+1} \geq (1 + x + kx + kx^2)$. ... (3)

Here k is a natural number and $x^2 \geq 0$ so that $kx^2 \geq 0$. Therefore

$$(1 + x + kx + kx^2) \geq (1 + x + kx),$$

and so we obtain

$$(1 + x)^{k+1} \geq (1 + x + kx)$$

i.e. $(1 + x)^{k+1} \geq [1 + (1 + k)x]$

Thus, the statement in (2) is established. Hence, by the principle of mathematical induction, P(n) is true for all natural numbers.

Example 6 Prove that

$$2.7^n + 3.5^n - 5 \text{ is divisible by 24, for all } n \in \mathbf{N}.$$

Solution Let the statement P(n) be defined as

$$P(n) : 2.7^n + 3.5^n - 5 \text{ is divisible by 24.}$$

We note that P(n) is true for $n = 1$, since $2.7 + 3.5 - 5 = 24$, which is divisible by 24.

Assume that P(k) is true

i.e. $2.7^k + 3.5^k - 5 = 24q$, when $q \in \mathbf{N}$ ... (1)

Now, we wish to prove that P(k + 1) is true whenever P(k) is true.

We have

$$
\begin{aligned}
2.7^{k+1} + 3.5^{k+1} - 5 &= 2.7^k . 7^1 + 3.5^k . 5^1 - 5 \\
&= 7 [2.7^k + 3.5^k - 5 - 3.5^k + 5] + 3.5^k . 5 - 5 \\
&= 7 [24q - 3.5^k + 5] + 15.5^k - 5 \\
&= 7 \times 24q - 21.5^k + 35 + 15.5^k - 5 \\
&= 7 \times 24q - 6.5^k + 30 \\
&= 7 \times 24q - 6 (5^k - 5) \\
&= 7 \times 24q - 6 (4p) \quad [(5^k - 5) \text{ is a multiple of 4 (why?)}] \\
&= 7 \times 24q - 24p \\
&= 24 (7q - p) \\
&= 24 \times r; \; r = 7q - p, \text{ is some natural number.} \qquad \text{... (2)}
\end{aligned}
$$

The expresion on the R.H.S. of (1) is divisible by 24. Thus P(k + 1) is true whenever P(k) is true.

Hence, by principle of mathematical induction, P(n) is true for all $n \in$ N.

Example 7 Prove that

$$1^2 + 2^2 + ... + n^2 > \frac{n^3}{3}, n \in \mathbf{N}$$

Solution Let P(n) be the given statement.

i.e., $P(n) : 1^2 + 2^2 + ... + n^2 > \frac{n^3}{3}, \ n \in \mathbf{N}$

We note that P(n) is true for $n = 1$ since $1^2 > \frac{1^3}{3}$

Assume that P(k) is true

i.e. $$P(k) : 1^2 + 2^2 + ... + k^2 > \frac{k^3}{3} \quad ...(1)$$

We shall now prove that P(k + 1) is true whenever P(k) is true.

We have $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2$

$$= \left(1^2 + 2^2 + ... + k^2\right) + (k+1)^2 > \frac{k^3}{3} + (k+1)^2 \quad \text{[by (1)]}$$

$$= \frac{1}{3}\,[k^3 + 3k^2 + 6k + 3]$$

$$= \frac{1}{3}\,[(k + 1)^3 + 3k + 2] > \frac{1}{3}\,(k + 1)^3$$

Therefore, P(k+ 1) is also true whenever P(k) is true. Hence, by mathematical induction P(n) is true for all $n \in \mathbf{N}$.

Example 8 Prove the rule of exponents $(ab)^n = a^nb^n$
by using principle of mathematical induction for every natural number.

Solution Let P(n) be the given statement
i.e. $P(n) : (ab)^n = a^nb^n$.

We note that P(n) is true for $n = 1$ since $(ab)^1 = a^1b^1$.

Let P(k) be true, i.e.,

$$(ab)^k = a^kb^k \quad ... (1)$$

We shall now prove that P(k + 1) is true whenever P(k) is true.

Now, we have

$(ab)^{k+1} = (ab)^k\,(ab)$

$$= (a^k\ b^k)\ (ab) \qquad\qquad [\text{by } (1)]$$

$$= (a^k\ .\ a^1)\ (b^k\ .\ b^1) = a^{k+1}\ .\ b^{k+1}$$

Therefore, P(k + 1) is also true whenever P(k) is true. Hence, by principle of mathematical induction, P(n) is true for all $n \in$ N.

EXERCISE 4.1

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$:

1. $1 + 3 + 3^2 + ... + 3^{n-1} = \dfrac{(3^n - 1)}{2}$.

2. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$.

3. $1 + \dfrac{1}{(1+2)} + \dfrac{1}{(1+2+3)} + ... + \dfrac{1}{(1+2+3+...n)} = \dfrac{2n}{(n+1)}$.

4. $1.2.3 + 2.3.4 + \ldots + n(n+1)\ (n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

5. $1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \dfrac{(2n-1)3^{n+1} + 3}{4}$.

6. $1.2 + 2.3 + 3.4 + \ldots + n.(n+1) = \left[\dfrac{n(n+1)(n+2)}{3}\right]$.

7. $1.3 + 3.5 + 5.7 + \ldots + (2n-1)\ (2n+1) = \dfrac{n(4n^2 + 6n - 1)}{3}$.

8. $1.2 + 2.2^2 + 3.2^2 + ... + n.2^n = (n-1)\ 2^{n+1} + 2$.

9. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ... + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$.

10. $\dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + ... + \dfrac{1}{(3n-1)(3n+2)} = \dfrac{n}{(6n+4)}$.

11. $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} + ... + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$.

12. $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$.

13. $\left(1+\dfrac{3}{1}\right)\left(1+\dfrac{5}{4}\right)\left(1+\dfrac{7}{9}\right)\ldots\left(1+\dfrac{(2n+1)}{n^2}\right) = (n+1)^2$.

14. $\left(1+\dfrac{1}{1}\right)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\ldots\left(1+\dfrac{1}{n}\right) = (n+1)$.

15. $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$.

16. $\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \ldots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{(3n+1)}$.

17. $\dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + \ldots + \dfrac{1}{(2n+1)(2n+3)} = \dfrac{n}{3(2n+3)}$.

18. $1 + 2 + 3 + \ldots + n < \dfrac{1}{8}(2n+1)^2$.

19. $n(n+1)(n+5)$ is a multiple of 3.

20. $10^{2n-1} + 1$ is divisible by 11.

21. $x^{2n} - y^{2n}$ is divisible by $x + y$.

22. $3^{2n+2} - 8n - 9$ is divisible by 8.

23. $41^n - 14^n$ is a multiple of 27.

24. $(2n + 7) < (n + 3)^2$.

Summary

- One key basis for mathematical thinking is deductive reasoning. In contrast to deduction, inductive reasoning depends on working with different cases and developing a conjective by observing incidences till we have observed each and every case. Thus, in simple language we can say the word 'induction' means the generalisation from particular cases or facts.
- The principle of mathematical induction is one such tool which can be used to prove a wide variety of mathematical statements. Each such statement is assumed as P(n) associated with positive integer n, for which the correctness

for the case $n = 1$ is examined. Then assuming the truth of P(k) for some positive integer k, the truth of P (k+1) is established.

Historical Note

Unlike other concepts and methods, proof by mathematical induction is not the invention of a particular individual at a fixed moment. It is said that the principle of mathematical induction was known by the Phythagoreans.

The French mathematician Blaise Pascal is credited with the origin of the principle of mathematical induction.

The name induction was used by the English mathematician John Wallis.

Later the principle was employed to provide a proof of the binomial theorem.

De Morgan contributed many accomplishments in the field of mathematics on many different subjects. He was the first person to define and name "mathematical induction" and developed De Morgan's rule to determine the convergence of a mathematical series.

G. Peano undertook the task of deducing the properties of natural numbers from a set of explicitly stated assumptions, now known as Peano's axioms.The principle of mathematical induction is a restatement of one of the Peano's axioms.

❖

Chapter 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖ *Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics. – GAUSS* ❖

5.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution as $x^2 + 1 = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation $x^2 = -1$. In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.

W. R. Hamilton
(1805 1865)

5.2 Complex Numbers

Let us denote $\sqrt{-1}$ by the symbol *i*. Then, we have $i^2 = -1$. This means that *i* is a solution of the equation $x^2 + 1 = 0$.

A number of the form $a + ib$, where a and b are real numbers, is defined to be a complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4 + i\left(\frac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by Re z and b is called the *imaginary part* denoted by Im z *of the complex number* z. For example, if $z = 2 + i5$, then Re $z = 2$ and Im $z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Example 1 If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then find the values of x and y.

Solution We have

$$4x + i(3x - y) = 3 + i(-6) \quad \text{... (1)}$$

Equating the real and the imaginary parts of (1), we get

$$4x = 3,\ 3x - y = -6,$$

which, on solving simultaneously, give $x = \frac{3}{4}$ and $y = \frac{33}{4}$.

5.3 Algebra of Complex Numbers

In this Section, we shall develop the algebra of complex numbers.

5.3.1 ***Addition of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + c) + i(b + d)$, which is again a complex number.

For example, $(2 + i3) + (-6 + i5) = (2 - 6) + i(3 + 5) = -4 + i\,8$

The addition of complex numbers satisfy the following properties:

(i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) *The commutative law* For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

(iii) *The associative law* For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iv) *The existence of additive identity* There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number,* such that, for every complex number z, $z + 0 = z$.

(v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of z*. We observe that $z + (-z) = 0$ (the additive identity).

5.3.2 ***Difference of two complex numbers*** Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows:

$$z_1 - z_2 = z_1 + (-z_2).$$

For example, $(6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and $(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

5.3.3 ***Multiplication of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

For example, $(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law** For any two complex numbers z_1 and z_2,
$$z_1 z_2 = z_2 z_1.$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3,
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i\,0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi(a \neq 0, b \neq 0)$, we have the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that

$$z.\frac{1}{z} = 1 \text{ (the multiplicative identity).}$$

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3,

(a) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

5.3.4 ***Division of two complex numbers*** Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

Then $$\frac{z_1}{z_2} = \left((6+3i) \times \frac{1}{2-i}\right) = (6+3i)\left(\frac{2}{2^2+(-1)^2} + i\frac{-(-1)}{2^2+(-1)^2}\right)$$

$$= (6+3i)\left(\frac{2+i}{5}\right) = \frac{1}{5}\left[12-3+i(6+6)\right] = \frac{1}{5}(9+12i)$$

5.3.5 ***Power of i*** we know that

$$i^3 = i^2 i = (-1)\, i = -i\,, \qquad i^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

$$i^5 = \left(i^2\right)^2 i = (-1)^2\, i = i\,, \qquad i^6 = \left(i^2\right)^3 = (-1)^3 = -1\,, \text{ etc.}$$

Also, we have $\quad i^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

5.3.6 ***The square roots of a negative real number***

Note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$

Therefore, the square roots of -1 are $i, -i$. However, by the symbol $\sqrt{-}$, we would mean i only.

Now, we can see that i and $-i$ both are the solutions of the equation $x^2 + 1 = 0$ or $x^2 = -1$.

Similarly $\quad \left(\sqrt{3}\, i\right)^2 = \left(\sqrt{3}\right)^2 i^2 = 3\,(-1) = -3$

$$\left(-\sqrt{3}\, i\right)^2 = \left(-\sqrt{3}\right)^2 i^2 = -3$$

Therefore, the square roots of -3 are $\sqrt{3}\, i$ and $-\sqrt{3}\, i$.

Again, the symbol $\sqrt{-3}$ is meant to represent $\sqrt{3}\, i$ only, i.e., $\sqrt{-3} = \sqrt{3}\, i$.

Generally, if a is a positive real number, $\sqrt{-a} = \sqrt{a}\,\sqrt{-1} = \sqrt{a}\, i$,

We already know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real number a and b. This result also holds true when either $a > 0, b < 0$ or $a < 0, b > 0$. What if $a < 0, b < 0$? Let us examine.

Note that

$i^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)}$ (by assuming $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all real numbers)

$= \sqrt{1} = 1$, which is a contradiction to the fact that $i \;\; = -$.

Therefore, $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers.

Further, if any of a and b is zero, then, clearly, $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

5.3.7 ***Identities*** We prove the following identity

$(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2$, for all complex numbers z_1 and z_2.

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$,

$$
\begin{aligned}
&= (z_1 + z_2)\, z_1 + (z_1 + z_2)\, z_2 && \text{(Distributive law)}\\
&= z_1^2 + z_2z_1 + z_1z_2 + z_2^2 && \text{(Distributive law)}\\
&= z_1^2 + z_2z_1 + z_1z_2 + z_2^2 && \text{(Commutative law of multiplicatoin)}\\
&= z_1^2 + 2z_1z_2 + z_2^2
\end{aligned}
$$

Similarly, we can prove the following identities:

(i) $(z_1 - z_2)^2 = z_1^2 - 2\,z_1\,z_2 + z_2^2$

(ii) $(z_1 + z_2)^3 = z_1^3 + 3\,z_1^2z_2 + 3z_1z_2^2 + z_2^3$

(iii) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$

(iv) $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

Example 2 Express the following in the form of $a + bi$:

(i) $(-5i)\left(\frac{1}{8}i\right)$ (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$

Solution (i) $(-5i)\left(\frac{1}{8}i\right) = \frac{-5}{8}i^2 = \frac{-5}{8}(-1) = \frac{5}{8} = \frac{5}{8} + i0$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3 = 2 \times \frac{1}{8 \times 8 \times 8} \times i^5 = \frac{1}{256}\left(i^2\right)^2\; i = \frac{1}{256}i$.

Example 3 Express $(5 - 3i)^3$ in the form $a + ib$.

Solution We have, $(5 - 3i)^3 = 5^3 - 3 \times 5^2 \times (3i) + 3 \times 5\ (3i)^2 - (3i)^3$

$$= 125 - 225i - 135 + 27i = -10 - 198i.$$

Example 4 Express $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right)$ in the form of $a + ib$

Solution We have, $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right) = \left(-\sqrt{3}+\sqrt{2}\,i\right)\left(2\sqrt{3}-i\right)$

$$= -6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}\,i^2 = \left(-6+\sqrt{2}\right)+\sqrt{3}\left(1+2\sqrt{2}\right)i$$

5.4 The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z, denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2+b^2}$, i.e., $|z| = \sqrt{a^2+b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $|3+i| = \sqrt{3^2+1^2} = \sqrt{10}$, $|2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$,

and $\overline{3+i} = 3-i$, $\overline{2-5i} = 2+5i$, $\overline{-3i-5} = 3i-5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2}+i\frac{-b}{a^2+b^2} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

or $\quad z\,\bar{z} = |z|^2$

Furthermore, the following results can easily be derived.

For any two compex numbers z_1 and z_2, we have

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$.

Example 5 Find the multiplicative inverse of $2 - 3i$.

Solution Let $z = 2 - 3i$

Then $\bar{z} = 2 + 3i$ and $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of $2 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$z^{-1} = \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)}$$

$$= \frac{2+3i}{2^2-(3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

Example 6 Express the following in the form $a + ib$

(i) $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (ii) i^{-35}

Solution (i) We have, $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} = \dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \dfrac{1+\sqrt{2}i}{1+\sqrt{2}i} = \dfrac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-\left(\sqrt{2}i\right)^2}$

$$= \frac{3+6\sqrt{2}i}{1+2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i.$$

(ii) $i^{-35} = \dfrac{1}{i^{35}} = \dfrac{1}{\left(i^2\right)^{17} i} = \dfrac{1}{-i} \times \dfrac{i}{i} = \dfrac{i}{-i^2} = i$

EXERCISE 5.1

Express each of the complex number given in the Exercises 1 to 10 in the form $a + ib$.

1. $(5i)\left(-\frac{3}{5}i\right)$ **2.** $i \;\; + i$ **3.** i^{-}

4. $3(7 + i7) + i\,(7 + i7)$ 5. $(1 - i) - (-1 + i6)$

6. $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$ 7. $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

8. $(1 - i)^4$ 9. $\left(\frac{1}{3}+3i\right)^3$ 10. $\left(-2-\frac{1}{3}i\right)^3$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 - 3i$ 12. $\sqrt{5}+3i$ 13. $-i$

14. Express the following expression in the form of $a + ib$:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}\,i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

5.5 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa.

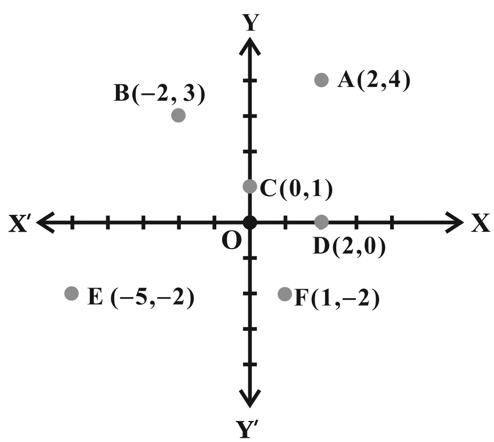


Fig 5.1

Some complex numbers such as $2 + 4i, -2 + 3i, 0 + 1i, 2 + 0i, -5 -2i$ and $1 - 2i$ which correspond to the ordered pairs (2, 4), (– 2, 3), (0, 1), (2, 0), (–5, –2), and (1, – 2), respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 5.1.

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.

Obviously, in the Argand plane, the modulus of the complex number

$x + iy = \sqrt{x^2 + y^2}$ is the distance between the point P(x, y) to the origin O (0, 0) (Fig 5.2). The points on the x-axis corresponds to the complex numbers of the form $a + i\,0$ and the points on the y-axis corresponds to the complex numbers of the form

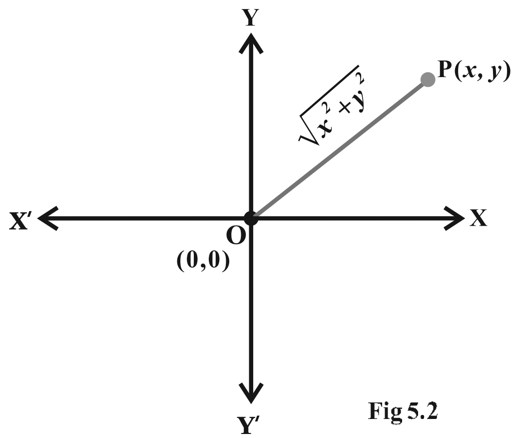


Fig 5.2

$0 + i\,b$. The x-axis and y-axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number $z = x + iy$ and its conjugate $z = x - iy$ in the Argand plane are, respectively, the points P (x, y) and Q (x, $-y$).

Geometrically, the point (x, $-y$) is the mirror image of the point (x, y) on the real axis (Fig 5.3).

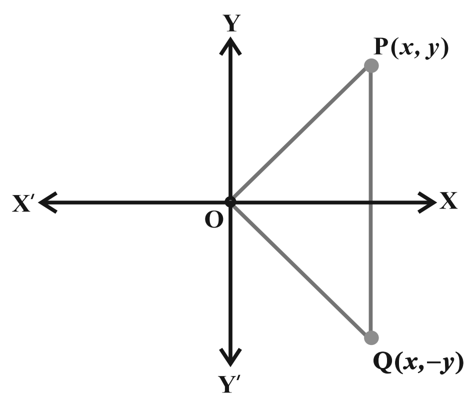


Fig 5.3

5.5.1 ***Polar representation of a complex number*** Let the point P represent the non-zero complex number $z = x + iy$. Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis (Fig 5.4).

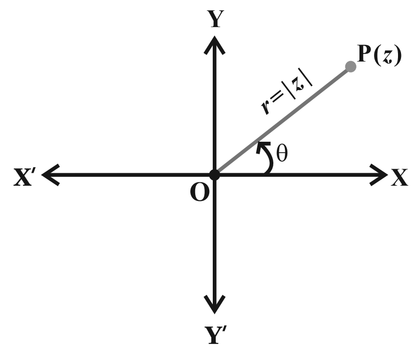


Fig 5.4

We may note that the point P is uniquely determined by the ordered pair of real numbers (r, θ), called the *polar coordinates of the point* P. We consider the origin as the pole and the positive direction of the x axis as the initial line.

We have, $x = r \cos \theta$, $y = r \sin \theta$ and therefore, $z = r (\cos \theta + i \sin \theta)$. The latter is said to be the *polar form of the complex number*. Here $r = \sqrt{x^2 + y^2} = |z|$ is the modus of z and θ is called the argument (or amplitude) of z which is denoted by arg z.

For any complex number $z \neq 0$, there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be such an interval.We shall take the value of θ such that $-\pi < \theta \leq \pi$, called **principal argument** of z and is denoted by arg z, unless specified otherwise. (Figs. 5.5 and 5.6)

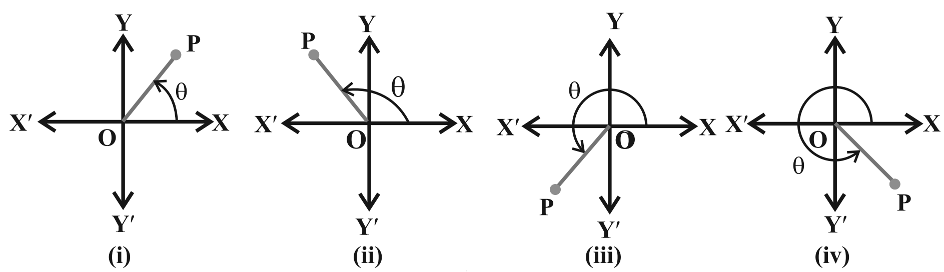


Fig 5.5 $(0 \leq \theta < 2\pi)$

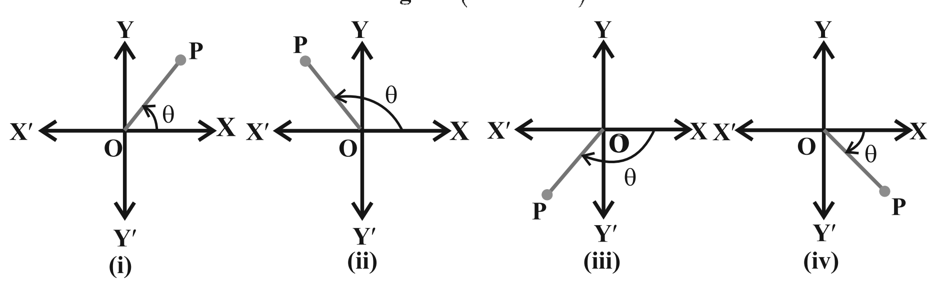


Fig 5.6 $(-\pi < \theta \leq \pi)$

Example 7 Represent the complex number $z = 1 + i\sqrt{3}$ in the polar form.

Solution Let $1 = r\cos\theta$, $\sqrt{\ } = r\sin\theta$

By squaring and adding, we get

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 4$$

i.e., $r = \sqrt{\ } =$ (conventionally, $r > 0$)

Therefore, $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$, which gives $\theta = \frac{\pi}{\ }$

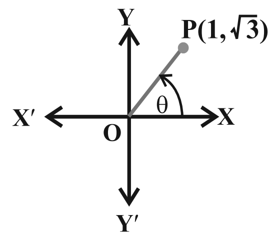


Fig 5.7

Therefore, required polar form is $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

The complex number $z = \ + i\sqrt{\ }$ is represented as shown in Fig 5.7.

Example 8 Convert the complex number $\frac{-16}{1 + i\sqrt{3}}$ into polar form.

Solution The given complex number $\frac{-16}{1 + i\sqrt{3}} = \frac{-16}{1 + i\sqrt{3}} \times \frac{1 - i\sqrt{3}}{1 - i\sqrt{3}}$

$$= \frac{-16\left(1 - i\sqrt{3}\right)}{1 - \left(i\sqrt{3}\right)^2} = \frac{-16\left(1 - i\sqrt{3}\right)}{1 + 3} = -\left(\ - i\sqrt{\ }\right) = -\ + i\ \sqrt{\ } \text{ (Fig 5.8).}$$

Let $-4 = r\cos\theta$, $\sqrt{\ } = r\sin\theta$

By squaring and adding, we get

$$16 + 48 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

which gives $r^2 = 64$, i.e., $r = 8$

Hence $\cos\theta = -\frac{\ }{\ }$, $\sin\theta = \frac{\sqrt{\ }}{\ }$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

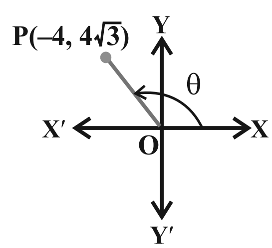


Fig 5.8

Thus, the required polar form is $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

EXERCISE 5.2

Find the modulus and the arguments of each of the complex numbers in Exercises 1 to 2.

1. $z = -1 - i\sqrt{\ }$ **2.** $z = -\sqrt{\ } + i$

Convert each of the complex numbers given in Exercises 3 to 8 in the polar form:

3. $1 - i$ **4.** $-1 + i$ **5.** $-1 - i$

6. -3 **7.** $\sqrt{\ } + i$ **8.** i

5.6 Quadratic Equations

We are already familiar with the quadratic equations and have solved them in the set of real numbers in the cases where discriminant is non-negative, i.e., ≥ 0,

Let us consider the following quadratic equation:

$ax^2 + bx + c = 0$ with real coefficients a, b, c and $a \neq 0$.

Also, let us assume that the $b^2 - 4ac < 0$.

Now, we know that we can find the square root of negative real numbers in the set of complex numbers. Therefore, the solutions to the above equation are available in the set of complex numbers which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

Note At this point of time, some would be interested to know as to how many roots does an equation have? In this regard, the following theorem known as the *Fundamental theorem of Algebra* is stated below (without proof).

"A polynomial equation has at least one root."

As a consequence of this theorem, the following result, which is of immense importance, is arrived at:

"A polynomial equation of degree *n* has *n* roots."

Example 9 Solve $x^2 + 2 = 0$

Solution We have, $x^2 + 2 = 0$

or $x^2 = -2$ i.e., $x = \pm\sqrt{-2} = \pm\sqrt{2}\, i$

Example 10 Solve $x^2 + x + 1 = 0$

Solution Here, $b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = 1 - 4 = -3$

Therefore, the solutions are given by $x = \dfrac{-1 \pm \sqrt{-3}}{2 \times 1} = \dfrac{-1 \pm \sqrt{3}i}{2}$

Example 11 Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$

Solution Here, the discriminant of the equation is

$$1^2 - 4 \times \sqrt{5} \times \sqrt{5} = 1 - 20 = -19$$

Therefore, the solutions are

$$\frac{-1 \pm \sqrt{-19}}{2\sqrt{5}} = \frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}.$$

EXERCISE 5.3

Solve each of the following equations:

1. $x^2 + 3 = 0$
2. $2x^2 + x + 1 = 0$
3. $x^2 + 3x + 9 = 0$
4. $-x^2 + x - 2 = 0$
5. $x^2 + 3x + 5 = 0$
6. $x^2 - x + 2 = 0$
7. $\sqrt{2}x^2 + x + \sqrt{2} = 0$
8. $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
9. $x^2 + x + \dfrac{1}{\sqrt{2}} = 0$
10. $x^2 + \dfrac{x}{\sqrt{2}} + 1 = 0$

Miscellaneous Examples

Example 12 Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Solution We have , $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$$= \frac{6+9i-4i+6}{2-i+4i+2} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i$$

Therefore, conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ is $\dfrac{63}{25} + \dfrac{16}{25}i$.

Example 13 Find the modulus and argument of the complex numbers:

(i) $\dfrac{1+i}{1-i}$, (ii) $\dfrac{1}{1+i}$

Solution (i) We have, $\dfrac{1+i}{1-i} = \dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i}=\dfrac{1-1+2i}{1+1} = i = 0 + i$

Now, let us put $0 = r\cos\theta$, $1 = r\sin\theta$

Squaring and adding, $r^2 = 1$ i.e., $r = 1$ so that

$$\cos\theta = 0,\ \sin\theta = 1$$

Therefore, $\theta = \dfrac{\pi}{2}$

Hence, the modulus of $\dfrac{1+i}{1-i}$ is 1 and the argument is $\dfrac{\pi}{2}$.

(ii) We have $\dfrac{1}{1+i} = \dfrac{1-i}{(1+i)(1-i)} = \dfrac{1-i}{1+1} = \dfrac{1}{2} - \dfrac{i}{2}$

Let $\dfrac{1}{2} = r\cos\theta,\ -\dfrac{1}{2} = r\sin\theta$

Proceeding as in part (i) above, we get $r = \dfrac{1}{\sqrt{2}};\ \cos\theta = \dfrac{1}{\sqrt{2}},\ \sin\theta = \dfrac{-1}{\sqrt{2}}$

Therefore $\theta = \dfrac{-\pi}{4}$

Hence, the modulus of $\dfrac{1}{1+i}$ is $\dfrac{1}{\sqrt{2}}$, argument is $\dfrac{-\pi}{4}$.

Example 14 If $x + iy = \dfrac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.

Solution We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2-b^2+2abi}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i$$

So that, $\qquad x - iy = \dfrac{a^2 - b^2}{a^2 + b^2} - \dfrac{2ab}{a^2 + b^2} i$

Therefore,

$$x^2 + y^2 = (x + iy)(x - iy) = \frac{(a^2 - b^2)^2}{(a^2 + b^2)^2} + \frac{4a^2b^2}{(a^2 + b^2)^2} = \frac{(a^2 + b^2)^2}{(a^2 + b^2)^2} = 1$$

Example 15 Find real θ such that

$$\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{ is purely real.}$$

Solution We have,

$$\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} = \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{(1 - 2i\sin\theta)(1 + 2i\sin\theta)}$$

$$= \frac{3 + 6i\sin\theta + 2i\sin\theta - 4\sin^2\theta}{1 + 4\sin^2\theta} = \frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} + \frac{8i\ \sin\theta}{1 + 4\sin^2\theta}$$

We are given the complex number to be real. Therefore

$$\frac{8\sin\theta}{1 + 4\sin^2\theta} = 0, \text{ i.e., } \sin\theta = 0$$

Thus $\qquad \theta = n\pi,\ n \in \mathbf{Z}$.

Example 16 Convert the complex number $z = \dfrac{i - 1}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$ in the polar form.

Solution We have, $z = \dfrac{i - 1}{\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i}$

$$= \frac{2(i - 1)}{1 + \sqrt{3}i} \times \frac{1 - \sqrt{3}i}{1 - \sqrt{3}i} = \frac{2\left(i + \sqrt{3} - 1 + \sqrt{3}i\right)}{1 + 3} = \frac{\sqrt{3} - 1}{2} + \frac{\sqrt{3} + 1}{2} i$$

Now, put $\qquad \dfrac{\sqrt{3} - 1}{2} = r\cos\theta,\ \dfrac{\sqrt{3} + 1}{2} = r\sin\theta$

Squaring and adding, we obtain

$$r^2 = \left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2 = \frac{2\left(\left(\sqrt{3}\right)^2+1\right)}{4} = \frac{2\times 4}{4} = 2$$

Hence, $r=\sqrt{2}$ which gives $\quad \cos\theta = \dfrac{\sqrt{3}-1}{2\sqrt{2}}, \quad \sin\theta = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Therefore, $\theta = \dfrac{\pi}{4}+\dfrac{\pi}{6} = \dfrac{5\pi}{12}$ (Why?)

Hence, the polar form is

$$\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

Miscellaneous Exercise on Chapter 5

1. Evaluate: $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$.

2. For any two complex numbers z_1 and z_2, prove that
Re $(z_1 z_2)$ = Re z_1 Re z_2 – Imz_1 Imz_2.

3. Reduce $\left(\dfrac{1}{1-4i}-\dfrac{2}{1+i}\right)\left(\dfrac{3-4i}{5+i}\right)$ to the standard form .

4. If $x-iy=\sqrt{\dfrac{a-ib}{c-id}}$ prove that $x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$.

5. Convert the following in the polar form:

(i) $\dfrac{1+7i}{(2-i)^2}$, $\qquad$ (ii) $\dfrac{1+3i}{1-2i}$

Solve each of the equation in Exercises 6 to 9.

6. $3x^2-4x+\dfrac{20}{3}=0$ $\qquad$ **7.** $x^2-2x+\dfrac{3}{2}=0$

8. $27x^2-10x+1=0$

9. $21x^2 - 28x + 10 = 0$

10. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$.

11. If $a + ib = \dfrac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \dfrac{(x^2+1)^2}{\left(2x^2+1\right)^2}$.

12. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\mathrm{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$, (ii) $\mathrm{Im}\left(\dfrac{1}{z_1 \bar{z}_1}\right)$.

13. Find the modulus and argument of the complex number $\dfrac{1+2i}{1-3i}$.

14. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

15. Find the modulus of $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$.

16. If $(x + iy)^3 = u + iv$, then show that $\dfrac{u}{x} + \dfrac{v}{y} = 4(x^2 - y^2)$.

17. If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\dfrac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$.

18. Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$.

19. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

20. If $\left(\dfrac{1+i}{1-i}\right)^m = 1$, then find the least integral value of m.

Summary

- A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*, a is called the *real part* and b is called the *imaginary* part of the complex number.
- Let $z_1 = a + ib$ and $z_2 = c + id$. Then
  (i) $z_1 + z_2 = (a + c) + i\,(b + d)$
  (ii) $z_1 z_2 = (ac - bd) + i\,(ad + bc)$
- For any non-zero complex number $z = a + ib$ ($a \neq 0$, $b \neq 0$), there exists the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$, denoted by $\frac{1}{z}$ or z^{-1}, called the *multiplicative inverse* of z such that $(a + ib)\left(\frac{a^2}{a^2+b^2} + i\frac{-b}{a^2+b^2}\right) = 1 + i0 = 1$
- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- The conjugate of the complex number $z = a + ib$, denoted by $\bar{z}$, is given by $\bar{z} = a - ib$.
- The polar form of the complex number $z = x + iy$ is r (cosθ + i sinθ), where $r = \sqrt{x^2+y^2}$ (the modulus of z) and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$. (θ is known as the argument of z. The value of θ, such that $-\pi < \theta \leq \pi$, is called the *principal argument* of z.
- A polynomial equation of n degree has n roots.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in$ R, $a \neq 0$, $b^2 - 4ac < 0$, are given by $x = \frac{-b \pm \sqrt{4ac - b^2}\,i}{2a}$.

Historical Note

The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician *Mahavira* (850 A.D.) who first stated this difficulty clearly. "He mentions in his work '*Ganitasara Sangraha*' as in the nature of things a negative (quantity) is not a square (quantity)', it has, therefore, no square root". *Bhaskara*, another Indian mathematician, also writes in his work *Bijaganita*, written in 1150. A.D. "There is no square root of a negative quantity, for it is not a square." *Cardan* (1545 A.D.) considered the problem of solving

$$x + y = 10, xy = 40.$$

He obtained $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$ as the solution of it, which was discarded by him by saying that these numbers are 'useless'. *Albert Girard* (about 1625 A.D.) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. *Euler* was the first to introduce the symbol i for $\sqrt{-1}$ and *W.R. Hamilton* (about 1830 A.D.) regarded the complex number $a + ib$ as an ordered pair of real numbers (a, b) thus giving it a purely mathematical definition and avoiding use of the so called '*imaginary numbers*'.

❖

Chapter 6

LINEAR INEQUALITIES

❖*Mathematics is the art of saying many things in many different ways. – MAXWELL*❖

6.1 Introduction

In earlier classes, we have studied equations in one variable and two variables and also solved some statement problems by translating them in the form of equations. Now a natural question arises: 'Is it always possible to translate a statement problem in the form of an equation? For example, the height of all the students in your class is less than 160 cm. Your classroom can occupy atmost 60 tables or chairs or both. Here we get certain statements involving a sign '<' (less than), '>' (greater than), '≤' (less than or equal) and ≥ (greater than or equal) which are known as *inequalities*.

In this Chapter, we will study linear inequalities in one and two variables. The study of inequalities is very useful in solving problems in the field of science, mathematics, statistics, optimisation problems, economics, psychology, etc.

6.2 Inequalities

Let us consider the following situations:

(i) Ravi goes to market with Rs 200 to buy rice, which is available in packets of 1kg. The price of one packet of rice is Rs 30. If x denotes the number of packets of rice, which he buys, then the total amount spent by him is Rs $30x$. Since, he has to buy rice in packets only, he may not be able to spend the entire amount of Rs 200. (Why?) Hence

$$30x < 200 \quad \text{... (1)}$$

Clearly the statement (i) is not an equation as it does not involve the sign of equality.

(ii) Reshma has Rs 120 and wants to buy some registers and pens. The cost of one register is Rs 40 and that of a pen is Rs 20. In this case, if x denotes the number of registers and y, the number of pens which Reshma buys, then the total amount spent by her is Rs $(40x + 20y)$ and we have

$$40x + 20y \leq 120 \quad \text{... (2)}$$

Since in this case the total amount spent may be upto Rs 120. Note that the statement (2) consists of two statements

$$40x + 20y < 120 \qquad \text{... (3)}$$

and $$40x + 20y = 120 \qquad \text{... (4)}$$

Statement (3) is not an equation, i.e., it is an inequality while statement (4) is an equation.

Definition 1 Two real numbers or two algebraic expressions related by the symbol '<', '>', '≤' or '≥' form an *inequality*.

Statements such as (1), (2) and (3) above are inequalities.

$3 < 5$; $7 > 5$ are the examples of *numerical inequalities* while

$x < 5$; $y > 2$; $x \geq 3$, $y \leq 4$ are the examples of *literal inequalities*.

$3 < 5 < 7$ (read as 5 is greater than 3 and less than 7), $3 \leq x < 5$ (read as x is greater than or equal to 3 and less than 5) and $2 < y \leq 4$ are the examples of *double inequalities*.

Some more examples of inequalities are:

$ax + b < 0$ ... (5)

$ax + b > 0$ ... (6)

$ax + b \leq 0$ ... (7)

$ax + b \geq 0$ ... (8)

$ax + by < c$ ... (9)

$ax + by > c$ ... (10)

$ax + by \leq c$ ... (11)

$ax + by \geq c$ ... (12)

$ax^2 + bx + c \leq 0$ ... (13)

$ax^2 + bx + c > 0$ ... (14)

Inequalities (5), (6), (9), (10) and (14) are *strict inequalities* while inequalities (7), (8), (11), (12), and (13) are *slack inequalities*. Inequalities from (5) to (8) are *linear inequalities* in one variable x when $a \neq 0$, while inequalities from (9) to (12) *are linear inequalities in two variables x and y* when $a \neq 0$, $b \neq 0$.

Inequalities (13) and (14) are not linear *(in fact, these are quadratic inequalities in one variable x when* $a \neq 0$*)*.

In this Chapter, we shall confine ourselves to the study of linear inequalities in one and two variables only.

6.3 Algebraic Solutions of Linear Inequalities in One Variable and their Graphical Representation

Let us consider the inequality (1) of Section 6.2, viz, $30x < 200$
Note that here x denotes the number of packets of rice.

Obviously, x cannot be a negative integer or a fraction. Left hand side (L.H.S.) of this inequality is $30x$ and right hand side (RHS) is 200. Therefore, we have

For $x = 0$, L.H.S. = 30 (0) = 0 < 200 (R.H.S.), which is true.

For $x = 1$, L.H.S. = 30 (1) = 30 < 200 (R.H.S.), which is true.

For $x = 2$, L.H.S. = 30 (2) = 60 < 200, which is true.

For $x = 3$, L.H.S. = 30 (3) = 90 < 200, which is true.

For $x = 4$, L.H.S. = 30 (4) = 120 < 200, which is true.

For $x = 5$, L.H.S. = 30 (5) = 150 < 200, which is true.

For $x = 6$, L.H.S. = 30 (6) = 180 < 200, which is true.

For $x = 7$, L.H.S. = 30 (7) = 210 < 200, which is false.

In the above situation, we find that the values of x, which makes the above inequality a true statement, are 0,1,2,3,4,5,6. These values of x, which make above inequality a true statement, are called *solutions* of inequality and the set {0,1,2,3,4,5,6} is called its *solution set.*

Thus, any solution of an inequality in one variable is a value of the variable which makes it a true statement.

We have found the solutions of the above inequality by *trial and error* method which is not very efficient. Obviously, this method is time consuming and sometimes not feasible. We must have some better or systematic techniques for solving inequalities. Before that we should go through some more properties of numerical inequalities and follow them as rules while solving the inequalities.

You will recall that while solving linear equations, we followed the following rules:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an equation.

Rule 2 Both sides of an equation may be multiplied (or divided) by the same non-zero number.

In the case of solving inequalities, we again follow the same rules except with a difference that in Rule 2, the sign of inequality is reversed (i.e., '<' becomes '>', ≤' becomes '≥' and so on) whenever we multiply (or divide) both sides of an inequality by a negative number. It is evident from the facts that

$$3 > 2 \text{ while } -3 < -2,$$
$$-8 < -7 \text{ while } (-8)(-2) > (-7)(-2) \text{ , i.e., } 16 > 14.$$

Thus, we state the following rules for solving an inequality:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of inequality.

Rule 2 Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of inequality is *reversed.*

Now, let us consider some examples.

Example 1 Solve $30\,x < 200$ when

(i) x is a natural number, (ii) x is an integer.

Solution We are given $30\,x < 200$

or $\frac{30x}{30} < \frac{200}{30}$ (Rule 2), i.e., $x < 20 / 3$.

(i) When x is a natural number, in this case the following values of x make the statement true.

1, 2, 3, 4, 5, 6.

The solution set of the inequality is $\{1,2,3,4,5,6\}$.

(ii) When x is an integer, the solutions of the given inequality are

$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$

The solution set of the inequality is $\{\ldots,-3, -2,-1, 0, 1, 2, 3, 4, 5, 6\}$

Example 2 Solve $5x - 3 < 3x + 1$ when

(i) x is an integer, (ii) x is a real number.

Solution We have, $5x - 3 < 3x + 1$

or $5x - 3 + 3 < 3x + 1 + 3$ (Rule 1)

or $5x < 3x + 4$

or $5x - 3x < 3x + 4 - 3x$ (Rule 1)

or $2x < 4$

or $x < 2$ (Rule 2)

(i) When x is an integer, the solutions of the given inequality are

$\ldots, -4, -3, -2, -1, 0, 1$

(ii) When x is a real number, the solutions of the inequality are given by $x < 2$, i.e., all real numbers x which are less than 2. Therefore, the solution set of the inequality is $x \in (-\infty, 2)$.

We have considered solutions of inequalities in the set of natural numbers, set of integers and in the set of real numbers. Henceforth, unless stated otherwise, we shall solve the inequalities in this Chapter in the set of real numbers.

Example 3 Solve $4x + 3 < 6x + 7$.

Solution We have, $4x + 3 < 6x + 7$

or $\quad 4x - 6x < 6x + 4 - 6x$

or $\quad -2x < 4 \quad$ or $\quad x > -2$

i.e., all the real numbers which are greater than -2, are the solutions of the given inequality. Hence, the solution set is $(-2, \infty)$.

Example 4 Solve $\frac{5-2x}{3} \leq \frac{x}{6} - 5$.

Solution We have

$$\frac{5-2x}{3} \leq \frac{x}{6} - 5$$

or $\quad 2(5-2x) \leq x - 30.$

or $\quad 10 - 4x \leq x - 30$

or $\quad -5x \leq -40$, i.e., $x \geq 8$

Thus, all real numbers x which are greater than or equal to 8 are the solutions of the given inequality, i.e., $x \in [8, \infty)$.

Example 5 Solve $7x + 3 < 5x + 9$. Show the graph of the solutions on number line.

Solution We have $7x + 3 < 5x + 9$ or

$$2x < 6 \text{ or } x < 3$$

The graphical representation of the solutions are given in Fig 6.1.

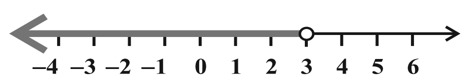


Fig 6.1

Example 6 Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Solution We have

$$\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$$

or $\quad \frac{3x-4}{2} \geq \frac{x-3}{4}$

or $\quad 2(3x-4) \geq (x-3)$

or $\quad 6x - 8 \geq x - 3$

or $\quad 5x \geq 5$ or $x \geq 1$

The graphical representation of solutions is given in Fig 6.2.

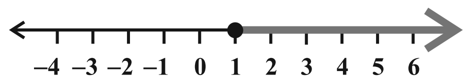


Fig 6.2

Example 7 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the number of minimum marks he should get in the annual examination to have an average of at least 60 marks.

Solution Let x be the marks obtained by student in the annual examination. Then

$$\frac{62+48+x}{3} \geq 60$$

or $\quad 110 + x \geq 180$

or $\quad x \geq 70$

Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks.

Example 8 Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.

Solution Let x be the smaller of the two consecutive odd natural number, so that the other one is $x + 2$. Then, we should have

$$x > 10 \quad \text{... (1)}$$

and $\quad x + (x + 2) < 40 \quad$ ... (2)

Solving (2), we get

$$2x + 2 < 40$$

i.e., $\quad x < 19 \quad$ ... (3)

From (1) and (3), we get

$$10 < x < 19$$

Since x is an odd number, x can take the values 11, 13, 15, and 17. So, the required possible pairs will be

$$(11, 13), (13, 15), (15, 17), (17, 19)$$

EXERCISE 6.1

1. Solve $24x < 100$, when
   (i) x is a natural number. (ii) x is an integer.
2. Solve $-12x > 30$, when
   (i) x is a natural number. (ii) x is an integer.
3. Solve $5x - 3 < 7$, when
   (i) x is an integer. (ii) x is a real number.
4. Solve $3x + 8 > 2$, when
   (i) x is an integer. (ii) x is a real number.

Solve the inequalities in Exercises 5 to 16 for real x.

5. $4x + 3 < 6x + 7$
6. $3x - 7 > 5x - 1$
7. $3(x - 1) \leq 2(x - 3)$
8. $3(2 - x) \geq 2(1 - x)$
9. $x + \frac{x}{2} + \frac{x}{3} < 11$
10. $\frac{x}{3} > \frac{x}{2} + 1$
11. $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$
12. $\frac{1}{2}\left(\frac{3x}{5} + 4\right) \geq \frac{1}{3}(x - 6)$
13. $2(2x + 3) - 10 < 6(x - 2)$
14. $37 - (3x + 5) \geq 9x - 8(x - 3)$
15. $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$
16. $\frac{(2x-1)}{3} \geq \frac{(3x-2)}{4} - \frac{(2-x)}{5}$

Solve the inequalities in Exercises 17 to 20 and show the graph of the solution in each case on number line

17. $3x - 2 < 2x + 1$
18. $5x - 3 \geq 3x - 5$
19. $3(1 - x) < 2(x + 4)$
20. $\frac{x}{2} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$
21. Ravi obtained 70 and 75 marks in first two unit test. Find the number if minimum marks he should get in the third test to have an average of at least 60 marks.
22. To receive Grade 'A' in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.
23. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.
24. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

25. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.
26. A man wants to cut three lengths from a single piece of board of length 91cm. The second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second?
[*Hint:* If x is the length of the shortest board, then x , $(x + 3)$ and $2x$ are the lengths of the second and third piece, respectively. Thus, $x + (x + 3) + 2x \leq 91$ and $2x \geq (x + 3) + 5$].

6.4 Graphical Solution of Linear Inequalities in Two Variables

In earlier section, we have seen that a graph of an inequality in one variable is a visual representation and is a convenient way to represent the solutions of the inequality. Now, we will discuss graph of a linear inequality in two variables.

We know that a line divides the Cartesian plane into two parts. Each part is known as a half plane. A vertical line will divide the plane in left and right half planes and a non-vertical line will divide the plane into lower and upper half planes (Figs. 6.3 and 6.4).

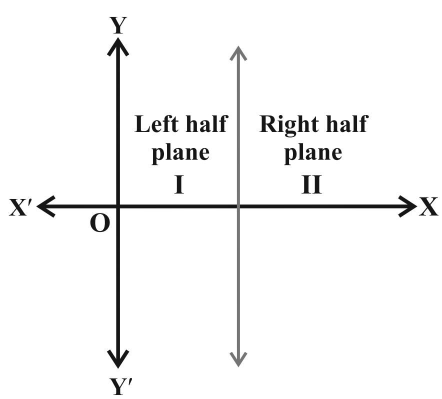


Fig 6.3

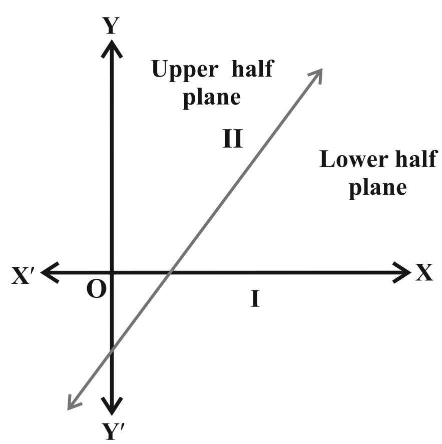


Fig 6.4

A point in the Cartesian plane will either lie on a line or will lie in either of the half planes I or II. We shall now examine the relationship, if any, of the points in the plane and the inequalities $ax + by < c$ or $ax + by > c$.

Let us consider the line

$$ax + by = c, a \neq 0, b \neq 0 \qquad \text{... (1)}$$

There are three possibilities namely:

(i) $ax + by = c$ (ii) $ax + by > c$ (iii) $ax + by < c$.

In case (i), clearly, all points (x, y) satisfying (i) lie on the line it represents and conversely. Consider case (ii), let us first assume that $b > 0$. Consider a point P (α,β) on the line $ax + by = c$, $b > 0$, so that $a\alpha + b\beta = c$. Take an arbitrary point Q (α, γ) in the half plane II (Fig 6.5).

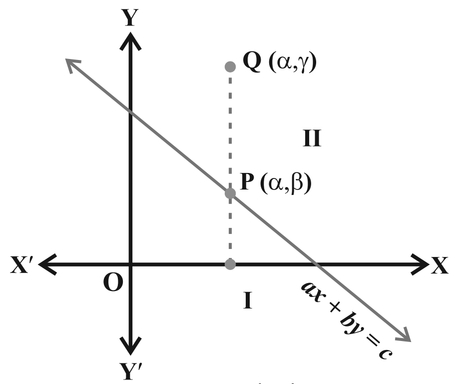


Fig 6.5

Now, from Fig 6.5, we interpret,

$\gamma > \beta$ (Why?)

or $b\gamma > b\beta$ or $a\alpha + b\gamma > a\alpha + b\beta$ (Why?)

or $a\alpha + b\gamma > c$

i.e., $Q(\alpha, \gamma)$ satisfies the inequality $ax + by > c$.

Thus, all the points lying in the half plane II above the line $ax + by = c$ satisfies the inequality $ax + by > c$. Conversely, let (α, β) be a point on line $ax + by = c$ and an arbitrary point $Q(\alpha, \gamma)$ satisfying

$ax + by > c$

so that $a\alpha + b\gamma > c$

$\Rightarrow$ $a\alpha + b\gamma > a\alpha + b\beta$ (Why?)

$\Rightarrow$ $\gamma > \beta$ (as $b > 0$)

This means that the point (α, γ) lies in the half plane II.

Thus, any point in the half plane II satisfies $ax + by > c$, and conversely any point satisfying the inequality $ax + by > c$ lies in half plane II.

In case $b < 0$, we can similarly prove that any point satisfying $ax + by > c$ lies in the half plane I, and conversely.

Hence, we deduce that all points satisfying $ax + by > c$ lies in one of the half planes II or I according as $b > 0$ or $b < 0$, and conversely.

Thus, graph of the inequality $ax + by > c$ will be one of the half plane (called *solution region*) and represented by shading in the corresponding half plane.

Note 1 The region containing all the solutions of an inequality is called the *solution region*.

2. In order to identify the half plane represented by an inequality, it is just sufficient to take any point (a, b) (not online) and check whether it satisfies the inequality or not. If it satisfies, then the inequality represents the half plane and shade the region

which contains the point, otherwise, the inequality represents that half plane which does not contain the point within it. For convenience, the point (0, 0) is preferred.

3. If an inequality is of the type $ax + by \geq c$ or $ax + by \leq c$, then the points on the line $ax + by = c$ are also included in the solution region. So draw a dark line in the solution region.

4. If an inequality is of the form $ax + by > c$ or $ax + by < c$, then the points on the line $ax + by = c$ are not to be included in the solution region. So draw a broken or dotted line in the solution region.

In Section 6.2, we obtained the following linear inequalities in two variables *x and y*: $40x + 20y \leq 120$ ... (1)

while translating the word problem of purchasing of registers and pens by Reshma.

Let us now solve this inequality keeping in mind that x and y can be only whole numbers, since the number of articles cannot be a fraction or a negative number. In this case, we find the pairs of values of x and y, which make the statement (1) true. In fact, the set of such pairs will be the *solution set* of the inequality (1).

To start with, let $x = 0$. Then L.H.S. of (1) is

$$40x + 20y = 40\,(0) + 20y = 20y.$$

Thus, we have

$$20y \leq 120 \text{ or } y \leq 6 \qquad \text{... (2)}$$

For $x = 0$, the corresponding values of y can be 0, 1, 2, 3, 4, 5, 6 only. In this case, the solutions of (1) are (0, 0), (0, 1), (0,2), (0,3), (0,4), (0, 5) and (0, 6).

Similarly, other solutions of (1), when $x = 1$, 2 and 3 are: (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 0), (2, 1), (2, 2), (3, 0)

This is shown in Fig 6.6.

Let us now extend the domain of x and y from whole numbers to real numbers, and see what will be the solutions of (1) in this case. You will see that the graphical method of solution will be very convenient in this case. For this purpose, let us consider the (corresponding) equation and draw its graph.

$$40x + 20y = 120 \qquad \text{... (3)}$$

In order to draw the graph of the inequality (1), we take one point say (0, 0), in half plane I and check whether values of x and y satisfy the inequality or not.

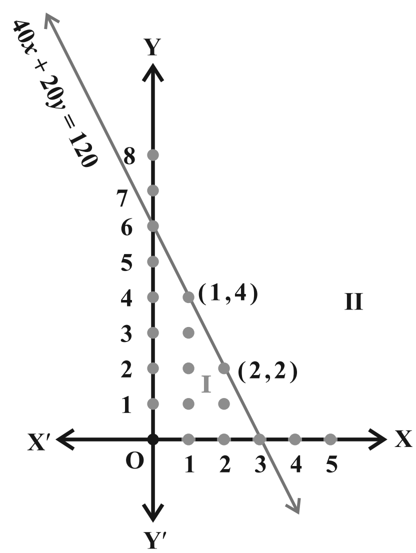


Fig 6.6

We observe that $x = 0$, $y = 0$ satisfy the inequality. Thus, we say that the half plane I is the graph (Fig 6.7) of the inequality. Since the points on the line also satisfy the inequality (1) above, the line is also a part of the graph.

Thus, the graph of the given inequality is half plane I including the line itself. Clearly half plane II is not the part of the graph. Hence, *solutions* of inequality (1) will consist of all the points of its graph (half plane I including the line).

We shall now consider some examples to explain the above procedure for solving a linear inequality involving two variables.

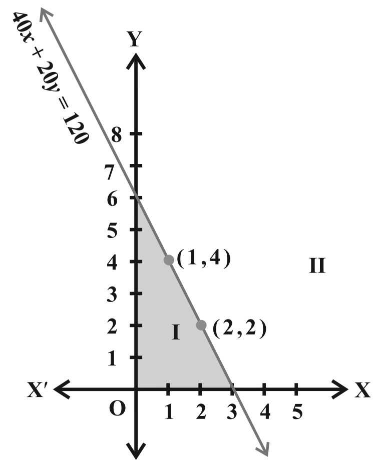


Fig 6.7

Example 9 Solve $3x + 2y > 6$ graphically.

Solution Graph of $3x + 2y = 6$ is given as dotted line in the Fig 6.8.

This line divides the xy-plane in two half planes I and II. We select a point (not on the line), say (0, 0), which lies in one of the half planes (Fig 6.8) and determine if this point satisfies the given inequality, we note that

$3 (0) + 2 (0) > 6$

or $0 > 6$, which is false.

Hence, half plane I is not the solution region of the given inequality. Clearly, any point on the line does not satisfy the given strict inequality. In other words, the shaded half plane II excluding the points on the line is the solution region of the inequality.

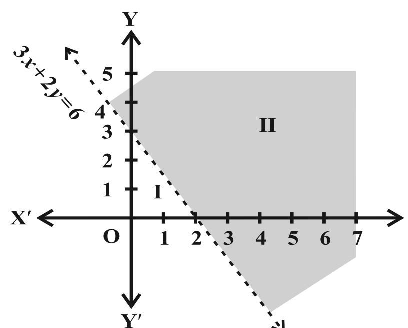


Fig 6.8

Example 10 Solve $3x - 6 \geq 0$ graphically in two dimensional plane.

Solution Graph of $3x - 6 = 0$ is given in the Fig 6.9.

We select a point, say (0, 0) and substituting it in given inequality, we see that:

$3 (0) - 6 \geq 0$ or $- 6 \geq 0$ which is false.

Thus, the solution region is the shaded region on the right hand side of the line $x = 2$.

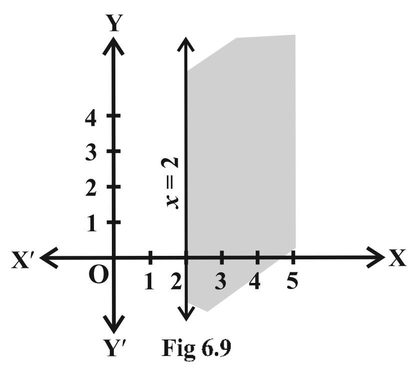


Fig 6.9

Example 11 Solve $y < 2$ graphically.

Solution Graph of $y = 2$ is given in the Fig 6.10.

Let us select a point, (0, 0) in lower half plane I and putting $y = 0$ in the given inequality, we see that

$1 \times 0 < 2$ or $0 < 2$ which is true.

Thus, the solution region is the shaded region below the line $y = 2$. Hence, every point below the line (excluding all the points on the line) determines the solution of the given inequality.

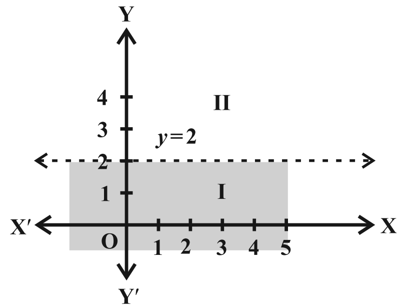


Fig 6.10

EXERCISE 6.2

Solve the following inequalities graphically in two-dimensional plane:

1. $x + y < 5$
2. $2x + y \geq 6$
3. $3x + 4y \leq 12$
4. $y + 8 \geq 2x$
5. $x - y \leq 2$
6. $2x - 3y > 6$
7. $-3x + 2y \geq -6$
8. $3y - 5x < 30$
9. $y < -2$
10. $x > -3$.

6.5 Solution of System of Linear Inequalities in Two Variables

In previous Section, you have learnt how to solve linear inequality in one or two variables graphically. We will now illustrate the method for solving a system of linear inequalities in two variables graphically through some examples.

Example 12 Solve the following system of linear inequalities graphically.

$$x + y \geq 5 \quad ... (1)$$

$$x - y \leq 3 \quad ... (2)$$

Solution The graph of linear equation

$$x + y = 5$$

is drawn in Fig 6.11.

We note that solution of inequality (1) is represented by the shaded region above the line $x + y = 5$, including the points on the line.

On the same set of axes, we draw the graph of the equation $x - y = 3$ as shown in Fig 6.11. Then we note that inequality (2) represents the shaded region above

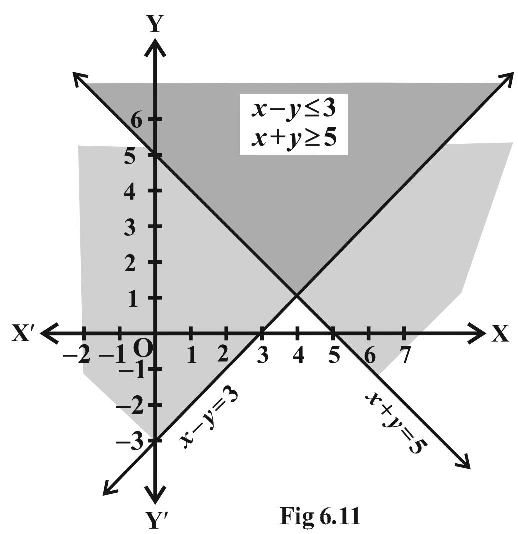


Fig 6.11

the line $x - y = 3$, including the points on the line.

Clearly, the double shaded region, common to the above two shaded regions is the required solution region of the given system of inequalities.

Example 13 Solve the following system of inequalities graphically

$$5x + 4y \leq 40 \quad \ldots (1)$$
$$x \geq 2 \quad \ldots (2)$$
$$y \geq 3 \quad \ldots (3)$$

Solution We first draw the graph of the line

$$5x + 4y = 40, \quad x = 2 \text{ and } y = 3$$

Then we note that the inequality (1) represents shaded region below the line $5x + 4y = 40$ and inequality (2) represents the shaded region right of line $x = 2$ but inequality (3) represents the shaded region above the line $y = 3$. Hence, shaded region (Fig 6.12) including all the point on the lines are also the solution of the given system of the linear inequalities.

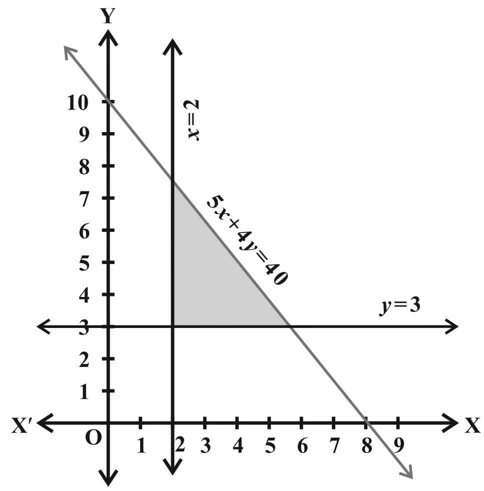


Fig 6.12

In many practical situations involving system of inequalities the variable x and y often represent quantities that cannot have negative values, for example, number of units produced, number of articles purchased, number of hours worked, etc. Clearly, in such cases, $x \geq 0$, $y \geq 0$ and the solution region lies only in the first quadrant.

Example 14 Solve the following system of inequalities

$$8x + 3y \leq 100 \quad \ldots (1)$$
$$x \geq 0 \quad \ldots (2)$$
$$y \geq 0 \quad \ldots (3)$$

Solution We draw the graph of the line

$$8x + 3y = 100$$

The inequality $8x + 3y \leq 100$ represents the shaded region below the line, including the points on the line $8x + 3y = 100$ (Fig 6.13).

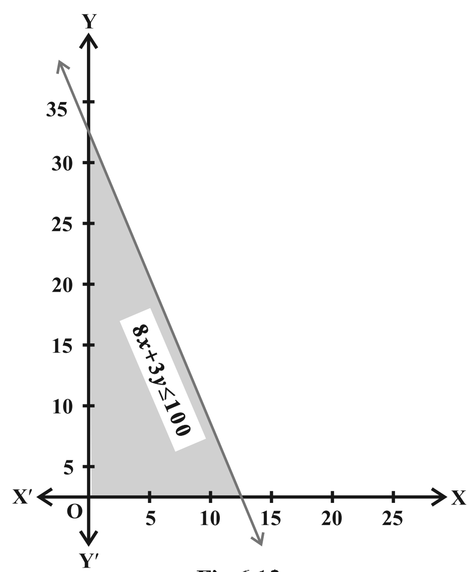


Fig 6.13

Since $x \geq 0, y \geq 0$, every point in the shaded region in the first quadrant, including the points on the line and the axes, represents the solution of the given system of inequalities.

Example 15 Solve the following system of inequalities graphically

$x + 2y \leq 8$ ... (1)
$2x + y \leq 8$ ... (2)
$x \geq 0$ ... (3)
$y \geq 0$ ... (4)

Solution We draw the graphs of the lines $x + 2y = 8$ and $2x + y = 8$. The inequality (1) and (2) represent the region below the two lines, including the point on the respective lines.

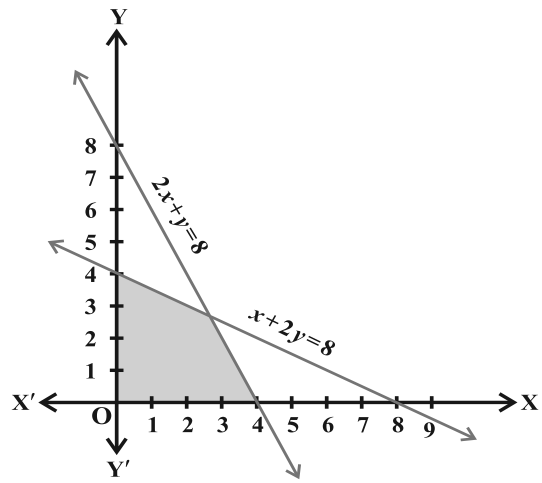


Fig 6.14

Since $x \geq 0, y \geq 0$, every point in the shaded region in the first quadrant represent a solution of the given system of inequalities (Fig 6.14).

EXERCISE 6.3

Solve the following system of inequalities graphically:

1. $x \geq 3, y \geq 2$
2. $3x + 2y \leq 12,\ x \geq 1, y \geq 2$
3. $2x + y \geq 6, 3x + 4y \leq 12$
4. $x + y > 4,\ 2x - y > 0$
5. $2x - y > 1, x - 2y < -1$
6. $x + y \leq 6,\ x + y \geq 4$
7. $2x + y \geq 8,\ x + 2y \geq 10$
8. $x + y \leq 9,\ y > x,\ x \geq 0$
9. $5x + 4y \leq 20,\ x \geq 1, y \geq 2$
10. $3x + 4y \leq 60, x + 3y \leq 30, x \geq 0,\ y \geq 0$
11. $2x + y \geq 4,\ x + y \leq 3,\ 2x - 3y \leq 6$
12. $x - 2y \leq 3, 3x + 4y \geq 12,\ x \geq 0, y \geq 1.$
13. $4x + 3y \leq 60, y \geq 2x,\ x \geq 3,\ x, y \geq 0$
14. $3x + 2y \leq 150, x + 4y \leq 80,\ x \leq 15,\ y \geq 0$
15. $x + 2y \leq 10, x + y \geq 1, x - y \leq 0, x \geq 0, y \geq 0$

Miscellaneous Examples

Example 16 Solve $-8 \le 5x - 3 < 7$.

Solution In this case, we have two inequalities, $-8 \le 5x - 3$ and $5x - 3 < 7$, which we will solve simultaneously. We have $-8 \le 5x - 3 < 7$

or $\quad -5 \le 5x < 10 \qquad$ or $\quad -1 \le x < 2$

Example 17 Solve $-5 \le \dfrac{5-3x}{2} \le 8$.

Solution We have $-5 \le \dfrac{5-3x}{2} \le 8$

or $\quad -10 \le 5 - 3x \le 16 \quad$ or $\quad -15 \le -3x \le 11$

or $\quad 5 \ge x \ge -\dfrac{11}{3}$

which can be written as $\dfrac{-11}{3} \le x \le 5$

Example 18 Solve the system of inequalities:

$$3x - 7 < 5 + x \qquad \ldots (1)$$

$$11 - 5x \le 1 \qquad \ldots (2)$$

and represent the solutions on the number line.

Solution From inequality (1), we have

$$3x - 7 < 5 + x$$

or $\quad x < 6 \qquad \ldots (3)$

Also, from inequality (2), we have

$$11 - 5x \le 1$$

or $\quad -5x \le -10$ i.e., $x \ge 2 \qquad \ldots (4)$

If we draw the graph of inequalities (3) and (4) on the number line, we see that the values of x, which are common to both, are shown by bold line in Fig 6.15.

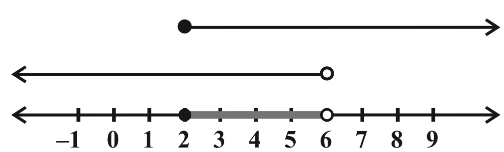


Fig 6.15

Thus, solution of the system are real numbers x lying between 2 and 6 including 2, i.e.,

$$2 \le x < 6$$

Example 19 In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by C = $\frac{5}{9}$ (F – 32), where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively.

Solution It is given that 30 < C < 35.

Putting $\quad C = \frac{5}{9}(F - 32)$, we get

$$30 < \frac{5}{9}(F - 32) < 35,$$

or $\quad \frac{9}{5} \times (30) < (F - 32) < \frac{9}{5} \times (35)$

or $\quad 54 < (F - 32) < 63$

or $\quad 86 < F < 95.$

Thus, the required range of temperature is between 86° F and 95° F.

Example 20 A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Solution Let x litres of 30% acid solution is required to be added. Then

Total mixture = $(x + 600)$ litres

Therefore $\quad 30\%\ x + 12\%$ of $600 > 15\%$ of $(x + 600)$

and $\quad 30\%\ x + 12\%$ of $600 < 18\%$ of $(x + 600)$

or $\quad \frac{30x}{100} + \frac{12}{100}(600) > \frac{15}{100}(x + 600)$

and $\quad \frac{30x}{100} + \frac{12}{100}(600) < \frac{18}{100}(x + 600)$

or $\quad 30x + 7200 > 15x + 9000$

and $\quad 30x + 7200 < 18x + 10800$

or $\quad 15x > 1800$ and $12x < 3600$

or $\quad x > 120$ and $x < 300$,

i.e. $\quad 120 < x < 300$

Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres.

Miscellaneous Exercise on Chapter 6

Solve the inequalities in Exercises 1 to 6.

1. $2 \le 3x - 4 \le 5$
2. $6 \le -3\,(2x - 4) < 12$
3. $-3 \le 4 - \dfrac{7x}{2} \le 18$
4. $-15 < \dfrac{3(x-2)}{5} \le 0$
5. $-12 < 4 - \dfrac{3x}{-5} \le 2$
6. $7 \le \dfrac{(3x+11)}{2} \le 11$.

Solve the inequalities in Exercises 7 to 11 and represent the solution graphically on number line.

7. $5x + 1 > -24, \quad 5x - 1 < 24$
8. $2\,(x - 1) < x + 5, \quad 3\,(x + 2) > 2 - x$
9. $3x - 7 > 2\,(x - 6)\,, \quad 6 - x > 11 - 2x$
10. $5\,(2x - 7) - 3\,(2x + 3) \le 0\,, \quad 2x + 19 \le 6x + 47$.
11. A solution is to be kept between 68° F and 77° F. What is the range in temperature in degree Celsius (C) if the Celsius / Fahrenheit (F) conversion formula is given by

$$F = \frac{9}{5}\,C + 32\ ?$$

12. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?
13. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?
14. IQ of a person is given by the formula

$$IQ = \frac{MA}{CA} \times 100,$$

where MA is mental age and CA is chronological age. If $80 \le IQ \le 140$ for a group of 12 years old children, find the range of their mental age.

Summary

- Two real numbers or two algebraic expressions related by the symbols $<$, $>$, $\leq$ or $\geq$ form an inequality.
- Equal numbers may be added to (or subtracted from) both sides of an inequality.
- Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied (or divided) by a negative number, then the inequality is reversed.
- The values of x, which make an inequality a true statement, are called *solutions of the inequality*.
- To represent $x < a$ (or $x > a$) on a number line, put a circle on the number a and dark line to the left (or right) of the number a.
- To represent $x \leq a$ (or $x \geq a$) on a number line, put a dark circle on the number a and dark the line to the left (or right) of the number x.
- If an inequality is having $\leq$ or $\geq$ symbol, then the points on the line are also included in the solutions of the inequality and the graph of the inequality lies left (below) or right (above) of the graph of the equality represented by dark line that satisfies an arbitrary point in that part.
- If an inequality is having $<$ or $>$ symbol, then the points on the line are not included in the solutions of the inequality and the graph of the inequality lies to the left (below) or right (above) of the graph of the corresponding equality represented by dotted line that satisfies an arbitrary point in that part.
- The solution region of a system of inequalities is the region which satisfies all the given inequalities in the system simultaneously.

Chapter 7

PERMUTATIONS AND COMBINATIONS

❖*Every body of discovery is mathematical in form because there is no other guidance we can have – DARWIN*❖

7.1 Introduction

Jacob Bernoulli
(1654 1705)

Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of these techniques.

7.2 Fundamental Principle of Counting

Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt.

Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig. 7.1.

Let us consider another problem of the same type.

Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 differnt ways.

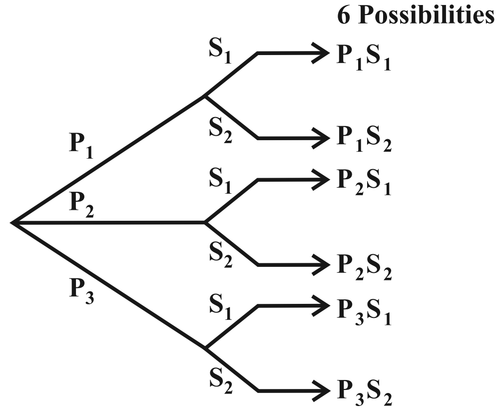


Fig 7.1

Hence, there are $6 \times 2 = 12$ different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2, these possibilities can be illustrated in the Fig. 7.2.

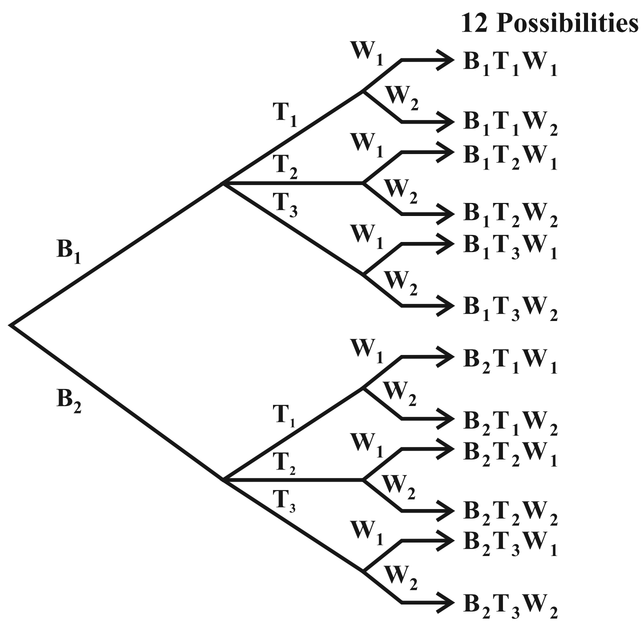


Fig 7.2

In fact, the problems of the above types are solved by applying the following principle known as the *fundamental principle of counting*, or, simply, the *multiplication principle*, which states that

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$*."*

The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows:

'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is $m \times n \times p$."

In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a pant

(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a school bag

(ii) the event of choosing a tiffin box

(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order.

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places $\square\ \square\ \square\ \square$ by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

Note If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$.

Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags.

The number of ways is $5 \times 4 \times 3 = 60$.

Continuing the same way, we find that

The number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$

and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

EXERCISE 7.1

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that
   (i) repetition of the digits is allowed?
   (ii) repetition of the digits is not allowed?
2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?
3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?
4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?
5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?
6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

7.3 Permutations

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a *permutation of* 4 *different letters taken all at a time*. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words $= 6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

Definition 1 A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

In the following sub Section, we shall obtain the formula needed to answer these questions immediately.

7.3.1 *Permutations when all the objects are distinct*

Theorem 1 The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by ${}^{n}P_{r}$.

Proof There will be as many permutations as there are ways of filling in r vacant places $\boxed{}\ \boxed{}\ \boxed{} \ldots \boxed{}$ by

$\leftarrow r$ vacant places $\rightarrow$

the n objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways,..., the rth place can be filled in $(n-(r-1))$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)(n-2)\ldots(n-(r-1))$ or $n(n-1)(n-2)\ldots(n-r+1)$

This expression for ${}^{n}P_{r}$ is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

7.3.2 ***Factorial notation*** The notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$. We read this symbol as 'n factorial'. Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

$$1 = 1!$$
$$1 \times 2 = 2!$$
$$1 \times 2 \times 3 = 3!$$
$$1 \times 2 \times 3 \times 4 = 4! \text{ and so on.}$$

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$
$= 5 \times 4 \times 3 \times 2 \times 1!$

Clearly, for a natural number n

$$\begin{aligned} n! &= n(n-1)! \\ &= n(n-1)(n-2)! && [\text{provided } (n \geq 2)] \\ &= n(n-1)(n-2)(n-3)! && [\text{provided } (n \geq 3)] \end{aligned}$$

and so on.

Example 5 Evaluate (i) 5 ! (ii) 7 ! (iii) 7 ! – 5!

Solution (i) 5 ! = 1 × 2 × 3 × 4 × 5 = 120

(ii) 7 ! = 1 × 2 × 3 × 4 × 5 × 6 ×7 = 5040

and (iii) 7 ! – 5! = 5040 – 120 = 4920.

Example 6 Compute (i) $\frac{7!}{5!}$ (ii) $\frac{12!}{(10!)\,(2!)}$

Solution (i) We have $\frac{7!}{5!} = \frac{7\times6\times5!}{5!} = 7 \times 6 = 42$

and (ii) $\frac{12!}{(10!)\,(2!)} = \frac{12\times11\times(10!)}{(10!)\times(2)} = 6 \times 11 = 66.$

Example 7 Evaluate $\frac{n!}{r!(n-r)!}$, when $n = 5$, $r = 2$.

Solution We have to evaluate $\frac{5!}{2!(5-2)!}$ (since $n = 5$, $r = 2$)

We have $\frac{5!}{2\,!(5-2)!} = \frac{5!}{2!\times3!} = \frac{4\times5}{2} = 10$.

Example 8 If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

Solution We have $\frac{1}{8!} + \frac{1}{9\times8!} = \frac{x}{10\times9\times8!}$

Therefore $1 + \frac{1}{9} = \frac{x}{10\times9}$ or $\frac{10}{9} = \frac{x}{10\times9}$

So $x = 100$.

EXERCISE 7.2

1. Evaluate

(i) 8 ! (ii) 4 ! – 3 !

2. Is 3 ! + 4 ! = 7 ! ? 3. Compute $\frac{8!}{6!\times 2!}$ 4. If $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$, find x

5. Evaluate $\frac{n!}{(n-r)!}$, when

(i) $n = 6$, $r = 2$ (ii) $n = 9$, $r = 5$.

7.3.3 *Derivation of the formula for* ^nP_r

$$^n\mathrm{P}_r = \frac{n!}{(n-r)!}, \ 0 \le r \le n$$

Let us now go back to the stage where we had determined the following formula:

$$^n\mathrm{P}_r = n\,(n-1)\,(n-2)\ldots\,(n-r+1)$$

Multiplying numerator and denomirator by $(n - r)\,(n - r - 1)\ldots 3 \times 2 \times 1$, we get

$$^n\mathrm{P}_r = \frac{n(n-1)\,(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3\times 2\times 1}{(n-r)(n-r-1)\ldots 3\times 2\times 1} = \frac{n!}{(n-r)!},$$

Thus $\quad ^n\mathrm{P}_r = \frac{n!}{(n-r)!}$, where $0 < r \le n$

This is a much more convenient expression for $^n\mathrm{P}_r$ than the previous one.

In particular, when $r = n$, $^n\mathrm{P}_n = \frac{n!}{0!} = n!$

Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have

$$^n\mathrm{P}_0 = 1 = \frac{n!}{n!} = \frac{n!}{(n-0)!} \qquad \ldots (1)$$

Therefore, the formula (1) is applicable for $r = 0$ also.

Thus $\quad ^n\mathrm{P}_r = \frac{n!}{(n-r)!}, \ 0 \le r \le n$.

Theorem 2 The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r.

Proof is very similar to that of Theorem 1 and is left for the reader to arrive at.

Here, we are solving some of the problems of the pervious Section using the formula for ${}^nP_r$ to illustrate its usefulness.

In Example 1, the required number of words = ${}^4P_4 = 4! = 24$. Here repetition is not allowed. If repeation is allowed, the required number of words would be $4^4 = 256$.

The number of 3-letter words which can be formed by the letters of the word NUMBER = ${}^6P_3 = \frac{6!}{3!} = 4 \times 5 \times 6 = 120$. Here, in this case also, the repetition is not allowed. If the repetition is allowed,the required number of words would be $6^3 = 216$.

The number of ways in which a Chairman and a Vice-Chairman can be chosen from amongst a group of 12 persons assuming that one person can not hold more than one position, clearly ${}^{12}P_2 = \frac{12!}{10!} = 11 \times 12 = 132$.

7.3.4 ***Permutations when all the objects are not distinct objects*** Suppose we have to find the number of ways of rearranging the letters of the word ROOT. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O_1 and O_2. The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO_1O_2T. Corresponding to this permutation,we have 2 ! permutations RO_1O_2T and RO_2O_1T which will be exactly the same permutation if O_1 and O_2 are not treated as different, i.e., if O_1 and O_2 are the same O at both places.

Therefore, the required number of permutations = $\frac{4!}{2!} = 3 \times 4 = 12$.

Permutations when O_1, O_2 are different.		Permutations when O_1, O_2 are the same O.
$\left.\begin{matrix} RO_1O_2T \\ RO_2O_1T \end{matrix}\right]$	$\longrightarrow$	R O O T
$\left.\begin{matrix} TO_1O_2R \\ TO_2O_1R \end{matrix}\right]$	$\longrightarrow$	T O O R

$$\left.\begin{matrix} R\,O_1T\,O_2 \\ R\,O_2T\,O_1 \end{matrix}\right] \longrightarrow R\,O\,T\,O$$

$$\left.\begin{matrix} T\,O_1R\,O_2 \\ T\,O_2R\,O_1 \end{matrix}\right] \longrightarrow T\,O\,R\,O$$

$$\left.\begin{matrix} R\,T\,O_1\,O_2 \\ R\,T\,O_2\,O_1 \end{matrix}\right] \longrightarrow R\,T\,O\,O$$

$$\left.\begin{matrix} T\,R\,O_1\,O_2 \\ T\,R\,O_2\,O_1 \end{matrix}\right] \longrightarrow T\,R\,O\,O$$

$$\left.\begin{matrix} O_1\;O_2\;R\,T \\ O_2\;O_1\;T\,R \end{matrix}\right] \longrightarrow O\,O\,R\,T$$

$$\left.\begin{matrix} O_1\,R\,O_2\,T \\ O_2\,R\,O_1\,T \end{matrix}\right] \longrightarrow O\,R\,O\,T$$

$$\left.\begin{matrix} O_1\;T\,O_2\;R \\ O_2\;T\,O_1\;R \end{matrix}\right] \longrightarrow O\,T\,O\,R$$

$$\left.\begin{matrix} O_1\;R\,T\,O_2 \\ O_2\;R\,T\,O_1 \end{matrix}\right] \longrightarrow O\,R\,T\,O$$

$$\left.\begin{matrix} O_1\;T\,R\,O_2 \\ O_2\;T\,R\,O_1 \end{matrix}\right] \longrightarrow O\,T\,R\,O$$

$$\left.\begin{matrix} O_1\;O_2T\,R \\ O_2\;O_1\,T\,R \end{matrix}\right] \longrightarrow O\,O\,T\,R$$

Let us now find the number of ways of rearranging the letters of the word INSTITUTE. In this case there are 9 letters, in which I appears 2 times and T appears 3 times.

Temporarily, let us treat these letters different and name them as I_1, I_2, T_1, T_2, T_3. The number of permutations of 9 different letters, in this case, taken all at a time is 9 !. Consider one such permutation, say, $I_1\,NT_1\,SI_2\,T_2\,U\,E\,T_3$. Here if I_1, I_2 are not same and T_1, T_2, T_3 are not same, then I_1, I_2 can be arranged in 2! ways and T_1, T_2, T_3 can be arranged in 3! ways. Therefore, 2! × 3! permutations will be just the same permutation corresponding to this chosen permutation $I_1NT_1SI_2T_2UET_3$. Hence, total number of different permutations will be $\frac{9!}{2!\,3!}$

We can state (without proof) the following theorems:

Theorem 3 The number of permutations of n objects, where p objects are of the same kind and rest are all different $= \frac{n!}{p!}$.

In fact, we have a more general theorem.

Theorem 4 The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of different kind is $\frac{n!}{p_1!\,p_2!\ldots p_k!}$.

Example 9 Find the number of permutations of the letters of the word ALLAHABAD.

Solution Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements $= \frac{9!}{4!\,2!} = \frac{5\times6\times7\times8\times9}{2} = 7560$

Example 10 How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution Here order matters for example 1234 and 1324 are two different numbers. Therefore, there will be as many 4 digit numbers as there are permutations of 9 different digits taken 4 at a time.

Therefore, the required 4 digit numbers $= {}^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024.$

Example 11 How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

Solution Every number between 100 and 1000 is a 3-digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be ${}^6P_3$. But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from ${}^6P_3$ to get the required number. To get the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is ${}^5P_2$. So

The required number $= {}^6P_3 - {}^5P_2 = \frac{6!}{3!} - \frac{5!}{3!}$

$= 4 \times 5 \times 6 - 4 \times 5 = 100$

Example 12 Find the value of n such that

(i) ${}^nP_5 = 42\ {}^nP_3,\ n > 4$ (ii) $\frac{{}^nP_4}{{}^{n-1}P_4} = \frac{5}{3},\ n > 4$

Solution (i) Given that

${}^nP_5 = 42\ {}^nP_3$

or $n(n-1)(n-2)(n-3)(n-4) = 42\, n(n-1)(n-2)$

Since $n > 4$ so $n(n-1)(n-2) \neq 0$

Therefore, by dividing both sides by $n(n-1)(n-2)$, we get

$(n - 3\ (n - 4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30$

or $(n - 10)(n + 3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$.

(ii) Given that $\frac{{}^nP_4}{{}^{n-1}P_4} = \frac{5}{3}$

Therefore $3n(n-1)(n-2)(n-3) = 5(n-1)(n-2)(n-3)(n-4)$

or $3n = 5(n-4)$ [as $(n-1)(n-2)(n-3) \neq 0,\ n > 4$]

or $n = 10$.

Example 13 Find r, if $5\ {}^4P_r = 6\ {}^5P_{r-1}$.

Solution We have $5\ {}^4P_r = 6\ {}^5P_{r-1}$

or $$5 \times \frac{4!}{(4-r)!} = 6 \times \frac{5!}{(5-r+1)!}$$

or $$\frac{5!}{(4-r)!} = \frac{6 \times 5!}{(5-r+1)(5-r)(5-r-1)!}$$

or $(6 - r)(5 - r) = 6$

or $r^2 - 11r + 24 = 0$

or $r^2 - 8r - 3r + 24 = 0$

or $(r - 8)(r - 3) = 0$

or $r = 8$ or $r = 3$.

Hence $r = 8, 3$.

Example 14 Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that

(i) all vowels occur together (ii) all vowels do not occur together.

Solution (i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be ${}^6P_6 = 6!$. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time . Hence, by the multiplication principle the required number of permutations $= 6! \times 3! = 4320$.

(ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangments of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together.

Therefore, the required number $8! - 6! \times 3! = 6!\,(7\times8 - 6)$

$= 2 \times 6!\,(28 - 3)$

$= 50 \times 6! = 50 \times 720 = 36000$

Example 15 In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ?

Solution Total number of discs are $4 + 3 + 2 = 9$. Out of 9 discs, 4 are of the first kind (red), 3 are of the second kind (yellow) and 2 are of the third kind (green).

Therefore, the number of arrangements $\frac{9!}{4!\,3!\,2!} = 1260$.

Example 16 Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

(i) do the words start with P
(ii) do all the vowels always occur together
(iii) do the vowels never occur together
(iv) do the words begin with I and end in P?

Solution There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore

The required number of arrangements $= \frac{12!}{3!\,4!\,2!} = 1663200$

(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required of words starting with P are

$$= \frac{11!}{3!\,2!\,4!} = 138600.$$

(ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object $\boxed{}$ for the time being. This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2 Ds, can be rearranged in $\frac{8!}{3!\,2!}$ ways. Corresponding to each of these arrangements, the 5 vowels E, E, E, E and I can be rearranged in $\frac{5!}{4!}$ ways. Therefore, by multiplication principle the required number of arrangements

$$= \frac{8!}{3!\,2!} \times \frac{5!}{4!} = 16800$$

(iii) The required number of arrangements
= the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.

$$= 1663200 - 16800 = 1646400$$

(iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end). We are left with 10 letters.
Hence, the required number of arrangements

$$= \frac{10!}{3!\,2!\,4!} = 12600$$

EXERCISE 7.3

1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?
2. How many 4-digit numbers are there with no digit repeated?
3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?
4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?
5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position?
6. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.
7. Find r if (i) ${}^5P_r = 2\,{}^6P_{r-1}$ (ii) ${}^5P_r = {}^6P_{r-1}$.
8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?
9. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.
   (i) 4 letters are used at a time, (ii) all letters are used at a time,
   (iii) all letters are used but first letter is a vowel?
10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?
11. In how many ways can the letters of the word PERMUTATIONS be arranged if the
   (i) words start with P and end with S, (ii) vowels are all together,
   (iii) there are always 4 letters between P and S?

7.4 Combinations

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

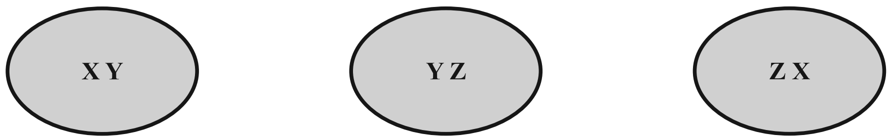


Fig. 7.3

These are XY, YZ and ZX (Fig 7.3).

Here, each selection is called a *combination of 3 different objects taken 2 at a time.* In a combination, the order is not important.

Now consider some more illustrations.

Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand shakes. X shaking hands with Y and Y with X will not be two different hand shakes. Here, order is not important. There will be as many hand shakes as there are combinations of 12 different things taken 2 at a time.

Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by ${}^nC_r$..

Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations = ${}^4C_2 \times 2!$.

On the other hand, the number of permutations of 4 different things taken 2 at a time = ${}^4P_2$.

Therefore $${}^4P_2 = {}^4C_2 \times 2! \quad \text{or} \quad \frac{4!}{(4-2)!\,2!} = {}^4C_2$$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these ${}^5C_3$ combinations, there are 3! permutations, because, the three objects in each combination can be rearranged in 3 ! ways. Therefore, the total of permutations = ${}^5C_3 \times 3!$

Therefore $\quad {}^5P_3 = {}^5C_3 \times 3! \;$ or $\; \dfrac{5!}{(5-3)!\;3!} = {}^5C_3$

These examples suggest the following theorem showing relationship between permutaion and combination:

Theorem 5 ${}^nP_r = {}^nC_r \;\; r!,\; 0 < r \le n.$

Proof Corresponding to each combination of ${}^nC_r$ we have r ! permutations, because r objects in every combination can be rearranged in r ! ways.

Hence, the total number of permutations of n different things taken r at a time is ${}^nC_r \times r!$. On the other hand, it is ${}^n\;\,{}_r$. Thus

$$ {}^nP_r = {}^nC_r \times r!,\; 0 < r \le n. $$

Remarks **1.** From above $\dfrac{n!}{(n-r)!} = {}^nC_r \times r!$, i.e., $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$.

In particular, if $r = n$, ${}^nC_n = \dfrac{n!}{n!\;0!} = 1$.

2. We define ${}^nC_0 = 1$, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define ${}^nC_0 = 1$.

3. As $\dfrac{n!}{0!(n-0)!} = 1 = {}^nC_0$, the formula ${}^n\;\,{}_r = \dfrac{n}{r\;\;(n-r)}$ is applicable for $r = 0$ also.

Hence

$$ {}^n\;\,{}_r = \frac{n}{r\;\;(n-r)},\; 0 \le r \le n. $$

4. ${}^nC_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r$, i.e., selecting r objects out of n objects is same as rejecting $(n - r)$ objects.

5. ${}^nC_a = {}^nC_b \Rightarrow a = b$ or $a = n - b$, i.e., $n = a + b$

Theorem 6 ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Proof We have $${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r \times (r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!} \times \frac{n-r+1+r}{r(n-r+1)} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r$$

Example 17 If ${}^nC_9 = {}^nC_8$, find ^{n} .

Solution We have ${}^nC_9 = {}^nC_8$

i.e., $$\frac{n!}{9!(n-9)!} = \frac{n!}{(n-8)!\,8!}$$

or $\frac{1}{9} = \frac{1}{n-8}$ or $n - 8 = 9$ or $n = 17$

Therefore ${}^nC_{17} = {}^{17}C_{17} = 1$.

Example 18 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways $= {}^5C_3 = \frac{5!}{3!\,2!} = \frac{4 \times 5}{2} = 10$.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways. Therefore, the required number of committees

$$= {}^{2}C_1 \times {}^{3}C_2 = \frac{2!}{1!\ 1!} \times \frac{3!}{2!\ 1!} = 6.$$

Example 19 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,
(ii) four cards belong to four different suits,
(iii) are face cards,
(iv) two are red cards and two are black cards,
(v) cards are of the same colour?

Solution There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

The required number of ways $= {}^{52}C_4 = \frac{52!}{4!\ 48!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4}$

$= 270725$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_4$ ways of choosing 4 clubs, ${}^{13}C_4$ ways of choosing 4 spades and ${}^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$.

$= 4 \times \frac{13!}{4!\ 9!} = 2860$

(ii) There are13 cards in each suit.

Therefore, there are ${}^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards and 4 are to be slected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways $= \frac{12!}{4!\ 8!} = 495$.

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

$$= \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards on ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ways.

Therefore, the required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$$= 2\times\frac{26!}{4!\ 22!} = 29900.$$

EXERCISE 7.4

1. If ${}^nC_8 = {}^nC_2$, find ${}^nC_2$.
2. Determine n if
   (i) ${}^{2n}C_2 : {}^nC_2 = 12 : 1$ (ii) ${}^{2n}C_3 : {}^nC_3 = 11 : 1$
3. How many chords can be drawn through 21 points on a circle?
4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?
5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.
6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.
7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?
8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.
9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Miscellaneous Examples

Example 20 How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE ?

Solution In the word INVOLUTE, there are 4 vowels, namely, I,O,E,Uand 4 consonants, namely, N, V, L and T.

The number of ways of selecting 3 vowels out of 4 = ${}^4C_3 = 4$.
The number of ways of selecting 2 consonants out of 4 = ${}^4C_2 = 6$.

Therefore, the number of combinations of 3 vowels and 2 consonants is $4 \times 6 = 24$.

Now, each of these 24 combinations has 5 letters which can be arranged among themselves in 5 ! ways. Therefore, the required number of different words is $24 \times 5\,! = 2880$.

Example 21 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl ? (ii) at least one boy and one girl ? (iii) at least 3 girls ?

Solution (i) Since, the team will not include any girl, therefore, only boys are to be selected. 5 boys out of 7 boys can be selected in ${}^7C_5$ ways. Therefore, the required

number of ways $= {}^7C_5 = \dfrac{7!}{5!\ 2!} = \dfrac{6 \times 7}{2} = 21$

(ii) Since, at least one boy and one girl are to be there in every team. Therefore, the team can consist of

(a) 1 boy and 4 girls (b) 2 boys and 3 girls

(c) 3 boys and 2 girls (d) 4 boys and 1 girl.

1 boy and 4 girls can be selected in ${}^7C_1 \times {}^4C_4$ ways.

2 boys and 3 girls can be selected in ${}^7C_2 \times {}^4C_3$ ways.

3 boys and 2 girls can be selected in ${}^7C_3 \times {}^4C_2$ ways.

4 boys and 1 girl can be selected in ${}^7C_4 \times {}^4C_1$ ways.

Therefore, the required number of ways

$= {}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$

$= 7 + 84 + 210 + 140 = 441$

(iii) Since, the team has to consist of at least 3 girls, the team can consist of
(a) 3 girls and 2 boys, or (b) 4 girls and 1 boy.

Note that the team cannot have all 5 girls, because, the group has only 4 girls.

3 girls and 2 boys can be selected in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy can be selected in ${}^4C_4 \times {}^7C_1$ ways.

Therefore, the required number of ways

$= {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 84 + 7 = 91$

Example 22 Find the number of words with or without meaning which can be made using all the letters of the word AGAIN. If these words are written as in a dictionary, what will be the 50^{th} word?

Solution There are 5 letters in the word AGAIN, in which A appears 2 times. Therefore, the required number of words = $\frac{5!}{2!} = 60$.

To get the number of words starting with A, we fix the letter A at the extreme left position, we then rearrange the remaining 4 letters taken all at a time. There will be as many arrangements of these 4 letters taken 4 at a time as there are permutations of 4 different things taken 4 at a time. Hence, the number of words starting with A = $4! = 24$. Then, starting with G, the number of words $= \frac{4!}{2!} = 12$ as after placing G at the extreme left position, we are left with the letters A, A, I and N. Similarly, there are 12 words starting with the next letter I. Total number of words so far obtained = 24 + 12 + 12 =48.

The 49^{th} word is NAAGI. The 50^{th} word is NAAIG.

Example 23 How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

Solution Since, 1000000 is a 7-digit number and the number of digits to be used is also 7. Therefore, the numbers to be counted will be 7-digit only. Also, the numbers have to be greater than 1000000, so they can begin either with 1, 2 or 4.

The number of numbers beginning with 1 = $\frac{6!}{3!\,2!} = \frac{4 \times 5 \times 6}{2} = 60$, as when 1 is fixed at the extreme left position, the remaining digits to be rearranged will be 0, 2, 2, 2, 4, 4, in which there are 3, 2*s* and 2, 4*s*.

Total numbers begining with 2

$$= \frac{6!}{2!\,2!} = \frac{3 \times 4 \times 5 \times 6}{2} = 180$$

and total numbers begining with 4 $= \frac{6!}{3!} = 4 \times 5 \times 6 = 120$

Therefore, the required number of numbers $= 60 + 180 + 120 = 360$.

Alternative Method

The number of 7-digit arrangements, clearly, $\frac{7!}{3!\ 2!} = 420$. But, this will include those numbers also, which have 0 at the extreme left position. The number of such arrangements $\frac{6!}{3!\ 2!}$ (by fixing 0 at the extreme left position) = 60.

Therefore, the required number of numbers $= 420 - 60 = 360$.

> **Note** If one or more than one digits given in the list is repeated, it will be understood that in any number, the digits can be used as many times as is given in the list, e.g., in the above example 1 and 0 can be used only once whereas 2 and 4 can be used 3 times and 2 times, respectively.

Example 24 In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?

Solution Let us first seat the 5 girls. This can be done in 5! ways. For each such arrangement, the three boys can be seated only at the cross marked places.

$\times\ G \times G \times G \times G \times G \times$.

There are 6 cross marked places and the three boys can be seated in ${}^6P_3$ ways. Hence, by multiplication principle, the total number of ways

$$= 5! \times {}^6P_3 = 5! \times \frac{6!}{3!}$$

$$= 4 \times 5 \times 2 \times 3 \times 4 \times 5 \times 6 = 14400.$$

Miscellaneous Exercise on Chapter 7

1. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER ?
2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together?
3. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:

   (i) exactly 3 girls ? (ii) atleast 3 girls ? (iii) atmost 3 girls ?
4. If the different permutations of all the letter of the word EXAMINATION are

listed as in a dictionary, how many words are there in this list before the first word starting with E ?

5. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9 which are divisible by 10 and no digit is repeated ?
6. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?
7. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions ?
8. Determine the number of 5-card combinations out of a deck of 52 cards if each selection of 5 cards has exactly one king.
9. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?
10. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen ?
11. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together ?

Summary

- *Fundamental principle of counting* If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.
- The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ${}^nP_r$ and is given by ${}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$.
- $n! = 1 \times 2 \times 3 \times ... \times n$
- $n! = n \times (n - 1)\,!$
- The number of permutations of n different things, taken r at a time, where repeatition is allowed, is n^r.
- The number of permutations of n objects taken all at a time, where p_1 objects

are of first kind, p_2 objects are of the second kind, ..., p_k objects are of the k^{th} kind and rest, if any, are all different is $\dfrac{n!}{p_1!\, p_2!\ldots p_k!}$.

- The number of combinations of n different things taken r at a time, denoted by ^nC_r, is given by $^nC_r = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$.

Historical Note

The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. The credit, however, goes to the Jains who treated its subject matter as a self-contained topic in mathematics, under the name *Vikalpa.*

Among the Jains, *Mahavira*, (around 850 A.D.) is perhaps the world's first mathematician credited with providing the general formulae for permutations and combinations.

In the 6th century B.C., *Sushruta,* in his medicinal work, *Sushruta Samhita*, asserts that 63 combinations can be made out of 6 different tastes, taken one at a time, two at a time, etc. *Pingala*, a Sanskrit scholar around third century B.C., gives the method of determining the number of combinations of a given number of letters, taken one at a time, two at a time, etc. in his work *Chhanda Sutra*. *Bhaskaracharya* (born 1114 A.D.) treated the subject matter of permutations and combinations under the name *Anka Pasha* in his famous work *Lilavati.* In addition to the general formulae for ^nC_r and ^nP_r already provided by *Mahavira, Bhaskaracharya* gives several important theorems and results concerning the subject.

Outside India, the subject matter of permutations and combinations had its humble beginnings in China in the famous book I–King (Book of changes). It is difficult to give the approximate time of this work, since in 213 B.C., the emperor had ordered all books and manuscripts in the country to be burnt which fortunately was not completely carried out. Greeks and later Latin writers also did some scattered work on the theory of permutations and combinations.

Some Arabic and Hebrew writers used the concepts of permutations and combinations in studying astronomy. *Rabbi ben Ezra,* for instance, determined the number of combinations of known planets taken two at a time, three at a time and so on. This was around 1140 A.D. It appears that *Rabbi ben Ezra* did not know the formula for ${}^nC_r$. However, he was aware that ${}^nC_r = {}^nC_{n-r}$ for specific values n and r. In 1321 A.D., *Levi Ben Gerson*, another Hebrew writer came up with the formulae for ${}^nP_r$, ${}^nP_n$ *and the general formula for* ${}^nC_r$.

The first book which gives a complete treatment of the subject matter of permutations and combinations is Ars Conjectandi written by a Swiss, *Jacob Bernoulli* (1654 – 1705 A.D.), posthumously published in 1713 A.D. This book contains essentially the theory of permutations and combinations as is known today.

❖

Chapter 8

BINOMIAL THEOREM

❖ *Mathematics is a most exact science and its conclusions are capable of absolute proofs. – C.P. STEINMETZ* ❖

8.1 Introduction

Blaise Pascal (1623-1662)

In earlier classes, we have learnt how to find the squares and cubes of binomials like $a + b$ and $a - b$. Using them, we could evaluate the numerical values of numbers like $(98)^2 = (100 - 2)^2$, $(999)^3 = (1000 - 1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a + b)^n$, where n is an integer or a rational number. In this Chapter, we study binomial theorem for positive integral indices only.

8.2 Binomial Theorem for Positive Integral Indices

Let us have a look at the following identities done earlier:

$$(a+ b)^0 = 1 \qquad a + b \neq 0$$
$$(a+ b)^1 = a + b$$
$$(a+ b)^2 = a^2 + 2ab + b^2$$
$$(a+ b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+ b)^4 = (a + b)^3 (a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In these expansions, we observe that

(i) The total number of terms in the expansion is one more than the index. For example, in the expansion of $(a + b)^2$, number of terms is 3 whereas the index of $(a + b)^2$ is 2.

(ii) Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a + b$.

We now arrange the coefficients in these expansions as follows (Fig 8.1):

Index	**Coefficients**
0	**1**
1	**1 1**
2	**1 2 1**
3	**1 3 3 1**
4	**1 4 6 4 1**

Fig 8.1

Do we observe any pattern in this table that will help us to write the next row? Yes we do. It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1, 2 and 2, 1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Also, 1 is present at the beginning and at the end of each row. This can be continued till any index of our interest.

We can extend the pattern given in Fig 8.2 by writing a few more rows.

Index	**Coefficients**
0	**1**
1	**1 1**
2	**1 2 1**
3	**1 3 3 1**
4	**1 4 6 4 1**

Fig 8.2

Pascal's Triangle

The structure given in Fig 8.2 looks like a triangle with 1 at the top vertex and running down the two slanting sides. This array of numbers is known as *Pascal's triangle*, after the name of French mathematician Blaise Pascal. It is also known as *Meru Prastara* by Pingla.

Expansions for the higher powers of a binomial are also possible by using Pascal's triangle. Let us expand $(2x + 3y)^5$ by using Pascal's triangle. The row for index 5 is

1 5 10 10 5 1

Using this row and our observations (i), (ii) and (iii), we get

$$(2x + 3y)^5 = (2x)^5 + 5(2x)^4 (3y) + 10(2x)^3 (3y)^2 + 10 (2x)^2 (3y)^3 + 5(2x)(3y)^4 + (3y)^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.$$

Now, if we want to find the expansion of $(2x + 3y)^{12}$, we are first required to get the row for index 12. This can be done by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy process. The process, as you observe, will become more difficult, if we need the expansions involving still larger powers.

We thus try to find a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle, that come before the row of the desired index.

For this, we make use of the concept of combinations studied earlier to rewrite the numbers in the Pascal's triangle. We know that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$ and n is a non-negative integer. Also, ${}^nC_0 = 1 = {}^nC_n$

The Pascal's triangle can now be rewritten as (Fig 8.3)

Index	**Coefficients**
0	${}^0C_0$ (=1)
1	${}^1C_0$ (=1) ${}^1C_1$ (=1)
2	${}^2C_0$ (=1) ${}^2C_1$ (=2) ${}^2C_2$ (=1)
3	${}^3C_0$ (=1) ${}^3C_1$ (=3) ${}^3C_2$ (=3) ${}^3C_3$ (=1)
4	${}^4C_0$ (=1) ${}^4C_1$ (=4) ${}^4C_2$ (=6) ${}^4C_3$ (=4) ${}^4C_4$ (=1)
5	${}^5C_0$ (=1) ${}^5C_1$ (=5) ${}^5C_2$ (=10) ${}^5C_3$ (=10) ${}^5C_4$ (=5) ${}^5C_5$ (=1)

Fig 8.3 Pascal's triangle

Observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example, for the index 7 the row would be

$$ {}^7C_0 \quad {}^7C_1 \quad {}^7C_2 \quad {}^7C_3 \quad {}^7C_4 \quad {}^7C_5 \quad {}^7C_6 \quad {}^7C_{7.} $$

Thus, using this row and the observations (i), (ii) and (iii), we have

$$(a + b)^7 = {}^7C_0\, a^7 + 7C_1a^6b + {}^7C_2a^5b^2 + {}^7C_3a^4b^3 + 7C_4a^3b^4 + {}^7C_5a^2b^5 + {}^7C_6ab^6 + {}^7C_7b^7$$

An expansion of a binomial to any positive integral index say n can now be visualised using these observations. We are now in a position to write the expansion of a binomial to any positive integral index.

8.2.1 ***Binomial theorem for any positive integer n,***

$$(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

Proof The proof is obtained by applying principle of mathematical induction.

Let the given statement be

$$P(n) : (a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

For $n = 1$, we have

$$P(1) : (a + b)^1 = {}^1C_0a^1 + {}^1C_1b^1 = a + b$$

Thus, P (1) is true.

Suppose P (k) is true for some positive integer k, i.e.

$$(a + b)^k = {}^kC_0a^k + {}^kC_1a^{k-1}b + {}^kC_2a^{k-2}b^2 + ...+ {}^kC_kb^k \quad ... (1)$$

We shall prove that P(k + 1) is also true, i.e.,

$$(a + b)^{k+1} = {}^{k+1}C_0\,a^{k+1} + {}^{k+1}C_1\,a^kb + {}^{k+1}C_2\,a^{k-1}b^2 + ...+ {}^{k+1}C_{k+1}\,b^{k+1}$$

Now, $(a + b)^{k+1} = (a + b)\,(a + b)^k$

$$= (a + b)\,({}^kC_0\,a^k + {}^kC_1a^{k-1}\,b + {}^kC_2\,a^{k-2}\,b^2 + ...+ {}^kC_{k-1}\,ab^{k-1} + {}^kC_k\,b^k)$$

[from (1)]

$$= {}^kC_0\,a^{k+1} + {}^kC_1\,a^kb + {}^kC_2a^{k-1}b^2 + ...+ {}^kC_{k-1}\,a^2b^{k-1} + {}^kC_k\,ab^k + {}^kC_0\,a^kb$$
$$+ {}^kC_1a^{k-1}b^2 + {}^kC_2a^{k-2}b^3 + ...+ {}^kC_{k-1}ab^k + {}^kC_kb^{k+1}$$

[by actual multiplication]

$$= {}^kC_0a^{k+1} + ({}^kC_1 + {}^kC_0)\,a^kb + ({}^kC_2 + {}^kC_1)a^{k-1}b^2 + ...$$
$$+ ({}^kC_k + {}^kC_{k-1})\,ab^k + {}^kC_kb^{k+1} \qquad \text{[grouping like terms]}$$

$$= {}^{k+1}C_0a^{k+1} + {}^{k+1}C_1a^kb + {}^{k+1}C_2\,a^{k-1}b^2 + ...+ {}^{k+1}C_kab^k + {}^{k+1}C_{k+1}\,b^{k+1}$$

(by using ${}^{k+1}C_0 = 1$, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$ and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$)

Thus, it has been proved that P (k + 1) is true whenever P(k) is true. Therefore, by principle of mathematical induction, P(n) is true for every positive integer n.

We illustrate this theorem by expanding $(x + 2)^6$:

$$(x + 2)^6 = {}^6C_0x^6 + {}^6C_1x^5.2 + {}^6C_2x^42^2 + {}^6C_3x^3.2^3 + {}^6C_4x^2.2^4 + {}^6C_5x.2^5 + {}^6C_6.2^6.$$
$$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

Thus $(x + 2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64.$

Observations

1. The notation $\sum_{k=0}^{n} {}^nC_k\, a^{n-k} b^k$ stands for

   ${}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + ... + {}^nC_r a^{n-r} b^r + ... + {}^nC_n a^{n-n} b^n$, where $b^0 = 1 = a^{n-n}$.

Hence the theorem can also be stated as

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k\, a^{n-k} b^k .$$

2. The coefficients ${}^nC_r$ occuring in the binomial theorem are known as binomial coefficients.
3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e., one more than the index.
4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n–1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.
5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n + 0 = n$ in the first term, $(n – 1) + 1 = n$ in the second term and so on $0 + n = n$ in the last term. Thus, it can be seen that the sum of the indices of a and b is n in every term of the expansion.

8.2.2 ***Some special cases*** In the expansion of $(a + b)^n$,

(i) Taking $a = x$ and $b = – y$, we obtain

$$\begin{aligned}(x-y)^n &= [x + (-y)]^n \\ &= {}^nC_0 x^n + {}^nC_1 x^{n-1}(-y) + {}^nC_2 x^{n-2}(-y)^2 + {}^nC_3 x^{n-3}(-y)^3 + ... + {}^nC_n (-y)^n \\ &= {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 - {}^nC_3 x^{n-3} y^3 + ... + (-1)^n\, {}^nC_n\, y^n\end{aligned}$$

Thus $(x-y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + ... + (-1)^n\, {}^nC_n\, y^n$

Using this, we have $(x-2y)^5 = {}^5C_0 x^5 - {}^5C_1 x^4 (2y) + {}^5C_2 x^3 (2y)^2 - {}^5C_3 x^2 (2y)^3 + {}^5C_4 x(2y)^4 - {}^5C_5 (2y)^5$

$$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5.$$

(ii) Taking $a = 1$, $b = x$, we obtain

$$\begin{aligned}(1+x)^n &= {}^nC_0 (1)^n + {}^nC_1 (1)^{n-1} x + {}^nC_2 (1)^{n-2} x^2 + ... + {}^nC_n x^n \\ &= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ... + {}^nC_n x^n\end{aligned}$$

Thus $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ... + {}^nC_n x^n$

In particular, for $x = 1$, we have

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n.$$

(iii) Taking $a = 1$, $b = -x$, we obtain

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - ... + (-1)^n\ {}^nC_nx^n$$

In particular, for $x = 1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n\ {}^nC_n$$

Example 1 Expand $\left(x^2 + \frac{3}{x}\right)^4$, $x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0(x^2)^4 + {}^4C_1(x^2)^3\left(\frac{3}{x}\right) + {}^4C_2(x^2)^2\left(\frac{3}{x}\right)^2 + {}^4C_3(x^2)\left(\frac{3}{x}\right)^3 + {}^4C_4\left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 . \frac{3}{x} + 6.x^4 . \frac{9}{x^2} + 4.x^2 . \frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example 2 Compute $(98)^5$.

Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem.

Write $98 = 100 - 2$

Therefore, $(98)^5 = (100 - 2)^5$

$$= {}^5C_0\ (100)^5 - {}^5C_1\ (100)^4.2 + {}^5C_2\ (100)^3 2^2$$
$$- {}^5C_3\ (100)^2\ (2)^3 + {}^5C_4\ (100)\ (2)^4 - {}^5C_5\ (2)^5$$
$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000$$
$$\times 8 + 5 \times 100 \times 16 - 32$$
$$= 10040008000 - 1000800032 = 9039207968.$$

Example 3 Which is larger $(1.01)^{1000000}$ or 10,000?

Solution Splitting 1.01 and using binomial theorem to write the first few terms we have

$$(1.01)^{1000000} = (1 + 0.01)^{1000000}$$

$$= {}^{1000000}C_0 + {}^{1000000}C_1(0.01) + \text{other positive terms}$$

$$= 1 + 1000000 \times 0.01 + \text{other positive terms}$$

$$= 1 + 10000 + \text{other positive terms}$$

$$> 10000$$

Hence $(1.01)^{1000000} > 10000$

Example 4 Using binomial theorem, prove that $6^n – 5n$ always leaves remainder 1 when divided by 25.

Solution For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that $6^n – 5n$ leaves remainder 1 when divided by 25, we prove that $6^n – 5n = 25k + 1$, where k is some natural number.

We have

$$(1 + a)^n = {}^nC_0 + {}^nC_1a + {}^nC_2a^2 + ... + {}^nC_na^n$$

For $a = 5$, we get

$$(1 + 5)^n = {}^nC_0 + {}^nC_15 + {}^nC_25^2 + ... + {}^nC_n5^n$$

i.e. $(6)^n = 1 + 5n + 5^2.{}^nC_2 + 5^3.{}^nC_3 + ... + 5^n$

i.e. $6^n – 5n = 1+5^2 ({}^nC_2 + {}^nC_35 + ... + 5^{n-2})$

or $6^n – 5n = 1+ 25 ({}^nC_2 + 5 .{}^nC_3 + ... + 5^{n-2})$

or $6^n – 5n = 25k+1$ where $k = {}^nC_2 + 5 .{}^nC_3 + ... + 5^{n–2}$.

This shows that when divided by 25, $6^n – 5n$ leaves remainder 1.

EXERCISE 8.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1–2x)^5$ **2.** $\left(\dfrac{2}{x} - \dfrac{x}{2}\right)^5$ **3.** $(2x – 3)^6$

4. $\left(\dfrac{x}{3}+\dfrac{1}{x}\right)^5$ 5. $\left(x+\dfrac{1}{x}\right)^6$

Using binomial theorem, evaluate each of the following:

6. $(96)^3$ 7. $(102)^5$ 8. $(101)^4$
9. $(99)^5$
10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.
11. Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3} \quad \sqrt{2})^4$.
12. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.
13. Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.
14. Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$.

8.3 General and Middle Terms

1. In the binomial expansion for $(a + b)^n$, we observe that the first term is ${}^nC_0a^n$, the second term is ${}^nC_1a^{n-1}b$, the third term is ${}^nC_2a^{n-2}b^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r + 1)^{th}$ term is ${}^nC_ra^{n-r}b^r$. The $(r + 1)^{th}$ term is also called the *general term* of the expansion $(a + b)^n$. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\, a^{n-r}b^r$.
2. Regarding the middle term in the expansion $(a + b)^n$, we have

   (i) If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even so $n + 1$ is odd. Therefore, the middle term is $\left(\dfrac{n+1+1}{2}\right)^{th}$, i.e., $\left(\dfrac{n}{2}+1\right)^{th}$ term.

   For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\dfrac{8}{2}+1\right)^{th}$ i.e., 5^{th} term.

   (ii) If n is odd, then $n + 1$ is even, so there will be two middle terms in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term. So in the expansion $(2x-y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$, i.e., 4^{th} and $\left(\frac{7+1}{2}+1\right)^{th}$, i.e., 5^{th} term.

3. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$, i.e., $(n+1)^{th}$ term, as $2n$ is even.

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant).

This term is called the *term independent* of x or the constant term.

Example 5 Find a if the 17^{th} and 18^{th} terms of the expansion $(2+a)^{50}$ are equal.

Solution The $(r+1)^{th}$ term of the expansion $(x+y)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$.

For the 17^{th} term, we have, $r+1 = 17$, i.e., $r = 16$

Therefore, $T_{17} = T_{16+1} = {}^{50}C_{16}\,(2)^{50-16}\,a^{16}$

$= {}^{50}C_{16}\,2^{34}\,a^{16}.$

Similarly, $T_{18} = {}^{50}C_{17}\,2^{33}\,a^{17}$

Given that $T_{17} = T_{18}$

So ${}^{50}C_{16}\,(2)^{34}\,a^{16} = {}^{50}C_{17}\,(2)^{33}\,a^{17}$

Therefore $$\frac{{}^{50}C_{16}\,.2^{34}}{{}^{50}C_{17}\,.2^{33}} = \frac{a^{17}}{a^{16}}$$

i.e., $$a = \frac{{}^{50}C_{16}\times 2}{{}^{50}C_{17}} = \frac{50!}{16!\,34!}\times\frac{17!\;.\;33!}{50!}\times 2 = 1$$

Example 6 Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}\,2n\,x^n$, where n is a positive integer.

Solution As $2n$ is even, the middle term of the expansion $(1 + x)^{2n}$ is $\left(\frac{2n}{2}+1\right)^{\text{th}}$, i.e., $(n + 1)^{\text{th}}$ term which is given by,

$$\mathrm{T}_{n+1} = {}^{2n}\mathrm{C}_n(1)^{2n-n}(x)^n = {}^{2n}\mathrm{C}_n x^n = \frac{(2n)!}{n!\ n!}x^n$$

$$= \frac{2n\,(2n-1)\ \ (2n-2)\ \ ...4.3.2.1}{n!\ \ n!}x^n$$

$$= \frac{1.2.3.4...(2n-2)\,(2n-1)\,(2n)}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n\ \ \ 1)][2.4.6...(2n)]}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)]2^n\,[1.2.3...n]}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)]\,n!}{n!\ \ n!}2^n.\,x^n$$

$$= \frac{1.3.5...(2n-1)}{n!}\ 2^n\ x^n$$

Example 7 Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$.

Solution Suppose x^6y^3 occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(x + 2y)^9$.

Now $\qquad \mathrm{T}_{r+1} = {}^9\mathrm{C}_r\, x^{9-r}\,(2y)^r = {}^9\mathrm{C}_r\, 2^r\,.\,x^{9-r}\,.\,y^r\,.$

Comparing the indices of x as well as y in x^6y^3 and in T_{r+1}, we get $r = 3$.

Thus, the coefficient of x^6y^3 is

$${}^9\mathrm{C}_3\, 2^3 = \frac{9!}{3!\,6!}.2^3 = \frac{9.8.7}{3.2}.2^3 = 672.$$

Example 8 The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080, respectively. Find x, a and n.

Solution Given that second term $\mathrm{T}_2 = 240$

We have $\quad T_2 = {}^nC_1 x^{n-1} \,.\, a$

So $\quad {}^nC_1 x^{n-1} \,.\, a = 240 \qquad \text{... (1)}$

Similarly $\quad {}^nC_2 x^{n-2}\, a^2 = 720 \qquad \text{... (2)}$

and $\quad {}^nC_3 x^{n-3}\, a^3 = 1080 \qquad \text{... (3)}$

Dividing (2) by (1), we get

$$\frac{{}^nC_2 x^{n-2} a^2}{{}^nC_1 x^{n-1} a} = \frac{720}{240} \quad \text{i.e.,} \quad \frac{(n-1)!}{(n-2)!}.\frac{a}{x} = 6$$

or $$\frac{a}{x} = \frac{6}{(n-1)} \qquad \text{... (4)}$$

Dividing (3) by (2), we have

$$\frac{a}{x} = \frac{9}{2(n-2)} \qquad \text{... (5)}$$

From (4) and (5),

$$\frac{6}{n-1} = \frac{9}{2(n-2)}. \qquad \text{Thus, } n = 5$$

Hence, from (1), $5x^4a = 240$, and from (4), $\dfrac{a}{x} = \dfrac{3}{2}$

Solving these equations for a and x, we get $x = 2$ and $a = 3$.

Example 9 The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio1: 7 : 42. Find n.

Solution Suppose the three consecutive terms in the expansion of $(1 + a)^n$ are $(r - 1)^{\text{th}}$, r^{th} and $(r + 1)^{\text{th}}$ terms.

The $(r - 1)^{\text{th}}$ term is ${}^nC_{r-2}\, a^{r-2}$, and its coefficient is ${}^nC_{r-2}$. Similarly, the coefficients of r^{th} and $(r + 1)^{\text{th}}$ terms are ${}^nC_{r-1}$ and ${}^nC_r$, respectively.

Since the coefficients are in the ratio 1 : 7 : 42, so we have,

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7}, \text{ i.e., } n - 8r + 9 = 0 \qquad \text{... (1)}$$

and $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{7}{42}, \text{ i.e., } n - 7r + 1 = 0 \qquad \text{... (2)}$$

Solving equations(1) and (2), we get, $n = 55$.

EXERCISE 8.2

Find the coefficient of

1. x^5 in $(x + 3)^8$
2. a^5b^7 in $(a - 2b)^{12}$.

Write the general term in the expansion of

3. $(x^2 - y)^6$
4. $(x^2 - yx)^{12}$, $x \neq 0$.
5. Find the 4th term in the expansion of $(x - 2y)^{12}$.
6. Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$.

Find the middle terms in the expansions of

7. $\left(3 - \frac{x^3}{6}\right)^7$
8. $\left(\frac{x}{3} + 9y\right)^{10}$.
9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.
10. The coefficients of the $(r - 1)^{\text{th}}$, r^{th} and $(r + 1)^{\text{th}}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.
11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.
12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Miscellaneous Examples

Example 10 Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$.

Solution We have
$$T_{r+1} = {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(-\frac{1}{3x}\right)^r$$

$$= {}^6C_r \left(\frac{3}{2}\right)^{6-r} \left(x^2\right)^{6-r} (-1)^r \left(\frac{1}{x}\right)^r \left(\frac{1}{3^r}\right)$$

$$= (-1)^r \; {}^6C_r \; \frac{(3)^{6-2r}}{(2)^{6-r}} \; x^{12-3r}$$

The term will be independent of x if the index of x is zero, i.e., $12 - 3r = 0$. Thus, $r = 4$

Hence 5^{th} term is independent of x and is given by $(-1)^4 \; {}^6C_4 \; \frac{(3)^{6-8}}{(2)^{6-4}} = \frac{5}{12}$.

Example 11 If the coefficients of a^{r-1}, a^r and a^{r+1} in the expansion of $(1 + a)^n$ are in arithmetic progression, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

Solution The $(r + 1)^{\text{th}}$ term in the expansion is ${}^nC_r a^r$. Thus it can be seen that a^r occurs in the $(r + 1)^{\text{th}}$ term, and its coefficient is ${}^nC_r$. Hence the coefficients of a^{r-1}, a^r and a^{r+1} are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$, respectively. Since these coefficients are in arithmetic progression, so we have, ${}^nC_{r-1} + {}^nC_{r+1} = 2.{}^nC_r$. This gives

$$\frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r+1)!(n-r-1)!} = 2 \times \frac{n!}{r!(n-r)!}$$

i.e. $$\frac{1}{(r-1)!(n-r+1)(n-r)(n-r-1)!} + \frac{1}{(r+1)(r)(r-1)!(n-r-1)!}$$

$$= 2 \times \frac{1}{r(r-1)!(n-r)(n-r-1)!}$$

or $$\frac{1}{(r-1)!\;(n-r-1)!}\left[\frac{1}{(n-r)(n-r+1)} + \frac{1}{(r+1)\;(r)}\right]$$

$$= 2 \times \frac{1}{(r-1)!\;(n-r-1)![r(n-r)]}$$

i.e. $$\frac{1}{(n-r+1)(n-r)} + \frac{1}{r(r+1)} = \frac{2}{r(n-r)},$$

or $$\frac{r(r+1) + (n-r)(n-r+1)}{(n-r)(n-r+1)r(r+1)} = \frac{2}{r(n-r)}$$

or $$r(r + 1) + (n - r)(n - r + 1) = 2(r + 1)(n - r + 1)$$

or $$r^2 + r + n^2 - nr + n - nr + r^2 - r = 2(nr - r^2 + r + n - r + 1)$$

or $\quad n^2 - 4nr - n + 4r^2 - 2 = 0$

i.e., $\quad n^2 - n(4r + 1) + 4r^2 - 2 = 0$

Example 12 Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

Solution As $2n$ is even so the expansion $(1 + x)^{2n}$ has only one middle term which is

$\left(\frac{2n}{2}+1\right)^{\text{th}}$ i.e., $(n + 1)^{\text{th}}$ term.

The $(n + 1)^{\text{th}}$ term is ${}^{2n}C_n x^n$. The coefficient of x^n is ${}^{2n}C_n$

Similarly, $(2n - 1)$ being odd, the other expansion has two middle terms,

$\left(\frac{2n-1+1}{2}\right)^{\text{th}}$ and $\left(\frac{2n-1+1}{2}+1\right)^{\text{th}}$ i.e., n^{th} and $(n + 1)^{\text{th}}$ terms. The coefficients of these terms are ${}^{2n-1}C_{n-1}$ and ${}^{2n-1}C_n$, respectively.

Now

$${}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{2n}C_n \qquad [\text{As } {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]. \text{ as required.}$$

Example 13 Find the coefficient of a^4 in the product $(1 + 2a)^4 (2 - a)^5$ using binomial theorem.

Solution We first expand each of the factors of the given product using Binomial Theorem. We have

$$(1 + 2a)^4 = {}^4C_0 + {}^4C_1 (2a) + {}^4C_2 (2a)^2 + {}^4C_3 (2a)^3 + {}^4C_4 (2a)^4$$

$$= 1 + 4(2a) + 6(4a^2) + 4(8a^3) + 16a^4.$$

$$= 1 + 8a + 24a^2 + 32a^3 + 16a^4$$

and $$(2 - a)^5 = {}^5C_0 (2)^5 - {}^5C_1 (2)^4 (a) + {}^5C_2 (2)^3 (a)^2 - {}^5C_3 (2)^2 (a)^3 + {}^5C_4 (2)(a)^4 - {}^5C_5 (a)^5$$

$$= 32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5$$

Thus $(1 + 2a)^4 (2 - a)^5$

$$= (1 + 8a + 24a^2 + 32a^3 + 16a^4)(32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5)$$

The complete multiplication of the two brackets need not be carried out. We write only those terms which involve a^4. This can be done if we note that $a^r . a^{4-r} = a^4$. The terms containing a^4 are

$$1(10a^4) + (8a)(-40a^3) + (24a^2)(80a^2) + (32a^3)(-80a) + (16a^4)(32) = -438a^4$$

Thus, the coefficient of a^4 in the given product is – 438.

Example 14 Find the r^{th} term from the end in the expansion of $(x + a)^n$.

Solution There are $(n + 1)$ terms in the expansion of $(x + a)^n$. Observing the terms we can say that the first term from the end is the last term, i.e., $(n + 1)^{th}$ term of the expansion and $n + 1 = (n + 1) - (1 - 1)$. The second term from the end is the n^{th} term of the expansion, and $n = (n + 1) - (2 - 1)$. The third term from the end is the $(n - 1)^{th}$ term of the expansion and $n - 1 = (n + 1) - (3 - 1)$ and so on. Thus r^{th} term from the end will be term number $(n + 1) - (r - 1) = (n - r + 2)$ of the expansion. And the $(n - r + 2)^{th}$ term is ${}^nC_{n-r+1}\, x^{r-1}\, a^{n-r+1}$.

Example 15 Find the term independent of x in the expansion of $\left(\sqrt[3]{x} + \frac{1}{2\sqrt[3]{x}}\right)^{18}, x > 0$.

Solution We have $T_{r+1} = {}^{18}C_r \left(\sqrt[3]{x}\right)^{18-r} \left(\frac{1}{2\sqrt[3]{x}}\right)^r$

$$= {}^{18}C_r\, x^{\frac{18-r}{3}} \cdot \frac{1}{2^r . x^{\frac{r}{3}}} = {}^{18}C_2 \frac{1}{2^r} . x^{\frac{18-2r}{3}}$$

Since we have to find a term independent of x, i.e., term not having x, so take $\frac{18-2r}{3} = 0$.

We get $r = 9$. The required term is ${}^{18}C_9 \frac{1}{2^9}$.

Example 16 The sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^m$, $x \neq 0$, m being a natural number, is 559. Find the term of the expansion containing x^3.

Solution The coefficients of the first three terms of $\left(x - \frac{3}{x^2}\right)^m$ are ${}^mC_0$, $(-3)\, {}^mC_1$ and $9\, {}^mC_2$. Therefore, by the given condition, we have

$${}^mC_0 - 3\, {}^mC_1 + 9\, {}^mC_2 = 559, \text{ i.e., } 1 - 3m + \frac{9m(m-1)}{2} = 559$$

which gives $m = 12$ (m being a natural number).

Now $$T_{r+1} = {}^{12}C_r\, x^{12-r}\left(-\frac{3}{x^2}\right)^r = {}^{12}C_r\,(-3)^r \cdot x^{12-3r}$$

Since we need the term containing x^3, so put $12 - 3r = 3$ i.e., $r = 3$.

Thus, the required term is ${}^{12}C_3\,(-3)^3\,x^3$, i.e., $-5940\,x^3$.

Example 17 If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are equal, find r.

Solution The coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms of the expansion $(1+x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$, respectively. Since they are equal so ${}^{34}C_{r-6} = {}^{34}C_{2r-2}$

Therefore, either $r - 6 = 2r - 2$ or $r - 6 = 34 - (2r - 2)$

[Using the fact that if ${}^nC_r = {}^nC_p$, then either $r = p$ or $r = n - p$]

So, we get $r = -4$ or $r = 14$. r being a natural number, $r = -4$ is not possible. So, $r = 14$.

Miscellaneous Exercise on Chapter 8

1. Find a, b and n in the expansion of $(a+b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.
2. Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.
3. Find the coefficient of x^5 in the product $(1+2x)^6\,(1-x)^7$ using binomial theorem.
4. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

   [**Hint** write $a^n = (a - b + b)^n$ and expand]
5. Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$.
6. Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$.
7. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.
8. Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$.

9. Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$, $x \neq 0$.

10. Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Summary

- The expansion of a binomial for any positive integral n is given by Binomial Theorem, which is $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$.
- The coefficients of the expansions are arranged in an array. This array is called *Pascal's triangle*.
- The general term of an expansion $(a + b)^n$ is $T_{r+1} = {}^nC_ra^{n-r}.b^r$.
- In the expansion $(a + b)^n$, if n is even, then the middle term is the $\left(\frac{n}{2}+1\right)^{th}$ term. If n is odd, then the middle terms are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Historical Note

The ancient Indian mathematicians knew about the coefficients in the expansions of $(x + y)^n$, $0 \leq n \leq 7$. The arrangement of these coefficients was in the form of a diagram called *Meru-Prastara*, provided by Pingla in his book *Chhanda shastra* (200B.C.). This triangular arrangement is also found in the work of Chinese mathematician Chu-shi-kie in 1303 A.D. The term binomial coefficients was first introduced by the German mathematician, Michael Stipel (1486-1567A.D.) in approximately 1544 A.D. Bombelli (1572 A.D.) also gave the coefficients in the expansion of $(a + b)^n$, for $n = 1,2 ...,7$ and Oughtred (1631 A.D.) gave them for $n = 1, 2,..., 10$. The arithmetic triangle, popularly known as *Pascal's triangle* and similar to the *Meru-Prastara* of Pingla was constructed by the French mathematician Blaise Pascal (1623-1662 A.D.) in 1665.

The present form of the binomial theorem for integral values of n appeared in *Trate du triange arithmetic*, written by Pascal and published posthumously in 1665 A.D.

Chapter 9

SEQUENCES AND SERIES

❖ *Natural numbers are the product of human spirit. – DEDEKIND* ❖

9.1 Introduction

Fibonacci
(1175 1250)

In mathematics, the word, "*sequence*" is used in much the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on. For example, population of human beings or bacteria at different times form a sequence. The amount of money deposited in a bank, over a number of years form a sequence. Depreciated values of certain commodity occur in a sequence. Sequences have important applications in several spheres of human activities.

Sequences, following specific patterns are called *progressions*. In previous class, we have studied about *arithmetic progression* (A.P). In this Chapter, besides discussing more about A.P.; *arithmetic mean, geometric mean, relationship between A.M. and G.M., special series in forms of sum to n terms of consecutive natural numbers, sum to n terms of squares of natural numbers and sum to n terms of cubes of natural numbers* will also be studied.

9.2 Sequences

Let us consider the following examples:

Assume that there is a generation gap of 30 years, we are asked to find the number of ancestors, i.e., parents, grandparents, great grandparents, etc. that a person might have over 300 years.

Here, the total number of generations $= \dfrac{300}{30} = 10$

The number of person's ancestors for the first, second, third, …, tenth generations are 2, 4, 8, 16, 32, …, 1024. These numbers form what we call a *sequence*.

Consider the successive quotients that we obtain in the division of 10 by 3 at different steps of division. In this process we get 3,3.3,3.33,3.333, ... and so on. These quotients also form a sequence. The various numbers occurring in a sequence are called its *terms*. We denote the terms of a sequence by $a_1, a_2, a_3, ..., a_n, …$, etc., the subscripts denote the position of the term. The n^{th} term is the number at the n^{th} position of the sequence and is denoted by a_n. The n^{th} term is also called the *general term* of the sequence.

Thus, the terms of the sequence of person's ancestors mentioned above are:

$$a_1 = 2, a_2 = 4, a_3 = 8, ..., a_{10} = 1024.$$

Similarly, in the example of successive quotients

$$a_1 = 3, a_2 = 3.3, a_3 = 3.33, …, a_6 = 3.33333, \text{etc.}$$

A sequence containing finite number of terms is called a *finite sequence*. For example, sequence of ancestors is a finite sequence since it contains 10 terms (a fixed number).

A sequence is called *infinite*, if it is not a finite sequence. For example, the sequence of successive quotients mentioned above is an *infinite sequence*, infinite in the sense that it never ends.

Often, it is possible to express the rule, which yields the various terms of a sequence in terms of algebraic formula. Consider for instance, the sequence of even natural numbers 2, 4, 6, …

Here $\quad a_1 = 2 = 2 \times 1 \qquad a_2 = 4 = 2 \times 2$

$$a_3 = 6 = 2 \times 3 \qquad a_4 = 8 = 2 \times 4$$

$$.... \quad .... \quad .... \qquad .... \quad .... \quad ....$$

$$.... \quad .... \quad .... \qquad .... \quad .... \quad ....$$

$$a_{23} = 46 = 2 \times 23, \; a_{24} = 48 = 2 \times 24, \text{ and so on.}$$

In fact, we see that the n^{th} term of this sequence can be written as $a_n = 2n$, where n is a natural number. Similarly, in the sequence of odd natural numbers 1,3,5, …, the n^{th} term is given by the formula, $a_n = 2n - 1$, where n is a natural number.

In some cases, an arrangement of numbers such as 1, 1, 2, 3, 5, 8,.. has no visible pattern, but the sequence is generated by the recurrence relation given by

$$a_1 = a_2 = 1$$

$$a_3 = a_1 + a_2$$

$$a_n = a_{n-2} + a_{n-1}, \; n > 2$$

This sequence is called *Fibonacci sequence.*

In the sequence of primes 2,3,5,7,…, we find that there is no formula for the n^{th} prime. Such sequence can only be described by verbal description.

In every sequence, we should not expect that its terms will necessarily be given by a specific formula. However, we expect a theoretical scheme or a rule for generating the terms $a_1, a_2, a_3, \ldots, a_n, \ldots$ in succession.

In view of the above, *a sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it of the type* $\{1, 2, 3...k\}$. *Sometimes, we use the functional notation a(n) for* a_n.

9.3 Series

Let $a_1, a_2, a_3, \ldots, a_n$, be a given sequence. Then, the expression

$$a_1 + a_2 + a_3 +, \ldots + a_n + ...$$

is called the *series associated with the given sequence* .The series is finite or infinite according as the given sequence is finite or infinite. Series are often represented in compact form, called *sigma notation*, using the Greek letter $\sum$ (sigma) as means of indicating the summation involved. Thus, the series $a_1 + a_2 + a_3 + ... + a_n$ is abbreviated as $\sum_{k=1}^{n} a_k$.

Remark When the series is used, it refers to the indicated sum not to the sum itself. For example, 1 + 3 + 5 + 7 is a finite series with four terms. When we use the phrase "*sum of a series*," we will mean the number that results from adding the terms, the sum of the series is 16.

We now consider some examples.

Example 1 Write the first three terms in each of the following sequences defined by the following:

(i) $a_n = 2n + 5$, (ii) $a_n = \dfrac{n-3}{4}$.

Solution (i) Here $a_n = 2n + 5$

Substituting n = 1, 2, 3, we get

$$a_1 = 2(1) + 5 = 7, a_2 = 9, a_3 = 11$$

Therefore, the required terms are 7, 9 and 11.

(ii) Here $a_n = \dfrac{n-3}{4}$. Thus, $a_1 = \dfrac{1-3}{4} = -\dfrac{1}{2}, a_2 = -\dfrac{1}{4}, a_3 = 0$

Hence, the first three terms are $-\frac{1}{2}$, $-\frac{1}{4}$ and 0.

Example 2 What is the 20th term of the sequence defined by

$$a_n = (n - 1)(2 - n)(3 + n) ?$$

Solution Putting $n = 20$, we obtain

$$a_{20} = (20 - 1)(2 - 20)(3 + 20)$$
$$= 19 \times (-18) \times (23) = -7866.$$

Example 3 Let the sequence a_n be defined as follows:

$$a_1 = 1, \; a_n = a_{n-1} + 2 \text{ for } n \geq 2.$$

Find first five terms and write corresponding series.

Solution We have

$$a_1 = 1, a_2 = a_1 + 2 = 1 + 2 = 3, a_3 = a_2 + 2 = 3 + 2 = 5,$$
$$a_4 = a_3 + 2 = 5 + 2 = 7, a_5 = a_4 + 2 = 7 + 2 = 9.$$

Hence, the first five terms of the sequence are 1,3,5,7 and 9. The corresponding series is 1 + 3 + 5 + 7 + 9 +...

EXERCISE 9.1

Write the first five terms of each of the sequences in Exercises 1 to 6 whose n^{th} terms are:

1. $a_n = n(n + 2)$ **2.** $a_n = \frac{n}{n+1}$ **3.** $a_n = 2^n$

4. $a_n = \frac{2n-3}{6}$ **5.** $a_n = (-1)^{n-1} 5^{n+1}$ **6.** $a_n = n\frac{n^2+5}{4}$.

Find the indicated terms in each of the sequences in Exercises 7 to 10 whose n^{th} terms are:

7. $a_n = 4n - 3$; a_{17}, a_{24} **8.** $a_n = \frac{n^2}{2^n}$; a_7

9. $a_n = (-1)^{n-1}n^3$; a_9 **10.** $a_n = \frac{n(n-2)}{n+3}$; a_{20}.

Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series:

11. $a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$

12. $a_1 = -1, a_n = \dfrac{a_{n-1}}{n}, n \geq 2$

13. $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$

14. The Fibonacci sequence is defined by

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2}, n > 2.$$

Find $\dfrac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

9.4 Arithmetic Progression (A.P.)

Let us recall some formulae and properties studied earlier.

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called *arithmetic sequence or arithmetic progression* if $a_{n+1} = a_n + d, n \in$ N, where a_1 is called the *first term* and the constant term d is called the *common difference* of the A.P.

Let us consider an A.P. (in its standard form) with first term a and common difference d, i.e., $a, a + d, a + 2d, \ldots$

Then the n^{th} term (*general term*) of the A.P. is $a_n = a + (n - 1) d$.

We can verify the following simple properties of an A.P. :

(i) If a constant is added to each term of an A.P., the resulting sequence is also an A.P.

(ii) If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P.

(iii) If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P.

(iv) If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

Here, we shall use the following notations for an arithmetic progression:

a = the first term, l = the last term, d = common difference,

n = the number of terms.

S_n = the sum to n terms of A.P.

Let $a, a + d, a + 2d, \ldots, a + (n - 1) d$ be an A.P. Then

$$l = a + (n - 1) d$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

We can also write, $S_n = \frac{n}{2}\left[a + l\right]$

Let us consider some examples.

Example 4 In an A.P. if m^{th} term is n and the n^{th} term is m, where $m \neq n$, find the pth term.

Solution We have $a_m = a + (m - 1)\, d = n$, ... (1)

and $a_n = a + (n - 1)\, d = m$ ... (2)

Solving (1) and (2), we get

$(m - n)\, d = n - m$, or $d = -1$, ... (3)

and $a = n + m - 1$ ... (4)

Therefore $a_p = a + (p - 1)d$

$= n + m - 1 + (p - 1)(-1) = n + m - p$

Hence, the p^{th} term is $n + m - p$.

Example 5 If the sum of n terms of an A.P. is $nP + \frac{1}{2}n(n-1)Q$, where P and Q are constants, find the common difference.

Solution Let $a_1, a_2, \ldots a_n$ be the given A.P. Then

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n = nP + \frac{1}{2}n\,(n - 1)\,Q$$

Therefore $S_1 = a_1 = P$, $S_2 = a_1 + a_2 = 2P + Q$

So that $a_2 = S_2 - S_1 = P + Q$

Hence, the common difference is given by $d = a_2 - a_1 = (P + Q) - P = Q$.

Example 6 The sum of n terms of two arithmetic progressions are in the ratio $(3n + 8) : (7n + 15)$. Find the ratio of their 12^{th} terms.

Solution Let a_1, a_2 and d_1, d_2 be the first terms and common difference of the first and second arithmetic progression, respectively. According to the given condition, we have

$$\frac{\text{Sum to } n \text{ terms of first A.P.}}{\text{Sum to } n \text{ terms of second A.P.}} = \frac{3n+8}{7n+15}$$

or $$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]}=\frac{3n+8}{7n+15}$$

or $$\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{3n+8}{7n+15} \quad \text{... (1)}$$

Now $$\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P}}=\frac{a_1+11d_1}{a_2+11d_2}$$

$$\frac{2a_1+22d_1}{2a_2+22d_2}=\frac{3\times 23+8}{7\times 23+15} \quad \text{[By putting } n = 23 \text{ in (1)]}$$

Therefore $$\frac{a_1+11d_1}{a_2+11d_2}=\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P.}}=\frac{7}{16}$$

Hence, the required ratio is 7 : 16.

Example 7 The income of a person is Rs. 3,00,000, in the first year and he receives an increase of Rs.10,000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.

Solution Here, we have an A.P. with $a = 3,00,000$, $d = 10,000$, and $n = 20$.
Using the sum formula, we get,

$$S_{20}=\frac{20}{2}[600000+19\times 10000]=10\,(790000)=79,00,000.$$

Hence, the person received Rs. 79,00,000 as the total amount at the end of 20 years.

9.4.1 ***Arithmetic mean*** Given two numbers a and b. We can insert a number A between them so that a, A, b is an A.P. Such a number A is called the *arithmetic mean* (A.M.) of the numbers a and b. Note that, in this case, we have

$$A - a = b - A, \quad \text{i.e., } A = \frac{a+b}{2}$$

We may also interpret the A.M. between two numbers a and b as their average $\frac{a+b}{2}$. For example, the A.M. of two numbers 4 and 16 is 10. We have, thus constructed an A.P. 4, 10, 16 by inserting a number 10 between 4 and 16. The natural question now arises : Can we insert two or more numbers between given two numbers so that the resulting sequence comes out to be an A.P. ? Observe that two numbers 8 and 12 can be inserted between 4 and 16 so that the resulting sequence 4, 8, 12, 16 becomes an A.P.

More generally, given any two numbers a and b, we can insert as many numbers as we like between them such that the resulting sequence is an A.P.

Let $A_1, A_2, A_3, \ldots, A_n$ be n numbers between a and b such that $a, A_1, A_2, A_3, \ldots, A_n, b$ is an A.P.

Here, b is the $(n + 2)^{\text{th}}$ term, i.e., $b = a + [(n + 2) - 1]d = a + (n + 1) d$.

This gives $$d = \frac{b-a}{n+1}.$$

Thus, n numbers between a and b are as follows:

$$A_1 = a + d = a + \frac{b-a}{n+1}$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$$

$$A_3 = a + 3d = a + \frac{3(b-a)}{n+1}$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$A_n = a + nd = a + \frac{n(b-a)}{n+1}.$$

Example 8 Insert 6 numbers between 3 and 24 such that the resulting sequence is an A.P.

Solution Let A_1, A_2, A_3, A_4, A_5 and A_6 be six numbers between 3 and 24 such that

$$3, A_1, A_2, A_3, A_4, A_5, A_6, 24 \text{ are in A.P. Here, } a = 3, b = 24, n = 8.$$

Therefore, $24 = 3 + (8 - 1) d$, so that $d = 3$.

Thus $\quad A_1 = a + d = 3 + 3 = 6; \quad A_2 = a + 2d = 3 + 2 \times 3 = 9;$

$A_3 = a + 3d = 3 + 3 \times 3 = 12; \quad A_4 = a + 4d = 3 + 4 \times 3 = 15;$

$A_5 = a + 5d = 3 + 5 \times 3 = 18; \quad A_6 = a + 6d = 3 + 6 \times 3 = 21.$

Hence, six numbers between 3 and 24 are 6, 9, 12, 15, 18 and 21.

EXERCISE 9.2

1. Find the sum of odd integers from 1 to 2001.
2. Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.
3. In an A.P., the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20^{th} term is -112.
4. How many terms of the A.P. $-6, -\frac{11}{2}, -5, \ldots$ are needed to give the sum -25?
5. In an A.P., if p^{th} term is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of first pq terms is $\frac{1}{2}(pq+1)$, where $p \neq q$.
6. If the sum of a certain number of terms of the A.P. 25, 22, 19, … is 116. Find the last term.
7. Find the sum to n terms of the A.P., whose k^{th} term is $5k + 1$.
8. If the sum of n terms of an A.P. is $(pn + qn^2)$, where p and q are constants, find the common difference.
9. The sums of n terms of two arithmetic progressions are in the ratio $5n + 4 : 9n + 6$. Find the ratio of their 18^{th} terms.
10. If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then find the sum of the first $(p + q)$ terms.
11. Sum of the first p, q and r terms of an A.P are. a, b and c, respectively.

    Prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$
12. The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of m^{th} and n^{th} term is $(2m - 1) : (2n - 1)$.
13. If the sum of n terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, find the value of m.
14. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P.
15. If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between a and b, then find the value of n.
16. Between 1 and 31, m numbers have been inserted in such a way that the resulting sequence is an A. P. and the ratio of 7^{th} and $(m - 1)^{\text{th}}$ numbers is 5 : 9. Find the value of m.

17. A man starts repaying a loan as first instalment of Rs. 100. If he increases the instalment by Rs 5 every month, what amount he will pay in the 30th instalment?
18. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

9.5 Geometric Progression (G. P.)

Let us consider the following sequences:

$$\text{(i) } 2,4,8,16,\ldots,\quad \text{(ii) } \frac{1}{9},\frac{-1}{27},\frac{1}{81},\frac{-1}{243}\ldots\quad \text{(iii) } .01,.0001,.000001,\ldots$$

In each of these sequences, how their terms progress? We note that each term, except the first progresses in a definite order.

In (i), we have $\frac{a_2}{a_1}=2, \frac{a_3}{a_2}=2, \frac{a_4}{a_3}=2$ and so on.

In (ii), we observe, $a_1=\frac{1}{9}, \frac{a_2}{a_1}=\frac{-1}{3}, \frac{a_3}{a_2}=\frac{-1}{3}, \frac{a_4}{a_3}=\frac{-1}{3}$ and so on.

Similarly, state how do the terms in (iii) progress? It is observed that in each case, every term except the first term bears a constant ratio to the term immediately preceding it. In (i), this constant ratio is 2; in (ii), it is $-\frac{1}{3}$ and in (iii), the constant ratio is 0.01. Such sequences are called *geometric sequence* or *geometric progression* abbreviated as G.P.

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called *geometric progression*, if each term is non-zero and $\frac{a_{k+1}}{a_k} = r$ (constant), for $k \geq 1$.

By letting $a_1 = a$, we obtain a geometric progression, $a, ar, ar^2, ar^3, \ldots$, where a is called the *first term* and r is called the *common ratio* of the G.P. Common ratio in geometric progression (i), (ii) and (iii) above are 2, $-\frac{1}{3}$ and 0.01, respectively.

As in case of arithmetic progression, the problem of finding the n^{th} term or sum of n terms of a geometric progression containing a large number of terms would be difficult without the use of the formulae which we shall develop in the next Section. We shall use the following notations with these formulae:

a = the first term, r = the common ratio, l = the last term,

n = the numbers of terms,

S_n = the sum of n terms.

9.5.1 General term of a G.P. Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'. Write a few terms of it. The second term is obtained by multiplying a by r, thus $a_2 = ar$. Similarly, third term is obtained by multiplying a_2 by r. Thus, $a_3 = a_2 r = ar^2$, and so on.

We write below these and few more terms.

1st term $= a_1 = a = ar^{1-1}$, 2nd term $= a_2 = ar = ar^{2-1}$, 3rd term $= a_3 = ar^2 = ar^{3-1}$
4th term $= a_4 = ar^3 = ar^{4-1}$, 5th term $= a_5 = ar^4 = ar^{5-1}$
Do you see a pattern? What will be 16th term?

$$a_{16} = ar^{16-1} = ar^{15}$$

Therefore, the pattern suggests that the n^{th} term of a G.P. is given by

$a_n = ar^{n-1}$.

Thus, a, G.P. can be written as $a, ar, ar^2, ar^3, \ldots ar^{n-1}$; $a, ar, ar^2, \ldots, ar^{n-1} \ldots$;according as G.P. is *finite* or *infinite*, respectively.

The series $a + ar + ar^2 + \ldots + ar^{n-1}$ or $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$are called *finite* or *infinite geometric series*, respectively.

9.5.2. Sum to n terms of a G.P. Let the first term of a G.P. be a and the common ratio be r. Let us denote by S_n the sum to first n terms of G.P. Then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad \ldots (1)$$

Case 1 If $r = 1$, we have $S_n = a + a + a + \ldots + a$ (n terms) $= na$

Case 2 If $r \neq 1$, multiplying (1) by r, we have

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \qquad \ldots (2)$$

Subtracting (2) from (1), we get $(1 - r)\, S_n = a - ar^n = a(1 - r^n)$

This gives $S_n = \dfrac{a(1-r^n)}{1-r}$ or $S_n = \dfrac{a(r^n-1)}{r-1}$

Example 9 Find the 10th and n^{th} terms of the G.P. 5, 25,125,… .

Solution Here $a = 5$ and $r = 5$. Thus, $a_{10} = 5(5)^{10-1} = 5(5)^9 = 5^{10}$
and $a_n = ar^{n-1} = 5(5)^{n-1} = 5^n$.

Example10 Which term of the G.P., 2,8,32, ... up to n terms is 131072?

Solution Let 131072 be the n^{th} term of the given G.P. Here $a = 2$ and $r = 4$.
Therefore $131072 = a_n = 2(4)^{n-1}$ or $65536 = 4^{n-1}$

This gives $4^8 = 4^{n-1}$.

So that $n - 1 = 8$, i.e., $n = 9$. Hence, 131072 is the 9th term of the G.P.

Example11 In a G.P., the 3rd term is 24 and the 6th term is 192.Find the 10th term.

Solution Here, $a_3 = ar^2 = 24$ ... (1)

and $a_6 = ar^5 = 192$ ... (2)

Dividing (2) by (1), we get $r = 2$. Substituting $r = 2$ in (1), we get $a = 6$.

Hence $a_{10} = 6\ (2)^9 = 3072$.

Example12 Find the sum of first n terms and the sum of first 5 terms of the geometric series $1+\frac{2}{3}+\frac{4}{9}+...$

Solution Here $a = 1$ and $r = \frac{2}{3}$. Therefore

$$S_n = \frac{a\,(1-r^n)}{1-r} = \frac{\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

In particular, $S_5 = 3\left[1-\left(\frac{2}{3}\right)^5\right] = 3\times\frac{211}{243} = \frac{211}{81}$.

Example 13 How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, ...$ are needed to give the sum $\frac{3069}{512}$?

Solution Let n be the number of terms needed. Given that $a = 3$, $r = \frac{1}{2}$ and $S_n = \frac{3069}{512}$

Since $$S_n = \frac{a\,(1-r^n)}{1-r}$$

Therefore $$\frac{3069}{512} = \frac{3(1-\frac{1}{2^n})}{1-\frac{1}{2}} = 6\left(1-\frac{1}{2^n}\right)$$

or $$\frac{3069}{3072} = 1-\frac{1}{2^n}$$

or $$\frac{1}{2^n} = 1-\frac{3069}{3072} = \frac{3}{3072} = \frac{1}{1024}$$

or $$2^n = 1024 = 2^{10}, \text{ which gives } n = 10.$$

Example 14 The sum of first three terms of a G.P. is $\frac{13}{12}$ and their product is – 1. Find the common ratio and the terms.

Solution Let $\frac{a}{r}$, a, ar be the first three terms of the G.P. Then

$$\frac{a}{r} + ar + a = \frac{13}{12} \qquad \text{... (1)}$$

and $$\left(\frac{a}{r}\right)(a)(ar) = -1 \qquad \text{... (2)}$$

From (2), we get $a^3 = -1$, i.e., $a = -1$ (considering only real roots)

Substituting $a = -1$ in (1), we have

$$-\frac{1}{r} - 1 - r = \frac{13}{12} \text{ or } 12r^2 + 25r + 12 = 0.$$

This is a quadratic in r, solving, we get $r = -\frac{3}{4}$ or $-\frac{4}{3}$.

Thus, the three terms of G.P. are : $\frac{4}{3}, -1, \frac{3}{4}$ for $r = \frac{-3}{4}$ and $\frac{3}{4}, -1, \frac{4}{3}$ for $r = \frac{-4}{3}$,

Example15 Find the sum of the sequence 7, 77, 777, 7777, ... to n terms.

Solution This is not a G.P., however, we can relate it to a G.P. by writing the terms as

$$S_n = 7 + 77 + 777 + 7777 + \ldots \text{ to } n \text{ terms}$$

$$= \frac{7}{9}[9 + 99 + 999 + 9999 + \ldots\text{to } n \text{ term}]$$

$$= \frac{7}{9}[(10-1) + (10^2-1) + (10^3-1) + (10^4-1) + \ldots n \text{ terms}]$$

$$= \frac{7}{9}[(10 + 10^2 + 10^3 + ...n \text{ terms}) - (1+1+1+...n \text{ terms})]$$

$$= \frac{7}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right] = \frac{7}{9}\left[\frac{10\,(10^n - 1)}{9} - n\right].$$

Example 16 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.

Solution Here $a = 2$, $r = 2$ and $n = 10$

Using the sum formula $\quad S_n = \dfrac{a\,(r^n - 1)}{r - 1}$

We have $\quad S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

9.5.3 ***Geometric Mean (G.M.)*** The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. Therefore, the geometric mean of 2 and 8 is 4. We observe that the three numbers 2,4,8 are consecutive terms of a G.P. This leads to a generalisation of the concept of geometric means of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P.

Let $G_1, G_2, ..., G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, G_3, ..., G_n, b$ is a G.P. Thus, b being the $(n + 2)^{th}$ term, we have

$$b = ar^{n+1}, \quad \text{or} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $\quad G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \quad G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example17 Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.

Solution Let G_1, G_2, G_3 be three numbers between 1 and 256 such that $1, G_1, G_2, G_3, 256$ is a G.P.

Therefore $\quad 256 = r^4$ giving $r = \pm 4$ (Taking real roots only)

For $r = 4$, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$, $G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are -4,16 and -64.

Hence, we can insert, 4, 16, 64 or -4, 16, -64, between 1 and 256 so that the resulting sequences are in G.P.

9.6 Relationship Between A.M. and G.M.

Let A and G be A.M. and G.M. of two given positive real numbers a and b, respectively. Then

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

Thus, we have

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \qquad \ldots (1)$$

From (1), we obtain the relationship $A \geq G$.

Example 18 If A.M. and G.M. of two positive numbers a and b are 10 and 8, respectively, find the numbers.

Solution Given that $\quad \text{A.M.} = \frac{a+b}{2} = 10 \qquad \ldots (1)$

and $\quad \text{G.M.} = \sqrt{ab} = 8 \qquad \ldots (2)$

From (1) and (2), we get

$$a + b = 20 \qquad \ldots (3)$$

$$ab = 64 \qquad \ldots (4)$$

Putting the value of a and b from (3), (4) in the identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a - b)^2 = 400 - 256 = 144$$

or $\quad a - b = \pm 12 \qquad \ldots (5)$

Solving (3) and (5), we obtain

$$a = 4, b = 16 \text{ or } a = 16, b = 4$$

Thus, the numbers a and b are 4, 16 or 16, 4 respectively.

EXERCISE 9.3

1. Find the 20th and nth terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$
2. Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.
3. The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.
4. The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term.
5. Which term of the following sequences:

   (a) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? (b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729 ?

   (c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?

6. For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10:

7. 0.15, 0.015, 0.0015, ... 20 terms.
8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ n terms.
9. $1, -a, a^2, -a^3, \ldots$ n terms (if $a \neq -1$).
10. $x^3, x^5, x^7, \ldots$ n terms (if $x \neq \pm 1$).
11. Evaluate $\sum_{k=1}^{11} (2 + 3^k)$.
12. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.
13. How many terms of G.P. $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?
14. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.
15. Given a G.P. with $a = 729$ and 7th term 64, determine S_7.
16. Find a G.P. for which sum of the first two terms is – 4 and the fifth term is 4 times the third term.
17. If the 4th, 10th and 16th terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

18. Find the sum to n terms of the sequence, 8, 88, 888, 8888… .

19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

20. Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and A, AR, AR^2, … AR^{n-1} form a G.P, and find the common ratio.

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that

$$a^{q-r}\, b^{r-p} c^{P-q} = 1.$$

23. If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

24. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

25. If a, b, c and d are in G.P. show that
$(a^2 + b^2 + c^2)\,(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

26. Insert two number between 3 and 81 so that the resulting sequence is G.P.

27. Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

28. The sum of two numbers is 6 times their geometric means, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

29. If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and n^{th} hour ?

31. What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

9.7 Sum to *n* Terms of Special Series

We shall now find the sum of first n terms of some special series, namely;

(i) $1 + 2 + 3 + \ldots + n$ (sum of first n natural numbers)

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2$(sum of squares of the first n natural numbers)

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3$(sum of cubes of the first n natural numbers).

Let us take them one by one.

(i) $S_n = 1 + 2 + 3 + \ldots + n$, then $S_n = \dfrac{n(n+1)}{2}$ (See Section 9.4)

(ii) Here $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$

We consider the identity $k^3 - (k-1)^3 = 3k^2 - 3k + 1$

Putting $k = 1, 2\ldots$ successively, we obtain

$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$
$$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$$
$$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$$

Adding both sides, we get

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$$

$$n^3 = 3\sum_{k-1}^{n} k^2 - 3\sum_{k-1}^{n} k + n$$

By (i), we know that $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Hence $\displaystyle S_n = \sum_{k=1}^{n} k^2 = \frac{1}{3}\left[n^3 + \frac{3n(n+1)}{2} - n\right] = \frac{1}{6}(2n^3 + 3n^2 + n)$

$$= \frac{n(n+1)(2n+1)}{6}$$

(iii) Here $S_n = 1^3 + 2^3 + \ldots + n^3$

We consider the identity, $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$

Putting $k = 1, 2, 3\ldots n$, we get

$2^4 - 1^4 = 4(1)^3 + 6(1)^2 + 4(1) + 1$

$3^4 - 2^4 = 4(2)^3 + 6(2)^2 + 4(2) + 1$

$4^4 - 3^4 = 4(3)^3 + 6(3)^2 + 4(3) + 1$

....

....

....

$(n-1)^4 - (n-2)^4 = 4(n-2)^3 + 6(n-2)^2 + 4(n-2) + 1$

$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$

$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$

Adding both sides, we get

$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + 3^3 + ... + n^3) + 6(1^2 + 2^2 + 3^2 + ... + n^2) +$
$4(1 + 2 + 3 + ... + n) + n$

$$= 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n \qquad ... (1)$$

From parts (i) and (ii), we know that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting these values in equation (1), we obtain

$$4\sum_{k=1}^{n} k^3 = n^4 + 4n^3 + 6n^2 + 4n - \frac{6n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} - n$$

or $\qquad 4S_n = n^4 + 4n^3 + 6n^2 + 4n - n(2n^2 + 3n + 1) - 2n(n+1) - n$

$\qquad = n^4 + 2n^3 + n^2$

$\qquad = n^2(n+1)^2.$

Hence, $\qquad S_n = \dfrac{n^2(n+1)^2}{4} = \dfrac{[n(n+1)]^2}{4}$

Example 19 Find the sum to n terms of the series: $5 + 11 + 19 + 29 + 41\ldots$

Solution Let us write

$$S_n = 5 + 11 + 19 + 29 + ... + a_{n-1} + a_n$$

or $\qquad S_n = \qquad 5 + 11 + 19 + ... + a_{n-2} + a_{n-1} + a_n$

On subtraction, we get

$$0 = 5 + [6 + 8 + 10 + 12 + ...(n - 1) \text{ terms}] - a_n$$

or $$a_n = 5 + \frac{(n-1)[12+(n-2)\times 2]}{2}$$

$$= 5 + (n - 1)(n + 4) = n^2 + 3n + 1$$

Hence $$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(k^2 + 3k + 1) = \sum_{k=1}^{n} k^2 + 3\sum_{1}^{n} k + n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n = \frac{n(n+2)(n+4)}{3}.$$

Example 20 Find the sum to n terms of the series whose n^{th} term is $n(n+3)$.

Solution Given that $a_n = n(n + 3) = n^2 + 3n$

Thus, the sum to n terms is given by

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} = \frac{n(n+1)(n+5)}{3}.$$

EXERCISE 9.4

Find the sum to n terms of each of the series in Exercises 1 to 7.

1. $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$
2. $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + ...$
3. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + ...$
4. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ...$
5. $5^2 + 6^2 + 7^2 + ... + 20^2$
6. $3 \times 8 + 6 \times 11 + 9 \times 14 + ...$
7. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$

Find the sum to n terms of the series in Exercises 8 to 10 whose n^{th} terms is given by

8. $n(n+1)(n+4)$.
9. $n^2 + 2^n$
10. $(2n - 1)^2$

Miscellaneous Examples

Example21 If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P, then show that $(p - q)$, $(q - r)$, $(r - s)$ are also in G.P.

Solution Here

$$a_p = a + (p - 1)\, d \qquad \ldots (1)$$

$$a_q = a + (q - 1)\, d \qquad \ldots (2)$$

$$a_r = a + (r - 1)\, d \qquad \ldots (3)$$

$$a_s = a + (s - 1)\, d \qquad \ldots (4)$$

Given that a_p, a_q, a_r and a_s are in G.P.,

So $$\frac{a_q}{a_p} = \frac{a_r}{a_q} = \frac{a_q - a_r}{a_p - a_q} = \frac{q - r}{p - q} \text{ (why ?)} \qquad \ldots (5)$$

Similarly $$\frac{a_r}{a_q} = \frac{a_s}{a_r} = \frac{a_r - a_s}{a_q - a_r} = \frac{r - s}{q - r} \text{ (why ?)} \qquad \ldots (6)$$

Hence, by (5) and (6)

$$\frac{q - r}{p - q} = \frac{r - s}{q - r}$$, i.e., $p - q$, $q - r$ and $r - s$ are in G.P.

Example 22 If a, b, c are in G.P. and $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, prove that x, y, z are in A.P.

Solution Let $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k$ Then

$$a = k^x,\ b = k^y \text{ and } c = k^z. \qquad \ldots (1)$$

Since a, b, c are in G.P., therefore,

$$b^2 = ac \qquad \ldots (2)$$

Using (1) in (2), we get

$k^{2y} = k^{x+z}$, which gives $2y = x + z$.

Hence, x, y and z are in A.P.

Example 23 If a, b, c, d and p are different real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \le 0$, then show that a, b, c and d are in G.P.

Solution Given that

$$(a^2 + b^2 + c^2)\, p^2 - 2\,(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \le 0 \qquad \ldots (1)$$

But L.H.S.

$= (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2)$,

which gives $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \geq 0$ ... (2)

Since the sum of squares of real numbers is non negative, therefore, from (1) and (2), we have, $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$

or $ap - b = 0,\ bp - c = 0,\ cp - d = 0$

This implies that $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = p$

Hence a, b, c and d are in G.P.

Example 24 If p,q,r are in G.P. and the equations, $px^2 + 2qx + r = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in A.P.

Solution The equation $px^2 + 2qx + r = 0$ has roots given by

$$x = \frac{-2q \pm \sqrt{4q^2 - 4rp}}{2p}$$

Since p, q, r are in G.P. $q^2 = pr$. Thus $x = \frac{-q}{p}$ but $\frac{-q}{p}$ is also root of $dx^2 + 2ex + f = 0$ (Why ?). Therefore

$$d\left(\frac{-q}{p}\right)^2 + 2e\left(\frac{-q}{p}\right) + f = 0,$$

or $dq^2 - 2eqp + fp^2 = 0$ ... (1)

Dividing (1) by pq^2 and using $q^2 = pr$, we get

$$\frac{d}{p} - \frac{2e}{q} + \frac{fp}{pr} = 0, \text{ or } \frac{2e}{q} = \frac{d}{p} + \frac{f}{r}$$

Hence $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in A.P.

Miscellaneous Exercise On Chapter 9

1. Show that the sum of $(m + n)^{th}$ and $(m - n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.
2. If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.
3. Let the sum of n, $2n$, $3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3(S_2 - S_1)$
4. Find the sum of all numbers between 200 and 400 which are divisible by 7.
5. Find the sum of integers from 1 to 100 that are divisible by 2 or 5.
6. Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.
7. If f is a function satisfying $f(x+y) = f(x)\,f(y)$ for all $x, y \in \mathbf{N}$ such that

   $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.
8. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.
9. The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.
10. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.
11. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.
12. The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.
13. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} (x \neq 0)$, then show that a, b, c and d are in G.P.
14. Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$.
15. The p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c, respectively. Show that
$$(q - r)a + (r - p)b + (p - q)c = 0$$
16. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.
17. If a, b, c, d are in G.P, prove that $(a^n + b^n)$, $(b^n + c^n)$, $(c^n + d^n)$ are in G.P.
18. If a and b are the roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d form a G.P. Prove that $(q + p) : (q - p) = 17:15$.

19. The ratio of the A.M. and G.M. of two positive numbers a and b, *is* $m : n$. Show that $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

20. If a, b, c are in A.P.; b, c, d are in G.P. and $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P. prove that a, c, e are in G.P.

21. Find the sum of the following series up to n terms:
(i) $5 + 55 + 555 + \ldots$ (ii) $.6 +. 66 +. 666+\ldots$

22. Find the 20th term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + ... + n$ terms.

23. Find the sum of the first n terms of the series: $3+ 7 +13 +21 +31 +\ldots$

24. If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3 (1 + 8S_1)$.

25. Find the sum of the following series up to n terms:

$$\frac{1^3}{1}+\frac{1^3+2^2}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$$

26. Show that $$\frac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)}=\frac{3n+5}{3n+1}.$$

27. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

28. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

29. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed.

30. A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years.

31. A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

32. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on.

It took 8 more days to finish the work. Find the number of days in which the work was completed.

Summary

◆ By a *sequence*, we mean an arrangement of a number in a definite order according to some rule. Also, we define a sequence as a function whose domain is the set of natural numbers or some subsets of the type {1, 2, 3....*k*). A sequence containing a finite number of terms is called a *finite sequence*. A sequence is called *infinite* if it is not a finite sequence.

◆ Let $a_1, a_2, a_3, ...$ be the sequence, then the sum expressed as $a_1 + a_2 + a_3 + ...$ is called *series*. A series is called *finite series* if it has got finite number of terms.

◆ An arithmetic progression (A.P.) is a sequence in which terms increase or decrease regularly by the same constant. This constant is called *common difference of the* A.P. Usually, we denote the first terms of A.P. by a, the common difference by d and the last term by l. The *general term* or the n^{th} *term* of the A.P. is given by $a_n = a + (n - 1)\,d$.

The sum S_n of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a+l).$$

◆ The *arithmetic mean A* of any two numbers a and b is given by $\frac{a+b}{2}$ i.e., the sequence a, A, b is in A.P.

◆ A sequence is said to be a *geometric progression* or *G.P.*, if the ratio of any term to its preceding term is same throughout. This constant factor is called the *common ratio*. Usually, we denote the first term of a G.P. by a and its common ratio by r. The general or the n^{th} term of G.P. is given by $a_n = ar^{n-1}$.

The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1} \text{ or } \frac{a\left(1 - r^n\right)}{1 - r}, \text{ if } r \neq 1$$

- The geometric mean (G.M.) of any two positive numbers a and b is given by $\sqrt{ab}$ i.e., the sequence a, G, b is G.P.

Historical Note

Evidence is found that Babylonians, some 4000 years ago, knew of arithmetic and geometric sequences. According to Boethius (510 A.D.), arithmetic and geometric sequences were known to early Greek writers. Among the Indian mathematician, Aryabhatta (476 A.D.) was the first to give the formula for the sum of squares and cubes of natural numbers in his famous work Aryabhatiyam, written around 499 A.D. He also gave the formula for finding the sum to n terms of an arithmetic sequence starting with p^{th} term. Noted Indian mathematicians Brahmgupta (598 A.D.), Mahavira (850 A.D.) and Bhaskara (1114-1185 A.D.) also considered the sum of squares and cubes. Another specific type of sequence having important applications in mathematics, called *Fibonacci sequence*, was discovered by Italian mathematician Leonardo Fibonacci (1170-1250 A.D.). Seventeenth century witnessed the classification of series into specific forms. In 1671 A.D. James Gregory used the term infinite series in connection with infinite sequence. It was only through the rigorous development of algebraic and set theoretic tools that the concepts related to sequence and series could be formulated suitably.

❖

Chapter 10

STRAIGHT LINES

❖ Geometry, as a logical system, is a means and even the most powerful means to make children feel the strength of the human spirit that is of their own spirit. – H. FREUDENTHAL ❖

10.1 Introduction

René Descartes
(1596 1650)

We are familiar with two-dimensional *coordinate geometry* from earlier classes. Mainly, it is a combination of *algebra* and *geometry*. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes, in his book 'La Géométry, published in 1637. This book introduced the notion of the equation of a curve and related analytical methods into the study of geometry. The resulting combination of analysis and geometry is referred now as *analytical geometry.* In the earlier classes, we initiated the study of coordinate geometry, where we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, section formule, etc. All these concepts are the basics of coordinate geometry.

Let us have a brief recall of coordinate geometry done in earlier classes. To recapitulate, the location of the points (6, – 4) and (3, 0) in the XY-plane is shown in Fig 10.1.

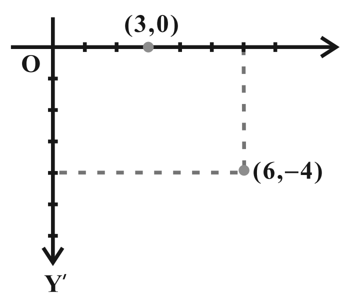


Fig 10.1

We may note that the point (6, – 4) is at 6 units distance from the y-axis measured along the positive x-axis and at 4 units distance from the x-axis measured along the negative y-axis. Similarly, the point (3, 0) is at 3 units distance from the y-axis measured along the positive x-axis and has zero distance from the x-axis.

We also studied there following important formulae:

I. Distance between the points P (x_1, y_1) and Q (x_2, y_2) is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, distance between the points (6, – 4) and (3, 0) is

$$\sqrt{(3-6)^2 + (0+4)^2} = \sqrt{9+16} = 5 \text{ units.}$$

II. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally, in the ratio $m: n$ are $\left(\dfrac{m x_2 + n x_1}{m+n}, \dfrac{m y_2 + n y_1}{m+n}\right)$.

For example, the coordinates of the point which divides the line segment joining A (1, –3) and B (–3, 9) internally, in the ratio 1: 3 are given by $x = \dfrac{1.(-3) + 3.1}{1+3} = 0$

and $y = \dfrac{1.9 + 3.(-3)}{1+3} = 0.$

III. In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

IV. Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right| .$$

For example, the area of the triangle, whose vertices are (4, 4), (3, – 2) and (– 3, 16) is

$$\frac{1}{2}\left|4(-2-16) + 3(16-4) + (-3)(4+2)\right| = \frac{|-54|}{2} = 27.$$

Remark If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

In the this Chapter, we shall continue the study of coordinate geometry to study properties of the simplest geometric figure – *straight line.* Despite its simplicity, the line is a vital concept of geometry and enters into our daily experiences in numerous interesting and useful ways. Main focus is on representing the line algebraically, for which *slope* is most essential.

10.2 Slope of a Line

A line in a coordinate plane forms two angles with the x-axis, which are supplementary.

The angle (say) θ made by the line l with positive direction of x-axis and measured anti clockwise is called the *inclination of the line*. Obviously $0° \leq \theta \leq 180°$ (Fig 10.2).

We observe that lines parallel to x-axis, or coinciding with x-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with y-axis) is 90°.

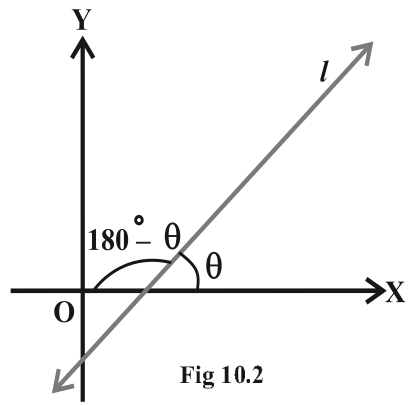


Fig 10.2

Definition 1 If θ is the inclination of a line l, then tan θ is called the *slope* or *gradient* of the line l.

The slope of a line whose inclination is 90° is not defined.

The slope of a line is denoted by m.

Thus, $m = \tan\theta$, $\theta \neq 90°$

It may be observed that the slope of x-axis is zero and slope of y-axis is not defined.

10.2.1 ***Slope of a line when coordinates of any two points on the line are given***

We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on non-vertical line l whose inclination is θ. Obviously, $x_1 \neq x_2$, otherwise the line will become perpendicular to x-axis and its slope will not be defined. The inclination of the line l may be acute or obtuse. Let us take these two cases.

Draw perpendicular QR to x-axis and PM perpendicular to RQ as shown in Figs. 10.3 (i) and (ii).

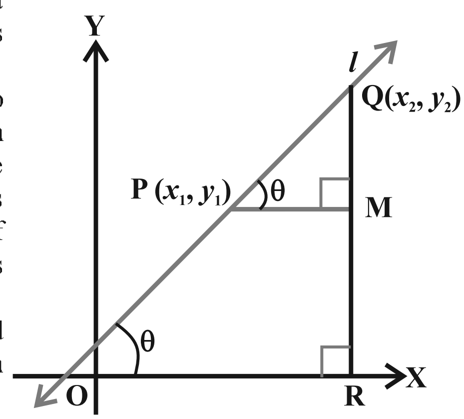


Fig 10. 3 (i)

Case 1 When angle θ is acute:

In Fig 10.3 (i), $\angle MPQ = \theta$. ... (1)

Therefore, slope of line $l = m = \tan\theta$.

But in ΔMPQ, we have $\tan\theta = \dfrac{MQ}{MP} = \dfrac{y_2 - y_1}{x_2 - x_1}$. ... (2)

From equations (1) and (2), we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

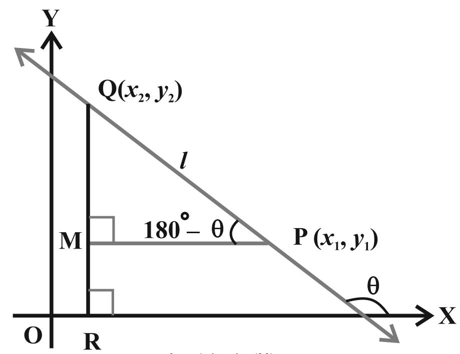


Fig 10. 3 (ii)

Case II When angle θ is obtuse:
In Fig 10.3 (ii), we have
$\angle$MPQ = 180° – θ.

Therefore, θ = 180° – $\angle$MPQ.

Now, slope of the line l

$$m = \tan\theta$$
$$= \tan(180° - \angle \text{MPQ}) = -\tan \angle \text{MPQ}$$
$$= -\frac{\text{MQ}}{\text{MP}} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Consequently, we see that in both the cases the slope m of the line through the points

(x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

10.2.2 ***Conditions for parallelism and perpendicularity of lines in terms of their slopes*** In a coordinate plane, suppose that non-vertical *lines* l_1 and l_2 have slopes m_1 *and* m_2, respectively. Let their inclinations be α and β, respectively.

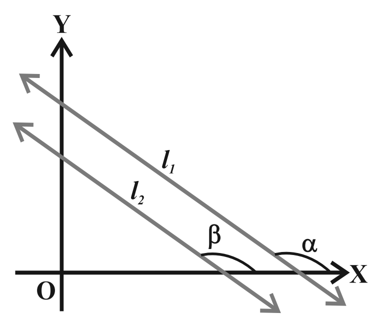


Fig 10. 4

If the line l_1 is parallel to l_2 (Fig 10.4), then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan\alpha = \tan\beta$$

Therefore $m_1 = m_2$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

$$m_1 = m_2.$$

Then $\tan\alpha = \tan\beta$.

By the property of tangent function (between 0° and 180°), α = β.

Therefore, the lines are parallel.

Hence, two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

If the lines l_1 and l_2 are perpendicular (Fig 10.5), then $\beta = \alpha + 90°$.

Therefore, $\tan\beta = \tan(\alpha + 90°)$

$$= -\cot\alpha = -\frac{1}{\tan\alpha}$$

i.e., $m_2 = -\frac{1}{m_1}$ or $m_1 m_2 = -1$

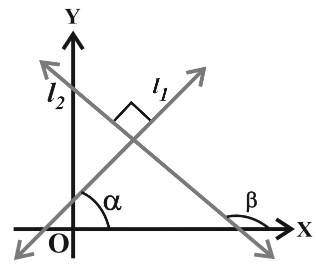


Fig 10. 5

Conversely, if $m_1 m_2 = -1$, i.e., $\tan\alpha \tan\beta = -1$.

Then $\tan\alpha = -\cot\beta = \tan(\beta + 90°)$ or $\tan(\beta - 90°)$

Therefore, α and β differ by 90°.

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

i.e., $m_2 = -\frac{1}{m_1}$ or, $m_1 m_2 = -1$.

Let us consider the following example.

Example 1 Find the slope of the lines:

(a) Passing through the points (3, – 2) and (–1, 4),

(b) Passing through the points (3, – 2) and (7, – 2),

(c) Passing through the points (3, – 2) and (3, 4),

(d) Making inclination of 60° with the positive direction of x-axis.

Solution (a) The slope of the line through (3, – 2) and (– 1, 4) is

$$m = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}.$$

(b) The slope of the line through the points (3, – 2) and (7, – 2) is

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0.$$

(c) The slope of the line through the points (3, – 2) and (3, 4) is

$$m = \frac{4-(-2)}{3-3} = \frac{6}{0}, \text{ which is not defined.}$$

(d) Here inclination of the line $\alpha = 60°$. Therefore, slope of the line is

$$m = \tan 60° = \sqrt{3}.$$

10.2.3 ***Angle between two lines*** When we think about more than one line in a plane, then we find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2, respectively. Then

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2.$$

We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Fig10.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90°.$$

Therefore $\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1 \tan\alpha_2} = \dfrac{m_2 - m_1}{1+m_1 m_2}$ (as $1 + m_1 m_2 \neq 0$)

and $\phi = 180° - \theta$ so that

$$\tan\phi = \tan(180° - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1+m_1 m_2}, \text{ as } 1 + m_1 m_2 \neq 0$$

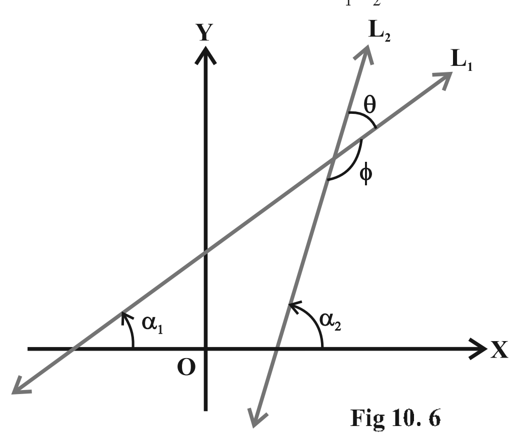


Fig 10. 6

Now, there arise two cases:

Case I If $\dfrac{m_2 - m_1}{1 + m_1 m_2}$ is positive, then tan θ will be positive and tan φ will be negative, which means θ will be acute and φ will be obtuse.

Case II If $\dfrac{m_2 - m_1}{1 + m_1 m_2}$ is negative, then tan θ will be negative and tan φ will be positive, which means that θ will be obtuse and φ will be acute.

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \text{ as } \quad 1 + m_1 m_2 \neq 0 \qquad \text{... (1)}$$

The obtuse angle (say φ) can be found by using $\phi = 180^0 - \theta$.

Example 2 If the angle between two lines is $\dfrac{\pi}{4}$ and slope of one of the lines is $\dfrac{1}{2}$, find the slope of the other line.

Solution We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \text{... (1)}$$

Let $m_1 = \dfrac{1}{2}$, $m_2 = m$ and $\theta = \dfrac{\pi}{4}$.

Now, putting these values in (1), we get

$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \quad \text{or} \quad 1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|,$$

which gives

$$\frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = 1 \quad \text{or} \quad -\frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = -1.$$

Therefore $\quad m = 3 \quad$ or $\quad m = -\dfrac{1}{3}$.

Hence, slope of the other line is 3 or $-\frac{1}{3}$. Fig 10.7 explains the reason of two answers.

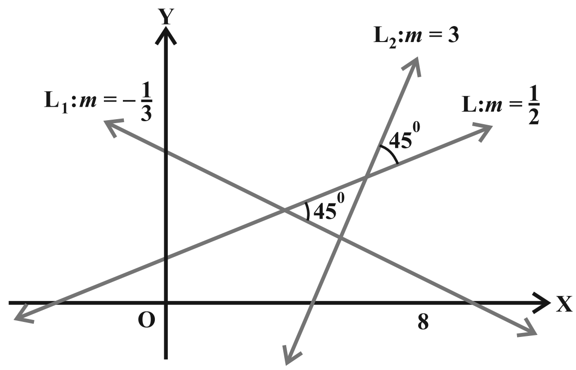


Fig 10. 7

Example 3 Line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

Solution Slope of the line through the points (– 2, 6) and (4, 8) is

$$m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Slope of the line through the points (8, 12) and (x, 24) is

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since two lines are perpendicular,
$m_1 m_2 = -1$, which gives

$$\frac{1}{3} \times \frac{12}{x-8} = -1 \text{ or } x = 4.$$

10.2.4 ***Collinearity of three points*** We know that slopes of two parallel lines are equal. If two lines having the same slope pass through a common point, then two lines will coincide. Hence, if A, B and C are three points in the XY-plane, then they will lie on a line, i.e., three points are collinear (Fig 10.8) if and only if slope of AB = slope of BC.

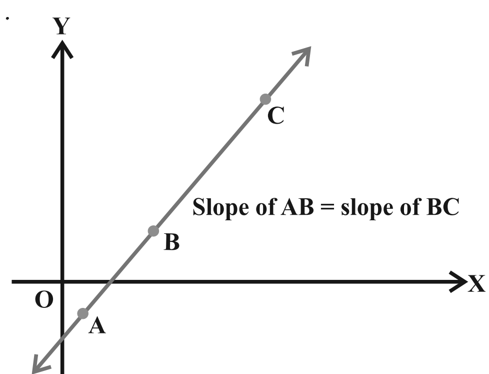


Fig 10. 8

Example 4 Three points P (h, k), Q (x_1, y_1) and R (x_2, y_2) lie on a line. Show that

$$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

Solution Since points P, Q and R are collinear, we have

$$\text{Slope of PQ} = \text{Slope of QR, i.e., } \frac{y_1 - k}{x_1 - h} = \frac{y_2 - y_1}{x_2 - x_1}$$

or $$\frac{k - y_1}{h - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

or $$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

Example 5 In Fig 10.9, time and distance graph of a linear motion is given. Two positions of time and distance are recorded as, when T = 0, D = 2 and when T = 3, D = 8. Using the concept of slope, find law of motion, i.e., how distance depends upon time.

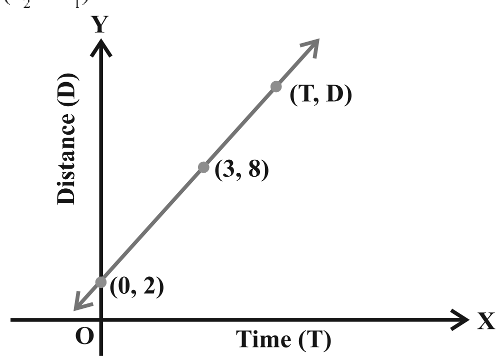


Fig 10.9

Solution Let (T, D) be any point on the line, where D denotes the distance at time T. Therefore, points (0, 2), (3, 8) and (T, D) are collinear so that

$$\frac{8-2}{3-0} = \frac{\text{D}-8}{\text{T}-3} \quad \text{or} \quad 6(\text{T}-3) = 3(\text{D}-8)$$

or $\text{D} = 2(\text{T} + 1)$,

which is the required relation.

EXERCISE 10.1

1. Draw a quadrilateral in the Cartesian plane, whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, –2). Also, find its area.
2. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.
3. Find the distance between P (x_1, y_1) and Q (x_2, y_2) when : (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.
4. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).
5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0).

6. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle.
7. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.
8. Find the value of x for which the points (x, – 1), (2, 1) and (4, 5) are collinear.
9. Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.
10. Find the angle between the x-axis and the line joining the points (3,–1) and (4,–2).
11. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
12. A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that
$$k - y_1 = m\,(h - x_1).$$
13. If three points $(h, 0)$, (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
14. Consider the following population and year graph (Fig 10.10), find the slope of the line AB and using it, find what will be the population in the year 2010?

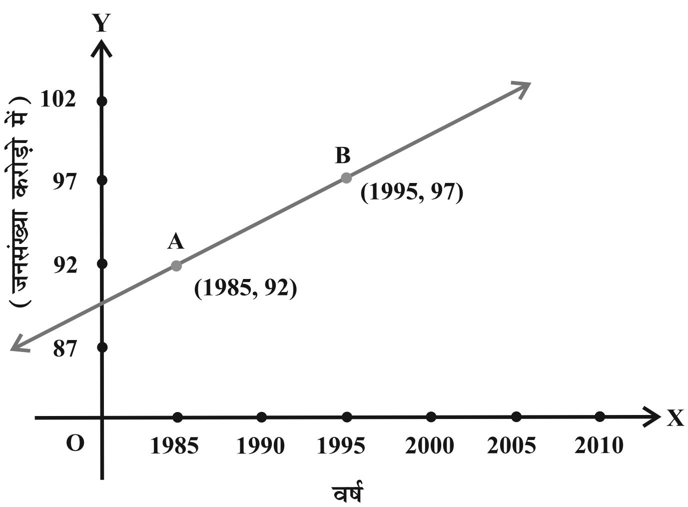


Fig 10.10

10.3 Various Forms of the Equation of a Line

We know that every line in a plane contains infinitely many points on it. This relationship between line and points leads us to find the solution of the following problem:

How can we say that a given point lies on the given line? Its answer may be that for a given line we should have a definite condition on the points lying on the line. Suppose P (x, y) is an arbitrary point in the XY-plane and L is the given line. For the equation of L, we wish to construct a *statement* or *condition* for the point P that is true, when P is on L, otherwise false. Of course the statement is merely an algebraic equation involving the variables x and y. Now, we will discuss the equation of a line under different conditions.

10.3.1 *Horizontal and vertical lines* If a horizontal line L is at a distance a from the x-axis then ordinate of every point lying on the line is either a or $-a$ [Fig 10.11 (a)]. Therefore, equation of the line L is either $y = a$ or $y = -a$. Choice of sign will depend upon the position of the line according as the line is above or below the y-axis. Similarly, the equation of a vertical line at a distance b from the x-axis is either $x = b$ or $x = -b$ [Fig 10.11(b)].

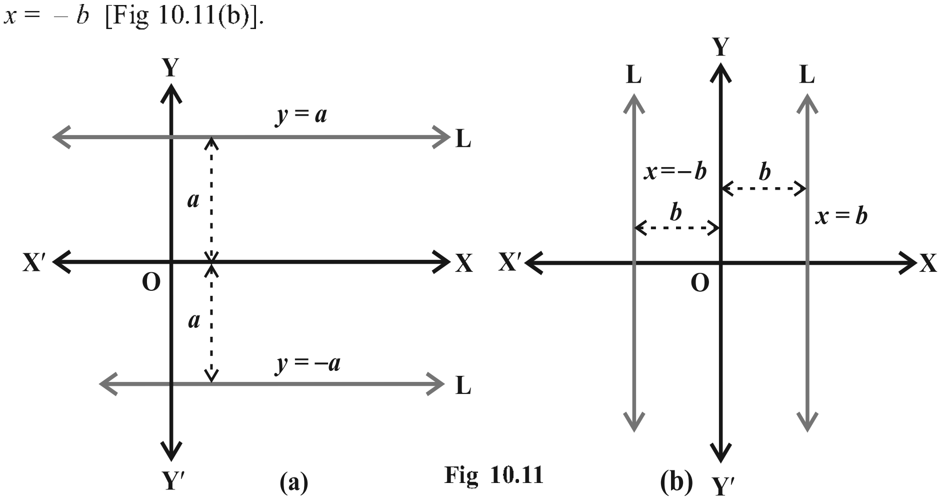


Fig 10.11

Example 6 Find the equations of the lines parallel to axes and passing through (– 2, 3).

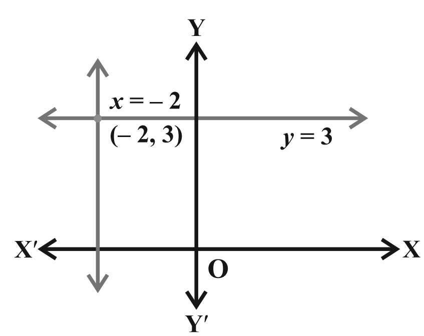


Fig 10.12

Solution Position of the lines is shown in the Fig 10.12. The y-coordinate of every point on the line parallel to x-axis is 3, therefore, equation of the line parallel to x-axis and passing through (– 2, 3) is $y = 3$. Similarly, equation of the line parallel to y-axis and passing through (– 2, 3) is $x = -2$.

10.3.2 *Point-slope form* Suppose that P_0 (x_0, y_0) is a fixed point on a non-vertical line L, whose slope is m. Let P (x, y) be an arbitrary point on L (Fig 10.13).

Then, by the definition, the slope of L is given by

$$m = \frac{y - y_0}{x - x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \quad ...(1)$$

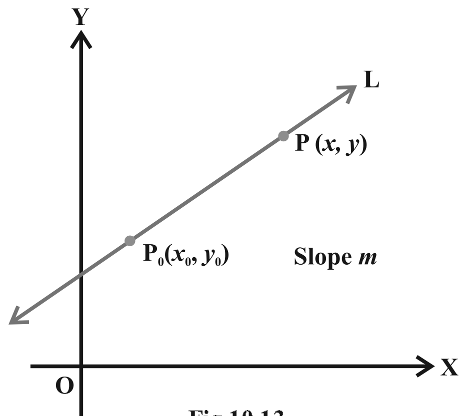


Fig 10.13

Since the point P_0 (x_0, y_0) along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L.

Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), if and only if, its coordinates satisfy the equation

$$y - y_0 = m\ (x - x_0)$$

Example 7 Find the equation of the line through (– 2, 3) with slope – 4.

Solution Here $m = -4$ and given point (x_0, y_0) is (– 2, 3).

By slope-intercept form formula (1) above, equation of the given line is

$y - 3 = -4\ (x + 2)$ or $4x + y + 5 = 0$, which is the required equation.

10.3.3 *Two-point form* Let the line L passes through two given points P_1 (x_1, y_1) and P_2 (x_2, y_2). Let P (x, y) be a general point on L (Fig 10.14).

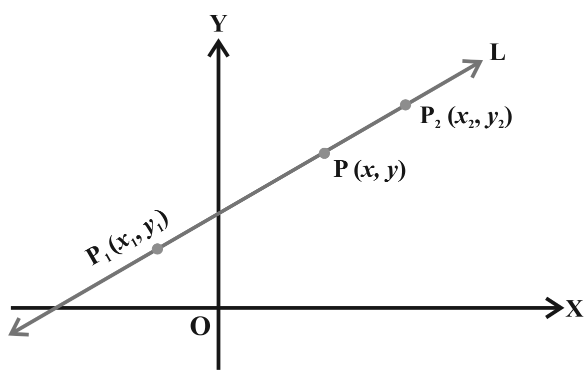


Fig 10.14

The three points P_1, P_2 and P are collinear, therefore, we have

slope of P_1P = slope of P_1P_2

$$\text{i.e.,} \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{or} \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Thus, equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{... (2)}$$

Example 8 Write the equation of the line through the points (1, –1) and (3, 5).

Solution Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5 - (-1)}{3 - 1}(x - 1)$$

or $\qquad -3x + y + 4 = 0$, which is the required equation.

10.3.4 ***Slope-intercept form*** Sometimes a line is known to us with its slope and an intercept on one of the axes. We will now find equations of such lines.

Case I Suppose a line L with slope m cuts the y-axis at a distance c from the origin (Fig10.15). The distance c is called the *y-intercept* of the line L. Obviously, coordinates of the point where the line meet the y-axis are $(0, c)$. Thus, L has slope m and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

$$y - c = m(x - 0) \quad \text{or} \quad y = mx + c$$

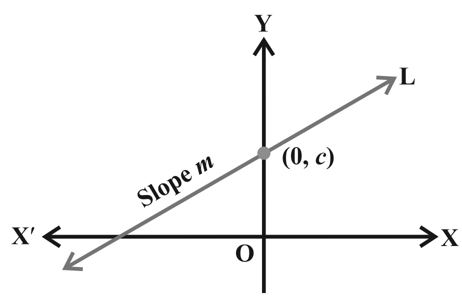


Fig 10.15

Thus, the point (x, y) on the line with slope m and y-intercept c lies on the line if and only if

$$y = mx + c \qquad \text{...(3)}$$

Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y-axis, respectively.

Case II Suppose line L with slope m makes x-intercept d. Then equation of L is

$$y = m(x - d) \qquad \text{... (4)}$$

Students may derive this equation themselves by the same method as in Case I.

Example 9 Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and (i) y-intercept is $-\frac{3}{2}$ (ii) x-intercept is 4.

Solution (i) Here, slope of the line is $m = \tan\theta = \frac{1}{2}$ and y - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \text{ or } 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan\theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x - 4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

10.3.5 ***Intercept - form*** Suppose a line L makes x-intercept a and y-intercept b on the axes. Obviously L meets x-axis at the point $(a, 0)$ and y-axis at the point $(0, b)$ (Fig.10.16). By two-point form of the equation of the line, we have

$$y - 0 = \frac{b-0}{0-a}(x-a) \quad \text{or} \quad ay = -bx + ab,$$

i.e., $$\frac{x}{a} + \frac{y}{b} = 1.$$

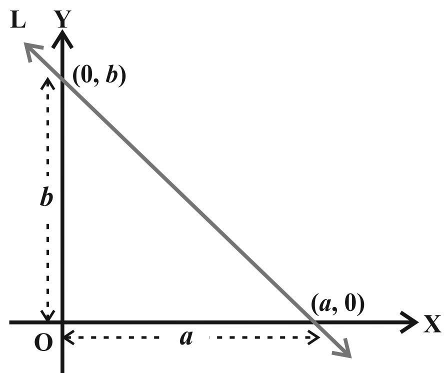


Fig 10.16

Thus, equation of the line making intercepts a and b on x-and y-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{... (5)}$$

Example 10 Find the equation of the line, which makes intercepts –3 and 2 on the x- and y-axes respectively.

Solution Here $a = -3$ and $b = 2$. By intercept form (5) above, equation of the line is

$$\frac{x}{-3} + \frac{y}{2} = 1 \quad \text{or} \quad 2x - 3y + 6 = 0.$$

10.3.6 *Normal form* Suppose a non-vertical line is known to us with following data:

(i) Length of the perpendicular (normal) from origin to the line.

(ii) Angle which normal makes with the positive direction of x-axis.

Let L be the line, whose perpendicular distance from origin O be OA $= p$ and the angle between the positive x-axis and OA be $\angle XOA = \omega$. The possible positions of line L in the Cartesian plane are shown in the Fig 10.17. Now, our purpose is to find slope of L and a point on it. Draw perpendicular AM on the x-axis in each case.

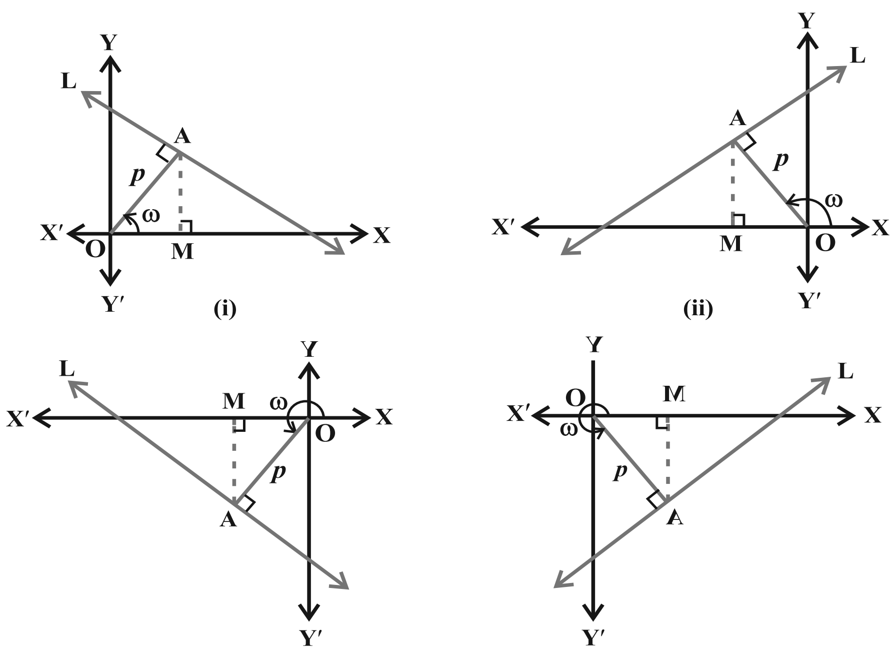


Fig 10.17

In each case, we have OM $= p \cos \omega$ and MA $= p \sin \omega$, so that the coordinates of the point A are $(p \cos \omega, p \sin \omega)$.

Further, line L is perpendicular to OA. Therefore

$$\text{The slope of the line L} = -\frac{1}{\text{slope of OA}} = -\frac{1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

Thus, the line L has slope $-\dfrac{\cos \omega}{\sin \omega}$ and point A $\left(p \cos \omega, p \sin \omega\right)$ on it. Therefore, by point-slope form, the equation of the line L is

$$y - p\sin\omega = -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega) \quad \text{or} \quad x\cos\omega + y\sin\omega = p(\sin^2\omega + \cos^2\omega)$$

or $$x\cos\omega + y\sin\omega = p.$$

Hence, the equation of the line having normal distance p from the origin and angle ω which the normal makes with the positive direction of x-axis is given by

$$x\cos\omega + y\sin\omega = p \qquad \text{... (6)}$$

Example 11 Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15°.

Solution Here, we are given $p = 4$ and $\omega = 15^0$ (Fig10.18).

Now $$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

and $$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{(Why?)}$$

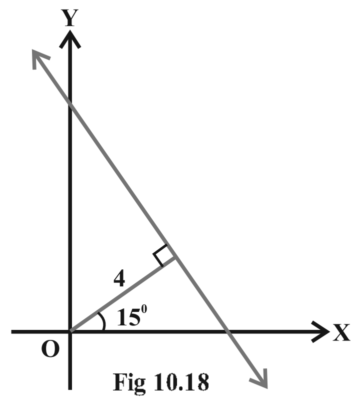


Fig 10.18

By the normal form (6) above, the equation of the line is

$$x\cos 15^0 + y\sin 15^0 = 4 \text{ or } \frac{\sqrt{3}+1}{2\sqrt{2}}x + \frac{\sqrt{3}-1}{2\sqrt{2}}y = 4 \quad \text{or} \quad \left(\sqrt{3}+1\right)x + \left(\sqrt{3}-1\right)y = 8\sqrt{2}.$$

This is the required equation.

Example 12 The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F, when K = 0.

Solution Assuming F along x-axis and K along y-axis, we have two points (32, 273) and (212, 373) in XY-plane. By two-point form, the point (F, K) satisfies the equation

$$\mathrm{K} - 273 = \frac{373-273}{212-32}(\mathrm{F}-32) \quad \text{or} \quad \mathrm{K} - 273 = \frac{100}{180}(\mathrm{F}-32)$$

or $$\mathrm{K} = \frac{5}{9}(\mathrm{F}-32) + 273 \qquad \text{... (1)}$$

which is the required relation.

When K = 0, Equation (1) gives

$$0 = \frac{5}{9}(F - 32) + 273 \quad \text{or} \quad F - 32 = -\frac{273 \times 9}{5} = -\ 491.4 \quad \text{or} \quad F = -459.4.$$

Alternate method We know that simplest form of the equation of a line is $y = mx + c$. Again assuming F along *x-axis* and K along *y-axis*, we can take equation in the form

$$K = mF + c \qquad \dots (1)$$

Equation (1) is satisfied by (32, 273) and (212, 373). Therefore

$$273 = 32m + c \qquad \dots (2)$$

and

$$373 = 212m + c \qquad \dots (3)$$

Solving (2) and (3), we get

$$m = \frac{5}{9} \text{ and } c = \frac{2297}{9}.$$

Putting the values of m and c in (1), we get

$$K = \frac{5}{9}F + \frac{2297}{9} \qquad \dots (4)$$

which is the required relation. When K = 0, (4) gives F = – 459.4.

Note We know, that the equation $y = mx + c$, contains two constants, namely, m and c. For finding these two constants, we need two conditions satisfied by the equation of line. In all the examples above, we are given two conditions to determine the equation of the line.

EXERCISE 10.2

In Exercises 1 to 8, find the equation of the line which satisfy the given conditions:

1. Write the equations for the x-and y-axes.
2. Passing through the point (– 4, 3) with slope $\frac{1}{2}$.
3. Passing through (0, 0) with slope m.
4. Passing through $(2, 2\sqrt{3})$ and inclined with the x-axis at an angle of 75°.
5. Intersecting the x-axis at a distance of 3 units to the left of origin with slope –2.
6. Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.
7. Passing through the points (–1, 1) and (2, – 4).

8. Perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive x-axis is 30^0.
9. The vertices of Δ PQR are P (2, 1), Q (–2, 3) and R (4, 5). Find equation of the median through the vertex R.
10. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).
11. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: n. Find the equation of the line.
12. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).
13. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.
14. Find equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.
15. The perpendicular from the origin to a line meets it at the point (–2, 9), find the equation of the line.
16. The length L (in centimetrs) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L= 125.134 when C = 110, express L in terms of C.
17. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?
18. P (a, b) is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a}+\frac{y}{b}=2$.
19. Point R (h, k) divides a line segment between the axes in the ratio 1: 2. Find equation of the line.
20. By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

10.4 General Equation of a Line

In earlier classes, we have studied general equation of first degree in two variables, Ax + By + C = 0, where A, B and C are real constants such that A and B are not zero simultaneously. Graph of the equation Ax + By + C = 0 is always a straight line.

Therefore, any equation of the form Ax + By + C = 0, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

10.4.1 *Different forms of* Ax + By + *C* = 0 The general equation of a line can be reduced into various forms of the equation of a line, by the following procedures:

(a) *Slope-intercept form* If B ≠ 0, then Ax + By + C = 0 can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \text{ or } y = mx + c \qquad \text{... (1)}$$

where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.

We know that Equation (1) is the slope-intercept form of the equation of a line whose slope is $-\frac{A}{B}$, and y-intercept is $-\frac{C}{B}$.

If B = 0, then $x = -\frac{C}{A}$,which is a vertical line whose slope is undefined and x-intercept is $-\frac{C}{A}$.

(b) Intercept form If C ≠ 0, then Ax + By + C = 0 can be written as

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \text{ or } \frac{x}{a} + \frac{y}{b} = 1$$

where $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$.

We know that equation (1) is intercept form of the equation of a line whose x-intercept is $-\frac{C}{A}$ and y-intercept is $-\frac{C}{B}$.

If C = 0, then Ax + By + C = 0 can be written as Ax + By = 0, which is a line passing through the origin and, therefore, has zero intercepts on the axes.

(c) Normal form Let $x \cos \omega + y \sin \omega = p$ be the normal form of the line represented by the equation Ax + By + C = 0 or Ax + By = – C. Thus, both the equations are same and therefore, $\frac{A}{\cos \omega} = \frac{B}{\sin \omega} = -\frac{C}{p}$

which gives $\cos\omega = -\frac{Ap}{C}$ and $\sin\omega = -\frac{Bp}{C}$.

Now $\sin^2\omega + \cos^2\omega = \left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$

or $p^2 = \frac{C^2}{A^2+B^2}$ or $p = \pm\frac{C}{\sqrt{A^2+B^2}}$

Therefore $\cos\omega = \pm\frac{A}{\sqrt{A^2+B^2}}$ and $\sin\omega = \pm\frac{B}{\sqrt{A^2+B^2}}$.

Thus, the normal form of the equation $Ax + By + C = 0$ is

$$x\cos\omega + y\sin\omega = p,$$

where $\cos\omega = \pm\frac{A}{\sqrt{A^2+B^2}}$, $\sin\omega = \pm\frac{B}{\sqrt{A^2+B^2}}$ and $p = \pm\frac{C}{\sqrt{A^2+B^2}}$.

Proper choice of signs is made so that p should be positive.

Example 13 Equation of a line is $3x - 4y + 10 = 0$. Find its (i) slope, (ii) x - and y-intercepts.

Solution (i) Given equation $3x - 4y + 10 = 0$ can be written as

$$y = \frac{3}{4}x + \frac{5}{2} \qquad \text{... (1)}$$

Comparing (1) with $y = mx + c$, we have slope of the given line as $m = \frac{3}{4}$.

(ii) Equation $3x - 4y + 10 = 0$ can be written as

$$3x - 4y = -10 \quad \text{or} \quad \frac{x}{-\frac{10}{3}} + \frac{y}{\frac{5}{2}} = 1 \qquad \text{... (2)}$$

Comparing (2) with $\frac{x}{a} + \frac{y}{b} = 1$, we have x-intercept as $a = -\frac{10}{3}$ and y-intercept as $b = \frac{5}{2}$.

Example 14 Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of p and ω.

Solution Given equation is

$$\sqrt{3}x + y - 8 = 0 \qquad \text{... (1)}$$

Dividing (1) by $\sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2$, we get

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4 \quad \text{or} \quad \cos 30° x + \sin 30° y = 4 \qquad \text{... (2)}$$

Comparing (2) with $x \cos \omega + y \sin \omega = p$, we get $p = 4$ and $\omega = 30°$.

Example15 Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

Solution Given lines are

$$y - \sqrt{3}x - 5 = 0 \text{ or } y = \sqrt{3}x + 5 \qquad \text{... (1)}$$

and

$$\sqrt{3}y - x + 6 = 0 \text{ or } y = \frac{1}{\sqrt{3}}x - 2\sqrt{3} \qquad \text{... (2)}$$

Slope of line (1) is $m_1 = \sqrt{3}$ and slope of line (2) is $m_2 = \frac{1}{\sqrt{3}}$.

The acute angle (say) θ between two lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \text{... (3)}$$

Putting the values of m_1 and m_2 in (3), we get

$$\tan\theta = \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}}\right| = \left|\frac{1-3}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

which gives $\theta = 30°$. Hence, angle between two lines is either $30°$ or $180° - 30° = 150°$.

Example 16 Show that two lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, where $b_1, b_2 \neq 0$ are:

(i) Parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and (ii) Perpendicular if $a_1a_2 + b_1b_2 = 0$.

Solution Given lines can be written as

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \qquad \text{... (1)}$$

and
$$y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2} \qquad \text{... (2)}$$

Slopes of the lines (1) and (2) are $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$, respectively. Now

(i) Lines are parallel, if $m_1 = m_2$, which gives

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2} \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

(ii) Lines are perpendicular, if $m_1.m_2 = -1$, which gives

$$\frac{a_1}{b_1}.\frac{a_2}{b_2} = -1 \text{ or } a_1a_2 + b_1b_2 = 0$$

Example 17 Find the equation of a line perpendicular to the line $x - 2y + 3 = 0$ and passing through the point $(1, -2)$.

Solution Given line $x - 2y + 3 = 0$ can be written as

$$y = \frac{1}{2}x + \frac{3}{2} \qquad \text{...(1)}$$

Slope of the line (1) is $m_1 = \frac{1}{2}$. Therefore, slope of the line perpendicular to line (1) is

$$m_2 = -\frac{1}{m_1} = -2$$

Equation of the line with slope -2 and passing through the point $(1, -2)$ is

$$y - (-2) = -2(x - 1) \quad \text{or} \quad y = -2x,$$

which is the required equation.

10.5 Distance of a Point From a Line

The distance of a point from a line is the length of the perpendicular drawn from the point to the line. Let L : Ax + By + C = 0 be a line, whose distance from the point P (x_1, y_1) is d. Draw a perpendicular PM from the point P to the line L (Fig10.19). If

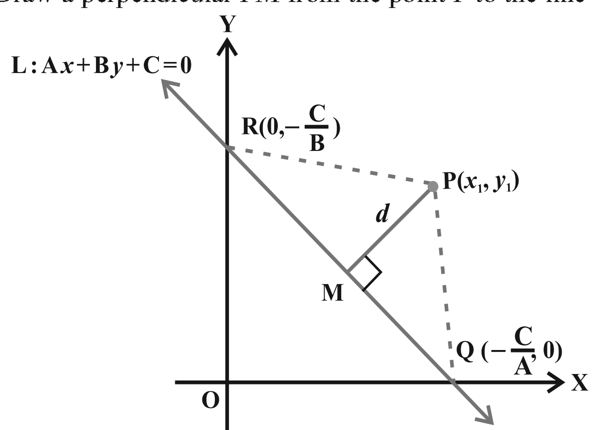


Fig10.19

the line meets the x-and y-axes at the points Q and R, respectively. Then, coordinates of the points are $Q\left(-\frac{C}{A},\ 0\right)$ and $R\left(0,\ -\frac{C}{B}\right)$. Thus, the area of the triangle PQR is given by

$$\text{area }(\Delta PQR) = \frac{1}{2}PM.QR\text{, which gives } PM = \frac{2\ \text{area }(\Delta PQR)}{QR} \qquad \ldots (1)$$

Also, area $$(\Delta PQR) = \frac{1}{2}\left| x_1\left(0 + \frac{C}{B}\right) + \left(-\frac{C}{A}\right)\left(-\frac{C}{B} - y_1\right) + 0\left(y_1 - 0\right)\right|$$

$$= \frac{1}{2}\left| x_1\frac{C}{B} + y_1\frac{C}{A} + \frac{C^2}{AB}\right|$$

or $\quad 2\ \text{area }(\Delta PQR) = \left|\frac{C}{AB}\right|.\ \left|Ax_1 + By_1 + C\right|$, and

$$QR = \sqrt{\left(0 + \frac{C}{A}\right)^2 + \left(\frac{C}{B} - 0\right)^2} = \left|\frac{C}{AB}\right| \sqrt{A^2 + B^2}$$

Substituting the values of area (ΔPQR) and QR in (1), we get

$$\text{PM} = \frac{|\text{A}x_1 + \text{B}y_1 + \text{C}|}{\sqrt{\text{A}^2 + \text{B}^2}}$$

or $$d = \frac{|\text{A}x_1 + \text{B}y_1 + \text{C}|}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

Thus, the perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x_1, y_1) is given by

$$d = \frac{|\text{A}x_1 + \text{B}y_1 + \text{C}|}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

10.5.1 ***Distance between two parallel lines*** We know that slopes of two parallel lines are equal. Therefore, two parallel lines can be taken in the form

$$y = mx + c_1 \qquad \text{... (1)}$$

and $$y = mx + c_2 \qquad \text{... (2)}$$

Line (1) will intersect x-axis at the point A $\left(-\frac{c_1}{m},\ 0\right)$ as shown in Fig10.20.

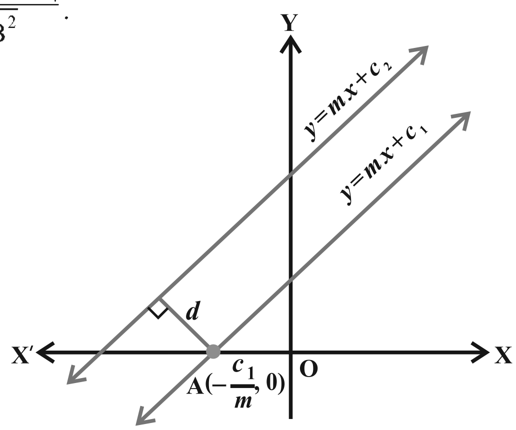


Fig10.20

Distance between two lines is equal to the length of the perpendicular from point A to line (2). Therefore, distance between the lines (1) and (2) is

$$\frac{\left|(-m)\left(-\frac{c_1}{m}\right) + (-c_2)\right|}{\sqrt{1+m^2}} \quad \text{or} \quad d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

Thus, the distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

If lines are given in general form, i.e., Ax + By + C_1 = 0 and Ax + By + C_2 = 0, then above formula will take the form $d = \dfrac{|\ C_1 - C_2\ |}{\sqrt{A^2 + B^2}}$

Students can derive it themselves.

Example 18 Find the distance of the point (3, – 5) from the line $3x - 4y - 26 = 0$.

Solution Given line is $3x - 4y - 26 = 0$ ... (1)

Comparing (1) with general equation of line $Ax + By + C = 0$, we get

$$A = 3,\ B = -4 \text{ and } C = -26.$$

Given point is $(x_1, y_1) = (3, -5)$. The distance of the given point from given line is

$$d = \frac{|\ Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3.3 + (-4)(-5) - 26|}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}.$$

Example 19 Find the distance between the parallel lines $3x - 4y + 7 = 0$ and

$$3x - 4y + 5 = 0$$

Solution Here $A = 3$, $B = -4$, $C_1 = 7$ and $C_2 = 5$. Therefore, the required distance is

$$d = \frac{|\ 7 - 5|}{\sqrt{3^2 + (-4)^2}} = \frac{2}{5}.$$

EXERCISE 10.3

1. Reduce the following equations into slope - intercept form and find their slopes and the y - intercepts.
   (i) $x + 7y = 0$, (ii) $6x + 3y - 5 = 0$, (iii) $y = 0$.
2. Reduce the following equations into intercept form and find their intercepts on the axes.
   (i) $3x + 2y - 12 = 0$, (ii) $4x - 3y = 6$, (iii) $3y + 2 = 0$.
3. Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis.
   (i) $x - \sqrt{3}y + 8 = 0$, (ii) $y - 2 = 0$, (iii) $x - y = 4$.
4. Find the distance of the point (–1, 1) from the line $12(x + 6) = 5(y - 2)$.
5. Find the points on the x-axis, whose distances from the line $\dfrac{x}{3} + \dfrac{y}{4} = 1$ are 4 units.
6. Find the distance between parallel lines
   (i) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$ (ii) $l(x + y) + p = 0$ and $l(x + y) - r = 0$.

7. Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (–2, 3).
8. Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3.
9. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.
10. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$. at right angle. Find the value of h.
11. Prove that the line through the point (x_1, y_1) and parallel to the line $Ax + By + C = 0$ is
$$A(x - x_1) + B(y - y_1) = 0.$$
12. Two lines passing through the point (2, 3) intersects each other at an angle of 60°. If slope of one line is 2, find equation of the other line.
13. Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).
14. Find the coordinates of the foot of perpendicular from the point (–1, 3) to the line $3x - 4y - 16 = 0$.
15. The perpendicular from the origin to the line $y = mx + c$ meets it at the point (–1, 2). Find the values of m and c.
16. If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x\sec\theta + y\,\text{cosec}\,\theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$.
17. In the triangle ABC with vertices A (2, 3), B (4, –1) and C (1, 2), find the equation and length of altitude from the vertex A.
18. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Miscellaneous Examples

Example 20 If the lines $2x + y - 3 = 0$, $5x + ky - 3 = 0$ and $3x - y - 2 = 0$ are concurrent, find the value of k.

Solution Three lines are said to be concurrent, if they pass through a common point, i.e., point of intersection of any two lines lies on the third line. Here given lines are

$$2x + y - 3 = 0 \qquad \ldots (1)$$
$$5x + ky - 3 = 0 \qquad \ldots (2)$$

$$3x - y - 2 = 0 \qquad \text{... (3)}$$

Solving (1) and (3) by cross-multiplication method, we get

$$\frac{x}{-2-3} = \frac{y}{-9+4} = \frac{1}{-2-3} \text{ or } x = 1,\ y = 1.$$

Therefore, the point of intersection of two lines is (1, 1). Since above three lines are concurrent, the point (1, 1) will satisfy equation (3) so that

$$5.1 + k\,.1 - 3 = 0 \text{ or } k = -2.$$

Example 21 Find the distance of the line $4x - y = 0$ from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis.

Solution Given line is $4x - y = 0$ ... (1)

In order to find the distance of the line (1) from the point P (4, 1) along another line, we have to find the point of intersection of both the lines. For this purpose, we will first find the equation of the second line (Fig 10.21). Slope of second line is tan 135° = –1. Equation of the line with slope – 1 through the point P (4, 1) is

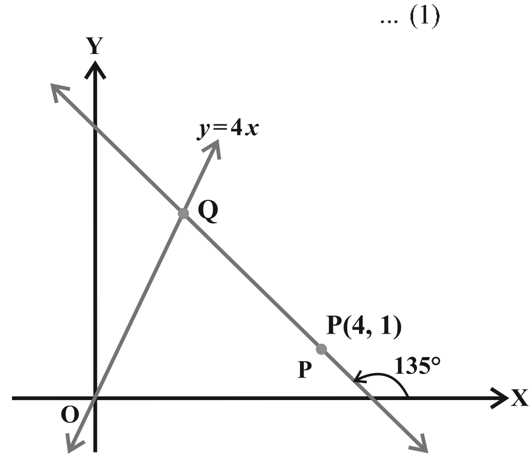


Fig 10.21

$$y - 1 = -1\ (x - 4) \text{ or } x + y - 5 = 0 \qquad \text{... (2)}$$

Solving (1) and (2), we get $x = 1$ and $y = 4$ so that point of intersection of the two lines is Q (1, 4). Now, distance of line (1) from the point P (4, 1) along the line (2)

= The distance between the points P (4, 1) and Q (1, 4).

$$= \sqrt{(1-4)^2 + (4-1)^2} = 3\sqrt{2} \text{ units.}$$

Example 22 Assuming that straight lines work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

Solution Let Q (h, k) is the image of the point P (1, 2) in the line

$$x - 3y + 4 = 0 \qquad \text{... (1)}$$

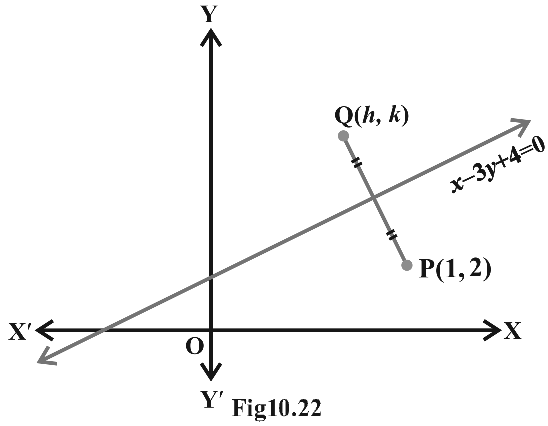


Fig10.22

Therefore, the line (1) is the perpendicular bisector of line segment PQ (Fig 10.22).

Hence $\qquad$ Slope of line PQ $= \dfrac{-1}{\text{Slope of line } \quad x - 3y + 4 = 0}$,

so that $\qquad \dfrac{k-2}{h-1} = \dfrac{-1}{\frac{1}{3}}$ or $3h + k = 5$ $\qquad$ ... (2)

and the mid-point of PQ, i.e., point $\left(\dfrac{h+1}{2}, \dfrac{k+2}{2}\right)$ will satisfy the equation (1) so that

$$\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0 \text{ or } h - 3k = -3 \qquad \text{... (3)}$$

Solving (2) and (3), we get $h = \dfrac{6}{5}$ and $k = \dfrac{7}{5}$.

Hence, the image of the point (1, 2) in the line (1) is $\left(\dfrac{6}{5}, \dfrac{7}{5}\right)$.

Example 23 Show that the area of the triangle formed by the lines

$y = m_1 x + c_1, y = m_2 x + c_2$ and $x = 0$ is $\dfrac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

Solution Given lines are

$y = m_1 x + c_1$ ... (1)

$y = m_2 x + c_2$ ... (2)

$x = 0$ ... (3)

We know that line $y = mx + c$ meets the line $x = 0$ (y-axis) at the point $(0, c)$. Therefore, two vertices of the triangle formed by lines (1) to (3) are P $(0, c_1)$ and Q $(0, c_2)$ (Fig 10. 23).

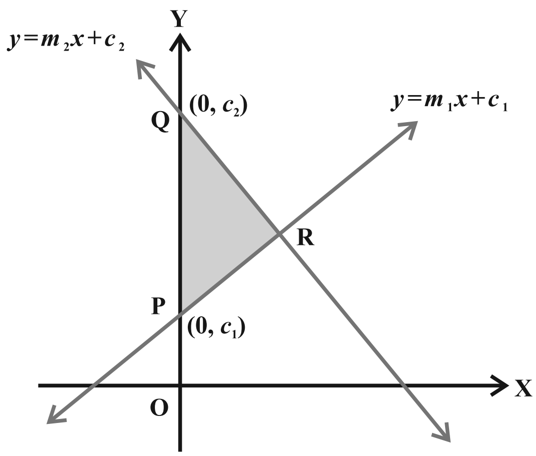


Fig 10.23

Third vertex can be obtained by solving equations (1) and (2). Solving (1) and (2), we get

$$x = \frac{(c_2 - c_1)}{(m_1 - m_2)} \text{ and } y = \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}$$

Therefore, third vertex of the triangle is R $\left(\frac{(c_2 - c_1)}{(m_1 - m_2)}, \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}\right)$.

Now, the area of the triangle is

$$= \frac{1}{2}\left| 0 \left(\frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} - c_2\right) + \frac{c_2 - c_1}{m_1 - m_2}(c_2 - c_1) + 0 \left(c_1 - \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}\right)\right| = \frac{(c_2 - c_1)^2}{2|m_1 - m_2|}$$

Example 24 A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.

Solution Given lines are

$5x - y + 4 = 0$ ... (1)

$3x + 4y - 4 = 0$ ... (2)

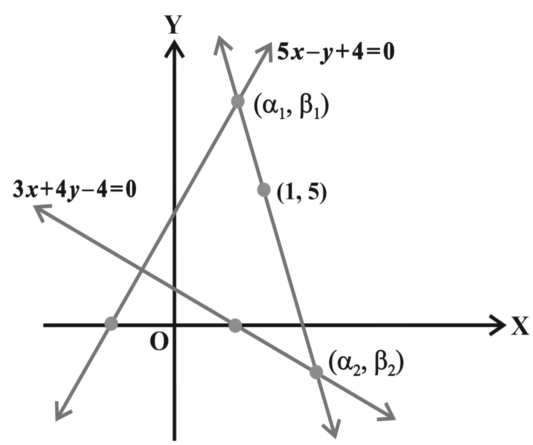


Fig 10.24

Let the required line intersects the lines (1) and (2) at the points , (α_1, β_1) and (α_2, β_2), respectively (Fig10.24). Therefore

$5\alpha_1 - \beta_1 + 4 = 0$ and

$3\alpha_2 + 4\beta_2 - 4 = 0$

or $\quad \beta_1 = 5\alpha_1 + 4$ and $\beta_2 = \dfrac{4-3\alpha_2}{4}$.

We are given that the mid point of the segment of the required line between (α_1, β_1) and (α_2, β_2) is (1, 5). Therefore

$$\frac{\alpha_1+\alpha_2}{2} = 1 \text{ and } \frac{\beta_1+\beta_2}{2} = 5,$$

or
$$\alpha_1 + \alpha_2 = 2 \text{ and } \frac{5\alpha_1 + 4 + \dfrac{4-3\alpha_2}{4}}{2} = 5,$$

or $\quad \alpha_1 + \alpha_2 = 2$ and $20\alpha_1 - 3\alpha_2 = 20 \qquad$ ... (3)

Solving equations in (3) for α_1 and α_2, we get

$$\alpha_1 = \frac{26}{23} \text{ and } \alpha_2 = \frac{20}{23} \text{ and hence, } \beta_1 = 5.\frac{26}{23} + 4 = \frac{222}{23}.$$

Equation of the required line passing through (1, 5) and (α_1, β_1) is

$$y - 5 = \frac{\beta_1 - 5}{\alpha_1 - 1}(x-1) \text{ or } y - 5 = \frac{\dfrac{223}{23} - 5}{\dfrac{26}{23} - 1}(x-1)$$

or $\quad 107x - 3y - 92 = 0,$

which is the equation of required line.

Example 25 Show that the path of a moving point such that its distances from two lines $3x - 2y = 5$ and $3x + 2y = 5$ are equal is a straight line.

Solution Given lines are

$$3x - 2y = 5 \qquad \text{... (1)}$$

and
$$3x + 2y = 5 \qquad \text{... (2)}$$

Let (h, k) is any point, whose distances from the lines (1) and (2) are equal. Therefore

$$\frac{|3h - 2k - 5|}{\sqrt{9+4}} = \frac{|3h + 2k - 5|}{\sqrt{9+4}} \text{ or } |3h - 2k - 5| = |3h + 2k - 5|,$$

which gives $3h - 2k - 5 = 3h + 2k - 5$ or $-(3h - 2k - 5) = 3h + 2k - 5$.

Solving these two relations we get $k = 0$ or $h = \frac{5}{3}$. Thus, the point (h, k) satisfy the equations $y = 0$ or $x = \frac{5}{3}$, which represent straight lines. Hence, path of the point equidistant from the lines (1) and (2) is a straight line.

Miscellaneous Exercise on Chapter 10

1. Find the values of k for which the line $(k-3)\,x - (4 - k^2)\,y + k^2 - 7k + 6 = 0$ is
   (a) Parallel to the x-axis,
   (b) Parallel to the y-axis,
   (c) Passing through the origin.
2. Find the values of θ and p, if the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}\,x + y + 2 = 0$.
3. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and – 6, respectively.
4. What are the points on the y-axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.
5. Find perpendicular distance from the origin of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.
6. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.
7. Find the equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point, where it meets the y-axis.
8. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.
9. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.
10. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.
11. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$.
12. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

13. Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ *is* $\dfrac{y}{x} = \pm\dfrac{m + \tan\theta}{1 - m\tan\theta}$.

14. In what ratio, the line joining $(-1, 1)$ and $(5, 7)$ is divided by the line $x + y = 4$?

15. Find the distance of the line $4x + 7y + 5 = 0$ from the point $(1, 2)$ along the line $2x - y = 0$.

16. Find the direction in which a straight line must be drawn through the point $(-1, 2)$ so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

17. The hypotenuse of a right angled triangle has its ends at the points $(1, 3)$ and $(-4, 1)$. Find the equation of the legs (perpendicular sides) of the triangle.

18. Find the image of the point $(3, 8)$ with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

19. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.

20. If sum of the perpendicular distances of a variable point P (x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line.

21. Find equation of the line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

22. A ray of light passing through the point $(1, 2)$ reflects on the x-axis at point A and the reflected ray passes through the point $(5, 3)$. Find the coordinates of A.

23. Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2 - b^2}, 0\right)$ and $\left(-\sqrt{a^2 - b^2}, 0\right)$ to the line $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ is b^2.

24. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Summary

- *Slope* (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$, $x_1 \neq x_2$.
- If a line makes an angle á with the positive direction of x-axis, then the slope of the line is given by $m = \tan\alpha$, $\alpha \neq 90°$.
- Slope of horizontal line is zero and slope of vertical line is undefined.

◆ An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|, 1 + m_1 m_2 \neq 0$.

◆ Two lines are *parallel* if and only if their slopes are equal.

◆ Two lines are *perpendicular* if and only if product of their slopes is -1.

◆ Three points A, B and C are collinear, if and only if slope of AB = slope of BC.

◆ Equation of the horizontal line having distance a from the x-axis is either $y = a$ or $y = -a$.

◆ Equation of the vertical line having distance b from the y-axis is either $x = b$ or $x = -b$.

◆ The point (x, y) lies on the line with slope m and through the fixed point (x_o, y_o), if and only if its coordinates satisfy the equation $y - y_o = m(x - x_o)$.

◆ Equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

◆ The point (x, y) on the line with slope m and y-intercept c lies on the line if and only if $y = mx + c$.

◆ If a line with slope m makes x-intercept d. Then equation of the line is $y = m(x - d)$.

◆ Equation of a line making intercepts a and b on the x-and y-axis, respectively, is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

◆ The equation of the line having normal distance from origin p and angle between normal and the positive x-axis ω is given by $x\cos\omega + y\sin\omega = p$.

◆ Any equation of the form $Ax + By + C = 0$, with A and B are not zero, simultaneously, is called the *general linear equation* or *general equation of a line*.

◆ The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

◆ Distance between the parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, is given by $d = \dfrac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$.

Chapter 11

CONIC SECTIONS

❖ *Let the relation of knowledge to real life be very visible to your pupils and let them understand how by knowledge the world could be transformed. – BERTRAND RUSSELL* ❖

11.1 Introduction

Apollonius
(262 B.C. 190 B.C.)

In the preceding Chapter 10, we have studied various forms of the equations of a line. In this Chapter, we shall study about some other curves, viz., circles, ellipses, parabolas and hyperbolas. The names parabola and hyperbola are given by Apollonius. These curves are in fact, known as *conic sections* or more commonly *conics* because they can be obtained as intersections of a plane with a double napped right circular cone. These curves have a very wide range of applications in fields such as planetary motion, design of telescopes and antennas, reflectors in flashlights and automobile headlights, etc. Now, in the subsequent sections we will see how the intersection of a plane with a double napped right circular cone results in different types of curves.

11.2 Sections of a Cone

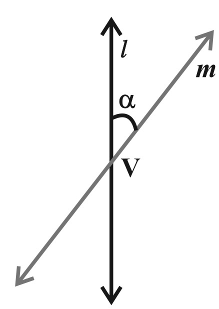


Fig 11. 1

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α (Fig11.1).

Suppose we rotate the line m around the line l in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as

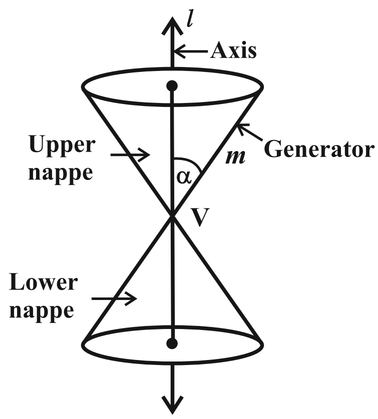


Fig 11. 2

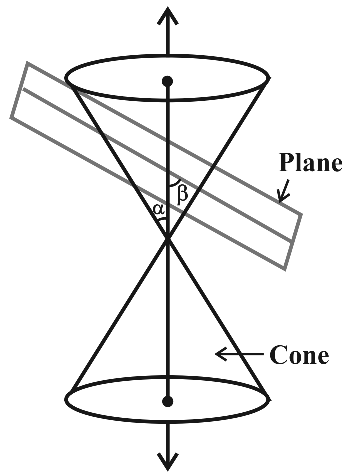


Fig 11. 3

cone and extending indefinitely far in both directions (Fig11.2).

The point V is called the *vertex*; the line *l* is the *axis* of the cone. The rotating line *m* is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane.

We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and by the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone (Fig11.3).

The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.

11.2.1 ***Circle, ellipse, parabola and hyperbola*** When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

(a) When $\beta = 90^\circ$, the section is a *circle* (Fig11.4).

(b) When $\alpha < \beta < 90^\circ$, the section is an *ellipse* (Fig11.5).

(c) When $\beta = \alpha$; the section is a *parabola* (Fig11.6).

(In each of the above three situations, the plane cuts entirely across one nappe of the cone).

(d) When $0 \leq \beta < \alpha$; the plane cuts through both the nappes and the curves of intersection is a *hyperbola* (Fig11.7).

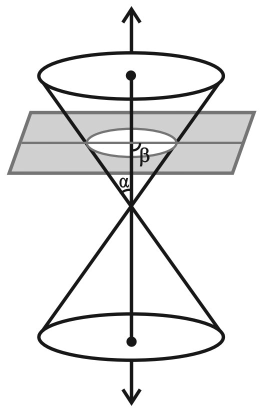


Fig 11. 4

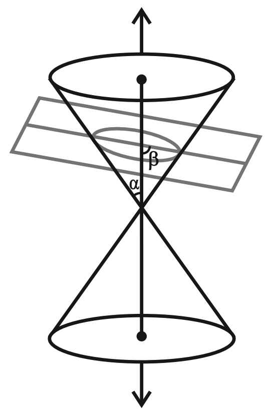


Fig 11. 5

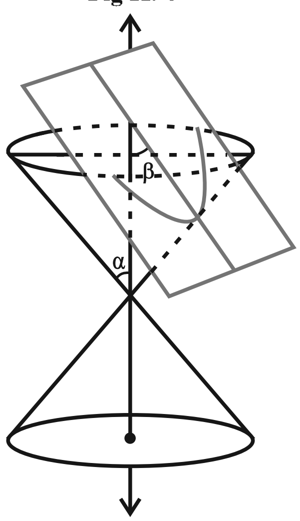


Fig 11. 6

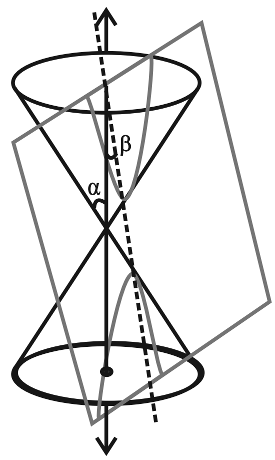


Fig 11. 7

11.2.2 *Degenerated conic sections*

When the plane cuts at the vertex of the cone, we have the following different cases:

(a) When $\alpha < \beta \le 90^o$, then the section is a point (Fig11.8).

(b) When $\beta = \alpha$, the plane contains a generator of the cone and the section is a straight line (Fig11.9).

It is the degenerated case of a parabola.

(c) When $0 \le \beta < \alpha$, the section is a pair of intersecting straight lines (Fig11.10). It is the degenerated case of a *hyperbola*.

In the following sections, we shall obtain the equations of each of these conic sections in standard form by defining them based on geometric properties.

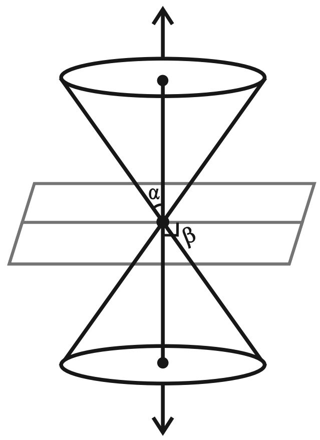


Fig 11. 8

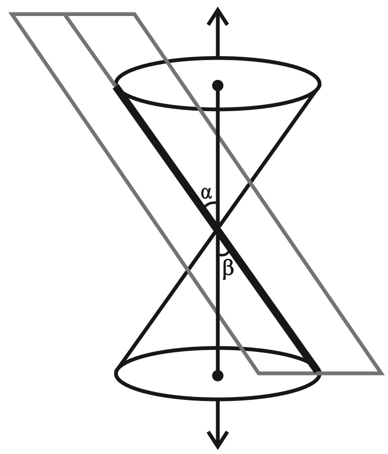
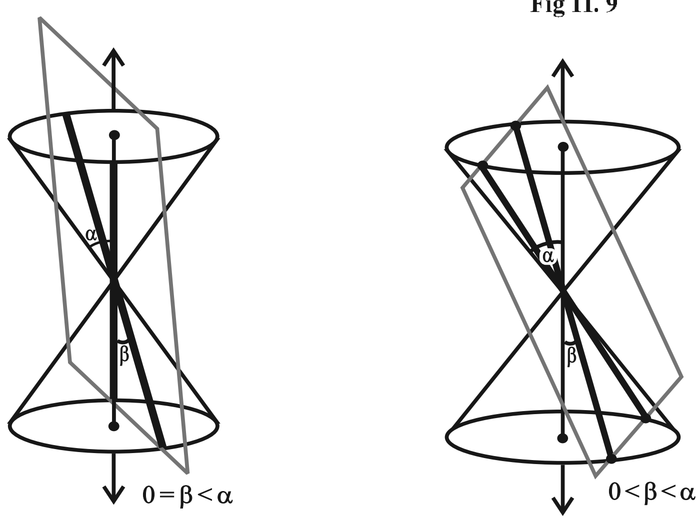


Fig 11. 9



(a) Fig 11. 10 (b)

11.3 Circle

Definition 1 A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.

The fixed point is called the *centre of the circle* and the distance from the centre to a point on the circle is called the *radius* of the circle (Fig 11.11).

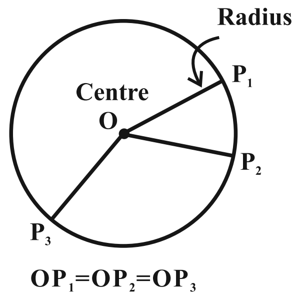


Fig 11. 11

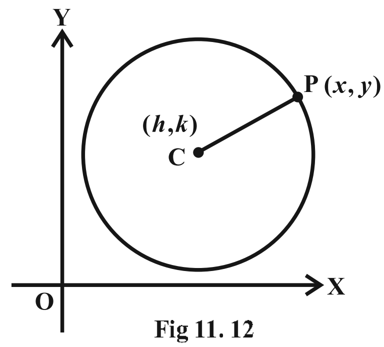


Fig 11. 12

The equation of the circle is simplest if the centre of the circle is at the origin. However, we derive below the equation of the circle with a given centre and radius (Fig 11.12).

Given C (h, k) be the centre and r the radius of circle. Let $P(x, y)$ be any point on the circle (Fig11.12). Then, by the definition, $|\ CP\ | = r$. By the distance formula, we have

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

i.e. $$(x-h)^2+(y-k)^2=r^2$$

This is the required equation of the circle with centre at (h,k) and radius r .

Example 1 Find an equation of the circle with centre at (0,0) and radius r.

Solution Here $h = k = 0$. Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

Example 2 Find the equation of the circle with centre (–3, 2) and radius 4.

Solution Here $h = -3$, $k = 2$ and $r = 4$. Therefore, the equation of the required circle is

$$(x+3)^2+(y-2)^2=16$$

Example 3 Find the centre and the radius of the circle $x^2 + y^2 + 8x + 10y - 8 = 0$

Solution The given equation is

$$(x^2+8x)+(y^2+10y)=8$$

Now, completing the squares within the parenthesis, we get

$$(x^2+8x+16)+(y^2+10y+25)=8+16+25$$

i.e. $$(x+4)^2+(y+5)^2=49$$

i.e. $$\{x-(-4)\}^2+\{y-(-5)\}^2=7^2$$

Therefore, the given circle has centre at (– 4, –5) and radius 7.

Example 4 Find the equation of the circle which passes through the points (2, – 2), and (3,4) and whose centre lies on the line $x + y = 2$.

Solution Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Since the circle passes through (2, – 2) and (3,4), we have

$$(2 - h)^2 + (-2 - k)^2 = r^2 \quad \text{... (1)}$$

and $$(3 - h)^2 + (4 - k)^2 = r^2 \quad \text{... (2)}$$

Also since the centre lies on the line $x + y = 2$, we have

$$h + k = 2 \quad \text{... (3)}$$

Solving the equations (1), (2) and (3), we get

$$h = 0.7, \quad k = 1.3 \quad \text{and } r^2 = 12.58$$

Hence, the equation of the required circle is

$$(x - 0.7)^2 + (y - 1.3)^2 = 12.58.$$

EXERCISE 11.1

In each of the following Exercises 1 to 5, find the equation of the circle with

1. centre (0,2) and radius 2
2. centre (–2,3) and radius 4
3. centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$
4. centre (1,1) and radius $\sqrt{2}$
5. centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.

In each of the following Exercises 6 to 9, find the centre and radius of the circles.

6. $(x + 5)^2 + (y - 3)^2 = 36$
7. $x^2 + y^2 - 4x - 8y - 45 = 0$
8. $x^2 + y^2 - 8x + 10y - 12 = 0$
9. $2x^2 + 2y^2 - x = 0$
10. Find the equation of the circle passing through the points (4,1) and (6,5) and whose centre is on the line $4x + y = 16$.
11. Find the equation of the circle passing through the points (2,3) and (–1,1) and whose centre is on the line $x - 3y - 11 = 0$.
12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2,3).
13. Find the equation of the circle passing through (0,0) and making intercepts a and b on the coordinate axes.
14. Find the equation of a circle with centre (2,2) and passes through the point (4,5).
15. Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

11.4 Parabola

Definition 2 A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point (not on the line) in the plane.

The fixed line is called the *directrix* of the parabola and the fixed point F is called the *focus* (Fig 11.13). ('Para' means 'for' and 'bola' means 'throwing', i.e., the shape described when you throw a ball in the air).

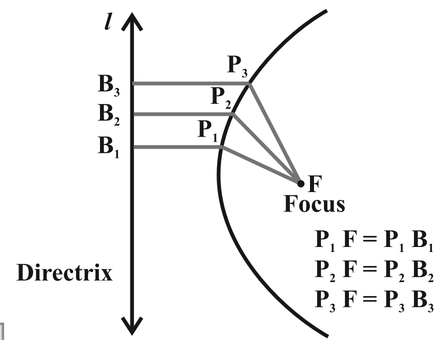


Fig 11. 13

> **Note** If the fixed point lies on the fixed line, then the set of points in the plane, which are equidistant from the fixed point and the fixed line is the straight line through the fixed point and perpendicular to the fixed line. We call this straight line as *degenerate case* of the parabola.

A line through the focus and perpendicular to the *directrix* is called the *axis* of the parabola. The point of intersection of parabola with the axis is called the vertex of the parabola (Fig11.14).

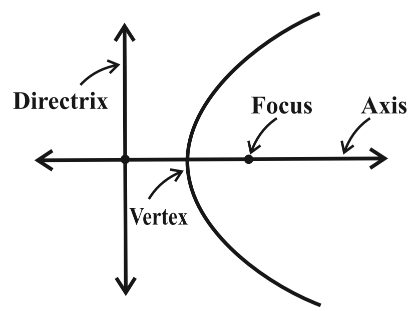
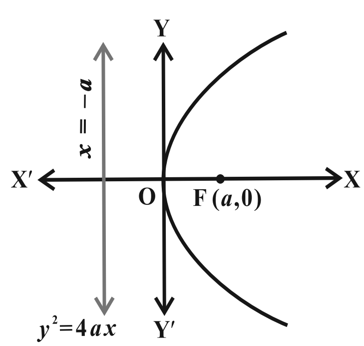
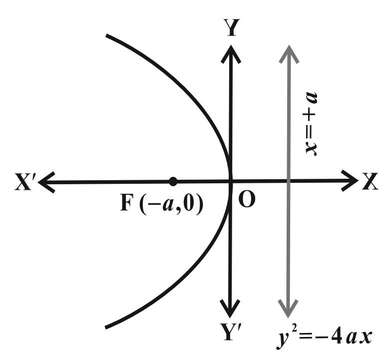


Fig 11.14

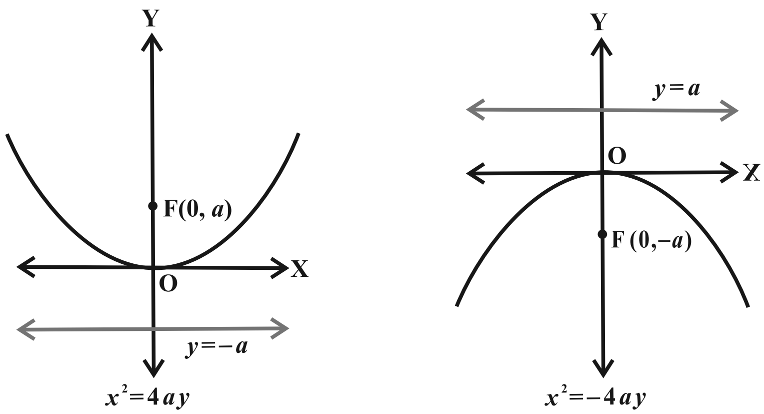
11.4.1 ***Standard equations of parabola*** The equation of a *parabola* is simplest if the vertex is at the origin and the axis of symmetry is along the x-axis or y-axis. The four possible such orientations of parabola are shown below in Fig11.15 (a) to (d).



(a)



(b)



(c) Fig 11.15 (a) to (d) (d)

We will derive the equation for the parabola shown above in Fig 11.15 (a) with *focus* at $(a, 0)$ $a > 0$; and directricx $x = -a$ as below:

Let F be the *focus* and l the *directrix*. Let FM be perpendicular to the *directrix* and bisect FM at the point O. Produce MO to X. By the definition of parabola, the mid-point O is on the parabola and is called the *vertex* of the parabola. Take O as origin, OX the x-axis and OY perpendicular to it as the y-axis. Let the distance from the directrix to the focus be $2a$. Then, the coordinates of the *focus* are $(a, 0)$, and the equation of the *directrix* is $x + a = 0$ as in Fig11.16.

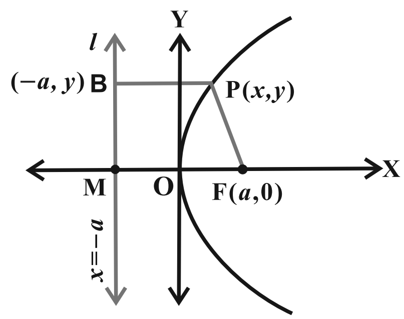


Fig 11.16

Let $P(x, y)$ be any point on the parabola such that

$$PF = PB, \quad \text{... (1)}$$

where PB is perpendicular to l. The coordinates of B are $(-a, y)$. By the distance formula, we have

$$PF = \sqrt{(x-a)^2 + y^2} \text{ and } PB = \sqrt{(x+a)^2}$$

Since PF = PB, we have

$$\sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2}$$

i.e. $(x - a)^2 + y^2 = (x + a)^2$

or $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

or $y^2 = 4ax \ (a > 0)$.

Hence, any point on the parabola satisfies

$$y^2 = 4ax. \qquad \text{... (2)}$$

Conversely, let P(x, y) satisfy the equation (2)

$$\text{PF} = \sqrt{(x-a)^2 + y^2} = \sqrt{(x-a)^2 + 4ax}$$

$$= \sqrt{(x+a)^2} = \text{PB} \qquad \text{... (3)}$$

and so P(x,y) lies on the parabola.

Thus, from (2) and (3) we have proved that the equation to the parabola with vertex at the origin, focus at (a,0) and directrix $x = -a$ is $y^2 = 4ax$.

Discussion In equation (2), since $a > 0$, x can assume any positive value or zero but no negative value and the curve extends indefinitely far into the first and the fourth quadrants. The axis of the parabola is the positive x-axis.

Similarly, we can derive the equations of the parabolas in:

Fig 11.15 (b) as $y^2 = -4ax$,

Fig 11.15 (c) as $x^2 = 4ay$,

Fig 11.15 (d) as $x^2 = -4ay$,

These four equations are known as *standard equations* of parabolas.

> **Note** The standard equations of parabolas have focus on one of the coordinate axis; vertex at the *origin* and thereby the directrix is parallel to the other coordinate axis. However, the study of the equations of parabolas with focus at any point and any line as directrix is beyond the scope here.

From the standard equations of the parabolas, Fig11.15, we have the following observations:

1. Parabola is symmetric with respect to the axis of the parabola.If the equation has a y^2 term, then the axis of symmetry is along the x-axis and if the equation has an x^2 term, then the axis of symmetry is along the y-axis.
2. When the axis of symmetry is along the x-axis the parabola opens to the
   (a) right if the coefficient of x is positive,
   (b) left if the coefficient of x is negative.
3. When the axis of symmetry is along the y-axis the parabola opens
   (c) upwards if the coefficient of y is positive.
   (d) downwards if the coefficient of y is negative.

11.4.2 *Latus rectum*

Definition 3 Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola (Fig11.17).

To find the Length of the latus rectum of the parabola $y^2 = 4ax$ (Fig 11.18).

By the definition of the parabola, AF = AC.

But $\quad$ AC = FM = $2a$

Hence $\quad$ AF = $2a$.

And since the parabola is symmetric with respect to x-axis AF = FB and so

AB = Length of the latus rectum = $4a$.

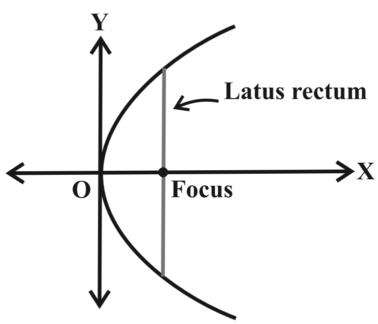


Fig 11.17

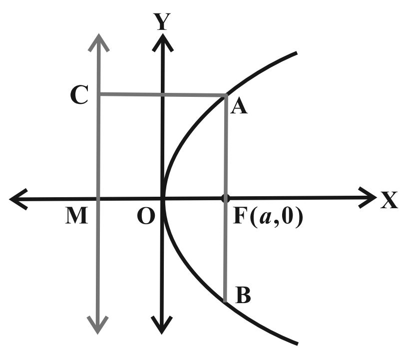


Fig 11.18

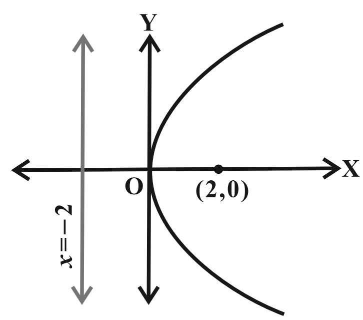


Fig 11.19

Example 5 Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

Solution The given equation involves y^2, so the axis of symmetry is along the x-axis.

The coefficient of x is positive so the parabola opens to the right. Comparing with the given equation $y^2 = 4ax$, we find that $a = 2$.

Thus, the focus of the parabola is (2, 0) and the equation of the directrix of the parabola is $x = -2$ (Fig 11.19).

Length of the latus rectum is $4a = 4 \times 2 = 8$.

Example 6 Find the equation of the parabola with focus (2,0) and directrix $x = -2$.

Solution Since the focus (2,0) lies on the x-axis, the x-axis itself is the axis of the parabola. Hence the equation of the parabola is of the form either $y^2 = 4ax$ or $y^2 = -4ax$. Since the directrix is $x = -2$ and the focus is (2,0), the parabola is to be of the form $y^2 = 4ax$ with $a = 2$. Hence the required equation is

$$y^2 = 4(2)x = 8x$$

Example 7 Find the equation of the parabola with vertex at (0, 0) and focus at (0, 2).

Solution Since the vertex is at (0,0) and the focus is at (0,2) which lies on y-axis, the y-axis is the axis of the parabola. Therefore, equation of the parabola is of the form $x^2 = 4ay$. thus, we have

$$x^2 = 4(2)y, \text{ i.e., } x^2 = 8y.$$

Example 8 Find the equation of the parabola which is symmetric about the y-axis, and passes through the point (2,–3).

Solution Since the parabola is symmetric about y-axis and has its vertex at the origin, the equation is of the form $x^2 = 4ay$ or $x^2 = -4ay$, where the sign depends on whether the parabola opens upwards or downwards. But the parabola passes through (2,–3) which lies in the fourth quadrant, it must open downwards. Thus the equation is of the form $x^2 = -4ay$.

Since the parabola passes through (2,–3), we have

$$2^2 = -4a(-3), \text{ i.e., } a = \frac{1}{3}$$

Therefore, the equation of the parabola is

$$x^2 = -4\left(\frac{1}{3}\right)y, \text{ i.e., } 3x^2 = -4y.$$

EXERCISE 11.2

In each of the following Exercises 1 to 6, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum.

1. $y^2 = 12x$
2. $x^2 = 6y$
3. $y^2 = -8x$
4. $x^2 = -16y$
5. $y^2 = 10x$
6. $x^2 = -9y$

In each of the Exercises 7 to 12, find the equation of the parabola that satisfies the given conditions:

7. Focus (6,0); directrix $x = -6$

8. Focus (0,–3); directrix $y = 3$

9. Vertex (0,0); focus (3,0)

10. Vertex (0,0); focus (–2,0)

11. Vertex (0,0) passing through (2,3) and axis is along x-axis.

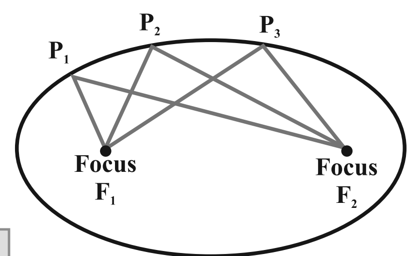
12. Vertex (0,0), passing through (5,2) and symmetric with respect to y-axis.

11. 5 Ellipse

Definition 4 An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.

The two fixed points are called the *foci* (plural of '*focus*') of the ellipse (Fig11.20).

> **Note** The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points.



$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2$

Fig 11.20

The mid point of the line segment joining the foci is called the *centre* of the ellipse. The line segment through the foci of the ellipse is called the *major axis* and the line segment through the centre and perpendicular to the major axis is called the *minor axis*. The end points of the major axis are called the *vertices* of the ellipse(Fig 11.21).

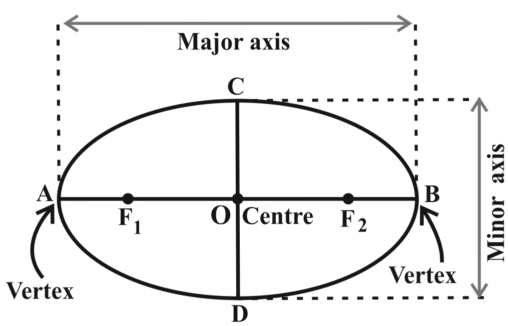


Fig 11.21

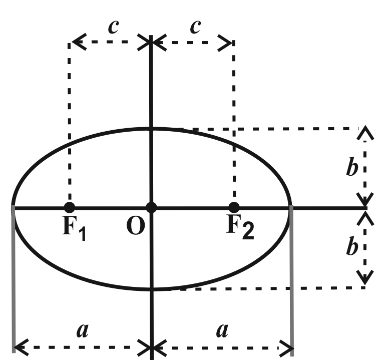


Fig 11.22

We denote the length of the major axis by $2a$, the length of the minor axis by $2b$ and the distance between the foci by $2c$. Thus, the length of the semi major axis is a and semi-minor axis is b (Fig11.22).

11.5.1 ***Relationship between semi-major axis, semi-minor axis and the distance of the focus from the centre of the ellipse (Fig 11.23).***

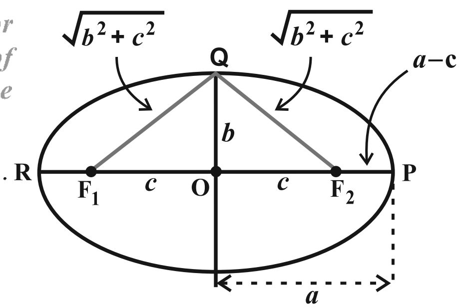


Fig 11.23

Take a point P at one end of the major axis.

Sum of the distances of the point P to the foci is $F_1P + F_2P = F_1O + OP + F_2P$

(Since, $F_1P = F_1O + OP$)

$= c + a + a - c = 2a$

Take a point Q at one end of the minor axis.

Sum of the distances from the point Q to the foci is

$$F_1Q + F_2Q = \sqrt{b^2 + c^2} + \sqrt{b^2 + c^2} = 2\sqrt{b^2 + c^2}$$

Since both P and Q lies on the ellipse.

By the definition of ellipse, we have

$$2\sqrt{b^2 + c^2} = 2a, \text{ i.e.,} \qquad a = \sqrt{b^2 + c^2}$$

or $$a^2 = b^2 + c^2, \text{ i.e.,} \qquad c = \sqrt{a^2 - b^2}.$$

11.5.2 ***Special cases of an ellipse*** In the equation $c^2 = a^2 - b^2$ obtained above, if we keep a fixed and vary c from 0 to a, the resulting ellipses will vary in shape.

Case (i) When $c = 0$, both foci merge together with the centre of the ellipse and $a^2 = b^2$, i.e., $a = b$, and so the ellipse becomes circle (Fig11.24). Thus, circle is a special case of an ellipse which is dealt in Section 11.3.

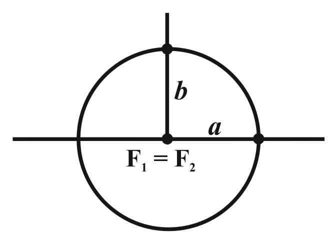


Fig 11.24

Case (ii) When $c = a$, then $b = 0$. The ellipse reduces to the line segment F_1F_2 joining the two foci (Fig11.25).

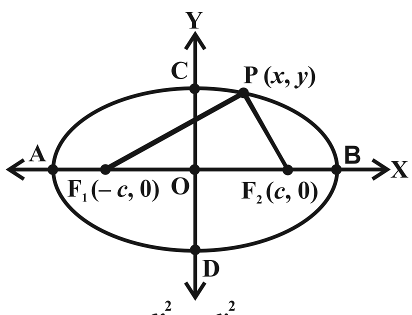
Fig 11.25

11.5.3 ***Eccentricity***

Definition 5 The eccentricity of an ellipse is the ratio of the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse (eccentricity is denoted by e) i.e., $e = \frac{c}{a}$.

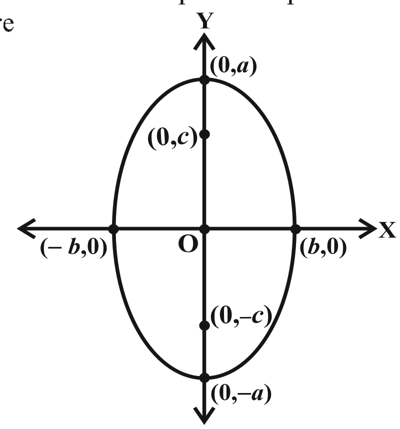
Then since the focus is at a distance of c from the centre, in terms of the eccentricity the focus is at a distance of ae from the centre.

11.5.4 ***Standard equations of an ellipse*** The equation of an ellipse is simplest if the centre of the ellipse is at the origin and the foci are on the x-axis or y-axis. The two such possible orientations are shown in Fig 11.26.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(a)



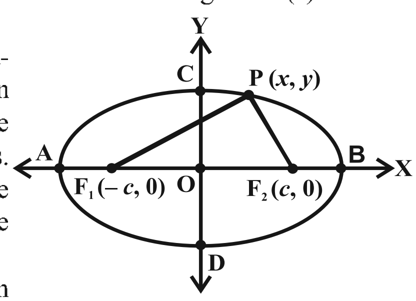
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

(b)

Fig 11.26

We will derive the equation for the ellipse shown above in Fig 11.26 (a) with foci on the x-axis.

Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line from O through F_2 be the positive x-axis and that through F_1 as the negative x-axis. Let, the line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F_1 be $(-c, 0)$ and F_2 be $(c, 0)$ (Fig 11.27).



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Fig 11.27

Let P(x, y) be any point on the ellipse such that the sum of the distances from P to the two foci be $2a$ so given

$$PF_1 + PF_2 = 2a. \qquad \text{... (1)}$$

Using the distance formula, we have

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

i.e., $\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$

Squaring both sides, we get

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

which on simplification gives

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x$$

Squaring again and simplifying, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

i.e.,
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(Since } c^2 = a^2 - b^2\text{)}$$

Hence any point on the ellipse satisfies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{... (2)}$$

Conversely, let P (x, y) satisfy the equation (2) with $0 < c < a$. Then

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

Therefore, $\text{PF}_1 = \sqrt{(x+c)^2 + y^2}$

$$= \sqrt{(x+c)^2 + b^2\left(\frac{a^2 - x^2}{a^2}\right)}$$

$$= \sqrt{(x+c)^2 + (a^2 - c^2)\left(\frac{a^2 - x^2}{a^2}\right)} \qquad \text{(since } b^2 = a^2 - c^2\text{)}$$

$$= \sqrt{\left(a + \frac{cx}{a}\right)^2} = a + \frac{c}{a}x$$

Similarly $\qquad \text{PF}_2 = a - \frac{c}{a}x$

Hence $$\text{PF}_1 + \text{PF}_2 = a + \frac{c}{a}x + a - \frac{c}{a}x = 2a \qquad \ldots (3)$$

So, any point that satisfies $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, satisfies the geometric condition and so P(x, y) lies on the ellipse.

Hence from (2) and (3), we proved that the equation of an ellipse with centre of the origin and major axis along the x-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Discussion From the equation of the ellipse obtained above, it follows that for every point P (x, y) on the ellipse, we have

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \leq 1, \text{ i.e., } x^2 \leq a^2, \text{ so } -a \leq x \leq a.$$

Therefore, the ellipse lies between the lines $x = -a$ and $x = a$ and touches these lines.

Similarly, the ellipse lies between the lines $y = -b$ and $y = b$ and touches these lines.

Similarly, we can derive the equation of the ellipse in Fig 11.26 (b) as $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

These two equations are known as *standard equations* of the ellipses.

> **Note** The standard equations of ellipses have centre at the origin and the major and minor axis are coordinate axes. However, the study of the ellipses with centre at any other point, and any line through the centre as major and the minor axes passing through the centre and perpendicular to major axis are beyond the scope here.

From the standard equations of the ellipses (Fig11.26), we have the following observations:

1. Ellipse is symmetric with respect to both the coordinate axes since if (x, y) is a point on the ellipse, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the ellipse.

2. The foci always lie on the major axis. The major axis can be determined by finding the intercepts on the axes of symmetry. That is, major axis is along the x-axis if the coefficient of x^2 has the larger denominator and it is along the y-axis if the coefficient of y^2 has the larger denominator.

11.5.5 *Latus rectum*

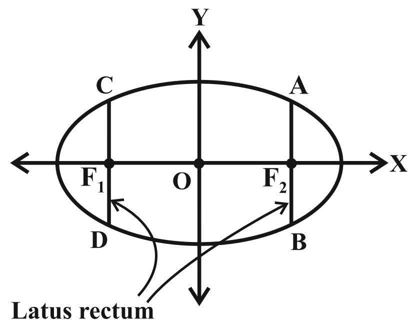


Fig 11. 28

Definition 6 Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse (Fig 11.28).

To find the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let the length of AF_2 be l.

Then the coordinates of A are (c, l), i.e., (ae, l)

Since A lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$\frac{(ae)^2}{a^2}-\frac{l^2}{b^2}=1$$

$$\Rightarrow l^2 = b^2\,(1 - e^2)$$

But $$e^2=\frac{c^2}{a^2}=\frac{a^2-b^2}{a^2}=1-\frac{b^2}{a^2}$$

Therefore $$l^2=\frac{b^4}{a^2}\text{, i.e., } l=\frac{b^2}{a}$$

Since the ellipse is symmetric with respect to y-axis (of course, it is symmetric w.r.t. both the coordinate axes), $AF_2 = F_2B$ and so length of the latus rectum is $\frac{2b^2}{a}$.

Example 9 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

Solution Since denominator of $\frac{x^2}{25}$ is larger than the denominator of $\frac{y^2}{9}$, the major axis is along the x-axis. Comparing the given equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$a = 5 \text{ and } b = 3. \text{ Also}$$

$$c=\sqrt{a^2-b^2}=\sqrt{25-9}=4$$

Therefore, the coordinates of the foci are (– 4,0) and (4,0), vertices are (– 5, 0) and (5, 0). Length of the major axis is 10 units length of the minor axis $2b$ is 6 units and the eccentricity is $\frac{4}{5}$ and latus rectum is $\frac{2b^2}{a}=\frac{18}{5}$.

Example 10 Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse $9x^2 + 4y^2 = 36$.

Solution The given equation of the ellipse can be written in standard form as

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Since the denominator of $\frac{y^2}{9}$ is larger than the denominator of $\frac{x^2}{4}$, the major axis is along the y-axis. Comparing the given equation with the standard equation

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, \text{ we have } b = 2 \text{ and } a = 3.$$

Also $$c=\sqrt{a^2-b^2} = \sqrt{9-4}=\sqrt{5}$$

and $$e=\frac{c}{a}=\frac{\sqrt{5}}{3}$$

Hence the foci are (0, $\sqrt{5}$) and (0, – $\sqrt{5}$), vertices are (0,3) and (0, –3), length of the major axis is 6 units, the length of the minor axis is 4 units and the eccentricity of the ellipse is $\frac{\sqrt{5}}{3}$.

Example 11 Find the equation of the ellipse whose vertices are (± 13, 0) and foci are (± 5, 0).

Solution Since the vertices are on x-axis, the equation will be of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } a \text{ is the semi-major axis.}$$

Given that $a = 13$, $c = \pm 5$.
Therefore, from the relation $c^2 = a^2 - b^2$, we get

$$25 = 169 - b^2 \text{, i.e., } b = 12$$

Hence the equation of the ellipse is $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

Example 12 Find the equation of the ellipse, whose length of the major axis is 20 and foci are $(0, \pm 5)$.

Solution Since the foci are on y-axis, the major axis is along the y-axis. So, equation of the ellipse is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Given that

$$a = \text{semi-major axis} = \frac{20}{2} = 10$$

and the relation $c^2 = a^2 - b^2$ gives

$$5^2 = 10^2 - b^2 \quad \text{i.e., } b^2 = 75$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{75} + \frac{y^2}{100} = 1$$

Example 13 Find the equation of the ellipse, with major axis along the x-axis and passing through the points (4, 3) and (– 1,4).

Solution The standard form of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since the points (4, 3) and (–1, 4) lie on the ellipse, we have

$$\frac{16}{a^2} + \frac{9}{b^2} = 1 \qquad \text{... (1)}$$

and

$$\frac{1}{a^2} + \frac{16}{b^2} = 1 \qquad \text{....(2)}$$

Solving equations (1) and (2), we find that $a^2 = \frac{247}{7}$ and $b^2 = \frac{247}{15}$.

Hence the required equation is

$$\frac{x^2}{\left(\frac{247}{7}\right)}+\frac{y^2}{\frac{247}{15}}=1\text{, i.e., } 7x^2 + 15y^2 = 247.$$

EXERCISE 11.3

In each of the Exercises 1 to 9, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

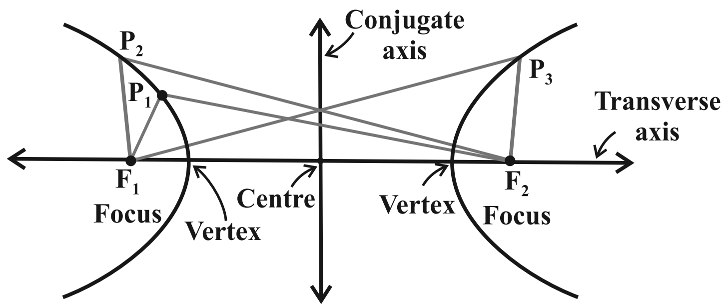
1. $\frac{x^2}{36}+\frac{y^2}{16}=1$
2. $\frac{x^2}{4}+\frac{y^2}{25}=1$
3. $\frac{x^2}{16}+\frac{y^2}{9}=1$
4. $\frac{x^2}{25}+\frac{y^2}{100}=1$
5. $\frac{x^2}{49}+\frac{y^2}{36}=1$
6. $\frac{x^2}{100}+\frac{y^2}{400}=1$
7. $36x^2 + 4y^2 = 144$
8. $16x^2 + y^2 = 16$
9. $4x^2 + 9y^2 = 36$

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

10. Vertices (± 5, 0), foci (± 4, 0)
11. Vertices (0, ± 13), foci (0, ± 5)
12. Vertices (± 6, 0), foci (± 4, 0)
13. Ends of major axis (± 3, 0), ends of minor axis (0, ± 2)
14. Ends of major axis $(0, \pm\sqrt{5})$, ends of minor axis (± 1, 0)
15. Length of major axis 26, foci (± 5, 0)
16. Length of minor axis 16, foci (0, ± 6).
17. Foci (± 3, 0), $a = 4$
18. $b = 3$, $c = 4$, centre at the origin; foci on a x axis.
19. Centre at (0,0), major axis on the y-axis and passes through the points (3, 2) and (1,6).
20. Major axis on the x-axis and passes through the points (4,3) and (6,2).

11.6 Hyperbola

Definition 7 A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.



$$P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

Fig 11.29

The term "*difference*" that is used in the definition means the distance to the further point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The mid-point of the line segment joining the foci is called the *centre of the hyperbola*. The line through the foci is called the *transverse axis* and the line through the centre and perpendicular to the transverse axis is called the *conjugate axis*. The points at which the hyperbola intersects the transverse axis are called the *vertices of the hyperbola* (Fig 11.29).

We denote the distance between the two foci by $2c$, the distance between two vertices (the length of the transverse axis) by $2a$ and we define the quantity b as

$$b = \sqrt{c^2 - a^2}$$

Also $2b$ is the length of the conjugate axis (Fig 11.30).

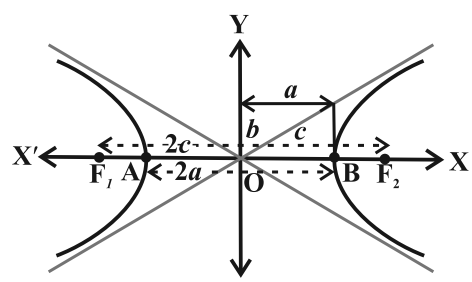


Fig 11.30

To find the constant $P_1F_2 - P_1F_1$:

By taking the point P at A and B in the Fig 11.30, we have

$BF_1 - BF_2 = AF_2 - AF_1$ (by the definition of the hyperbola)

$BA + AF_1 - BF_2 = AB + BF_2 - AF_1$

i.e., $AF_1 = BF_2$

So that, $BF_1 - BF_2 = BA + AF_1 - BF_2 = BA = 2a$

11.6.1 *Eccentricity*

Definition 8 Just like an ellipse, the ratio $e = \frac{c}{a}$ is called the *eccentricity of the hyperbola*. Since $c \geq a$, the eccentricity is never less than one. In terms of the eccentricity, the foci are at a distance of ae from the centre.

11.6.2 ***Standard equation of Hyperbola*** The equation of a hyperbola is simplest if the centre of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two such possible orientations are shown in Fig11.31.

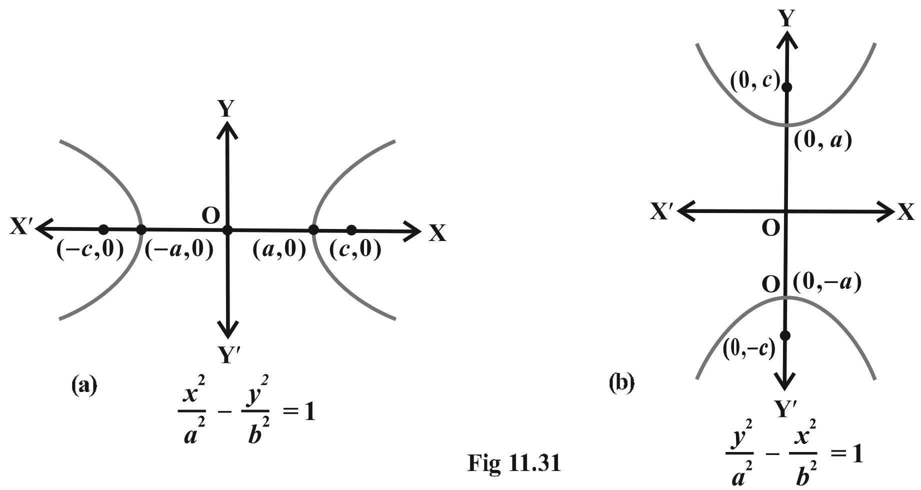


Fig 11.31

We will derive the equation for the hyperbola shown in Fig 11.31(a) with *foci* on the x-axis.

Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line through O through F_2 be the positive x-axis and that through F_1 as the negative x-axis. The line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F_1 be $(-c,0)$ and F_2 be $(c,0)$ (Fig 11.32).

Let $P(x, y)$ be any point on the hyperbola such that the difference of the distances from P to the farther point minus the closer point be $2a$. So given, $PF_1 - PF_2 = 2a$

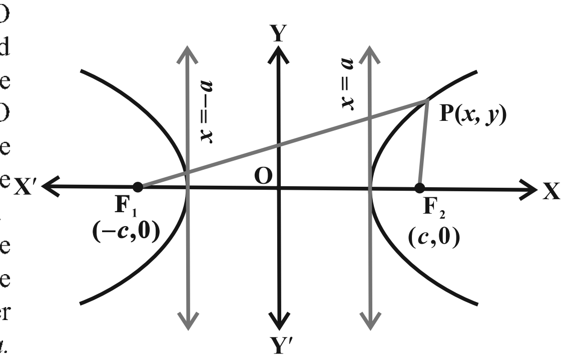


Fig 11.32

Using the distance formula, we have

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

i.e.,
$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Squaring both side, we get

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

and on simplifying, we get

$$\frac{cx}{a}-a=\sqrt{(x-c)^2+y^2}$$

On squaring again and further simplifying, we get

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

i.e.,
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (\text{Since } c^2-a^2=b^2)$$

Hence any point on the hyperbola satisfies $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Conversely, let P(x, y) satisfy the above equation with $0 < a < c$. Then

$$y^2 = b^2\left(\frac{x^2-a^2}{a^2}\right)$$

Therefore,
$$\text{PF}_1 = +\sqrt{(x+c)^2+y^2}$$
$$= +\sqrt{(x+c)^2+b^2\left(\frac{x^2-a^2}{a^2}\right)} = a+\frac{c}{a}x$$

Similarly,
$$\text{PF}_2 = a-\frac{a}{c}x$$

In hyperbola $c > a$; and since P is to the right of the line $x = a$, $x > a$, $\frac{c}{a}x > a$. Therefore,

$a-\frac{c}{a}x$ becomes negative. Thus, $\text{PF}_2 = \frac{c}{a}x - a$.

Therefore $\quad PF_1 - PF_2 = a + \frac{c}{a}x - \frac{cx}{a} + a = 2a$

Also, note that if P is to the left of the line $x = -a$, then

$$PF_1 = -\left(a + \frac{c}{a}x\right), \; PF_2 = a - \frac{c}{a}x.$$

In that case $PF_2 - PF_1 = 2a$. So, any point that satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, lies on the hyperbola.

Thus, we proved that the equation of hyperbola with origin (0,0) and transverse axis along x-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Note A hyperbola in which $a = b$ is called an *equilateral hyperbola*.

Discussion From the equation of the hyperbola we have obtained, it follows that, we have for every point (x, y) on the hyperbola, $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$.

i.e, $\left|\frac{x}{a}\right| \geq 1$, i.e., $x \leq -a$ or $x \geq a$. Therefore, no portion of the curve lies between the lines $x = +a$ and $x = -a$, (i.e. no real intercept on the conjugate axis).

Similarly, we can derive the equation of the hyperbola in Fig 11.31 (b) as $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

These two equations are known as the *standard equations* of *hyperbolas*.

Note The standard equations of hyperbolas have transverse and conjugate axes as the coordinate axes and the centre at the origin. However, there are hyperbolas with any two perpendicular lines as transverse and conjugate axes, but the study of such cases will be dealt in higher classes.

From the standard equations of hyperbolas (Fig11.29), we have the following observations:

1. Hyperbola is symmetric with respect to both the axes, since if (x, y) is a point on the hyperbola, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the hyperbola.

2. The foci are always on the transverse axis. It is the positive term whose denominator gives the transverse axis. For example, $\frac{x^2}{9} - \frac{y^2}{16} = 1$ has transverse axis along x-axis of length 6, while $\frac{y^2}{25} - \frac{x^2}{16} = 1$ has transverse axis along y-axis of length 10.

11.6.3 *Latus rectum*

Definition 9 Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.

As in ellipse, it is easy to show that the length of the latus rectum in hyperbola is $\frac{2b^2}{a}$.

Example 14 Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas:

(i) $\frac{x^2}{9} - \frac{y^2}{16} = 1$, (ii) $y^2 - 16x^2 = 1$

Solution (i) Comparing the equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$ with the standard equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Here, $a = 3$, $b = 4$ and $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$

Therefore, the coordinates of the foci are $(\pm 5, 0)$ and that of vertices are $(\pm 3, 0)$.Also,

The eccentricity $e = \frac{c}{a} = \frac{5}{3}$. The latus rectum $= \frac{2b^2}{a} = \frac{32}{3}$

(ii) Dividing the equation by 16 on both sides, we have $\frac{y^2}{16} - \frac{x^2}{1} = 1$

Comparing the equation with the standard equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we find that

$a = 4$, $b = 1$ and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 1} = \sqrt{17}$.

Therefore, the coordinates of the foci are (0, $\pm\sqrt{17}$) and that of the vertices are (0, ± 4). Also,

The eccentricity $e = \frac{c}{a} = \frac{\sqrt{17}}{4}$. The latus rectum $= \frac{2b^2}{a^2} = \frac{1}{2}$.

Example 15 Find the equation of the hyperbola with foci (0, ± 3) and vertices $(0, \pm \frac{\sqrt{11}}{2})$.

Solution Since the foci is on y-axis, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Since vertices are $(0, \pm \frac{\sqrt{11}}{2})$, $a = \frac{\sqrt{11}}{2}$

Also, since foci are (0, ± 3); $c = 3$ and $b^2 = c^2 - a^2 = 25/4$.

Therefore, the equation of the hyperbola is

$$\frac{y^2}{\left(\frac{11}{4}\right)} - \frac{x^2}{\left(\frac{25}{4}\right)} = 1, \text{ i.e., } 100\,y^2 - 44\,x^2 = 275.$$

Example 16 Find the equation of the hyperbola where foci are (0, ±12) and the length of the latus rectum is 36.

Solution Since foci are (0, ± 12), it follows that $c = 12$.

Length of the latus rectum $= \frac{2b^2}{a} = 36$ or $b^2 = 18a$

Therefore $\quad c^2 = a^2 + b^2$; gives

$144 = a^2 + 18a$

i.e., $\quad a^2 + 18a - 144 = 0$,

So $\quad a = -24, 6$.

Since a cannot be negative, we take $a = 6$ and so $b^2 = 108$.

Therefore, the equation of the required hyperbola is $\frac{y^2}{36} - \frac{x^2}{108} = 1$, i.e., $3y^2 - x^2 = 108$

EXERCISE 11.4

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $\frac{x^2}{16}-\frac{y^2}{9}=1$
2. $\frac{y^2}{9}-\frac{x^2}{27}=1$
3. $9y^2 - 4x^2 = 36$
4. $16x^2 - 9y^2 = 576$
5. $5y^2 - 9x^2 = 36$
6. $49y^2 - 16x^2 = 784$.

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

7. Vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$
8. Vertices $(0, \pm 5)$, foci $(0, \pm 8)$
9. Vertices $(0, \pm 3)$, foci $(0, \pm 5)$
10. Foci $(\pm 5, 0)$, the transverse axis is of length 8.
11. Foci $(0, \pm 13)$, the conjugate axis is of length 24.
12. Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8.
13. Foci $(\pm 4, 0)$, the latus rectum is of length 12
14. vertices $(\pm 7,0)$, $e = \frac{4}{3}$.
15. Foci $(0, \pm\sqrt{10})$, passing through (2,3)

Miscellaneous Examples

Example 17 The focus of a parabolic mirror as shown in Fig 11.33 is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB (Fig 11.33).

Solution Since the distance from the focus to the vertex is 5 cm. We have, $a = 5$. If the origin is taken at the vertex and the axis of the mirror lies along the positive x-axis, the equation of the parabolic section is

$$y^2 = 4\,(5)\,x = 20\,x$$

Note that $x = 45$. Thus

$$y^2 = 900$$

Therefore $y = \pm 30$

Hence $\text{AB} = 2y = 2 \times 30 = 60$ cm.

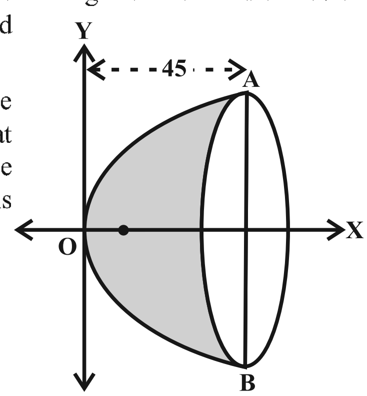


Fig 11.33

Example 18 A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

Solution Let the vertex be at the lowest point and the axis vertical. Let the coordinate axis be chosen as shown in Fig 11.34.

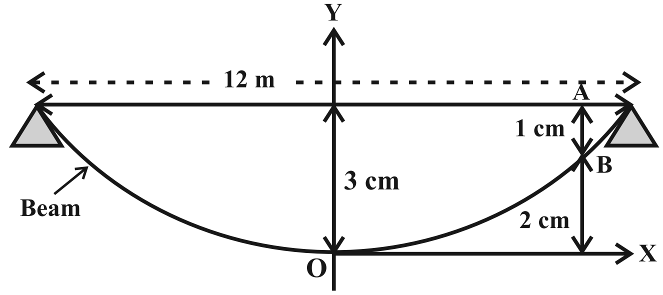


Fig 11.34

The equation of the parabola takes the form $x^2 = 4ay$. Since it passes through $\left(6, \frac{3}{100}\right)$, we have $(6)^2 = 4a\left(\frac{3}{100}\right)$, i.e., $a = \frac{36 \times 100}{12} = 300$ m

Let AB is the deflection of the beam which is $\frac{1}{100}$ m. Coordinates of B are $\left(x, \frac{2}{100}\right)$.

Therefore $\qquad x^2 = 4 \times 300 \times \frac{2}{100} = 24$

i.e. $\qquad x = \sqrt{24} = 2\sqrt{6}$ metres

Example 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point P(x, y) is taken on the rod in such a way that AP = 6 cm. Show that the locus of P is an ellipse.

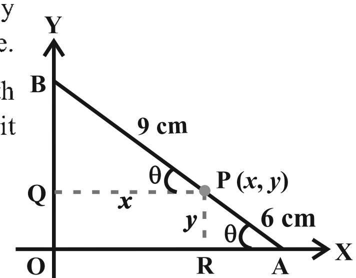


Fig 11.35

Solution Let AB be the rod making an angle θ with OX as shown in Fig 11.35 and P (x, y) the point on it such that AP = 6 cm.

Since AB = 15 cm, we have

PB = 9 cm.

From P draw PQ and PR perpendicular on y-axis and x-axis, respectively.

From Δ PBQ, $\cos\theta = \frac{x}{9}$

From Δ PRA, $\sin\theta = \frac{y}{6}$

Since $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

or $$\frac{x^2}{81} + \frac{y^2}{36} = 1$$

Thus the locus of P is an ellipse.

Miscellaneous Exercise on Chapter 11

1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.
2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?
3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.
4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.
5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.
6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.
7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.
8. An equilateral triangle is inscribed in the parabola $y^2 = 4\,ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Summary

In this Chapter the following concepts and generalisations are studied.

- A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.
- The equation of a circle with centre (h, k) and the radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane.
- The equation of the parabola with focus at $(a, 0)$ $a > 0$ and directrix $x = -a$ is

$$y^2 = 4ax.$$

- Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the hyperbola.
- Length of the latus rectum of the parabola $y^2 = 4ax$ is $4a$.
- An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.
- The equations of an ellipse with foci on the x-axis is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse.
- Length of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.
- The eccentricity of an ellipse is the ratio between the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse.
- A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.
- The equation of a hyperbola with foci on the x-axis is : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.
- Length of the latus rectum of the hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is : $\frac{2b^2}{a}$.
- The eccentricity of a hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola.

Historical Note

Geometry is one of the most ancient branches of mathematics. The Greek geometers investigated the properties of many curves that have theoretical and practical importance. Euclid wrote his treatise on geometry around 300 B.C. He was the first who organised the geometric figures based on certain axioms suggested by physical considerations. Geometry as initially studied by the ancient Indians and Greeks, who made essentially no use of the process of algebra. The synthetic approach to the subject of geometry as given by Euclid and in *Sulbasutras*, etc., was continued for some 1300 years. In the 200 B.C., Apollonius wrote a book called '*The Conic*' which was all about conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

Modern analytic geometry is called '*Cartesian*' after the name of Rene Descartes (1596-1650 A.D.) whose relevant 'La Geometrie' was published in 1637. But the fundamental principle and method of analytical geometry were already discovered by Pierre de Fermat (1601-1665 A.D.). Unfortunately, Fermats treatise on the subject, entitled *Ad Locus Planos et So LIDOS Isagoge* (Introduction to Plane and Solid Loci) was published only posthumously in 1679 A.D. So, Descartes came to be regarded as the unique inventor of the analytical geometry.

Isaac Barrow avoided using cartesian method. Newton used method of undetermined coefficients to find equations of curves. He used several types of coordinates including polar and bipolar. Leibnitz used the terms '*abscissa*', 'ordinate' and 'coordinate'. L' Hospital (about 1700 A.D.) wrote an important textbook on analytical geometry.

Clairaut (1729 A.D.) was the first to give the distance formula although in clumsy form. He also gave the intercept form of the linear equation. Cramer

(1750 A.D.) made formal use of the two axes and gave the equation of a circle as

$$(y - a)^2 + (b - x)^2 = r$$

He gave the best exposition of the analytical geometry of his time. Monge (1781 A.D.) gave the modern 'point-slope' form of equation of a line as

$$y - y' = a (x - x')$$

and the condition of perpendicularity of two lines as $aa' + 1 = 0$.

S.F. Lacroix (1765–1843 A.D.) was a prolific textbook writer, but his contributions to analytical geometry are found scattered. He gave the 'two-point' form of equation of a line as

$$y - \beta = \frac{\beta' - \beta}{\alpha' - \alpha}(x - \alpha)$$

and the length of the perpendicular from (α, β) on $y = ax + b$ as $\dfrac{(\beta - ax - b)}{\sqrt{1 + a^2}}$.

His formula for finding angle between two lines was $\tan\theta = \left(\dfrac{a' - a}{1 + aa'}\right)$. It is, of course, surprising that one has to wait for more than 150 years after the invention of analytical geometry before finding such essential basic formula. In 1818, C. Lame, a civil engineer, gave $mE + m'E' = 0$ as the curve passing through the points of intersection of two loci $E = 0$ and $E' = 0$.

Many important discoveries, both in Mathematics and Science, have been linked to the conic sections. The Greeks particularly Archimedes (287–212 B.C.) and Apollonius (200 B.C.) studied conic sections for their own beauty. These curves are important tools for present day exploration of outer space and also for research into behaviour of atomic particles.

❖

Chapter 12

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

❖*Mathematics is both the queen and the hand-maiden of all sciences – E.T. BELL*❖

12.1 Introduction

You may recall that to locate the position of a point in a plane, we need two intersecting mutually perpendicular lines in the plane. These lines are called the *coordinate axes* and the two numbers are called the *coordinates of the point with respect to the axes*. In actual life, we do not have to deal with points lying in a plane only. For example, consider the position of a ball thrown in space at different points of time or the position of an aeroplane as it flies from one place to another at different times during its flight.

Leonhard Euler (1707 1783)

Similarly, if we were to locate the position of the lowest tip of an electric bulb hanging from the ceiling of a room or the position of the central tip of the ceiling fan in a room, we will not only require the perpendicular distances of the point to be located from two perpendicular walls of the room but also the height of the point from the floor of the room. Therefore, we need not only two but three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. The three numbers representing the three distances are called the *coordinates of the point with reference to the three coordinate planes*. So, a point in space has three coordinates. In this Chapter, we shall study the basic concepts of geometry in three dimensional space.*

---

* For various activities in three dimensional geometry one may refer to the Book, "***A Hand Book for designing Mathematics Laboratory in Schools***", NCERT, 2005.

12.2 Coordinate Axes and Coordinate Planes in Three Dimensional Space

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other (Fig 12.1). These three planes intersect along the lines X′OX, Y′OY and Z′OZ, called the *x*, *y* and *z-axes*, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes XOY, YOZ and ZOX, called, respectively the XY-*plane*, YZ-*plane* and the ZX-*plane*, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line Z′OZ as perpendicular to the plane XOY. If the plane of the paper is considered as horizontal, then the line Z′OZ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ′ are taken as negative. Similarly, the distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY′ as negative, in front of the YZ-plane along OX as positive and to the back of it along OX′ as negative. The point O is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants could be named as XOYZ, X′OYZ, X′OY′Z, XOY′Z, XOYZ′, X′OYZ′, X′OY′Z′ and XOY′Z′. and denoted by I, II, III, ..., VIII , respectively.

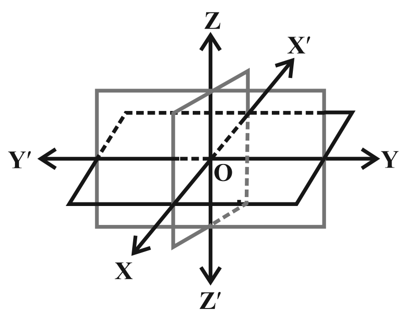


Fig 12.1

12.3 Coordinates of a Point in Space

Having chosen a fixed coordinate system in the space, consisting of coordinate axes, coordinate planes and the origin, we now explain, as to how, given a point in the space, we associate with it three coordinates (x,y,z) and conversely, given a triplet of three numbers (x, y, z), how, we locate a point in the space.

Given a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular (Fig 12.2). Then, from the point M, we draw a perpendicular ML to the x-axis, meeting it at L. Let OL be x, LM be y and MP be z. Then x,y and z are called the x, y and z *coordinates*, respectively, of the point P in the space. In Fig 12.2, we may note that the point P (x, y, z) lies in the octant XOYZ and so all x, y, z are positive. If P was in any other octant, the signs of x, y and z would change

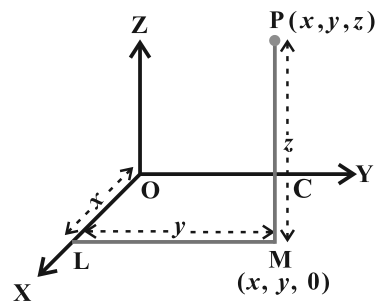


Fig 12.2

accordingly. Thus, to each point P in the space there corresponds an ordered triplet (x, y, z) of real numbers.

Conversely, given any triplet (x, y, z), we would first fix the point L on the x-axis corresponding to x, then locate the point M in the XY-plane such that (x, y) are the coordinates of the point M in the XY-plane. Note that LM is perpendicular to the x-axis or is parallel to the y-axis. Having reached the point M, we draw a perpendicular MP to the XY-plane and locate on it the point P corresponding to z. The point P so obtained has then the coordinates (x, y, z). Thus, there is a one to one correspondence between the points in space and ordered triplet (x, y, z) of real numbers.

Alternatively, through the point P in the space, we draw three planes parallel to the coordinate planes, meeting the x-axis, y-axis and z-axis in the points A, B and C, respectively (Fig 12.3). Let OA = x, OB = y and OC = z. Then, the point P will have the coordinates x, y and z and we write P (x, y, z). Conversely, given x, y and z, we locate the three points A, B and C on the three coordinate axes. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane, respectively. The point of interesection of these three planes, namely, ADPF, BDPE and CEPF is obviously the point P, corresponding to the ordered triplet (x, y, z). We observe that if P (x, y, z) is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

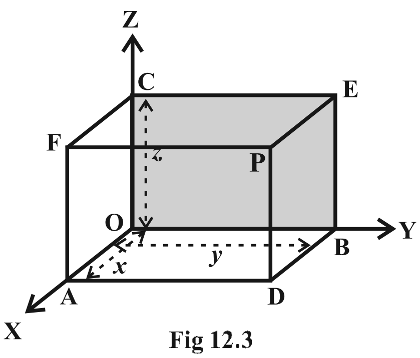


Fig 12.3

Note The coordinates of the origin O are (0,0,0). The coordinates of any point on the x-axis will be as $(x,0,0)$ and the coordinates of any point in the YZ-plane will be as $(0, y, z)$.

Remark The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Table 12.1

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

Example 1 In Fig 12.3, if P is (2,4,5), find the coordinates of F.

Solution For the point F, the distance measured along OY is zero. Therefore, the coordinates of F are (2,0,5).

Example 2 Find the octant in which the points (–3,1,2) and (–3,1,– 2) lie.

Solution From the Table 12.1, the point (–3,1, 2) lies in second octant and the point (–3, 1, – 2) lies in octant VI.

EXERCISE 12.1

1. A point is on the x-axis. What are its y-coordinate and z-coordinates?
2. A point is in the XZ-plane. What can you say about its y-coordinate?
3. Name the octants in which the following points lie:

   (1,. 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (– 4, 2, –5), (– 4, 2, 5), (–3, –1, 6) (2, – 4, –7).
4. Fill in the blanks:
   (i) The x-axis and y-axis taken together determine a plane known as ________.
   (ii) The coordinates of points in the XY-plane are of the form ________.
   (iii) Coordinate planes divide the space into ________ octants.

12.4 Distance between Two Points

We have studied about the distance between two points in two-dimensional coordinate system. Let us now extend this study to three-dimensional system.

Let P(x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelopiped with one diagonal PQ (Fig 12.4).

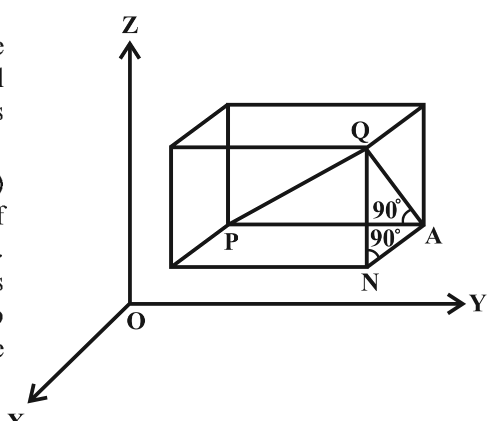


Fig 12.4

Now, since ∠PAQ is a right angle, it follows that, in triangle PAQ,

$$PQ^2 = PA^2 + AQ^2 \qquad \dots (1)$$

Also, triangle ANQ is right angle triangle with ∠ANQ a right angle.

Therefore $AQ^2 = AN^2 + NQ^2$ ... (2)

From (1) and (2), we have

$$PQ^2 = PA^2 + AN^2 + NQ^2$$

Now $PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Therefore $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{x_2^2+y_2^2+z_2^2}$, which gives the distance between the origin O and any point Q (x_2, y_2, z_2).

Example 3 Find the distance between the points P(1, –3, 4) and Q (– 4, 1, 2).

Solution The distance PQ between the points P (1,–3, 4) and Q (– 4, 1, 2) is

$$PQ = \sqrt{(-4-1)^2+(1+3)^2+(2-4)^2}$$

$$= \sqrt{25+16+4}$$

$$= \sqrt{45} = 3\sqrt{5} \text{ units}$$

Example 4 Show that the points P (–2, 3, 5), Q (1, 2, 3) and R (7, 0, –1) are collinear.

Solution We know that points are said to be collinear if they lie on a line.

Now, $PQ = \sqrt{(1+2)^2+(2-3)^2+(3-5)^2} = \sqrt{9+1+4} = \sqrt{14}$

$QR = \sqrt{(7-1)^2+(0-2)^2+(-1-3)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$

and $PR = \sqrt{(7+2)^2+(0-3)^2+(-1-5)^2} = \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}$

Thus, PQ + QR = PR. Hence, P, Q and R are collinear.

Example 5 Are the points A (3, 6, 9), B (10, 20, 30) and C (25, – 41, 5), the vertices of a right angled triangle?

Solution By the distance formula, we have

$$AB^2 = (10-3)^2 + (20-6)^2 + (30-9)^2$$

$$= 49 + 196 + 441 = 686$$

$$BC^2 = (25-10)^2 + (-41-20)^2 + (5-30)^2$$

$= 225 + 3721 + 625 = 4571$

$$\text{CA}^2 = (3 - 25)^2 + (6 + 41)^2 + (9 - 5)^2$$

$$= 484 + 2209 + 16 = 2709$$

We find that $\text{CA}^2 + \text{AB}^2 \neq \text{BC}^2$.

Hence, the triangle ABC is not a right angled triangle.

Example 6 Find the equation of set of points P such that $\text{PA}^2 + \text{PB}^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1, 3, –7), respectively.

Solution Let the coordinates of point P be (*x*, y, z).

Here $\text{PA}^2 = (x - 3)^2 + (y - 4)^2 + (z - 5)^2$

$\text{PB}^2 = (x + 1)^2 + (y - 3)^2 + (z + 7)^2$

By the given condition $\text{PA}^2 + \text{PB}^2 = 2k^2$, we have

$$(x - 3)^2 + (y - 4)^2 + (z - 5)^2 + (x + 1)^2 + (y - 3)^2 + (z + 7)^2 = 2k^2$$

i.e., $2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109.$

EXERCISE 12.2

1. Find the distance between the following pairs of points:
   (i) (2, 3, 5) and (4, 3, 1) (ii) (–3, 7, 2) and (2, 4, –1)
   (iii) (–1, 3, – 4) and (1, –3, 4) (iv) (2, –1, 3) and (–2, 1, 3).
2. Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.
3. Verify the following:
   (i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.
   (ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.
   (iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.
4. Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).
5. Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (– 4, 0, 0) is equal to 10.

12.5 Section Formula

In two dimensional geometry, we have learnt how to find the coordinates of a point dividing a line segment in a given ratio internally. Now, we extend this to three dimensional geometry as follows:

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let the point R (x, y, z)

divide PQ in the given ratio $m : n$ internally. Draw PL, QM and RN perpendicular to the XY-plane. Obviously PL || RN || QM and feet of these perpendiculars lie in a XY-plane. The points L, M and N will lie on a line which is the intersection of the plane containing PL, RN and QM with the XY-plane. Through the point R draw a line ST parallel to the line LM. Line ST will intersect the line LP externally at the point S and the line MQ at T, as shown in Fig 12.5.

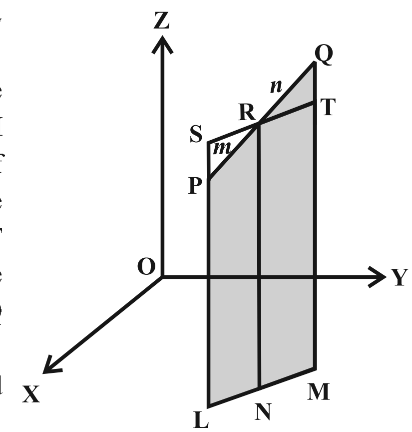


Fig 12.5

Also note that quadrilaterals LNRS and NMTR are parallelograms.

The triangles PSR and QTR are similar. Therefore,

$$\frac{m}{n} = \frac{\text{PR}}{\text{QR}} = \frac{\text{SP}}{\text{QT}} = \frac{\text{SL} - \text{PL}}{\text{QM} - \text{TM}} = \frac{\text{NR} - \text{PL}}{\text{QM} - \text{NR}} = \frac{z - z_1}{z_2 - z}$$

This implies $$z = \frac{mz_2 + nz_1}{m+n}$$

Similarly, by drawing perpendiculars to the XZ and YZ-planes, we get

$$y = \frac{my_2 + ny_1}{m + n} \quad \text{and} \quad x = \frac{mx_2 + nx_1}{m + n}$$

Hence, the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

If the point R divides PQ externally in the ratio $m : n$, then its coordinates are obtained by replacing n by $-n$ so that coordinates of point R will be

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

Case 1 Coordinates of the mid-point: In case R is the mid-point of PQ, then

$$m : n = 1 : 1 \text{ so that } x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2} \text{ and } z = \frac{z_1 + z_2}{2}.$$

These are the coordinates of the mid point of the segment joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2).

Case 2 The coordinates of the point R which divides PQ in the ratio $k : 1$ are obtained by taking $k = \frac{m}{n}$ which are as given below:

$$\left(\frac{kx_2+x_1}{1+k}, \frac{ky_2+y_1}{1+k}, \frac{kz_2+z_1}{1+k}\right)$$

Generally, this result is used in solving problems involving a general point on the line passing through two given points.

Example 7 Find the coordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2 : 3 (i) internally, and (ii) externally.

Solution (i) Let P (x, y, z) be the point which divides line segment joining A(1, – 2, 3) and B (3, 4, –5) internally in the ratio 2 : 3. Therefore

$$x = \frac{2(3)+3(1)}{2+3} = \frac{9}{5}, \quad y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5}, \quad z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$

(ii) Let P (x, y, z) be the point which divides segment joining A (1, –2, 3) and B (3, 4, –5) externally in the ratio 2 : 3. Then

$$x = \frac{2(3)+(-3)(1)}{2+(-3)} = -3, \quad y = \frac{2(4)+(-3)(-2)}{2+(-3)} = -14, \quad z = \frac{2(-5)+(-3)(3)}{2+(-3)} = 19$$

Therefore, the required point is (–3, –14, 19).

Example 8 Using section formula, prove that the three points (– 4, 6, 10), (2, 4, 6) and (14, 0, –2) are collinear.

Solution Let A (– 4, 6, 10), B (2, 4, 6) and C(14, 0, – 2) be the given points. Let the point P divides AB in the ratio $k : 1$. Then coordinates of the point P are

$$\left(\frac{2k-4}{k+1}, \frac{4k+6}{k+1}, \frac{6k+10}{k+1}\right)$$

Let us examine whether for some value of k, the point P coincides with point C.

On putting $\frac{2k-4}{k+1}=14$, we get $k=-\frac{3}{2}$

When $k=-\frac{3}{2}$, then $\frac{4k+6}{k+1}=\frac{4(-\frac{3}{2})+6}{-\frac{3}{2}+1}=0$

and $$\frac{6k+10}{k+1}=\frac{6(-\frac{3}{2})+10}{-\frac{3}{2}+1}=-2$$

Therefore, C (14, 0, –2) is a point which divides AB externally in the ratio 3 : 2 and is same as P.Hence A, B, C are collinear.

Example 9 Find the coordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Solution Let ABC be the triangle. Let the coordinates of the vertices A, B,C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), respectively. Let D be the mid-point of BC. Hence coordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

Let G be the centroid of the triangle. Therefore, it divides the median AD in the ratio 2 : 1. Hence, the coordinates of G are

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}, \frac{2\left(\frac{z_2+z_3}{2}\right)+z_1}{2+1}\right)$$

or $$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Example 10 Find the ratio in which the line segment joining the points (4, 8, 10) and (6, 10, – 8) is divided by the YZ-plane.

Solution Let YZ-plane divides the line segment joining A (4, 8, 10) and B (6, 10, – 8) at P (x, y, z) in the ratio k : 1. Then the coordinates of P are

$$\left(\frac{4+6k}{k+1}, \frac{8+10k}{k+1}, \frac{10-8k}{k+1}\right)$$

Since P lies on the YZ-plane, its x-coordinate is zero, i.e., $\frac{4+6k}{k+1}=0$

$$\text{or} \qquad k = -\frac{2}{3}$$

Therefore, YZ-plane divides AB externally in the ratio 2 : 3.

EXERCISE 12.3

1. Find the coordinates of the point which divides the line segment joining the points (– 2, 3, 5) and (1, – 4, 6) in the ratio (i) 2 : 3 internally, (ii) 2 : 3 externally.
2. Given that P (3, 2, – 4), Q (5, 4, – 6) and R (9, 8, –10) are collinear. Find the ratio in which Q divides PR.
3. Find the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8).
4. Using section formula, show that the points A (2, –3, 4), B (–1, 2, 1) and $C\left(0, \frac{1}{3}, 2\right)$ are collinear.
5. Find the coordinates of the points which trisect the line segment joining the points P (4, 2, – 6) and Q (10, –16, 6).

Miscellaneous Examples

Example 11 Show that the points A (1, 2, 3), B (–1, –2, –1), C (2, 3, 2) and D (4, 7, 6) are the vertices of a parallelogram ABCD, but it is not a rectangle.

Solution To show ABCD is a parallelogram we need to show opposite side are equal Note that.

$$AB = \sqrt{(-1-1)^2+(-2-2)^2+(-1-3)^2} = \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2+(3+2)^2+(2+1)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(4-2)^2+(7-3)^2+(6-2)^2} = \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2+(2-7)^2+(3-6)^2} = \sqrt{9+25+9}=\sqrt{43}$$

Since AB = CD and BC = AD, ABCD is a parallelogram.

Now, it is required to prove that ABCD is not a rectangle. For this, we show that diagonals AC and BD are unequal. We have

$$AC = \sqrt{(2-1)^2+(3-2)^2+(2-3)^2} = \sqrt{1+1+1}=\sqrt{3}$$

$$BD = \sqrt{(4+1)^2+(7+2)^2+(6+1)^2} = \sqrt{25+81+49}=\sqrt{155}.$$

Since AC ≠ BD, ABCD is not a rectangle.

Note We can also show that ABCD is a parallelogram, using the property that diagonals AC and BD bisect each other.

Example 12 Find the equation of the set of the points P such that its distances from the points A (3, 4, –5) and B (– 2, 1, 4) are equal.

Solution If P (x, y, z) be any point such that PA = PB.

Now $\sqrt{(x-3)^2+(y-4)^2+(z+5)^2} = \sqrt{(x+2)^2+(y-1)^2+(z-4)^2}$

or $(x-3)^2+(y-4)^2+(z+5)^2 = (x+2)^2+(y-1)^2+(z-4)^2$

or $10x+6y-18z-29=0$.

Example 13 The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, – 6), respectively, find the coordinates of the point C.

Solution Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be (1, 1, 1). Then

$$\frac{x+3-1}{3} = 1, \text{ i.e., } x = 1; \quad \frac{y-5+7}{3} = 1, \text{ i.e., } y = 1; \quad \frac{z+7-6}{3} = 1, \text{ i.e., } z = 2.$$

Hence, coordinates of C are (1, 1, 2).

Miscellaneous Exercise on Chapter 12

1. Three vertices of a parallelogram ABCD are A(3, – 1, 2), B (1, 2, – 4) and C (– 1, 1, 2). Find the coordinates of the fourth vertex.
2. Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0,4, 0) and (6, 0, 0).
3. If the origin is the centroid of the triangle PQR with vertices P $(2a, 2, 6)$, Q $(-4, 3b, -10)$ and R$(8, 14, 2c)$, then find the values of a, b and c.

4. Find the coordinates of a point on y-axis which are at a distance of $5\sqrt{2}$ from the point P (3, –2, 5).

5. A point R with x-coordinate 4 lies on the line segment joining the points P(2, –3, 4) and Q (8, 0, 10). Find the coordinates of the point R.

[**Hint** Suppose R divides PQ in the ratio $k : 1$. The coordinates of the point R are given by $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$].

6. If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Summary

- In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z-axes.
- The three planes determined by the pair of axes are the coordinate planes, called XY, YZ and ZX-*planes*.
- The three coordinate planes divide the space into eight parts known as *octants*.
- The coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances from the YZ, ZX and XY-planes.
- (i) Any point on x-axis is of the form $(x, 0, 0)$

  (ii) Any point on y-axis is of the form $(0, y, 0)$

  (iii) Any point on z-axis is of the form $(0, 0, z)$.
- Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The coordinates of the point R which divides the line segment joining two points $P(x_1\ y_1\ z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio $m : n$ are given by

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right) \text{ and } \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right),$$

respectively.
- The coordinates of the mid-point of the line segment joining two points

P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

- The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1) (x_2, y_2, z_2) and (x_3, y_3, z_3), are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+x_3}{3}\right)$.

Historical Note

Rene' Descartes (1596–1650 A.D.), the father of analytical geometry, essentially dealt with plane geometry only in 1637 A.D. The same is true of his co-inventor Pierre Fermat (1601-1665 A.D.) and La Hire (1640-1718 A.D.). Although suggestions for the three dimensional coordinate geometry can be found in their works but no details. Descartes had the idea of coordinates in three dimensions but did not develop it.

J.Bernoulli (1667-1748 A.D.) in a letter of 1715 A.D. to Leibnitz introduced the three coordinate planes which we use today. It was Antoinne Parent (1666-1716 A.D.), who gave a systematic development of analytical solid geometry for the first time in a paper presented to the French Academy in 1700 A.D.

L.Euler (1707-1783 A.D.) took up systematically the three dimensional coordinate geometry, in Chapter 5 of the appendix to the second volume of his "Introduction to Geometry" in 1748 A.D.

It was not until the middle of the nineteenth century that geometry was extended to more than three dimensions, the well-known application of which is in the Space-Time Continuum of Einstein's Theory of Relativity.

— ❖ —

Chapter 13

LIMITS AND DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature – WHITEHEAD* ❖

13.1 Introduction

This chapter is an introduction to Calculus. Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change. First, we give an intuitive idea of derivative (without actually defining it). Then we give a naive definition of limit and study some algebra of limits. Then we come back to a definition of derivative and study some algebra of derivatives. We also obtain derivatives of certain standard functions.

Sir Issac Newton (1642 1727)

13.2 Intuitive Idea of Derivatives

Physical experiments have confirmed that the body dropped from a tall cliff covers a distance of $4.9t^2$ metres in t seconds, i.e., distance s in metres covered by the body as a function of time t in seconds is given by $s = 4.9t^2$.

The adjoining Table 13.1 gives the distance travelled in metres at various intervals of time in seconds of a body dropped from a tall cliff.

The objective is to find the veloctiy of the body at time $t = 2$ seconds from this data. One way to approach this problem is to find the average velocity for various intervals of time ending at $t = 2$ seconds and hope that these throw some light on the velocity at $t = 2$ seconds.

Average velocity between $t = t_1$ and $t = t_2$ equals distance travelled between $t = t_1$ and $t = t_2$ seconds divided by $(t_2 - t_1)$. Hence the average velocity in the first two seconds

$$= \frac{\text{Distance travelled between } t_2 = 2 \text{ and } t_1 = 0}{\text{Time interval } (t_2 - t_1)}$$

$$= \frac{(19.6 - 0)m}{(2-0)s} = 9.8 \, m/s \, .$$

Similarly, the average velocity between $t = 1$ and $t = 2$ is

$$\frac{(19.6 - 4.9)m}{(2-1)s} = 14.7 \ m/s$$

Likewise we compute the average velocitiy between $t = t_1$ and $t = 2$ for various t_1. The following Table 13.2 gives the average velocity (v), $t = t_1$ seconds and $t = 2$ seconds.

Table 13.1

t	s
0	0
1	4.9
1.5	11.025
1.8	15.876
1.9	17.689
1.95	18.63225
2	19.6
2.05	20.59225
2.1	21.609
2.2	23.716
2.5	30.625
3	44.1
4	78.4

Table 13.2

t_1	0	1	1.5	1.8	1.9	1.95	1.99
v	9.8	14.7	17.15	18.62	19.11	19.355	19.551

From Table 13.2, we observe that the average velocity is gradually increasing. As we make the time intervals ending at $t = 2$ smaller, we see that we get a better idea of the velocity at $t = 2$. Hoping that nothing really dramatic happens between 1.99 seconds and 2 seconds, we conclude that the average velocity at $t = 2$ seconds is just above 19.551m/s.

This conclusion is somewhat strengthened by the following set of computation. Compute the average velocities for various time intervals starting at $t = 2$ seconds. As before the average velocity v between $t = 2$ seconds and $t = t_2$ seconds is

$$= \frac{\text{Distance travelled between 2 seconds and } t_2 \text{ seconds}}{t_2 - 2}$$

$$= \frac{\text{Distance travelled in } t_2 \text{ seconds } - \text{ Distance travelled in 2 seconds}}{t_2 - 2}$$

$$= \frac{\text{Distance travelled in } t_2 \text{ seconds } - 19.6}{t_2 - 2}$$

The following Table 13.3 gives the average velocity v in metres per second between $t = 2$ seconds and t_2 seconds.

Table 13.3

t_2	4	3	2.5	2.2	2.1	2.05	2.01
v	29.4	24.5	22.05	20.58	20.09	19.845	19.649

Here again we note that if we take smaller time intervals starting at $t = 2$, we get better idea of the velocity at $t = 2$.

In the first set of computations, what we have done is to find average velocities in increasing time intervals ending at $t = 2$ and then hope that nothing dramatic happens just before $t = 2$. In the second set of computations, we have found the average velocities decreasing in time intervals ending at $t = 2$ and then hope that nothing dramatic happens just after $t = 2$. Purely on the physical grounds, both these sequences of average velocities must approach a common limit. We can safely conclude that the velocity of the body at $t = 2$ is between 19.551*m/s* and 19.649 *m/s*. Technically, we say that the instantaneous velocity at $t = 2$ is between 19.551 *m/s* and 19.649 *m/s*. As is well-known, *velocity is the rate of change of distance*. Hence what we have accomplished is the following. From the given data of distance covered at various time instants we have estimated the rate of change of the distance at a given instant of time. We say that the *derivative* of the distance function $s = 4.9t^2$ at $t = 2$ is between 19.551 and 19.649.

An alternate way of viewing this limiting process is shown in Fig 13.1. This is a plot of distance s of the body from the top of the cliff versus the time t elapsed. In the limit as the sequence of time intervals $h_1, h_2, ...$, approaches zero, the sequence of average velocities approaches the same limit as does the sequence of ratios

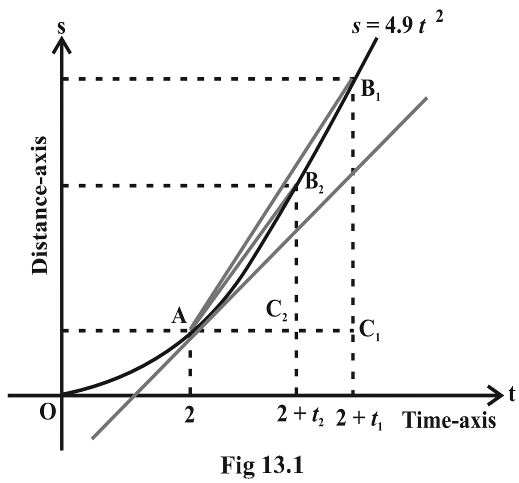


Fig 13.1

$$\frac{C_1B_1}{AC_1}, \frac{C_2B_2}{AC_2}, \frac{C_3B_3}{AC_3}, \ldots$$

where $C_1B_1 = s_1 - s_0$ is the distance travelled by the body in the time interval $h_1 = AC_1$, etc. From the Fig 13.1 it is safe to conclude that this latter sequence approaches the slope of the tangent to the curve at point A. In other words, the instantaneous velocity $v(t)$ of a body at time $t = 2$ is equal to the slope of the tangent of the curve $s = 4.9t^2$ at $t = 2$.

13.3 Limits

The above discussion clearly points towards the fact that we need to understand limiting process in greater clarity. We study a few illustrative examples to gain some familiarity with the concept of limits.

Consider the function $f(x) = x^2$. Observe that as x takes values very close to 0, the value of $f(x)$ also moves towards 0 (See Fig 2.10 Chapter 2). We say

$$\lim_{x \to 0} f(x) = 0$$

(to be read as limit of $f(x)$ as x tends to zero equals zero). The limit of $f(x)$ as x tends to zero is to be thought of as the value $f(x)$ should assume at $x = 0$.

In general as $x \to a, f(x) \to l$, then l is called *limit of the function* $f(x)$ which is symbolically written as $\lim_{x \to a} f(x) = l$.

Consider the following function $g(x) = |x|, x \neq 0$. Observe that $g(0)$ is not defined. Computing the value of $g(x)$ for values of x very near to 0, we see that the value of $g(x)$ moves towards 0. So, $\lim_{x \to 0} g(x) = 0$. This is intuitively clear from the graph of $y = |x|$ for $x \neq 0$. (See Fig 2.13, Chapter 2).

Consider the following function.

$$h(x) = \frac{x^2 - 4}{x - 2}, x \neq 2.$$

Compute the value of $h(x)$ for values of x very near to 2 (but not at 2). Convince yourself that all these values are near to 4. This is somewhat strengthened by considering the graph of the function $y = h(x)$ given here (Fig 13.2).

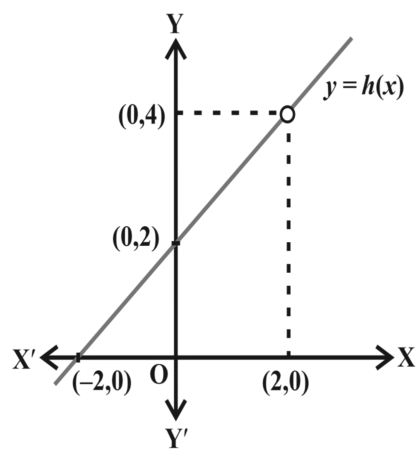


Fig 13.2

In all these illustrations the value which the function should assume at a given point $x = a$ did not really depend on how is x tending to a. Note that there are essentially two ways x could approach a number a either from left or from right, i.e., all the values of x near a could be less than a or could be greater than a. This naturally leads to two limits – the *right hand limit* and the *left hand limit*. *Right hand limit* of a function $f(x)$ is that value of $f(x)$ which is dictated by the values of $f(x)$ when x tends to a from the right. Similarly, the *left hand limit*. To illustrate this, consider the function

$$f(x) = \begin{cases} 1, & x \le 0 \\ 2, & x > 0 \end{cases}$$

Graph of this function is shown in the Fig 13.3. It is clear that the value of f at 0 dictated by values of $f(x)$ with $x \le 0$ equals 1, i.e., the left hand limit of $f(x)$ at 0 is

$$\lim_{x \to 0} f(x) = 1 .$$

Similarly, the value of f at 0 dictated by values of $f(x)$ with $x > 0$ equals 2., i.e., the right hand limit of $f(x)$ at 0 is

$$\lim_{x \to 0^+} f(x) = 2 .$$

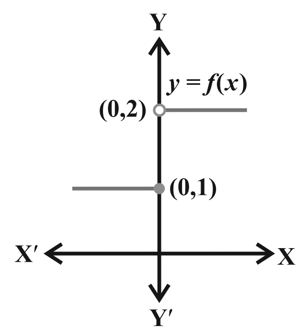


Fig 13.3

In this case the right and left hand limits are different, and hence we say that the limit of $f(x)$ as x tends to zero does not exist (even though the function is defined at 0).

Summary

We say $\lim\limits_{x \to a} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim\limits_{x \to a^+} f(x)$ is the expected value f at $x = a$ given the values of f near x to the right of a. This value is called the *right hand limit* of $f(x)$ at a.

If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x = a$ and denote it by $\lim\limits_{x \to a} f(x)$.

Illustration 1 Consider the function $f(x) = x + 10$. We want to find the limit of this function at $x = 5$. Let us compute the value of the function $f(x)$ for x very near to 5. Some of the points near and to the left of 5 are 4.9, 4.95, 4.99, 4.995. . ., etc. Values of the function at these points are tabulated below. Similarly, the real number 5.001, 5.01, 5.1 are also points near and to the right of 5. Value of the function at these points are also given in the Table 13.4.

Table 13.4

x	4.9	4.95	4.99	4.995	5.001	5.01	5.1
$f(x)$	14.9	14.95	14.99	14.995	15.001	15.01	15.1

From the Table 13.4, we deduce that value of $f(x)$ at $x = 5$ should be greater than 14.995 and less than 15.001 assuming nothing dramatic happens between $x = 4.995$ and 5.001. It is reasonable to assume that the value of the $f(x)$ at $x = 5$ as dictated by the numbers to the left of 5 is 15, i.e.,

$$\lim_{x \to 5} f(x) = 15 .$$

Similarly, when x approaches 5 from the right, $f(x)$ should be taking value 15, i.e.,

$$\lim_{x \to 5^+} f(x) = 15 .$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 15. Thus,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = 15 .$$

This conclusion about the limit being equal to 15 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.16, Chapter 2. In this figure, we note that as x appraches 5 from either right or left, the graph of the function $f(x) = x + 10$ approaches the point (5, 15).

We observe that the value of the function at $x = 2$ also happens to be equal to 12.

Illustration 2 Consider the function $f(x) = x^3$. Let us try to find the limit of this function at $x = 1$. Proceeding as in the previous case, we tabulate the value of $f(x)$ at x near 1. This is given in the Table 13.5.

Table 13.5

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.729	0.970299	0.997002999	1.003003001	1.030301	1.331

From this table, we deduce that value of $f(x)$ at $x = 1$ should be greater than 0.997002999 and less than 1.003003001 assuming nothing dramatic happens between

$x = 0.999$ and 1.001. It is reasonable to assume that the value of the $f(x)$ at $x = 1$ as dictated by the numbers to the left of 1 is 1, i.e.,

$$\lim_{x \to 1^-} f(x) = 1.$$

Similarly, when x approaches 1 from the right, $f(x)$ should be taking value 1., i.e.,

$$\lim_{x \to 1^+} f(x) = 1.$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 1. Thus,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1.$$

This conclusion about the limit being equal to 1 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.11, Chapter 2. In this figure, we note that as x approaches 1 from either right or left, the graph of the function $f(x) = x^3$ approaches the point (1, 1).

We observe, again, that the value of the function at $x = 1$ also happens to be equal to 1.

Illustration 3 Consider the function $f(x) = 3x$. Let us try to find the limit of this function at $x = 2$. The following Table 13.6 is now self-explanatory.

Table 13.6

x	1.9	1.95	1.99	1.999	2.001	2.01	2.1
$f(x)$	5.7	5.85	5.97	5.997	6.003	6.03	6.3

As before we observe that as x approaches 2 from either left or right, the value of $f(x)$ seem to approach 6. We record this as

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 6$$

Its graph shown in Fig 13.4 strengthens this fact.

Here again we note that the value of the function at $x = 2$ coincides with the limit at $x = 2$.

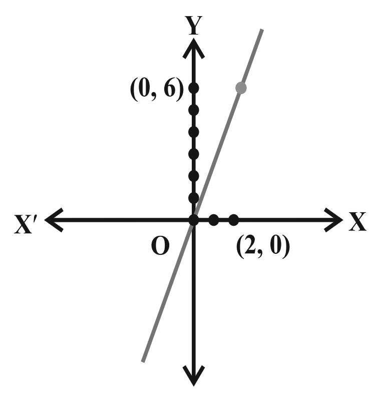


Fig 13.4

Illustration 4 Consider the constant function $f(x) = 3$. Let us try to find its limit at $x = 2$. This function being the constant function takes the same value (3, in this case) everywhere, i.e., its value at points close to 2 is 3. Hence

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 3$$

Graph of $f(x) = 3$ is anyway the line parallel to x-axis passing through (0, 3) and is shown in Fig 2.9, Chapter 2. From this also it is clear that the required limit is 3. In fact, it is easily observed that $\lim_{x \to a} f(x) = 3$ for any real number a.

Illustration 5 Consider the function $f(x) = x^2 + x$. We want to find $\lim_{x \to 1} f(x)$. We tabulate the values of $f(x)$ near $x = 1$ in Table 13.7.

Table 13.7

x	0.9	0.99	0.999	1.01	1.1	1.2
$f(x)$	1.71	1.9701	1.997001	2.0301	2.31	2.64

From this it is reasonable to deduce that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 2.$$

From the graph of $f(x) = x^2 + x$ shown in the Fig 13.5, it is clear that as x approaches 1, the graph approaches (1, 2).

Here, again we observe that the

$$\lim_{x \to 1} f(x) = f(1)$$

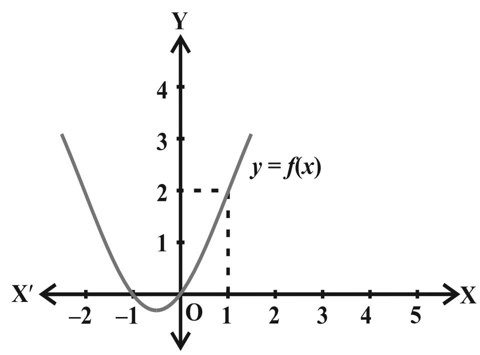


Fig 13.5

Now, convince yourself of the following three facts:

$$\lim_{x \to 1} x^2 = 1, \quad \lim_{x \to 1} x = 1 \text{ and } \lim_{x \to 1} x + 1 = 2$$

Then $$\lim_{x \to 1} x^2 + \lim_{x \to 1} x = 1 + 1 = 2 = \lim_{x \to 1}\left[x^2 + x\right].$$

Also $$\lim_{x \to 1} x. \lim_{x \to 1}(x+1) = 1.2 = 2 = \lim_{x \to 1}\left[x(x+1)\right] = \lim_{x \to 1}\left[x^2 + x\right].$$

Illustration 6 Consider the function $f(x) = \sin x$. We are interested in $\lim\limits_{x \to \frac{\pi}{2}} \sin x$, where the angle is measured in radians.

Here, we tabulate the (approximate) value of $f(x)$ near $\frac{\pi}{2}$ (Table 13.8). From this, we may deduce that

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}} f(x) = 1.$$

Further, this is supported by the graph of $f(x) = \sin x$ which is given in the Fig 3.8 (Chapter 3). In this case too, we observe that $\lim\limits_{x \to \frac{\pi}{2}} \sin x = 1$.

Table 13.8

x	$\frac{\pi}{2} - 0.1$	$\frac{\pi}{2} - 0.01$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.1$
$f(x)$	0.9950	0.9999	0.9999	0.9950

Illustration 7 Consider the function $f(x) = x + \cos x$. We want to find the $\lim\limits_{x \to 0} f(x)$.

Here we tabulate the (approximate) value of $f(x)$ near 0 (Table 13.9).

Table 13.9

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	0.9850	0.98995	0.9989995	1.0009995	1.00995	1.0950

From the Table 13.9, we may deduce that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 1$$

In this case too, we observe that $\lim\limits_{x \to 0} f(x) = f(0) = 1$.

Now, can you convince yourself that

$$\lim_{x \to 0} [x + \cos x] = \lim_{x \to 0} x + \lim_{x \to 0} \cos x \text{ is indeed true?}$$

Illustration 8 Consider the function $f(x)=\frac{1}{x^2}$ for $x>0$. We want to know $\lim_{x\to 0} f(x)$.

Here, observe that the domain of the function is given to be all positive real numbers. Hence, when we tabulate the values of $f(x)$, it does not make sense to talk of x approaching 0 from the left. Below we tabulate the values of the function for positive x close to 0 (in this table n denotes any positive integer).

From the Table 13.10 given below, we see that as x tends to 0, $f(x)$ becomes larger and larger. What we mean here is that the value of $f(x)$ may be made larger than any given number.

Table 13.10

x	1	0.1	0.01	10^{-n}
$f(x)$	1	100	10000	10^{2n}

Mathematically, we say

$$\lim_{x\to 0} f(x)=+\infty$$

We also remark that we will not come across such limits in this course.

Illustration 9 We want to find $\lim_{x\to 0} f(x)$, where

$$f(x)=\begin{cases} x-2, & x<0 \\ 0\ , & x=0 \\ x+2, & x>0 \end{cases}$$

As usual we make a table of x near 0 with $f(x)$. Observe that for negative values of x we need to evaluate $x-2$ and for positive values, we need to evaluate $x+2$.

Table 13.11

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	– 2.1	– 2.01	– 2.001	2.001	2.01	2.1

From the first three entries of the Table 13.11, we deduce that the value of the function is decreasing to –2 and hence.

$$\lim_{x\to 0^-} f(x)=-2$$

From the last three entires of the table we deduce that the value of the function is increasing from 2 and hence

$$\lim_{x \to 0^+} f(x) = 2$$

Since the left and right hand limits at 0 do not coincide, we say that the limit of the function at 0 does not exist.

Graph of this function is given in the Fig13.6. Here, we remark that the value of the function at $x = 0$ is well defined and is, indeed, equal to 0, but the limit of the function at $x = 0$ is not even defined.

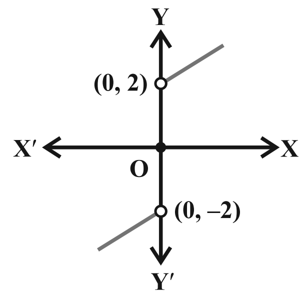


Fig 13.6

Illustration 10 As a final illustration, we find $\lim_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} x+2 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Table 13.12

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.9	2.99	2.999	3.001	3.01	3.1

As usual we tabulate the values of $f(x)$ for x near 1. From the values of $f(x)$ for x less than 1, it seems that the function should take value 3 at $x = 1$., i.e.,

$$\lim_{x \to 1^-} f(x) = 3.$$

Similarly, the value of $f(x)$ should be 3 as dictated by values of $f(x)$ at x greater than 1. i.e.

$$\lim_{x \to 1^+} f(x) = 3.$$

But then the left and right hand limits coincide and hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 3.$$

Graph of function given in Fig 13.7 strengthens our deduction about the limit. Here, we

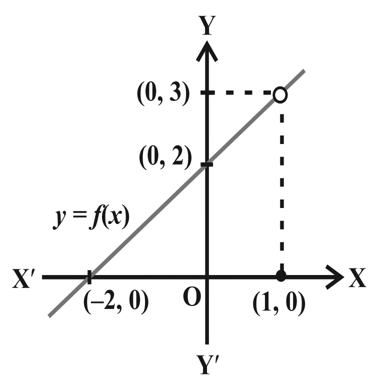


Fig 13.7

note that in general, at a given point the value of the function and its limit may be different (even when both are defined).

13.3.1 ***Algebra of limits*** In the above illustrations, we have observed that the limiting process respects addition, subtraction, multiplication and division as long as the limits and functions under consideration are well defined. This is not a coincidence. In fact, below we formalise these as a theorem without proof.

Theorem 1 Let f and g be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist.

Then

(i) Limit of sum of two functions is sum of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

(ii) Limit of difference of two functions is difference of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x).$$

(iii) Limit of product of two functions is product of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) \,.\, g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x).$$

(iv) Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$$

> **Note** In particular as a special case of (iii), when g is the constant function such that $g(x) = \lambda$, for some real number λ, we have

$$\lim_{x \to a} \left[(\lambda . f)(x)\right] = \lambda . \lim_{x \to a} f(x).$$

In the next two subsections, we illustrate how to exploit this theorem to evaluate limits of special types of functions.

13.3.2 ***Limits of polynomials and rational functions*** A function f is said to be a polynomial function if $f(x)$ is zero function or if $f(x) = a_0 + a_1x + a_2x^2 + . . . + a_nx^n$, where a_is are real numbers such that $a_n \neq 0$ for some natural number n.

We know that $\lim\limits_{x \to a} x = a$. Hence

$$\lim_{x \to a} x^2 = \lim_{x \to a} (x.x) = \lim_{x \to a} x . \lim_{x \to a} x = a.\ a = a^2$$

An easy exercise in induction on n tells us that

$$\lim_{x \to a} x^n = a^n$$

Now, let $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ be a polynomial function. Thinking of each of $a_0, a_1\ x, a_2 x^2, ..., a_n\ x^n$ as a function, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left[a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \right] \\
&= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + ... + \lim_{x \to a} a_n x^n \\
&= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + ... + a_n \lim_{x \to a} x^n \\
&= a_0 + a_1 a + a_2 a^2 + ... + a_n a^n \\
&= f(a)
\end{aligned}$$

(Make sure that you understand the justification for each step in the above!)

A function f is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim\limits_{x \to a} g(x)}{\lim\limits_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a) = 0$, there are two scenarios – (i) when $g(a) \neq 0$ and (ii) when $g(a) = 0$. In the former case we say that the limit does not exist. In the latter case we can write $g(x) = (x - a)^k g_1(x)$, where k is the maximum of powers of $(x - a)$ in $g(x)$ Similarly, $h(x) = (x - a)^l h_1(x)$ as $h(a) = 0$. Now, if $k \geq l$, we have

$$\lim_{x \to a} f(x) = \frac{\lim\limits_{x \to a} g(x)}{\lim\limits_{x \to a} h(x)} = \frac{\lim\limits_{x \to a} (x-a)^k g_1(x)}{\lim\limits_{x \to a} (x-a)^l h_1(x)}$$

$$= \frac{\lim\limits_{x \to a} (x-a)^{(k-l)} g_1(x)}{\lim\limits_{x \to a} h_1(x)} = \frac{0.g_1(a)}{h_1(a)} = 0$$

If $k < l$, the limit is not defined.

Example 1 Find the limits: (i) $\lim\limits_{x \to 1} \left[x^3 - x^2 + 1\right]$ (ii) $\lim\limits_{x \to 3} \left[x(x+1)\right]$

(iii) $\lim\limits_{x \to -1} \left[1 + x + x^2 + ... + x^{10}\right]$.

Solution The required limits are all limits of some polynomial functions. Hence the limits are the values of the function at the prescribed points. We have

(i) $\lim\limits_{x \to 1} [x^3 - x^2 + 1] = 1^3 - 1^2 + 1 = 1$

(ii) $\lim\limits_{x \to 3} \left[x(x+1)\right] = 3(3+1) = 3(4) = 12$

(iii) $\lim\limits_{x \to -1} \left[1 + x + x^2 + ... + x^{10}\right] = 1 + (-1) + (-1)^2 + ... + (-1)^{10}$

$= 1 - 1 + 1 ... + 1 = 1$.

Example 2 Find the limits:

(i) $\lim\limits_{x \to 1} \left[\dfrac{x^2 + 1}{x + 100}\right]$ (ii) $\lim\limits_{x \to 2} \left[\dfrac{x^3 - 4x^2 + 4x}{x^2 - 4}\right]$

(iii) $\lim\limits_{x \to 2} \left[\dfrac{x^2 - 4}{x^3 - 4x^2 + 4x}\right]$ (iv) $\lim\limits_{x \to 2} \left[\dfrac{x^3 - 2x^2}{x^2 - 5x + 6}\right]$

(v) $\lim\limits_{x \to 1} \left[\dfrac{x-2}{x^2 - x} - \dfrac{1}{x^3 - 3x^2 + 2x}\right]$.

Solution All the functions under consideration are rational functions. Hence, we first evaluate these functions at the prescribed points. If this is of the form $\frac{0}{0}$, we try to rewrite the function cancelling the factors which are causing the limit to be of the form $\frac{0}{0}$.

(i) We have $\lim\limits_{x \to 1} \dfrac{x^2+1}{x+100} = \dfrac{1^2+1}{1+100} = \dfrac{2}{101}$

(ii) Evaluating the function at 2, it is of the form $\dfrac{0}{0}$.

$$\text{Hence } \lim_{x \to 2} \frac{x^3-4x^2+4x}{x^2-4} = \lim_{x \to 2} \frac{x(x-2)^2}{(x+2)(x-2)}$$

$$= \lim_{x \to 2} \frac{x(x-2)}{(x+2)} \qquad \text{as } x \neq 2$$

$$= \frac{2(2-2)}{2+2} = \frac{0}{4} = 0.$$

(iii) Evaluating the function at 2, we get it of the form $\dfrac{0}{0}$.

$$\text{Hence } \lim_{x \to 2} \frac{x^2-4}{x^3-4x^2+4x} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x+2)}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0}$$

which is not defined.

(iv) Evaluating the function at 2, we get it of the form $\dfrac{0}{0}$.

$$\text{Hence } \lim_{x \to 2} \frac{x^3-2x^2}{x^2-5x+6} = \lim_{x \to 2} \frac{x^2(x-2)}{(x-2)(x-3)}$$

$$= \lim_{x \to 2} \frac{x^2}{(x-3)} = \frac{(2)^2}{2-3} = \frac{4}{-1} = -4.$$

(v) First, we rewrite the function as a rational function.

$$\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\left[\frac{x-2}{x(x-1)}-\frac{1}{x\left(x^2-3x+2\right)}\right]$$

$$=\left[\frac{x-2}{x(x-1)}-\frac{1}{x(x-1)(x-2)}\right]$$

$$=\left[\frac{x^2-4x+4-1}{x(x-1)(x-2)}\right]$$

$$=\frac{x^2-4x+3}{x(x-1)(x-2)}$$

Evaluating the function at 1, we get it of the form $\frac{0}{0}$.

Hence $$\lim_{x\to 1}\left[\frac{x^2-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\lim_{x\to 1}\frac{x^2-4x+3}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{(x-3)(x-1)}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{x-3}{x(x-2)}=\frac{1-3}{1(1-2)}=2.$$

We remark that we could cancel the term $(x-1)$ in the above evaluation because $x\neq 1$.

Evaluation of an important limit which will be used in the sequel is given as a theorem below.

Theorem 2 For any positive integer *n*,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}.$$

Remark The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

Proof Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + x\,a^{n-2} + a^{n-1})$$

Thus, $$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + x\,a^{n-2} + a^{n-1})$$

$$= a^{n-1} + a\,a^{n-2} + . . . + a^{n-2}(a) + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + ... + a^{n-1} + a^{n-1} \ (n \text{ terms})$$

$$= na^{n-1}$$

Example 3 Evaluate:

(i) $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$ (ii) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

Solution (i) We have

$$\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1} = \lim_{x \to 1} \left[\frac{x^{15} - 1}{x - 1} \div \frac{x^{10} - 1}{x - 1} \right]$$

$$= \lim_{x \to 1} \left[\frac{x^{15} - 1}{x - 1} \right] \div \lim_{x \to 1} \left[\frac{x^{10} - 1}{x - 1} \right]$$

$$= 15\,(1)^{14} \div 10(1)^{9} \ \text{(by the theorem above)}$$

$$= 15 \div 10 = \frac{3}{2}$$

(ii) Put $y = 1 + x$, so that $y \to 1$ as $x \to 0$.

Then $$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{y \to 1} \frac{\sqrt{y} - 1}{y - 1}$$

$$= \lim_{y \to 1} \frac{y^{\frac{1}{2}} - 1^{\frac{1}{2}}}{y - 1}$$

$$= \frac{1}{2}(1)^{\frac{1}{2} - 1} \ \text{(by the remark above)} = \frac{1}{2}$$

13.4 Limits of Trigonometric Functions

The following facts (stated as theorems) about functions in general come in handy in calculating limits of some trigonometric functions.

Theorem 3 Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition, For some a, if both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$. This is illustrated in Fig 13.8.

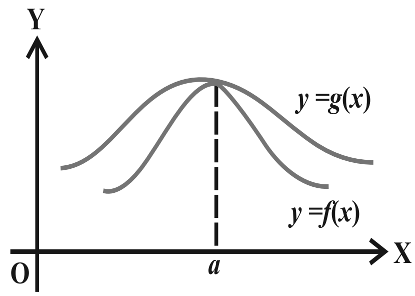


Fig 13.8

Theorem 4 (Sandwich Theorem) Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For some real number a, if $\lim\limits_{x \to a} f(x) = l = \lim\limits_{x \to a} h(x)$, then $\lim\limits_{x \to a} g(x) = l$. This is illustrated in Fig 13.9.

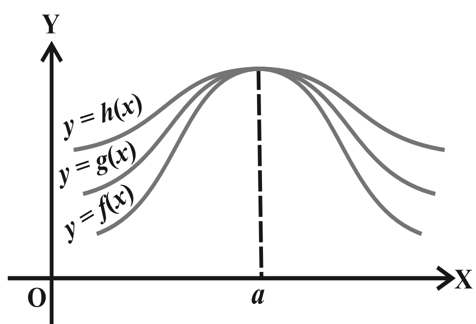


Fig 13.9

Given below is a beautiful geometric proof of the following important inequality relating trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2} \qquad (*)$$

Proof We know that sin (– x) = – sin x and cos(– x) = cos x. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$.

In the Fig 13.10, O is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$. Line segments B A and CD are perpendiculars to OA. Further, join AC. Then

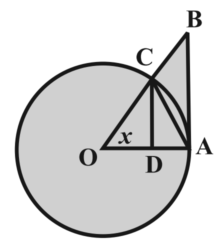


Fig 13.10

Area of ΔOAC < Area of sector OAC < Area of Δ OAB.

i.e., $\frac{1}{2} \text{OA.CD} < \frac{x}{2\pi}.\pi.(\text{OA})^2 < \frac{1}{2}\text{OA.AB}$.

i.e., CD < x . OA < AB.

From Δ OCD,

$\sin x = \frac{\text{CD}}{\text{OA}}$ (since OC = OA) and hence CD = OA sin x. Also $\tan x = \frac{\text{AB}}{\text{OA}}$ and

hence AB = OA. tan x. Thus

OA sin x < OA. x < OA. tan x.

Since length OA is positive, we have

sin x < x < tan x.

Since $0 < x < \frac{\pi}{2}$, sin$x$ is positive and thus by dividing throughout by sin x, we have

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$. Taking reciprocals throughout, we have

$\cos x < \frac{\sin x}{x} < 1$

which complete the proof.

Proposition 5 The following are two important limits.

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$. (ii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$.

Proof (i) The inequality in (*) says that the function $\frac{\sin x}{x}$ is sandwiched between the function cos x and the constant function which takes value 1.

Further, since $\lim\limits_{x \to 0} \cos x = 1$, we see that the proof of (i) of the theorem is complete by sandwich theorem.

To prove (ii), we recall the trigonometric identity $1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right)$.

Then

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} . \sin\left(\frac{x}{2}\right)$$

$$= \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} . \lim_{x \to 0} \sin\left(\frac{x}{2}\right) = 1.0 = 0$$

Observe that we have implicitly used the fact that $x \to 0$ is equivalent to $\frac{x}{2} \to 0$. This may be justified by putting $y = \frac{x}{2}$.

Example 4 Evaluate: (i) $\lim\limits_{x \to 0} \frac{\sin 4x}{\sin 2x}$ (ii) $\lim\limits_{x \to 0} \frac{\tan x}{x}$

Solution (i) $\lim\limits_{x \to 0} \frac{\sin 4x}{\sin 2x} = \lim\limits_{x \to 0} \left[\frac{\sin 4x}{4x} . \frac{2x}{\sin 2x} . 2\right]$

$$= 2 . \lim_{x \to 0} \left[\frac{\sin 4x}{4x}\right] \div \left[\frac{\sin 2x}{2x}\right]$$

$$= 2 . \lim_{4x \to 0} \left[\frac{\sin 4x}{4x}\right] \div \lim_{2x \to 0} \left[\frac{\sin 2x}{2x}\right]$$

$= 2.1.1 = 2$ (as $x \to 0$, $4x \to 0$ and $2x \to 0$)

(ii) We have $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\dfrac{\sin x}{x\cos x} = \lim\limits_{x\to 0}\dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0}\dfrac{1}{\cos x} = 1.1 = 1$

A general rule that needs to be kept in mind while evaluating limits is the following.

Say, given that the limit $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ exists and we want to evaluate this. First we check the value of $f(a)$ and $g(a)$. If both are 0, then we see if we can get the factor which is causing the terms to vanish, i.e., see if we can write $f(x) = f_1(x)\, f_2(x)$ so that $f_1(a) = 0$ and $f_2(a) \neq 0$. Similarly, we write $g(x) = g_1(x)\, g_2(x)$, where $g_1(a) = 0$ and $g_2(a) \neq 0$. Cancel out the common factors from $f(x)$ and $g(x)$ (if possible) and write

$$\frac{f(x)}{g(x)} = \frac{p(x)}{q(x)}, \text{ where } q(x) \neq 0.$$

Then
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{p(a)}{q(a)}.$$

EXERCISE 13.1

Evaluate the following limits in Exercises 1 to 22.

1. $\lim\limits_{x\to 3} x+3$
2. $\lim\limits_{x\to \pi}\left(x-\dfrac{22}{7}\right)$
3. $\lim\limits_{r\to 1}\pi r^2$
4. $\lim\limits_{x\to 4}\dfrac{4x+3}{x-2}$
5. $\lim\limits_{x\to -1}\dfrac{x^{10}+x^5+1}{x-1}$
6. $\lim\limits_{x\to 0}\dfrac{(x+1)^5-1}{x}$
7. $\lim\limits_{x\to 2}\dfrac{3x^2-x-10}{x^2-4}$
8. $\lim\limits_{x\to 3}\dfrac{x^4-81}{2x^2-5x-3}$
9. $\lim\limits_{x\to 0}\dfrac{ax+b}{cx+1}$
10. $\lim\limits_{z\to 1}\dfrac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$
11. $\lim\limits_{x\to 1}\dfrac{ax^2+bx+c}{cx^2+bx+a},\ a+b+c\neq 0$
12. $\lim\limits_{x\to -2}\dfrac{\frac{1}{x}+\frac{1}{2}}{x+2}$
13. $\lim\limits_{x\to 0}\dfrac{\sin ax}{bx}$
14. $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx},\ a,b\neq 0$

15. $\lim\limits_{x \to \pi} \dfrac{\sin(\pi - x)}{\pi(\pi - x)}$

16. $\lim\limits_{x \to 0} \dfrac{\cos x}{\pi - x}$

17. $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{\cos x - 1}$

18. $\lim\limits_{x \to 0} \dfrac{ax + x\cos x}{b\sin x}$

19. $\lim\limits_{x \to 0} x\sec x$

20. $\lim\limits_{x \to 0} \dfrac{\sin ax + bx}{ax + \sin bx}\ a, b, a + b \neq 0$,

21. $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x)$

22. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \dfrac{\pi}{2}}$

23. Find $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases}$

24. Find $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$

25. Evaluate $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

26. Find $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

27. Find $\lim\limits_{x \to 5} f(x)$, where $f(x) = |x| - 5$

28. Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and if $\lim\limits_{x \to 1} f(x) = f(1)$ what are possible values of a and b?

29. Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2)\ldots(x - a_n).$$

What is $\lim_{x \to a_1} (x)$? For some $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x \to a} f(x)$.

30. If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$.

For what value (s) of a does $\lim_{x \to a} f(x)$ exists?

31. If the function $f(x)$ satisfies $\lim_{x \to 1} \dfrac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$.

32. If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m, & x > 1 \end{cases}$. For what integers m and n does both $\lim_{x \to 0} f(x)$

and $\lim_{x \to 1} f(x)$ exist?

13.5 Derivatives

We have seen in the Section 13.2, that by knowing the position of a body at various time intervals it is possible to find the rate at which the position of the body is changing. It is of very general interest to know a certain parameter at various instants of time and try to finding the rate at which it is changing. There are several real life situations where such a process needs to be carried out. For instance, people maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time, Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times. Financial institutions need to predict the changes in the value of a particular stock knowing its present value. In these, and many such cases it is desirable to know how a particular parameter is changing with respect to some other parameter. The heart of the matter is derivative of a function at a given point in its domain of definition.

Definition 1 *Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. Derivative of f (x) at a is denoted by f′(a).

Observe that $f'(a)$ quantifies the change in $f(x)$ at a with respect to x.

Example 5 Find the derivative at $x = 2$ of the function $f(x) = 3x$.

Solution We have

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{3(2+h) - 3(2)}{h}$$

$$= \lim_{h \to 0} \frac{6 + 3h - 6}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3 .$$

The derivative of the function $3x$ at $x = 2$ is 3.

Example 6 Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also prove that $f'(0) + 3f'(-1) = 0$.

Solution We first find the derivatives of $f(x)$ at $x = -1$ and at $x = 0$. We have

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(-1+h)^2 + 3(-1+h) - 5\right] - \left[2(-1)^2 + 3(-1) - 5\right]}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 - h}{h} = \lim_{h \to 0} (2h - 1) = 2(0) - 1 = -1$$

and $$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(0+h)^2 + 3(0+h) - 5\right] - \left[2(0)^2 + 3(0) - 5\right]}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + 3h}{h} = \lim_{h\to 0} (2h+3) = 2(0) + 3 = 3$$

Clearly $\qquad f'(0) + 3f'(-1) = 0$

Remark At this stage note that evaluating derivative at a point involves effective use of various rules, limits are subjected to. The following illustrates this.

Example 7 Find the derivative of sin x at $x = 0$.

Solution Let $f(x) = \sin x$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$$

Example 8 Find the derivative of $f(x) = 3$ at $x = 0$ and at $x = 3$.

Solution Since the derivative measures the change in function, intuitively it is clear that the derivative of the constant function must be zero at every point. This is indeed, supported by the following computation.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{3-3}{h} = \lim_{h\to 0} \frac{0}{h} = 0 .$$

Similarly $\qquad f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{3-3}{h} = 0 .$

We now present a geometric interpretation of derivative of a function at a point. Let $y = f(x)$ be a function and let P = $(a, f(a))$ and Q = $(a + h, f(a + h))$ be two points close to each other on the graph of this function. The Fig 13.11 is now self explanatory.

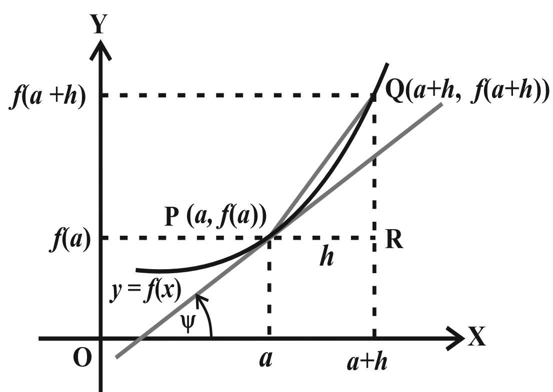


Fig 13.11

We know that $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

From the triangle PQR, it is clear that the ratio whose limit we are taking is precisely equal to tan(QPR) which is the slope of the chord PQ. In the limiting process, as h tends to 0, the point Q tends to P and we have

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{\mathrm{Q} \to \mathrm{P}} \frac{\mathrm{QR}}{\mathrm{PR}}$$

This is equivalent to the fact that the chord PQ tends to the tangent at P of the curve $y = f(x)$. Thus the limit turns out to be equal to the slope of the tangent. Hence

$$f'(a) = \tan \psi .$$

For a given function f we can find the derivative at every point. If the derivative exists at every point, it defines a new function called the derivative of f. Formally, we define derivative of a function as follows.

Definition 2 *Suppose f is a real valued function, the function defined by*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative.

Thus $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Clearly the domain of definition of $f'(x)$ is wherever the above limit exists. There are different notations for derivative of a function. Sometimes $f'(x)$ is denoted by $\frac{d}{dx}(f(x))$ or if $y = f(x)$, it is denoted by $\frac{dy}{dx}$. This is referred to as derivative of $f(x)$ or y with respect to x. It is also denoted by D $(f(x))$. Further, derivative of f at $x = a$ is also denoted by $\left.\frac{d}{dx} f(x)\right|_a$ or $\left.\frac{df}{dx}\right|_a$ or even $\left(\frac{df}{dx}\right)_{x=a}$.

Example 9 Find the derivative of $f(x) = 10x$.

Solution Since $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{10(x+h) - 10(x)}{h}$$

$$= \lim_{h \to 0} \frac{10h}{h} = \lim_{h \to 0} (10) = 10.$$

Example 10 Find the derivative of $f(x) = x^2$.

Solution We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} = \lim_{h \to 0} (h + 2x) = 2x$$

Example 11 Find the derivative of the constant function $f(x) = a$ for a fixed real number a.

Solution We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{a - a}{h} = \lim_{h \to 0} \frac{0}{h} = 0 \text{ as } h \neq 0$$

Example 12 Find the derivative of $f(x) = \frac{1}{x}$

Solution We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x - (x+h)}{x(x+h)} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{x(x+h)} \right] = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

13.5.1 ***Algebra of derivative of functions*** Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules for derivatives to follow closely that of limits. We collect these in the following theorem.

Theorem 5 Let f and g be two functions such that their derivatives are defined in a common domain. Then

(i) Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$

(ii) Derivative of difference of two functions is difference of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x).$$

(iii) Derivative of product of two functions is given by the following *product rule.*

$$\frac{d}{dx}\left[f(x)\,.\,g(x)\right]=\frac{d}{dx}f(x)\,.\,g(x)+f(x)\,.\,\frac{d}{dx}g(x)$$

(iv) Derivative of quotient of two functions is given by the following *quotient rule* (whenever the denominator is non–zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}f(x)\,.\,g(x)-f(x)\frac{d}{dx}g(x)}{\left(g(x)\right)^2}$$

The proofs of these follow essentially from the analogous theorem for limits. We will not prove these here. As in the case of limits this theorem tells us how to compute derivatives of special types of functions. The last two statements in the theorem may be restated in the following fashion which aids in recalling them easily:

Let $u=f(x)$ and $v=g(x)$. Then

$$(uv)' = u'v+uv'$$

This is referred to a Leibnitz rule for differentiating product of functions or the product rule. Similarly, the quotient rule is

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Now, let us tackle derivatives of some standard functions.

It is easy to see that the derivative of the function $f(x) = x$ is the constant function 1. This is because $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h}$

$$= \lim_{h \to 0} 1 = 1 .$$

We use this and the above theorem to compute the derivative of $f(x) = 10x = x + ... + x$ (ten terms). By (1) of the above theorem

$$\frac{df(x)}{dx} = \frac{d}{dx}\left(x + ... + x\right) \text{ (ten terms)}$$

$$= \frac{d}{dx}x + . . . + \frac{d}{dx}x \quad \text{(ten terms)}$$

$$= 1 + ... + 1 \text{ (ten terms)} = 10.$$

We note that this limit may be evaluated using product rule too. Write $f(x) = 10x = uv$, where u is the constant function taking value 10 everywhere and $v(x) = x$. Here, $f(x) = 10x = uv$ we know that the derivative of u equals 0. Also derivative of $v(x) = x$ equals 1. Thus by the product rule we have

$$f'(x) = (10x)' = (uv)' = u'v + uv' = 0.x + 10.1 = 10$$

On similar lines the derivative of $f(x) = x^2$ may be evaluated. We have $f(x) = x^2 = x .x$ and hence

$$\frac{df}{dx} = \frac{d}{dx}(x.x) = \frac{d}{dx}(x).x + x.\frac{d}{dx}(x)$$

$$= 1.x + x.1 = 2x \cdot$$

More generally, we have the following theorem.

Theorem 6 Derivative of $f(x) = x^n$ is nx^{n-1} for any positive integer n.

Proof By definition of the derivative function, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} .$$

Binomial theorem tells that $(x+h)^n = \left({}^nC_0\right)x^n + \left({}^nC_1\right)x^{n-1}h + ... + \left({}^nC_n\right)h^n$ and hence $(x+h)^n - x^n = h(nx^{n-1} + ... + h^{n-1})$. Thus

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{h\left(nx^{n-1} + ... + h^{n-1}\right)}{h}$$

$$= \lim_{h\to 0}\left(nx^{n-1} + ... + h^{n-1}\right), = nx^{n-1}.$$

Alternatively, we may also prove this by induction on n and the product rule as follows. The result is true for $n = 1$, which has been proved earlier. We have

$$\frac{d}{dx}\left(x^n\right) = \frac{d}{dx}\left(x.x^{n-1}\right)$$

$$= \frac{d}{dx}(x).\left(x^{n-1}\right) + x.\frac{d}{dx}\left(x^{n-1}\right) \text{ (by product rule)}$$

$$= 1.x^{n-1} + x.\left((n-1)x^{n-2}\right) \text{ (by induction hypothesis)}$$

$$= x^{n-1} + (n-1)x^{n-1} = nx^{n-1}.$$

Remark The above theorem is true for all powers of x, i.e., n can be any real number (but we will not prove it here).

13.4.2 *Derivative of polynomials and trigonometric functions* We start with the following theorem which tells us the derivative of a polynomial function.

Theorem 7 Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ be a polynomial function, where $a_i s$ are all real numbers and $a_n \neq 0$. Then, the derivative function is given by

$$\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{x-2} + ... + 2a_2 x + a_1.$$

Proof of this theorem is just putting together part (i) of Theorem 5 and Theorem 6.

Example 13 Compute the derivative of $6x^{100} - x^{55} + x$.

Solution A direct application of the above theorem tells that the derivative of the above function is $600x^{99} - 55x^{54} + 1$.

Example 14 Find the derivative of $f(x) = 1 + x + x^2 + x^3 + ... + x^{50}$ at $x = 1$.

Solution A direct application of the above Theorem 7 tells that the derivative of the above function is $1 + 2x + 3x^2 + . . . + 50x^{49}$. At $x = 1$ the value of this function equals

$$1 + 2(1) + 3(1)^2 + .. . + 50(1)^{49} = 1 + 2 + 3 + . . . + 50 = \frac{(50)(51)}{2} = 1275.$$

Example 15 Find the derivative of $f(x) = \dfrac{x+1}{x}$

Solution Clearly this function is defined everywhere except at $x = 0$. We use the quotient rule with $u = x + 1$ and $v = x$. Hence $u' = 1$ and $v' = 1$. Therefore

$$\frac{df(x)}{dx} = \frac{d}{dx}\left(\frac{x+1}{x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} = \frac{1(x)-(x+1)1}{x^2} = -\frac{1}{x^2}$$

Example 16 Compute the derivative of $\sin x$.

Solution Let $f(x) = \sin x$. Then

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \quad \text{(using formula for sin A – sin B)}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right).\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x.1 = \cos x.$$

Example 17 Compute the derivative of $\tan x$.

Solution Let $f(x) = \tan x$. Then

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h \to 0} \left[\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x} \right]$$

$$= \lim_{h \to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x} \quad \text{(using formula for sin (A + B))}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} . \lim_{h \to 0} \frac{1}{\cos(x+h)\cos x}$$

$$= 1. \frac{1}{\cos^2 x} = \sec^2 x \, .$$

Example 18 Compute the derivative of $f(x) = \sin^2 x$.

Solution We use the Leibnitz product rule to evaluate this.

$$\frac{df(x)}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$$= (\sin x)' \sin x + \sin x (\sin x)'$$

$$= (\cos x)\sin x + \sin x(\cos x)$$

$$= 2\sin x \cos x = \sin 2x \cdot$$

EXERCISE 13.2

1. Find the derivative of $x^2 - 2$ at $x = 10$.
2. Find the derivative of $99x$ at $x = 100$.
3. Find the derivative of x at $x = 1$.
4. Find the derivative of the following functions from first principle.

   (i) $x^3 - 27$ (ii) $(x-1)(x-2)$

   (iii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

5. For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \ldots + \frac{x^2}{2} + x + 1 \, .$$

Prove that $f'(1) = 100 f'(0)$.

6. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$ for some fixed real number a.
7. For some constants a and b, find the derivative of

   (i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$ (iii) $\dfrac{x-a}{x-b}$

8. Find the derivative of $\dfrac{x^n - a^n}{x-a}$ for some constant a.
9. Find the derivative of

   (i) $2x - \dfrac{3}{4}$ (ii) $(5x^3+3x-1)(x-1)$

   (iii) $x^{-3}(5+3x)$ (iv) $x^5(3-6x^{-9})$

   (v) $x^{-4}(3-4x^{-5})$ (vi) $\dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

10. Find the derivative of cos x from first principle.
11. Find the derivative of the following functions:

   (i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5\sec x + 4\cos x$

   (iv) $\operatorname{cosec} x$ (v) $3\cot x + 5\operatorname{cosec} x$

   (vi) $5\sin x - 6\cos x + 7$ (vii) $2\tan x - 7\sec x$

Miscellaneous Examples

Example 19 Find the derivative of f from the first principles, where f is given by

(i) $f(x) = \dfrac{2x+3}{x-2}$ (ii) $f(x) = x + \dfrac{1}{x}$

Solution (i) Note that function is not defined at $x = 2$. But, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{2(x+h)+3}{x+h-2} - \dfrac{2x+3}{x-2}}{h}$$

$$= \lim_{h\to 0} \frac{(2x+2h+3)(x-2)-(2x+3)(x+h-2)}{h(x-2)(x+h-2)}$$

$$= \lim_{h\to 0} \frac{(2x+3)(x-2)+2h(x-2)-(2x+3)(x-2)-h(2x+3)}{h(x-2)(x+h-2)}$$

$$= \lim_{h\to 0} \frac{-7}{(x-2)\,(x+h-2)} = -\frac{7}{(x-2)^2}$$

Again, note that the function f' is also not defined at $x = 2$.

(ii) The function is not defined at $x = 0$. But, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(x+h+\frac{1}{x+h}\right)-\left(x+\frac{1}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[h+\frac{1}{x+h}-\frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[h+\frac{x-x-h}{x(x+h)}\right] = \lim_{h\to 0} \frac{1}{h}\left[h\left(1-\frac{1}{x(x+h)}\right)\right]$$

$$= \lim_{h\to 0} \left[1-\frac{1}{x(x+h)}\right] = 1-\frac{1}{x^2}$$

Again, note that the function f' is not defined at $x = 0$.

Example 20 Find the derivative of $f(x)$ from the first principles, where $f(x)$ is

(i) $\sin x + \cos x$ (ii) $x \sin x$

Solution (i) we have $f'(x) = \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sin(x+h)+\cos(x+h)-\sin x-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,\cos h+\cos x\,\sin h+\cos x\cos h-\sin x\,\sin h-\sin x-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{\sin h(\cos x - \sin x) + \sin x(\cos h - 1) + \cos x(\cos h - 1)}{h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h}(\cos x - \sin x) + \lim_{h\to 0} \sin x \frac{(\cos h - 1)}{h} \quad + \lim_{h\to 0} \cos x \frac{(\cos h - 1)}{h}$$

$$= \cos x - \sin x$$

(ii) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)\sin(x+h) - x\sin x}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(\sin x \cos h + \sin h \cos x) - x\sin x}{h}$$

$$= \lim_{h\to 0} \frac{x\sin x(\cos h - 1) + x\cos x \sin h + h(\sin x \cos h + \sin h \cos x)}{h}$$

$$= \lim_{h\to 0} \frac{x\sin x(\cos h - 1)}{h} + \lim_{h\to 0} x\cos x \frac{\sin h}{h} \quad + \lim_{h\to 0}(\sin x \cos h + \sin h \cos x)$$

$$= x\cos x + \sin x$$

Example 21 Compute derivative of

(i) $f(x) = \sin 2x$ (ii) $g(x) = \cot x$

Solution (i) Recall the trigonometric formula $\sin 2x = 2 \sin x \cos x$. Thus

$$\frac{df(x)}{dx} = \frac{d}{dx}(2\sin x\cos x) = 2\frac{d}{dx}(\sin x \cos x)$$

$$= 2\left[(\sin x)' \cos x + \sin x(\cos x)'\right]$$

$$= 2\left[(\cos x)\cos x + \sin x(-\sin x)\right]$$

$$= 2(\cos^2 x - \sin^2 x)$$

(ii) By definition, $g(x) = \cot x = \dfrac{\cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined. $\dfrac{dg}{dx} = \dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

$$= \frac{(\cos x)'(\sin x) - (\cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Alternatively, this may be computed by noting that $\cot x = \frac{1}{\tan x}$. Here, we use the fact that the derivative of *tan x* is *sec² x* which we saw in Example 17 and also that the derivative of the constant function is 0.

$$\frac{dg}{dx} = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{(1)'(\tan x) - (1)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(0)(\tan x) - (\sec x)^2}{(\tan x)^2}$$

$$= \frac{-\sec^2 x}{\tan^2 x} = -\operatorname{cosec}^2 x$$

Example 22 Find the derivative of

(i) $\dfrac{x^5 - \cos x}{\sin x}$ (ii) $\dfrac{x + \cos x}{\tan x}$

Solution (i) Let $h(x) = \frac{x^5 - \cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined.

$$h'(x) = \frac{(x^5 - \cos x)'\sin x - (x^5 - \cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(5x^4 + \sin x)\sin x - (x^5 - \cos x)\cos x}{\sin^2 x}$$

$$= \frac{-x^5 \cos x + 5x^4 \sin x + 1}{(\sin x)^2}$$

(ii) We use quotient rule on the function $\frac{x + \cos x}{\tan x}$ wherever it is defined.

$$h'(x) = \frac{(x + \cos x)' \tan x - (x + \cos x)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(1 - \sin x)\tan x - (x + \cos x)\sec^2 x}{(\tan x)^2}$$

Miscellaneous Exercise on Chapter 13

1. Find the derivative of the following functions from first principles:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x + 1)$ (iv) $\cos\left(x - \frac{\pi}{8}\right)$

Find the derivative of the following functions (it is to be understood that *a, b, c, d, p, q, r* and *s* are fixed non-zero constants and *m* and *n* are integers):

2. $(x + a)$ **3.** $(px + q)\left(\frac{r}{x} + s\right)$ **4.** $(ax + b)(cx + d)^2$

5. $\frac{ax + b}{cx + d}$ **6.** $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$ **7.** $\frac{1}{ax^2 + bx + c}$

8. $\frac{ax + b}{px^2 + qx + r}$ **9.** $\frac{px^2 + qx + r}{ax + b}$ **10.** $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

11. $4\sqrt{x} - 2$ **12.** $(ax + b)^n$ **13.** $(ax + b)^n (cx + d)^m$

14. $\sin(x + a)$ **15.** $\operatorname{cosec} x \cot x$ **16.** $\frac{\cos x}{1 + \sin x}$

17. $\dfrac{\sin x + \cos x}{\sin x - \cos x}$ 18. $\dfrac{\sec x - 1}{\sec x + 1}$ 19. $\sin^n x$

20. $\dfrac{a + b\sin x}{c + d\cos x}$ 21. $\dfrac{\sin(x+a)}{\cos x}$ 22. $x^4(5\sin x - 3\cos x)$

23. $\left(x^2+1\right)\cos x$ 24. $\left(ax^2 + \sin x\right)\left(p + q\cos x\right)$

25. $(x + \cos x)(x - \tan x)$ 26. $\dfrac{4x + 5\sin x}{3x + 7\cos x}$ 27. $\dfrac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$

28. $\dfrac{x}{1 + \tan x}$ 29. $(x + \sec x)(x - \tan x)$ 30. $\dfrac{x}{\sin^n x}$

Summary

- The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point. Similarly the right hand limit.
- Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- For a function f and a real number a, $\lim\limits_{x\to a} f(x)$ and $f(a)$ may not be same (In fact, one may be defined and not the other one).
- For functions f and g the following holds:

$$\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[f(x).g(x)\right] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

- Following are some of the standard limits

$$\lim_{x\to a}\frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

- The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- For functions u and v the following holds:

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$ provided all are defined.

- Following are some of the standard derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Historical Note

In the history of mathematics two names are prominent to share the credit for inventing calculus, Issac Newton (1642 – 1727) and G.W. Leibnitz (1646 – 1717). Both of them independently invented calculus around the seventeenth century. After the advent of calculus many mathematicians contributed for further development of calculus. The rigorous concept is mainly attributed to the great mathematicians, A.L. Cauchy, J.L.Lagrange and Karl Weierstrass. Cauchy gave the foundation of calculus as we have now generally accepted in our textbooks. Cauchy used D' Alembert's limit concept to define the derivative of a function. Starting with definition of a limit, Cauchy gave examples such as the limit of $\frac{\sin\alpha}{\alpha}$ for $\alpha = 0$. He wrote $\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}$, and called the limit for $i \to 0$, the "function derive'e, y' for $f'(x)$".

Before 1900, it was thought that calculus is quite difficult to teach. So calculus became beyond the reach of youngsters. But just in 1900, John Perry and others in England started propagating the view that essential ideas and methods of calculus were simple and could be taught even in schools. F.L. Griffin, pioneered the teaching of calculus to first year students. This was regarded as one of the most daring act in those days.

Today not only the mathematics but many other subjects such as Physics, Chemistry, Economics and Biological Sciences are enjoying the fruits of calculus.

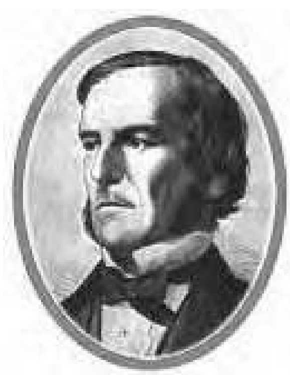
❖

Chapter 14

MATHEMATICAL REASONING

❖ *There are few things which we know which are not capable of mathematical reasoning and when these can not, it is a sign that our knowledge of them is very small and confused and where a mathematical reasoning can be had, it is as great a folly to make use of another, as to grope for a thing in the dark when you have a candle stick standing by you. – ARTHENBOT* ❖

14.1 Introduction



George Boole (1815 1864)

In this Chapter, we shall discuss about some basic ideas of Mathematical Reasoning. All of us know that human beings evolved from the lower species over many millennia. The main asset that made humans "*superior*" to other species was the ability to reason. How well this ability can be used depends on each person's power of reasoning. How to develop this power? Here, we shall discuss the process of reasoning especially in the context of mathematics.

In mathematical language, there are two kinds of reasoning – inductive and deductive. We have already discussed the inductive reasoning in the context of mathematical induction. In this Chapter, we shall discuss some fundamentals of deductive reasoning.

14.2 Statements

The basic unit involved in mathematical reasoning is a *mathematical statement*.

Let us start with two sentences:

In 2003, the president of India was a woman.

An elephant weighs more than a human being.

When we read these sentences, we immediately decide that the first sentence is false and the second is correct. There is no confusion regarding these. In mathematics such sentences are called *statements*.

On the other hand, consider the sentence:

Women are more intelligent than men.

Some people may think it is true while others may disagree. Regarding this sentence we cannot say whether it is always true or false . That means this sentence is ambiguous. Such a sentence is not acceptable as a statement in mathematics.

A sentence is called a mathematically acceptable statement if it is either true or false but not both. Whenever we mention a statement here, it is a "*mathematically acceptable*" statement.

While studying mathematics, we come across many such sentences. Some examples are:

Two plus two equals four.

The sum of two positive numbers is positive.

All prime numbers are odd numbers.

Of these sentences, the first two are *true* and the third one is *false*. There is no ambiguity regarding these sentences. Therefore, they are statements.

Can you think of an example of a sentence which is vague or ambiguous? Consider the sentence:

The sum of x and y is greater than 0

Here, we are not in a position to determine whether it is true or false, unless we know what x and y are. For example, it is false where $x = 1$, $y = -3$ and true when $x = 1$ and $y = 0$. Therefore, this sentence is not a statement. But the sentence:

For any natural numbers x and y, the sum of x and y is greater than 0

is a statement.

Now, consider the following sentences :

How beautiful!

Open the door.

Where are you going?

Are they statements? No, because the first one is an exclamation, the second an order and the third a question. None of these is considered as a statement in mathematical language. Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements. This is because it is not known what time is referred here. For example, the sentence

Tomorrow is Friday

is not a statement. The sentence is correct (true) on a Thursday but not on other days. The same argument holds for sentences with pronouns unless a particular person is referred to and for variable places such as "here", "there" etc., For example, the sentences

She is a mathematics graduate.

Kashmir is far from here.

are not statements.

Here is another sentence

There are 40 days in a month.

Would you call this a statement? Note that the period mentioned in the sentence above is a "variable time" that is any of 12 months. But we know that the sentence is always false (irrespective of the month) since the maximum number of days in a month can never exceed 31. Therefore, this sentence is a statement. So, what makes a sentence a statement is the fact that the sentence is either true or false but not both.

While dealing with statements, we usually denote them by small letters p, q, r,... For example, we denote the statement "*Fire is always hot*" by p. This is also written as

p: Fire is always hot.

Example 1 Check whether the following sentences are statements. Give reasons for your answer.

(i) 8 is less than 6.
(ii) Every set is a finite set.
(iii) The sun is a star.
(iv) Mathematics is fun.
(v) There is no rain without clouds.
(vi) How far is Chennai from here?

Solution (i) This sentence is false because 8 is greater than 6. Hence it is a statement.

(ii) This sentence is also false since there are sets which are not finite. Hence it is a statement.

(iii) It is a scientifically established fact that sun is a star and, therefore, this sentence is always true. Hence it is a statement.

(iv) This sentence is subjective in the sense that for those who like mathematics, it may be fun but for others it may not be. This means that this sentence is not always true. Hence it is not a statement.

(v) It is a scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence it is a statement.

(vi) This is a question which also contains the word "Here". Hence it is not a statement.

The above examples show that whenever we say that a sentence is a statement we should always say why it is so. This "why" of it is more important than the answer.

EXERCISE 14.1

1. Which of the following sentences are statements? Give reasons for your answer.
   - (i) There are 35 days in a month.
   - (ii) Mathematics is difficult.
   - (iii) The sum of 5 and 7 is greater than 10.
   - (iv) The square of a number is an even number.
   - (v) The sides of a quadrilateral have equal length.
   - (vi) Answer this question.
   - (vii) The product of (-1) and 8 is 8.
   - (viii) The sum of all interior angles of a triangle is $180°$.
   - (ix) Today is a windy day.
   - (x) All real numbers are complex numbers.
2. Give three examples of sentences which are not statements. Give reasons for the answers.

14.3 New Statements from Old

We now look into method for producing new statements from those that we already have. An English mathematician, "George Boole" discussed these methods in his book "The laws of Thought" in 1854. Here, we shall discuss two techniques.

As a first step in our study of statements, we look at an important technique that we may use in order to deepen our understanding of mathematical statements. This technique is to ask not only what it means to say that a given statement is true but also what it would mean to say that the given statement is not true.

14.3.1 ***Negation of a statement*** The denial of a statement is called the *negation* of the statement.

Let us consider the statement:

p: New Delhi is a city

The negation of this statement is

It is not the case that New Delhi is a city

This can also be written as

It is false that New Delhi is a city.

This can simply be expressed as

New Delhi is not a city.

Definition 1 If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$, and read as 'not p'.

Note While forming the negation of a statement, phrases like, "It is not the case" or "It is false that" are also used.

Here is an example to illustrate how, by looking at the negation of a statement, we may improve our understanding of it.

Let us consider the statement

p: Everyone in Germany speaks German.

The denial of this sentence tells us that not everyone in Germany speaks German. This does not mean that no person in Germany speaks German. It says merely that at least one person in Germany does not speak German.

We shall consider more examples.

Example 2 Write the negation of the following statements.

(i) Both the diagonals of a rectangle have the same length.

(ii) $\sqrt{7}$ is rational.

Solution (i) This statement says that in a rectangle, both the diagonals have the same length. This means that if you take any rectangle, then both the diagonals have the same length. The negation of this statement is

It is false that both the diagonals in a rectangle have the same length

This means the statement

There is atleast one rectangle whose both diagonals do not have the same length.

(ii) The negation of the statement in (ii) may also be written as

It is not the case that $\sqrt{7}$ *is rational.*

This can also be rewritten as

$\sqrt{7}$ *is not rational.*

Example 3 Write the negation of the following statements and check whether the resulting statements are true,

(i) Australia is a continent.

(ii) There does not exist a quadrilateral which has all its sides equal.

(iii) Every natural number is greater than 0.

(iv) The sum of 3 and 4 is 9.

Solution (i) The negation of the statement is

It is false that Australia is a continent.

This can also be rewritten as

Australia is not a continent.

We know that this statement is false.

(ii) The negation of the statement is

It is not the case that there does not exist a quadrilateral which has all its sides equal.

This also means the following:

There exists a quadrilateral which has all its sides equal.

This statement is true because we know that square is a quadrilaterial such that its four sides are equal.

(iii) The negation of the statement is

It is false that every natural number is greater than 0.

This can be rewritten as

There exists a natural number which is not greater than 0.

This is a false statement.

(iv) The negation is

It is false that the sum of 3 *and* 4 *is* 9.

This can be written as

The sum of 3 and 4 is not equal to 9.

This statement is true.

14.3.2 ***Compound statements*** Many mathematical statements are obtained by combining one or more statements using some connecting words like "and", "or", etc. Consider the following statement

p: There is something wrong with the bulb or with the wiring.

This statement tells us that there is something wrong with the bulb or there is something wrong with the wiring. That means the given statement is actually made up of two smaller statements:

q: There is something wrong with the bulb.

r: There is something wrong with the wiring.

connected by “or”

Now, suppose two statements are given as below:

p: 7 is an odd number.

q: 7 is a prime number.

These two statements can be combined with “and”

r: 7 is both odd and prime number.

This is a compound statement.

This leads us to the following definition:

Definition 2 **A Compound Statement** is a statement which is made up of two or more statements. In this case, each statement is called a ***component statement.***
Let us consider some examples.

Example 4 Find the component statements of the following compound statements.

(i) The sky is blue and the grass is green.

(ii) It is raining and it is cold.

(iii) All rational numbers are real and all real numbers are complex.

(iv) 0 is a positive number or a negative number.

Solution Let us consider one by one

(i) The component statements are

p: The sky is blue.

q: The grass is green.

The connecting word is ‘and’.

(ii) The component statements are

p: It is raining.

q: It is cold.

The connecting word is ‘and’.

(iii) The component statements are

p: All rational numbers are real.

q: All real numbers are complex.

The connecting word is ‘and’.

(iv) The component statements are

p: 0 is a positive number.

q: 0 is a negative number.

The connecting word is 'or'.

Example 5 Find the component statements of the following and check whether they are true or not.

(i) A square is a quadrilateral and its four sides equal.

(ii) All prime numbers are either even or odd.

(iii) A person who has taken Mathematics or Computer Science can go for MCA.

(iv) Chandigarh is the capital of Haryana and UP.

(v) $\sqrt{2}$ is a rational number or an irrational number.

(vi) 24 is a multiple of 2, 4 and 8.

Solution (i) The component statements are

p: A square is a quadrilateral.

q: A square has all its sides equal.

We know that both these statements are true. Here the connecting word is 'and'.

(ii) The component statements are

p: All prime numbers are odd number.

q: All prime numbers are even number.

Both these statements are false and the connecting word is 'or'.

(iii) The component statements are

p: A person who has taken Mathematics can go for MCA.

q: A person who has taken computer science can go for MCA.

Both these statements are true. Here the connecting word is 'or'.

(iv) The component statements are

p: Chandigarh is the capital of Haryana.

q: Chandigarh is the capital of UP.

The first statement is true but the second is false. Here the connecting word is 'and'.

(v) The component statements are

p: $\sqrt{2}$ *is a rational number.*

q: $\sqrt{2}$ *is an irrational number.*

The first statement is false and second is true. Here the connecting word is ‘or’.

(vi) The component statements are

p: 24 *is a multiple of* 2.

q: 24 *is a multiple of* 4.

r: 24 *is a multiple of* 8.

All the three statements are true. Here the connecting words are ‘and’.

Thus, we observe that compound statements are actually made-up of two or more statements connected by the words like “and”, “or”, etc. These words have special meaning in mathematics. We shall discuss this mattter in the following section.

EXERCISE 14.2

1. Write the negation of the following statements:
   (i) Chennai is the capital of Tamil Nadu.
   (ii) $\sqrt{2}$ is not a complex number
   (iii) All triangles are not equilateral triangle.
   (iv) The number 2 is greater than 7.
   (v) Every natural number is an integer.
2. Are the following pairs of statements negations of each other:
   (i) The number x is not a rational number.
   The number x is not an irrational number.
   (ii) The number x is a rational number.
   The number x is an irrational number.
3. Find the component statements of the following compound statements and check whether they are true or false.
   (i) Number 3 is prime or it is odd.
   (ii) All integers are positive or negative.
   (iii) 100 is divisible by 3, 11 and 5.

14.4 Special Words/Phrases

Some of the connecting words which are found in compound statements like **“And”,**

"Or", etc. are often used in Mathematical Statements. These are called connectives. When we use these compound statements, it is necessary to understand the role of these words. We discuss this below.

14.4.1 ***The word "And"*** Let us look at a compound statement with "And".

p: A point occupies a position and its location can be determined.

The statement can be broken into two component statements as

q: A point occupies a position.

r: Its location can be determined.

Here, we observe that both statements are true.

Let us look at another statement.

p: 42 *is divisible by* 5, 6 *and* 7.

This statement has following component statements

q: 42 is divisible by 5.

r: 42 is divisible by 6.

s: 42 is divisible by 7.

Here, we know that the first is false while the other two are true.

We have the following rules regarding the connective "And"

1. The compound statement with 'And' is true if all its component statements are true.
2. The component statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).

Example 6 Write the component statements of the following compound statements and check whether the compound statement is true or false.

(i) A line is straight and extends indefinitely in both directions.

(ii) 0 is less than every positive integer and every negative integer.

(iii) All living things have two legs and two eyes.

Solution (i) The component statements are

p: A line is straight.

q: A line extends indefinitely in both directions.

Both these statements are true, therefore, the compound statement is true.

(ii) The component statements are

p: 0 is less than every positive integer.

q: 0 is less than every negative integer.

The second statement is false. Therefore, the compound statement is false.

(iii) The two component statements are

p: All living things have two legs.

q: All living things have two eyes.

Both these statements are false. Therefore, the compound statement is false.

Now, consider the following statement.

p: A mixture of alcohol and water can be separated by chemical methods.

This sentence cannot be considered as a compound statement with "And". Here the word "And" refers to two things – alcohol and water.

This leads us to an important note.

Note Do not think that a statement with "And" is always a compound statement as shown in the above example. Therefore, the word "And" is not used as a conjunction.

14.4.2 ***The word "Or"*** Let us look at the following statement.

p: Two lines in a plane either intersect at one point or they are parallel.

We know that this is a true statement. What does this mean? This means that if two lines in a plane intersect, then they are not parallel. Alternatively, if the two lines are not parallel, then they intersect at a point. That is this statement is true in both the situations.

In order to understand statements with "Or" we first notice that the word "Or" is used in two ways in English language. Let us first look at the following statement.

p: An ice cream or pepsi is available with a Thali in a restaurant.

This means that a person who does not want ice cream can have a pepsi along with *Thali* or one does not want pepsi can have an ice cream along with *Thali*. That is, who do not want a pepsi can have an ice cream. A person cannot have both ice cream and pepsi. This is called an ***exclusive "Or".***

Here is another statement.

A student who has taken biology or chemistry can apply for M.Sc. microbiology programme.

Here we mean that the students who have taken both biology and chemistry can apply for the microbiology programme, as well as the students who have taken only one of these subjects. In this case, we are using **inclusive "Or".**

It is important to note the difference between these two ways because we require this when we check whether the statement is true or not.

Let us look at an example.

Example 7 For each of the following statements, determine whether an **inclusive "Or"** or **exclusive "Or"** is used. Give reasons for your answer.

(i) To enter a country, you need a passport or a voter registration card.

(ii) The school is closed if it is a holiday or a Sunday.

(iii) Two lines intersect at a point or are parallel.

(iv) Students can take French or Sanskrit as their third language.

Solution (i) Here "Or" is inclusive since a person can have both a passport and a voter registration card to enter a country.

(ii) Here also "Or" is inclusive since school is closed on holiday as well as on Sunday.

(iii) Here "Or" is exclusive because it is not possible for two lines to intersect and parallel together.

(iv) Here also "Or" is exclusive because a student cannot take both French and Sanskrit.

Rule for the compound statement with 'Or'

1. A compound statement with an 'Or' is true when one component statement is true or both the component statements are true.
2. A compound statement with an 'Or' is false when both the component statements are false.

For example, consider the following statement.

p: Two lines intersect at a point or they are parallel

The component statements are

q: *Two lines intersect at a point.*

r: *Two lines are parallel.*

Then, when q is true r is false and when r is true q is false. Therefore, the compound statement p is true.

Consider another statement.

p: 125 *is a multiple of* 7 *or* 8.

Its component statements are

q: 125 *is a multiple of* 7.

r: 125 *is a multiple of* 8.

Both q and r are false. Therefore, the compound statement p is false.

Again, consider the following statement:

p: The school is closed, if there is a holiday or Sunday.

The component statements are

q: School is closed if there is a holiday.

r: School is closed if there is a Sunday.

Both q and r are true, therefore, the compound statement is true.
Consider another statement.

p: Mumbai is the capital of Kolkata or Karnataka.

The component statements are

q: Mumbai is the capital of Kolkata.

r: Mumbai is the capital of Karnataka.

Both these statements are false. Therefore, the compound statement is false.
Let us consider some examples.

Example 8 Identify the type of "Or" used in the following statements and check whether the statements are true or false:

(i) $\sqrt{2}$ is a rational number or an irrational number.

(ii) To enter into a public library children need an identity card from the school or a letter from the school authorities.

(iii) A rectangle is a quadrilateral or a 5-sided polygon.

Solution (i) The component statements are

p: $\sqrt{2}$ *is a rational number.*

q: $\sqrt{2}$ *is an irrational number.*

Here, we know that the first statement is false and the second is true and "Or" is exclusive. Therefore, the compound statement is true.

(ii) The component statements are

p: To get into a public library children need an identity card.

q: To get into a public library children need a letter from the school authorities.

Children can enter the library if they have either of the two, an identity card or the letter, as well as when they have both. Therefore, it is inclusive "Or" the compound statement is also true when children have both the card and the letter.

(iii) Here "Or" is exclusive. When we look at the component statements, we get that the statement is true.

14.4.3 ***Quantifiers*** Quantifiers are phrases like, "There exists" and "For all". Another phrase which appears in mathematical statements is "there exists". For example, consider the statement. p: *There exists a rectangle whose all sides are equal.* This means that there is atleast one rectangle whose all sides are equal.

A word closely connected with "there exists" is "for every" (or for all). Consider a statement.

p: For every prime number p, $\sqrt{p}$ *is an irrational number.*

This means that if S denotes the set of all prime numbers, then for all the members p of the set S, $\sqrt{p}$ is an irrational number.

In general, a mathematical statement that says "for every" can be interpreted as saying that all the members of the given set S where the property applies must satisfy that property.

We should also observe that it is important to know precisely where in the sentence a given connecting word is introduced. For example, compare the following two sentences:

1. For every positive number x there exists a positive number y such that $y < x$.
2. There exists a positive number y such that for every positive number x, we have $y < x$.

Although these statements may look similar, they do not say the same thing. As a matter of fact, (1) is true and (2) is false. Thus, in order for a piece of mathematical writing to make sense, all of the symbols must be carefully introduced and each symbol must be introduced at precisely the right place – not too early and not too late.

The words "And" and "Or" are called *connectives* and "There exists" and "For all" are called *quantifiers.*

Thus, we have seen that many mathematical statements contain some special words and it is important to know the meaning attached to them, especially when we have to check the validity of different statements.

EXERCISE 14.3

1. For each of the following compound statements first identify the connecting words and then break it into component statements.
   (i) All rational numbers are real and all real numbers are not complex.
   (ii) Square of an integer is positive or negative.
   (iii) The sand heats up quickly in the Sun and does not cool down fast at night.
   (iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

2. Identify the quantifier in the following statements and write the negation of the statements.
   (i) There exists a number which is equal to its square.
   (ii) For every real number x, x is less than $x + 1$.
   (iii) There exists a capital for every state in India.
3. Check whether the following pair of statements are negation of each other. Give reasons for your answer.
   (i) $x + y = y + x$ is true for every real numbers x and y.
   (ii) There exists real numbers x and y for which $x + y = y + x$.
4. State whether the "Or" used in the following statements is "exclusive "or" inclusive. Give reasons for your answer.
   (i) Sun rises or Moon sets.
   (ii) To apply for a driving licence, you should have a ration card or a passport.
   (iii) All integers are positive or negative.

14.5 Implications

In this Section, we shall discuss the implications of "if-then", "only if" and "if and only if".

The statements with "if-then" are very common in mathematics. For example, consider the statement.

r: If you are born in some country, then you are a citizen of that country.

When we look at this statement, we observe that it corresponds to two statements p and q given by

p : you are born in some country.

q : you are citizen of that country.

Then the sentence "if p then q" says that in the event if p is true, then q must be true.

One of the most important facts about the sentence "if p then q" is that it does not say any thing (or places no demand) on q when p is false. For example, if you are not born in the country, then you cannot say anything about q. To put it in other words" not happening of p has no effect on happening of q.

Another point to be noted for the statement "if p then q" is that the statement does not imply that p happens.

There are several ways of understanding "if p then q" statements. We shall illustrate these ways in the context of the following statement.

r: If a number is a multiple of 9, then it is a multiple of 3.

Let p and q denote the statements

p : a number is a multiple of 9.

q: a number is a multiple of 3.

Then, if p then q is the same as the following:

1. p **implies** q is denoted by $p \Rightarrow q$. The symbol $\Rightarrow$ stands for implies.
   This says that a number is a multiple of 9 implies that it is a multiple of 3.
2. p is a sufficient condition for q.
   This says that knowing that a number as a multiple of 9 is sufficient to conclude that it is a multiple of 3.
3. p only if q.
   This says that a number is a multiple of 9 only if it is a multiple of 3.
4. q is a necessary condition for p.
   This says that when a number is a multiple of 9, it is necessarily a multiple of 3.
5. $\sim q$ implies $\sim p$.
   This says that if a number is not a multiple of 3, then it is not a multiple of 9.

14.5.1 ***Contrapositive and converse*** Contrapositive and converse are certain other statements which can be formed from a given statement with "if-then".

For example, let us consider the following "if-then" statement.

If the physical environment changes, then the biological environment changes.

Then the contrapositive of this statement is

If the biological environment does not change, then the physical environment does not change.

Note that both these statements convey the same meaning.

To understand this, let us consider more examples.

Example 9 Write the contrapositive of the following statement:

(i) If a number is divisible by 9, then it is divisible by 3.

(ii) If you are born in India, then you are a citizen of India.

(iii) If a triangle is equilateral, it is isosceles.

Solution The contrapositive of the these statements are

(i) If a number is not divisible by 3, it is not divisible by 9.

(ii) If you are not a citizen of India, then you were not born in India.

(iii) If a triangle is not isosceles, then it is not equilateral.

The above examples show the contrapositive of the statement if p, then q is "if $\sim q$, then $\sim p$".

Next, we shall consider another term called *converse*.

The converse of a given statement "if p, then q" is if q, then p.

For example, the converse of the statement

p: If a number is divisible by 10*, it is divisible by* 5 *is*

q: If a number is divisible by 5*, then it is divisible by* 10.

Example 10 Write the converse of the following statements.

(i) If a number n is even, then n^2 is even.

(ii) If you do all the exercises in the book, you get an A grade in the class.

(iii) If two integers a and b are such that $a > b$, then $a - b$ is always a positive integer.

Solution The converse of these statements are

(i) If a number n^2 is even, then n is even.

(ii) If you get an A grade in the class, then you have done all the exercises of the book.

(iii) If two integers a and b are such that $a - b$ is always a positive integer, then $a > b$.

Let us consider some more examples.

Example 11 For each of the following compound statements, first identify the corresponding component statements. Then check whether the statements are true or not.

(i) If a triangle ABC is equilateral, then it is isosceles.

(ii) If a and b are integers, then ab is a rational number.

Solution (i) The component statements are given by

p : Triangle ABC is equilateral.

q : Triangle ABC is Isosceles.

Since an equilateral triangle is isosceles, we infer that the given compound statement is true.

(ii) The component statements are given by

p : a and b are integers.

q : ab is a rational number.

since the product of two integers is an integer and therefore a rational number, the compound statement is true.

'If and only if', represented by the symbol '$\Leftrightarrow$' means the following equivalent forms for the given statements p and q.

(i) p if and only if q

(ii) q if and only if p

(iii) p is necessary and sufficient condition for q and vice-versa

(iv) $p \Leftrightarrow q$

Consider an example.

Example 12 Given below are two pairs of statements. Combine these two statements using "if and only if ".

(i) *p:* If a rectangle is a square, then all its four sides are equal.

q: If all the four sides of a rectangle are equal, then the rectangle is a square.

(ii) *p:* If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q: If a number is divisible by 3, then the sum of its digits is divisible by 3.

Solution (i) A rectangle is a square if and only if all its four sides are equal.

(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

EXERCISE 14.4

1. Rewrite the following statement with "if-then" in five different ways conveying the same meaning.

If a natural number is odd, then its square is also odd.

2. Write the contrapositive and converse of the following statements.

(i) If x is a prime number, then x is odd.

(ii) If the two lines are parallel, then they do not intersect in the same plane.

(iii) Something is cold implies that it has low temperature.

(iv) You cannot comprehend geometry if you do not know how to reason deductively.

(v) x is an even number implies that x is divisible by 4.

3. Write each of the following statements in the form "if-then"

(i) You get a job implies that your credentials are good.

(ii) The Bannana trees will bloom if it stays warm for a month.

(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

(iv) To get an A^+ in the class, it is necessary that you do all the exercises of the book.

4. Given statements in (a) and (b). Identify the statements given below as contrapositive or converse of each other.

(a) If you live in Delhi, then you have winter clothes.

(i) If you do not have winter clothes, then you do not live in Delhi.

(ii) If you have winter clothes, then you live in Delhi.

(b) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

(ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

14.6 Validating Statements

In this Section, we will discuss when a statement is true. To answer this question, one must answer all the following questions.

What does the statement mean? What would it mean to say that this statement is true and when this statement is not true?

The answer to these questions depend upon which of the special words and phrases "and", "or", and which of the implications "if and only", "if-then", and which of the quantifiers "for every", "there exists", appear in the given statement.

Here, we shall discuss some techniques to find when a statement is valid.

We shall list some general rules for checking whether a statement is true or not.

Rule 1 ***If p and q are mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed.***

Step-1 Show that the statement p is true.

Step-2 Show that the statement q is true.

Rule 2 ***Statements with "Or"***

If p and q are mathematical statements , then in order to show that the statement "p or q" is true, one must consider the following.

Case 1 By assuming that p is false, show that q must be true.

Case 2 By assuming that q is false, show that p must be true.

Rule 3 ***Statements with "If-then"***

In order to prove the statement "if p then q" we need to show that *any one* of the following case is true.

Case 1 By assuming that p is true, prove that q must be true.(Direct method)

Case 2 By assuming that q is false, prove that p must be false.(Contrapositive method)

Rule 4 ***Statements with "if and only if "***

In order to prove the statement "p if and only if q", we need to show.

(i) *If p is true, then q is true* and (ii) *If q is true, then p is true*

Now we consider some examples.

Example 13 Check whether the following statement is true or not.
If $x, y \in \mathbf{Z}$ are such that x and y are odd, then xy is odd.

Solution Let $p : x, y \in \mathbf{Z}$ such that x and y are odd

$$q : xy \text{ is odd}$$

To check the validity of the given statement, we apply Case 1 of Rule 3. That is assume that if p is true, then q is true.
p is true means that x and y are odd integers. Then

$x = 2m + 1$, for some integer m. $y = 2n + 1$, for some integer n. Thus

$$\begin{aligned} xy &= (2m + 1)(2n + 1) \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

This shows that xy is odd. Therefore, the given statement is true.

Suppose we want to check this by using Case 2 of Rule 3, then we will proceed as follows.

We assume that q is not true. This implies that we need to consider the negation of the statement q. This gives the statement

$$\sim q : \text{Product } xy \text{ is even.}$$

This is possible only if either x or y is even. This shows that p is not true. Thus we have shown that

$$\sim q \Rightarrow \sim p$$

Note The above example illustrates that to prove $p \Rightarrow q$, it is enough to show $\sim q \Rightarrow \sim p$ which is the contrapositive of the statement $p \Rightarrow q$.

Example 14 Check whether the following statement is true or false by proving its contrapositive. If $x, y \in \mathbf{Z}$ such that xy is odd, then both x and y are odd.

Solution Let us name the statements as below *p : xy is odd.*

q : both x and y are odd.

We have to check whether the statement $p \Rightarrow q$ is true or not, that is, by checking its contrapositive statement i.e., $\sim q \Rightarrow \sim p$

Now $\sim q$: It is false that both x and y are odd. This implies that x (or y) is even.

Then $x = 2n$ for some integer n.

Therefore, $xy = 2ny$ for some integer n. This shows that xy is even. That is $\sim p$ is true. Thus, we have shown that $\sim q \Rightarrow \sim p$ and hence the given statement is true.

Now what happens when we combine an implication and its converse? Next, we shall discuss this.

Let us consider the following statements.

p : A tumbler is half empty.

q : A tumbler is half full.

We know that if the first statement happens, then the second happens and also if the second happens, then the first happens. We can express this fact as

If a tumbler is half empty, then it is half full.

If a tumbler is half full, then it is half empty.

We combine these two statements and get the following:

A tumbler is half empty if and only if it is half full.

Now, we discuss another method.

14.5.1 ***By Contradiction*** Here to check whether a statement p is true, we assume that p is not true i.e. $\sim p$ is true. Then, we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.

Example 15 Verify by the method of contradiction.

p: $\sqrt{7}$ is irrational

Solution In this method, we assume that the given statement is false. That is we assume that $\sqrt{7}$ is rational. This means that there exists positive integers a and b such that $\sqrt{7} = \frac{a}{b}$, where a and b have no common factors. Squaring the equation, we get $7 = \frac{a^2}{b^2} \Rightarrow a^2 = 7b^2 \Rightarrow 7$ divides a. Therefore, there exists an integer c such that $a = 7c$. Then $a^2 = 49c^2$ and $a^2 = 7b^2$

Hence, $7b^2 = 49c^2 \Rightarrow b^2 = 7c^2 \Rightarrow 7$ divides b. But we have already shown that 7 divides a. This implies that 7 is a common factor of both of a and b which contradicts our earlier assumption that a and b have no common factors. This shows that the assumption $\sqrt{7}$ is rational is wrong. Hence, the statement $\sqrt{7}$ is irrational is true.

Next, we shall discuss a method by which we may show that a statement is false. The method involves giving an ***example of a situation where the statement is not valid.*** Such an example is called a ***counter example***. The name itself suggests that this is an example to counter the given statement.

Example 16 By giving a counter example, show that the following statement is false. If n is an odd integer, then n is prime.

Solution The given statement is in the form "if p then q" we have to show that this is false. For this purpose we need to show that if p then $\sim q$. To show this we look for an odd integer n which is not a prime number. 9 is one such number. So $n = 9$ is a counter example. Thus, we conclude that the given statement is false.

In the above, we have discussed some techniques for checking whether a statement is true or not.

Note In mathematics, counter examples are used to disprove the statement. However, generating examples in favour of a statement do not provide validity of the statement.

EXERCISE 14.5

1. Show that the statement
   p: "If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by
   (i) direct method, (ii) method of contradiction, (iii) method of contrapositive
2. Show that the statement "For any real numbers a and b, $a^2 = b^2$ implies that $a = b$" is not true by giving a counter-example.
3. Show that the following statement is true by the method of contrapositive.
   p: *If x is an integer and x^2 is even, then x is also even.*
4. By giving a counter example, show that the following statements are not true.
   (i) p: If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.
   (ii) q: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

5. Which of the following statements are true and which are false? In each case give a valid reason for saying so.
   (i) *p:* Each radius of a circle is a chord of the circle.
   (ii) *q:* The centre of a circle bisects each chord of the circle.
   (iii) *r:* Circle is a particular case of an ellipse.
   (iv) *s:* If x and y are integers such that $x > y$, then $-x < -y$.
   (v) t : $\sqrt{11}$ is a rational number.

Miscellaneous Examples

Example 17 Check whether "Or" used in the following compound statement is exclusive or inclusive? Write the component statements of the compound statements and use them to check whether the compound statement is true or not. Justify your answer.

t: you are wet when it rains or you are in a river.

Solution "Or" used in the given statement is inclusive because it is possible that it rains and you are in the river.

The component statements of the given statement are

p : you are wet when it rains.

q : You are wet when you are in a river.

Here both the component statements are true and therefore, the compound statement is true.

Example 18 Write the negation of the following statements:

(i) *p:* For every real number x, $x^2 > x$.

(ii) *q:* There exists a rational number x such that $x^2 = 2$.

(iii) *r:* All birds have wings.

(iv) *s:* All students study mathematics at the elementary level.

Solution (i) The negation of p is "It is false that p is" which means that the condition $x^2 > x$ does not hold for all real numbers. This can be expressed as

$\sim p$: There exists a real number x such that $x^2 < x$.

(ii) Negation of q is "it is false that q", Thus $\sim q$ is the statement.

$\sim q$: There does not exist a rational number x such that $x^2 = 2$.

This statement can be rewritten as

$\sim q$: For all real numbers x, $x^2 \neq 2$

(iii) The negation of the statement is

$\sim r$: There exists a bird which have no wings.

(iv) The negation of the given statement is $\sim s$: There exists a student who does not study mathematics at the elementary level.

Example 19 Using the words "necessary and sufficient" rewrite the statement "The integer n is odd if and only if n^2 is odd". Also check whether the statement is true.

Solution The necessary and sufficient condition that the integer n be odd is n^2 must be odd. Let p and q denote the statements

p : the integer n is odd.

q : n^2 is odd.

To check the validity of "p if q", we have to check whether "if p then q" and "if q then p" is true.

Case 1 If p, then q

If p, then q is the statement:

If the integer n is odd, then n^2 is odd. We have to check whether this statement is true. Let us assume that n is odd. Then $n = 2k + 1$ when k is an integer. Thus

$$n^2 = (2k + 1)^2$$

$$= 4k^2 + 4k + 1$$

Therefore, n^2 is one more than an even number and hence is odd.

Case 2 If q, then p

If q, then p is the statement

If n is an integer and n^2 is odd, then n is odd.

We have to check whether this statement is true. We check this by contrapositive method. The contrapositive of the given statement is:

If n is an even integer, then n^2 is an even integer

n is even implies that $n = 2k$ for some k. Then $n^2 = 4k^2$. Therefore, n^2 is even.

Example 20 For the given statements identify the necessary and sufficient conditions.

t: If you drive over 80 km per hour, then you will get a fine.

Solution Let p and q denote the statements:

p : you drive over 80 km per hour.

q : you will get a fine.

The implication if p, then q indicates that p is sufficient for q. That is driving over 80 km per hour is sufficient to get a fine.

Here the sufficient condition is "driving over 80 km per hour":

Similarly, if p, then q also indicates that q is necessary for p. That is

When you drive over 80 km per hour, you will necessarily get a fine.
Here the necessary condition is "getting a fine".

Miscellaneous Exercise on Chapter 14

1. Write the negation of the following statements:
   (i) p: For every positive real number x, the number $x - 1$ is also positive.
   (ii) q: All cats scratch.
   (iii) r: For every real number x, either $x > 1$ or $x < 1$.
   (iv) s: There exists a number x such that $0 < x < 1$.
2. State the converse and contrapositive of each of the following statements:
   (i) p: A positive integer is prime only if it has no divisors other than 1 and itself.
   (ii) q: I go to a beach whenever it is a sunny day.
   (iii) r: If it is hot outside, then you feel thirsty.
3. Write each of the statements in the form "if p, then q"
   (i) p: It is necessary to have a password to log on to the server.
   (ii) q: There is traffic jam whenever it rains.
   (iii) r: You can access the website only if you pay a subsciption fee.
4. Rewrite each of the following statements in the form "p if and only if q"
   (i) p: If you watch television, then your mind is free and if your mind is free, then you watch television.
   (ii) q: For you to get an A grade, it is necessary and sufficient that you do all the homework regularly.
   (iii) r: If a quadrilateral is equiangular, then it is a rectangle and if a quadrilateral is a rectangle, then it is equiangular.
5. Given below are two statements

p : *25 is a multiple of 5.*

q : *25 is a multiple of 8.*

Write the compound statements connecting these two statements with "And" and "Or". In both cases check the validity of the compound statement.

6. Check the validity of the statements given below by the method given against it.
   (i) p: The sum of an irrational number and a rational number is irrational (by contradiction method).
   (ii) q: If n is a real number with $n > 3$, then $n^2 > 9$ (by contradiction method).
7. Write the following statement in five different ways, conveying the same meaning.
   p: *If a triangle is equiangular, then it is an obtuse angled triangle.*

Summary

- A mathematically acceptable statement is a sentence which is either true or false.
- Explained the terms:
  - Negation of a statement p: If p denote a statement, then the negation of p is denoted by $\sim p$.
  - Compound statements and their related component statements:
    A statement is a compound statement if it is made up of two or more smaller statements. The smaller statements are called component statements of the compound statement.
  - The role of "And", "Or", "There exists" and "For every" in compound statements.
  - The meaning of implications "If ", "only if ", " if and only if ".
    A sentence with if p, then q can be written in the following ways.
  - p implies q (denoted by $p \Rightarrow q$)
  - p is a sufficient condition for q
  - q is a necessary condition for p
  - p only if q
  - $\sim q$ implies $\sim p$
  - The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$. The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.
    $p \Rightarrow q$ together with its converse, gives p if and only if q.
- The following methods are used to check the validity of statements:
  (i) direct method
  (ii) contrapositive method
  (iii) method of contradiction
  (iv) using a counter example.

Historical Note

The first treatise on logic was written by *Aristotle* (384 B.C.-322 B.C.). It was a collection of rules for deductive reasoning which would serve as a basis for the study of every branch of knowledge. Later, in the seventeenth century, German mathematician G. W. Leibnitz (1646 – 1716 A.D.) conceived the idea of using symbols in logic to mechanise the process of deductive reasoning. His idea was realised in the nineteenth century by the English mathematician *George Boole* (1815–1864 A.D.) and *Augustus De Morgan* (1806–1871 A.D.) , who founded the modern subject of symbolic logic.

Chapter 15

STATISTICS

❖ *"Statistics may be rightly called the science of averages and their estimates." – A.L.BOWLEY & A.L. BODDINGTON* ❖

15.1 Introduction

Karl Pearson (1857 1936)

We know that statistics deals with data collected for specific purposes. We can make decisions about the data by analysing and interpreting it. In earlier classes, we have studied methods of representing data graphically and in tabular form. This representation reveals certain salient features or characteristics of the data. We have also studied the methods of finding a representative value for the given data. This value is called the measure of central tendency. Recall mean (arithmetic mean), median and mode are three measures of central tendency. A *measure of central tendency* gives us a rough idea where data points are centred. But, in order to make better interpretation from the data, we should also have an idea how the data are scattered or how much they are bunched around a measure of central tendency.

Consider now the runs scored by two batsmen in their last ten matches as follows:

Batsman A : 30, 91, 0, 64, 42, 80, 30, 5, 117, 71
Batsman B : 53, 46, 48, 50, 53, 53, 58, 60, 57, 52

Clearly, the mean and median of the data are

	Batsman A	Batsman B
Mean	53	53
Median	53	53

Recall that, we calculate the mean of a data (denoted by $\bar{x}$) by dividing the sum of the observations by the number of observations, i.e.,

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Also, the median is obtained by first arranging the data in ascending or descending order and applying the following rule.

If the number of observations is odd, then the median is $\left(\frac{n+1}{2}\right)^{\text{th}}$ observation.

If the number of observations is even, then median is the mean of $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2}+1\right)^{\text{th}}$ observations.

We find that the mean and median of the runs scored by both the batsmen A and B are same i.e., 53. Can we say that the performance of two players is same? Clearly No, because the variability in the scores of batsman A is from 0 (minimum) to 117 (maximum). Whereas, the range of the runs scored by batsman B is from 46 to 60.

Let us now plot the above scores as dots on a number line. We find the following diagrams:

For batsman A

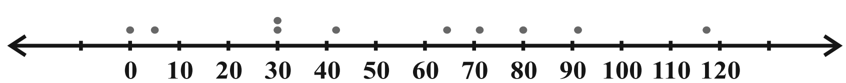


Fig 15.1

For batsman B

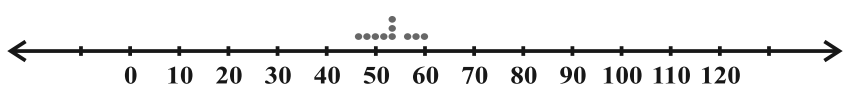


Fig 15.2

We can see that the dots corresponding to batsman B are close to each other and are clustering around the measure of central tendency (mean and median), while those corresponding to batsman A are scattered or more spread out.

Thus, the measures of central tendency are not sufficient to give complete information about a given data. Variability is another factor which is required to be studied under statistics. Like '*measures of central tendency*' we want to have a single number to describe variability. This single number is called a '*measure of dispersion*'. In this Chapter, we shall learn some of the important measures of dispersion and their methods of calculation for ungrouped and grouped data.

15.2 Measures of Dispersion

The dispersion or scatter in a data is measured on the basis of the observations and the types of the measure of central tendency, used there. There are following measures of dispersion:

(i) Range, (ii) Quartile deviation, (iii) Mean deviation, (iv) Standard deviation.

In this Chapter, we shall study all of these measures of dispersion except the quartile deviation.

15.3 Range

Recall that, in the example of runs scored by two batsmen A and B, we had some idea of variability in the scores on the basis of minimum and maximum runs in each series. To obtain a single number for this, we find the difference of maximum and minimum values of each series. This difference is called the 'Range' of the data.

In case of batsman A, Range = 117 – 0 = 117 and for batsman B, Range = 60 – 46 = 14. Clearly, Range of A > Range of B. Therefore, the scores are scattered or dispersed in case of A while for B these are close to each other.

Thus, Range of a series = Maximum value – Minimum value.

The range of data gives us a rough idea of variability or scatter but does not tell about the dispersion of the data from a measure of central tendency. For this purpose, we need some other measure of variability. Clearly, such measure must depend upon the difference (or deviation) of the values from the central tendency.

The important measures of dispersion, which depend upon the deviations of the observations from a central tendency are mean deviation and standard deviation. Let us discuss them in detail.

15.4 Mean Deviation

Recall that the deviation of an observation x from a fixed value *'a'* is the difference $x - a$. In order to find the dispersion of values of x from a central value *'a'*, we find the deviations about a. An absolute measure of dispersion is the mean of these deviations. To find the mean, we must obtain the sum of the deviations. But, we know that a measure of central tendency lies between the maximum and the minimum values of the set of observations. Therefore, some of the deviations will be negative and some positive. Thus, the sum of deviations may vanish. Moreover, the sum of the deviations from mean ($\bar{x}$) is zero.

Also $$\text{Mean of deviations} = \frac{\text{Sum of deviations}}{\text{Number of observations}} = \frac{0}{n} = 0$$

Thus, finding the mean of deviations about mean is not of any use for us, as far as the measure of dispersion is concerned.

Remember that, in finding a suitable measure of dispersion, we require the distance of each value from a central tendency or a fixed number 'a'. Recall, that the absolute value of the difference of two numbers gives the distance between the numbers when represented on a number line. Thus, to find the measure of dispersion from a fixed number 'a' we may take the mean of the absolute values of the deviations from the central value. This mean is called the '*mean deviation*'. Thus mean deviation about a central value 'a' is the mean of the absolute values of the deviations of the observations from 'a'. The mean deviation from 'a' is denoted as M.D. (a). Therefore,

$$\text{M.D.}(a) = \frac{\text{Sum of absolute values of deviations from '}a\text{'}}{\text{Number of observations}}.$$

Remark Mean deviation may be obtained from any measure of central tendency. However, mean deviation from mean and median are commonly used in statistical studies.

Let us now learn how to calculate mean deviation about mean and mean deviation about median for various types of data

15.4.1 ***Mean deviation for ungrouped data*** Let n observations be $x_1, x_2, x_3, ..., x_n$. The following steps are involved in the calculation of mean deviation about mean or median:

Step 1 Calculate the measure of central tendency about which we are to find the mean deviation. Let it be 'a'.

Step 2 Find the deviation of each x_i from a, i.e., $x_1 - a, x_2 - a, x_3 - a, . . . , x_n - a$

Step 3 Find the absolute values of the deviations, i.e., drop the minus sign (–), if it is there, i.e., $|x_1 - a|, |x_2 - a|, |x_3 - a|, ..., |x_n - a|$

Step 4 Find the mean of the absolute values of the deviations. This mean is the mean deviation about a, i.e.,

$$\text{M.D.}(a) = \frac{\sum_{i=1}^{n} |x_i - a|}{n}$$

Thus $\quad \text{M.D.}(\bar{x}) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}|$, where $\bar{x}$ = Mean

and $\quad \text{M.D.}(\text{M}) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \text{M}|$, where M = Median

Note In this Chapter, we shall use the symbol M to denote median unless stated otherwise. Let us now illustrate the steps of the above method in following examples.

Example 1 Find the mean deviation about the mean for the following data:

6, 7, 10, 12, 13, 4, 8, 12

Solution We proceed step-wise and get the following:

Step 1 Mean of the given data is

$$\bar{x} = \frac{6+7+10+12+13+4+8+12}{8} = \frac{72}{8} = 9$$

Step 2 The deviations of the respective observations from the mean $\bar{x}$, i.e., $x_i - \bar{x}$ are

6 – 9, 7 – 9, 10 – 9, 12 – 9, 13 – 9, 4 – 9, 8 – 9, 12 – 9,

or –3, –2, 1, 3, 4, –5, –1, 3

Step 3 The absolute values of the deviations, i.e., $|x_i - \bar{x}|$ are

3, 2, 1, 3, 4, 5, 1, 3

Step 4 The required mean deviation about the mean is

$$\text{M.D.}\ (\bar{x}) = \frac{\sum_{i=1}^{8} |x_i - \bar{x}|}{8}$$

$$= \frac{3+2+1+3+4+5+1+3}{8} = \frac{22}{8} = 2.75$$

Note Instead of carrying out the steps every time, we can carry on calculation, step-wise without referring to steps.

Example 2 Find the mean deviation about the mean for the following data :

12, 3, 18, 17, 4, 9, 17, 19, 20, 15, 8, 17, 2, 3, 16, 11, 3, 1, 0, 5

Solution We have to first find the mean ($\bar{x}$) of the given data

$$\bar{x} = \frac{1}{20}\sum_{i=1}^{20} x_i = \frac{200}{20} = 10$$

The respective absolute values of the deviations from mean, i.e., $|x_i - \bar{x}|$ are

2, 7, 8, 7, 6, 1, 7, 9, 10, 5, 2, 7, 8, 7, 6, 1, 7, 9, 10, 5

Therefore $$\sum_{i=1}^{20}|x_i - \bar{x}| = 124$$

and $$\text{M.D.}\ (\bar{x}) = \frac{124}{20} = 6.2$$

Example 3 Find the mean deviation about the median for the following data:

3, 9, 5, 3, 12, 10, 18, 4, 7, 19, 21.

Solution Here the number of observations is 11 which is odd. Arranging the data into ascending order, we have 3, 3, 4, 5, 7, 9, 10, 12, 18, 19, 21

Now $$\text{Median} = \left(\frac{11 + 1}{2}\right)^{\text{th}} \text{ or } 6^{\text{th}} \text{ observation} = 9$$

The absolute values of the respective deviations from the median, i.e., $|x_i - \text{M}|$ are

6, 6, 5, 4, 2, 0, 1, 3, 9, 10, 12

Therefore $$\sum_{i=1}^{11}|x_i - \text{M}| = 58$$

and $$\text{M.D.}\left(\text{M}\right) = \frac{1}{11}\sum_{i=1}^{11}|x_i - \text{M}| = \frac{1}{11}\times 58 = 5.27$$

15.4.2 ***Mean deviation for grouped data*** We know that data can be grouped into two ways :

(a) Discrete frequency distribution,

(b) Continuous frequency distribution.

Let us discuss the method of finding mean deviation for both types of the data.

(a) Discrete frequency distribution Let the given data consist of n distinct values $x_1, x_2, ..., x_n$ occurring with frequencies $f_1, f_2, ..., f_n$ respectively. This data can be represented in the tabular form as given below, and is called *discrete frequency distribution*:

$$x : x_1 \quad x_2 \quad x_3 \ \ldots \ x_n$$
$$f : f_1 \quad f_2 \quad f_3 \ \ldots \ f_n$$

(i) Mean deviation about mean

First of all we find the mean $\bar{x}$ of the given data by using the formula

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} x_i f_i ,$$

where $\sum_{i=1}^{n} x_i f_i$ denotes the sum of the products of observations x_i with their respective frequencies f_i and $N = \sum_{i=1}^{n} f_i$ is the sum of the frequencies.

Then, we find the deviations of observations x_i from the mean $\bar{x}$ and take their absolute values, i.e., $|x_i - \bar{x}|$ for all i =1, 2,..., n.

After this, find the mean of the absolute values of the deviations, which is the required mean deviation about the mean. Thus

$$\text{M.D.}(\bar{x}) = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - \bar{x}|$$

(ii) Mean deviation about median To find mean deviation about median, we find the median of the given discrete frequency distribution. For this the observations are arranged in ascending order. After this the cumulative frequencies are obtained. Then, we identify the observation whose cumulative frequency is equal to or just greater than $\frac{N}{2}$, where N is the sum of frequencies. This value of the observation lies in the middle of the data, therefore, it is the required median. After finding median, we obtain the mean of the absolute values of the deviations from median.Thus,

$$\text{M.D.}(M) = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - M|$$

Example 4 Find mean deviation about the mean for the following data :

x_i	2	5	6	8	10	12
f_i	2	8	10	7	8	5

Solution Let us make a Table 15.1 of the given data and append other columns after calculations.

Table 15.1

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
2	2	4	5.5	11
5	8	40	2.5	20
6	10	60	1.5	15
8	7	56	0.5	3.5
10	8	80	2.5	20
12	5	60	4.5	22.5
	40	300		92

$$\mathrm{N} = \sum_{i=1}^{6} f_i = 40, \quad \sum_{i=1}^{6} f_i x_i = 300, \quad \sum_{i=1}^{6} f_i \lvert x_i - \bar{x} \rvert = 92$$

Therefore $\bar{x} = \dfrac{1}{\mathrm{N}} \sum_{i=1}^{6} f_i x_i = \dfrac{1}{40} \times 300 = 7.5$

and $\mathrm{M.D.}\ (\bar{x}) = \dfrac{1}{\mathrm{N}} \sum_{i=1}^{6} f_i \ \lvert x_i - \bar{x} \rvert = \dfrac{1}{40} \times 92 = 2.3$

Example 5 Find the mean deviation about the median for the following data:

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3

Solution The given observations are already in ascending order. Adding a row corresponding to cumulative frequencies to the given data, we get (Table 15.2).

Table 15.2

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3
c.f.	3	7	12	14	18	23	27	30

Now, N=30 which is even.

Median is the mean of the 15^{th} and 16^{th} observations. Both of these observations lie in the cummulative freqeuncy 18, for which the corresponding observation is 13.

$$\text{Therefore, Median M} = \frac{15^{\text{th}} \text{ observation} + 16^{\text{th}} \text{ observation}}{2} = \frac{13+13}{2} = 13$$

Now, absolute values of the deviations from median, i.e., $|x_i - \text{M}|$ are shown in Table 15.3.

Table 15.3

$\lvert x_i - \text{M}\rvert$	10	7	4	1	0	2	8	9
f_i	3	4	5	2	4	5	4	3
$f_i \lvert x_i - \text{M}\rvert$	30	28	20	2	0	10	32	27

$$\sum_{i=1}^{8} f_i = 30 \quad \text{and} \quad \sum_{i=1}^{8} f_i \left|x_i - \text{M}\right| = 149$$

Therefore $$\text{M. D. (M)} = \frac{1}{\text{N}} \sum_{i=1}^{8} f_i |x_i - \text{M}|$$

$$= \frac{1}{30} \times 149 = 4.97.$$

(b) Continuous frequency distribution A continuous frequency distribution is a series in which the data are classified into different class-intervals without gaps alongwith their respective frequencies.

For example, marks obtained by 100 students are presented in a continuous frequency distribution as follows :

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
Number of Students	12	18	27	20	17	6

(i) Mean deviation about mean While calculating the mean of a continuous frequency distribution, we had made the assumption that the frequency in each class is centred at its mid-point. Here also, we write the mid-point of each given class and proceed further as for a discrete frequency distribution to find the mean deviation.

Let us take the following example.

Example 6 Find the mean deviation about the mean for the following data.

Marks obtained	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	2	3	8	14	8	3	2

Solution We make the following Table 15.4 from the given data :

Table 15.4

Marks obtained	Number of students f_i	Mid-points x_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
10-20	2	15	30	30	60
20-30	3	25	75	20	60
30-40	8	35	280	10	80
40-50	14	45	630	0	0
50-60	8	55	440	10	80
60-70	3	65	195	20	60
70-80	2	75	150	30	60
	40		1800		400

Here $\quad \mathrm{N} = \sum_{i=1}^{7} f_i = 40, \sum_{i=1}^{7} f_i x_i = 1800, \sum_{i=1}^{7} f_i \lvert x_i - \bar{x} \rvert = 400$

Therefore $\quad \bar{x} = \frac{1}{\mathrm{N}} \sum_{i=1}^{7} f_i x_i = \frac{1800}{40} = 45$

and $\quad \mathrm{M.D.}(\bar{x}) = \frac{1}{\mathrm{N}} \sum_{i=1}^{7} f_i \lvert x_i - \bar{x} \rvert = \frac{1}{40} \times 400 = 10$

Shortcut method for calculating mean deviation about mean We can avoid the tedious calculations of computing $\bar{x}$ by following step-deviation method. Recall that in this method, we take an assumed mean which is in the middle or just close to it in the data. Then deviations of the observations (or mid-points of classes) are taken from the assumed mean. This is nothing but the shifting of origin from zero to the assumed mean on the number line, as shown in Fig 15.3

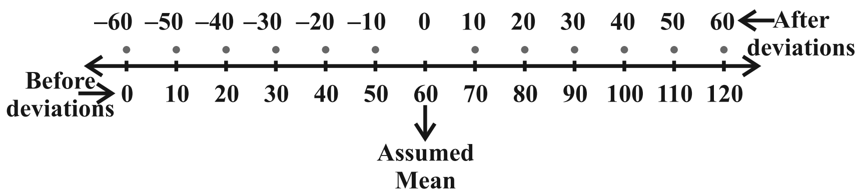


Fig 15.3

If there is a common factor of all the deviations, we divide them by this common factor to further simplify the deviations. These are known as step-deviations. The process of taking step-deviations is the change of scale on the number line as shown in Fig 15.4

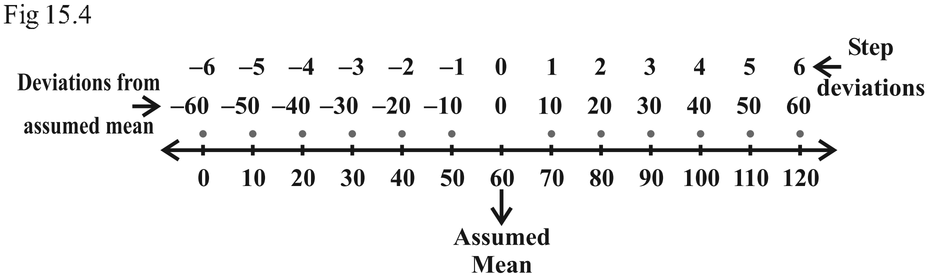


Fig 15.4

The deviations and step-deviations reduce the size of the observations, so that the computations viz. multiplication, etc., become simpler. Let, the new variable be denoted by $d_i = \dfrac{x_i - a}{h}$, where 'a' is the assumed mean and h is the common factor. Then, the mean $\bar{x}$ by step-deviation method is given by

$$\bar{x} = a + \frac{\sum_{i=1}^{n} f_i\, d_i}{\text{N}} \times h$$

Let us take the data of Example 6 and find the mean deviation by using step-deviation method.

Take the assumed mean $a = 45$ and $h = 10$, and form the following Table 15.5.

Table 15.5

Marks obtained	Number of students	Mid-points	$d_i = \dfrac{x_i - 45}{10}$	$f_i d_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
	f_i	x_i				
10-20	2	15	– 3	– 6	30	60
20-30	3	25	– 2	– 6	20	60
30-40	8	35	– 1	– 8	10	80
40-50	14	45	0	0	0	0
50-60	8	55	1	8	10	80
60-70	3	65	2	6	20	60
70-80	2	75	3	6	30	60
	40			0		400

Therefore $$\bar{x} = a + \frac{\sum_{i=1}^{7} f_i d_i}{N} \times h$$

$$= 45 + \frac{0}{40} \times 10 = 45$$

and $$\text{M.D. } (\bar{x}) = \frac{1}{N} \sum_{i=1}^{7} f_i \lvert x_i - \bar{x}\rvert = \frac{400}{40} = 10$$

Note The step deviation method is applied to compute $\bar{x}$. Rest of the procedure is same.

(ii) Mean deviation about median The process of finding the mean deviation about median for a continuous frequency distribution is similar as we did for mean deviation about the mean. The only difference lies in the replacement of the mean by median while taking deviations.

Let us recall the process of finding median for a continuous frequency distribution.

The data is first arranged in ascending order. Then, the median of continuous frequency distribution is obtained by first identifying the class in which median lies (median class) and then applying the formula

$$\text{Median } = l + \frac{\frac{N}{2} - C}{f} \times h$$

where median class is the class interval whose cumulative frequency is just greater than or equal to $\frac{N}{2}$, N is the sum of frequencies, l, f, h and C are, respectively the lower limit, the frequency, the width of the median class and C the cumulative frequency of the class just preceding the median class. After finding the median, the absolute values of the deviations of mid-point x_i of each class from the median i.e., $|x_i - M|$ are obtained.

Then $$\text{M.D. (M)} = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - M|$$

The process is illustrated in the following example:

Example 7 Calculate the mean deviation about median for the following data :

Class	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	6	7	15	16	4	2

Solution Form the following Table 15.6 from the given data :

Table 15.6

Class	Frequency	Cummulative frequency	Mid-points	$\lvert x_i - \text{Med.}\rvert$	$f_i \lvert x_i - \text{Med.}\rvert$
	f_i	$(c.f.)$	x_i		
0-10	6	6	5	23	138
10-20	7	13	15	13	91
20-30	15	28	25	3	45
30-40	16	44	35	7	112
40-50	4	48	45	17	68
50-60	2	50	55	27	54
	50				508

The class interval containing $\frac{N}{2}^{th}$ or 25th item is 20-30. Therefore, 20–30 is the median class. We know that

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h$$

Here $l = 20$, $C = 13$, $f = 15$, $h = 10$ and $N = 50$

Therefore, $\text{Median} = 20 + \frac{25-13}{15} \times 10 = 20 + 8 = 28$

Thus, Mean deviation about median is given by

$$\text{M.D. (M)} = \frac{1}{N}\sum_{i=1}^{6} f_i |x_i - M| = \frac{1}{50} \times 508 = 10.16$$

EXERCISE 15.1

Find the mean deviation about the mean for the data in Exercises 1 and 2.

1. 4, 7, 8, 9, 10, 12, 13, 17
2. 38, 70, 48, 40, 42, 55, 63, 46, 54, 44

Find the mean deviation about the median for the data in Exercises 3 and 4.

3. 13, 17, 16, 14, 11, 13, 10, 16, 11, 18, 12, 17
4. 36, 72, 46, 42, 60, 45, 53, 46, 51, 49

Find the mean deviation about the mean for the data in Exercises 5 and 6.

5.

x_i	5	10	15	20	25
f_i	7	4	6	3	5

6.

x_i	10	30	50	70	90
f_i	4	24	28	16	8

Find the mean deviation about the median for the data in Exercises 7 and 8.

7.

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

8.

x_i	15	21	27	30	35
f_i	3	5	6	7	8

Find the mean deviation about the mean for the data in Exercises 9 and 10.

9.

Income per day	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Number of persons	4	8	9	10	7	5	4	3

10.

Height in cms	95-105	105-115	115-125	125-135	135-145	145-155
Number of boys	9	13	26	30	12	10

11. Find the mean deviation about median for the following data :

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of Girls	6	8	14	16	4	2

12. Calculate the mean deviation about median age for the age distribution of 100 persons given below:

Age	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55
Number	5	6	12	14	26	12	16	9

[**Hint** Convert the given data into continuous frequency distribution by subtracting 0.5 from the lower limit and adding 0.5 to the upper limit of each class interval]

15.4.3 ***Limitations of mean deviation*** In a series, where the degree of variability is very high, the median is not a representative central tendency. Thus, the mean deviation about median calculated for such series can not be fully relied.

The sum of the deviations from the mean (minus signs ignored) is more than the sum of the deviations from median. Therefore, the mean deviation about the mean is not very scientific.Thus, in many cases, mean deviation may give unsatisfactory results. Also mean deviation is calculated on the basis of absolute values of the deviations and therefore, cannot be subjected to further algebraic treatment. This implies that we must have some other measure of dispersion. Standard deviation is such a measure of dispersion.

15.5 Variance and Standard Deviation

Recall that while calculating mean deviation about mean or median, the absolute values of the deviations were taken. The absolute values were taken to give meaning to the mean deviation, otherwise the deviations may cancel among themselves.

Another way to overcome this difficulty which arose due to the signs of deviations, is to take squares of all the deviations. Obviously all these squares of deviations are non-negative. Let $x_1, x_2, x_3, ..., x_n$ be n observations and $\bar{x}$ be their mean. Then

$$(x_1-\bar{x})^2 + (x_2 - \bar{x})^2 + ...+ (x_n-\bar{x})^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 .$$

If this sum is zero, then each $(x_i - \bar{x})$ has to be zero. This implies that there is no dispersion at all as all observations are equal to the mean $\bar{x}$.

If $\sum_{i=1}^{n}(x_i-\bar{x})^2$ is small , this indicates that the observations $x_1, x_2, x_3,...,x_n$ are close to the mean $\bar{x}$ and therefore, there is a lower degree of dispersion. On the contrary, if this sum is large, there is a higher degree of dispersion of the observations from the mean $\bar{x}$. Can we thus say that the sum $\sum_{i=1}^{n}(x_i-\bar{x})^2$ is a reasonable indicator of the degree of dispersion or scatter?

Let us take the set A of six observations 5, 15, 25, 35, 45, 55. The mean of the observations is $\bar{x} = 30$. The sum of squares of deviations from $\bar{x}$ for this set is

$$\sum_{i=1}^{6}(x_i-\bar{x})^2 = (5-30)^2 + (15-30)^2 + (25-30)^2 + (35-30)^2 + (45-30)^2 + (55-30)^2$$

$$= 625 + 225 + 25 + 25 + 225 + 625 = 1750$$

Let us now take another set B of 31 observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45. The mean of these observations is $\bar{y} = 30$

Note that both the sets A and B of observations have a mean of 30.

Now, the sum of squares of deviations of observations for set B from the mean $\bar{y}$ is given by

$$\sum_{i=1}^{31}(y_i-\bar{y})^2 = (15-30)^2 + (16-30)^2 + (17-30)^2 + ...+ (44-30)^2 + (45-30)^2$$

$$= (-15)^2 + (-14)^2 + ...+ (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + ...+ 14^2 + 15^2$$

$$= 2\,[15^2 + 14^2 + ... + 1^2]$$

$$= 2\times\frac{15\times(15+1)\,(30+1)}{6} = 5 \times 16 \times 31 = 2480$$

(Because sum of squares of first n natural numbers $= \dfrac{n\,(n+1)\,(2n+1)}{6}$. Here $n = 15$)

If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is simply our measure of dispersion or scatter about mean, we will tend to say that the set A of six observations has a lesser dispersion about the mean than the set B of 31 observations, even though the observations in set A are more scattered from the mean (the range of deviations being from –25 to 25) than in the set B (where the range of deviations is from –15 to 15).

This is also clear from the following diagrams.

For the set A, we have

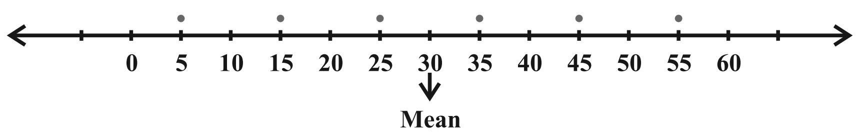


Fig 15.5

For the set B, we have

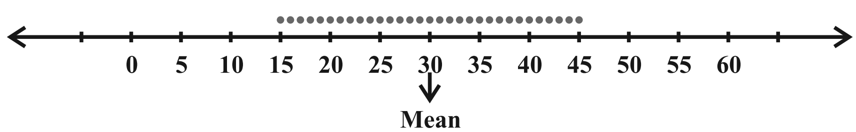


Fig 15.6

Thus, we can say that the sum of squares of deviations from the mean is not a proper measure of dispersion. To overcome this difficulty we take the mean of the squares of the deviations, i.e., we take $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. In case of the set A, we have

Mean $= \frac{1}{6} \times 1750 = 291.67$ and in case of the set B, it is $\frac{1}{31} \times 2480 = 80$.

This indicates that the scatter or dispersion is more in set A than the scatter or dispersion in set B, which confirms with the geometrical representation of the two sets.

Thus, we can take $\frac{1}{n}\sum(x_i - \bar{x})^2$ as a quantity which leads to a proper measure of dispersion. This number, i.e., mean of the squares of the deviations from mean is called the ***variance*** and is denoted by σ^2 (read as sigma square). Therefore, the variance of n observations $x_1, x_2, ..., x_n$ is given by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

15.5.1 ***Standard Deviation*** In the calculation of variance, we find that the units of individual observations x_i and the unit of their mean $\bar{x}$ are different from that of variance, since variance involves the sum of squares of $(x_i - \bar{x})$. For this reason, the proper measure of dispersion about the mean of a set of observations is expressed as positive square-root of the variance and is called *standard deviation.* Therefore, the standard deviation, usually denoted by σ, is given by

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \text{... (1)}$$

Let us take the following example to illustrate the calculation of variance and hence, standard deviation of ungrouped data.

Example 8 Find the Variance of the following data:

6, 8, 10, 12, 14, 16, 18, 20, 22, 24

Solution From the given data we can form the following Table 15.7. The mean is calculated by step-deviation method taking 14 as assumed mean. The number of observations is $n = 10$

Table 15.7

x_i	$d_i = \frac{x_i - 14}{2}$	Deviations from mean $(x_i - \bar{x})$	$(x_i - \bar{x})$
6	–4	–9	81
8	–3	–7	49
10	–2	–5	25
12	–1	–3	9
14	0	–1	1
16	1	1	1
18	2	3	9
20	3	5	25
22	4	7	49
24	5	9	81
	5		330

Therefore Mean $\bar{x}$ = assumed mean + $\dfrac{\sum_{i=1}^{n} d_i}{n} \times h = 14 + \dfrac{5}{10} \times 2 = 15$

and Variance $(\sigma^2) = \dfrac{1}{n}\sum_{i=1}^{10}(x_i - \bar{x})^2 = \dfrac{1}{10} \times 330 = 33$

Thus Standard deviation $(\sigma) = \sqrt{33} = 5.74$

15.5.2 ***Standard deviation of a discrete frequency distribution*** Let the given discrete frequency distribution be

$$x: \quad x_1, \quad x_2, \quad x_3, \ldots, x_n$$
$$f: \quad f_1, \quad f_2, \quad f_3, \ldots, f_n$$

In this case standard deviation $(\sigma) = \sqrt{\dfrac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$ ... (2)

where $N = \sum_{i=1}^{n} f_i$.

Let us take up following example.

Example 9 Find the variance and standard deviation for the following data:

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

Solution Presenting the data in tabular form (Table 15.8), we get

Table 15.8

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	12	–10	100	300
8	5	40	–6	36	180
11	9	99	–3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
	30	420			1374

$$\mathrm{N} = 30,\ \sum_{i=1}^{7} f_i x_i = 420,\ \sum_{i=1}^{7} f_i \left(x_i - \bar{x}\right)^2 = 1374$$

Therefore $$\bar{x} = \frac{\sum_{i=1}^{7} f_i x_i}{\mathrm{N}} = \frac{1}{30} \times 420 = 14$$

Hence $$\text{variance } (\sigma^2) = \frac{1}{\mathrm{N}} \sum_{i=1}^{7} f_i (x_i - \bar{x})^2$$

$$= \frac{1}{30} \times 1374 = 45.8$$

and $$\text{Standard deviation } (\sigma) = \sqrt{45.8} = 6.77$$

15.5.3 ***Standard deviation of a continuous frequency distribution*** The given continuous frequency distribution can be represented as a discrete frequency distribution by replacing each class by its mid-point. Then, the standard deviation is calculated by the technique adopted in the case of a discrete frequency distribution.

If there is a frequency distribution of n classes each class defined by its mid-point x_i with frequency f_i, the standard deviation will be obtained by the formula

$$\sigma = \sqrt{\frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2},$$

where $\bar{x}$ is the mean of the distribution and $\mathrm{N} = \sum_{i=1}^{n} f_i$.

Another formula for standard deviation We know that

$$\text{Variance } (\sigma^2) = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i^2 + \bar{x}^2 - 2\bar{x}\, x_i)$$

$$= \frac{1}{\mathrm{N}} \left[\sum_{i=1}^{n} f_i x_i^2 + \sum_{i=1}^{n} \bar{x}^2 f_i - \sum_{i=1}^{n} 2\bar{x}\, f_i x_i \right]$$

$$= \frac{1}{\mathrm{N}} \left[\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 \sum_{i=1}^{n} f_i - 2\bar{x} \sum_{i=1}^{n} x_i f_i \right]$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i x_i + \bar{x}^2 N - 2\bar{x} \, . \, N\bar{x}\right] \quad \left[\text{Here } \frac{1}{N}\sum_{i=1}^{n} x_i f_i = \bar{x} \text{ or } \sum_{i=1}^{n} x_i f_i = N\bar{x}\right]$$

$$= \frac{1}{N} \sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 - 2\bar{x}^2 = \frac{1}{N} \sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2$$

or $$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2 = \frac{1}{N^2}\left[N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2\right]$$

Thus, standard deviation $(\sigma) = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$ ... (3)

Example 10 Calculate the mean, variance and standard deviation for the following distribution:

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Solution From the given data, we construct the following Table 15.9.

Table 15.9

Class	Frequency (f_i)	Mid-point (x_i)	$f_i x_i$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
30-40	3	35	105	729	2187
40-50	7	45	315	289	2023
50-60	12	55	660	49	588
60-70	15	65	975	9	135
70-80	8	75	600	169	1352
80-90	3	85	255	529	1587
90-100	2	95	190	1089	2178
	50		3100		10050

Thus $\qquad$ Mean $\bar{x} = \frac{1}{N}\sum_{i=1}^{7} f_i x_i = \frac{3100}{50} = 62$

$$\text{Variance } \left(\sigma^2\right) = \frac{1}{N}\sum_{i=1}^{7} f_i (x_i - \bar{x})^2$$

$$= \frac{1}{50} \times 10050 = 201$$

and $\qquad$ Standard deviation $(\sigma) = \sqrt{201} = 14.18$

Example 11 Find the standard deviation for the following data :

x_i	3	8	13	18	23
f_i	7	10	15	10	6

Solution Let us form the following Table 15.10:

Table 15.10

x_i	f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
3	7	21	9	63
8	10	80	64	640
13	15	195	169	2535
18	10	180	324	3240
23	6	138	529	3174
48	614			9652

Now, by formula (3), we have

$$\sigma = \frac{1}{N}\sqrt{N\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

$$= \frac{1}{48}\sqrt{48 \times 9652 - (614)^2}$$

$$= \frac{1}{48}\sqrt{463296 - 376996}$$

$$= \frac{1}{48} \times 293.77 = 6.12$$

Therefore, Standard deviation $(\sigma) = 6.12$

15.5.4. ***Shortcut method to find variance and standard deviation*** Sometimes the values of x_i in a discrete distribution or the mid points x_i of different classes in a continuous distribution are large and so the calculation of mean and variance becomes tedious and time consuming. By using step-deviation method, it is possible to simplify the procedure.

Let the assumed mean be 'A' and the scale be reduced to $\frac{1}{h}$ times (h being the width of class-intervals). Let the step-deviations or the new values be y_i.

i.e. $y_i = \frac{x_i - \text{A}}{h}$ or $x_i = \text{A} + hy_i$ ... (1)

We know that $\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\text{N}}$ ... (2)

Replacing x_i from (1) in (2), we get

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i(\text{A} + hy_i)}{\text{N}}$$

$$= \frac{1}{\text{N}}\left(\sum_{i=1}^{n} f_i\text{A} + \sum_{i=1}^{n} h\, f_i y_i\right) = \frac{1}{\text{N}}\left(\text{A}\sum_{i=1}^{n} f_i + h\sum_{i=1}^{n} f_i y_i\right)$$

$$= \text{A}\,.\frac{\text{N}}{\text{N}} + h\frac{\sum_{i=1}^{n} f_i y_i}{\text{N}} \qquad \left(\text{because} \sum_{i=1}^{n} f_i = \text{N}\right)$$

Thus $\bar{x} = \text{A} + h\,\bar{y}$ ... (3)

Now Variance of the variable x, $\sigma_x^{\,2} = \frac{1}{\text{N}}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2$

$$= \frac{1}{\text{N}}\sum_{i=1}^{n} f_i\,(\text{A} + hy_i - \text{A} - h\,\bar{y})^2 \qquad \text{(Using (1) and (3))}$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\, h^2\, (y_i - \overline{y})^2$$

$$= \frac{h^2}{N}\sum_{i=1}^{n} f_i\, (y_i - \overline{y})^2 = h^2 \times \text{variance of the variable } y_i$$

i.e. $\sigma_x^{\,2} = h^2\sigma_y^{\,2}$

or $\sigma_x = h\sigma_y$ ... (4)

From (3) and (4), we have

$$\sigma_x = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^{\,2} - \left(\sum_{i=1}^{n} f_i y_i\right)^2} \quad \text{... (5)}$$

Let us solve Example 11 by the short-cut method and using formula (5)

Examples 12 Calculate mean, Variance and Standard Deviation for the following distribution.

Classes	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Solution Let the assumed mean A = 65. Here $h = 10$

We obtain the following Table 15.11 from the given data :

Table 15.11

Class	Frequency	Mid-point	$y_i = \frac{x_i - 65}{10}$	$y_i^{\,2}$	$f_i\, y_i$	$f_i\, y_i^{\,2}$
	f_i	x_i				
30-40	3	35	– 3	9	– 9	27
40-50	7	45	– 2	4	– 14	28
50-60	12	55	– 1	1	– 12	12
60-70	15	65	0	0	0	0
70-80	8	75	1	1	8	8
80-90	3	85	2	4	6	12
9 0-100	2	95	3	9	6	18
	N=50				– 15	105

Therefore $\bar{x} = A + \frac{\sum f_i y_i}{50} \times h = 65 - \frac{15}{50} \times 10 = 62$

Variance $\sigma^2 = \frac{h^2}{N^2}\left[N\Sigma f_i y_i^2 - \left(\Sigma f_i y_i\right)^2\right]$

$$= \frac{(10)^2}{(50)^2}\left[50 \times 105 - (-15)^2\right]$$

$$= \frac{1}{25}[5250 - 225] = 201$$

and standard deviation $(\sigma) = \sqrt{201} = 14.18$

EXERCISE 15.2

Find the mean and variance for each of the data in Exercies 1 to 5.

1. 6, 7, 10, 12, 13, 4, 8, 12
2. First n natural numbers
3. First 10 multiples of 3
4.

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

5.

x_i	92	93	97	98	102	104	109
f_i	3	2	3	2	6	3	3

6. Find the mean and standard deviation using short-cut method.

x_i	60	61	62	63	64	65	66	67	68
f_i	2	1	12	29	25	12	10	4	5

Find the mean and variance for the following frequency distributions in Exercises 7 and 8.

7.

Classes	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequencies	2	3	5	10	3	5	2

8.

Classes	0-10	10-20	20-30	30-40	40-50
Frequencies	5	8	15	16	6

9. Find the mean, variance and standard deviation using short-cut method

Height in cms	70-75	75-80	80-85	85-90	90-95	95-100	100-105	105-110	110-115
No. of children	3	4	7	7	15	9	6	6	3

10. The diameters of circles (in mm) drawn in a design are given below:

Diameters	33-36	37-40	41-44	45-48	49-52
No. of circles	15	17	21	22	25

Calculate the standard deviation and mean diameter of the circles.

[**Hint** First make the data continuous by making the classes as 32.5-36.5, 36.5-40.5, 40.5-44.5, 44.5 - 48.5, 48.5 - 52.5 and then proceed.]

15.6 Analysis of Frequency Distributions

In earlier sections, we have studied about some types of measures of dispersion. The mean deviation and the standard deviation have the same units in which the data are given. Whenever we want to compare the variability of two series with same mean, which are measured in different units, we do not merely calculate the measures of dispersion but we require such measures which are independent of the units. The measure of variability which is independent of units is called coefficient of variation (denoted as C.V.)

The coefficient of variation is defined as

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100 \quad , \bar{x} \neq 0,$$

where σ and $\bar{x}$ are the standard deviation and mean of the data.

For comparing the variability or dispersion of two series, we calculate the coefficient of variance for each series. The series having greater C.V. is said to be more variable than the other. The series having lesser C.V. is said to be more consistent than the other.

15.6.1 ***Comparison of two frequency distributions with same mean*** Let $\bar{x}_1$ and σ_1 be the mean and standard deviation of the first distribution, and $\bar{x}_2$ and σ_2 be the mean and standard deviation of the second distribution.

Then C.V. (1st distribution) = $\frac{\sigma_1}{\bar{x}_1} \times 100$

and C.V. (2nd distribution) = $\frac{\sigma_2}{\bar{x}_2} \times 100$

Given $\bar{x}_1 = \bar{x}_2 = \bar{x}$ (say)

Therefore C.V. (1st distribution) = $\frac{\sigma_1}{\bar{x}} \times 100$ ... (1)

and C.V. (2nd distribution) = $\frac{\sigma_2}{\bar{x}} \times 100$ ... (2)

It is clear from (1) and (2) that the two C.Vs. can be compared on the basis of values of σ_1 and σ_2 only.

Thus, we say that for two series with equal means, the series with greater standard deviation (or variance) is called more variable or dispersed than the other. Also, the series with lesser value of standard deviation (or variance) is said to be more consistent than the other.

Let us now take following examples:

Example 13 Two plants A and B of a factory show following results about the number of workers and the wages paid to them.

	A	B
No. of workers	5000	6000
Average monthly wages	Rs 2500	Rs 2500
Variance of distribution of wages	81	100

In which plant, A or B is there greater variability in individual wages?

Solution The variance of the distribution of wages in plant A (σ_1^2) = 81

Therefore, standard deviation of the distribution of wages in plant A (σ_1) = 9

Also, the variance of the distribution of wages in plant B (σ_2^2) = 100

Therefore, standard deviation of the distribution of wages in plant B (σ_2) = 10

Since the average monthly wages in both the plants is same, i.e., Rs.2500, therefore, the plant with greater standard deviation will have more variability.
Thus, the plant B has greater variability in the individual wages.

Example 14 Coefficient of variation of two distributions are 60 and 70, and their standard deviations are 21 and 16, respectively. What are their arithmetic means.

Solution Given C.V. (1st distribution) = 60, $\sigma_1 = 21$

C.V. (2nd distribution) = 70, $\sigma_2 = 16$

Let $\bar{x}_1$ and $\bar{x}_2$ be the means of 1st and 2nd distribution, respectively. Then

$$\text{C.V. (1st distribution)} = \frac{\sigma_1}{\bar{x}_1} \times 100$$

Therefore $$60 = \frac{21}{\bar{x}_1} \times 100 \text{ or } \bar{x}_1 = \frac{21}{60} \times 100 = 35$$

and $$\text{C.V. (2nd distribution)} = \frac{\sigma_2}{\bar{x}_2} \times 100$$

i.e. $$70 = \frac{16}{\bar{x}_2} \times 100 \text{ or } \bar{x}_2 = \frac{16}{70} \times 100 = 22.85$$

Example 15 The following values are calculated in respect of heights and weights of the students of a section of Class XI :

	Height	Weight
Mean	162.6 cm	52.36 kg
Variance	127.69 cm^2	23.1361 kg^2

Can we say that the weights show greater variation than the heights?

Solution To compare the variability, we have to calculate their coefficients of variation.

Given Variance of height = 127.69cm^2

Therefore Standard deviation of height = $\sqrt{127.69}$cm = 11.3 cm

Also Variance of weight = 23.1361 kg^2

Therefore Standard deviation of weight = $\sqrt{23.1361}$ kg = 4.81 kg

Now, the coefficient of variations (C.V.) are given by

$$\text{(C.V.) in heights} = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100$$

$$= \frac{11.3}{162.6} \times 100 = 6.95$$

and $$\text{(C.V.) in weights} = \frac{4.81}{52.36} \times 100 = 9.18$$

Clearly C.V. in weights is greater than the C.V. in heights

Therefore, we can say that weights show more variability than heights.

EXERCISE 15.3

1. From the data given below state which group is more variable, A or B?

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Group A	9	17	32	33	40	10	9
Group B	10	20	30	25	43	15	7

2. From the prices of shares X and Y below, find out which is more stable in value:

X	35	54	52	53	56	58	52	50	51	49
Y	108	107	105	105	106	107	104	103	104	101

3. An analysis of monthly wages paid to workers in two firms A and B, belonging to the same industry, gives the following results:

	Firm A	Firm B
No. of wage earners	586	648
Mean of monthly wages	Rs 5253	Rs 5253
Variance of the distribution of wages	100	121

(i) Which firm A or B pays larger amount as monthly wages?

(ii) Which firm, A or B, shows greater variability in individual wages?

4. The following is the record of goals scored by team A in a football session:

No. of goals scored	0	1	2	3	4
No. of matches	1	9	7	5	3

For the team B, mean number of goals scored per match was 2 with a standard deviation 1.25 goals. Find which team may be considered more consistent?

5. The sum and sum of squares corresponding to length x (in cm) and weight y (in gm) of 50 plant products are given below:

$$\sum_{i=1}^{50} x_i = 212,\ \sum_{i=1}^{50} x_i^2 = 902.8,\ \sum_{i=1}^{50} y_i = 261,\ \sum_{i=1}^{50} y_i^2 = 1457.6$$

Which is more varying, the length or weight?

Miscellaneous Examples

Example 16 The variance of 20 observations is 5. If each observation is multiplied by 2, find the new variance of the resulting observations.

Solution Let the observations be $x_1, x_2, ..., x_{20}$ and $\bar{x}$ be their mean. Given that variance = 5 and $n = 20$. We know that

$$\text{Variance } (\sigma^2) = \frac{1}{n}\sum_{i=1}^{20}(x_i - \bar{x})^2 \text{, i.e., } 5 = \frac{1}{20}\sum_{i=1}^{20}(x_i - \bar{x})^2$$

or $$\sum_{i=1}^{20}(x_i - \bar{x})^2 = 100 \qquad ... (1)$$

If each observation is multiplied by 2, and the new resulting observations are y_i, then

$$y_i = 2x_i \text{ i.e., } x_i = \frac{1}{2}y_i$$

Therefore $$\bar{y} = \frac{1}{n}\sum_{i=1}^{20} y_i = \frac{1}{20}\sum_{i=1}^{20} 2x_i = 2 \cdot \frac{1}{20}\sum_{i=1}^{20} x_i$$

i.e. $$\bar{y} = 2\bar{x} \quad \text{or} \quad \bar{x} = \frac{1}{2}\bar{y}$$

Substituting the values of x_i and $\bar{x}$ in (1), we get

$$\sum_{i=1}^{20}\left(\frac{1}{2}y_i - \frac{1}{2}\overline{y}\right)^2 = 100 \text{, i.e., } \sum_{i=1}^{20}(y_i - \overline{y})^2 = 400$$

Thus the variance of new observations $= \frac{1}{20} \times 400 = 20 = 2^2 \times 5$

Note The reader may note that if each observation is multiplied by a constant k, the variance of the resulting observations becomes k^2 times the original variance.

Example17 The mean of 5 observations is 4.4 and their variance is 8.24. If three of the observations are 1, 2 and 6, find the other two observations.

Solution Let the other two observations be x and y.
Therefore, the series is 1, 2, 6, x, y.

Now Mean $\overline{x} = 4.4 = \frac{1+2+6+x+y}{5}$

or $22 = 9 + x + y$

Therefore $x + y = 13$ ... (1)

Also variance $= 8.24 = \frac{1}{n}\sum_{i=1}^{5}(x_i - \overline{x})^2$

i.e. $8.24 = \frac{1}{5}\left[(3.4)^2 + (2.4)^2 + (1.6)^2 + x^2 + y^2 - 2\times 4.4\,(x+y) + 2\times(4.4)^2\right]$

or $41.20 = 11.56 + 5.76 + 2.56 + x^2 + y^2 - 8.8 \times 13 + 38.72$

Therefore $x^2 + y^2 = 97$ ... (2)

But from (1), we have

$x^2 + y^2 + 2xy = 169$ ... (3)

From (2) and (3), we have

$2xy = 72$ ... (4)

Subtracting (4) from (2), we get

$x^2 + y^2 - 2xy = 97 - 72$ i.e. $(x - y)^2 = 25$

or $x - y = \pm 5$ ... (5)

So, from (1) and (5), we get

$x = 9, y = 4$ when $x - y = 5$

or $x = 4, y = 9$ when $x - y = -5$

Thus, the remaining observations are 4 and 9.

Example 18 If each of the observation $x_1, x_2, ..., x_n$ is increased by 'a', where a is a negative or positive number, show that the variance remains unchanged.

Solution Let $\bar{x}$ be the mean of $x_1, x_2, ..., x_n$. Then the variance is given by

$$\sigma_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

If 'a is added to each observation, the new observations will be

$$y_i = x_i + a \qquad ... (1)$$

Let the mean of the new observations be $\bar{y}$. Then

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(x_i + a)$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} a\right] = \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{na}{n} = \bar{x} + a$$

i.e. $$\bar{y} = \bar{x} + a \qquad ... (2)$$

Thus, the variance of the new observations

$$\sigma_2^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i + a - \bar{x} - a)^2 \qquad \text{[Using (1) and (2)]}$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sigma_1^2$$

Thus, the variance of the new observations is same as that of the original observations.

Note We may note that adding (or subtracting) a positive number to (or from) each observation of a group does not affect the variance.

Example 19 The mean and standard deviation of 100 observations were calculated as 40 and 5.1, respectively by a student who took by mistake 50 instead of 40 for one observation. What are the correct mean and standard deviation?

Solution Given that number of observations (n) = 100

Incorrect mean ($\bar{x}$) = 40,
Incorrect standard deviation (σ) = 5.1

We know that $$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

i.e. $$40 = \frac{1}{100}\sum_{i=1}^{100} x_i \quad \text{or} \quad \sum_{i=1}^{100} x_i = 4000$$

i.e. Incorrect sum of observations = 4000

Thus the correct sum of observations = Incorrect sum – 50 + 40

$$= 4000 - 50 + 40 = 3990$$

Hence Correct mean $= \dfrac{\text{correct sum}}{100} = \dfrac{3990}{100} = 39.9$

Also Standard deviation $\sigma = \sqrt{\dfrac{1}{n}\sum_{i=1}^{n} x_i^2 - \dfrac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2}$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\bar{x})^2}$$

i.e. $5.1 = \sqrt{\dfrac{1}{100} \times \text{Incorrect} \sum_{i=1}^{n} x_i^2 - (40)^2}$

or $26.01 = \dfrac{1}{100} \times \text{Incorrect} \sum_{i=1}^{n} x_i^2 - 1600$

Therefore $\text{Incorrect} \sum_{i=1}^{n} x_i^2 = 100\,(26.01 + 1600) = 162601$

Now $\text{Correct} \sum_{i=1}^{n} x_i^2 = \text{Incorrect} \sum_{i=1}^{n} x_i^2 - (50)^2 + (40)^2$

$$= 162601 - 2500 + 1600 = 161701$$

Therefore Correct standard deviation

$$= \sqrt{\frac{\text{Correct} \sum x_i^2}{n} - (\text{Correct mean})^2}$$

$$= \sqrt{\frac{161701}{100} - (39.9)^2}$$

$$= \sqrt{1617.01 - 1592.01} = \sqrt{25} = 5$$

Miscellaneous Exercise On Chapter 15

1. The mean and variance of eight observations are 9 and 9.25, respectively. If six of the observations are 6, 7, 10, 12, 12 and 13, find the remaining two observations.
2. The mean and variance of 7 observations are 8 and 16, respectively. If five of the observations are 2, 4, 10, 12, 14. Find the remaining two observations.
3. The mean and standard deviation of six observations are 8 and 4, respectively. If each observation is multiplied by 3, find the new mean and new standard deviation of the resulting observations.
4. Given that $\bar{x}$ is the mean and σ^2 is the variance of n observations $x_1, x_2, ...,x_n$. Prove that the mean and variance of the observations $ax_1, ax_2, ax_3, ..., ax_n$ are $a\bar{x}$ and $a^2\sigma^2$, respectively, $(a \neq 0)$.
5. The mean and standard deviation of 20 observations are found to be 10 and 2, respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:
   (i) If wrong item is omitted. (ii) If it is replaced by 12.
6. The mean and standard deviation of marks obtained by 50 students of a class in three subjects, Mathematics, Physics and Chemistry are given below:

Subject	**Mathematics**	**Physics**	**Chemistry**
Mean	42	32	40.9
Standard deviation	12	15	20

   which of the three subjects shows the highest variability in marks and which shows the lowest?
7. The mean and standard deviation of a group of 100 observations were found to be 20 and 3, respectively. Later on it was found that three observations were incorrect, which were recorded as 21, 21 and 18. Find the mean and standard deviation if the incorrect observations are omitted.

Summary

- **Measures of dispersion** Range, Quartile deviation, mean deviation, variance, standard deviation are measures of dispersion.
  Range = Maximum Value – Minimum Value
- **Mean deviation for ungrouped data**

$$\text{M.D.}(\bar{x}) = \frac{\sum(x_i - \bar{x})}{n}, \qquad \text{M.D. (M)} = \frac{\sum(x_i - \text{M})}{n}$$

- **Mean deviation for grouped data**

$$\text{M.D.}(\bar{x}) = \frac{\sum f_i(x_i - \bar{x})}{\text{N}}, \qquad \text{M.D.}(\text{M}) = \frac{\sum f_i(x_i - \text{M})}{\text{N}}, \text{ where N} = \sum f_i$$

- **Variance and standard deviation for ungrouped data**

$$\sigma^2 = \frac{1}{n}\sum (x_i - \bar{x})^2, \qquad \sigma = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2}$$

- **Variance and standard deviation of a discrete frequency distribution**

$$\sigma^2 = \frac{1}{\text{N}}\sum f_i(x_i - \bar{x})^2, \qquad \sigma = \sqrt{\frac{1}{\text{N}}\sum f_i(x_i - \bar{x})^2}$$

- **Variance and standard deviation of a continuous frequency distribution**

$$\sigma^2 = \frac{1}{\text{N}}\sum f_i(x_i - \bar{x})^2, \qquad \sigma = \frac{1}{\text{N}}\sqrt{\text{N}\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

- **Shortcut method to find variance and standard deviation.**

$$\sigma^2 = \frac{h^2}{\text{N}^2}\left[\text{N}\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2\right], \quad \sigma = \frac{h}{\text{N}}\sqrt{\text{N}\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2},$$

where $y_i = \dfrac{x_i - \text{A}}{h}$

- **Coefficient of variation (C.V.)** $= \dfrac{\sigma}{\bar{x}} \times 100, \bar{x} \neq 0.$

For series with equal means, the series with lesser standard deviation is more consistent or less scattered.

Historical Note

'Statistics' is derived from the Latin word 'status' which means a political state. This suggests that statistics is as old as human civilisation. In the year 3050 B.C., perhaps the first census was held in Egypt. In India also, about 2000 years ago, we had an efficient system of collecting administrative statistics, particularly, during the regime of Chandra Gupta Maurya (324-300 B.C.). The system of collecting data related to births and deaths is mentioned in Kautilya's *Arthshastra* (around 300 B.C.) A detailed account of administrative surveys conducted during Akbar's regime is given in *Ain-I-Akbari* written by Abul Fazl.

Captain John Graunt of London (1620-1674) is known as father of vital statistics due to his studies on statistics of births and deaths. Jacob Bernoulli (1654-1705) stated the Law of Large numbers in his book "Ars Conjectandi', published in 1713.

The theoretical development of statistics came during the mid seventeenth century and continued after that with the introduction of theory of games and chance (i.e., probability). Francis Galton (1822-1921), an Englishman, pioneered the use of statistical methods, in the field of Biometry. Karl Pearson (1857-1936) contributed a lot to the development of statistical studies with his discovery of *Chi square test* and foundation of *statistical laboratory* in England (1911). Sir Ronald A. Fisher (1890-1962), known as the Father of modern statistics, applied it to various diversified fields such as Genetics, Biometry, Education, Agriculture, etc.

❖

Chapter 16

PROBABILITY

❖ *Where a mathematical reasoning can be had, it is as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle in your hand. – JOHN ARBUTHNOT* ❖

16.1 Introduction

In earlier classes, we studied about the concept of probability as a measure of uncertainty of various phenomenon. We have obtained the probability of getting an even number in throwing a die as $\frac{3}{6}$ i.e., $\frac{1}{2}$. Here the total possible outcomes are 1,2,3,4,5 and 6 (six in number). The outcomes in favour of the event of 'getting an even number' are 2,4,6 (i.e., three in number). In general, to obtain the probability of an event, we find the ratio of the number of outcomes favourable to the event, to the total number of equally likely outcomes. This theory of probability is known as *classical theory of probability*.

Kolmogorove (1903 1987)

In Class IX, we learnt to find the probability on the basis of observations and collected data. This is called *statistical approach of probability*.

Both the theories have some serious difficulties. For instance, these theories can not be applied to the activities/experiments which have infinite number of outcomes. In classical theory we assume all the outcomes to be equally likely. Recall that the outcomes are called equally likely when we have no reason to believe that one is more likely to occur than the other. In other words, we assume that all outcomes have equal chance (probability) to occur. Thus, to define probability, we used equally likely or equally probable outcomes. This is logically not a correct definition. Thus, another theory of probability was developed by A.N. Kolmogorov, a Russian mathematician, in 1933. He laid down some axioms to interpret probability, in his book 'Foundation of Probability' published in 1933. In this Chapter, we will study about this approach called *axiomatic approach of probability*. To understand this approach we must know about few basic terms viz. random experiment, sample space, events, etc. Let us learn about these all, in what follows next.

16.2 Random Experiments

In our day to day life, we perform many activities which have a fixed result no matter any number of times they are repeated. For example given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

We also perform many experimental activities, where the result may not be same, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called *random experiments*.

An experiment is called random experiment if it satisfies the following two conditions:

(i) It has more than one possible outcome.

(ii) It is not possible to predict the outcome in advance.

Check whether the experiment of tossing a die is random or not?

In this chapter, we shall refer the random experiment by experiment only unless stated otherwise.

16.2.1 ***Outcomes and sample space*** A possible result of a random experiment is called its *outcome*.

Consider the experiment of rolling a die. The outcomes of this experiment are 1, 2, 3, 4, 5, or 6, if we are interested in the number of dots on the upper face of the die.

The set of outcomes $\{1, 2, 3, 4, 5, 6\}$ is called the *sample space of the experiment*.

Thus, the set of all possible outcomes of a random experiment is called the *sample space* associated with the experiment. Sample space is denoted by the symbol S.

Each element of the sample space is called a *sample point*. In other words, each outcome of the random experiment is also called *sample point*.

Let us now consider some examples.

Example 1 Two coins (a one rupee coin and a two rupee coin) are tossed once. Find a sample space.

Solution Clearly the coins are distinguishable in the sense that we can speak of the first coin and the second coin. Since either coin can turn up Head (H) or Tail(T), the possible outcomes may be

Heads on both coins = (H,H) = HH
Head on first coin and Tail on the other = (H,T) = HT
Tail on first coin and Head on the other = (T,H) = TH
Tail on both coins = (T,T) = TT
Thus, the sample space is S = {HH, HT, TH, TT}

Note The outcomes of this experiment are ordered pairs of H and T. For the sake of simplicity the commas are omitted from the ordered pairs.

Example 2 Find the sample space associated with the experiment of rolling a pair of dice (one is blue and the other red) once. Also, find the number of elements of this sample space.

Solution Suppose 1 appears on blue die and 2 on the red die. We denote this outcome by an ordered pair (1,2). Simlarly, if '3' appears on blue die and '5' on red, the outcome is denoted by the ordered pair (3,5).

In general each outcome can be denoted by the ordered pair (x, y), where x is the number appeared on the blue die and y is the number appeared on the red die. Therefore, this sample space is given by

S = $\{(x, y)$: x is the number on the blue die and y is the number on the red die$\}$.

The number of elements of this sample space is $6 \times 6 = 36$ and the sample space is given below:

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

Example 3 In each of the following experiments specify appropriate sample space

(i) A boy has a 1 rupee coin, a 2 rupee coin and a 5 rupee coin in his pocket. He takes out two coins out of his pocket, one after the other.

(ii) A person is noting down the number of accidents along a busy highway during a year.

Solution (i) Let Q denote a 1 rupee coin, H denotes a 2 rupee coin and R denotes a 5 rupee coin. The first coin he takes out of his pocket may be any one of the three coins Q, H or R. Corresponding to Q, the second draw may be H or R. So the result of two draws may be QH or QR. Similarly, corresponding to H, the second draw may be Q or R.

Therefore, the outcomes may be HQ or HR. Lastly, corresponding to R, the second draw may be H or Q.

So, the outcomes may be RH or RQ.

Thus, the sample space is S={QH, QR, HQ, HR, RH, RQ}

(ii) The number of accidents along a busy highway during the year of observation can be either 0 (for no accident) or 1 or 2, or some other positive integer.

Thus, a sample space associated with this experiment is S= {0,1,2,...}

Example 4 A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.

Solution Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is

$$S = \{ HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6\}.$$

Here HB_i means head on the coin and ball B_i is drawn, HW_i means head on the coin and ball W_i is drawn. Similarly, Ti means tail on the coin and the number i on the die.

Example 5 Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Solution In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on till head is obtained. Hence, the desired sample space is

S= {H, TH, TTH, TTTH, TTTTH,...}

EXERCISE 16.1

In each of the following Exercises 1 to 7, describe the sample space for the indicated experiment.

1. A coin is tossed three times.
2. A die is thrown two times.
3. A coin is tossed four times.
4. A coin is tossed and a die is thrown.
5. A coin is tossed and then a die is rolled only in case a head is shown on the coin.
6. 2 boys and 2 girls are in Room X, and 1 boy and 3 girls in Room Y. Specify the sample space for the experiment in which a room is selected and then a person.
7. One die of red colour, one of white colour and one of blue colour are placed in a bag. One die is selected at random and rolled, its colour and the number on its uppermost face is noted. Describe the sample space.
8. An experiment consists of recording boy–girl composition of families with 2 children.
   (i) What is the sample space if we are interested in knowing whether it is a boy or girl in the order of their births?

(ii) What is the sample space if we are interested in the number of girls in the family?

9. A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.
10. An experiment consists of tossing a coin and then throwing it second time if a head occurs. If a tail occurs on the first toss, then a die is rolled once. Find the sample space.
11. Suppose 3 bulbs are selected at random from a lot. Each bulb is tested and classified as defective (D) or non – defective(N). Write the sample space of this experiment.
12. A coin is tossed. If the out come is a head, a die is thrown. If the die shows up an even number, the die is thrown again. What is the sample space for the experiment?
13. The numbers 1, 2, 3 and 4 are written separatly on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.
14. An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.
15. A coin is tossed. If it shows a tail, we draw a ball from a box which contains 2 red and 3 black balls. If it shows head, we throw a die. Find the sample space for this experiment.
16. A die is thrown repeatedly untill a six comes up. What is the sample space for this experiment?

16.3 Event

We have studied about random experiment and sample space associated with an experiment. The sample space serves as an universal set for all questions concerned with the experiment.

Consider the experiment of tossing a coin two times. An associated sample space is S = {HH, HT, TH, TT}.

Now suppose that we are interested in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set E = { HT, TH}

We know that the set E is a subset of the sample space S . Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tails is exactly 2	A = {TT}
Number of tails is atleast one	B = {HT, TH, TT}
Number of heads is atmost one	C = {HT, TH, TT}
Second toss is not head	D = { HT, TT}
Number of tails is atmost two	S = {HH, HT, TH, TT}
Number of tails is more than two	ϕ

The above discussion suggests that a subset of sample space is associated with an event and an event is associated with a subset of sample space. In the light of this we define an event as follows.

Definition Any subset E of a sample space S is called *an event*.

16.3.1 ***Occurrence of an event*** Consider the experiment of throwing a die. Let E denotes the event " a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred. As a matter of fact if outcomes are 2 or 3, we say that event E has occurred

Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in$ E. If the outcome ω is such that $\omega \notin$ E, we say that the event E has not occurred.

16.3.2 ***Types of events*** Events can be classified into various types on the basis of the elements they have.

1. Impossible and Sure Events The empty set ϕ and the sample space S describe events. In fact ϕ is called an *impossible event* and S, i.e., the whole sample space is called the *sure event*.

To understand these let us consider the experiment of rolling a die. The associated sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let E be the event " the number appears on the die is a multiple of 7". Can you write the subset associated with the event E?

Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event E = ϕ is an impossible event.

Now let us take up another event F "the number turns up is odd or even". Clearly

$F = \{1, 2, 3, 4, 5, 6,\} = S$, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event F = S is a sure event.

2. Simple Event If an event E has only one sample point of a sample space, it is called a *simple* (or *elementary*) *event.*

In a sample space containing n distinct elements, there are exactly n simple events.

For example in the experiment of tossing two coins, a sample space is

S={HH, HT, TH, TT}

There are four simple events corresponding to this sample space. These are

E_1= {HH}, E_2={HT}, E_3= { TH} and E_4={TT}.

3. Compound Event If an event has more than one sample point, it is called a *Compound event.*

For example, in the experiment of "tossing a coin thrice" the events

E: 'Exactly one head appeared'

F: 'Atleast one head appeared'

G: 'Atmost one head appeared' etc.

are all compound events. The subsets of S associated with these events are

E={HTT,THT,TTH}

F={HTT,THT, TTH, HHT, HTH, THH, HHH}

G= {TTT, THT, HTT, TTH}

Each of the above subsets contain more than one sample point, hence they are all compound events.

16.3.3 ***Algebra of events*** In the Chapter on Sets, we have studied about different ways of combining two or more sets, viz, union, intersection, difference, complement of a set etc. Like-wise we can combine two or more events by using the analogous set notations.

Let A, B, C be events associated with an experiment whose sample space is S.

1. Complementary Event For every event A, there corresponds another event A' called the complementary event to A. It is also called the *event 'not* A'.

For example, take the experiment 'of tossing three coins'. An associated sample space is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let A={HTH, HHT, THH} be the event 'only one tail appears'

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.

Thus the complementary event 'not A' to the event A is

$$A' = \{HHH, HTT, THT, TTH, TTT\}$$

or $\quad A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A.$

2. The Event 'A or B' Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both.

When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.

Therefore $\quad$ Event 'A or B' $= A \cup B$

$$= \{\omega : \omega \in A \text{ or } \omega \in B\}$$

3. The Event 'A and B' We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B. i.e., which belong to both 'A and B'.

If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

Thus, $\quad A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

For example, in the experiment of 'throwing a die twice' Let A be the event 'score on the first throw is six' and B is the event 'sum of two scores is atleast 11' then

$$A = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}, \text{ and } B = \{(5,6), (6,5), (6,6)\}$$

so $\quad A \cap B = \{(6,5), (6,6)\}$

Note that the set $A \cap B = \{(6,5), (6,6)\}$ may represent the event 'the score on the first throw is six and the sum of the scores is atleast 11'.

4. The Event 'A but not B' We know that A–B is the set of all those elements which are in A but not in B. Therefore, the set A–B may denote the event 'A but not B'.We know that

$$A - B = A \cap B'$$

Example 6 Consider the experiment of rolling a die. Let A be the event 'getting a prime number', B be the event 'getting an odd number'. Write the sets representing the events (i) Aor B (ii) A and B (iii) A but not B (iv) 'not A'.

Solution Here $\quad S = \{1, 2, 3, 4, 5, 6\}, A = \{2, 3, 5\}$ and $B = \{1, 3, 5\}$

Obviously

(i) 'A or B' $= A \cup B = \{1, 2, 3, 5\}$

(ii) 'A and B' $= A \cap B = \{3,5\}$

(iii) 'A but not B' $= A - B = \{2\}$

(iv) 'not A' $= A' = \{1,4,6\}$

16.3.4 ***Mutually exclusive events*** In the experiment of rolling a die, a sample space is S = {1, 2, 3, 4, 5, 6}. Consider events, A 'an odd number appears' and B 'an even number appears'

Clearly the event A excludes the event B and vice versa. In other words, there is no outcome which ensures the occurrence of events A and B simultaneously. Here

A = {1, 3, 5} and B = {2, 4, 6}

Clearly $A \cap B = \phi$, i.e., A and B are disjoint sets.

In general, two events A and B are called *mutually exclusive* events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they can not occur simultaneously. In this case the sets A and B are disjoint.

Again in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears'

Obviously A = {1, 3, 5} and B = {1, 2, 3}

Now $3 \in A$ as well as $3 \in B$

Therefore, A and B are not mutually exclusive events.

Remark Simple events of a sample space are always mutually exclusive.

16.3.5 ***Exhaustive events*** Consider the experiment of throwing a die. We have S = {1, 2, 3, 4, 5, 6}. Let us define the following events

A: 'a number less than 4 appears',
B: 'a number greater than 2 but less than 5 appears'
and C: 'a number greater than 4 appears'.

Then A = {1, 2, 3}, B = {3,4} and C = {5, 6}. We observe that

$$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S.$$

Such events A, B and C are called exhaustive events. In general, if $E_1, E_2, ..., E_n$ are n events of a sample space S and if

$$E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = \bigcup_{i=1}^{n} E_i = S$$

then $E_1, E_2, ..., E_n$ are called *exhaustive events*.In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if atleast one of them necessarily occurs whenever the experiment is performed.

Further, if $E_i \cap E_j = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, ..., E_n$ are called *mutually exclusive and exhaustive events*.

We now consider some examples.

Example 7 Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events associated with this experiment

A: 'the sum is even'.
B: 'the sum is a multiple of 3'.
C: 'the sum is less than 4'.
D: 'the sum is greater than 11'.

Which pairs of these events are mutually exclusive?

Solution There are 36 elements in the sample space S = $\{(x, y): x, y = 1, 2, 3, 4, 5, 6\}$. Then

A = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

B = {(1, 2), (2, 1), (1, 5), (5, 1), (3, 3), (2, 4), (4, 2), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)}

C = {(1, 1), (2, 1), (1, 2)} and D = {(6, 6)}

We find that

$$A \cap B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)\} \neq \phi$$

Therefore, A and B are not mutually exclusive events.

Similarly $A \cap C \neq \phi$, $A \cap D \neq \phi$, $B \cap C \neq \phi$ and $B \cap D \neq \phi$.

Thus, the pairs, (A, C), (A, D), (B, C), (B, D) are not mutually exclusive events.

Also $C \cap D = \phi$ and so C and D are mutually exclusive events.

Example 8 A coin is tossed three times, consider the following events.

A: 'No head appears', B: 'Exactly one head appears' and C: 'Atleast two heads appear'.

Do they form a set of mutually exclusive and exhaustive events?

Solution The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

and A = {TTT}, B = {HTT, THT, TTH}, C = {HHT, HTH, THH, HHH}

Now

$A \cup B \cup C$ = {TTT, HTT, THT, TTH, HHT, HTH, THH, HHH} = S

Therefore, A, B and C are exhaustive events.

Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$

Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive.

Hence, A, B and C form a set of mutually exclusive and exhaustive events.

EXERCISE 16.2

1. A die is rolled. Let E be the event "die shows 4" and F be the event "die shows even number". Are E and F mutually exclusive?
2. A die is thrown. Describe the following events:

   (i) A: a number less than 7 (ii) B: a number greater than 7
   (iii) C: a multiple of 3 (iv) D: a number less than 4
   (v) E: an even number greater than 4 (vi) F: a number not less than 3

   Also find $A \cup B$, $A \cap B$, $E \cup F$, $D \cap E$, $A - C$, $D - E$, F', $E \cap F'$,
3. An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events:
   A: the sum is greater than 8, B: 2 occurs on either die
   C: the sum is at least 7 and a multiple of 3.
   Which pairs of these events are mutually exclusive?
4. Three coins are tossed once. Let A denote the event 'three heads show", B denote the event "two heads and one tail show", C denote the event" three tails show and D denote the event 'a head shows on the first coin". Which events are
   (i) mutually exclusive? (ii) simple? (iii) Compound?
5. Three coins are tossed. Describe
   (i) Two events which are mutually exclusive.
   (ii) Three events which are mutually exclusive and exhaustive.
   (iii) Two events, which are not mutually exclusive.
   (iv) Two events which are mutually exclusive but not exhaustive.
   (v) Three events which are mutually exclusive but not exhaustive.
6. Two dice are thrown. The events A, B and C are as follows:
   A: getting an even number on the first die.
   B: getting an odd number on the first die.
   C: getting the sum of the numbers on the dice ≤ 5.
   Describe the events
   (i) A' (ii) not B (iii) A or B
   (iv) A and B (v) A but not C (vi) B or C
   (vii) B and C (viii) $A \cap B' \cap C'$
7. Refer to question 6 above, state true or false: (give reason for your answer)
   (i) A and B are mutually exclusive
   (ii) A and B are mutually exclusive and exhaustive
   (iii) $A = B'$

(iv) A and C are mutually exclusive

(v) A and B′ are mutually exclusive.

(vi) A′, B′, C are mutually exclusive and exhaustive.

16.4 Axiomatic Approach to Probability

In earlier sections, we have considered random experiments, sample space and events associated with these experiments. In our day to day life we use many words about the chances of occurrence of events. Probability theory attempts to quantify these chances of occurrence or non occurrence of events.

In earlier classes, we have studied some methods of assigning probability to an event associated with an experiment having known the number of total outcomes.

Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0,1] satisfying the following axioms

(i) For any event E, $P(E) \geq 0$ (ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$.

It follows from (iii) that $P(\phi) = 0$. To prove this, we take $F = \phi$ and note that E and ϕ are disjoint events. Therefore, from axiom (iii), we get

$P(E \cup \phi) = P(E) + P(\phi)$ or $P(E) = P(E) + P(\phi)$ i.e. $P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, ..., \omega_n$, i.e.,

$$S = \{\omega_1, \omega_2, ..., \omega_n\}$$

It follows from the axiomatic definition of probability that

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + ... + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \sum P(\omega_i)$, $\omega_i \in A$.

Note It may be noted that the singleton $\{\omega_i\}$ is called elementary event and for notational convenience, we write $P(\omega_i)$ for $P(\{\omega_i\})$.

For example, in 'a coin tossing' experiment we can assign the number — to each of the outcomes H and T.

i.e. P(H) = — and P(T) = — (1)

Clearly this assignment satisfies both the conditions i.e., each number is neither less than zero nor greater than 1 and

$$P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$$

Therefore, in this case we can say that probability of H = $\frac{1}{2}$, and probability of T = $\frac{1}{2}$

If we take $\quad P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$ ... (2)

Does this assignment satisfy the conditions of axiomatic approach?

Yes, in this case, probability of H = — and probability of T = $\frac{3}{4}$.

We find that both the assignments (1) and (2) are valid for probability of H and T.

In fact, we can assign the numbers p and $(1 - p)$ to both the outcomes such that $0 \le p \le 1$ and $P(H) + P(T) = p + (1 - p) = 1$

This assignment, too, satisfies both conditions of the axiomatic approach of probability. Hence, we can say that there are many ways (rather infinite) to assign probabilities to outcomes of an experiment. We now consider some examples.

Example 9 Let a sample space be $S = \{\omega_1, \omega_2, ..., \omega_6\}$. Which of the following assignments of probabilities to each outcome are valid?

Outcomes	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
(a)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
(b)	1	0	0	0	0	0
(c)	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{3}$
(d)	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{2}$
(e)	0.1	0.2	0.3	0.4	0.5	0.6

Solution (a) Condition (i): Each of the number $p(\omega_i)$ is positive and less than one.

Condition (ii): Sum of probabilities

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Therefore, the assignment is valid

(b) Condition (i): Each of the number $p(\omega_i)$ is either 0 or 1.
Condition (ii) Sum of the probabilities = 1 + 0 + 0 + 0 + 0 + 0 = 1
Therefore, the assignment is valid

(c) Condition (i) Two of the probabilities $p(\omega_5)$ and $p(\omega_6)$ are negative, the assignment is not valid

(d) Since $p(\omega_6) = \frac{3}{2} > 1$, the assignment is not valid

(e) Since, sum of probabilities = 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 = 2.1, the assignment is not valid.

16.4.1 ***Probability of an event*** Let S be a sample space associated with the experiment 'examining three consecutive pens produced by a machine and classified as Good (non-defective) and bad (defective)'. We may get 0, 1, 2 or 3 defective pens as result of this examination.

A sample space associated with this experiment is

$$S = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\},$$

where B stands for a defective or bad pen and G for a non – defective or good pen.

Let the probabilities assigned to the outcomes be as follows

Sample point:	BBB	BBG	BGB	GBB	BGG	GBG	GGB	GGG
Probability:	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Let event A: there is exactly one defective pen and event B: there are atleast two defective pens.

Hence A = {BGG, GBG, GGB} and B = {BBG, BGB, GBB, BBB}

Now $P(A) = \sum P(\omega_i), \forall \omega_i \in A$

$$= P(BGG) + P(GBG) + P(GGB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

and $P(B) = \sum P(\omega_i), \forall \omega_i \in B$

$$= P(BBG) + P(BGB) + P(GBB) + P(BBB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

Let us consider another experiment of 'tossing a coin "twice"

The sample space of this experiment is S = {HH, HT, TH, TT}

Let the following probabilities be assigned to the outcomes

$$P(HH) = \frac{1}{4},\ P(HT) = \frac{1}{7},\ P(TH) = \frac{2}{7},\ P(TT) = \frac{9}{28}$$

Clearly this assignment satisfies the conditions of axiomatic approach. Now, let us find the probability of the event E: 'Both the tosses yield the same result'.

Here $\quad E = \{HH, TT\}$

Now $\quad P(E) = \Sigma\, P(w_i)$, for all $w_i \in E$

$$= P(HH) + P(TT) = \frac{1}{4} + \frac{9}{28} = \frac{4}{7}$$

For the event F: 'exactly two heads', we have F = {HH}

and $\quad P(F) = P(HH) = -$

16.4.2 ***Probabilities of equally likely outcomes*** Let a sample space of an experiment be

$$S = \{\omega_1, \omega_2, ..., \omega_n\}.$$

Let all the outcomes are equally likely to occur, i.e., the chance of occurrence of each simple event must be same.

i.e. $\quad P(\omega_i) = p$, for all $\omega_i \in S$ where $0 \le p \le 1$

Since $\quad \sum_{i=1}^{n} P(\omega_i) = 1$ i.e., $p + p + ... + p$ (n times) = 1

or $\quad np = 1$ i.e., $p = \frac{1}{n}$

Let S be a sample space and E be an event, such that $n(S) = n$ and $n(E) = m$. If each out come is equally likely, then it follows that

$$P(E) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to E}}{\text{Total possible outcomes}}$$

16.4.3 ***Probability of the event 'A or B'*** Let us now find the probability of event 'A or B', i.e., $P(A \cup B)$

Let $\quad A = \{HHT, HTH, THH\}$ and $B = \{HTH, THH, HHH\}$ be two events associated with 'tossing of a coin thrice'

Clearly $\quad A \cup B = \{HHT, HTH, THH, HHH\}$

Now $\quad P(A \cup B) = P(HHT) + P(HTH) + P(THH) + P(HHH)$

If all the outcomes are equally likely, then

$$P(A \cup B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Also $\quad P(A) = P(HHT) + P(HTH) + P(THH) = \frac{3}{8}$

and $\quad P(B) = P(HTH) + P(THH) + P(HHH) = \frac{3}{8}$

Therefore $\quad P(A) + P(B) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8}$

It is clear that $\quad P(A \cup B) \neq P(A) + P(B)$

The points HTH and THH are common to both A and B . In the computation of P(A) + P(B) the probabilities of points HTH and THH, i.e., the elements of $A \cap B$ are included twice. Thus to get the probability $P(A \cup B)$ we have to subtract the probabilities of the sample points in $A \cap B$ from P(A) + P(B)

i.e. $\quad P(A \cup B) = P(A) + P(B) - \Sigma P(\omega_i), \forall \omega_i \in A \cap B$

$$= P(A) + P(B) - P(A \cap B)$$

Thus we observe that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In general, if A and B are any two events associated with a random experiment, then by the definition of probability of an event, we have

$$P(A \cup B) = \Sigma p(\omega_i), \forall \omega_i \in A \cup B.$$

Since $\quad A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$,

we have

$P(A \cup B) = \left[\Sigma P(\omega_i) \forall \omega_i \in (A-B)\right] + \left[\Sigma P(\omega_i) \forall \omega_i \in A \cap B\right] + \left[\Sigma P(\omega_i) \forall \omega_i \in B - A\right]$

(because A–B, $A \cap B$ and B – A are mutually exclusive) ... (1)

Also $P(A) + P(B) = \left[\Sigma p(\omega_i) \ \forall \omega_i \in A\right] + \left[\Sigma p(\omega_i) \ \forall \omega_i \in B\right]$

$$= \left[\Sigma P(\omega_i) \forall \omega_i \in (A-B) \cup (A \cap B)\right] + \left[\Sigma P(\omega_i) \forall \omega_i \in (B - A) \cup (A \cap B)\right]$$

$$= \left[\Sigma P(\omega_i) \forall \omega_i \in (A - B)\right] + \left[\Sigma P(\omega_i) \forall \omega_i \in (A \cap B)\right] + \left[\Sigma P(\omega_i) \forall \omega_i \in (B-A)\right] + \left[\Sigma P(\omega_i) \forall \omega_i \in (A \cap B)\right]$$

$$= P(A \cup B) + \left[\Sigma P(\omega_i) \forall \omega_i \in A \cap B\right] \quad \text{[using (1)]}$$

$$= P(A \cup B) + P(A \cap B).$$

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Alternatively, it can also be proved as follows:

$A \cup B = A \cup (B - A)$, where A and B – A are mutually exclusive,

and $B = (A \cap B) \cup (B - A)$, where $A \cap B$ and B – A are mutually exclsuive.

Using Axiom (iii) of probability, we get

$$P(A \cup B) = P(A) + P(B - A) \quad \ldots (2)$$

and

$$P(B) = P(A \cap B) + P(B - A) \quad \ldots (3)$$

Subtracting (3) from (2) gives

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

or

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The above result can further be verified by observing the Venn Diagram (Fig 16.1)

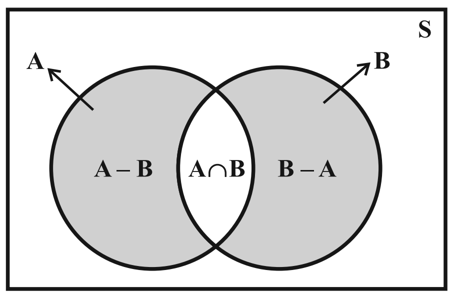


Fig 16.1

If A and B are disjoint sets, i.e., they are mutually exclusive events, then $A \cap B = \phi$

Therefore $P(A \cap B) = P(\phi) = 0$

Thus, for mutually exclusive events A and B, we have

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom (iii) of probability.

16.4.4 ***Probability of event 'not A'*** Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10}

If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability of each outcome is $\frac{1}{10}$

Now $\quad P(A) = P(2) + P(4) + P(6) + P(8)$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

Also event 'not A' = $A' = \{1, 3, 5, 7, 9, 10\}$

Now $\quad P(A') = P(1) + P(3) + P(5) + P(7) + P(9) + P(10)$

$$= \frac{6}{10} = \frac{3}{5}$$

Thus, $\quad P(A') = \frac{3}{5} = 1 - \frac{2}{5} = 1 - P(A)$

Also, we know that A' and A are mutually exclusive and exhaustive events i.e.,

$A \cap A' = \phi$ and $A \cup A' = S$

or $\quad P(A \cup A') = P(S)$

Now $\quad P(A) + P(A') = 1$, $\quad$ by using axioms (ii) and (iii).

or $\quad P(A') = P(\text{not } A) = 1 - P(A)$

We now consider some examples and exercises having equally likely outcomes unless stated otherwise.

Example 10 One card is drawn from a well shuffled deck of 52 cards. If each outcome is equally likely, calculate the probability that the card will be

(i) a diamond $\qquad$ (ii) not an ace

(iii) a black card (i.e., a club or, a spade) $\qquad$ (iv) not a diamond

(v) not a black card.

Solution When a card is drawn from a well shuffled deck of 52 cards, the number of possible outcomes is 52.

(i) Let A be the event 'the card drawn is a diamond'

Clearly the number of elements in set A is 13.

Therefore, $\quad P(A) = \frac{13}{52} = \frac{1}{4}$

i.e. Probability of a diamond card = —

(ii) We assume that the event 'Card drawn is an ace' is B

Therefore 'Card drawn is not an ace' should be B'.

We know that $P(B') = 1 - P(B) = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$

(iii) Let C denote the event 'card drawn is black card'

Therefore, number of elements in the set C = 26

i.e. $P(C) = \frac{26}{52} = \frac{1}{2}$

Thus, Probability of a black card = $\frac{1}{2}$.

(iv) We assumed in (i) above that A is the event 'card drawn is a diamond',

so the event 'card drawn is not a diamond' may be denoted as A' or 'not A'

Now $P(\text{not } A) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$

(v) The event 'card drawn is not a black card' may be denoted as C′ or 'not C'.

We know that $P(\text{not } C) = 1 - P(C) = 1 - \frac{1}{2} = \frac{1}{2}$

Therefore, Probability of not a black card = $\frac{1}{2}$

Example 11 A bag contains 9 discs of which 4 are red, 3 are blue and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be (i) red, (ii) yellow, (iii) blue, (iv) not blue, (v) either red or yellow.

Solution There are 9 discs in all so the total number of possible outcomes is 9.

Let the events A, B, C be defined as

A: 'the disc drawn is red'

B: 'the disc drawn is yellow'

C: 'the disc drawn is blue'.

(i) The number of red discs = 4, i.e., $n(A) = 4$

Hence $P(A) = \frac{4}{9}$

(ii) The number of yelow discs = 2, i.e., $n(B) = 2$

Therefore, $P(B) = \frac{2}{9}$

(iii) The number of blue discs = 3, i.e., $n(C) = 3$

Therefore, $P(C) = \frac{3}{9} = \frac{1}{3}$

(iv) Clearly the event 'not blue' is 'not C'. We know that P(not C) = 1 – P(C)

Therefore $P(\text{not } C) = 1 - \frac{1}{3} = \frac{2}{3}$

(v) The event 'either red or yellow' may be described by the set 'A or C'

Since, A and C are mutually exclusive events, we have

$$P(A \text{ or } C) = P(A \cup C) = P(A) + P(C) = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}$$

Example 12 Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination.

(b) Atleast one of them will not qualify the examination and

(c) Only one of them will qualify the examination.

Solution Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that

$$P(E) = 0.05,\ P(F) = 0.10 \text{ and } P(E \cap F) = 0.02.$$

Then

(a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as $E' \cap F'$.

Since, E' is 'not E', i.e., Anil will not qualify the examination and F' is 'not F', i.e., Ashima will not qualify the examination.

Also $E' \cap F' = (E \cup F)'$ (by Demorgan's Law)

Now $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $P(E \cup F) = 0.05 + 0.10 - 0.02 = 0.13$

Therefore $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) P (atleast one of them will not qualify)

$= 1 -$ P(both of them will qualify)

$= 1 - 0.02 = 0.98$

(c) The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) i.e., $E \cap F'$ or $E' \cap F$, where $E \cap F'$ and $E' \cap F$ are mutually exclusive.

Therefore, P(only one of them will qualify) $= P(E \cap F' \text{ or } E' \cap F)$

$= P(E \cap F') + P(E' \cap F) = P(E) - P(E \cap F) + P(F) - P(E \cap F)$

$= 0.05 - 0.02 + 0.10 - 0.02 = 0.11$

Example 13 A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man? (b) one man? (c) two men?

Solution The total number of persons = 2 + 2 = 4. Out of these four person, two can be selected in ${}^4C_2$ ways.

(a) No men in the committee of two means there will be two women in the committee. Out of two women, two can be selected in ${}^2C_2 = 1$ way.

Therefore $$P(\text{no man}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1 \times 2 \times 1}{4 \times 3} = \frac{1}{6}$$

(b) One man in the committee means that there is one woman. One man out of 2 can be selected in ${}^2C_1$ ways and one woman out of 2 can be selected in ${}^2C_1$ ways. Together they can be selected in ${}^2C_1 \times {}^2C_1$ ways.

Therefore $$P(\text{One man}) = \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \frac{2 \times 2}{2 \times 3} = \frac{2}{3}$$

(c) Two men can be selected in ${}^2C_2$ way.

Hence $$P(\text{Two men}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{{}^4C_2} = \frac{1}{6}$$

EXERCISE 16.3

1. Which of the following can not be valid assignment of probabilities for outcomes of sample Space S $= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}$

Assignment	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
(a)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(b)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
(c)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
(d)	– 0.1	0.2	0.3	0.4	– 0.2	0.1	0.3
(e)	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{15}{14}$

2. A coin is tossed twice, what is the probability that atleast one tail occurs?
3. A die is thrown, find the probability of following events:
   (i) A prime number will appear,
   (ii) A number greater than or equal to 3 will appear,
   (iii) A number less than or equal to one will appear,
   (iv) A number more than 6 will appear,
   (v) A number less than 6 will appear.
4. A card is selected from a pack of 52 cards.
   (a) How many points are there in the sample space?
   (b) Calculate the probability that the card is an ace of spades.
   (c) Calculate the probability that the card is (i) an ace (ii) black card.
5. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. find the probability that the sum of numbers that turn up is (i) 3 (ii) 12
6. There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?
7. A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up.
   From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.
8. Three coins are tossed once. Find the probability of getting
   (i) 3 heads (ii) 2 heads (iii) atleast 2 heads
   (iv) atmost 2 heads (v) no head (vi) 3 tails
   (vii) exactly two tails (viii) no tail (ix) atmost two tails
9. If $\frac{2}{11}$ is the probability of an event, what is the probability of the event 'not A'.
10. A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (i) a vowel (ii) a consonant

11. In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of Winning the prize in the game. [**Hint** order of the numbers is not important.]
12. Check whether the following probabilities P(A) and P(B) are consistently defined
    (i) $P(A) = 0.5,\ P(B) = 0.7,\ P(A \cap B) = 0.6$
    (ii) $P(A) = 0.5, P(B) = 0.4, P(A \cup B) = 0.8$
13. Fill in the blanks in following table:

	P(A)	**P(B)**	**P(A ∩ B)**	**P(A ∪ B)**
(i)	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$	. . .
(ii)	0.35	. . .	0.25	0.6
(iii)	0.5	0.35	. . .	0.7

14. Given $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find P(A or B), if A and B are mutually exclusive events.
15. If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and P(E and F) = $\frac{1}{8}$, find (i) P(E or F), (ii) P(not E and not F).
16. Events E and F are such that P(not E or not F) = 0.25, State whether E and F are mutually exclusive.
17. A and B are events such that P(A) = 0.42, P(B) = 0.48 and P(A and B) = 0.16. Determine (i) P(not A), (ii) P(not B) and (iii) P(A or B)
18. In Class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology.
19. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?
20. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Hindi examination?

21. In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(i) The student opted for NCC or NSS.
(ii) The student has opted neither NCC nor NSS.
(iii) The student has opted NSS but not NCC.

Miscellaneous Examples

Example 14 On her vacations Veena visits four cities (A, B, C and D) in a random order. What is the probability that she visits

(i) A before B? (ii) A before B and B before C?
(iii) A first and B last? (iv) A either first or second?
(v) A just before B?

Solution The number of arrangements (orders) in which Veena can visit four cities A, B, C, or D is 4! i.e., 24.Therefore, n (S) = 24.
Since the number of elements in the sample space of the experiment is 24 all of these outcomes are considered to be equally likely. A sample space for the experiment is

S = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB
BACD, BADC, BDAC, BDCA, BCAD, BCDA
CABD, CADB, CBDA, CBAD, CDAB, CDBA
DABC, DACB, DBCA, DBAC, DCAB, DCBA}

(i) Let the event 'she visits A before B' be denoted by E

Therefore, E = {ABCD, CABD, DABC, ABDC, CADB, DACB
ACBD, ACDB, ADBC, CDAB, DCAB, ADCB}

Thus $$\mathrm{P}(\mathrm{E}) = \frac{n(\mathrm{E})}{n(\mathrm{S})} = \frac{12}{24} = \frac{1}{2}$$

(ii) Let the event 'Veena visits A before B and B before C' be denoted by F.
Here F = {ABCD, DABC, ABDC, ADBC}

Therefore, $$\mathrm{P}(\mathrm{F}) = \frac{n(\mathrm{F})}{n(\mathrm{S})} = \frac{4}{24} = \frac{1}{6}$$

Students are advised to find the probability in case of (iii), (iv) and (v).

Example 15 Find the probability that when a hand of 7 cards is drawn from a well shuffled deck of 52 cards, it contains (i) all Kings (ii) 3 Kings (iii) atleast 3 Kings.

Solution Total number of possible hands = ${}^{52}C_7$

(i) Number of hands with 4 Kings = ${}^{4}C_4 \times {}^{48}C_3$ (other 3 cards must be chosen from the rest 48 cards)

Hence P (a hand will have 4 Kings) = $\dfrac{{}^{4}C_4 \times {}^{48}C_3}{{}^{52}C_7} = \dfrac{1}{7735}$

(ii) Number of hands with 3 Kings and 4 non-King cards = ${}^{4}C_3 \times {}^{48}C_4$

Therefore P (3 Kings) = $\dfrac{{}^{4}C_3 \times {}^{48}C_4}{{}^{52}C_7} = \dfrac{9}{1547}$

(iii) P(atleast 3 King) = P(3 Kings or 4 Kings)
= P(3 Kings) + P(4 Kings)

$$= \frac{9}{1547} + \frac{1}{7735} = \frac{46}{7735}$$

Example 16 If A, B, C are three events associated with a random experiment, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Solution Consider E = B ∪ C so that

$$P(A \cup B \cup C) = P(A \cup E)$$
$$= P(A) + P(E) - P(A \cap E) \quad \text{... (1)}$$

Now

$$P(E) = P(B \cup C)$$
$$= P(B) + P(C) - P(B \cap C) \quad \text{... (2)}$$

Also $A \cap E = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [using distribution property of intersection of sets over the union]. Thus

$$P(A \cap E) = P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[A \cap B \cap C] \quad \ldots (3)$$

Using (2) and (3) in (1), we get

$$P[A \cup B \cup C] = P(A) + P(B) + P(C) - P(B \cap C)$$

$$- P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Example 17 In a relay race there are five teams A, B, C, D and E.

(a) What is the probability that A, B and C finish first, second and third, respectively.

(b) What is the probability that A, B and C are first three to finish (in any order) (Assume that all finishing orders are equally likely)

Solution If we consider the sample space consisting of all finishing orders in the first three places, we will have ${}^5P_3$, i.e., $\frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60$ sample points, each with a probability of $\frac{1}{60}$.

(a) A, B and C finish first, second and third, respectively. There is only one finishing order for this, i.e., ABC.

Thus P(A, B and C finish first, second and third respectively) = $\frac{1}{60}$.

(b) A, B and C are the first three finishers. There will be 3! arrangements for A, B and C. Therefore, the sample points corresponding to this event will be 3! in number.

So P (A, B and C are first three to finish) $= \frac{3!}{60} = \frac{6}{60} = \frac{1}{10}$

Miscellaneous Exercise on Chapter 16

1. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that
(i) all will be blue? (ii) atleast one will be green?
2. 4 cards are drawn from a well – shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?

3. A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(i) P(2) (ii) P(1 or 3) (iii) P(not 3)

4. In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets.

5. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?
(b) you both enter the different sections?

6. Three letters are dictated to three persons and an evelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope.

7. A and B are two events such that P(A) = 0.54, P(B) = 0.69 and $P(A \cap B) = 0.35$. Find (i) $P(A \cup B)$ (ii) $P(A' \cap B')$ (iii) $P(A \cap B')$ (iv) $P(B \cap A')$

8. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. Particulars of five persons are as follows:

S. No.	**Name**	**Sex**	**Age in years**
1.	Harish	M	30
2.	Rohan	M	33
3.	Sheetal	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokesperson. What is the probability that the spokesperson will be either male or over 35 years?

9. If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when, (i) the digits are repeated? (ii) the repetition of digits is not allowed?

10. The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e., from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase?

Summary

In this Chapter, we studied about the axiomatic approach of probability. The main features of this Chapter are as follows:

- ***Sample space***: The set of all possible outcomes
- ***Sample points***: Elements of sample space
- ***Event***: A subset of the sample space
- ***Impossible event*** : The empty set
- ***Sure event***: The whole sample space
- ***Complementary event or 'not event'*** : The set A′ or S – A
- ***Event* A *or* B**: The set $A \cup B$
- ***Event* A *and* B**: The set $A \cap B$
- ***Event* A *and* not B**: The set A – B
- ***Mutually exclusive event***: A and B are mutually exclusive if $A \cap B = \phi$
- ***Exhaustive and mutually exclusive events***: Events $E_1, E_2,..., E_n$ are mutually exclusive and exhaustive if $E_1 \cup E_2 \cup ... \cup E_n = S$ and $E_i \cap E_j = \phi \ \forall\ i \neq j$
- ***Probability***: Number P (ω_i) associated with sample point ω_i such that

  (i) $0 \leq P(\omega_i) \leq 1$ (ii) $\sum P(\omega_i)$ for all $\omega_i \in S = 1$

  (iii) $P(A) = \sum P(\omega_i)$ for all $\omega_i \in A$. The number P (ω_i) is called *probability of the outcome* ω_i.
- ***Equally likely outcomes***: All outcomes with equal probability
- ***Probability of an event***: For a finite sample space with equally likely outcomes

  Probability of an event $P(A) = \dfrac{n(A)}{n(S)}$, where n(A) = number of elements in the set A, n(S) = number of elements in the set S.
- If A and B are any two events, then

  P(A or B) = P(A) + P(B) – P(A and B)

  equivalently, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are mutually exclusive, then P(A or B) = P(A) + P(B)
- If A is any event, then

  P(not A) = 1 – P(A)

Historical Note

Probability theory like many other branches of mathematics, evolved out of practical consideration. It had its origin in the 16th century when an Italian physician and mathematician Jerome Cardan (1501–1576) wrote the first book on the subject "Book on Games of Chance" (Biber de Ludo Aleae). It was published in 1663 after his death.

In 1654, a gambler Chevalier de Metre approached the well known French Philosoher and Mathematician Blaise Pascal (1623–1662) for certain dice problem. Pascal became interested in these problems and discussed with famous French Mathematician Pierre de Fermat (1601–1665). Both Pascal and Fermat solved the problem independently. Besides, Pascal and Fermat, outstanding contributions to probability theory were also made by Christian Huygenes (1629–1665), a Dutchman, J. Bernoulli (1654–1705), De Moivre (1667–1754), a Frenchman Pierre Laplace (1749–1827), A Frenchman and the Russian P.L Chebyshev (1821–1897), A. A Markov (1856–1922) and A. N Kolmogorove (1903–1987). Kolmogorove is credited with the axiomatic theory of probability. His book 'Foundations of Probability' published in 1933, introduces probability as a set function and is considered a classic.

Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

Much of our progress in the last few centuries has made it necessary to apply mathematical methods to real-life problems arising from different fields – be it Science, Finance, Management etc. The use of Mathematics in solving real-world problems has become widespread especially due to the increasing computational power of digital computers and computing methods, both of which have facilitated the handling of lengthy and complicated problems. The process of translation of a real-life problem into a mathematical form can give a better representation and solution of certain problems. The process of translation is called Mathematical Modelling.

Here we shall familiaries you with the steps involved in this process through examples. We shall first talk about what a mathematical model is, then we discuss the steps involved in the process of modelling.

A.2.2 Preliminaries

Mathematical modelling is an essential tool for understanding the world. In olden days the Chinese, Egyptians, Indians, Babylonians and Greeks indulged in understanding and predicting the natural phenomena through their knowledge of mathematics. The architects, artisans and craftsmen based many of their works of art on geometric prinicples.

Suppose a surveyor wants to measure the height of a tower. It is physically very difficult to measure the height using the measuring tape. So, the other option is to find out the factors that are useful to find the height. From his knowledge of trigonometry, he knows that if he has an angle of elevation and the distance of the foot of the tower to the point where he is standing, then he can calculate the height of the tower.

So, his job is now simplified to find the angle of elevation to the top of the tower and the distance from the foot of the tower to the point where he is standing. Both of which are easily measurable. Thus, if he measures the angle of elevation as 40° and the distance as 450m, then the problem can be solved as given in Example 1.

Example 1 The angle of elevation of the top of a tower from a point O on the ground, which is 450 m away from the foot of the tower, is 40°. Find the height of the tower.

Solution We shall solve this in different steps.

Step 1 We first try to understand the real problem. In the problem a tower is given and its height is to be measured. Let h denote the height. It is given that the horizontal distance of the foot of the tower from a particular point O on the ground is 450 m. Let d denotes this distance. Then d = 450m. We also know that the angle of elevation, denoted by θ, is 40°.

The real problem is to find the height h of the tower using the known distance d and the angle of elevation θ.

Step 2 The three quantities mentioned in the problem are height, distance and angle of elevation.

So we look for a relation connecting these three quantities. This is obtained by expressing it geometrically in the following way (Fig 1).

AB denotes the tower. OA gives the horizontal distance from the point O to foot of the tower. $\angle$AOB is the angle of elevation. Then we have

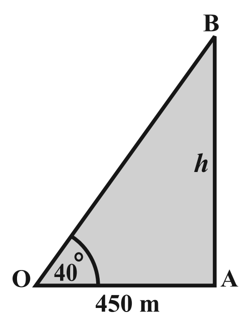


Fig 1

$$\tan\theta = \frac{h}{d} \text{ or } h = d\tan\theta \qquad \ldots (1)$$

This is an equation connecting θ, h and d.

Step 3 We use Equation (1) to solve h. We have θ = 40°. and d = 450m. Then we get $h = \tan 40° \times 450 = 450 \times 0.839 = 377.6$m

Step 4 Thus we got that the height of the tower approximately 378m.

Let us now look at the different steps used in solving the problem. In step 1, we have studied the real problem and found that the problem involves three parameters height, distance and angle of elevation. That means in this step we have *studied the real-life problem and identified the parameters.*

In the Step 2, we used some geometry and found that the problem can be represented geometrically as given in Fig 1. Then we used the trigonometric ratio for the "tangent" function and found the relation as

$$h = d\tan\theta$$

So, in this step we formulated the problem mathematically. That means we found an equation representing the real problem.

In Step 3, we solved the mathematical problem and got that h = 377.6m. That is we found

Solution of the problem.

In the last step, we interpreted the solution of the problem and stated that the height of the tower is approximately 378m. We call this as

Interpreting the mathematical solution to the real situation

In fact these are the steps mathematicians and others use to study various real-life situations. We shall consider the question, "why is it necessary to use mathematics to solve different situations."

Here are some of the examples where mathematics is used effectively to study various situations.

1. Proper flow of blood is essential to transmit oxygen and other nutrients to various parts of the body in humanbeings as well as in all other animals. Any constriction in the blood vessel or any change in the characteristics of blood vessels can change the flow and cause damages ranging from minor discomfort to sudden death. The problem is to find the relationship between blood flow and physiological characteristics of blood vessel.
2. In cricket a third umpire takes decision of a LBW by looking at the trajectory of a ball, simulated, assuming that the batsman is not there. Mathematical equations are arrived at, based on the known paths of balls before it hits the batsman's leg. This simulated model is used to take decision of LBW.
3. Meteorology department makes weather predictions based on mathematical models. Some of the parameters which affect change in weather conditions are temperature, air pressure, humidity, wind speed, etc. The instruments are used to measure these parameters which include thermometers to measure temperature, barometers to measure airpressure, hygrometers to measure humidity, anemometers to measure wind speed. Once data are received from many stations around the country and feed into computers for further analysis and interpretation.
4. Department of Agriculture wants to estimate the yield of rice in India from the standing crops. Scientists identify areas of rice cultivation and find the average yield per acre by cutting and weighing crops from some representative fields. Based on some statistical techniques decisions are made on the average yield of rice.

   How do mathematicians help in solving such problems? They sit with experts in the area, for example, a physiologist in the first problem and work out a mathematical equivalent of the problem. This equivalent consists of one or more equations or inequalities etc. which are called the mathematical models. Then solve the model and interpret the solution in terms of the original problem. Before we explain the process, we shall discuss what a mathematical model is.

A mathematical model is a representation which comprehends a situation.

An interesting geometric model is illustrated in the following example.

Example 2 (Bridge Problem) Konigsberg is a town on the Pregel River, which in the 18th century was a German town, but now is Russian. Within the town are two river islands that are connected to the banks with seven bridges as shown in (Fig 2).

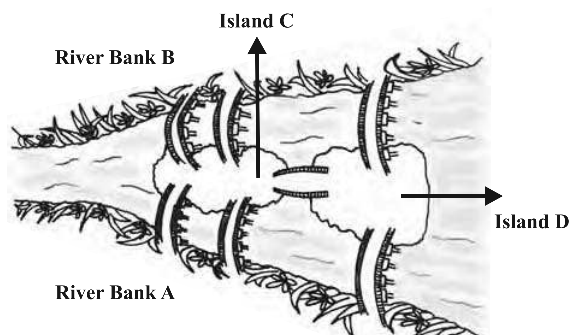


Fig 2

People tried to walk around the town in a way that only crossed each bridge once, but it proved to be difficult problem. Leonhard Euler, a Swiss mathematician in the service of the Russian empire Catherine the Great, heard about the problem. In 1736 Euler proved that the walk was not possible to do. He proved this by inventing a kind of diagram called a network, that is made up of vertices (dots where lines meet) and arcs (lines) (Fig3).

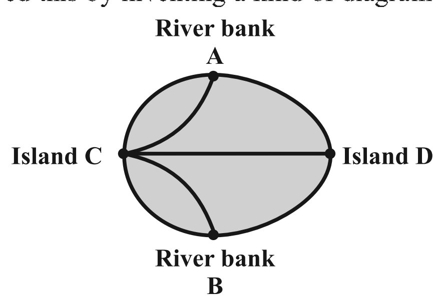


Fig 3

He used four dots (vertices) for the two river banks and the two islands. These have been marked A, B and C, D. The seven lines (arcs) are the seven bridges. You can see that 3 bridges (arcs) join to riverbank, A, and 3 join to riverbank B. 5 bridges (arcs) join to island C, and 3 join to island D. This means that all the vertices have an odd number of arcs, so they are called odd vertices (An even vertex would have to have an even number of arcs joining to it).

Remember that the problem was to travel around town crossing each bridge only once. On Euler's network this meant tracing over each arc only once, visiting all the vertices. Euler proved it could not be done because he worked out that, to have an odd vertex you would have to begin or end the trip at that vertex. (Think about it). Since there can only be one beginning and one end, there can only be two odd vertices if you are to trace over each arc only once. Since the bridge problem has 4 odd vertices, it just not possible to do!

After Euler proved his Theorem, much water has flown under the bridges in Konigsberg. In 1875, an extra bridge was built in Konigsberg, joining the land areas A and D (Fig 4). Is it possible now for the Konigsbergians to go round the city, using each bridge only once?

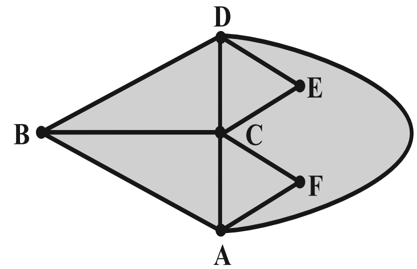


Fig 4

Here the situation will be as in Fig 4. After the addition of the new edge, both the vertices A and D have become even degree vertices. However, B and C still have odd degree. So, it is possible for the Konigsbergians to go around the city using each bridge exactly once.

The invention of networks began a new theory called graph theory which is now used in many ways, including planning and mapping railway networks (Fig 4).

A.2.3 What is Mathematical Modelling?

Here, we shall define what mathematical modelling is and illustrate the different processes involved in this through examples.

Definition Mathematical modelling is an attempt to study some part (or form) of the real-life problem in mathematical terms.

Conversion of physical situation into mathematics with some suitable conditions is known as mathematical modelling. Mathematical modelling is nothing but a technique and the pedagogy taken from fine arts and not from the basic sciences. Let us now understand the different processes involved in Mathematical Modelling. Four steps are involved in this process. As an illustrative example, we consider the modelling done to study the motion of a simple pendulum.

Understanding the problem

This involves, for example, understanding the process involved in the motion of simple pendulum. All of us are familiar with the simple pendulum. This pendulum is simply a mass (known as bob) attached to one end of a string whose other end is fixed at a point. We have studied that the motion of the simple pendulum is periodic. The period depends upon the length of the string and acceleration due to gravity. So, what we need to find is the period of oscillation. Based on this, we give a precise statement of the problem as

Statement How do we find the period of oscillation of the simple pendulum?
The next step is formulation.

Formulation Consists of two main steps.

1. Identifying the relevant factors In this, we find out what are the factors/

parameters involved in the problem. For example, in the case of pendulum, the factors are period of oscillation (T), the mass of the bob (m), effective length (l) of the pendulum which is the distance between the point of suspension to the centre of mass of the bob. Here, we consider the length of string as effective length of the pendulum and acceleration due to gravity (g), which is assumed to be constant at a place.

So, we have identified four parameters for studying the problem. Now, our purpose is to find T. For this we need to understand what are the parameters that affect the period which can be done by performing a simple experiment.

We take two metal balls of two different masses and conduct experiment with each of them attached to two strings of equal lengths. We measure the period of oscillation. We make the observation that there is no appreciable change of the period with mass. Now, we perform the same experiment on equal mass of balls but take strings of different lengths and observe that there is clear dependence of the period on the length of the pendulum.

This indicates that the mass m is not an *essential parameter* for finding period whereas the length l is an essential parameter.

This process of searching the **essential parameters** is necessary before we go to the next step.

2. Mathematical description This involves finding an equation, inequality or a geometric figure using the parameters already identified.

In the case of simple pendulum, experiments were conducted in which the values of period T were measured for different values of l. These values were plotted on a graph which resulted in a curve that resembled a parabola. It implies that the relation between T and l could be expressed

$$T^2 = kl \qquad \ldots (1)$$

It was found that $k = \dfrac{4\pi^2}{g}$. This gives the equation

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \ldots (2)$$

Equation (2) gives the mathematical formulation of the problem.

Finding the solution The mathematical formulation rarely gives the answer directly. Usually we have to do some operation which involves solving an equation, calculation or applying a theorem etc. In the case of simple pendulums the solution involves applying the formula given in Equation (2).

The period of oscillation calculated for two different pendulums having different lengths is given in Table 1

Table 1

l	225 cm	275cm
T	3.04 sec	3.36 sec

The table shows that for l = 225 cm, T = 3.04 sec and for l = 275 cm, T = 3.36 sec.

Interpretation/Validation

A mathematical model is an attempt to study, the essential characteristic of a real life problem. Many times model equations are obtained by assuming the situation in an idealised context. The model will be useful only if it explains all the facts that we would like it to explain. Otherwise, we will reject it, or else, improve it, then test it again. In other words, ***we measure the effectiveness of the model by comparing the results obtained from the mathematical model, with the known facts about the real problem. This process is called validation of the model.*** In the case of simple pendulum, we conduct some experiments on the pendulum and find out period of oscillation. The results of the experiment are given in Table 2.

Table 2

Periods obtained experimentally for four different pendulums

Mass (gms)	Length (cms)	Time (secs)
385	275 225	3.371 3.056
230	275 225	3.352 3.042

Now, we compare the measured values in Table 2 with the calculated values given in Table 1.

The difference in the observed values and calculated values gives the error. For example, for l = 275 cm, and mass m = 385 gm,

$$\text{error} = 3.371 - 3.36 = 0.011$$

which is small and the model is accepted.

Once we accept the model, we have to interpret the model. ***The process of describing the solution in the context of the real situation is called interpretation of the model.*** In this case, we can interpret the solution in the following way:

(a) The period is directly proportional to the square root of the length of the pendulum.

(b) It is inversely proportional to the square root of the acceleration due to gravity.

Our validation and interpretation of this model shows that the mathematical model is in good agreement with the practical (or observed) values. But we found that there is some error in the calculated result and measured result. This is because we have neglected the mass of the string and resistance of the medium. So, in such situation we look for a better model and this process continues.

This leads us to an important observation. The real world is far too complex to understand and describe completely. We just pick one or two main factors to be completely accurate that may influence the situation. Then try to obtain a simplified model which gives some information about the situation. We study the simple situation with this model expecting that we can obtain a better model of the situation.

Now, we summarise the main process involved in the modelling as

(a) Formulation (b) Solution (c) Interpretation/Validation

The next example shows how modelling can be done using the techniques of finding graphical solution of inequality.

Example 3 A farm house uses atleast 800 kg of special food daily. The special food is a mixture of corn and soyabean with the following compositions

Table 3

Material	Nutrients present per Kg Protein	Nutrients present per Kg Fibre	Cost per Kg
Corn	.09	.02	Rs 10
Soyabean	.60	.06	Rs 20

The dietary requirements of the special food stipulate atleast 30% protein and at most 5% fibre. Determine the daily minimum cost of the food mix.

Solution Step 1 Here the objective is to minimise the total daily cost of the food which is made up of corn and soyabean. So the variables (factors) that are to be considered are

x = the amount of corn

y = the amount of soyabean

z = the cost

Step 2 The last column in Table 3 indicates that z, x, y are related by the equation

$$z = 10x + 20y \qquad \text{... (1)}$$

The problem is to minimise z with the following constraints:

(a) The farm used atleast 800 kg food consisting of corn and soyabean

i.e., $x + y \geq 800$ ... (2)

(b) The food should have atleast 30% protein dietary requirement in the proportion as given in the first column of Table 3. This gives

$0.09x + 0.6y \geq 0.3\ (x + y)$ ... (3)

(c) Similarly the food should have atmost 5% fibre in the proportion given in 2nd column of Table 3. This gives

$0.02x + 0.06\ y \leq 0.05\ (x + y)$ ... (4)

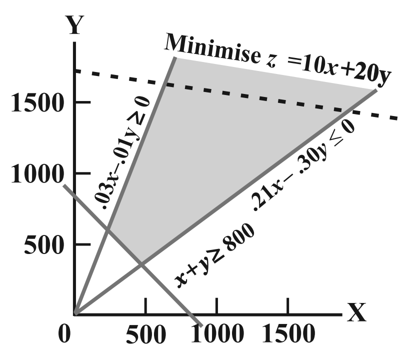
We simplify the constraints given in (2), (3) and (4) by grouping all the coefficients of x, y.

Then the problem can be restated in the following mathematical form. **Statement** Minimise z subject to

$$x + y \geq 800$$
$$0.21x - .30y \leq 0$$
$$0.03x - .01y \geq 0$$

This gives the formulation of the model.

Step 3 This can be solved graphically. The shaded region in Fig 5 gives the possible solution of the equations. From the graph it is clear that the minimum value is got at the point (470.6,329.4) i.e., $x = 470.6$ and $y = 329.4$.



Optimum: $x = 470.6$ **kg**
$y = 329.4$ **kg**
$z =$ **Rs 11294**

Fig 5

This gives the value of z as $z = 10 \times 470.6 + 20 \times 329.4 = 11294$

This is the mathematical solution.

Step 4 The solution can be interpreted as saying that, "The minimum cost of the special food with corn and soyabean having the required portion of nutrient contents, protein and fibre is Rs 11294 and we obtain this minimum cost if we use 470.6 kg of corn and 329.4 kg of soyabean."

In the next example, we shall discuss how modelling is used to study the population of a country at a particular time.

Example 4 Suppose a population control unit wants to find out "how many people will be there in a certain country after 10 years"

Step 1 Formulation We first observe that the population changes with time and it increases with birth and decreases with deaths.

We want to find the population at a particular time. Let t denote the time in years. Then t takes values 0, 1, 2, ..., $t = 0$ stands for the present time, $t = 1$ stands for the next year etc. For any time t, let $p(t)$ denote the population in that particular year.

Suppose we want to find the population in a particular year, say $t_0 = 2006$. How will we do that. We find the population by Jan. 1st, 2005. Add the number of births in that year and subtract the number of deaths in that year. Let B(t) denote the number of births in the one year between t and $t + 1$ and D(t) denote the number of deaths between t and $t + 1$. Then we get the relation

$$\mathrm{P}(t+1) = \mathrm{P}(t) + \mathrm{B}(t) - \mathrm{D}(t)$$

Now we make some assumptions and definitions

1. $\dfrac{\mathrm{B}(t)}{\mathrm{P}(t)}$ is called the *birth rate* for the time interval t to $t + 1$.

2. $\dfrac{\mathrm{D}(\mathrm{t})}{\mathrm{P}(\mathrm{t})}$ is called the *death rate* for the time interval t to $t + 1$.

Assumptions

1. The birth rate is the same for all intervals. Likewise, the death rate is the same for all intervals. This means that there is a constant b, called the birth rate, and a constant d, called the death rate so that, for all $t \geq 0$,

$$b = \frac{\mathrm{B}(t)}{\mathrm{P}(t)} \quad \text{and} \quad d = \frac{\mathrm{D}(t)}{\mathrm{P}(t)} \qquad \text{... (1)}$$

2. There is no migration into or out of the population; i.e., the only source of population change is birth and death.

As a result of assumptions 1 and 2, we deduce that, for $t \geq 0$,

$$\begin{aligned} P(t+1) &= P(t) + B(t) - D(t) \\ &= P(t) + bP(t) - dP(t) \\ &= (1+b-d)\,P(t) \qquad \ldots (2) \end{aligned}$$

Setting $t = 0$ in (2) gives

$$P(1) = (1 + b - d)P(0) \qquad \ldots (3)$$

Setting $t = 1$ in Equation (2) gives

$$\begin{aligned} P(2) &= (1+b-d)\,P(1) \\ &= (1+b-d)(1+b-d)\,P(0) \qquad \text{(Using equation 3)} \\ &= (1+b-d)^2\,P(0) \end{aligned}$$

Continuing this way, we get

$$P(t) = (1 + b - d)^t\,P(0) \qquad \ldots (4)$$

for $t = 0, 1, 2, \ldots$ The constant $1 + b - d$ is often abbreviated by r and called the *growth rate or*, in more high-flown language, the *Malthusian parameter,* in honor of Robert Malthus who first brought this model to popular attention. In terms of r, Equation (4) becomes

$$P(t) = P(0)r^t \quad , \quad t = 0, 1, 2, \ldots \qquad \ldots (5)$$

$P(t)$ is an example of an *exponential function.* Any function of the form cr^t, where c and r are constants, is an exponential function.
Equation (5) gives the mathematical formulation of the problem.

Step 2 – Solution

Suppose the current population is 250,000,000 and the rates are $b = 0.02$ and $d = 0.01$. What will the population be in 10 years? Using the formula, we calculate P(10).

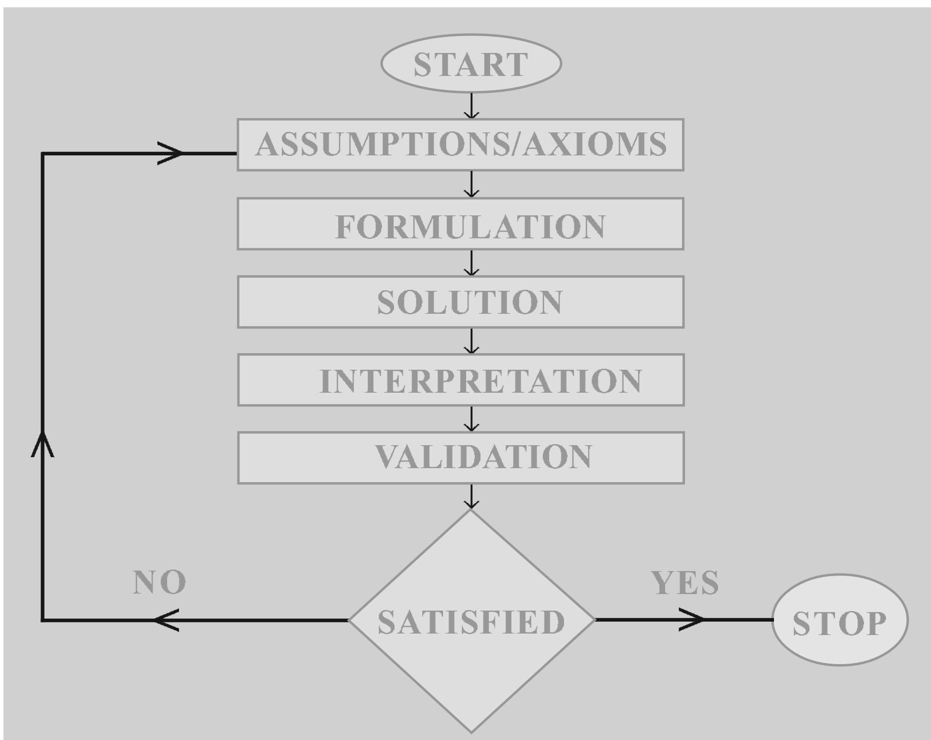
$$\begin{aligned} P(10) &= (1.01)^{10}\,(250{,}000{,}000) \\ &= (1.104622125)\,(250{,}000{,}000) \\ &= 276{,}155{,}531.25 \end{aligned}$$

Step 3 Interpretation and Validation

Naturally, this result is absurd, since one can't have 0.25 of a person.
So, we do some approximation and conclude that the population is 276,155,531 (approximately). Here, we are not getting the exact answer because of the assumptions that we have made in our mathematical model.

The above examples show how modelling is done in variety of situations using different mathematical techniques.

Since a mathematical model is a simplified representation of a real problem, by its very nature, has built-in assumptions and approximations. Obviously, the most important question is to decide whether our model is a good one or not i.e., when the obtained results are interpreted physically whether or not the model gives reasonable answers. If a model is not accurate enough, we try to identify the sources of the shortcomings. It may happen that we need a new formulation, new mathematical manipulation and hence a new evaluation. Thus mathematical modelling can be a cycle of the modelling process as shown in the flowchart given below:



❖

ANSWERS

EXERCISE 1.1

1. (i), (iv), (v), (vi), (vii) and (viii) are sets.
2. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (vi) $\in$ (v) $\in$ (vi) $\notin$
3. (i) A = {–3, –2, –1, 0, 1, 2, 3, 4, 5, 6 } (ii) B = {1, 2, 3, 4, 5}
   (iii) C = {17, 26, 35, 44, 53, 62, 71, 80} (iv) D = {2, 3, 5}
   (v) E = {T, R, I, G, O, N, M, E, Y} (vi) F = {B, E, T, R,}
4. (i) $\{ x : x = 3n \text{ and } 1 \leq n \leq 4 \}$ (ii) $\{ x : x = 2^n \text{ and } 1 \leq n \leq 5 \}$
   (iii) $\{ x : x = 5^n \text{ and } 1 \leq n \leq 4 \}$ (iv) $\{ x : x$ is an even natural number$\}$
   (v) $\{ x : x = n^2 \text{ and } 1 \leq n \leq 10 \}$
5. (i) A = {1, 3, , 5, . . . } (ii) B = {0, 1, 2, 3, 4 }
   (iii) C = { –2, –1, 0, 1, 2 } (iv) D = { L, O, Y, A }
   (v) E = { February, April, June, September, November }
   (vi) F = $\{b, c, d, f, g, h, j \}$
6. (i) $\leftrightarrow$ (c) (ii) $\leftrightarrow$ (a) (iii) $\leftrightarrow$ (d) (iv) $\leftrightarrow$ (b)

EXERCISE 1.2

1. (i), (iii), (iv)
2. (i) Finite (ii) Infinite (iii) Finite (iv) Infinite (v) Finite
3. (i) Infinite (ii) Finite (iii) Infinite (iv) Finite (v) Infinite
4. (i) Yes (ii) No (iii) Yes (iv) No
5. (i) No (ii) Yes 6. B= D, E = G

EXERCISE 1.3

1. (i) $\subset$ (ii) $\not\subset$ (iii) $\subset$ (iv) $\not\subset$ (v) $\not\subset$ (vi) $\subset$
   (vii) $\subset$
2. (i) False (ii) True (iii) False (iv) True (v) False (vi) True
3. (i), (v), (vii), (viii), (ix), (xi)
4. (i) ϕ $\{ a \}$, (ii) ϕ, $\{ a \}$, $\{ b \}$ $\{ a, b \}$
   (iii) ϕ, { 1 }, { 2 }, { 3 }, { 1, 2 }, { 1, 3 }, { 2, 3 } { 1, 2, 3 } (iv) ϕ
5. 1
6. (i) (– 4, 6] (ii) (– 12, –10) (iii) [0, 7)
   (iv) [3, 4]
7. (i) $\{ x : x \in \mathbf{R}, -3 < x < 0 \}$ (ii) $\{ x : x \in \mathbf{R}, 6 \leq x \leq 12 \}$
   (iii) $\{ x : x \in \mathbf{R}, 6 < x \leq 12 \}$ (iv) $\{ x \ \mathbf{R} : -23 \leq x < 5 \}$ 9. (iii)

EXERCISE 1.4

1. (i) $X \cup Y = \{1, 2, 3, 5\}$ (ii) $A \cup B = \{a, b, c, e, i, o, u\}$
   (iii) $A \cup B = \{x : x = 1, 2, 4, 5$ or a multiple of $3\}$
   (iv) $A \cup B = \{x : 1 < x < 10, x \in N\}$ (v) $A \cup B = \{1, 2, 3\}$
2. Yes, A B = $\{a, b, c\}$ 3. B
4. (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (iii) $\{3, 4, 5, 6, 7, 8\}$
   (iv) $\{3, 4, 5, 6, 7, 8, 9, 10)$ (v) $\{1, 2, 3, 4, 5, 6, 7, 8\}$
   (vi) $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (vii) $\{3, 4, 5, 6, 7, 8, 9, 10)$
5. (i) $X \cap Y = \{1, 3\}$ (ii) $A \cap B = \{a\}$ (iii) $\{3\}$
6. (i) $\{7, 9, 11\}$ (ii) $\{11, 13\}$ (iii) ϕ (iv) $\{11\}$
   (v) ϕ (vi) $\{7, 9, 11\}$ (vii) ϕ
   (viii) $\{7, 9, 11\}$ (ix) $\{7, 9, 11\}$ (x) $\{7, 9, 11, 15\}$
7. (i) B (ii) C (iii) D (iv) ϕ
   (v) $\{2\}$ (vi) $\{x : x$ is an odd prime number $\}$ 8. (iii)
9. (i) $\{3, 6, 9, 15, 18, 21\}$ (ii) $\{3, 9, 15, 18, 21\}$ (iii) $\{3, 6, 9, 12, 18, 21\}$
   (iv) $\{4, 8, 16, 20)$ (v) $\{2, 4, 8, 10, 14, 16\}$ (vi) $\{5, 10, 20\}$
   (vii) $\{20)$ (viii) $\{4, 8, 12, 16\}$ (ix) $\{2, 6, 10, 14\}$
   (x) $\{5, 10, 15\}$ (xi) $\{2, 4, 6, 8, 12, 14, 16\}$ (xii) $\{5, 15, 20\}$
10. (i) $\{a, c\}$ (ii) $\{f, g\}$ (iii) $\{b, d\}$
11. Set of irrational numbers 12. (i) F (ii) F (iii) T (iv) T

EXERCISE 1.5

1. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{7, 8, 9\}$
   (iv) $\{5, 7, 9)$ (v) $\{1, 2, 3, 4\}$ (vi) $\{1, 3, 4, 5, 6, 7, 9\}$
2. (i) $\{d, e, f, g, h\}$ (ii) $\{a, b, c, h\}$ (iii) $\{b, d, f, h\}$
   (iv) $\{b, c, d, e)$
3. (i) $\{x : x$ is an odd natural number $\}$
   (ii) $\{x : x$ is an even natural number $\}$
   (iii) $\{x : x \in \mathbf{N}$ and x is not a multiple of $3\}$
   (iv) $\{x : x$ is a positive composit number and $x = 1]$

(v) $\{x : x$ is a positive integer which is not divisible by 3 or not divisible by 5$\}$
(vi) $\{x : x \in \mathbf{N}$ and x is not a perfect square $\}$
(vii) $\{x : x \in \mathbf{N}$ and x is not a perfect cube $\}$
(viii) $\{x : x \in \mathbf{N}$ and $x = 3\}$ (ix) $\{x : x \in \mathbf{N}$ and $x = 2\}$
(x) $\{x : x \in \mathbf{N}$ and $x < 7\}$ (xi) $\{x : x \in \mathbf{N}$ and $x > \frac{9}{2}\}$

6. is the set of all equilateral triangles.
7. (i) U (ii) A (iii) ϕ (iv) ϕ

EXERCISE 1.6

1. 2 2. 5 3. 50 4. 42
5. 30 6. 19 7. 25, 35 8. 60

Miscellaneous Exercise on Chapter 1

1. $A \subset B$, $A \subset C$, $B \subset C$, $D \subset A$, $D \subset B$, $D \subset C$
2. (i) False (ii) False (iii) True (iv) False (v) False
(vi) True
7. False 12. We may take $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$
13. 325 14. 125 15. 52, 30 16. 11

EXERCISE 2.1

1. $x = 2$ and $y = 1$ 2. The number of elements in $A \times B$ is 9.
3. $G \times H = \{(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)\}$
$H \times G = \{(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)\}$
4. (i) False
$P \times Q = \{(m, n)\ (m, m)\ (n, n), (n, m)\}$
(ii) False
$A \times B$ is a non empty set of ordered pairs (x, y) such that $x \in A$ and $y \in B$
(iii) True
5. $A \times A = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$
$A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1),$
$(1, 1, -1), (1, 1, 1)\}$
6. $A = \{a, b\}$, $B = \{x, y\}$
8. $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
$A \times B$ will have $2^4 = 16$ subsets.
9. $A = \{x, y, z\}$ and $B = \{1, 2\}$

10. A = {–1, 0, 1}, remaining elements of
A × A are (–1, –1), (–1, 1), (0, –1), (0, 0), (1, –1), (1, 0), (1, 1)

EXERCISE 2.2

1. R = {(1, 3), (2, 6), (3, 9), (4, 12)}
Domain of R = {1, 2, 3, 4}
Range of R = {3, 6, 9, 12}
Co domain of R = {1, 2, ..., 14}

2. R = {(1, 6), (2, 7), (3, 8)}
Domain of R = {1, 2, 3}
Range of R = {6, 7, 8}

3. R = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)}

4. (i) R = $\{(x, y) : y = x - 2 \text{ for } x = 5, 6, 7\}$
(ii) R = {(5,3), (6,4), (7,5)}. Domain of R = {5, 6, 7}, Range of R = {3, 4, 5}

5. (i) R = {(1, 1), (1,2), (1, 3), (1, 4), (1, 6), (2 4), (2, 6), (2, 2), (4, 4), (6, 6), (3, 3), (3, 6)}
(ii) Domain of R = {1, 2, 3, 4, 6}
(iii) Range of R = {1, 2, 3, 4, 6}

6. Domian of R = {0, 1, 2, 3, 4, 5,}
Range of R = {5, 6, 7, 8, 9, 10}

7. R = {(2, 8), (3, 27), (5, 125), (7, 343)}

8. No. of relations from A into B = 2^6

9. Domain of R = **Z**
Range of R = **Z**

EXERCISE 2.3

1. (i) yes, Domain = {2, 5, 8, 11, 14, 17}, Range = {1}
(ii) yes, Domain = (2, 4, 6, 8, 10, 12, 14}, Range = {1, 2, 3, 4, 5, 6, 7}
(iii) No.

2. (i) Domain = **R**, Range = $(-\infty, 0]$
(ii) Domain of Function = $\{x : -3 \le x \le 3\}$
(iii) Range of Function = $\{x : 0 \le x \le 3\}$

3. (i) $f(0) = -5$ (ii) $f(7) = 9$ (iii) $f(-3) = -11$

4. (i) $t(0) = 32$ (ii) $t(28) = \dfrac{412}{5}$ (iii) $t(-10) = 14$ (iv) 100

5. (i) Range = $(-\infty, 2)$ (ii) Range = $[2, \infty)$ (iii) Range = **R**

Miscellaneous Exercise on Chapter 2

2. 2.1 **3.** Domain of function is set of real numbers except 6 and 2.

4. Domain $= [1, \infty)$, Range $= [0, \infty)$

5. Domain = **R**, Range = non-negative real numbers

6. Range = Any positive real number x such that $0 \le x < 1$

7. $(f + g)\ x = 3x - 2$ **8.** $a = 2,\ b = -1$ **9.** (i) No (ii) No (iii) No

$(f - g)\ x = -x + 4$

$$\left(\frac{f}{g}\right)x = \frac{x+1}{2x-3},\quad x \neq \frac{3}{2}$$

10. (i) Yes, (ii) No **11.** No **12.** Range of $f = \{3, 5, 11, 13\}$

EXERCISE 3.1

1. (i) $\frac{5\pi}{36}$ (ii) $-\frac{19\pi}{72}$ (iii) $\frac{4\pi}{3}$ (iv) $\frac{26\pi}{9}$

2. (i) $39^\circ\ 22'\ 30''$ (ii) $-229^\circ\ 5'\ 29''$ (iii) 300° (iv) 210°

3. 12π **4.** $12^\circ\ 36'$ **5.** $\frac{20\pi}{3}$ **6.** $5 : 4$

7. (i) $\frac{2}{15}$ (ii) $\frac{1}{5}$ (iii) $\frac{7}{25}$

EXERCISE 3.2

1. $\sin x = -\frac{\sqrt{3}}{2}$, $\operatorname{cosec} x = -\frac{2}{\sqrt{3}}$, $\sec x = -2$, $\tan x = \sqrt{3}$, $\cot x = \frac{1}{\sqrt{3}}$

2. $\operatorname{cosec} x = \frac{5}{3}$, $\cos x = -\frac{4}{5}$, $\sec x = -\frac{5}{4}$, $\tan x = -\frac{3}{4}$, $\cot x = -\frac{4}{3}$

3. $\sin x = -\frac{4}{5}$, $\operatorname{cosec} x = -\frac{5}{4}$, $\cos x = -\frac{3}{5}$, $\sec x = -\frac{5}{3}$, $\tan x = \frac{4}{3}$

4. $\sin x = -\frac{12}{13}$, $\operatorname{cosec} x = -\frac{13}{12}$, $\cos x = \frac{5}{13}$, $\tan x = -\frac{12}{5}$, $\cot x = -\frac{5}{12}$

5. $\sin x = \frac{5}{13}$, $\text{cosec}\, x = \frac{13}{5}$, $\cos x = -\frac{12}{13}$, $\sec x = -\frac{13}{12}$, $\cot x = -\frac{12}{5}$

6. $\frac{1}{\sqrt{2}}$ 7. 2 8. $\sqrt{3}$ 9. $\frac{\sqrt{3}}{2}$ 10. 1

EXERCISE 3.3

5. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $2-\sqrt{3}$

EXERCISE 3.4

1. $\frac{\pi}{3}, \frac{4\pi}{3}, n\pi + \frac{\pi}{3}, n \in \mathbf{Z}$

2. $\frac{\pi}{3}, \frac{5\pi}{3}, 2n\pi \pm \frac{\pi}{3}, n \in \mathbf{Z}$

3. $\frac{5\pi}{6}, \frac{11\pi}{6}, n\pi \pm \frac{5\pi}{6}, n \in \mathbf{Z}$

4. $\frac{7\pi}{6}, \frac{11\pi}{6}, n\pi + (-1)^n \frac{7\pi}{6}, n \in \mathbf{Z}$

5. $x = \frac{n\pi}{3}$ or $x = n\pi$, $n \in \mathbf{Z}$

6. $x = (2n+1)\frac{\pi}{4}$, or $2n\pi \pm \frac{\pi}{3}$, $n \in \mathbf{Z}$

7. $x = n\pi + (-1)^n \frac{7\pi}{6}$ or $(2n+1)\frac{\pi}{2}$, $n \in \mathbf{Z}$

8. $x = \frac{n\pi}{2}$, or $\frac{n\pi}{2} + \frac{3\pi}{8}$, $n \in \mathbf{Z}$

9. $x = \frac{n\pi}{3}$, or $n\pi \pm \frac{\pi}{3}$, $n \in \mathbf{Z}$

Miscellaneous Exercise on Chapter 3

8. $\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, 2$

9. $\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}, -\sqrt{2}$

10. $\frac{\sqrt{8+2\sqrt{15}}}{4}, \frac{\sqrt{8-2\sqrt{15}}}{4}, 4+\sqrt{15}$

EXERCISE 5.1

1. 3
2. 0
3. i
4. $14 + 28\,i$
5. $2 - 7\,i$
6. $-\dfrac{19}{5} - \dfrac{21i}{10}$
7. $\dfrac{17}{3} = i\dfrac{5}{3}$
8. -4
9. $-\dfrac{242}{27} - 26i$
10. $\dfrac{-22}{3} - i\dfrac{107}{27}$
11. $\dfrac{4}{25} + i\dfrac{3}{25}$
12. $\dfrac{\sqrt{5}}{14} - i\dfrac{3}{14}$
13. i
14. $\dfrac{-7\sqrt{2}}{2}\,i$

EXERCISE 5.2

1. $2, \dfrac{-2\pi}{3}$
2. $2, \dfrac{5\pi}{6}$
3. $\sqrt{2}\left(\cos\dfrac{-\pi}{4} + i\sin\dfrac{-\pi}{4}\right)$
4. $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$
5. $\sqrt{2}\left(\cos\dfrac{-3\pi}{4} + i\sin\dfrac{-3\pi}{4}\right)$
6. $3\,(\cos\pi + i\sin\pi)$
7. $2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$
8. $\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$

EXERCISE 5.3

1. $\pm\sqrt{3}\,i$
2. $\dfrac{-1 \pm \sqrt{7}\,i}{4}$
3. $\dfrac{-3 \pm 3\sqrt{3}\,i}{2}$
4. $\dfrac{-1 \pm \sqrt{7}i}{-2}$
5. $\dfrac{-3 \pm \sqrt{11}\,i}{2}$
6. $\dfrac{1 \pm \sqrt{7}\,i}{2}$
7. $\dfrac{-1 \pm \sqrt{7}\,i}{2\sqrt{2}}$
8. $\dfrac{\sqrt{2} \pm \sqrt{34}\,i}{2\sqrt{3}}$
9. $\dfrac{-1 \pm \sqrt{\left(4 - \sqrt{2}\right)}\,i}{2}$
10. $\dfrac{-1 \pm \sqrt{7}\,i}{2\sqrt{2}}$

Miscellaneous Exercise on Chapter 5

1. $2-2i$ **3.** $\dfrac{307+599i}{442}$

5. (i) $\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$, (ii) $\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

6. $\dfrac{2}{3}\pm\dfrac{4}{3}i$ **7.** $1\pm\dfrac{\sqrt{2}}{2}i$ **8.** $\dfrac{5}{27}\pm\dfrac{\sqrt{2}}{27}i$ **9.** $\dfrac{14}{21}\pm\dfrac{\sqrt{14}}{21}i$

10. $\dfrac{4\sqrt{5}}{5}$ **12.** (i) $\dfrac{-2}{5}$, (ii) 0 **13.** $\dfrac{1}{\sqrt{2}}, \dfrac{3\pi}{4}$ **14.** $x=3, y=-3$

15. 2 **17.** 1 **18.** 0 **20.** 4

EXERCISE 6.1

1. (i) $\{1, 2, 3, 4\}$ (ii) $\{\ldots -3, -2, -1, 0, 1,2,3,4,\}$

2. (i) No Solution (ii) $\{\ldots -4, -3\}$

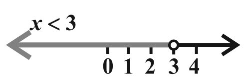
3. (i) $\{\ldots -2, -1, 0, 1\}$ (ii) $(-\infty, 2)$

4. (i) $\{-1, 0, 1, 2, 3, \ldots\}$ (ii) $(-2, \infty)$

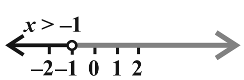
5. $(-2, \infty)$ **6.** $(-\infty, -3)$ **7.** $(-\infty, -3]$ **8.** $(-\infty, 4]$

9. $(-\infty, 6)$ **10.** $(-\infty, -6)$ **11.** $(-\infty, 2]$ **12.** $(-\infty, 120]$

13. $(4, \infty)$ **14.** $(-\infty, 2]$ **15.** $(4, \infty)$ **16.** $(-\infty, 2]$

17. $x<3$, 

18. $x\geq -1$, 

19. $x>-1$, 

20. $x<-\dfrac{2}{7}$, 

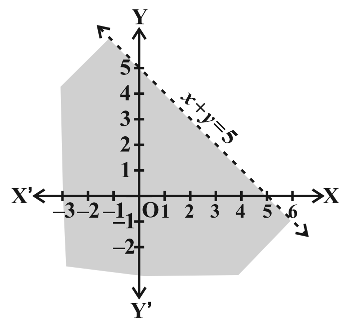
21. More than or equal to 35 **22.** Greater than or equal to 82

23. (5,7), (7,9) **24.** (6,8), (8,10), (10,12)

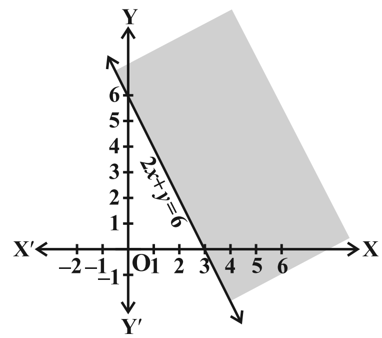
25. 9 cm **26.** Greater than or equal to 8 but less than or equal to 22

EXERCISE 6.2

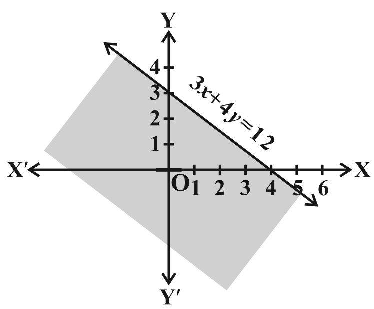
1.



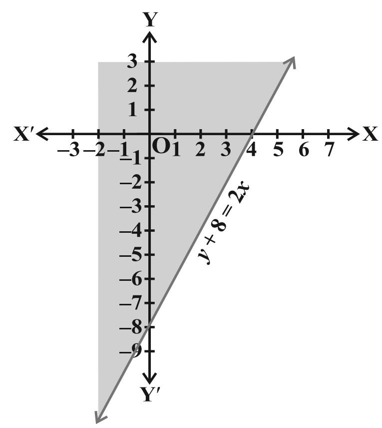
2.



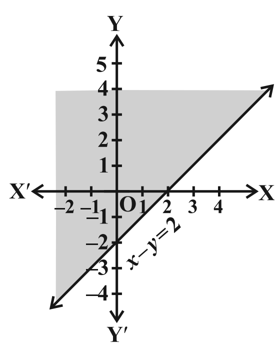
3.



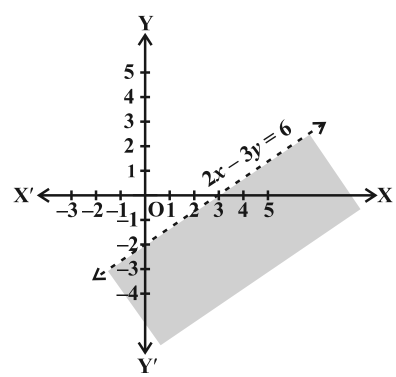
4.



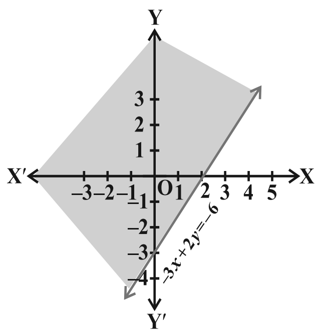
5.



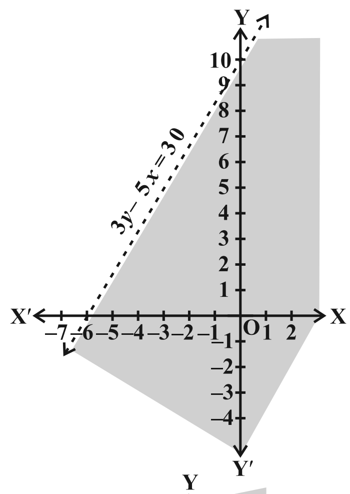
6.



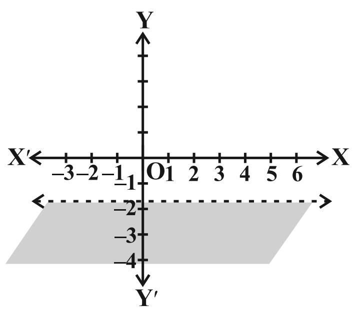
7.



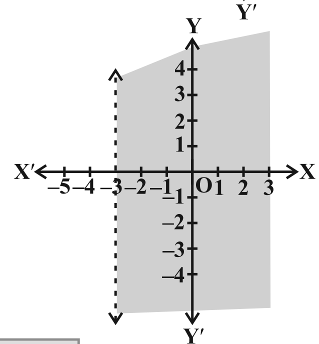
8.



9.

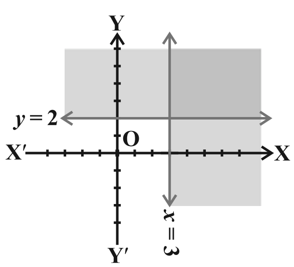


10.

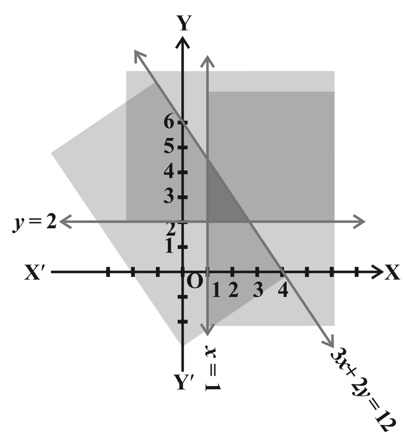


EXERCISE 6.3

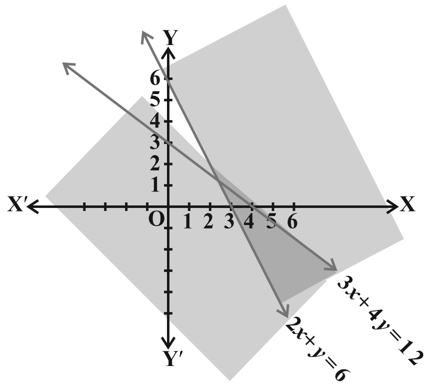
1.



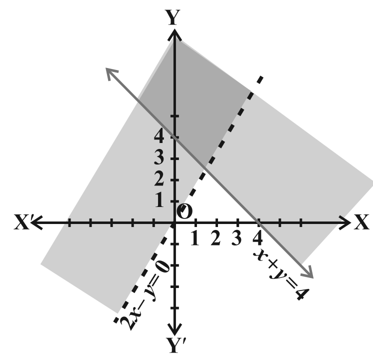
2.



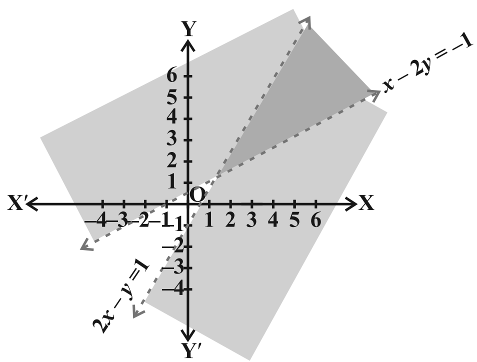
3.


Y
X′
X
O
Y′
3x + 4y = 12
2x + y = 6

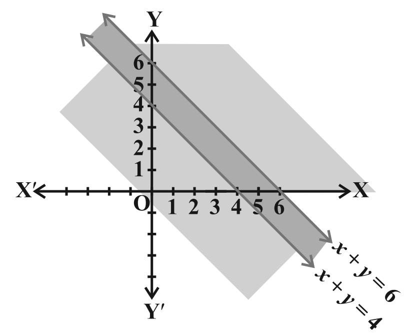
4.


Y
X′
X
O
Y′
2x – y = 0
x + y = 4

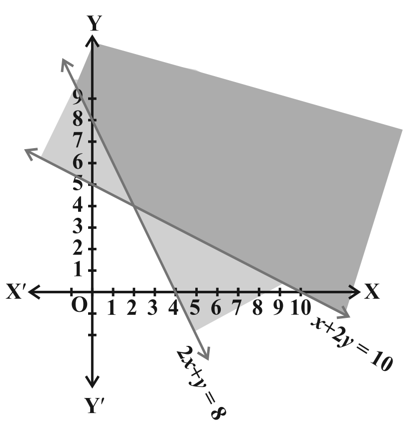
5.


Y
X′
X
O
Y′
x – 2y = –1
2x – y = 1

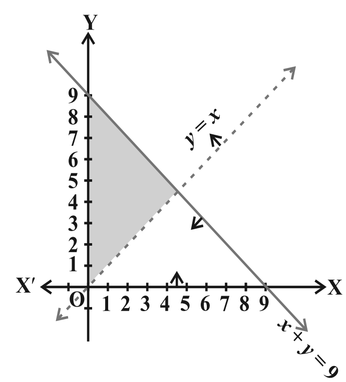
6.


Y
X′
X
O
Y′
x + y = 6
x + y = 4

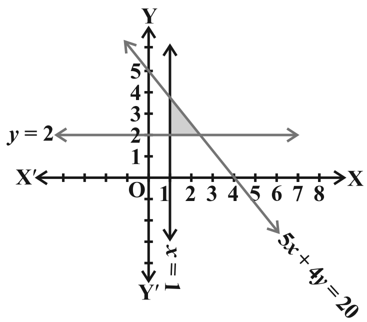
7.


Y
X′
X
O
Y′
x+2y = 10
2x+y = 8

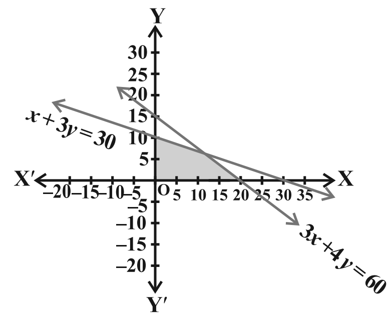
8.


Y
X′
X
O
y = x
x + y = 9

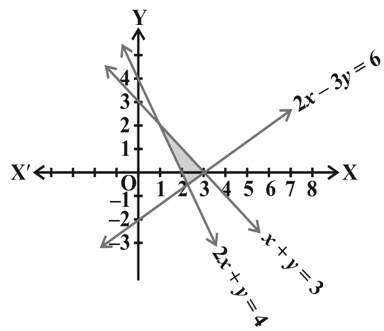
9.



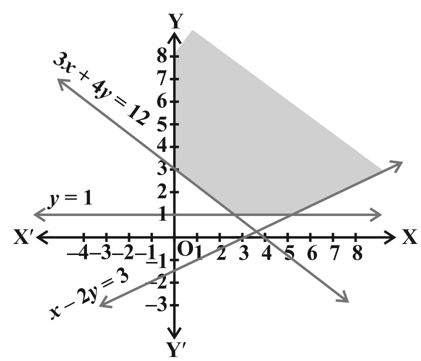
10.



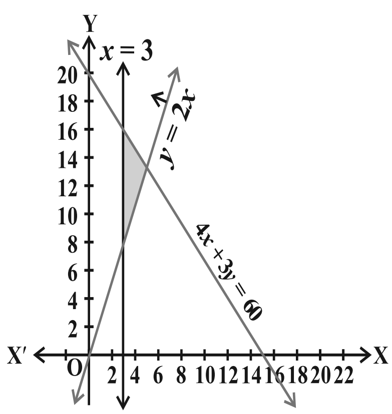
11.



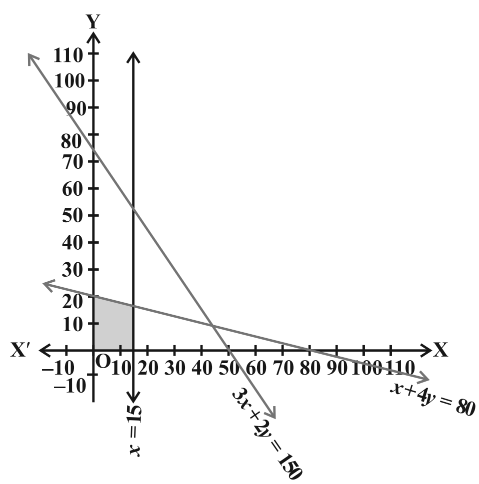
12.

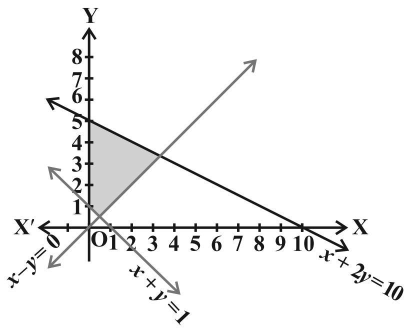


13.



14.

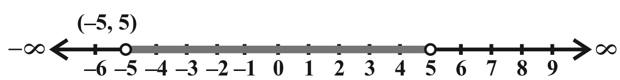


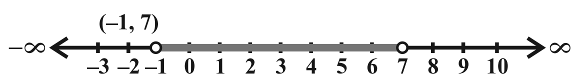
15. 

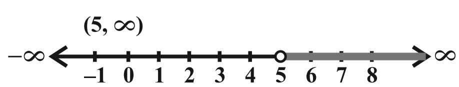
Miscellaneous Exercise on Chapter 6

1. [2, 3] 2. (0, 1] 3. [– 4, 2]

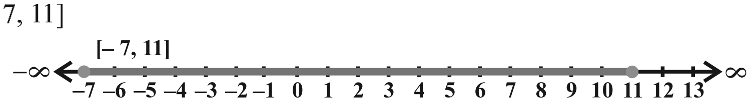
4. (– 23, 2) 5. $\left(\frac{-80}{3}, \frac{-10}{3}\right]$ 6. $\left[1, \frac{11}{3}\right]$

7. (–5, 5) 

8. (–1, 7) 

9. (5, ∞) 

10. [– 7, 11]



11. Between 20°C and 25°C
12. More than 320 litres but less than 1280 litres.
13. More than 562.5 litres but less than 900 litres.
14. Atleast 9.6 but more than 16.8.

EXERCISE 7.1

1. (i) 125, (ii) 60. 2. 108 3. 5040 4. 336
5. 8 6. 20

EXERCISE 7.2

1. (i) 40320, (ii) 18 2. 30, No 3. 28 4. 64
5. (i) 30, (ii) 15120

EXERCISE 7.3

1. 504 2. 4536 3. 60 4. 120, 48
5. 56 6. 9 7. (i) 3, (ii) 4 8. 40320
9. (i) 360, (ii) 720, (iii) 240 10. 33810
11. (i) 1814400, (ii) 2419200, (iii) 25401600

EXERCISE 7.4

1. 45 2. (i) 5, (ii) 6 3. 210 4. 40
5. 2000 6. 778320 7. 3960 8. 200
9. 35

Miscellaneous Exercise on Chapter 7

1. 3600 2. 1440 3. (i) 504, (ii) 588, (iii) 1632
4. 907200 5. 120 6. 50400 7. 420
8. ${}^{4}C_{1}\times{}^{48}C_{4}$ 9. 2880 10. ${}^{22}C_{7}+{}^{22}C_{10}$ 11. 151200

EXERCISE 8.1

1. $1-10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$
2. $\frac{32}{x^5}-\frac{40}{x^3}+\frac{20}{x}-5x+\frac{5}{8}x^3-\frac{x^5}{32}$
3. $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$
4. $\frac{x^5}{243}+\frac{5x^2}{81}+\frac{10}{27}x+\frac{10}{9x}+\frac{5}{3x^3}+\frac{1}{x^5}$
5. $x^6+6x^4+15x^2+20+\frac{15}{x^2}+\frac{6}{x^4}+\frac{1}{x^6}$
6. 884736 7. 11040808032 8. 104060401
9. 9509900499 10. $(1.1)^{10000} > 1000$ 11. $8(a^3b + ab^3)$; $40\sqrt{6}$
12. $2(x^6 + 15x^4 + 15x^2 + 1)$, 198

EXERCISE 8.2

1. 1512
2. – 101376
3. $(-1)^r\ {}^6C_r.x^{12-2r}.y^r$
4. $(-1)^r\ {}^{12}C_r.x^{24-r}.y^r$
5. – 1760 x^9y^3
6. 18564
7. $\frac{-105}{8}x^9;\ \frac{35}{48}x^{12}$
8. 61236 x^5y^5
10. $n = 7;\ r = 3$
12. $m = 4$

Miscellaneous Exercise on Chapter 8

1. $a = 3;\ b = 5;\ n = 6$
2. $n = 7, 14$
3. $a = \frac{9}{7}$
5. $396\sqrt{6}$
6. $2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$
7. 0.9510
8. $n = 10$
9. $\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5$
10. $27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

EXERCISE 9.1

1. 3, 8, 15, 24, 35
2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$
3. 2, 4, 8, 16 and 32
4. $-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$ and $\frac{7}{6}$
5. 25, –125, 625, –3125, 15625
6. $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}, 21$ and $\frac{75}{2}$
7. 65, 93
8. $\frac{49}{128}$
9. 729
10. $\frac{360}{23}$
11. 3, 11, 35, 107, 323; 3 + 11 + 35 + 107 + 323 + ...
12. $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120};\ -1+\left(\frac{-1}{2}\right)+\left(\frac{-1}{6}\right)+\left(\frac{-1}{24}\right)+\left(\frac{-1}{120}\right)+....$

13. 2, 2, 1, 0, –1; $2 + 2 + 1 + 0 + (-1) + ...$ **14.** $1, 2, \frac{3}{2}, \frac{5}{3}$ and $\frac{8}{5}$

EXERCISE 9.2

1. 1002001 **2.** 98450 **4.** 5 or 20 **6.** 4

7. $\frac{n}{2}(5n+7)$ **8.** $2q$ **9.** $\frac{179}{321}$ **10.** 0

13. 27 **14.** 11, 14, 17, 20 and 23 **15.** 1

16. 14 **17.** Rs 245 **18.** 9

EXERCISE 9.3

1. $\frac{5}{2^{20}}, \frac{5}{2^n}$ **2.** 3072 **4.** -2187

5. (a) 13th, (b) 12th, (c) 9th **6.** ± 1 **7.** $\frac{1}{6}\left[1-(0.1)^{20}\right]$

8. $\frac{\sqrt{7}}{2}\left(\sqrt{3}+1\right)\left(3^{\frac{n}{2}}-1\right)$ **9.** $\frac{\left[1-(-a)^n\right]}{1+a}$ **10.** $\frac{x^3\left(1-x^{2n}\right)}{1-x^2}$

11. $22+\frac{3}{2}\left(3^{11}-1\right)$ **12.** $r=\frac{5}{2}$ or $\frac{2}{5}$; Terms are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$

13. 4 **14.** $\frac{16}{7}; 2; \frac{16}{7}\left(2^n-1\right)$ **15.** 2059

16. $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, ...$ or $4, -8, 16, -32, 64, ..$ **18.** $\frac{80}{81}\left(10^n-1\right)-\frac{8}{9}n$

19. 496 **20.** rR **21.** 3, –6, 12, –24 **26.** 9 and 27

27. $n=\frac{-1}{2}$ **30.** 120, 480, 30 (2^n) **31.** Rs 500 $(1.1)^{10}$

32. $x^2 - 16x + 25 = 0$

EXERCISE 9.4

1. $\frac{n}{3}(n+1)(n+2)$ **2.** $\frac{n(n+1)(n+2)(n+3)}{4}$

3. $\frac{n}{6}(n+1)(3n^2+5n+1)$ 4. $\frac{n}{n+1}$ 5. 2840

6. $3n(n+1)(n+3)$ 7. $\frac{n(n+1)^2(n+2)}{12}$

8. $\frac{n(n+1)}{12}(3n^2+23n+34)$

9. $\frac{n}{6}(n+1)(2n+1)+2(2^n-1)$ 10. $\frac{n}{3}(2n+1)(2n-1)$

Miscellaneous Exercise on Chapter 9

2. 5, 8, 11 4. 8729 5. 3050 6. 1210
7. 4 8. 160; 6 9. ± 3 10. 8, 16, 32
11. 4 12. 11

21. (i) $\frac{50}{81}(10^n-1)-\frac{5n}{9}$, (ii) $\frac{2n}{3}-\frac{2}{27}(1-10^{-n})$ 22. 1680

23. $\frac{n}{3}(n^2+3n+5)$ 25. $\frac{n}{24}(2n^2+9n+13)$

27. Rs 16680 28. Rs 39100 29. Rs 43690 30. Rs 17000; 295000
31. Rs 5120 32. 25 days

EXERCISE 10.1

1. $\frac{121}{2}$ square unit.

2. (0, a), (0, – a) and $(-\sqrt{3}a, 0)$ or (0, a), (0, – a), and $(\sqrt{3}a, 0)$

3. (i) $|y_2-y_1|$, (ii) $|x_2-x_1|$ 4. $\left(\frac{15}{2}, 0\right)$ 5. $-\frac{1}{2}$

7. $-\sqrt{3}$ 8. $x = 1$ 10. 135°

11. 1 and 2, or $\frac{1}{2}$ and 1, or – 1 and –2, or $-\frac{1}{2}$ and – 1 14. $\frac{1}{2}$, 104.5 Crores

EXERCISE 10.2

1. $y = 0$ and $x = 0$ 2. $x - 2y + 10 = 0$ 3. $y = mx$

4. $\left(\sqrt{3}+1\right)x - \left(\sqrt{3}-1\right)y = 4\left(\sqrt{3}-1\right)$ 5. $2x + y + 6 = 0$

6. $x - \sqrt{3}y + 2\sqrt{3} = 0$ 7. $5x + 3y + 2 = 0$

8. $\sqrt{3}x + y = 10$ 9. $3x - 4y + 8 = 0$ 10. $5x - y + 20 = 0$

11. $(1 + n)x + 3(1 + n)y = n + 11$ 12. $x + y = 5$

13. $x + 2y - 6 = 0$, $2x + y - 6 = 0$

14. $\sqrt{3}x + y - 2 = 0$ and $\sqrt{3}x + y + 2 = 0$ 15. $2x - 9y + 85 = 0$

16. $L = \frac{.192}{90}\left(C - 20\right) + 124.942$ 17. 1340 litres. 19. $2kx + hy = 3kh.$

EXERCISE 10.3

1. (i) $y = -\frac{1}{7}x + 0, -\frac{1}{7}, 0$; (ii) $y = -2x + \frac{5}{3}, -2, \frac{5}{3}$; (iii) $y = 0x + 0, 0, 0$

2. (i) $\frac{x}{4} + \frac{y}{6} = 1, 4, 6;$ (ii) $\frac{x}{\frac{3}{2}} + \frac{y}{-2} = 1, \frac{3}{2}, -2;$

(iii) $y = -\frac{2}{3}$, intercept with y-axis $= -\frac{2}{3}$ and no intercept with x-axis.

3. (i) $x \cos 120° + y \sin 120° = 4$, 4, 120° (ii) $x \cos 90° + y \sin 90° = 2$, 2, 90°;

(iii) $x \cos 315° + y \sin 315° = 2\sqrt{2}$, $2\sqrt{2}$, 315° 4. 5 units

5. (– 2, 0) and (8, 0) 6. (i) $\frac{65}{17}$ units, (ii) $\frac{1}{\sqrt{2}}\left|\frac{p+r}{l}\right|$ units.

7. $3x - 4y + 18 = 0$ 8. $y + 7x = 21$ 9. 30° and 150°

10. $\frac{22}{9}$

12. $\left(\sqrt{3}+2\right)x + \left(2\sqrt{3}-1\right)y = 8\sqrt{3}+1$ or $\left(\sqrt{3}-2\right)x + \left(1+2\sqrt{3}\right)y = -1+8\sqrt{3}$

13. $2x + y = 5$ 14. $\left(\frac{68}{25}, -\frac{49}{25}\right)$ 15. $m = \frac{1}{2}, c = \frac{5}{2}$

17. $y - x = 1, \sqrt{2}$

Miscellaneous Exercise on Chapter 10

1. (a) 3, (b) ± 2, (c) 6 or 1 2. $\frac{7\pi}{6}, 1$

3. $2x - 3y = 6, -3x + 2y = 6$ 4. $\left(0, -\frac{8}{3}\right), \left(0, \frac{32}{3}\right)$

5. $\frac{|\sin(\phi - \theta)|}{2\left|\sin\frac{\phi - \theta}{2}\right|}$ 6. $x = -\frac{5}{22}$ 7. $2x - 3y + 18 = 0$

8. k^2 square units 9. 5 11. $3x - y = 7, \quad x + 3y = 9$

12. $13x + 13y = 6$ 14. 1 : 2 15. $\frac{23\sqrt{5}}{18}$ units

16. Slope of the line is zero i.e. line is parallel to x - axis

17. $x = 1, \quad y = 1.$ 18. $(-1, -4)$. 19. $\frac{1 \pm 5\sqrt{2}}{7}$

21. $18x + 12y + 11 = 0$ 22. $\left(\frac{13}{5}, 0\right)$ 24. $119x + 102y = 205$

EXERCISE 11.1

1. $x^2 + y^2 - 4y = 0$ 2. $x^2 + y^2 + 4x - 6y - 3 = 0$
3. $36x^2 + 36y^2 - 36x - 18y + 11 = 0$ 4. $x^2 + y^2 - 2x - 2y = 0$
5. $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$ 6. $c(-5, 3), r = 6$

7. $c(2, 4), r = \sqrt{65}$ 8. $c(4, -5), r = \sqrt{53}$ 9. $c\left(\frac{1}{4}, 0\right); r = \frac{1}{4}$

10. $x^2 + y^2 - 6x - 8y + 15 = 0$ 11. $x^2 + y^2 - 7x + 5y - 14 = 0$

12. $x^2 + y^2 + 4x - 21 = 0$ & $x^2 + y^2 - 12x + 11 = 0$

13. $x^2 + y^2 - ax - by = 0$ **14.** $x^2 + y^2 - 4x - 4y = 5$

15. Inside the circle; since the distance of the point to the centre of the circle is less than the radius of the circle.

EXERCISE 11.2

1. F (3, 0), axis - x - axis, directrix $x = -3$, length of the Latus rectum = 12
2. F $(0, \frac{3}{2})$, axis - y - axis, directrix $y = -\frac{3}{2}$, length of the Latus rectum = 6
3. F (–2, 0), axis - x - axis, directrix $x = 2$, length of the Latus rectum = 8
4. F (0, –4), axis - y - axis, directrix $y = 4$, length of the Latus rectum = 16
5. F $(\frac{5}{2}, 0)$ axis - x - axis, directrix $x = -\frac{5}{2}$, length of the Latus rectum = 10
6. F $(0, \frac{-9}{4})$, axis - y - axis, directrix $y = \frac{9}{4}$, length of the Latus rectum = 9
7. $y^2 = 24x$ 8. $x^2 = -12y$ 9. $y^2 = 12x$
10. $y^2 = -8x$ 11. $2y^2 = 9x$ 12. $2x^2 = 25y$

EXERCISE 11.3

1. F $(\pm\sqrt{20}, 0)$; V $(\pm 6, 0)$; Major axis = 12; Minor axis = 8, $e = \frac{\sqrt{20}}{6}$,

   Latus rectum = $\frac{16}{3}$

2. F $(0, \pm\sqrt{21})$; V $(0, \pm 5)$; Major axis = 10; Minor axis = 4, $e = \frac{\sqrt{21}}{5}$;

   Latus rectum = $\frac{8}{5}$

3. F $(\pm\sqrt{7}, 0)$; V $(\pm 4, 0)$; Major axis = 8; Minor axis = 6, $e = \frac{\sqrt{7}}{4}$;

   Latus rectum = $\frac{9}{2}$

4. F $(0, \pm\sqrt{75})$; V $(0, \pm 10)$; Major axis = 20; Minor axis = 10 , $e = \frac{\sqrt{3}}{2}$;
Latus rectum = 5

5. F $(\pm\sqrt{13}, 0)$; V $(\pm 7, 0)$; Major axis =14 ; Minor axis = 12 , $e = \frac{\sqrt{13}}{7}$;
Latus rectum = $\frac{72}{7}$

6. F $(0, \pm 10\sqrt{3})$; V $(0, \pm 20)$; Major axis =40 ; Minor axis = 20 , $e = \frac{\sqrt{3}}{2}$;
Latus rectum = 10

7. F $(0, \pm 4\sqrt{2})$; V $(0, \pm 6)$; Major axis =12 ; Minor axis = 4 , $e = \frac{2\sqrt{2}}{3}$;
Latus rectum $=\frac{4}{3}$

8. $F\left(0, \pm\sqrt{15}\right)$; V $(0, \pm 4)$; Major axis = 8 ; Minor axis = 2 , $e = \frac{\sqrt{15}}{4}$;
Latus rectum $=\frac{1}{2}$

9. F $(\pm\sqrt{5}, 0)$; V $(\pm 3, 0)$; Major axis = 6 ; Minor axis = 4 , $e = \frac{\sqrt{5}}{3}$;
Latus rectum $=\frac{8}{3}$

10. $\frac{x^2}{25}+\frac{y^2}{9}=1$ 11. $\frac{x^2}{144}+\frac{y^2}{169}=1$ 12. $\frac{x^2}{36}+\frac{y^2}{20}=1$

13. $\frac{x^2}{9}+\frac{y^2}{4}=1$ 14. $\frac{x^2}{1}+\frac{y^2}{5}=1$ 15. $\frac{x^2}{169}+\frac{y^2}{144}=1$

16. $\frac{x^2}{64}+\frac{y^2}{100}=1$ 17. $\frac{x^2}{16}+\frac{y^2}{7}=1$ 18. $\frac{x^2}{25}+\frac{y^2}{9}=1$

19. $\frac{x^2}{10}+\frac{y^2}{40}=1$

20. $x^2 + 4y^2 = 52$ or $\frac{x^2}{52}+\frac{y^2}{13}=1$

EXERCISE 11.4

1. Foci ($\pm$ 5, 0), Vertices ($\pm$ 4, 0); $e = \frac{5}{4}$; Latus rectum $= \frac{9}{2}$
2. Foci (0 $\pm$ 6), Vertices (0, $\pm$ 3); $e = 2$; Latus rectum = 18
3. Foci (0, $\pm\sqrt{13}$), Vertices (0, $\pm$ 2); $e = \frac{\sqrt{13}}{2}$; Latus rectum $= 9$
4. Foci ($\pm$ 10, 0), Vertices ($\pm$ 6, 0); $e = \frac{5}{3}$; Latus rectum $= \frac{64}{3}$
5. Foci $\left(0, \pm\frac{2\sqrt{14}}{\sqrt{5}}\right)$, Vertices $\left(0, \pm\frac{6}{\sqrt{5}}\right)$; $e = \frac{\sqrt{14}}{3}$; Latus rectum $= \frac{4\sqrt{5}}{3}$
6. Foci (0, $\pm\sqrt{65}$), Vertices (0, $\pm$ 4); $e = \frac{\sqrt{65}}{4}$; Latus rectum $= \frac{49}{2}$
7. $\frac{x^2}{4}-\frac{y^2}{5}=1$
8. $\frac{y^2}{25}-\frac{x^2}{39}=1$
9. $\frac{y^2}{9}-\frac{x^2}{16}=1$
10. $\frac{x^2}{16}-\frac{y^2}{9}=1$
11. $\frac{y^2}{25}-\frac{x^2}{144}=1$
12. $\frac{x^2}{25}-\frac{y^2}{20}=1$
13. $\frac{x^2}{4}-\frac{y^2}{12}=1$
14. $\frac{x^2}{49}-\frac{9y^2}{343}=1$
15. $\frac{x^2}{5}-\frac{y^2}{5}=1$

Miscellaneous Exercise on Chapter 11

1. Focus is at the mid-point of the given diameter.
2. 2.23 m (approx.)
3. 9.11 m (approx.)
4. 1.56m (approx.)
5. $\frac{x^2}{81}+\frac{y^2}{9}=1$
6. 18 sq units
7. $\frac{x^2}{25}+\frac{y^2}{9}=1$
8. $8\sqrt{3}a$

EXERCISE 12.1

1. y and z - coordinate are zero
2. y - coordinate is zero
3. I, IV, VIII, V, VI, II, III, VII
4. (i) XY - plane (ii) $(x, y, 0)$ (iii) Eight regions

EXERCISE 12.2

1. (i) $2\sqrt{5}$ (ii) $\sqrt{43}$ (iii) $2\sqrt{26}$ (iv) $2\sqrt{5}$
4. $x - 2z = 0$ **5.** $9x^2 + 25y^2 + 25z^2 - 225 = 0$

EXERCISE 12.3

1. (i) $\left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right)$, (ii) $(-8, 17, 3)$
2. 1 : 2
3. 2 : 3
5. (6, – 4, – 2), (8, – 10, 2)

Miscellaneous Exercise on Chapter 12

1. (1, – 2, 8)
2. $7\sqrt{34}, 7$
3. $a = -2,\ b = -\frac{16}{3},\ c = 2$
4. (0, 2, 0) and (0, – 6, 0)
5. (4, – 2, 6)
6. $x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}$

EXERCISE 13.1

1. 6
2. $\left(\pi - \frac{22}{7}\right)$
3. π
4. $\frac{19}{2}$
5. $-\frac{1}{2}$
6. 5
7. $\frac{11}{4}$
8. $\frac{108}{7}$
9. b
10. 2
11. 1
12. $-\frac{1}{4}$
13. $\frac{a}{b}$
14. $\frac{a}{b}$
15. $\frac{1}{\pi}$
16. $\frac{1}{\pi}$

17. 4 **18.** $\dfrac{a+1}{b}$ **19.** 0 **20.** 1

21. 0 **22.** 2 **23.** 3, 6

24. Limit does not exist at $x = 1$

25. Limit does not exist at $x = 0$ **26.** Limit does not exist at $x = 0$

27. 0 **28.** $a=0, b=4$

29. $\lim\limits_{x\to a_1} f(x) = 0$ and $\lim\limits_{x\to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_x)$

30. $\lim\limits_{x\to a} f(x)$ exists for all $a \neq 0$. **31.** 2

32. For $\lim\limits_{x\to 0} f(x)$ to exists, we need $m = n$; $\lim\limits_{x\to 1} f(x)$ exists for any integral value of m and n.

EXERCISE 13.2

1. 20 **2.** 99 **3.** 1

4. (i) $3x^2$ (ii) $2x - 3$ (iii) $\dfrac{-2}{x^3}$ (iv) $\dfrac{-2}{(x-1)^2}$

6. $nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1}$

7. (i) $2x - a - b$ (ii) $4ax\left(ax^2 + b\right)$ (iii) $\dfrac{a-b}{(x-b)^2}$

8. $\dfrac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$

9. (i) 2 (ii) $20x^3 - 15x^2 + 6x - 4$ (iii) $\dfrac{-3}{x^4}(5+2x)$ (iv) $15x^4 + \dfrac{24}{x^5}$

(v) $\dfrac{-12}{x^5} + \dfrac{36}{x^{10}}$ (vi) $\dfrac{-2}{(x+1)^2} - \dfrac{x(3x-2)}{(3x-1)^2}$ **10.** $-\sin x$

11. (i) $\cos 2x$ (ii) $\sec x \tan x$

(iii) $5\sec x \tan x - 4\sin x$ (iv) $-\operatorname{cosec} x \cot x$

(v) $-3\operatorname{cosec}^2 x - 5 \operatorname{cosec} x \cot x$ (vi) $5\cos x + 6\sin x$

(vii) $2\sec^2 x - 7\sec x \tan x$

Miscellaneous Exercise on Chapter 13

1. (i) – 1 (ii) $\frac{1}{x^2}$ (iii) cos $(x + 1)$ (vi) $-\sin\left(x-\frac{\pi}{8}\right)$ **2.** 1

3. $\frac{-qr}{x^2}+ps$ **4.** $2c\ (ax+b)\ (cx + d) + a\ (cx + d)^2$

5. $\frac{ad-bc}{(cx+d)^2}$ **6.** $\frac{-2}{(x-1)^2},\ x\neq 0,1$ **7.** $\frac{-(2ax+b)}{(ax^2+bx+c)^2}$

8. $\frac{-apx^2-2bpx+ar-bq}{(px^2+2x+r)^2}$ **9.** $\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$ **10.** $\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$

11. $\frac{2}{\sqrt{x}}$ **12.** $na(ax+b)^{n-1}$

13. $(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]$ **14.** $\cos\ (x+a)$

15. $-\ \text{cosec}^3\ x - \text{cosec}\ x\ \cot^2 x$ **16.** $\frac{-1}{1+\sin x}$

17. $\frac{-2}{(\sin x-\cos x)^2}$ **18.** $\frac{2\sec x\tan x}{(\sec x+1)^2}$ **19.** $n\ \sin^{n-1}x\quad \cos x$

20. $\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$ **21.** $\frac{\cos\alpha}{\cos^2 x}$

22. $x^3(5x\cos x+3x\sin x+20\sin x-12\cos x)$

23. $-x^2\sin x-\sin x+2x\cos x$

24. $-q\ sin\,x\left(ax^2+\sin x\right)+\left(p+q\cos x\right)\left(2a\,x+\cos x\right)$

25. $-\tan^2 x\left(x+\cos x\right)+\left(x-\tan x\right)\left(1-\sin x\right)$

26. $\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$

27. $\dfrac{x\cos\frac{\pi}{4}\left(2\sin x - x\cos x\right)}{\sin^2 x}$ **28.** $\dfrac{1+\tan x - x\sec^2 x}{\left(1+\tan x\right)^2}$

29. $\left(x+\sec x\right)\left(1-\sec^2 x\right)+\left(x-\tan x\right).\left(1+\sec x\tan x\right)$

30. $\dfrac{\sin x - n\,x\cos x}{\sin^{n+1} x}$

EXERCISE 14.1

1. (i) This sentence is always false because the maximum number of days in a month is 31. Therefore, it is a statement.

(ii) This is not a statement because for some people mathematics can be easy and for some others it can be difficult.

(iii) This sentence is always true because the sum is 12 and it is greater than 10. Therefore, it is a statement.

(iv) This sentence is sometimes true and sometimes not true. For example the square of 2 is even number and the square of 3 is an odd number. Therefore, it is not a statement.

(v) This sentence is sometimes true and sometimes false. For example, squares and rhombus have equal length whereas rectangles and trapezium have unequal length. Therefore, it is not a statement.

(vi) It is an order and therefore, is not a statement.

(vii) This sentence is false as the product is (–8). Therefore, it is a statement.

(viii) This sentence is always true and therefore, it is a statement.

(ix) It is not clear from the context which day is referred and therefore, it is not a statement.

(x) This is a true statement because all real numbers can be written in the form $a + i \times 0$.

2. The three examples can be:

(i) Everyone in this room is bold. This is not a statement because from the context it is not clear which room is reffered here and the term bold is not precisely defined.

(ii) She is an engineering student. This is also not a statement because who ‘she’ is.

(iii) “$\cos^2\theta$ is always greater than 1/2”. Unless, we know what θ is, we cannot say whether the sentence is true or not.

EXERCISES 14.2

1. (i) Chennai is not the capital of Tamil Nadu.
   (ii) $\sqrt{2}$ is a complex number.
   (iii) All triangles are equilateral tringles.
   (iv) The number 2 is not greater than 7.
   (v) Every natural number is not an integer.
2. (i) The negation of the first statement is "the number x is a rational number." which is the same as the second statement" This is because when a number is not irrational, it is a rational. Therefore, the given pairs are negations of each other.
   (ii) The negation of the first statement is "x is an irrational number" which is the same as the second statement. Therefore, the pairs are negations of each other.
3. (i) Number 3 is prime; number 3 is odd (True).
   (ii) All integers are positive; all integers are negative (False).
   (iii) 100 is divisible by 3,100 is divisible by 11 and 100 is divisible by 5 (False).

EXERCISE 14.3

1. (i) "And". The component statements are:
   All rational numbers are real.
   All real numbers are not complex.
   (ii) "Or". The component statements are:
   Square of an integer is positive.
   Square of an integer is negative.
   (iii) "And". the component statements are:
   The sand heats up quickily in the sun.
   The sand does not cool down fast at night.
   (iv) "And". The component statements are:
   $x = 2$ is a root of the equation $3x^2 - x - 10 = 0$
   $x = 3$ is a root of the equation $3x^2 - x - 10 = 0$
2. (i) "There exists". The negation is
   There does not exist a number which is equal to its square.
   (ii) "For every". The negation is
   There exists a real number x such that x is not less than $x + 1$.
   (iii) "There exists". The negation is
   There exists a state in India which does not have a capital.

3. No. The negation of the statement in (i) is "There exists real number x and y for which $x + y \neq y + x$", instead of the statement given in (ii).

4. (i) Exclusive
 (ii) Inclusive
 (iii) Exclusive

EXERCISE 14.4

1. (i) A natural number is odd implies that its square is odd.
 (ii) A natural number is odd only if its square is odd.
 (iii) For a natural number to be odd it is necessary that its square is odd.
 (iv) For the square of a natural number to be odd, it is sufficient that the number is odd
 (v) If the square of a natural number is not odd, then the natural number is not odd.

2. (i) The contrapositive is
 If a number x is not odd, then x is not a prime number.
 The converse is
 If a number x in odd, then it is a prime number.
 (ii) The contrapositive is
 If two lines intersect in the same plane, then they are not parallel
 The converse is
 If two lines do not interesect in the same plane, then they are parallel
 (iii) The contrapositive is
 If something is not at low temperature, then it is not cold
 The converse is
 If something is at low temperature, then it is cold
 (iv) The contrapositive is
 If you know how to reason deductively, then you can comprehend geometry.
 The converse is
 If you do not know how to reason deductively, then you can not comprehend geometry.
 (v) This statement can be written as "If x is an even number, then x is divisible by 4".
 The contrapositive is, If x is not divisible by 4, then x is not an even number.
 The converse is, If x is divisible by 4, then x is an even number.

3. (i) If you get a job, then your credentials are good.
 (ii) If the banana tree stays warm for a month, then it will bloom.

(iii) If diagonals of a quadrilateral bisect each other, then it is a parallelogram.
(iv) If you get A^+ in the class, then you do all the exercises in the book.

4. **a** (i) Contrapositive
(ii) Converse
b (i) Contrapositive
(ii) Converse

EXERCISE 14.5

5. (i) False. By definition of the chord, it should intersect the circle in two points.
(ii) False. This can be shown by giving a counter example. A chord which is not a dimaeter gives the counter example.
(iii) True. In the equation of an ellipse if we put $a = b$, then it is a circle (Direct Method)
(iv) True, by the rule of inequality
(v) False. Since 11 is a prime number, therefore $\sqrt{11}$ is irrational.

Miscellaneous Exercise on Chapter 14

1. (i) There exists a positive real number x such that $x–1$ is not positive.
(ii) There exists a cat which does not scratch.
(iii) There exists a real number x such that neither $x > 1$ nor $x < 1$.
(iv) There does not exist a number x such that $0 < x < 1$.

2. (i) The statement can be written as "If a positive integer is prime, then it has no divisors other than 1 and itself.
The converse of the statement is
If a positive integer has no divisors other than 1 and itself, then it is a prime.
The contrapositive of the statement is
If positive integer has divisors other than 1 and itself then it is not prime.
(ii) The given statement can be written as "If it is a sunny day, then I go to a beach.
The converse of the statement is
If I go to beach, then it is a sunny day.
The contrapositive is
If I do not go to a beach, then it is not a sunny day.
(iii) The converse is
If you feel thirsty, then it is hot outside.
The contrapositive is
If you do not feel thirsty, then it is not hot outside.

3. (i) If there is log on to the server, then you have a password.
   (ii) If it rains, then there is traffic jam.
   (iii) If you can access the website, then you pay a subscription fee.
4. (i) You watch television if and only if your mind in free.
   (ii) You get an A grade if and only you do all the homework regularly.
   (iii) A quadrilateral is equiangular if and only if it is a rectangle.
5. The compound statement with "And" is 25 is a multiple of 5 and 8
   This is a false statement.
   The compound statement with "Or" is 25 is a multiple of 5 or 8
   This is true statement.
7. Same as Q1 in Exercise 14.4

EXERCISE 15.1

1. 3	2. 8.4	3. 2.33	4. 7
5. 6.32	6. 16	7. 3.23	8. 5.1
9. 157.92	10. 11.28	11. 10.34	12. 7.35

EXERCISE 15.2

1. 9, 9.25	2. $\frac{n+1}{2}, \frac{n^2-1}{12}$	3. 16.5, 74.25	4. 19, 43.4
5. 100, 29.09	6. 64, 1.69	7. 107, 2276	8. 27, 132
9. 93, 105.52, 10.27		10. 5.55, 43.5	

EXERCISE 15.3

1. B	2. Y	3. (i) B, (ii) B
4. A	5. Weight	

Miscellaneous Exercise on Chapter 15

1. 4, 8   2. 6, 8   3. 24, 12
5. (i) 10.1, 1.99 (ii) 10.2, 1.98
6. Highest Chemistry and lowest Mathematics   7. 20, 3.036

EXERCISE 16.1

1. {HHH, HHT, HTH, THH, TTH, HTT, THT, TTT}
2. $\{(x, y) : x, y = 1,2,3,4,5,6\}$

or {(1,1), (1,2), (1,3), ..., (1,6), (2,1), (2,2), ..., (2,6), ..., (6, 1), (6, 2), ..., (6,6)}

3. {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}
4. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}
5. {H1, H2, H3, H4, H5, H6, T}
6. $\{XB_1, XB_2, XG_1, XG_2, YB_3, YG_3, YG_4, YG_5\}$
7. {R1, R2, R3, R4, R5, R6, W1, W2, W3, W4, W5, W6, B1, B2, B3, B4, B5, B6}
8. (i) {BB, BG, GB, GG} (ii) {0, 1, 2}
9. {RW, WR, WW}
10. [HH, HT, T1, T2, T3, T4, T5, T6}
11. {DDD, DDN, DND, NDD, DNN, NDN, NND, NNN}
12. {T, H1, H3, H5, H21, H22, H23, H24, H25, H26, H41, H42, H43, H44, H45, H46, H61, H62, H63, H64, H65, H66}
13. {(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)}
14. {1HH, 1HT, 1TH, 1TT, 2H, 2T, 3HH, 3HT, 3TH, 3TT, 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T}
15. $\{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5, H6\}$
16. {6, (1,6), (2,6), (3,6), (4,6), (5,6), (1,1,6), (1,2,6), ..., (1,5,6), (2,1,6). (2,2,6), ..., (2,5,6), ..., (5,1,6), (5,2,6), ... }

EXERCISE 16.2

1. No.
2. (i) {1, 2, 3, 4, 5, 6} (ii) ϕ (iii) {3, 6} (iv) {1, 2, 3} (v) {6}

(vi) {3, 4, 5, 6}, $A\cup B = \{1, 2, 3, 4, 5, 6\}$, $A\cap B = \phi$, $B\cup C = \{3, 6\}$, $E\cap F = \{6\}$, $D\cap E = \phi$,

$A - C = \{1, 2,4,5\}$, $D - E = \{1,2,3\}$, $E\cap F' = \phi$, $F' = \{1, 2\}$

3. A = {(3,6), (4,5), (5, 4), (6,3), (4,6), (5,5), (6,4), (5,6), (6,5), (6,6)}
B = {(1,2), (2,2), (3, 2), (4,2), (5,2), (6,2), (2,1), (2,3), (2,4), (2,5), (2,6)}
C ={(3,6), (6,3), (5, 4), (4,5), (6,6)}
A and B, B and C are mutually exclusive.
4. (i) A and B; A and C; B and C; C and D (ii) A and C (iii) B and D
5. (i) "Getting at least two heads", and "getting at least two tails"
(ii) "Getting no heads", "getting exactly one head" and "getting at least two heads"

(iii) "Getting at most two tails", and "getting exactly two tails"
(iv) "Getting exactly one head" and "getting exactly two heads"
(v) "Getting exactly one tail", "getting exactly two tails", and getting exactly three tails"

Note There may be other events also as answer to the above question.

6. A = {(2, 1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

B = {(1, 1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)}

C = {(1, 1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)}

(i) A′ = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)} = B

(ii) B′ = {(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} = A

(iii) A∪B = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (2,1), (2,2), (2,3), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} = S

(iv) $A \cap B = \phi$

(v) A – C = {(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

(vi) B ∪ C = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)}

(vii) B ∩ C = {(1,1), (1,2), (1,3), (1,4), (3,1), (3,2)}

(viii) A ∩ B′ ∩ C′ = {(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

7. (i) True (ii) True (iii) True (iv) False (v) False (vi) False

EXERCISE 16.3

1. (a) Yes (b) Yes (c) No (d) No (e) No **2.** $\frac{3}{4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{6}$ (iv) 0 (v) $\frac{5}{6}$ **4.** (a) 52 (b) $\frac{1}{52}$ (c) (i) $\frac{1}{13}$ (ii) $\frac{1}{2}$

5. (i) $\frac{1}{12}$ (ii) $\frac{1}{12}$ **6.** $\frac{3}{5}$

7. Rs 4.00 gain, Rs 1.50 gain, Re 1.00 loss, Rs 3.50 loss, Rs 6.00 loss.

P (Winning Rs 4.00) $=\frac{1}{16}$, P(Winning Rs 1.50) $=\frac{1}{4}$, P (Losing Re. 1.00) $=\frac{3}{8}$

P (Losing Rs 3.50) $=\frac{1}{4}$, P (Losing Rs 6.00) $=\frac{1}{16}$.

8. (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$ (iv) $\frac{7}{8}$ (v) $\frac{1}{8}$ (vi) $\frac{1}{8}$ (vii) $\frac{3}{8}$ (viii) $\frac{1}{8}$ (ix) $\frac{7}{8}$

9. $\frac{9}{11}$ 10. (i) $\frac{6}{13}$ (ii) $\frac{7}{13}$ 11. $\frac{1}{38760}$

12. (i) No, because P($A \cap B$) must be less than or equal to P(A) and P(B), (ii) Yes

13. (i) $\frac{7}{15}$ (ii) 0.5 (iii) 0.15 14. $\frac{4}{5}$

15. (i) $\frac{5}{8}$ (ii) $\frac{3}{8}$ 16. No 17. (i) 0.58 (ii) 0.52 (iii) 0.74

18. 0.6 19. 0.55 20. 0.65

21. (i) $\frac{19}{30}$ (ii) $\frac{11}{30}$ (iii) $\frac{2}{15}$

Miscellaneous Exercise on Chapter 16

1. (i) $\frac{^{20}C_5}{^{60}C_5}$ (ii) $1-\frac{^{30}C_5}{^{60}C_5}$ 2. $\frac{^{13}C_3 \cdot {}^{13}C_1}{^{52}C_4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{5}{6}$ 4. (a) $\frac{999}{1000}$ (b) $\frac{^{9990}C_2}{^{10000}C_2}$ (c) $\frac{^{9990}C_{10}}{^{10000}C_{10}}$

5. (a) $\frac{17}{33}$ (b) $\frac{16}{33}$ 6. $\frac{2}{3}$

7. (i) 0.88 (ii) 0.12 (iii) 0.19 (iv) 0.34 8. $\frac{4}{5}$

9. (i) $\frac{2}{5}$ (ii) $\frac{3}{8}$ 10. $\frac{1}{5040}$

Appendix 1

INFINITE SERIES

A.1.1 Introduction

As discussed in the Chapter 9 on Sequences and Series, a sequence $a_1, a_2, ..., a_n, ...$ having infinite number of terms is called *infinite sequence* and its indicated sum, i.e., $a_1 + a_2 + a_3 + ... + a_n + ...$ is called an *infinte series* associated with infinite sequence. This series can also be expressed in abbreviated form using the sigma notation, i.e.,

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$$

In this Chapter, we shall study about some special types of series which may be required in different problem situations.

A.1.2 Binomial Theorem for any Index

In Chapter 8, we discussed the Binomial Theorem in which the index was a positive integer. In this Section, we state a more general form of the theorem in which the index is not necessarily a whole number. It gives us a particular type of infinite series, called *Binomial Series*. We illustrate few applications, by examples.

We know the formula

$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + \ldots + {}^nC_n\, x^n$$

Here, n is non-negative integer. Observe that if we replace index n by negative integer or a fraction, then the combinations ${}^nC_r$ do not make any sense.

We now state (without proof), the Binomial Theorem, giving an infinite series in which the index is negative or a fraction and not a whole number.

Theorem The formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + ...$$

holds whenever $|x| < 1$.

Remark 1. Note carefully the condition $|x| < 1$, i.e., $-1 < x < 1$ is necessary when m is negative integer or a fraction. For example, if we take $x = -2$ and $m = -2$, we obtain

$$(1-2)^{-2} = 1 + (-2)(-2) + \frac{(-2)(-3)}{1.2}(-2)^2 + ...$$

or $\qquad 1 = 1 + 4 + 12 + \dots$

This is not possible

2. Note that there are infinite number of terms in the expansion of $(1+x)^m$, when m is a negative integer or a fraction

Consider
$$(a+b)^m = \left[a\left(1+\frac{b}{a}\right)\right]^m = a^m\left(1+\frac{b}{a}\right)^m$$

$$= a^m\left[1 + m\frac{b}{a} + \frac{m(m-1)}{1.2}\left(\frac{b}{a}\right)^2 + ...\right]$$

$$= a^m + ma^{m-1}b + \frac{m(m-1)}{1.2}a^{m-2}b^2 + ...$$

This expansion is valid when $\left|\frac{b}{a}\right| < 1$ or equivalently when $|b| < |a|$.

The general term in the expansion of $(a + b)^m$ is

$$\frac{m(m-1)(m-2)...(m-r+1)a^{m-r}b^r}{1.2.3...r}$$

We give below certain particular cases of Binomial Theorem, when we assume $|x| < 1$, these are left to students as exercises:

1. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$
2. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$
3. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
4. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Example 1 Expand $\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$, when $|x| < 2$.

Solution We have

$$\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = 1+\frac{\left(-\frac{1}{2}\right)}{1}\left(\frac{-x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2}\left(-\frac{x}{2}\right)^2+...$$

$$= 1+\frac{x}{4}+\frac{3x^2}{32}+...$$

A.1.3 Infinite Geometric Series

From Chapter 9, Section 9.3, a sequence a_1, a_2, a_3, ..., a_n is called G.P., if $\frac{a_{k+1}}{a_k} = r$ (constant) for $k = 1, 2, 3, \ldots, n-1$. Particularly, if we take $a_1 = a$, then the resulting sequence $a, ar, ar^2, \ldots, ar^{n-1}$ is taken as the standard form of G.P., where a is first term and r, the common ratio of G.P.

Earlier, we have discussed the formula to find the sum of finite series $a + ar + ar^2 + \ldots + ar^{n-1}$ which is given by

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

In this section, we state the formula to find the sum of infinite geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ and illustrate the same by examples.

Let us consider the G.P. $1, \frac{2}{3}, \frac{4}{9}, \ldots$

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right] \qquad \ldots (1)$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger.

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes sufficiently small. In other words, as $n \to \infty, \left(\frac{2}{3}\right)^n \to 0$. Consequently, we find that the sum of infinitely many terms is given by S = 3.

Thus, for infinite geometric progression $a, ar, ar^2, ...$, if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case, $r^n \to 0$ as $n \to \infty$ since $|r|<1$ and then $\frac{ar^n}{1-r} \to 0$. Therefore,

$$S_n \to \frac{a}{1-r} \text{ as } n \to \infty.$$

Symbolically, sum to infinity of infinite geometric series is denoted by S. Thus, we have $\mathrm{S} = \frac{a}{1-r}$

For example

(i) $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = \frac{1}{1-\frac{1}{2}} = 2$

(ii) $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + ... = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

Example 2 Find the sum to infinity of the G.P. ;

$$\frac{-5}{4}, \frac{5}{16}, \frac{-5}{64}, ...$$

Solution Here $a = \frac{-5}{4}$ and $r = -\frac{1}{4}$. Also $|r| < 1$.

Hence, the sum to infinity is $\dfrac{\frac{-5}{4}}{1+\frac{1}{4}} = \dfrac{\frac{-5}{4}}{\frac{5}{4}} = -1$.

A.1.4 Exponential Series

Leonhard Euler (1707 – 1783), the great Swiss mathematician introduced the number e in his calculus text in 1748. The number e is useful in calculus as π in the study of the circle.

Consider the following infinite series of numbers

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... \qquad ... (1)$$

The sum of the series given in (1) is denoted by the number e

Let us estimate the value of the number e.

Since every term of the series (1) is positive, it is clear that its sum is also positive.

Consider the two sums

$$\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ... + \frac{1}{n!} + ... \qquad ... (2)$$

and $$\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ... + \frac{1}{2^{n-1}} + ... \qquad ... (3)$$

Observe that

$$\frac{1}{3!} = \frac{1}{6} \text{ and } \frac{1}{2^2} = \frac{1}{4}, \text{ which gives } \frac{1}{3!} < \frac{1}{2^2}$$

$$\frac{1}{4!} = \frac{1}{24} \text{ and } \frac{1}{2^3} = \frac{1}{8}, \text{ which gives } \frac{1}{4!} < \frac{1}{2^3}$$

$$\frac{1}{5!} = \frac{1}{120} \text{ and } \frac{1}{2^4} = \frac{1}{16}, \text{ which gives } \frac{1}{5!} < \frac{1}{2^4}.$$

Therefore, by analogy, we can say that

$$\frac{1}{n!} < \frac{1}{2^{n-1}}, \text{ when } n > 2$$

We observe that each term in (2) is less than the corresponding term in (3),

Therefore $$\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...+\frac{1}{n!}\right) < \left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right) \qquad ... (4)$$

Adding $\left(1+\frac{1}{1!}+\frac{1}{2!}\right)$ on both sides of (4), we get,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...+\frac{1}{n!}+...\right)$$

$$< \left\{\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right)\right\} \qquad ... (5)$$

$$= \left\{1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right)\right\}$$

$$= 1+\frac{1}{1-\frac{1}{2}} = 1+2 \;= 3$$

Left hand side of (5) represents the series (1). Therefore $e < 3$ and also $e > 2$ and hence $2 < e < 3$.

Remark The exponential series involving variable x can be expressed as

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Example 3 Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x.

Solution In the exponential series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...$$

replacing x by $(2x + 3)$, we get

$$e^{2x+3} = 1+\frac{(2x+3)}{1!}+\frac{(2x+3)^2}{2!}+...$$

Here, the general term is $\frac{(2x+3)^n}{n!} = \frac{(3+2x)^n}{n!}$. This can be expanded by the Binomial Theorem as

$$\frac{1}{n!}\left[3^n + {}^nC_1 3^{n-1}(2x) + {}^nC_2 3^{n-2}(2x)^2 + ... + (2x)^n\right].$$

Here, the coefficient of x^2 is $\frac{{}^nC_2 3^{n-2} 2^2}{n!}$. Therefore, the coefficient of x^2 in the whole series is

$$\sum_{n=2}^{\infty} \frac{{}^nC_2 3^{n-2} 2^2}{n!} = 2\sum_{n=2}^{\infty}\frac{n(n-1)3^{n-2}}{n!}$$

$$= 2\sum_{n=2}^{\infty}\frac{3^{n-2}}{(n-2)!} \quad \text{[using } n! = n\,(n-1)\,(n-2)!\text{]}$$

$$= 2\left[1+\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+...\right]$$

$$= 2e^3 .$$

Thus $2e^3$ is the coefficient of x^2 in the expansion of e^{2x+3}.

Alternatively $e^{2x+3} = e^3 \cdot e^{2x}$

$$= e^3\left[1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+...\right]$$

Thus, the coefficient of x^2 in the expansion of e^{2x+3} is $e^3 . \frac{2^2}{2!} = 2e^3$

Example 4 Find the value of e^2, rounded off to one decimal place.

Solution Using the formula of exponential series involving x, we have

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Putting $x = 2$, we get

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \ldots$$

$$= 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + \frac{4}{45} + \ldots$$

$\geq$ the sum of first seven terms ≥ 7.355.

On the other hand, we have

$$e^2 < \left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) + \frac{2^5}{5!}\left(1 + \frac{2}{6} + \frac{2^2}{6^2} + \frac{2^3}{6^3} + \ldots\right)$$

$$= 7 + \frac{4}{15}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \ldots\right) = 7 + \frac{4}{15}\left(\frac{1}{1 - \frac{1}{3}}\right) = 7 + \frac{2}{5} = 7.4.$$

Thus, e^2 lies between 7.355 and 7.4. Therefore, the value of e^2, rounded off to one decimal place, is 7.4.

A.1.5 Logarithmic Series

Another very important series is logarithmic series which is also in the form of infinite series. We state the following result without proof and illustrate its application with an example.

Theorem If $| x | < 1$, then

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

The series on the right hand side of the above is called the *logarithmic series*.

Note The expansion of $\log_e (1+x)$ is valid for $x = 1$. Substituting $x = 1$ in the expansion of $\log_e (1+x)$, we get

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Example 5 If α, β are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log_e\left(1 + px + qx^2\right) = (\alpha - \beta)x - \frac{\alpha^2 + \beta^2}{2}x^2 + \frac{\alpha^3 + \beta^3}{3}x^3 - \ldots$$

Solution Right hand side $= \left[\alpha x - \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{3} - \ldots\right] + \left[\beta x - \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{3} - \ldots\right]$

$$= \log_e(1 + \alpha x) + \log(1 + \beta x)$$

$$= \log_e\left(1 + (\alpha + \beta)x + \alpha\beta x^2\right)$$

$$= \log_e\left(1 + px + qx^2\right) = \text{Left hand side.}$$

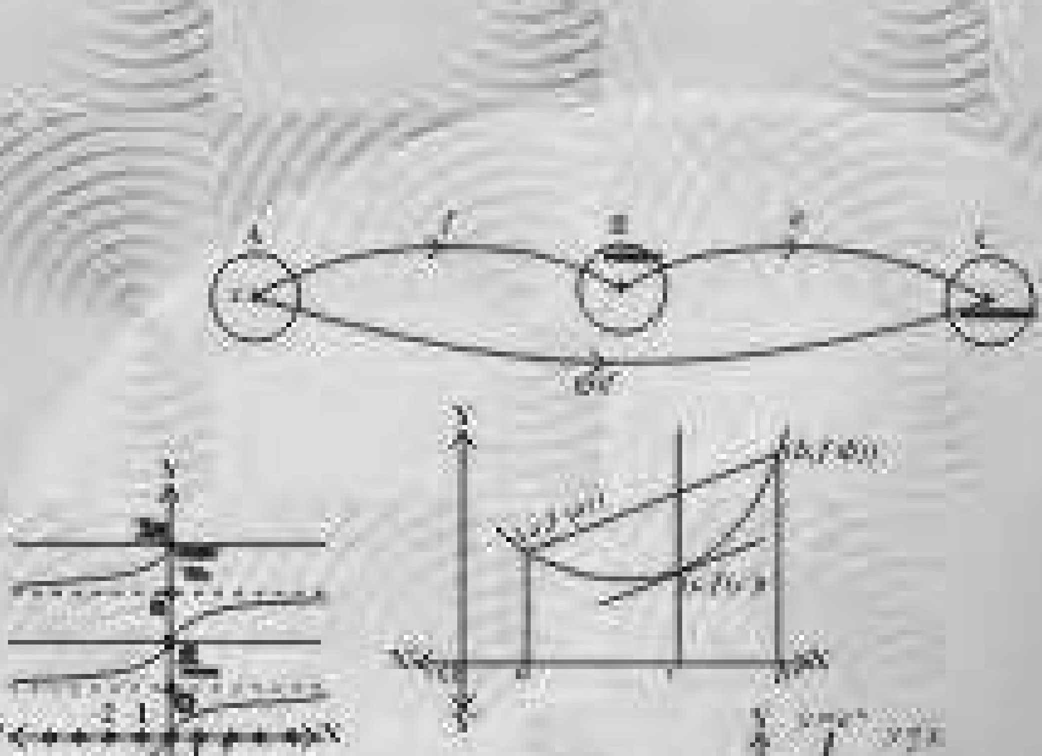
Here, we have used the facts $\alpha + \beta = p$ and $\alpha\beta = q$. We know this from the given roots of the quadratic equation. We have also assumed that both $|\alpha x| < 1$ and $|\beta x| < 1$.

❖

MATHEMATICS

Textbook for Class XII

Part I



MATHEMATICS Part I Class XII

Contents

Part I

Foreword v
Preface vii

1. Relations and Functions **1**
1.1 Introduction 1
1.2 Types of Relations 2
1.3 Types of Functions 7
1.4 Composition of Functions and Invertible Function 12
1.5 Binary Operations 19

2. Inverse Trigonometric Functions **33**
2.1 Introduction 33
2.2 Basic Concepts 33
2.3 Properties of Inverse Trigonometric Functions 42

3. Matrices **56**
3.1 Introduction 56
3.2 Matrix 56
3.3 Types of Matrices 61
3.4 Operations on Matrices 65
3.5 Transpose of a Matrix 83
3.6 Symmetric and Skew Symmetric Matrices 85
3.7 Elementary Operation (Transformation) of a Matrix 90
3.8 Invertible Matrices 91

4. Determinants **103**
4.1 Introduction 103
4.2 Determinant 103
4.3 Properties of Determinants 109
4.4 Area of a Triangle 121
4.5 Minors and Cofactors 123
4.6 Adjoint and Inverse of a Matrix 126
4.7 Applications of Determinants and Matrices 133

5. Continuity and Differentiability **147**
- 5.1 Introduction 147
- 5.2 Continuity 147
- 5.3 Differentiability 161
- 5.4 Exponential and Logarithmic Functions 170
- 5.5 Logarithmic Differentiation 174
- 5.6 Derivatives of Functions in Parametric Forms 179
- 5.7 Second Order Derivative 181
- 5.8 Mean Value Theorem 184

6. Application of Derivatives **194**
- 6.1 Introduction 194
- 6.2 Rate of Change of Quantities 194
- 6.3 Increasing and Decreasing Functions 199
- 6.4 Tangents and Normals 206
- 6.5 Approximations 213
- 6.6 Maxima and Minima 216

Appendix 1: Proofs in Mathematics **247**
- A.1.1 Introduction 247
- A.1.2 What is a Proof? 247

Appendix 2: Mathematical Modelling **256**
- A.2.1 Introduction 256
- A.2.2 Why Mathematical Modelling? 256
- A.2.3 Principles of Mathematical Modelling 257

Answers **268**

Chapter 1

RELATIONS AND FUNCTIONS

❖ *There is no permanent place in the world for ugly mathematics ... . It may be very hard to define mathematical beauty but that is just as true of beauty of any kind, we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it. — G. H. HARDY* ❖

1.1 Introduction

Lejeune Dirichlet (1805-1859)

Recall that the notion of relations and functions, domain, co-domain and range have been introduced in Class XI along with different types of specific real valued functions and their graphs. The concept of the term 'relation' in mathematics has been drawn from the meaning of relation in English language, according to which two objects or quantities are related if there is a recognisable connection or link between the two objects or quantities. Let A be the set of students of Class XII of a school and B be the set of students of Class XI of the same school. Then some of the examples of relations from A to B are

(i) $\{(a, b) \in A \times B$: a is brother of $b\}$,

(ii) $\{(a, b) \in A \times B$: a is sister of $b\}$,

(iii) $\{(a, b) \in A \times B$: age of a is greater than age of $b\}$,

(iv) $\{(a, b) \in A \times B$: total marks obtained by a in the final examination is less than the total marks obtained by b in the final examination$\}$,

(v) $\{(a, b) \in A \times B$: a lives in the same locality as $b\}$. However, abstracting from this, we define mathematically a relation R from A to B as an arbitrary subset of $A \times B$.

If $(a, b) \in R$, we say that a is related to b under the relation R and we write as a R b. In general, $(a, b) \in R$, we do not bother whether there is a recognisable connection or link between a and b. As seen in Class XI, functions are special kind of relations.

In this chapter, we will study different types of relations and functions, composition of functions, invertible functions and binary operations.

1.2 Types of Relations

In this section, we would like to study different types of relations. We know that a relation in a set A is a subset of A × A. Thus, the empty set ϕ and A × A are two extreme relations. For illustration, consider a relation R in the set A = {1, 2, 3, 4} given by R = $\{(a, b): a - b = 10\}$. This is the empty set, as no pair (a, b) satisfies the condition $a - b = 10$. Similarly, $R' = \{(a, b) : | a - b | \geq 0\}$ is the whole set A × A, as all pairs (a, b) in A × A satisfy $| a - b | \geq 0$. These two extreme examples lead us to the following definitions.

Definition 1 A relation R in a set A is called *empty relation*, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

Definition 2 A relation R in a set A is called *universal relation*, if each element of A is related to every element of A, i.e., $R = A \times A$.

Both the empty relation and the universal relation are some times called *trivial relations*.

Example 1 Let A be the set of all students of a boys school. Show that the relation R in A given by $R = \{(a, b) : a$ is sister of $b\}$ is the empty relation and $R' = \{(a, b) :$ the difference between heights of a and b is less than 3 meters$\}$ is the universal relation.

Solution Since the school is boys school, no student of the school can be sister of any student of the school. Hence, $R = \phi$, showing that R is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 meters. This shows that $R' = A \times A$ is the universal relation.

Remark In Class XI, we have seen two ways of representing a relation, namely roaster method and set builder method. However, a relation R in the set {1, 2, 3, 4} defined by $R = \{(a, b) : b = a + 1\}$ is also expressed as a R b if and only if $b = a + 1$ by many authors. We may also use this notation, as and when convenient.

If $(a, b) \in R$, we say that a is related to b and we denote it as a R b.

One of the most important relation, which plays a significant role in Mathematics, is an *equivalence relation*. To study equivalence relation, we first consider three types of relations, namely reflexive, symmetric and transitive.

Definition 3 A relation R in a set A is called

(i) *reflexive*, if $(a, a) \in R$, for every $a \in A$,

(ii) *symmetric*, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

(iii) *transitive*, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

Definition 4 A relation R in a set A is said to be an *equivalence relation* if R is reflexive, symmetric and transitive.

Example 2 Let T be the set of all triangles in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution R is reflexive, since every triangle is congruent to itself. Further, $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to $T_2 \Rightarrow T_2$ is congruent to $T_1 \Rightarrow (T_2, T_1) \in R$. Hence, R is symmetric. Moreover, $(T_1, T_2), (T_2, T_3) \in R \Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to $T_3 \Rightarrow T_1$ is congruent to $T_3 \Rightarrow (T_1, T_3) \in R$. Therefore, R is an equivalence relation.

Example 3 Let L be the set of all lines in a plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$. Show that R is symmetric but neither reflexive nor transitive.

Solution R is not reflexive, as a line L_1 can not be perpendicular to itself, i.e., $(L_1, L_1) \notin R$. R is symmetric as $(L_1, L_2) \in R$

$\Rightarrow$ L_1 is perpendicular to L_2

$\Rightarrow$ L_2 is perpendicular to L_1

$\Rightarrow$ $(L_2, L_1) \in R$.

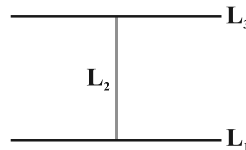


Fig 1.1

R is not transitive. Indeed, if L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 can never be perpendicular to L_3. In fact, L_1 is parallel to L_3, i.e., $(L_1, L_2) \in R$, $(L_2, L_3) \in R$ but $(L_1, L_3) \notin R$.

Example 4 Show that the relation R in the set {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is reflexive but neither symmetric nor transitive.

Solution R is reflexive, since (1, 1), (2, 2) and (3, 3) lie in R. Also, R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, R is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

Example 5 Show that the relation R in the set **Z** of integers given by

$$R = \{(a, b) : 2 \text{ divides } a - b\}$$

is an equivalence relation.

Solution R is reflexive, as 2 divides $(a - a)$ for all $a \in \mathbf{Z}$. Further, if $(a, b) \in R$, then 2 divides $a - b$. Therefore, 2 divides $b - a$. Hence, $(b, a) \in R$, which shows that R is symmetric. Similarly, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are divisible by 2. Now, $a - c = (a - b) + (b - c)$ is even (Why?). So, $(a - c)$ is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in **Z**.

In Example 5, note that all even integers are related to zero, as $(0, \pm 2)$, $(0, \pm 4)$ etc., lie in R and no odd integer is related to 0, as $(0, \pm 1)$, $(0, \pm 3)$ etc., do not lie in R. Similarly, all odd integers are related to one and no even integer is related to one. Therefore, the set E of all even integers and the set O of all odd integers are subsets of **Z** satisfying following conditions:

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $\mathbf{Z} = E \cup O$.

The subset E is called the *equivalence class containing zero* and is denoted by [0]. Similarly, O is the equivalence class containing 1 and is denoted by [1]. Note that $[0] \neq [1]$, $[0] = [2r]$ and $[1] = [2r + 1]$, $r \in \mathbf{Z}$. Infact, what we have seen above is true for an arbitrary equivalence relation R in a set X. Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets A_i called partitions or subdivisions of X satisfying:

(i) all elements of A_i are related to each other, for all i.

(ii) no element of A_i is related to any element of A_j, $i \neq j$.

(iii) $\cup A_j = X$ and $A_i \cap A_j = \phi$, $i \neq j$.

The subsets A_i are called *equivalence classes*. The interesting part of the situation is that we can go reverse also. For example, consider a subdivision of the set **Z** given by three mutually disjoint subsets A_1, A_2 and A_3 whose union is **Z** with

$$A_1 = \{x \in \mathbf{Z} : x \text{ is a multiple of } 3\} = \{..., -6, -3, 0, 3, 6, ...\}$$

$$A_2 = \{x \in \mathbf{Z} : x - 1 \text{ is a multiple of } 3\} = \{..., -5, -2, 1, 4, 7, ...\}$$

$$A_3 = \{x \in \mathbf{Z} : x - 2 \text{ is a multiple of } 3\} = \{..., -4, -1, 2, 5, 8, ...\}$$

Define a relation R in **Z** given by $R = \{(a, b) : 3 \text{ divides } a - b\}$. Following the arguments similar to those used in Example 5, we can show that R is an equivalence relation. Also, A_1 coincides with the set of all integers in **Z** which are related to zero, A_2 coincides with the set of all integers which are related to 1 and A_3 coincides with the set of all integers in **Z** which are related to 2. Thus, $A_1 = [0]$, $A_2 = [1]$ and $A_3 = [2]$. In fact, $A_1 = [3r]$, $A_2 = [3r + 1]$ and $A_3 = [3r + 2]$, for all $r \in \mathbf{Z}$.

Example 6 Let R be the relation defined in the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ by $R = \{(a, b) : \text{both } a \text{ and } b \text{ are either odd or even}\}$. Show that R is an equivalence relation. Further, show that all the elements of the subset {1, 3, 5, 7} are related to each other and all the elements of the subset {2, 4, 6} are related to each other, but no element of the subset {1, 3, 5, 7} is related to any element of the subset {2, 4, 6}.

Solution Given any element a in A, both a and a must be either odd or even, so that $(a, a) \in$ R. Further, $(a, b) \in$ R $\Rightarrow$ both a and b must be either odd or even $\Rightarrow (b, a) \in$ R. Similarly, $(a, b) \in$ R and $(b, c) \in$ R $\Rightarrow$ all elements a, b, c, must be either even or odd simultaneously $\Rightarrow (a, c) \in$ R. Hence, R is an equivalence relation. Further, all the elements of $\{1, 3, 5, 7\}$ are related to each other, as all the elements of this subset are odd. Similarly, all the elements of the subset $\{2, 4, 6\}$ are related to each other, as all of them are even. Also, no element of the subset $\{1, 3, 5, 7\}$ can be related to any element of $\{2, 4, 6\}$, as elements of $\{1, 3, 5, 7\}$ are odd, while elements of $\{2, 4, 6\}$ are even.

EXERCISE 1.1

1. Determine whether each of the following relations are reflexive, symmetric and transitive:
   (i) Relation R in the set A = $\{1, 2, 3, ..., 13, 14\}$ defined as
   $$R = \{(x, y) : 3x - y = 0\}$$
   (ii) Relation R in the set **N** of natural numbers defined as
   $$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$
   (iii) Relation R in the set A = $\{1, 2, 3, 4, 5, 6\}$ as
   $$R = \{(x, y) : y \text{ is divisible by } x\}$$
   (iv) Relation R in the set **Z** of all integers defined as
   $$R = \{(x, y) : x - y \text{ is an integer}\}$$
   (v) Relation R in the set A of human beings in a town at a particular time given by
   (a) R = $\{(x, y) : x$ and y work at the same place$\}$
   (b) R = $\{(x, y) : x$ and y live in the same locality$\}$
   (c) R = $\{(x, y) : x$ is exactly 7 cm taller than $y\}$
   (d) R = $\{(x, y) : x$ is wife of $y\}$
   (e) R = $\{(x, y) : x$ is father of $y\}$
2. Show that the relation R in the set **R** of real numbers, defined as
   R = $\{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.
3. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as
   R = $\{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.
4. Show that the relation R in **R** defined as R = $\{(a, b) : a \leq b\}$, is reflexive and transitive but not symmetric.
5. Check whether the relation R in **R** defined by R = $\{(a, b) : a \leq b^3\}$ is reflexive, symmetric or transitive.

6. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

7. Show that the relation R in the set A of all the books in a library of a college, given by R = {(x, y) : x and y have same number of pages} is an equivalence relation.

8. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by

R = {(a, b) : $|a - b|$ is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

9. Show that each of the relation R in the set $A = \{x \in \mathbf{Z} : 0 \le x \le 12\}$, given by

(i) R = {(a, b) : $|a - b|$ is a multiple of 4}

(ii) R = {(a, b) : $a = b$}

is an equivalence relation. Find the set of all elements related to 1 in each case.

10. Give an example of a relation. Which is

(i) Symmetric but neither reflexive nor transitive.

(ii) Transitive but neither reflexive nor symmetric.

(iii) Reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but not symmetric.

(v) Symmetric and transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

12. Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

13. Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) : P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

14. Let L be the set of all lines in XY plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is parallel to L_2}. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

15. Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

16. Let R be the relation in the set **N** given by R = $\{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.

(A) $(2, 4) \in R$ (B) $(3, 8) \in R$ (C) $(6, 8) \in R$ (D) $(8, 7) \in R$

1.3 Types of Functions

The notion of a function along with some special functions like identity function, constant function, polynomial function, rational function, modulus function, signum function etc. along with their graphs have been given in Class XI.

Addition, subtraction, multiplication and division of two functions have also been studied. As the concept of function is of paramount importance in mathematics and among other disciplines as well, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Consider the functions f_1, f_2, f_3 and f_4 given by the following diagrams.

In Fig 1.2, we observe that the images of distinct elements of X_1 under the function f_1 are distinct, but the image of two distinct elements 1 and 2 of X_1 under f_2 is same, namely b. Further, there are some elements like e and f in X_2 which are not images of any element of X_1 under f_1, while all elements of X_3 are images of some elements of X_1 under f_3. The above observations lead to the following definitions:

Definition 5 A function $f : X \to Y$ is defined to be *one-one* (or *injective*), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called *many-one*.

The function f_1 and f_4 in Fig 1.2 (i) and (iv) are one-one and the function f_2 and f_3 in Fig 1.2 (ii) and (iii) are many-one.

Definition 6 A function $f : X \to Y$ is said to be *onto* (or *surjective*), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

The function f_3 and f_4 in Fig 1.2 (iii), (iv) are onto and the function f_1 in Fig 1.2 (i) is not onto as elements e, f in X_2 are not the image of any element in X_1 under f_1.

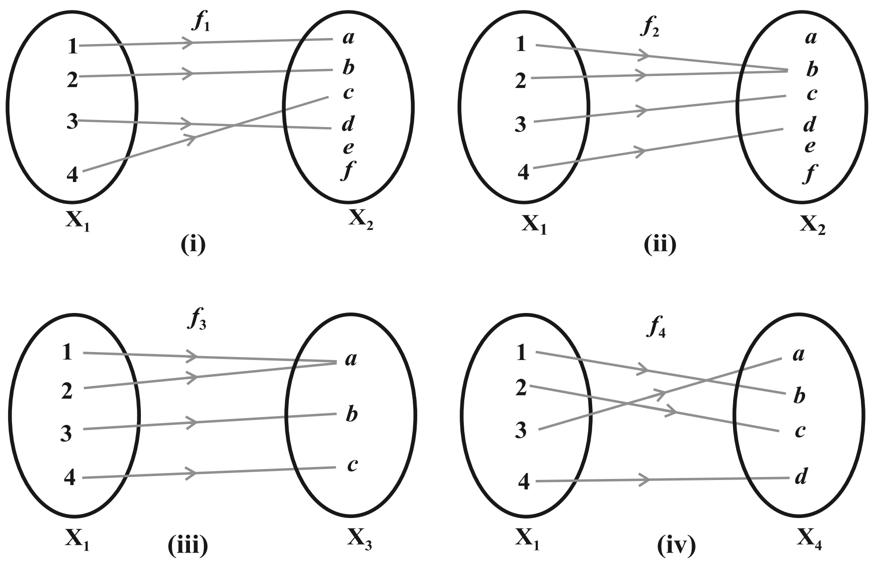


Fig 1.2 (i) to (iv)

Remark $f : X \to Y$ is onto if and only if Range of $f = Y$.

Definition 7 A function $f : X \to Y$ is said to be *one-one* and *onto* (or *bijective*), if f is both one-one and onto.

The function f_4 in Fig 1.2 (iv) is one-one and onto.

Example 7 Let A be the set of all 50 students of Class X in a school. Let $f : A \to \mathbf{N}$ be function defined by $f(x)$ = roll number of the student x. Show that f is one-one but not onto.

Solution No two different students of the class can have same roll number. Therefore, f must be one-one. We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in $\mathbf{N}$ is not roll number of any student of the class, so that 51 can not be image of any element of X under f. Hence, f is not onto.

Example 8 Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(x) = 2x$, is one-one but not onto.

Solution The function f is one-one, for $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Further, f is not onto, as for $1 \in \mathbf{N}$, there does not exist any x in $\mathbf{N}$ such that $f(x) = 2x = 1$.

Example 9 Prove that the function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = 2x$, is one-one and onto.

Solution f is one-one, as $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Also, given any real number y in R, there exists $\frac{y}{2}$ in R such that $f(\frac{y}{2}) = 2 \,.\, (\frac{y}{2}) = y$. Hence, f is onto.

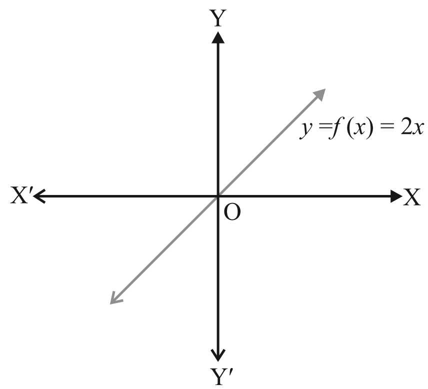


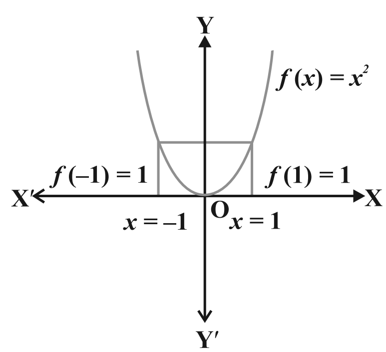
Fig 1.3

Example 10 Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$, for every $x > 2$, is onto but not one-one.

Solution f is not one-one, as $f(1) = f(2) = 1$. But f is onto, as given any $y \in \mathbf{N}$, $y \neq 1$, we can choose x as $y + 1$ such that $f(y + 1) = y + 1 - 1 = y$. Also for $1 \in \mathbf{N}$, we have $f(1) = 1$.

Example 11 Show that the function $f : \mathbf{R} \to \mathbf{R}$, defined as $f(x) = x^2$, is neither one-one nor onto.

Solution Since $f(-1) = 1 = f(1)$, f is not one-one. Also, the element -2 in the co-domain $\mathbf{R}$ is not image of any element x in the domain $\mathbf{R}$ (Why?). Therefore f is not onto.



The image of 1 and –1 under f is 1.

Fig 1.4

Example 12 Show that $f : \mathbf{N} \to \mathbf{N}$, given by

$$f(x) = \begin{cases} x+1, \text{if } x \text{ is odd,} \\ x-1, \text{if } x \text{ is even} \end{cases}$$

is both one-one and onto.

Solution Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 1 = x_2 - 1$, i.e., $x_2 - x_1 = 2$ which is impossible. Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd. Then $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. Similarly, if both x_1 and x_2 are even, then also $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$. Thus, f is one-one. Also, any odd number $2r + 1$ in the co-domain **N** is the image of $2r + 2$ in the domain **N** and any even number $2r$ in the co-domain **N** is the image of $2r - 1$ in the domain **N**. Thus, f is onto.

Example 13 Show that an onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.

Solution Suppose f is not one-one. Then there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same. Also, the image of 3 under f can be only one element. Therefore, the range set can have at the most two elements of the co-domain $\{1, 2, 3\}$, showing that f is not onto, a contradiction. Hence, f must be one-one.

Example 14 Show that a one-one function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.

Solution Since f is one-one, three elements of $\{1, 2, 3\}$ must be taken to 3 different elements of the co-domain $\{1, 2, 3\}$ under f. Hence, f has to be onto.

Remark The results mentioned in Examples 13 and 14 are also true for an arbitrary finite set X, i.e., a one-one function $f : X \to X$ is necessarily onto and an onto map $f : X \to X$ is necessarily one-one, for every finite set X. In contrast to this, Examples 8 and 10 show that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

EXERCISE 1.2

1. Show that the function $f : \mathbf{R}_* \to \mathbf{R}_*$ defined by $f(x) = \dfrac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by **N** with co-domain being same as $\mathbf{R}_*$?
2. Check the injectivity and surjectivity of the following functions:
   (i) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$
   (ii) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$
   (iii) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$
   (iv) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$
   (v) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$
3. Prove that the Greatest Integer Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

4. Show that the Modulus Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

5. Show that the Signum Function $f : \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$$

is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$

(ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$

8. Let A and B be sets. Show that $f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is bijective function.

9. Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, \text{ if } n \text{ is odd} \\ \dfrac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

10. Let A = $\mathbf{R}$ – {3} and B = $\mathbf{R}$ – {1}. Consider the function $f : A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

11. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto.

12. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto.

1.4 Composition of Functions and Invertible Function

In this section, we will study composition of functions and the inverse of a bijective function. Consider the set A of all students, who appeared in Class X of a Board Examination in 2006. Each student appearing in the Board Examination is assigned a roll number by the Board which is written by the students in the answer script at the time of examination. In order to have confidentiality, the Board arranges to deface the roll numbers of students in the answer scripts and assigns a fake code number to each roll number. Let $B \subset \mathbf{N}$ be the set of all roll numbers and $C \subset \mathbf{N}$ be the set of all code numbers. This gives rise to two functions $f : A \to B$ and $g : B \to C$ given by $f(a)$ = the roll number assigned to the student a and $g(b)$ = the code number assigned to the roll number b. In this process each student is assigned a roll number through the function f and each roll number is assigned a code number through the function g. Thus, by the combination of these two functions, each student is eventually attached a code number.

This leads to the following definition:

Definition 8 Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$gof(x) = g(f(x)), \forall\ x \in A.$$

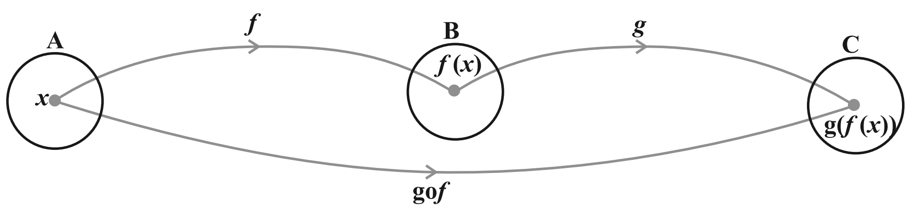


Fig 1.5

Example 15 Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Solution We have $gof(2) = g(f(2)) = g(3) = 7$, $gof(3) = g(f(3)) = g(4) = 7$, $gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(5) = 11$.

Example 16 Find gof and fog, if $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution We have $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$. Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, $gof \neq fog$.

Example 17 Show that if $f: \mathbf{R} - \left\{\frac{7}{5}\right\} \to \mathbf{R} - \left\{\frac{3}{5}\right\}$ is defined by $f(x) = \frac{3x+4}{5x-7}$ and $g: \mathbf{R} - \left\{\frac{3}{5}\right\} \to \mathbf{R} - \left\{\frac{7}{5}\right\}$ is defined by $g(x) = \frac{7x+4}{5x-3}$, then $fog = \mathrm{I}_\mathrm{A}$ and $gof = \mathrm{I}_\mathrm{B}$, where, $\mathrm{A} = \mathbf{R} - \left\{\frac{3}{5}\right\}$, $\mathrm{B} = \mathbf{R} - \left\{\frac{7}{5}\right\}$; $\mathrm{I}_\mathrm{A}(x) = x$, $\forall\, x \in \mathrm{A}$, $\mathrm{I}_\mathrm{B}(x) = x$, $\forall\, x \in \mathrm{B}$ are called identity functions on sets A and B, respectively.

Solution We have

$$gof(x) = g\left(\frac{3x+4}{5x-7}\right) = \frac{7\left(\frac{(3x+4)}{(5x-7)}\right)+4}{5\left(\frac{(3x+4)}{(5x-7)}\right)-3} = \frac{21x+28+20x-28}{15x+20-15x+21} = \frac{41x}{41} = x$$

Similarly, $$fog(x) = f\left(\frac{7x+4}{5x-3}\right) = \frac{3\left(\frac{(7x+4)}{(5x-3)}\right)+4}{5\left(\frac{(7x+4)}{(5x-3)}\right)-7} = \frac{21x+12+20x-12}{35x+20-35x+21} = \frac{41x}{41} = x$$

Thus, $gof(x) = x$, $\forall\, x \in \mathrm{B}$ and $fog(x) = x$, $\forall\, x \in \mathrm{A}$, which implies that $gof = \mathrm{I}_\mathrm{B}$ and $fog = \mathrm{I}_\mathrm{A}$.

Example 18 Show that if $f: \mathrm{A} \to \mathrm{B}$ and $g: \mathrm{B} \to \mathrm{C}$ are one-one, then $gof: \mathrm{A} \to \mathrm{C}$ is also one-one.

Solution Suppose $gof(x_1) = gof(x_2)$

$\Rightarrow$ $g(f(x_1)) = g(f(x_2))$

$\Rightarrow$ $f(x_1) = f(x_2)$, as g is one-one

$\Rightarrow$ $x_1 = x_2$, as f is one-one

Hence, gof is one-one.

Example 19 Show that if $f: \mathrm{A} \to \mathrm{B}$ and $g: \mathrm{B} \to \mathrm{C}$ are onto, then $gof: \mathrm{A} \to \mathrm{C}$ is also onto.

Solution Given an arbitrary element $z \in \mathrm{C}$, there exists a pre-image y of z under g such that $g(y) = z$, since g is onto. Further, for $y \in \mathrm{B}$, there exists an element x in A with $f(x) = y$, since f is onto. Therefore, $gof(x) = g(f(x)) = g(y) = z$, showing that gof is onto.

Example 20 Consider functions f and g such that composite gof is defined and is one-one. Are f and g both necessarily one-one.

Solution Consider $f: \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\}$ defined as $f(x) = x,\ \forall\, x$ and $g: \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ as $g(x) = x$, for $x = 1, 2, 3, 4$ and $g(5) = g(6) = 5$. Then, $gof(x) = x\ \forall\, x$, which shows that gof is one-one. But g is clearly not one-one.

Example 21 Are f and g both necessarily onto, if gof is onto?

Solution Consider $f: \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ and $g: \{1, 2, 3, 4\} \to \{1, 2, 3\}$ defined as $f(1) = 1, f(2) = 2, f(3) = f(4) = 3, g(1) = 1, g(2) = 2$ and $g(3) = g(4) = 3$. It can be seen that gof is onto but f is not onto.

Remark It can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.

Now, we would like to have close look at the functions f and g described in the beginning of this section in reference to a Board Examination. Each student appearing in Class X Examination of the Board is assigned a roll number under the function f and each roll number is assigned a code number under g. After the answer scripts are examined, examiner enters the mark against each code number in a mark book and submits to the office of the Board. The Board officials decode by assigning roll number back to each code number through a process reverse to g and thus mark gets attached to roll number rather than code number. Further, the process reverse to f assigns a roll number to the student having that roll number. This helps in assigning mark to the student scoring that mark. We observe that while composing f and g, to get gof, first f and then g was applied, while in the reverse process of the composite gof, first the reverse process of g is applied and then the reverse process of f.

Example 22 Let $f: \{1, 2, 3\} \to \{a, b, c\}$ be one-one and onto function given by $f(1) = a, f(2) = b$ and $f(3) = c$. Show that there exists a function $g: \{a, b, c\} \to \{1, 2, 3\}$ such that $gof = I_X$ and $fog = I_Y$, where, $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Solution Consider $g: \{a, b, c\} \to \{1, 2, 3\}$ as $g(a) = 1$, $g(b) = 2$ and $g(c) = 3$. It is easy to verify that the composite $gof = I_X$ is the identity function on X and the composite $fog = I_Y$ is the identity function on Y.

Remark The interesting fact is that the result mentioned in the above example is true for an arbitrary one-one and onto function $f: X \to Y$. Not only this, even the converse is also true , i.e., if $f: X \to Y$ is a function such that there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$, then f must be one-one and onto.

The above discussion, Example 22 and Remark lead to the following definition:

Definition 9 A function $f : X \to Y$ is defined to be *invertible*, if there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the *inverse of f* and is denoted by f^{-1}.

Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, specially when the actual inverse of f is not to be determined.

Example 23 Let $f : \mathbf{N} \to Y$ be a function defined as $f(x) = 4x + 3$, where, $Y = \{y \in \mathbf{N} : y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that f is invertible. Find the inverse.

Solution Consider an arbitrary element y of Y. By the definition of Y, $y = 4x + 3$, for some x in the domain $\mathbf{N}$. This shows that $x = \frac{(y-3)}{4}$. Define $g : Y \to \mathbf{N}$ by $g(y) = \frac{(y-3)}{4}$. Now, $gof(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3-3)}{4} = x$ and

$fog(y) = f(g(y)) = f\left(\frac{(y-3)}{4}\right) = \frac{4(y-3)}{4} + 3 = y - 3 + 3 = y$. This shows that $gof = I_N$ and $fog = I_Y$, which implies that f is invertible and g is the inverse of f.

Example 24 Let $Y = \{n^2 : n \in \mathbf{N}\} \subset \mathbf{N}$. Consider $f : \mathbf{N} \to Y$ as $f(n) = n^2$. Show that f is invertible. Find the inverse of f.

Solution An arbitrary element y in Y is of the form n^2, for some $n \in \mathbf{N}$. This implies that $n = \sqrt{y}$. This gives a function $g : Y \to \mathbf{N}$, defined by $g(y) = \sqrt{y}$. Now, $gof(n) = g(n^2) = \sqrt{n^2} = n$ and $fog(y) = f\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y$, which shows that $gof = I_N$ and $fog = I_Y$. Hence, f is invertible with $f^{-1} = g$.

Example 25 Let $f : \mathbf{N} \to \mathbf{R}$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : \mathbf{N} \to S$, where, S is the range of f, is invertible. Find the inverse of f.

Solution Let y be an arbitrary element of range f. Then $y = 4x^2 + 12x + 15$, for some x in $\mathbf{N}$, which implies that $y = (2x + 3)^2 + 6$. This gives $x = \frac{\left(\left(\sqrt{y-6}\right) - 3\right)}{2}$, as $y \geq 6$.

Let us define $g : S \to \mathbf{N}$ by $g(y) = \dfrac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}$.

Now $\quad gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g((2x+3)^2 + 6)$

$$= \frac{\left(\left(\sqrt{(2x+3)^2+6-6}\right)-3\right)}{2} = \frac{(2x+3-3)}{2} = x$$

and $\quad fog\ (y) = f\left(\dfrac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}\right) = \left(\dfrac{2\left(\left(\sqrt{y-6}\right)-3\right)}{2}+3\right)^2 + 6$

$$= \left(\left(\sqrt{y-6}\right)-3+3\right)^2 + 6 = \left(\sqrt{y-6}\right)^2 + 6 = y - 6 + 6 = y.$$

Hence, $\quad gof = I_N$ and $fog = I_S$. This implies that f is invertible with $f^{-1} = g$.

Example 26 Consider $f : \mathbf{N} \to \mathbf{N}$, $g : \mathbf{N} \to \mathbf{N}$ and $h : \mathbf{N} \to \mathbf{R}$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$, $\forall\, x, y$ and z in N. Show that $ho(gof) = (hog)\, of$.

Solution We have

$$ho(gof)\ (x) = h(gof\ (x)) = h(g(f(x))) = h(g(2x))$$
$$= h(3(2x) + 4) = h(6x + 4) = \sin(6x + 4)\ \forall\, x \in \mathbf{N}.$$

Also, $\quad ((hog) \circ f)\ (x) = (hog)\ (f(x)) = (hog)\ (2x) = h(g(2x))$

$$= h(3(2x) + 4) = h(6x + 4) = \sin(6x + 4),\ \forall\, x \in \mathbf{N}.$$

This shows that $ho(gof) = (hog) \circ f$.

This result is true in general situation as well.

Theorem 1 If $f : X \to Y$, $g : Y \to Z$ and $h : Z \to S$ are functions, then

$$ho(gof) = (hog) \circ f.$$

Proof We have

$$ho(gof)\ (x) = h(gof(x)) = h(g(f(x))),\ \forall\, x \text{ in X}$$

and $\quad (hog)\ of\ (x) = hog(f(x)) = h(g(f(x))),\ \forall\, x$ in X.

Hence, $\quad ho(gof) = (hog) \circ f.$

Example 27 Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ and $g : \{a, b, c\} \to$ {apple, ball, cat} defined as $f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a)$ = apple, $g(b)$ = ball and $g(c)$ = cat. Show that f, g and gof are invertible. Find out f^{-1}, g^{-1} and $(gof)^{-1}$ and show that $(gof)^{-1} = f^{-1} \circ g^{-1}$.

Solution Note that by definition, f and g are bijective functions. Let f^{-1}: $\{a, b, c\} \to (1, 2, 3\}$ and g^{-1} : {apple, ball, cat} $\to \{a, b, c\}$ be defined as $f^{-1}\{a\} = 1, f^{-1}\{b\} = 2$, $f^{-1}\{c\} = 3$, g^{-1}\{apple\} = a, g^{-1}\{ball\} = b and g^{-1}\{cat\} = c. It is easy to verify that $f^{-1}\text{o}f = I_{\{1, 2, 3\}}$, $f \text{ o } f^{-1} = I_{\{a, b, c\}}$, $g^{-1}\text{o}g = I_{\{a, b, c\}}$ and $g \text{ o } g^{-1} = I_D$, where, D = {apple, ball, cat}. Now, gof : {1, 2, 3} $\to$ {apple, ball, cat} is given by $gof(1)$ = apple, $gof(2)$ = ball, $gof(3)$ = cat. We can define

$(gof)^{-1}$: {apple, ball, cat} $\to$ {1, 2, 3} by $(gof)^{-1}$ (apple) = 1, $(gof)^{-1}$ (ball) = 2 and $(g \text{o} f)^{-1}$ (cat) = 3. It is easy to see that $(g \text{o} f)^{-1} \text{ o } (g \text{o} f) = I_{\{1, 2, 3\}}$ and $(gof) \text{ o } (gof)^{-1} = I_D$. Thus, we have seen that f, g and gof are invertible.

Now, $f^{-1}\text{o}g^{-1}$ (apple)= $f^{-1}(g^{-1}$(apple)) $= f^{-1}(a) = 1 = (gof)^{-1}$ (apple)

$f^{-1}\text{o}g^{-1}$ (ball) $= f^{-1}(g^{-1}$(ball)) $= f^{-1}(b) = 2 = (gof)^{-1}$ (ball) and

$f^{-1}\text{o}g^{-1}$ (cat) $= f^{-1}(g^{-1}$(cat)) $= f^{-1}(c) = 3 = (gof)^{-1}$ (cat).

Hence $(gof)^{-1} = f^{-1}\text{o}g^{-1}$.

The above result is true in general situation also.

Theorem 2 Let f : X $\to$ Y and g : Y $\to$ Z be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}\text{o}g^{-1}$.

Proof To show that gof is invertible with $(gof)^{-1} = f^{-1}\text{o}g^{-1}$, it is enough to show that $(f^{-1}\text{o}g^{-1})\text{o}(gof) = I_X$ and $(gof)\text{o}(f^{-1}\text{o}g^{-1}) = I_Z$.

$$
\begin{aligned}
\text{Now,} \quad (f^{-1}\text{o}g^{-1})\text{o}(gof) &= ((f^{-1}\text{o}g^{-1}) \text{ o}g) \text{ o}f, \text{ by Theorem 1}\\
&= (f^{-1}\text{o}(g^{-1}\text{o}g)) \text{ o}f, \text{ by Theorem 1}\\
&= (f^{-1} \text{o} I_Y) \text{ o}f, \text{ by definition of } g^{-1}\\
&= I_X.
\end{aligned}
$$

Similarly, it can be shown that $(gof)\text{ o }(f^{-1} \text{ o } g^{-1}) = I_Z$.

Example 28 Let S = {1, 2, 3}. Determine whether the functions f : S $\to$ S defined as below have inverses. Find f^{-1}, if it exists.

(a) $f = \{(1, 1), (2, 2), (3, 3)\}$
(b) $f = \{(1, 2), (2, 1), (3, 1)\}$
(c) $f = \{(1, 3), (3, 2), (2, 1)\}$

Solution

(a) It is easy to see that f is one-one and onto, so that f is invertible with the inverse f^{-1} of f given by $f^{-1} = \{(1, 1), (2, 2), (3, 3)\} = f$.

(b) Since $f(2) = f(3) = 1$, f is not one-one, so that f is not invertible.

(c) It is easy to see that f is one-one and onto, so that f is invertible with $f^{-1} = \{(3, 1), (2, 3), (1, 2)\}$.

EXERCISE 1.3

1. Let $f : \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g : \{1, 2, 5\} \rightarrow \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof.
2. Let f, g and h be functions from **R** to **R**. Show that
$$(f + g)oh = foh + goh$$
$$(f \, . \, g)oh = (foh) \, . \, (goh)$$
3. Find gof and fog, if
   (i) $f(x) = | x |$ and $g(x) = | 5x - 2 |$
   (ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$.
4. If $f(x) = \dfrac{(4x+3)}{(6x-4)}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of f?
5. State with reason whether following functions have inverse
   (i) $f : \{1, 2, 3, 4\} \rightarrow \{10\}$ with
   $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$
   (ii) $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with
   $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$
   (iii) $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with
   $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$
6. Show that $f : [-1, 1] \rightarrow \mathbf{R}$, given by $f(x) = \dfrac{x}{(x+2)}$ is one-one. Find the inverse of the function $f : [-1, 1] \rightarrow$ Range f.

   (Hint: For $y \in$ Range f, $y = f(x) = \dfrac{x}{x+2}$, for some x in $[-1, 1]$, i.e., $x = \dfrac{2y}{(1-y)}$)
7. Consider $f : \mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.
8. Consider $f : \mathbf{R}_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

9. Consider $f : \mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\dfrac{\left(\sqrt{y+6}\right)-1}{3}\right)$.

10. Let $f : X \to Y$ be an invertible function. Show that f has unique inverse.
(Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$,
$fog_1(y) = 1_Y(y) = fog_2(y)$. Use one-one ness of f).

11. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

12. Let f: X $\to$ Y be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

13. If f: $\mathbf{R} \to \mathbf{R}$ be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $x^{\frac{1}{3}}$ (B) x^3 (C) x (D) $(3 - x^3)$.

14. Let $f : \mathbf{R} - \left\{-\dfrac{4}{3}\right\} \to \mathbf{R}$ be a function defined as $f(x) = \dfrac{4x}{3x+4}$. The inverse of f is the map g : Range $f \to \mathbf{R} - \left\{-\dfrac{4}{3}\right\}$ given by

(A) $g(y) = \dfrac{3y}{3-4y}$ (B) $g(y) = \dfrac{4y}{4-3y}$

(C) $g(y) = \dfrac{4y}{3-4y}$ (D) $g(y) = \dfrac{3y}{4-3y}$

1.5 Binary Operations

Right from the school days, you must have come across four fundamental operations namely addition, subtraction, multiplication and division. The main feature of these operations is that given any two numbers a and b, we associate another number $a + b$ or $a - b$ or ab or $\dfrac{a}{b}$, $b \neq 0$. It is to be noted that only two numbers can be added or multiplied at a time. When we need to add three numbers, we first add two numbers and the result is then added to the third number. Thus, addition, multiplication, subtraction and division are examples of binary operation, as 'binary' means two. If we want to have a general definition which can cover all these four operations, then the set of numbers is to be replaced by an arbitrary set X and then general binary operation is nothing but association of any pair of elements a, b from X to another element of X. This gives rise to a general definition as follows:

Definition 10 A binary operation $*$ on a set A is a function $* : A \times A \to A$. We denote $* (a, b)$ by $a * b$.

Example 29 Show that addition, subtraction and multiplication are binary operations on **R**, but division is not a binary operation on **R**. Further, show that division is a binary operation on the set $\mathbf{R}_*$ of nonzero real numbers.

Solution $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by

$$(a, b) \to a + b$$

$- : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by

$$(a, b) \to a - b$$

$\times : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by

$$(a, b) \to ab$$

Since '+', '–' and '×' are functions, they are binary operations on **R**.

But $\div: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, given by $(a, b) \to \frac{a}{b}$, is not a function and hence not a binary operation, as for $b = 0$, $\frac{a}{b}$ is not defined.

However, $\div : \mathbf{R}_* \times \mathbf{R}_* \to \mathbf{R}_*$, given by $(a, b) \to \frac{a}{b}$ is a function and hence a binary operation on $\mathbf{R}_*$.

Example 30 Show that subtraction and division are not binary operations on **N**.

Solution $- : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, given by $(a, b) \to a - b$, is not binary operation, as the image of (3, 5) under '–' is $3 - 5 = -2 \notin \mathbf{N}$. Similarly, $\div : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, given by $(a, b) \to a \div b$ is not a binary operation, as the image of (3, 5) under $\div$ is $3 \div 5 = \frac{3}{5} \notin \mathbf{N}$.

Example 31 Show that $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to a + 4b^2$ is a binary operation.

Solution Since $*$ carries each pair (a, b) to a unique element $a + 4b^2$ in **R**, $*$ is a binary operation on **R**.

Example 32 Let P be the set of all subsets of a given set X. Show that $\cup : P \times P \to P$ given by $(A, B) \to A \cup B$ and $\cap : P \times P \to P$ given by $(A, B) \to A \cap B$ are binary operations on the set P.

Solution Since union operation $\cup$ carries each pair (A, B) in $P \times P$ to a unique element $A \cup B$ in P, $\cup$ is binary operation on P. Similarly, the intersection operation $\cap$ carries each pair (A, B) in $P \times P$ to a unique element $A \cap B$ in P, $\cap$ is a binary operation on P.

Example 33 Show that the $\vee : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to \max \{a, b\}$ and the $\wedge : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to \min \{a, b\}$ are binary operations.

Solution Since $\vee$ carries each pair (a, b) in $\mathbf{R} \times \mathbf{R}$ to a unique element namely maximum of a and b lying in **R**, $\vee$ is a binary operation. Using the similar argument, one can say that $\wedge$ is also a binary operation.

Remark $\vee (4, 7) = 7$, $\vee (4, -7) = 4$, $\wedge (4, 7) = 4$ and $\wedge (4, -7) = -7$.

When number of elements in a set A is small, we can express a binary operation $*$ on the set A through a table called the *operation table* for the operation $*$. For example consider $A = \{1, 2, 3\}$. Then, the operation $\vee$ on A defined in Example 33 can be expressed by the following operation table (Table 1.1) . Here, $\vee (1, 3) = 3$, $\vee (2, 3) = 3$, $\vee (1, 2) = 2$.

Table 1.1

V	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

Here, we are having 3 rows and 3 columns in the operation table with (i, j) the entry of the table being maximum of i^{th} and j^{th} elements of the set A. This can be generalised for general operation $* : A \times A \to A$. If $A = \{a_1, a_2, ..., a_n\}$. Then the operation table will be having n rows and n columns with $(i, j)^{\text{th}}$ entry being $a_i * a_j$. Conversely, given any operation table having n rows and n columns with each entry being an element of $A = \{a_1, a_2, ..., a_n\}$, we can define a binary operation $* : A \times A \to A$ given by $a_i * a_j =$ the entry in the i^{th} row and j^{th} column of the operation table.

One may note that 3 and 4 can be added in any order and the result is same, i.e., $3 + 4 = 4 + 3$, but subtraction of 3 and 4 in different order give different results, i.e., $3 - 4 \neq 4 - 3$. Similarly, in case of multiplication of 3 and 4, order is immaterial, but division of 3 and 4 in different order give different results. Thus, addition and multiplication of 3 and 4 are meaningful, but subtraction and division of 3 and 4 are meaningless. For subtraction and division we have to write 'subtract 3 from 4', 'subtract 4 from 3', 'divide 3 by 4' or 'divide 4 by 3'.

This leads to the following definition:

Definition 11 A binary operation $*$ on the set X is called *commutative*, if $a * b = b * a$, for every $a, b \in$ X.

Example 34 Show that $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\times : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are commutative binary operations, but $- : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\div : \mathbf{R}_* \times \mathbf{R}_* \to \mathbf{R}_*$ are not commutative.

Solution Since $a + b = b + a$ and $a \times b = b \times a$, $\forall\, a, b \in \mathbf{R}$, '+' and '×' are commutative binary operation. However, '–' is not commutative, since $3 - 4 \neq 4 - 3$. Similarly, $3 \div 4 \neq 4 \div 3$ shows that '÷' is not commutative.

Example 35 Show that $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by $a * b = a + 2b$ is not commutative.

Solution Since $3 * 4 = 3 + 8 = 11$ and $4 * 3 = 4 + 6 = 10$, showing that the operation $*$ is not commutative.

If we want to associate three elements of a set X through a binary operation on X, we encounter a natural problem. The expression $a * b * c$ may be interpreted as $(a * b) * c$ or $a * (b * c)$ and these two expressions need not be same. For example, $(8 - 5) - 2 \neq 8 - (5 - 2)$. Therefore, association of three numbers 8, 5 and 3 through the binary operation 'subtraction' is meaningless, unless bracket is used. But in case of addition, $8 + 5 + 2$ has the same value whether we look at it as $(8 + 5) + 2$ or as $8 + (5 + 2)$. Thus, association of 3 or even more than 3 numbers through addition is meaningful without using bracket. This leads to the following:

Definition 12 A binary operation $* : A \times A \to A$ is said to be *associative* if

$$(a * b) * c = a * (b * c),\ \forall\, a, b, c, \in A.$$

Example 36 Show that addition and multiplication are associative binary operation on **R**. But subtraction is not associative on **R**. Division is not associative on $\mathbf{R}_*$.

Solution Addition and multiplication are associative, since $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)\ \forall\ a, b, c \in$ R. However, subtraction and division are not associative, as $(8 - 5) - 3 \neq 8 - (5 - 3)$ and $(8 \div 5) \div 3 \neq 8 \div (5 \div 3)$.

Example 37 Show that $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $a * b \to a + 2b$ is not associative.

Solution The operation $*$ is not associative, since

$$(8 * 5) * 3 = (8 + 10) * 3 = (8 + 10) + 6 = 24,$$

while

$$8 * (5 * 3) = 8 * (5 + 6) = 8 * 11 = 8 + 22 = 30.$$

Remark Associative property of a binary operation is very important in the sense that with this property of a binary operation, we can write $a_1 * a_2 * \ldots * a_n$ which is not ambiguous. But in absence of this property, the expression $a_1 * a_2 * \ldots * a_n$ is ambiguous unless brackets are used. Recall that in the earlier classes brackets were used whenever subtraction or division operations or more than one operation occurred.

For the binary operation '+' on **R**, the interesting feature of the number zero is that $a + 0 = a = 0 + a$, i.e., any number remains unaltered by adding zero. But in case of multiplication, the number 1 plays this role, as $a \times 1 = a = 1 \times a$, $\forall$ a in **R**. This leads to the following definition:

Definition 13 Given a binary operation $* : A \times A \to A$, an element $e \in A$, if it exists, is called *identity* for the operation $*$, if $a * e = a = e * a$, $\forall$ $a \in A$.

Example 38 Show that zero is the identity for addition on **R** and 1 is the identity for multiplication on R. But there is no identity element for the operations

$$- : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \text{ and } \div : \mathbf{R}_* \times \mathbf{R}_* \to \mathbf{R}_*.$$

Solution $a + 0 = 0 + a = a$ and $a \times 1 = a = 1 \times a$, $\forall\, a \in$ **R** implies that 0 and 1 are identity elements for the operations '+' and '×' respectively. Further, there is no element e in **R** with $a - e = e - a$, $\forall\, a$. Similarly, we can not find any element e in $\mathbf{R}_*$ such that $a \div e = e \div a$, $\forall\, a$ in $\mathbf{R}_*$. Hence, '–' and '÷' do not have identity element.

Remark Zero is identity for the addition operation on **R** but it is not identity for the addition operation on **N**, as $0 \notin$ **N**. In fact the addition operation on **N** does not have any identity.

One further notices that for the addition operation $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, given any $a \in$ **R**, there exists $-a$ in **R** such that $a + (-a) = 0$ (identity for '+') $= (-a) + a$. Similarly, for the multiplication operation on **R**, given any $a \neq 0$ in **R**, we can choose $\frac{1}{a}$ in **R** such that $a \times \frac{1}{a} = 1$(identity for '×') $= \frac{1}{a} \times a$. This leads to the following definition:

Definition 14 Given a binary operation $* : A \times A \to A$ with the identity element e in A, an element $a \in A$ is said to be *invertible* with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the *inverse of* a and is denoted by a^{-1}.

Example 39 Show that $-a$ is the inverse of a for the addition operation '+' on **R** and $\frac{1}{a}$ is the inverse of $a \neq 0$ for the multiplication operation '×' on **R**.

Solution As $a + (-a) = a - a = 0$ and $(-a) + a = 0$, $-a$ is the inverse of a for addition. Similarly, for $a \neq 0$, $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$ implies that $\frac{1}{a}$ is the inverse of a for multiplication.

Example 40 Show that $-a$ is not the inverse of $a \in \mathbf{N}$ for the addition operation + on $\mathbf{N}$ and $\frac{1}{a}$ is not the inverse of $a \in \mathbf{N}$ for multiplication operation × on $\mathbf{N}$, for $a \neq 1$.

Solution Since $-a \notin \mathbf{N}$, $-a$ can not be inverse of a for addition operation on $\mathbf{N}$, although $-a$ satisfies $a + (-a) = 0 = (-a) + a$.

Similarly, for $a \neq 1$ in $\mathbf{N}$, $\frac{1}{a} \notin \mathbf{N}$, which implies that other than 1 no element of $\mathbf{N}$ has inverse for multiplication operation on $\mathbf{N}$.

Examples 34, 36, 38 and 39 show that addition on $\mathbf{R}$ is a commutative and associative binary operation with 0 as the identity element and $-a$ as the inverse of a in $\mathbf{R}$ $\forall a$.

EXERCISE 1.4

1. Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.
   (i) On $\mathbf{Z}^+$, define $*$ by $a * b = a - b$
   (ii) On $\mathbf{Z}^+$, define $*$ by $a * b = ab$
   (iii) On $\mathbf{R}$, define $*$ by $a * b = ab^2$
   (iv) On $\mathbf{Z}^+$, define $*$ by $a * b = |a - b|$
   (v) On $\mathbf{Z}^+$, define $*$ by $a * b = a$
2. For each binary operation $*$ defined below, determine whether $*$ is commutative or associative.
   (i) On $\mathbf{Z}$, define $a * b = a - b$
   (ii) On $\mathbf{Q}$, define $a * b = ab + 1$
   (iii) On $\mathbf{Q}$, define $a * b = \frac{ab}{2}$
   (iv) On $\mathbf{Z}^+$, define $a * b = 2^{ab}$
   (v) On $\mathbf{Z}^+$, define $a * b = a^b$
   (vi) On $\mathbf{R} - \{-1\}$, define $a * b = \frac{a}{b+1}$
3. Consider the binary operation $\wedge$ on the set $\{1, 2, 3, 4, 5\}$ defined by $a \wedge b = \min\{a, b\}$. Write the operation table of the operation $\wedge$.

4. Consider a binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ given by the following multiplication table (Table 1.2).
   (i) Compute $(2 * 3) * 4$ and $2 * (3 * 4)$
   (ii) Is $*$ commutative?
   (iii) Compute $(2 * 3) * (4 * 5)$.
   (Hint: use the following table)

Table 1.2

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

5. Let $*'$ be the binary operation on the set $\{1, 2, 3, 4, 5\}$ defined by $a *' b$ = H.C.F. of a and b. Is the operation $*'$ same as the operation $*$ defined in Exercise 4 above? Justify your answer.
6. Let $*$ be the binary operation on **N** given by $a * b$ = L.C.M. of a and b. Find
   (i) $5 * 7$, $20 * 16$ (ii) Is $*$ commutative?
   (iii) Is $*$ associative? (iv) Find the identity of $*$ in **N**
   (v) Which elements of **N** are invertible for the operation $*$?
7. Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b$ = L.C.M. of a and b a binary operation? Justify your answer.
8. Let $*$ be the binary operation on **N** defined by $a * b$ = H.C.F. of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on **N**?
9. Let $*$ be a binary operation on the set **Q** of rational numbers as follows:
   (i) $a * b = a - b$ (ii) $a * b = a^2 + b^2$
   (iii) $a * b = a + ab$ (iv) $a * b = (a - b)^2$
   (v) $a * b = \dfrac{ab}{4}$ (vi) $a * b = ab^2$

Find which of the binary operations are commutative and which are associative.

10. Show that none of the operations given above has identity.
11. Let A = **N** × **N** and $*$ be the binary operation on A defined by

$$(a, b) * (c, d) = (a + c, b + d)$$

Show that $*$ is commutative and associative. Find the identity element for $*$ on A, if any.

12. State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation $*$ on a set **N**, $a * a = a \ \forall a \in \mathbf{N}$.

(ii) If $*$ is a commutative binary operation on **N**, then $a * (b * c) = (c * b) * a$

13. Consider a binary operation $*$ on **N** defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is $*$ both associative and commutative?

(B) Is $*$ commutative but not associative?

(C) Is $*$ associative but not commutative?

(D) Is $*$ neither commutative nor associative?

Miscellaneous Examples

Example 41 If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Solution Since R_1 and R_2 are equivalence relations, $(a, a) \in R_1$, and $(a, a) \in R_2 \ \forall a \in A$. This implies that $(a, a) \in R_1 \cap R_2$, $\forall a$, showing $R_1 \cap R_2$ is reflexive. Further, $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2$, hence, $R_1 \cap R_2$ is symmetric. Similarly, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2 \Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2 \Rightarrow (a, c) \in R_1 \cap R_2$. This shows that $R_1 \cap R_2$ is transitive. Thus, $R_1 \cap R_2$ is an equivalence relation.

Example 42 Let R be a relation on the set A of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if $xv = yu$. Show that R is an equivalence relation.

Solution Clearly, (x, y) R (x, y), $\forall (x, y) \in A$, since $xy = yx$. This shows that R is reflexive. Further, (x, y) R $(u, v) \Rightarrow xv = yu \Rightarrow uy = vx$ and hence (u, v) R (x, y). This shows that R is symmetric. Similarly, (x, y) R (u, v) and (u, v) R $(a, b) \Rightarrow xv = yu$ and $ub = va \Rightarrow xv\frac{a}{u} = yu\frac{a}{u} \Rightarrow xv\frac{b}{v} = yu\frac{a}{u} \Rightarrow xb = ya$ and hence (x, y) R (a, b). Thus, R is transitive. Thus, R is an equivalence relation.

Example 43 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let R_1 be a relation in X given by $R_1 = \{(x, y) : x - y \text{ is divisible by } 3\}$ and R_2 be another relation on X given by $R_2 = \{(x, y): \{x, y\} \subset \{1, 4, 7\}\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$. Show that $R_1 = R_2$.

Solution Note that the characteristic of sets {1, 4, 7}, {2, 5, 8} and {3, 6, 9} is that difference between any two elements of these sets is a multiple of 3. Therefore, $(x, y) \in R_1 \Rightarrow x - y$ is a multiple of $3 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$. Hence, $R_1 \subset R_2$. Similarly, $\{x, y\} \in R_2 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow x - y$ is divisible by $3 \Rightarrow \{x, y\} \in R_1$. This shows that $R_2 \subset R_1$. Hence, $R_1 = R_2$.

Example 44 Let $f : X \to Y$ be a function. Define a relation R in X given by $R = \{(a, b): f(a) = f(b)\}$. Examine if R is an equivalence relation.

Solution For every $a \in X$, $(a, a) \in R$, since $f(a) = f(a)$, showing that R is reflexive. Similarly, $(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$. Therefore, R is symmetric. Further, $(a, b) \in R$ and $(b, c) \in R \Rightarrow f(a) = f(b)$ and $f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow (a, c) \in R$, which implies that R is transitive. Hence, R is an equivalence relation.

Example 45 Determine which of the following binary operations on the set **N** are associative and which are commutative.

(a) $a * b = 1 \ \forall \ a, b \in \mathbf{N}$ (b) $a * b = \dfrac{(a+b)}{2} \ \forall \ a, b \in \mathbf{N}$

Solution

(a) Clearly, by definition $a * b = b * a = 1$, $\forall a, b \in \mathbf{N}$. Also $(a * b) * c = (1 * c) = 1$ and $a * (b * c) = a * (1) = 1$, $\forall a, b, c \in \mathbf{N}$. Hence R is both associative and commutative.

(b) $a * b = \dfrac{a+b}{2} = \dfrac{b+a}{2} = b * a$, shows that $*$ is commutative. Further,

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c.$$

$$= \frac{\left(\frac{a+b}{2}\right) + c}{2} = \frac{a+b+2c}{4}.$$

But

$$a * (b * c) = a * \left(\frac{b+c}{2}\right)$$

$$= \frac{a + \frac{b+c}{2}}{2} = \frac{2a+b+c}{4} \neq \frac{a+b+2c}{4} \text{ in general.}$$

Hence, $*$ is not associative.

Example 46 Find the number of all one-one functions from set A = {1, 2, 3} to itself.

Solution One-one function from {1, 2, 3} to itself is simply a permutation on three symbols 1, 2, 3. Therefore, total number of one-one maps from {1, 2, 3} to itself is same as total number of permutations on three symbols 1, 2, 3 which is 3! = 6.

Example 47 Let A = {1, 2, 3}. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is four.

Solution The smallest relation R_1 containing (1, 2) and (2, 3) which is reflexive and transitive but not symmetric is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}. Now, if we add the pair (2, 1) to R_1 to get R_2, then the relation R_2 will be reflexive, transitive but not symmetric. Similarly, we can obtain R_3 and R_4 by adding (3, 2) and (3, 1) respectively, to R_1 to get the desired relations. However, we can not add any two pairs out of (2, 1), (3, 2) and (3, 1) to R_1 at a time, as by doing so, we will be forced to add the remaining third pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is four.

Example 48 Show that the number of equivalence relation in the set {1, 2, 3} containing (1, 2) and (2, 1) is two.

Solution The smallest equivalence relation R_1 containing (1, 2) and (2, 1) is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}. Now we are left with only 4 pairs namely (2, 3), (3, 2), (1, 3) and (3, 1). If we add any one, say (2, 3) to R_1, then for symmetry we must add (3, 2) also and now for transitivity we are forced to add (1, 3) and (3, 1). Thus, the only equivalence relation bigger than R_1 is the universal relation. This shows that the total number of equivalence relations containing (1, 2) and (2, 1) is two.

Example 49 Show that the number of binary operations on {1, 2} having 1 as identity and having 2 as the inverse of 2 is exactly one.

Solution A binary operation $*$ on {1, 2} is a function from {1, 2} × {1, 2} to {1, 2}, i.e., a function from {(1, 1), (1, 2), (2, 1), (2, 2)} $\rightarrow$ {1, 2}. Since 1 is the identity for the desired binary operation $*$, $*$ (1, 1) = 1, $*$ (1, 2) = 2, $*$ (2, 1) = 2 and the only choice left is for the pair (2, 2). Since 2 is the inverse of 2, i.e., $*$ (2, 2) must be equal to 1. Thus, the number of desired binary operation is only one.

Example 50 Consider the identity function $I_{\mathbf{N}} : \mathbf{N} \rightarrow \mathbf{N}$ defined as $I_{\mathbf{N}}(x) = x \ \forall x \in \mathbf{N}$. Show that although $I_{\mathbf{N}}$ is onto but $I_{\mathbf{N}} + I_{\mathbf{N}} : \mathbf{N} \rightarrow \mathbf{N}$ defined as

$$(I_{\mathbf{N}} + I_{\mathbf{N}})(x) = I_{\mathbf{N}}(x) + I_{\mathbf{N}}(x) = x + x = 2x \text{ is not onto.}$$

Solution Clearly $I_{\mathbf{N}}$ is onto. But $I_{\mathbf{N}} + I_{\mathbf{N}}$ is not onto, as we can find an element 3 in the co-domain **N** such that there does not exist any x in the domain **N** with $(I_{\mathbf{N}} + I_{\mathbf{N}})(x) = 2x = 3$.

Example 51 Consider a function $f : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one-one.

Solution Since for any two distinct elements x_1 and x_2 in $\left[0, \frac{\pi}{2}\right]$, $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$, both f and g must be one-one. But $(f + g)(0) = \sin 0 + \cos 0 = 1$ and $(f + g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$. Therefore, $f + g$ is not one-one.

Miscellaneous Exercise on Chapter 1

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 10x + 7$. Find the function $g : \mathbf{R} \to \mathbf{R}$ such that $g \circ f = f \circ g = 1_{\mathbf{R}}$.
2. Let $f : \mathrm{W} \to \mathrm{W}$ be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.
3. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.
4. Show that the function $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1 + |x|}$, $x \in \mathbf{R}$ is one one and onto function.
5. Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is injective.
6. Give examples of two functions $f : \mathbf{N} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ such that $g \circ f$ is injective but g is not injective.

   (Hint : Consider $f(x) = x$ and $g(x) = |x|$).
7. Give examples of two functions $f : \mathbf{N} \to \mathbf{N}$ and $g : \mathbf{N} \to \mathbf{N}$ such that $g \circ f$ is onto but f is not onto.

   (Hint : Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$
8. Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify your answer.

9. Given a non-empty set X, consider the binary operation $* : P(X) \times P(X) \to P(X)$ given by $A * B = A \cap B \;\; \forall$ A, B in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation $*$.

10. Find the number of all onto functions from the set $\{1, 2, 3, ..., n\}$ to itself.

11. Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

(i) $F = \{(a, 3), (b, 2), (c, 1)\}$ (ii) $F = \{(a, 2), (b, 1), (c, 1)\}$

12. Consider the binary operations $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $o : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined as $a * b = |a - b|$ and $a \text{ o } b = a, \;\forall\, a, b \in \mathbf{R}$. Show that $*$ is commutative but not associative, o is associative but not commutative. Further, show that $\forall\, a, b, c \in \mathbf{R}$, $a * (b \text{ o } c) = (a * b) \text{ o } (a * b)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

13. Given a non-empty set X, let $* : P(X) \times P(X) \to P(X)$ be defined as $A * B = (A - B) \cup (B - A), \;\forall$ A, B $\in$ P(X). Show that the empty set ϕ is the identity for the operation $*$ and all the elements A of P(X) are invertible with $A^{-1} = A$. (Hint : $(A - \phi) \cup (\phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \phi$).

14. Define a binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ as

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element a of the set is invertible with $6 - a$ being the inverse of a.

15. Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x,\; x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1,\; x \in A$. Are f and g equal? Justify your answer. (Hint: One may note that two functions $f : A \to B$ and $g : A \to B$ such that $f(a) = g(a) \;\forall\, a \in A$, are called equal functions).

16. Let $A = \{1, 2, 3\}$. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

17. Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

18. Let $f : \mathbf{R} \to \mathbf{R}$ be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g : \mathrm{R} \to \mathrm{R}$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then, does fog and gof coincide in $(0, 1]$?

19. Number of binary operations on the set $\{a, b\}$ are

(A) 10 (B) 16 (C) 20 (D) 8

Summary

In this chapter, we studied different types of relations and equivalence relation, composition of functions, invertible functions and binary operations. The main features of this chapter are as follows:

- *Empty relation* is the relation R in X given by $R = \phi \subset X \times X$.
- *Universal relation* is the relation R in X given by $R = X \times X$.
- *Reflexive relation* R in X is a relation with $(a, a) \in R \ \forall a \in X$.
- *Symmetric relation* R in X is a relation satisfying $(a, b) \in R$ implies $(b, a) \in R$.
- *Transitive relation* R in X is a relation satisfying $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.
- *Equivalence relation* R in X is a relation which is reflexive, symmetric and transitive.
- *Equivalence class* $[a]$ containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function $f : X \to Y$ is *one-one* (or *injective*) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$.
- A function $f : X \to Y$ is *onto* (or *surjective*) if given any $y \in Y$, $\exists\, x \in X$ such that $f(x) = y$.
- A function $f : X \to Y$ is *one-one and onto* (or *bijective*), if f is both one-one and onto.
- The *composition* of functions $f : A \to B$ and $g : B \to C$ is the function $gof : A \to C$ given by $gof(x) = g(f(x)) \ \forall \ x \in A$.
- A function $f : X \to Y$ is *invertible* if $\exists\, g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$.
- A function $f : X \to Y$ is *invertible* if and only if f is one-one and onto.

- Given a finite set X, a function $f : X \to X$ is one-one (respectively onto) if and only if f is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set
- A ***binary operation*** $*$ on a set A is a function $*$ from $A \times A$ to A.
- An element $e \in X$ is the ***identity*** element for binary operation $* : X \times X \to X$, if $a * e = a = e * a \ \forall a \in X$.
- An element $a \in X$ is ***invertible*** for binary operation $* : X \times X \to X$, if there exists $b \in X$ such that $a * b = e = b * a$ where, e is the identity for the binary operation $*$. The element b is called ***inverse*** of a and is denoted by a^{-1}.
- An operation $*$ on X is ***commutative*** if $a * b = b * a \ \forall a, b$ in X.
- An operation $*$ on X is ***associative*** if $(a * b) * c = a * (b * c) \forall a, b, c$ in X.

Historical Note

The concept of function has evolved over a long period of time starting from R. Descartes (1596-1650), who used the word 'function' in his manuscript "*Geometrie*" in 1637 to mean some positive integral power x^n of a variable x while studying geometrical curves like hyperbola, parabola and ellipse. James Gregory (1636-1675) in his work " *Vera Circuli et Hyperbolae Quadratura*" (1667) considered function as a quantity obtained from other quantities by successive use of algebraic operations or by any other operations. Later G. W. Leibnitz (1646-1716) in his manuscript "*Methodus tangentium inversa, seu de functionibus*" written in 1673 used the word 'function' to mean a quantity varying from point to point on a curve such as the coordinates of a point on the curve, the slope of the curve, the tangent and the normal to the curve at a point. However, in his manuscript "*Historia*" (1714), Leibnitz used the word 'function' to mean quantities that depend on a variable. He was the first to use the phrase 'function of x'. John Bernoulli (1667-1748) used the notation ϕx for the first time in 1718 to indicate a function of x. But the general adoption of symbols like f, F, ϕ, ψ ... to represent functions was made by Leonhard Euler (1707-1783) in 1734 in the first part of his manuscript "*Analysis Infinitorium*". Later on, Joeph Louis Lagrange (1736-1813) published his manuscripts "*Theorie des functions analytiques*" in 1793, where he discussed about analytic function and used the notion $f(x)$, $F(x)$, $\phi(x)$ etc. for different function of x. Subsequently, Lejeunne Dirichlet (1805-1859) gave the definition of function which was being used till the set theoretic definition of function presently used, was given after set theory was developed by Georg Cantor (1845-1918). The set theoretic definition of function known to us presently is simply an abstraction of the definition given by Dirichlet in a rigorous manner.

Chapter 2

INVERSE TRIGONOMETRIC FUNCTIONS

❖ *Mathematics, in general, is fundamentally the science of self-evident things. — FELIX KLEIN* ❖

2.1 Introduction

In Chapter 1, we have studied that the inverse of a function f, denoted by f^{-1}, exists if f is one-one and onto. There are many functions which are not one-one, onto or both and hence we can not talk of their inverses. In Class XI, we studied that trigonometric functions are not one-one and onto over their natural domains and ranges and hence their inverses do not exist. In this chapter, we shall study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverses and observe their behaviour through graphical representations. Besides, some elementary properties will also be discussed.

Arya Bhatta
(476-550 A. D.)

The inverse trigonometric functions play an important role in calculus for they serve to define many integrals. The concepts of inverse trigonometric functions is also used in science and engineering.

2.2 Basic Concepts

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e., sine : $\mathbf{R} \to [-1, 1]$

cosine function, i.e., cos : $\mathbf{R} \to [-1, 1]$

tangent function, i.e., tan : $\mathbf{R} - \{ x : x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R}$

cotangent function, i.e., cot : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R}$

secant function, i.e., sec : $\mathbf{R} - \{ x : x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

cosecant function, i.e., cosec : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

We have also learnt in Chapter 1 that if $f : X \to Y$ such that $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Here, the domain of g = range of f and the range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Further, g is also one-one and onto and inverse of g is f. Thus, $g^{-1} = (f^{-1})^{-1} = f$. We also have

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

and $$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$$

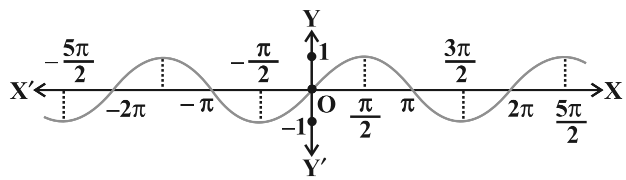
Since the domain of sine function is the set of all real numbers and range is the closed interval [–1, 1]. If we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto with range [– 1, 1]. Actually, sine function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc., is one-one and its range is [–1, 1]. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is [– 1, 1] and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a *branch* of the function $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal value branch, whereas other intervals as range give different branches of* $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is [–1, 1] and range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, if $y = \sin^{-1} x$, then $\sin y = x$.

Remarks

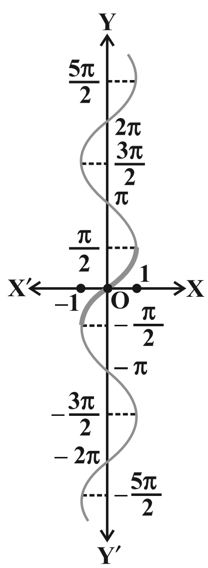
(i) We know from Chapter 1, that if $y = f(x)$ is an invertible function, then $x = f^{-1}(y)$. Thus, the graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging x and y axes, i.e., if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse of sine function. Thus, the graph of the function $y = \sin^{-1} x$ can be obtained from the graph of $y = \sin x$ by interchanging x and y axes. The graphs of $y = \sin x$ and $y = \sin^{-1} x$ are as given in Fig 2.1 (i), (ii), (iii). The dark portion of the graph of $y = \sin^{-1} x$ represent the principal value branch.

(ii) It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$. This can be visualised by looking the graphs of $y = \sin x$ and $y = \sin^{-1} x$ as given in the same axes (Fig 2.1 (iii)).



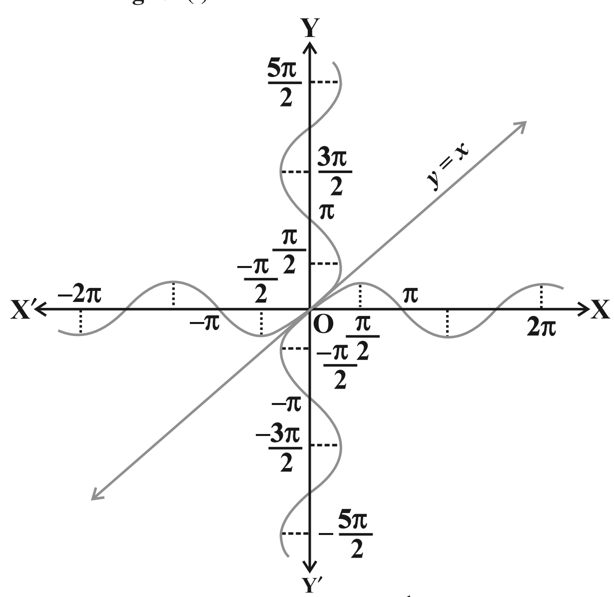
$y = \sin x$

Fig 2.1 (i)



$y = \sin^{-1} x$

Fig 2.1 (ii)



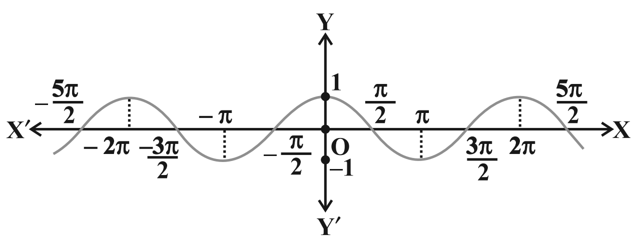
$y = \sin x$ and $y = \sin^{-1} x$

Fig 2.1 (iii)

Like sine function, the cosine function is a function whose domain is the set of all real numbers and range is the set [–1, 1]. If we restrict the domain of cosine function to [0, π], then it becomes one-one and onto with range [–1, 1]. Actually, cosine function restricted to any of the intervals $[-\pi, 0]$, $[0,\pi]$, $[\pi, 2\pi]$ etc., is bijective with range as $[-1, 1]$. We can, therefore, define the inverse of cosine function in each of these intervals. We denote the inverse of the cosine function by $\cos^{-1}$ (arc cosine function). Thus, $\cos^{-1}$ is a function whose domain is $[-1, 1]$ and range could be any of the intervals $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$ etc. Corresponding to each such interval, we get a branch of the function $\cos^{-1}$. The branch with range $[0, \pi]$ is called the *principal value branch* of the function $\cos^{-1}$. We write

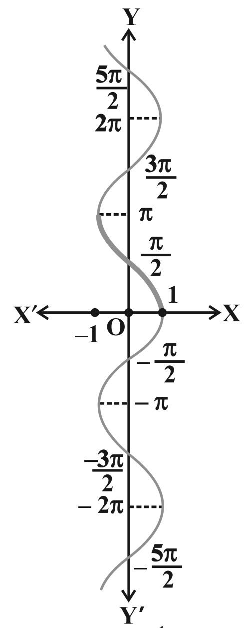
$$\cos^{-1} : [-1, 1] \to [0, \pi].$$

The graph of the function given by $y = \cos^{-1} x$ can be drawn in the same way as discussed about the graph of $y = \sin^{-1} x$. The graphs of $y = \cos x$ and $y = \cos^{-1} x$ are given in Fig 2.2 (i) and (ii).



$y = \cos x$

Fig 2.2 (i)



$y = \cos^{-1} x$

Fig 2.2 (ii)

Let us now discuss $\text{cosec}^{-1}x$ and $\sec^{-1}x$ as follows:

Since, $\text{cosec}\, x = \dfrac{1}{\sin x}$, the domain of the cosec function is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi,\ n \in \mathbf{Z}\}$ and the range is the set $\{y : y \in \mathbf{R},\ y \geq 1$ or $y \leq -1\}$ i.e., the set $\mathbf{R} - (-1, 1)$. It means that $y = \text{cosec}\, x$ assumes all real values except $-1 < y < 1$ and is not defined for integral multiple of π. If we restrict the domain of cosec function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, then it is one to one and onto with its range as the set $\mathbf{R} - (-1, 1)$. Actually, cosec function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ etc., is bijective and its range is the set of all real numbers $\mathbf{R} - (-1, 1)$.

Thus cosec^{-1} can be defined as a function whose domain is $\mathbf{R} - (-1, 1)$ and range could be any of the intervals $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ etc. The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the *principal value branch* of cosec^{-1}. We thus have principal branch as

$$\text{cosec}^{-1} : \mathbf{R} - (-1, 1) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

The graphs of $y = \text{cosec } x$ and $y = \text{cosec}^{-1} x$ are given in Fig 2.3 (i), (ii).

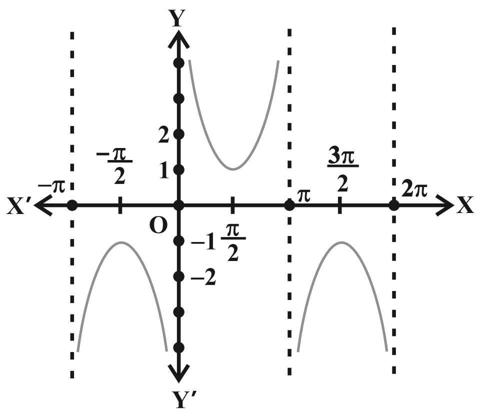


Fig 2.3 (i)

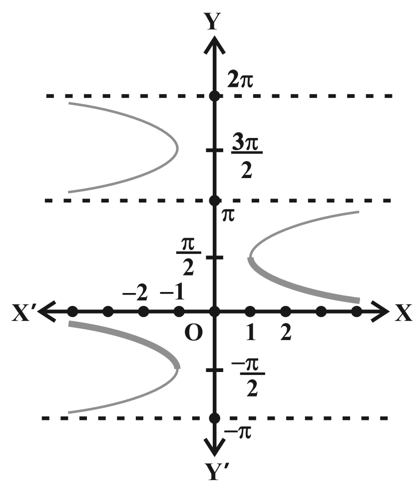


Fig 2.3 (ii)

Also, since $\sec x = \frac{1}{\cos x}$, the domain of $y = \sec x$ is the set $\mathbf{R} - \{x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\mathbf{R} - (-1, 1)$. It means that sec (secant function) assumes all real values except $-1 < y < 1$ and is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of secant function to $[0, \pi] - \{\frac{\pi}{2}\}$, then it is one-one and onto with its range as the set **R** – (–1, 1). Actually, secant function restricted to any of the intervals [–π, 0] – { $\frac{-\pi}{2}$ }, [0, π] – $\left\{\frac{\pi}{2}\right\}$, [π, 2π] – { $\frac{3\pi}{2}$ } etc., is bijective and its range is **R** – {–1, 1}. Thus $\sec^{-1}$ can be defined as a function whose domain is **R**– (–1, 1) and range could be any of the intervals [– π, 0] – { $\frac{-\pi}{2}$ }, [0, π] – { $\frac{\pi}{2}$ }, [π, 2π] – { $\frac{3\pi}{2}$ } etc. Corresponding to each of these intervals, we get different branches of the function $\sec^{-1}$. The branch with range [0, π] – { $\frac{\pi}{2}$ } is called the *principal value branch* of the function $\sec^{-1}$. We thus have

$$\sec^{-1} : \mathbf{R} - (-1,1) \to [0, \pi] - \{\frac{\pi}{2}\}$$

The graphs of the functions $y = \sec x$ and $y = \sec^{-1} x$ are given in Fig 2.4 (i), (ii).

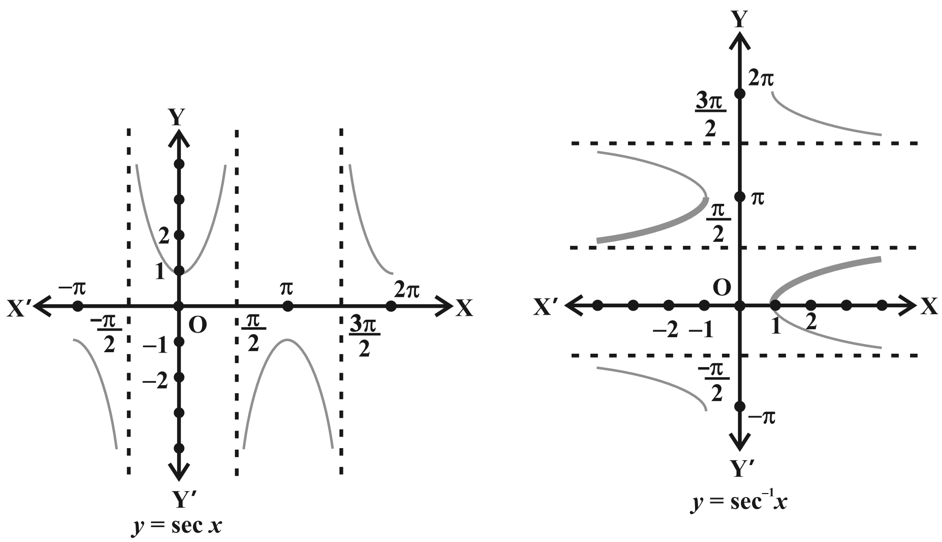


Fig 2.4 (i)

Fig 2.4 (ii)

Finally, we now discuss $\tan^{-1}$ and $\cot^{-1}$

We know that the domain of the tan function (tangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and the range is **R**. It means that tan function is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of tangent function to

$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then it is one-one and onto with its range as **R**. Actually, tangent function restricted to any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ etc., is bijective and its range is **R**. Thus $\tan^{-1}$ can be defined as a function whose domain is **R** and range could be any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and so on. These intervals give different branches of the function $\tan^{-1}$. The branch with range $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is called the *principal value branch* of the function $\tan^{-1}$.

We thus have

$$\tan^{-1} : \mathbf{R} \rightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the function $y = \tan x$ and $y = \tan^{-1}x$ are given in Fig 2.5 (i), (ii).

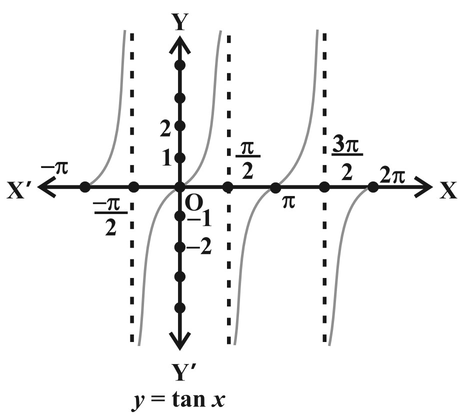


Fig 2.5 (i)

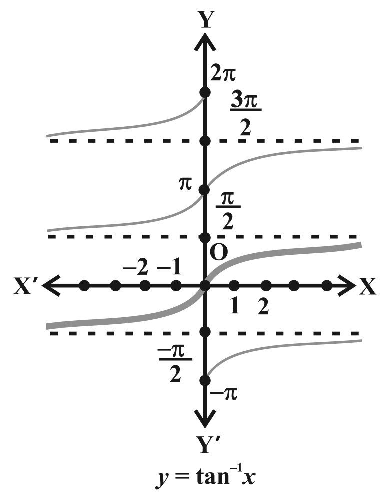


Fig 2.5 (ii)

We know that domain of the cot function (cotangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq n\pi, n \in \mathbf{Z}\}$ and range is **R**. It means that cotangent function is not defined for integral multiples of π. If we restrict the domain of cotangent function to (0, π), then it is bijective with and its range as **R**. In fact, cotangent function restricted to any of the intervals (–π, 0), (0, π), (π, 2π) etc., is bijective and its range is **R**. Thus $\cot^{-1}$ can be defined as a function whose domain is the **R** and range as any of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$ etc. These intervals give different branches of the function $\cot^{-1}$. The function with range $(0, \pi)$ is called the *principal value branch* of the function $\cot^{-1}$. We thus have

$$\cot^{-1} : \mathbf{R} \to (0, \pi)$$

The graphs of $y = \cot x$ and $y = \cot^{-1}x$ are given in Fig 2.6 (i), (ii).

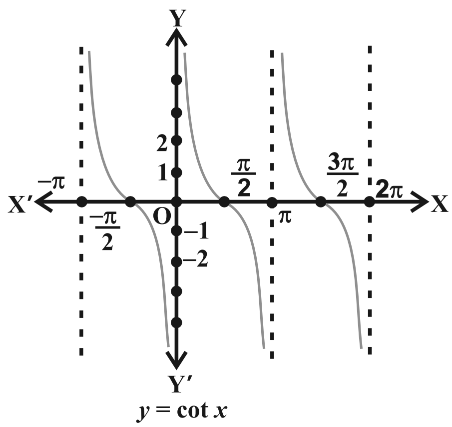


Fig 2.6 (i)

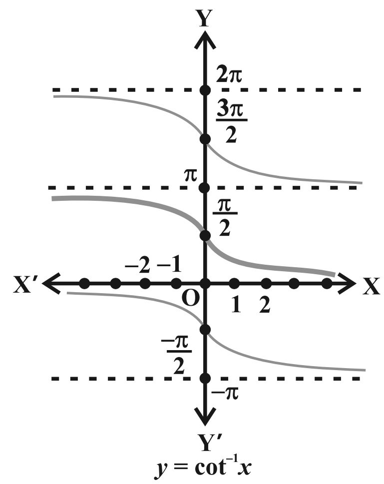


Fig 2.6 (ii)

The following table gives the inverse trigonometric function (principal value branches) along with their domains and ranges.

$\sin^{-1}$	:	$[-1, 1]$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\to$	$[0, \pi]$
cosec^{-1}	:	$\mathbf{R} - (-1,1)$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$\mathbf{R} - (-1, 1)$	$\to$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$	:	$\mathbf{R}$	$\to$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	$\mathbf{R}$	$\to$	$(0, \pi)$

Note

1. $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \dfrac{1}{\sin x}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

We now consider some examples:

Example 1 Find the principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$.

Solution Let $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = y$. Then, $\sin y = \dfrac{1}{\sqrt{2}}$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$. Therefore, principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ is $\dfrac{\pi}{4}$

Example 2 Find the principal value of $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

Solution Let $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = y$. Then,

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot\left(\frac{\pi}{3}\right) = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\left(\frac{2\pi}{3}\right)$$

We know that the range of principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\dfrac{2\pi}{3}\right) = \dfrac{-1}{\sqrt{3}}$. Hence, principal value of $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$ is $\dfrac{2\pi}{3}$

EXERCISE 2.1

Find the principal values of the following:

1. $\sin^{-1}\left(-\dfrac{1}{2}\right)$ **2.** $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ **3.** $\operatorname{cosec}^{-1}(2)$

4. $\tan^{-1}(-\sqrt{3})$ **5.** $\cos^{-1}\left(-\dfrac{1}{2}\right)$ **6.** $\tan^{-1}(-1)$

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **8.** $\cot^{-1}(\sqrt{3})$ **9.** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

10. $\text{cosec}^{-1}(-\sqrt{2})$

Find the values of the following:

11. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ **12.** $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π (B) $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

2.3 Properties of Inverse Trigonometric Functions

In this section, we shall prove some important properties of inverse trigonometric functions. It may be mentioned here that these results are valid within the principal value branches of the corresponding inverse trigonometric functions and wherever they are defined. Some results may not be valid for all values of the domains of inverse trigonometric functions. In fact, they will be valid only for some values of x for which inverse trigonometric functions are defined. We will not go into the details of these values of x in the domain as this discussion goes beyond the scope of this text book.

Let us recall that if $y = \sin^{-1}x$, then $x = \sin y$ and if $x = \sin y$, then $y = \sin^{-1}x$. This is equivalent to

$$\sin(\sin^{-1} x) = x,\ x \in [-1, 1] \text{ and } \sin^{-1}(\sin x) = x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Same is true for other five inverse trigonometric functions as well. We now prove some properties of inverse trigonometric functions.

1. (i) $\boldsymbol{\sin^{-1}\frac{1}{x} = \text{cosec}^{-1} x,\ x \ge 1 \text{ or } x \le -1}$

(ii) $\boldsymbol{\cos^{-1}\frac{1}{x} = \sec^{-1}x,\ x \ge 1 \text{ or } x \le -1}$

(iii) $\mathbf{\tan^{-1}\dfrac{1}{x} = \cot^{-1} x, x > 0}$

To prove the first result, we put $\text{cosec}^{-1} x = y$, i.e., $x = \text{cosec } y$

Therefore $\dfrac{1}{x} = \sin y$

Hence $\sin^{-1}\dfrac{1}{x} = y$

or $\sin^{-1}\dfrac{1}{x} = \text{cosec}^{-1} x$

Similarly, we can prove the other parts.

2. **(i)** $\mathbf{\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]}$

(ii) $\mathbf{\tan^{-1}(-x) = -\tan^{-1} x, x \in R}$

(iii) $\mathbf{\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x, |x| \geq 1}$

Let $\sin^{-1}(-x) = y$, i.e., $-x = \sin y$ so that $x = -\sin y$, i.e., $x = \sin(-y)$.

Hence $\sin^{-1} x = -y = -\sin^{-1}(-x)$

Therefore $\sin^{-1}(-x) = -\sin^{-1}x$

Similarly, we can prove the other parts.

3. **(i)** $\mathbf{\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]}$

(ii) $\mathbf{\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1}$

(iii) $\mathbf{\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R}$

Let $\cos^{-1}(-x) = y$ i.e., $-x = \cos y$ so that $x = -\cos y = \cos(\pi - y)$

Therefore $\cos^{-1} x = \pi - y = \pi - \cos^{-1}(-x)$

Hence $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly, we can prove the other parts.

4. **(i)** $\mathbf{\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}, x \in [-1, 1]}$

(ii) $\mathbf{\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}, x \in R}$

(iii) $\mathbf{\text{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}, |x| \geq 1}$

Let $\sin^{-1} x = y$. Then $x = \sin y = \cos\left(\dfrac{\pi}{2} - y\right)$

Therefore $\cos^{-1} x = \dfrac{\pi}{2} - y = \dfrac{\pi}{2} - \sin^{-1} x$

Hence $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

Similarly, we can prove the other parts.

5. (i) $\tan^{-1}x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}, xy < 1$

(ii) $\tan^{-1}x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}, xy > -1$

(iii) $2\tan^{-1}x = \tan^{-1} \dfrac{2x}{1-x^2}, |x| < 1$

Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$. Then $x = \tan \theta$, $y = \tan \phi$

Now $$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{x+y}{1-xy}$$

This gives $$\theta + \phi = \tan^{-1}\frac{x+y}{1-xy}$$

Hence $$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

In the above result, if we replace y by $-y$, we get the second result and by replacing y by x, we get the third result.

6. (i) $2\tan^{-1} x = \sin^{-1} \dfrac{2x}{1+x^2}, |x| \le 1$

(ii) $2\tan^{-1} x = \cos^{-1} \dfrac{1-x^2}{1+x^2}, x \ge 0$

(iii) $2 \tan^{-1} x = \tan^{-1} \dfrac{2x}{1-x^2}, -1 < x < 1$

Let $\tan^{-1} x = y$, then $x = \tan y$. Now

$$\sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} \frac{2\tan y}{1+\tan^2 y}$$
$$= \sin^{-1} (\sin 2y) = 2y = 2\tan^{-1} x$$

Also $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \cos^{-1}\dfrac{1-\tan^2 y}{1+\tan^2 y} = \cos^{-1}(\cos 2y) = 2y = 2\tan^{-1} x$

(iii) Can be worked out similarly.

We now consider some examples.

Example 3 Show that

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x, \ -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x, \ \dfrac{1}{\sqrt{2}} \le x \le 1$

Solution

(i) Let $x = \sin\theta$. Then $\sin^{-1} x = \theta$. We have

$$\begin{aligned}\sin^{-1}\left(2x\sqrt{1-x^2}\right) &= \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)\\ &= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta\\ &= 2\sin^{-1} x\end{aligned}$$

(ii) Take $x = \cos\theta$, then proceeding as above, we get, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x$

Example 4 Show that $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{2}{11} = \tan^{-1}\dfrac{3}{4}$

Solution By property 5 (i), we have

$$\text{L.H.S.} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}} = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4} = \text{R.H.S.}$$

Example 5 Express $\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right), \ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ in the simplest form.

Solution We write

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Alternatively,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right] = \tan^{-1}\left[\frac{\sin\left(\frac{\pi-2x}{2}\right)}{1-\cos\left(\frac{\pi-2x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi-2x}{4}\right)\cos\left(\frac{\pi-2x}{4}\right)}{2\sin^2\left(\frac{\pi-2x}{4}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{\pi-2x}{4}\right)\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\pi-2x}{4}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Example 6 Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $|x| > 1$ in the simplest form.

Solution Let $x = \sec\theta$, then $\sqrt{x^2-1} = \sqrt{\sec^2\theta-1} = \tan\theta$

Therefore, $\cot^{-1}\dfrac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1}x$, which is the simplest form.

Example 7 Prove that $\tan^{-1}x + \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$, $|x| < \dfrac{1}{\sqrt{3}}$

Solution Let $x = \tan\theta$. Then $\theta = \tan^{-1}x$. We have

$$\text{R.H.S.} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x = \tan^{-1}x + 2\tan^{-1}x$$

$$= \tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \text{L.H.S. (Why?)}$$

Example 8 Find the value of $\cos(\sec^{-1}x + \text{cosec}^{-1}x)$, $|x| \geq 1$

Solution We have $\cos(\sec^{-1}x + \text{cosec}^{-1}x) = \cos\left(\dfrac{\pi}{2}\right) = 0$

EXERCISE 2.2

Prove the following:

1. $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$, $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

2. $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$, $x \in \left[\dfrac{1}{2}, 1\right]$

3. $\tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$

4. $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$

Write the following functions in the simplest form:

5. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$, $x \neq 0$

6. $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$, $|x| > 1$

7. $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right)$, $x < \pi$

8. $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)$, $x < \pi$

9. $\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}}$, $|x| < a$

10. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$, $a > 0$; $\dfrac{-a}{\sqrt{3}} \le x \le \dfrac{a}{\sqrt{3}}$

Find the values of each of the following:

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right]$

12. $\cot(\tan^{-1}a + \cot^{-1}a)$

13. $\tan\dfrac{1}{2}\left[\sin^{-1}\dfrac{2x}{1+x^2}+\cos^{-1}\dfrac{1-y^2}{1+y^2}\right]$, $|x| < 1$, $y > 0$ and $xy < 1$

14. If $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x

15. If $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}$, then find the value of x

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$

17. $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

18. $\tan\left(\sin^{-1}\dfrac{3}{5}+\cot^{-1}\dfrac{3}{2}\right)$

19. $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ is equal to

(A) $\dfrac{7\pi}{6}$ (B) $\dfrac{5\pi}{6}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{6}$

20. $\sin\left(\dfrac{\pi}{3}-\sin^{-1}(-\dfrac{1}{2})\right)$ is equal to

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{4}$ (D) 1

21. $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$ is equal to

(A) π (B) $-\dfrac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Miscellaneous Examples

Example 9 Find the value of $\sin^{-1}(\sin\frac{3\pi}{5})$

Solution We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}(\sin\frac{3\pi}{5}) = \frac{3\pi}{5}$

But $\quad \frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1} x$

However $\quad \sin(\frac{3\pi}{5}) = \sin(\pi - \frac{3\pi}{5}) = \sin\frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore $\quad \sin^{-1}(\sin\frac{3\pi}{5}) = \sin^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$

Example 10 Show that $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Solution Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$

Therefore $\quad \sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$

Now $\quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$ (Why?)

and $\quad \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{289}} = \frac{15}{17}$

We have $\quad \cos(x-y) = \cos x \cos y + \sin x \sin y$

$$= \frac{4}{5}\times\frac{15}{17} + \frac{3}{5}\times\frac{8}{17} = \frac{84}{85}$$

Therefore $\quad x - y = \cos^{-1}\left(\frac{84}{85}\right)$

Hence $\quad \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Example 11 Show that $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Solution Let $\sin^{-1}\frac{12}{13}=x,\quad \cos^{-1}\frac{4}{5}=y, \tan^{-1}\frac{63}{16}=z$

Then $\sin x=\frac{12}{13},\quad \cos y=\frac{4}{5},\quad \tan z=\frac{63}{16}$

Therefore $\cos x=\frac{5}{13},\ \sin y=\frac{3}{5},\ \tan x=\frac{12}{5}$ and $\tan y=\frac{3}{4}$

We have $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}=\dfrac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}=-\frac{63}{16}$

Hence $\tan(x+y)=-\tan z$

i.e., $\tan(x+y)=\tan(-z)$ or $\tan(x+y)=\tan(\pi-z)$

Therefore $x+y=-z$ or $x+y=\pi-z$

Since x, y and z are positive, $x+y\neq -z$ (Why?)

Hence $x+y+z=\pi$ or $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Example 12 Simplify $\tan^{-1}\left[\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$, if $\frac{a}{b}\tan x>-1$

Solution We have,

$$\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]=\tan^{-1}\left[\frac{\dfrac{a\cos x-b\sin x}{b\cos x}}{\dfrac{b\cos x+a\sin x}{b\cos x}}\right]=\tan^{-1}\left[\frac{\dfrac{a}{b}-\tan x}{1+\dfrac{a}{b}\tan x}\right]$$

$$=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)=\tan^{-1}\frac{a}{b}-x$$

Example 13 Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

Solution We have $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

or $\tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right) = \frac{\pi}{4}$

i.e. $\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$

Therefore $\frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$

or $6x^2 + 5x - 1 = 0$ i.e., $(6x - 1)(x + 1) = 0$

which gives $x = \frac{1}{6}$ or $x = -1$.

Since $x = -1$ does not satisfy the equation, as the L.H.S. of the equation becomes negative, $x = \frac{1}{6}$ is the only solution of the given equation.

Miscellaneous Exercise on Chapter 2

Find the value of the following:

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Prove that

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

4. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

5. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

6. $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

7. $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

8. $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Prove that

9. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0, 1]$

10. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$

11. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$, $-\frac{1}{\sqrt{2}} \le x \le 1$ [Hint: Put $x = \cos 2\theta$]

12. $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solve the following equations:

13. $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\mathrm{cosec}\,x)$

14. $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x > 0)$

15. $\sin(\tan^{-1}x), |x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

16. $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

17. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{-3\pi}{4}$

Summary

- The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	**Domain**	**Range (Principal Value Branches)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \text{cosec}^{-1} x$	$\mathbf{R} - (-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$\mathbf{R} - (-1, 1)$	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \tan^{-1} x$	$\mathbf{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	$\mathbf{R}$	$(0, \pi)$

- $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
- The value of an inverse trigonometric functions which lies in its principal value branch is called the *principal value* of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = \sin^{-1} x \Rightarrow x = \sin y$
- $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin (\sin^{-1} x) = x$
- $\sin^{-1} (\sin x) = x$
- $\sin^{-1} \frac{1}{x} = \text{cosec}^{-1} x$
- $\cos^{-1} (-x) = \pi - \cos^{-1} x$
- $\cos^{-1} \frac{1}{x} = \sec^{-1}x$
- $\cot^{-1} (-x) = \pi - \cot^{-1} x$
- $\tan^{-1} \frac{1}{x} = \cot^{-1} x$
- $\sec^{-1} (-x) = \pi - \sec^{-1} x$

- $\sin^{-1}(-x) = -\sin^{-1} x$
- $\tan^{-1}(-x) = -\tan^{-1} x$
- $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$
- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$
- $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$
- $\text{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1} \dfrac{x+y}{1-xy}$
- $2\tan^{-1}x = \tan^{-1} \dfrac{2x}{1-x^2}$
- $\tan^{-1}x - \tan^{-1}y = \tan^{-1} \dfrac{x-y}{1+xy}$
- $2\tan^{-1} x = \sin^{-1} \dfrac{2x}{1+x^2} = \cos^{-1} \dfrac{1-x^2}{1+x^2}$

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476A.D.), Brahmagupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) got important results of trigonometry. All this knowledge went from India to Arabia and then from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents one of the main contribution of the *siddhantas* (Sanskrit astronomical works) to mathematics.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., were given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The name of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\mathrm{H}}{\mathrm{S}} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

Chapter 3

MATRICES

❖ *The essence of Mathematics lies in its freedom. — CANTOR* ❖

3.1 Introduction

The knowledge of matrices is necessary in various branches of mathematics. Matrices are one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evolution of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Matrices are also used in cryptography. This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

In this chapter, we shall find it interesting to become acquainted with the fundamentals of matrix and matrix algebra.

3.2 Matrix

Suppose we wish to express the information that Radha has 15 notebooks. We may express it as [15] with the understanding that the number inside [ ] is the number of notebooks that Radha has. Now, if we have to express that Radha has 15 notebooks and 6 pens. We may express it as [15 6] with the understanding that first number inside [ ] is the number of notebooks while the other one is the number of pens possessed by Radha. Let us now suppose that we wish to express the information of possession of notebooks and pens by Radha and her two friends Fauzia and Simran which is as follows:

Radha	has	15	notebooks	and	6 pens,
Fauzia	has	10	notebooks	and	2 pens,
Simran	has	13	notebooks	and	5 pens.

Now this could be arranged in the tabular form as follows:

	Notebooks	**Pens**
Radha	15	6
Fauzia	10	2
Simran	13	5

and this can be expressed as

$$\begin{array}{cl} \begin{bmatrix} 15 & 6 \\ 10 & 2 \\ 13 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \\ \leftarrow \text{Third row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{First} & \text{Second} \\ \text{Column} & \text{Column} \end{array} & \end{array}$$

or

	Radha	**Fauzia**	**Simran**
Notebooks	15	10	13
Pens	6	2	5

which can be expressed as:

$$\begin{array}{cl} \begin{bmatrix} 15 & 10 & 13 \\ 6 & 2 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{First} & \text{Second} & \text{Third} \\ \text{Column} & \text{Column} & \text{Column} \end{array} & \end{array}$$

In the first arrangement the entries in the first column represent the number of note books possessed by Radha, Fauzia and Simran, respectively and the entries in the second column represent the number of pens possessed by Radha, Fauzia and Simran, respectively. Similarly, in the second arrangement, the entries in the first row represent the number of notebooks possessed by Radha, Fauzia and Simran, respectively. The entries in the second row represent the number of pens possessed by Radha, Fauzia and Simran, respectively. An arrangement or display of the above kind is called a *matrix*. Formally, we define matrix as:

Definition 1 A *matrix* is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

We denote matrices by capital letters. The following are some examples of matrices:

$$A = \begin{bmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 2+i & 3 & -\frac{1}{2} \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & \frac{5}{7} \end{bmatrix}, \; C = \begin{bmatrix} 1+x & x^3 & 3 \\ \cos x & \sin x + 2 & \tan x \end{bmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, ***rows*** of the matrix and the vertical lines of elements are said to constitute, ***columns*** of the matrix. Thus A has 3 rows and 2 columns, B has 3 rows and 3 columns while C has 2 rows and 3 columns.

3.2.1 *Order of a matrix*

A matrix having m rows and n columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). So referring to the above examples of matrices, we have A as 3×2 matrix, B as 3×3 matrix and C as 2×3 matrix. We observe that A has $3 \times 2 = 6$ elements, B and C have 9 and 6 elements, respectively.

In general, an $m \times n$ matrix has the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

or $A = [a_{ij}]_{m \times n}$, $1 \le i \le m$, $1 \le j \le n$ $\quad i, j \in$ N

Thus the i^{th} row consists of the elements $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$, while the j^{th} column consists of the elements $a_{1j}, a_{2j}, a_{3j}, \dots, a_{mj}$,

In general a_{ij}, is an element lying in the i^{th} row and j^{th} column. We can also call it as the $(i, j)^{th}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn.

Note In this chapter

1. We shall follow the notation, namely A = $[a_{ij}]_{m \times n}$ to indicate that A is a matrix of order $m \times n$.
2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

We can also represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or $[x, y]$). For example point P(0, 1) as a matrix representation may be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0\ 1].$$

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (–1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} \end{array} \quad \text{or} \quad Y = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used as representation of vertices of geometrical figures in a plane.

Now, let us consider some examples.

Example 1 Consider the following information regarding the number of men and women workers in three factories I, II and III

	Men workers	**Women workers**
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

Solution The information is represented in the form of a 3×2 matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 2 If a matrix has 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1×8, 8×1, 4×2, 2×4

Example 3 Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution In general a 3×2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Now $\quad a_{ij} = \frac{1}{2}|i - 3j|$, $i = 1, 2, 3$ and $j = 1, 2$.

Therefore

$$a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1 \qquad a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$$

$$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2} \qquad a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$$

$$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0 \qquad a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$$

Hence the required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$.

3.3 Types of Matrices

In this section, we shall discuss different types of matrices.

(i) **Column matrix**

A matrix is said to be a *column matrix* if it has only one column.

For example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$ is a column matrix of order 4 × 1.

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(ii) **Row matrix**

A matrix is said to be a *row matrix* if it has only one row.

For example, $B = \begin{bmatrix} -\frac{1}{2} & \sqrt{5} & 2 & 3 \end{bmatrix}_{1\times 4}$ is a row matrix.

In general, $B = [b_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.

(iii) **Square matrix**

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.

For example $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

Note If $A = [a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, ..., a_{nn}$ are said to constitute the *diagonal*, of the matrix A. Thus, if $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$.

Then the elements of the diagonal of A are 1, 4, 6.

(iv) **Diagonal matrix**

A square matrix B = $[b_{ij}]_{m \times m}$ is said to be a *diagonal matrix* if all its non diagonal elements are zero, that is a matrix B = $[b_{ij}]_{m \times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

For example, $A = [4]$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, are diagonal matrices of order 1, 2, 3, respectively.

(v) **Scalar matrix**

A diagonal matrix is said to be a *scalar matrix* if its diagonal elements are equal, that is, a square matrix B = $[b_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$b_{ij} = 0, \quad \text{when } i \neq j$$
$$b_{ij} = k, \quad \text{when } i = j, \text{ for some constant } k.$$

For example

$$A = [3], \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

are scalar matrices of order 1, 2 and 3, respectively.

(vi) **Identity matrix**

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an *identity matrix*. In other words, the square matrix A = $[a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We denote the identity matrix of order n by I_n. When order is clear from the context, we simply write it as I.

For example [1], $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3, respectively.

Observe that a scalar matrix is an identity matrix when $k = 1$. But every identity matrix is clearly a scalar matrix.

(vii) **Zero matrix**

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero.

For example, [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, [0, 0] are all zero matrices. We denote zero matrix by O. Its order will be clear from the context.

3.3.1 *Equality of matrices*

Definition 2 Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if

(i) they are of the same order

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices. Symbolically, if two matrices A and B are equal, we write A = B.

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then $x = -1.5$, $y = 0$, $z = 2$, $a = \sqrt{6}$, $b = 3$, $c = 2$

Example 4 If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Find the values of a, b, c, x, y and z.

Solution As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$$x + 3 = 0, \qquad z + 4 = 6, \qquad 2y - 7 = 3y - 2$$

$$a - 1 = -3, \qquad 0 = 2c + 2 \qquad b - 3 = 2b + 4,$$

Simplifying, we get

$$a = -2,\ b = -7,\ c = -1,\ x = -3,\ y = -5,\ z = 2$$

Example 5 Find the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution By equality of two matrices, equating the corresponding elements, we get

$$2a + b = 4 \qquad 5c - d = 11$$
$$a - 2b = -3 \qquad 4c + 3d = 24$$

Solving these equations, we get

$$a = 1,\ b = 2,\ c = 3 \text{ and } d = 4$$

EXERCISE 3.1

1. In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

   (i) The order of the matrix, (ii) The number of elements,

   (iii) Write the elements a_{13}, a_{21}, a_{33}, a_{24}, a_{23}.

2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

4. Construct a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by:

   (i) $a_{ij} = \frac{(i+j)^2}{2}$ (ii) $a_{ij} = \frac{i}{j}$ (iii) $a_{ij} = \frac{(i+2j)^2}{2}$

5. Construct a 3×4 matrix, whose elements are given by:

   (i) $a_{ij} = \frac{1}{2}|-3i + j|$ (ii) $a_{ij} = 2i - j$

6. Find the values of x, y and z from the following equations:

   (i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

7. Find the value of a, b, c and d from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

8. A = $[a_{ij}]_{m \times n}$ is a square matrix, if

(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these

9. Which of the given values of x and y make the following pair of matrices equal

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(A) $x = \frac{-1}{3},\quad y = 7$ (B) Not possible to find

(C) $y = 7,\quad x = \frac{-2}{3}$ (D) $x = \frac{-1}{3},\quad y = \frac{-2}{3}$

10. The number of all possible matrices of order 3 × 3 with each entry 0 or 1 is:

(A) 27 (B) 18 (C) 81 (D) 512

3.4 Operations on Matrices

In this section, we shall introduce certain operations on matrices, namely, addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

3.4.1 *Addition of matrices*

Suppose Fatima has two factories at places A and B. Each factory produces sport shoes for boys and girls in three different price categories labelled 1, 2 and 3. The quantities produced by each factory are represented as matrices given below:

$$\begin{array}{c} \text{Factory at A} \\ \begin{array}{cc} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array} \end{array} \qquad \begin{array}{c} \text{Factory at B} \\ \begin{array}{cc} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 90 & 50 \\ 70 & 55 \\ 75 & 75 \end{bmatrix} \end{array} \end{array}$$

Suppose Fatima wants to know the total production of sport shoes in each price category. Then the total production

In category 1 : for boys (80 + 90), for girls (60 + 50)

In category 2 : for boys (75 + 70), for girls (65 + 55)

In category 3 : for boys (90 + 75), for girls (85 + 75)

This can be represented in the matrix form as $\begin{bmatrix} 80+90 & 60+50 \\ 75+70 & 65+55 \\ 90+75 & 85+75 \end{bmatrix}$.

This new matrix is the **sum** of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is a 2×3 matrix and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ is another 2×3 matrix. Then, we define $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$.

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$. Then, the sum of the two matrices A and B is *defined* as a matrix $C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, for all possible values of i and j.

Example 6 Given $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$, find A + B

Since A, B are of the same order 2×3. Therefore, addition of A and B is defined and is given by

$$A+B = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 1-1 \\ 2-2 & 3+3 & 0+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$$

Note

1. We emphasise that if A and B are not of the same order, then A + B is not defined. For example if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then $A + B$ is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

3.4.2 *Multiplication of a matrix by a scalar*

Now suppose that Fatima has doubled the production at a factory A in all categories (refer to 3.4.1).

Previously quantities (in standard units) produced by factory A were

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array}$$

Revised quantities produced by factory A are as given below:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 2\times 80 & 2\times 60 \\ 2\times 75 & 2\times 65 \\ 2\times 90 & 2\times 85 \end{bmatrix} \end{array}$$

This can be represented in the matrix form as $\begin{bmatrix} 160 & 120 \\ 150 & 130 \\ 180 & 170 \end{bmatrix}$. We observe that the new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, we may define *multiplication of a matrix* by a scalar as follows: if $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

In other words, $kA = k\,[a_{ij}]_{m \times n} = [k\,(a_{ij})]_{m \times n}$, that is, $(i, j)^{\text{th}}$ element of kA is ka_{ij} for all possible values of i and j.

For example, if $\quad A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3A = 3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is denoted by $-A$. We define $-A = (-1)\,A$.

For example, let $$A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}$$, then – A is given by

$$-A = (-1)A = (-1)\begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

Difference of matrices If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$, then difference A – B is defined as a matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j. In other words, D = A – B = A + (–1) B, that is sum of the matrix A and the matrix – B.

Example 7 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find 2A – B.

Solution We have

$$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

3.4.3 *Properties of matrix addition*

The addition of matrices satisfy the following properties:

(i) **Commutative Law** If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then A + B = B + A.

Now $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$
$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)
$= ([b_{ij}] + [a_{ij}]) = B + A$

(ii) **Associative Law** For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, (A + B) + C = A + (B + C).

Now $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$
$= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}]$
$= [a_{ij} + (b_{ij} + c_{ij})]$ (Why?)
$= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C)$

(iii) **Existence of additive identity** Let A = $[a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

(iv) **The existence of additive inverse** Let A = $[a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as – A = $[– a_{ij}]_{m \times n}$ such that A + (– A) = (– A) + A= O. So – A is the additive inverse of A or negative of A.

3.4.4 *Properties of scalar multiplication of a matrix*

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say $m \times n$, and k and l are scalars, then

(i) $k(\text{A} + \text{B}) = k\,\text{A} + k\text{B}$, (ii) $(k + l)\text{A} = k\,\text{A} + l\,\text{A}$

(ii) $k\,(\text{A} + \text{B}) = k\,([a_{ij}] + [b_{ij}])$

$$= k\,[a_{ij} + b_{ij}] = [k\,(a_{ij} + b_{ij})] = [(k\,a_{ij}) + (k\,b_{ij})]$$

$$= [k\,a_{ij}] + [k\,b_{ij}] = k\,[a_{ij}] + k\,[b_{ij}] = k\text{A} + k\text{B}$$

(iii) $(k + l)\,\text{A} = (k + l)\,[a_{ij}]$

$$= [(k + l)\,a_{ij}] + [k\,a_{ij}] + [l\,a_{ij}] = k\,[a_{ij}] + l\,[a_{ij}] = k\,\text{A} + l\,\text{A}$$

Example 8 If $\text{A} = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B.

Solution We have 2A + 3X = 5B

or	2A + 3X – 2A = 5B – 2A	
or	2A – 2A + 3X = 5B – 2A	(Matrix addition is commutative)
or	O + 3X = 5B – 2A	(– 2A is the additive inverse of 2A)
or	3X = 5B – 2A	(O is the additive identity)

or $$\text{X} = \frac{1}{3}\,(5\text{B} - 2\text{A})$$

or $$\text{X} = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Example 9 Find X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X + Y) + (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $$(X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$$

or $$X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

Also $$(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

or $$(X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

or $$Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

Example 10 Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

or $2x + 3 = 7$ and $2y - 4 = 14$ (Why?)

or $2x = 7 - 3$ and $2y = 18$

or $x = \frac{4}{2}$ and $y = \frac{18}{2}$

i.e. $x = 2$ and $y = 9$.

Example 11 Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in Rupees)

$$A = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

October Sales (in Rupees)

$$B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(i) Find the combined sales in September and October for each farmer in each variety.

(ii) Find the decrease in sales from September to October.

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Solution

(i) Combined sales in September and October for each farmer in each variety is given by

$$A + B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 15,000 & 30,000 & 36,000 \\ 70,000 & 40,000 & 20,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(ii) Change in sales from September to October is given by

$$A - B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 5000 & 10,000 & 24,000 \\ 30,000 & 20,000 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(iii) 2% of B $= \dfrac{2}{100} \times B = 0.02 \times B$

$$= 0.02 \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 100 & 200 & 120 \\ 400 & 200 & 200 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

Thus, in October Ramkishan receives Rs 100, Rs 200 and Rs 120 as profit in the sale of each variety of rice, respectively, and Grucharan Singh receives profit of Rs 400, Rs 200 and Rs 200 in the sale of each variety of rice, respectively.

3.4.5 *Multiplication of matrices*

Suppose Meera and Nadeem are two friends. Meera wants to buy 2 pens and 5 story books, while Nadeem needs 8 pens and 10 story books. They both go to a shop to enquire about the rates which are quoted as follows:

Pen – Rs 5 each, story book – Rs 50 each.

How much money does each need to spend? Clearly, Meera needs Rs $(5 \times 2 + 50 \times 5)$ that is Rs 260, while Nadeem needs $(8 \times 5 + 50 \times 10)$ Rs, that is Rs 540. In terms of matrix representation, we can write the above information as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 50 \end{bmatrix} \qquad \begin{bmatrix} 5\times 2 + 5\times 50 \\ 8\times 5 + 10\times 50 \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$$

Suppose that they enquire about the rates from another shop, quoted as follows:

pen – Rs 4 each, story book – Rs 40 each.

Now, the money required by Meera and Nadeem to make purchases will be respectively Rs $(4 \times 2 + 40 \times 5)$ = Rs 208 and Rs $(8 \times 4 + 10 \times 40)$ = Rs 432

Again, the above information can be represented as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 40 \end{bmatrix} \qquad \begin{bmatrix} 4\times 2+40\times 5 \\ 8\times 4+10\times 40 \end{bmatrix} = \begin{bmatrix} 208 \\ 432 \end{bmatrix}$$

Now, the information in both the cases can be combined and expressed in terms of matrices as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix} \qquad \begin{bmatrix} 5\times 2+5\times 50 & 4\times 2+40\times 5 \\ 8\times 5+10\times 50 & 8\times 4+10\times 40 \end{bmatrix}$$

$$= \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$$

The above is an example of multiplication of matrices. We observe that, for multiplication of two matrices A and B, the number of columns in A should be equal to the number of rows in B. Furthermore for getting the elements of the product matrix, we take rows of A and columns of B, multiply them element-wise and take the sum. Formally, we define multiplication of matrices as follows:

The *product* of two matrices A and B is *defined* if the number of columns of A is equal to the number of rows of B. Let A = $[a_{ij}]$ be an $m \times n$ matrix and B = $[b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{\text{th}}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them elementwise and take the sum of all these products. In other words, if A = $[a_{ij}]_{m \times n}$, B = $[b_{jk}]_{n \times p}$, then the i^{th} row of A is $[a_{i1}\ a_{i2}\ \ldots\ a_{in}]$ and the k^{th} column of

B is $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$, then $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \ldots + a_{in} b_{nk} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

The matrix C = $[c_{ik}]_{m \times p}$ is the product of A and B.

For example, if $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$, then the product CD is defined and is given by $CD = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$. This is a 2×2 matrix in which each entry is the sum of the products across some row of C with the corresponding entries down some column of D. These four computations are

Entry in first row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} (1)(2) + (-1)(-1) + (2)(5) & ? \\ ? & ? \end{bmatrix}$

Entry in first row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & (1)(7) + (-1)(1) + 2(-4) \\ ? & ? \end{bmatrix}$

Entry in second row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 0(2) + 3(-1) + 4(5) & ? \end{bmatrix}$

Entry in second row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 17 & 0(7) + 3(1) + 4(-4) \end{bmatrix}$

Thus $CD = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

Example 12 Find AB, if $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$.

Solution The matrix A has 2 columns which is equal to the number of rows of B. Hence AB is defined. Now

$$AB = \begin{bmatrix} 6(2) + 9(7) & 6(6) + 9(9) & 6(0) + 9(8) \\ 2(2) + 3(7) & 2(6) + 3(9) & 2(0) + 3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 63 & 36 + 81 & 0 + 72 \\ 4 + 21 & 12 + 27 & 0 + 24 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Remark If AB is defined, then BA need not be defined. In the above example, AB is defined but BA is not defined because B has 3 column while A has only 2 (and not 3) rows. If A, B are, respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined **if and only if** $n = k$ and $l = m$. In particular, if both A and B are square matrices of the same order, then both AB and BA are defined.

Non-commutativity of multiplication of matrices

Now, we shall see by an example that even if AB and BA are both defined, it is not necessary that AB = BA.

Example 13 If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB, BA. Show that AB ≠ BA.

Solution Since A is a 2×3 matrix and B is 3×2 matrix. Hence AB and BA are both defined and are matrices of order 2×2 and 3×3, respectively. Note that

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly AB ≠ BA

In the above example both AB and BA are of different order and so AB ≠ BA. But one may think that perhaps AB and BA could be the same if they were of the same order. But it is not so, here we give an example to show that even if AB and BA are of same order they may not be same.

Example 14 If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

and $$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ Clearly AB ≠ BA.

Thus matrix multiplication is not commutative.

Note This does not mean that AB ≠ BA for every pair of matrices A, B for which AB and BA, are defined. For instance,

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$

Observe that multiplication of diagonal matrices of same order will be commutative.

Zero matrix as the product of two non zero matrices

We know that, for real numbers a, b if $ab = 0$, then either $a = 0$ or $b = 0$. This need not be true for matrices, we will observe this through an example.

Example 15 Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Solution We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

3.4.6 *Properties of multiplication of matrices*

The multiplication of matrices possesses the following properties, which we state without proof.

1. **The associative law** For any three matrices A, B and C. We have

   (AB) C = A (BC), whenever both sides of the equality are defined.

2. **The distributive law** For three matrices A, B and C.

   (i) A (B+C) = AB + AC

   (ii) (A+B) C = AC + BC, whenever both sides of equality are defined.

3. **The existence of multiplicative identity** For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

Now, we shall verify these properties by examples.

Example 16 If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, find A(BC), (AB)C and show that (AB)C = A(BC).

Solution We have $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3+0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$

$$(AB)\ (C) = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

Now $\quad BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

Therefore $\quad A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-4 & -9+4-22 & -3-2+16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}.\ \text{Clearly, } (AB)\ C = A\ (BC)$$

Example 17 If $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Calculate AC, BC and (A + B)C. Also, verify that (A + B)C = AC + BC

Solution Now, $A + B = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}$

So $$(A + B)\,C = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-14+24 \\ -10+0+30 \\ 16+12+0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

Further $$AC = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-12+21 \\ -12+0+24 \\ 14+16+0 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$$

and $$BC = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-2+3 \\ 2+0+6 \\ 2-4+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}$$

So $$AC + BC = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

Clearly, $(A + B)\,C = AC + BC$

Example 18 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40\,I = O$

Solution We have $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

So $$A^3 = A\,A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now

$$A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 19 In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{array}{c} \textit{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{array}{c} \begin{array}{ccc} \text{Telephone} & \text{Housecall} & \text{Letter} \end{array} \\ \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \to X \\ \to Y \end{array}$$. Find the total amount spent by the group in the two cities X and Y.

Solution We have

$$BA = \begin{bmatrix} 40{,}000 + 50{,}000 + 250{,}000 \\ 120{,}000 + 100{,}000 + 500{,}000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

$$= \begin{bmatrix} 340{,}000 \\ 720{,}000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

So the total amount spent by the group in the two cities is 340,000 paise and 720,000 paise, i.e., Rs 3400 and Rs 7200, respectively.

EXERCISE 3.2

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

(i) A + B (ii) A – B (iii) 3A – C

(iv) AB (v) BA

2. Compute the following:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

3. Compute the indicated products.

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

4. If $A=\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{bmatrix}, B=\begin{bmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 2\\ 1 & -2 & 3\end{bmatrix}$, then compute (A+B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

5. If $A=\begin{bmatrix}\frac{2}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3}\\ \frac{7}{3} & 2 & \frac{2}{3}\end{bmatrix}$ and $B=\begin{bmatrix}\frac{2}{5} & \frac{3}{5} & 1\\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5}\\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5}\end{bmatrix}$, then compute 3A – 5B.

6. Simplify $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$

7. Find X and Y, if

(i) $X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

(ii) $2X+3Y=\begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$ and $3X+2Y=\begin{bmatrix}2 & -2\\ -1 & 5\end{bmatrix}$

8. Find X, if $Y=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$ and $2X+Y=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$

9. Find x and y, if $2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

10. Solve the equation for x, y, z and t, if $2\begin{bmatrix}x & z\\ y & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\ 4 & 6\end{bmatrix}$

11. If $x\begin{bmatrix}2\\ 3\end{bmatrix}+y\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}10\\ 5\end{bmatrix}$, find the values of x and y.

12. Given $3\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2w\end{bmatrix}+\begin{bmatrix}4 & x+y\\ z+w & 3\end{bmatrix}$, find the values of x, y, z and w.

13. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x)\,F(y) = F(x + y)$.

14. Show that

(i) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

15. Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

16. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$

17. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$

18. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

19. A trust fund has Rs 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

(a) Rs 1800 (b) Rs 2000

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs 80, Rs 60 and Rs 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

22. If $n = p$, then the order of the matrix 7X – 5Z is:

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

3.5. Transpose of a Matrix

In this section, we shall learn about transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

Definition 3 If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose of the matrix A is denoted by A′ or (A^T). In other words, if $A = [a_{ij}]_{m \times n}$, then $A' = [a_{ji}]_{n \times m}$. For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3 \times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2 \times 3}$$

3.5.1 Properties of transpose of the matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any matrices A and B of suitable orders, we have

(i) $(A')' = A$, (ii) $(kA)' = kA'$ (where k is any constant)

(iii) $(A + B)' = A' + B'$ (iv) $(A B)' = B' A'$

Example 20 If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, verify that

(i) $(A')' = A$, (ii) $(A + B)' = A' + B'$,

(iii) $(kB)' = kB'$, where k is any constant.

Solution

(i) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$$

Thus $(A')' = A$

(ii) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix},\ B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

Therefore $$(A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Now $$A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}, B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix},$$

So $$A' + B' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Thus $(A + B)' = A' + B'$

(iii) We have

$$kB = k\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$$

Then $$(kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$$

Thus $(kB)' = kB'$

Example 21 If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$$

then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2\ 4\ 5],\ B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

3.6 Symmetric and Skew Symmetric Matrices

Definition 4 A square matrix $A = [a_{ij}]$ is said to be *symmetric* if $A' = A$, that is, $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

For example $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ is a symmetric matrix as $A' = A$

Definition 5 A square matrix $A = [a_{ij}]$ is said to be *skew symmetric* matrix if $A' = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j. Now, if we put $i = j$, we have $a_{ii} = -a_{ii}$. Therefore $2a_{ii} = 0$ or $a_{ii} = 0$ for all i's.

This means that all the diagonal elements of a skew symmetric matrix are zero.

For example, the matrix $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$ is a skew symmetric matrix as $B' = -B$

Now, we are going to prove some results of symmetric and skew-symmetric matrices.

Theorem 1 For any square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Proof Let $B = A + A'$, then

$$
\begin{aligned}
B' &= (A + A')' \\
&= A' + (A')' \text{ (as } (A + B)' = A' + B') \\
&= A' + A \text{ (as } (A')' = A) \\
&= A + A' \text{ (as } A + B = B + A) \\
&= B
\end{aligned}
$$

Therefore $B = A + A'$ is a symmetric matrix

Now let $C = A - A'$

$$
\begin{aligned}
C' &= (A - A')' = A' - (A')' \quad \text{(Why?)} \\
&= A' - A \quad \text{(Why?)} \\
&= -(A - A') = -C
\end{aligned}
$$

Therefore $C = A - A'$ is a skew symmetric matrix.

Theorem 2 Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Proof Let A be a square matrix, then we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

From the Theorem 1, we know that $(A + A')$ is a symmetric matrix and $(A - A')$ is a skew symmetric matrix. Since for any matrix A, $(kA)' = kA'$, it follows that $\frac{1}{2}(A + A')$ is symmetric matrix and $\frac{1}{2}(A - A')$ is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Example 22 Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution Here

$$B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

Let $$P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

Now $$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, let $$Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then $$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus $\quad Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now
$$P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

EXERCISE 3.3

1. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$, (ii) $(A - B)' = A' - B'$

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$ (ii) $(A - B)' = A' - B'$

4. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$

5. For the matrices A and B, verify that $(AB)' = B'A'$, where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

6. If (i) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that A′ A = I

(ii) If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then verify that A′ A = I

7. (i) Show that the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.

(ii) Show that the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix.

8. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(i) (A + A′) is a symmetric matrix

(ii) (A – A′) is a skew symmetric matrix

9. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Choose the correct answer in the Exercises 11 and 12.

11. If A, B are symmetric matrices of same order, then AB – BA is a

(A) Skew symmetric matrix (B) Symmetric matrix

(C) Zero matrix (D) Identity matrix

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then $A + A' = I$, if the value of α is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$

(C) π (D) $\dfrac{3\pi}{2}$

3.7 Elementary Operation (Transformation) of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as *elementary operations* or *transformations*.

(i) *The interchange of any two rows or two columns*. Symbolically the interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example, applying $R_1 \leftrightarrow R_2$ to $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \end{bmatrix}$, we get $\begin{bmatrix} -1 & \sqrt{3} & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}$.

(ii) *The multiplication of the elements of any row or column by a non zero number*. Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \to kR_i$.

The corresponding column operation is denoted by $C_i \to kC_i$

For example, applying $C_3 \to \frac{1}{7}C_3$, to $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 & \frac{1}{7} \\ -1 & \sqrt{3} & \frac{1}{7} \end{bmatrix}$

(iii) *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.*

Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted by $R_i \to R_i + kR_j$.

The corresponding column operation is denoted by $C_i \to C_i + kC_j$.

For example, applying $R_2 \to R_2 - 2R_1$, to $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$.

3.8 Invertible Matrices

Definition 6 If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now
$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Note

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
2. If B is the inverse of A, then A is also the inverse of B.

Theorem 3 (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

Proof Let $A = [a_{ij}]$ be a square matrix of order m. If possible, let B and C be two inverses of A. We shall show that B = C.

Since B is the inverse of A

$$AB = BA = I \quad \text{... (1)}$$

Since C is also the inverse of A

$$AC = CA = I \quad \text{... (2)}$$

Thus
$$B = BI = B\,(AC) = (BA)\,C = IC = C$$

Theorem 4 If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof From the definition of inverse of a matrix, we have

$$(AB)\ (AB)^{-1} = 1$$

or $\quad A^{-1}\ (AB)\ (AB)^{-1} = A^{-1}I \qquad$ (Pre multiplying both sides by A^{-1})

or $\quad (A^{-1}A)\ B\ (AB)^{-1} = A^{-1} \qquad$ (Since $A^{-1}\ I = A^{-1}$)

or $\quad IB\ (AB)^{-1} = A^{-1}$

or $\quad B\ (AB)^{-1} = A^{-1}$

or $\quad B^{-1}\ B\ (AB)^{-1} = B^{-1}\ A^{-1}$

or $\quad I\ (AB)^{-1} = B^{-1}\ A^{-1}$

Hence $\quad (AB)^{-1} = B^{-1}\ A^{-1}$

3.8.1 *Inverse of a matrix by elementary operations*

Let X, A and B be matrices of, the same order such that X = AB. In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

Similarly, in order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply, these operations simultaneously on X and on the second matrix B of the product AB on RHS.

In view of the above discussion, we conclude that if A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations, write A = IA and apply a sequence of row operation on A = IA till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then, write A = AI and apply a sequence of column operations on A = AI till we get, I = AB.

Remark In case, after applying one or more elementary row (column) operations on A = IA (A = AI), if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

Example 23 By using elementary operations, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Solution In order to use elementary row operations we may write A = IA.

or $\quad \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$, then $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$ (applying $R_2 \to R_2 - 2R_1$)

or $$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A \quad \left(\text{applying } R_2 \to -\frac{1}{5} R_2\right)$$

or $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A \quad (\text{applying } R_1 \to R_1 - 2R_2)$$

Thus $$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Alternatively, in order to use elementary column operations, we write A = AI, i.e.,

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1$, we get

$$\begin{bmatrix} 1 & 0 \\ 2 & -5 \end{bmatrix} = A \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Now applying $C_2 \to -\frac{1}{5} C_2$, we have

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & \frac{-1}{5} \end{bmatrix}$$

Finally, applying $C_1 \to C_1 - 2C_2$, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Hence $$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Example 24 Obtain the inverse of the following matrix using elementary operations

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

Solution Write A = I A, i.e., $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{applying } R_1 \leftrightarrow R_2)$$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_3 \to R_3 - 3R_1)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_1 \to R_1 - 2R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_3 \to R_3 + 5R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{applying } R_3 \to \frac{1}{2} R_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{applying } R_1 \to R_1 + R_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{applying } R_2 \to R_2 - 2R_3)$$

Hence $$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Alternatively, write A = AI, i.e.,

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_1 \leftrightarrow C_2)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_3 \to C_3 - 2C_1)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_3 \to C_3 + C_2)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad (C_3 \to \frac{1}{2} C_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} -2 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (C_1 \to C_1 - 2C_2)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -4 & 0 & -1 \\ \frac{5}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad (C_1 \to C_1 + 5C_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} \quad (C_2 \to C_2 - 3C_3)$$

Hence $$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Example 25 Find P^{-1}, if it exists, given $P = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$.

Solution We have P = I P, i.e., $\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P$.

or $$\begin{bmatrix} 1 & \frac{-1}{5} \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} P \quad \left(\text{applying } R_1 \to \frac{1}{10} R_1\right)$$

or $$\begin{bmatrix} 1 & \frac{-1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} P \text{ (applying } R_2 \to R_2 + 5R_1\text{)}$$

We have all zeros in the second row of the left hand side matrix of the above equation. Therefore, P^{-1} does not exist.

EXERCISE 3.4

Using elementary transformations, find the inverse of each of the matrices, if it exists in Exercises 1 to 17.

1. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ **8.** $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$ **11.** $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$ **12.** $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. **15.** $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$ **17.** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

18. Matrices A and B will be inverse of each other only if

(A) AB = BA (B) AB = BA = 0

(C) AB = 0, BA = I (D) AB = BA = I

Miscellaneous Examples

Example 26 If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$.

Solution We shall prove the result by using principle of mathematical induction.

We have $\quad P(n)$: If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$

$$P(1): A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ so } A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, $\quad$ the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k): A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k + 1$

Now $\quad A^{k+1} = A \cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction, we have $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, holds for all natural numbers.

Example 27 If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is AB = BA.

Solution Since A and B are both symmetric matrices, therefore $A' = A$ and $B' = B$.

Let $\quad$ AB be symmetric, then $(AB)' = AB$

But $(AB)' = B'A' = BA$ (Why?)

Therefore $BA = AB$

Conversely, if AB = BA, then we shall show that AB is symmetric.

Now $(AB)' = B'A'$

$= B\ A$ (as A and B are symmetric)

$= AB$

Hence AB is symmetric.

Example 28 Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that CD – AB = O.

Solution Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then CD – AB = 0 gives

or $$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or $$\begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \qquad \text{... (1)}$$

$$3a + 8c - 43 = 0 \qquad \text{... (2)}$$

$$2b + 5d = 0 \qquad \text{... (3)}$$

and $$3b + 8d - 22 = 0 \qquad \text{... (4)}$$

Solving (1) and (2), we get $a = -191$, $c = 77$. Solving (3) and (4), we get $b = -110$, $d = 44$.

Therefore $$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Miscellaneous Exercise on Chapter 3

1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1} bA$, where I is the identity matrix of order 2 and $n \in \mathbf{N}$.

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$, $n \in \mathbf{N}$.

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

4. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

5. Show that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

6. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A'A = I$.

7. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

8. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

9. Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

10. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	**Products**		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x, y and z are Rs 2.50, Rs 1.50 and Rs 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are Rs 2.00, Rs 1.00 and 50 paise respectively. Find the gross profit.

11. Find the matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

12. If A and B are square matrices of the same order such that AB = BA, then prove by induction that $AB^n = B^nA$. Further, prove that $(AB)^n = A^nB^n$ for all $n \in$ N.

Choose the correct answer in the following questions:

13. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

14. If the matrix A is both symmetric and skew symmetric, then

(A) A is a diagonal matrix (B) A is a zero matrix

(C) A is a square matrix (D) None of these

15. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(A) A (B) I – A (C) I (D) 3A

Summary

- A matrix is an ordered rectangular array of numbers or functions.
- A matrix having m rows and n columns is called a matrix of order $m \times n$.
- $[a_{ij}]_{m \times 1}$ is a column matrix.
- $[a_{ij}]_{1 \times n}$ is a row matrix.
- An $m \times n$ matrix is a square matrix if $m = n$.
- $A = [a_{ij}]_{m \times m}$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

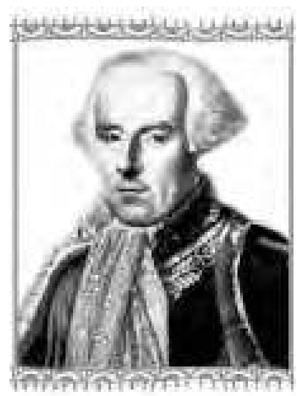
- $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$, $a_{ij} = k$, (k is some constant), when $i = j$.
- $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$, $a_{ij} = 0$, when $i \neq j$.
- A zero matrix has all its elements as zero.
- $A = [a_{ij}] = [b_{ij}] = B$ if (i) A and B are of same order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.
- $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$
- $-A = (-1)A$
- $A - B = A + (-1)B$
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$, where A, B and C are of same order.
- $k(A + B) = kA + kB$, where A and B are of same order, k is constant.
- $(k + l)A = kA + lA$, where k and l are constant.
- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$, then $AB = C = [c_{ik}]_{m \times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$
- (i) $A(BC) = (AB)C$, (ii) $A(B + C) = AB + AC$, (iii) $(A + B)C = AC + BC$
- If $A = [a_{ij}]_{m \times n}$, then A' or $A^T = [a_{ji}]_{n \times m}$
- (i) $(A')' = A$, (ii) $(kA)' = kA'$, (iii) $(A + B)' = A' + B'$, (iv) $(AB)' = B'A'$
- A is a symmetric matrix if $A' = A$.
- A is a skew symmetric matrix if $A' = -A$.
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- Elementary operations of a matrix are as follows:
  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
  (ii) $R_i \to kR_i$ or $C_i \to kC_i$
  (iii) $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$
- If A and B are two square matrices such that $AB = BA = I$, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B.
- Inverse of a square matrix, if it exists, is unique.

Chapter 4

DETERMINANTS

❖ *All Mathematical truths are relative and conditional. — C.P. STEINMETZ* ❖

4.1 Introduction



P.S. Laplace
(1749-1827)

In the previous chapter, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Now, this system of equations has a unique solution or not, is determined by the number $a_1 b_2 - a_2 b_1$. (Recall that if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or, $a_1 b_2 - a_2 b_1 \neq 0$, then the system of linear equations has a unique solution). The number $a_1 b_2 - a_2 b_1$ which determines uniqueness of solution is associated with the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called the determinant of A or det A. Determinants have wide applications in Engineering, Science, Economics, Social Science, etc.

In this chapter, we shall study determinants up to order three only with real entries. Also, we will study various properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equations in two or three variables using inverse of a matrix.

4.2 Determinant

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex). If M is the set of square matrices, K is the set of numbers (real or complex) and $f : M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A. It is also denoted by $|A|$ or det A or Δ.

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then determinant of A is written as } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$$

Remarks

(i) For matrix A, $|A|$ is read as determinant of A and not modulus of A.

(ii) Only square matrices have determinants.

4.2.1 *Determinant of a matrix of order one*

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a

4.2.2 *Determinant of a matrix of order two*

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ be a matrix of order 2 × 2,

then the determinant of A is defined as:

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1 Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1) = 4 + 4 = 8.$

Example 2 Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Solution We have

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$$

4.2.3 *Determinant of a matrix of order* 3 × 3

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order

3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below.

Consider the determinant of square matrix A = $[a_{ij}]_{3 \times 3}$

i.e.,
$$|A| = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along first Row (R_1)

Step 1 Multiply first element a_{11} of R_1 by $(-1)^{(1+1)}$ $[(-1)^{\text{sum of suffixes in } a_{11}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of | A | as a_{11} lies in R_1 and C_1,

i.e.,
$$(-1)^{1+1}\, a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Step 2 Multiply 2nd element a_{12} of R_1 by $(-1)^{1+2}$ $[(-1)^{\text{sum of suffixes in } a_{12}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and 2nd column (C_2) of | A | as a_{12} lies in R_1 and C_2,

i.e.,
$$(-1)^{1+2}\, a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Step 3 Multiply third element a_{13} of R_1 by $(-1)^{1+3}$ $[(-1)^{\text{sum of suffixes in } a_{13}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and third column (C_3) of | A | as a_{13} lies in R_1 and C_3,

i.e.,
$$(-1)^{1+3}\, a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Step 4 Now the expansion of determinant of A, that is, | A | written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

$$\det A = |A| = (-1)^{1+1}\, a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}\, a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or
$$|A| = a_{11}\,(a_{22}\, a_{33} - a_{32}\, a_{23}) - a_{12}\,(a_{21}\, a_{33} - a_{31}\, a_{23})$$
$$+ a_{13}\,(a_{21}\, a_{32} - a_{31}\, a_{22})$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{32}\, a_{23} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{31}\, a_{23} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \quad \ldots (1)$$

Note We shall apply all four steps together.

Expansion along second row ($\mathbf{R_2}$)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_2, we get

$$|A| = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{21}\,(a_{12}\, a_{33} - a_{32}\, a_{13}) + a_{22}\,(a_{11}\, a_{33} - a_{31}\, a_{13}) - a_{23}\,(a_{11}\, a_{32} - a_{31}\, a_{12})$$

$$|A| = -a_{21}\, a_{12}\, a_{33} + a_{21}\, a_{32}\, a_{13} + a_{22}\, a_{11}\, a_{33} - a_{22}\, a_{31}\, a_{13} - a_{23}\, a_{11}\, a_{32} + a_{23}\, a_{31}\, a_{12}$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \quad \ldots (2)$$

Expansion along first Column ($\mathbf{C_1}$)

$$|A| = \begin{vmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & a_{22} & a_{23} \\ \boldsymbol{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

By expanding along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{21}\,(a_{12}\, a_{33} - a_{13}\, a_{32}) + a_{31}\,(a_{12}\, a_{23} - a_{13}\, a_{22})$$

$$|A| = a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{21}\,a_{12}\,a_{33} + a_{21}\,a_{13}\,a_{32} + a_{31}\,a_{12}\,a_{23} - a_{31}\,a_{13}\,a_{22}$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{31}\,a_{22} \quad \ldots (3)$$

Clearly, values of $|A|$ in (1), (2) and (3) are equal. It is left as an exercise to the reader to verify that the values of $|A|$ by expanding along R_3, C_2 and C_3 are equal to the value of $|A|$ obtained in (1), (2) or (3).

Hence, expanding a determinant along any row or column gives same value.

Remarks

(i) For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or –1 according as $(i + j)$ is even or odd.

(iii) Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then, it is easy to verify that A = 2B. Also $|A| = 0 - 8 = -8$ and $|B| = 0 - 2 = -2$.

Observe that, $|A| = 4(-2) = 2^2|B|$ or $|A| = 2^n|B|$, where $n = 2$ is the order of square matrices A and B.

In general, if $A = kB$ where A and B are square matrices of order n, then $|A| = k^n |B|$, where $n = 1, 2, 3$

Example 3 Evaluate the determinant $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$.

Solution Note that in the third column, two entries are zero. So expanding along third column (C_3), we get

$$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 4(-1 - 12) - 0 + 0 = -52$$

Example 4 Evaluate $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$.

Solution Expanding along R_1, we get

$$\Delta = 0\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

$$= 0 - \sin\alpha\,(0 - \sin\beta\cos\alpha) - \cos\alpha\,(\sin\alpha\sin\beta - 0)$$
$$= \sin\alpha\sin\beta\cos\alpha - \cos\alpha\sin\alpha\sin\beta = 0$$

Example 5 Find values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

i.e. $\quad 3 - x^2 = 3 - 8$

i.e. $\quad x^2 = 8$

Hence $\quad x = \pm 2\sqrt{2}$

EXERCISE 4.1

Evaluate the determinants in Exercises 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

5. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find | A |

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ± 6 (C) – 6 (D) 0

4.3 Properties of Determinants

In the previous section, we have learnt how to expand the determinants. In this section, we will study some properties of determinants which simplifies its evaluation by obtaining maximum number of zeros in a row or a column. These properties are true for determinants of any order. However, we shall restrict ourselves upto determinants of order 3 only.

Property 1 The value of the determinant remains unchanged if its rows and columns are interchanged.

Verification Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

By interchanging the rows and columns of Δ, we get the determinant

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding Δ_1 along first column, we get

$$\Delta_1 = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Hence $\Delta = \Delta_1$

Remark It follows from above property that if A is a square matrix, then det (A) = det (A′), where A′ = transpose of A.

> **Note** If R_i = ith row and C_i = ith column, then for interchange of row and columns, we will symbolically write $C_i \leftrightarrow R_i$
>
> Let us verify the above property by example.

Example 6 Verify Property 1 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Solution Expanding the determinant along first row, we have

$$\Delta = 2\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$$

$$= 2 (0 - 20) + 3 (-42 - 4) + 5 (30 - 0)$$

$$= -40 - 138 + 150 = -28$$

By interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix} \quad \text{(Expanding along first column)}$$

$$= 2\begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 1 \\ 0 & 5 \end{vmatrix}$$

$$= 2 (0 - 20) + 3 (-42 - 4) + 5 (30 - 0)$$

$$= -40 - 138 + 150 = -28$$

Clearly $\Delta = \Delta_1$

Hence, Property 1 is verified.

Property 2 If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

Verification Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Interchanging first and third rows, the new determinant obtained is given by

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Expanding along third row, we get

$$\Delta_1 = a_1 (c_2 b_3 - b_2 c_3) - a_2 (c_1 b_3 - c_3 b_1) + a_3 (b_2 c_1 - b_1 c_2)$$

$$= - [a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)]$$

Clearly $\Delta_1 = - \Delta$

Similarly, we can verify the result by interchanging any two columns.

Note We can denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$.

Example 7 Verify Property 2 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$.

Solution $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = - 28$ (See Example 6)

Interchanging rows R_2 and R_3 i.e., $R_2 \leftrightarrow R_3$, we have

$$\Delta_1 = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 5 & -7 \\ 6 & 0 & 4 \end{vmatrix}$$

Expanding the determinant Δ_1 along first row, we have

$$\Delta_1 = 2\begin{vmatrix} 5 & -7 \\ 0 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -7 \\ 6 & 4 \end{vmatrix} + 5\begin{vmatrix} 1 & 5 \\ 6 & 0 \end{vmatrix}$$

$$= 2 (20 - 0) + 3 (4 + 42) + 5 (0 - 30)$$

$$= 40 + 138 - 150 = 28$$

Clearly $\Delta_1 = -\Delta$

Hence, Property 2 is verified.

Property 3 If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

Proof If we interchange the identical rows (or columns) of the determinant Δ, then Δ does not change. However, by Property 2, it follows that Δ has changed its sign

Therefore $\Delta = -\Delta$

or $\Delta = 0$

Let us verify the above property by an example.

Example 8 Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

Solution Expanding along first row, we get

$$\Delta = 3\,(6-6) - 2\,(6-9) + 3\,(4-6)$$
$$= 0 - 2\,(-3) + 3\,(-2) = 6 - 6 = 0$$

Here R_1 and R_3 are identical.

Property 4 If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

Verification Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and Δ_1 be the determinant obtained by multiplying the elements of the first row by k. Then

$$\Delta_1 = \begin{vmatrix} k\,a_1 & k\,b_1 & k\,c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along first row, we get

$$\Delta_1 = k\,a_1\,(b_2\,c_3 - b_3\,c_2) - k\,b_1\,(a_2\,c_3 - c_2\,a_3) + k\,c_1\,(a_2\,b_3 - b_2\,a_3)$$
$$= k\,[a_1\,(b_2\,c_3 - b_3\,c_2) - b_1\,(a_2\,c_3 - c_2\,a_3) + c_1\,(a_2\,b_3 - b_2\,a_3)]$$
$$= k\,\Delta$$

Hence $\begin{vmatrix} k\,a_1 & k\,b_1 & k\,c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Remarks

(i) By this property, we can take out any common factor from any one row or any one column of a given determinant.

(ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero. For example

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ k\,a_1 & k\,a_2 & k\,a_3 \end{vmatrix} = 0 \text{ (rows } R_1 \text{ and } R_2 \text{ are proportional)}$$

Example 9 Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Solution Note that $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$

(Using Properties 3 and 4)

Property 5 If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

For example, $\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Verification L.H.S. = $\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding the determinants along the first row, we get

$$\begin{aligned}\Delta &= (a_1 + \lambda_1)(b_2 c_3 - c_2 b_3) - (a_2 + \lambda_2)(b_1 c_3 - b_3 c_1) \\ &\quad + (a_3 + \lambda_3)(b_1 c_2 - b_2 c_1) \\ &= a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1) \\ &\quad + \lambda_1 (b_2 c_3 - c_2 b_3) - \lambda_2 (b_1 c_3 - b_3 c_1) + \lambda_3 (b_1 c_2 - b_2 c_1)\end{aligned}$$

(by rearranging terms)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{R.H.S.}$$

Similarly, we may verify Property 5 for other rows or columns.

Example 10 Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Solution We have $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

(by Property 5)

$= 0 + 0 = 0$ (Using Property 3 and Property 4)

Property 6 If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + k C_j$.

Verification

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_1 + k c_1 & a_2 + k c_2 & a_3 + k c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

where Δ_1 is obtained by the operation $R_1 \to R_1 + kR_3$.

Here, we have multiplied the elements of the third row (R_3) by a constant k and added them to the corresponding elements of the first row (R_1).

Symbolically, we write this operation as $R_1 \to R_1 + k R_3$.

Now, again

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k\,c_1 & k\,c_2 & k\,c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{(Using Property 5)}$$

$$= \Delta + 0 \qquad \text{(since } R_1 \text{ and } R_3 \text{ are proportional)}$$

Hence $\Delta = \Delta_1$

Remarks

(i) If Δ_1 is the determinant obtained by applying $R_i \to kR_i$ or $C_i \to kC_i$ to the determinant Δ, then $\Delta_1 = k\Delta$.

(ii) If more than one operation like $R_i \to R_i + kR_j$ is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.

Example 11 Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$.

Solution Applying operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ to the given determinant Δ, we have

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we get

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we obtain

$$\Delta = a\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0 + 0$$

$$= a\,(a^2 - 0) = a\,(a^2) = a^3$$

Example 12 Without expanding, prove that

$$\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution Applying $R_1 \to R_1 + R_2$ to Δ, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

Since the elements of R_1 and R_3 are proportional, $\Delta = 0$.

Example 13 Evaluate

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Solution Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking factors $(b - a)$ and $(c - a)$ common from R_2 and R_3, respectively, we get

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$$= (b - a)(c - a)[(- b + c)] \text{ (Expanding along first column)}$$

$$= (a - b)(b - c)(c - a)$$

Example 14 Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Solution Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$ to Δ, we get

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1, we obtain

$$\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2\,c\,(a\,b + b^2 - bc) - 2\,b\,(b\,c - c^2 - ac)$$

$$= 2\,a\,b\,c + 2\,cb^2 - 2\,bc^2 - 2\,b^2c + 2\,bc^2 + 2\,abc$$

$$= 4\,abc$$

Example 15 If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then show that $1 + xyz = 0$

Solution We have

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \quad \text{(Using Property 5)}$$

$$= (-1)^2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad \text{(Using } C_3 \leftrightarrow C_2 \text{ and then } C_1 \leftrightarrow C_2\text{)}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)$$

$$= (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \qquad \text{(Using } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

Taking out common factor $(y-x)$ from R_2 and $(z-x)$ from R_3, we get

$$\Delta = (1+xyz)\,(y-x)\,(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$$= (1+xyz)\,(y-x)\,(z-x)\,(z-y) \quad \text{(on expanding along } C_1\text{)}$$

Since $\Delta = 0$ and x, y, z are all different, i.e., $x-y \neq 0$, $y-z \neq 0$, $z-x \neq 0$, we get $1 + xyz = 0$

Example 16 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab$$

Solution Taking out factors a,b,c common from R_1, R_2 and R_3, we get

$$\text{L.H.S.} = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left[1(1-0)\right]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab = \text{R.H.S.}$$

Note Alternately try by applying $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - C_2$, then apply $C_1 \to C_1 - a\,C_3$.

EXERCISE 4.2

Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

2. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

4. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

5. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

6. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

7. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

By using properties of determinants, in Exercises 8 to 14, show that:

8. (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

9. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)\ (y-z)\ (z-x)\ (xy+yz+zx)$

10. (i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

11. (i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

12. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \left(1-x^3\right)^2$

13. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$

14. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

Choose the correct answer in Exercises 15 and 16.

15. Let A be a square matrix of order 3 × 3, then $|k\mathrm{A}|$ is equal to

(A) $k|\mathrm{A}|$ (B) $k^2|\mathrm{A}|$ (C) $k^3|\mathrm{A}|$ (D) $3k|\mathrm{A}|$

16. Which of the following is correct

(A) Determinant is a square matrix.

(B) Determinant is a number associated to a matrix.

(C) Determinant is a number associated to a square matrix.

(D) None of these

4.4 Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by the expression $\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$. Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{... (1)}$$

Remarks

(i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

Example 17 Find the area of the triangle whose vertices are (3, 8), (– 4, 2) and (5, 1).

Solution The area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3(2-1) - 8(-4-5) + 1(-4-10)\right]$$

$$= \frac{1}{2}(3+72-14) = \frac{61}{2}$$

Example 18 Find the equation of the line joining A(1, 3) and B (0, 0) using determinants and find k if D(k, 0) is a point such that area of triangle ABD is 3sq units.

Solution Let P (x, y) be any point on AB. Then, area of triangle ABP is zero (Why?). So

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

This gives $\frac{1}{2}(y-3x) = 0$ or $y = 3x$,

which is the equation of required line AB.

Also, since the area of the triangle ABD is 3 sq. units, we have

$$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

This gives, $\frac{-3k}{2} = \pm 3$, i.e., $k = \mp 2$.

EXERCISE 4.3

1. Find area of the triangle with vertices at the point given in each of the following :

(i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)

(iii) (–2, –3), (3, 2), (–1, –8)

2. Show that points
A $(a, b + c)$, B $(b, c + a)$, C $(c, a + b)$ are collinear.
3. Find values of k if area of triangle is 4 sq. units and vertices are
(i) $(k, 0), (4, 0), (0, 2)$ (ii) $(-2, 0), (0, 4), (0, k)$
4. (i) Find equation of line joining (1, 2) and (3, 6) using determinants.
(ii) Find equation of line joining (3, 1) and (9, 3) using determinants.
5. If area of triangle is 35 sq units with vertices $(2, -6)$, $(5, 4)$ and $(k, 4)$. Then k is
(A) 12 (B) –2 (C) –12, –2 (D) 12, –2

4.5 Minors and Cofactors

In this section, we will learn to write the expansion of a determinant in compact form using minors and cofactors.

Definition 1 Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Remark Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$.

Example 19 Find the minor of element 6 in the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution Since 6 lies in the second row and third column, its minor M_{23} is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6 \text{ (obtained by deleting } R_2 \text{ and } C_3 \text{ in } \Delta).$$

Definition 2 Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of } a_{ij}.$$

Example 20 Find minors and cofactors of all the elements of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Solution Minor of the element a_{ij} is M_{ij}

Here $a_{11} = 1$. So M_{11} = Minor of $a_{11} = 3$

M_{12} = Minor of the element $a_{12} = 4$

M_{21} = Minor of the element $a_{21} = -2$

M_{22} = Minor of the element $a_{22} = 1$

Now, cofactor of a_{ij} is A_{ij}. So

$A_{11} = (-1)^{1+1}\ M_{11} = (-1)^2\ (3) = 3$

$A_{12} = (-1)^{1+2}\ M_{12} = (-1)^3\ (4) = -4$

$A_{21} = (-1)^{2+1}\ M_{21} = (-1)^3\ (-2) = 2$

$A_{22} = (-1)^{2+2}\ M_{22} = (-1)^4\ (1) = 1$

Example 21 Find minors and cofactors of the elements a_{11}, a_{21} in the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution By definition of minors and cofactors, we have

Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{23}\, a_{32}$

Cofactor of $a_{11} = A_{11} = (-1)^{1+1}\ M_{11} = a_{22}\, a_{33} - a_{23}\, a_{32}$

Minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\, a_{33} - a_{13}\, a_{32}$

Cofactor of $a_{21} = A_{21} = (-1)^{2+1}\ M_{21} = (-1)\ (a_{12}\, a_{33} - a_{13}\, a_{32}) = -a_{12}\, a_{33} + a_{13}\, a_{32}$

Remark Expanding the determinant Δ, in Example 21, along R_1, we have

$$\Delta = (-1)^{1+1}\, a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}\, a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}\ a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}\, A_{11} + a_{12}\, A_{12} + a_{13}\, A_{13}$, where A_{ij} is cofactor of a_{ij}

= sum of product of elements of R_1 with their corresponding cofactors

Similarly, Δ can be calculated by other five ways of expansion that is along R_2, R_3, C_1, C_2 and C_3.

Hence Δ = sum of the product of elements of any row (or column) with their corresponding cofactors.

Note If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example,

$$\Delta = a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23}$$

$$= a_{11} (-1)^{1+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} (-1)^{1+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} (-1)^{1+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)}$$

Similarly, we can try for other rows and columns.

Example 22 Find minors and cofactors of the elements of the determinant

$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

Solution We have $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$; $A_{11} = (-1)^{1+1} (-20) = -20$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46; \qquad A_{12} = (-1)^{1+2} (-46) = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30; \qquad A_{13} = (-1)^{1+3} (30) = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4; \qquad A_{21} = (-1)^{2+1} (-4) = 4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19; \qquad A_{22} = (-1)^{2+2} (-19) = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13; \qquad A_{23} = (-1)^{2+3} (13) = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12; \qquad A_{31} = (-1)^{3+1} (-12) = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22; \qquad A_{32} = (-1)^{3+2}(-22) = 22$$

and $$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18; \qquad A_{33} = (-1)^{3+3}(18) = 18$$

Now $a_{11} = 2,\ a_{12} = -3,\ a_{13} = 5;\ A_{31} = -12,\ A_{32} = 22,\ A_{33} = 18$

So $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$

$= 2(-12) + (-3)(22) + 5(18) = -24 - 66 + 90 = 0$

EXERCISE 4.4

Write Minors and Cofactors of the elements of following determinants:

1. (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

3. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by

(A) $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$ (B) $a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31}$

(C) $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$ (D) $a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$

4.6 Adjoint and Inverse of a Matrix

In the previous chapter, we have studied inverse of a matrix. In this section, we shall discuss the condition for existence of inverse of a matrix.

To find inverse of a matrix A, i.e., A^{-1} we shall first define adjoint of a matrix.

4.6.1 *Adjoint of a matrix*

Definition 3 The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj* A.

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then
$$adj\, A = \text{Transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example 23 Find *adj* A for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution We have $A_{11} = 4, A_{12} = -1, A_{21} = -3, A_{22} = 2$

Hence
$$adj\, A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Remark For a square matrix of order 2, given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The *adj* A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e.,

$$adj\, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Change sign Interchange

We state the following theorem without proof.

Theorem 1 If A be any given square matrix of order n, then

$$A(adj\, A) = (adj\, A)\, A = |A|\, I,$$

where I is the identity matrix of order n

Verification

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then *adj* $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to $|A|$ and otherwise zero, we have

$$A\,(adj\,A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I$$

Similarly, we can show $(adj\,A)\,A = |A|\ I$

Hence $A\,(adj\,A) = (adj\,A)\,A = |A|\ I$

Definition 4 A square matrix A is said to be singular if $|A| = 0$.

For example, the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is zero

Hence A is a singular matrix.

Definition 5 A square matrix A is said to be non-singular if $|A| \neq 0$

Let $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$

Hence A is a nonsingular matrix

We state the following theorems without proof.

Theorem 2 If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

Theorem 3 The determinant of the product of matrices is equal to product of their respective determinants, that is, $|AB| = |A|\ |B|$, where A and B are square matrices of the same order

Remark We know that $(adj\,A)\,A = |A|\ I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$

Writing determinants of matrices on both sides, we have

$$|(adj\,A)\,A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

i.e. $$|(adj\ A)|\ |A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{(Why?)}$$

i.e. $|(adj\ A)|\ |A| = |A|^3$ (1)

i.e. $|(adj\ A)| = |A|^2$

In general, if A is *a* square matrix of order n, then $|adj(A)| = |A|^{n-1}$.

Theorem 4 A square matrix A is invertible if and only if A is nonsingular matrix.

Proof Let A be invertible matrix of order n and I be the identity matrix of order n. Then, there exists a square matrix B of order n such that AB = BA = I

Now AB = I. So $|AB| = |I|$ or $|A|\ |B| = 1$ (since $|I| = 1, |AB| = |A||B|$)

This gives $|A| \neq 0$. Hence A is nonsingular.

Conversely, let A be nonsingular. Then $|A| \neq 0$

Now $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$ (Theorem 1)

or $$A\left(\frac{1}{|A|}adj\,A\right) = \left(\frac{1}{|A|}adj\,A\right)A = I$$

or AB = BA = I, where $B = \frac{1}{|A|}adj\,A$

Thus A is invertible and $A^{-1} = \frac{1}{|A|}adj\,A$

Example 24 If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A *adj* A = |A| I. Also find A^{-1}.

Solution We have $|A| = 1\,(16 - 9) - 3\,(4 - 3) + 3\,(3 - 4) = 1 \neq 0$

Now $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

Therefore $$adj\ A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $$A\,(adj\ A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,.\,I$$

Also $$|A|^{-1} = \frac{1}{|A|}\,adj\,A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Example 25 If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution We have $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Since, $|AB| = -11 \neq 0$, $(AB)^{-1}$ exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|}\,adj\,(AB) = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further, $|A| = -11 \neq 0$ and $|B| = 1 \neq 0$. Therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore $\quad B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Hence $(AB)^{-1} = B^{-1}A^{-1}$

Example 26 Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using this equation, find A^{-1}.

Solution We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $\quad A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now $\quad A^2 - 4A + I = O$

Therefore $\quad A A - 4A = -I$

or $\quad A\ A(A^{-1}) - 4\,A A^{-1} = -I A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $\quad A(A A^{-1}) - 4I = -A^{-1}$

or $\quad AI - 4I = -A^{-1}$

or $\quad A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $\quad A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

EXERCISE 4.5

Find adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Verify A (*adj* A) = (*adj* A) A = |A| I in Exercises 3 and 4

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

16. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

17. Let A be a nonsingular square matrix of order 3 × 3. Then $|adj\,A|$ is equal to

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$

18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

(A) det (A) (B) $\dfrac{1}{\det(A)}$ (C) 1 (D) 0

4.7 Applications of Determinants and Matrices

In this section, we shall discuss application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent system A system of equations is said to be *consistent* if its solution (one or more) exists.

Inconsistent system A system of equations is said to be *inconsistent* if its solution does not exist.

> **Note** In this chapter, we restrict ourselves to the system of linear equations having unique solutions only.

4.7.1 *Solution of system of linear equations using inverse of a matrix*

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let $\quad A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equations can be written as, AX = B, i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I If A is a nonsingular matrix, then its inverse exists. Now

$$AX = B$$

or $\quad A^{-1}(AX) = A^{-1} B \quad$ (premultiplying by A^{-1})

or $\quad (A^{-1}A) X = A^{-1} B \quad$ (by associative property)

or $\quad I X = A^{-1} B$

or $\quad X = A^{-1} B$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II If A is a singular matrix, then $|A| = 0$.

In this case, we calculate (*adj* A) B.

If (*adj* A) B $\neq$ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (*adj* A) B = O, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Example 27 Solve the system of equations

$$2x + 5y = 1$$
$$3x + 2y = 7$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now, $|A| = -11 \neq 0$, Hence, A is nonsingular matrix and so has a unique solution.

Note that $$A^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore $$X = A^{-1}B = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence $$x = 3, y = -1$$

Example 28 Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3\,(2 - 3) + 2(4 + 4) + 3\,(-6 - 4) = -17 \neq 0$$

Hence, A is nonsingular and so its inverse exists. Now

$A_{11} = -1,$ $A_{12} = -8,$ $A_{13} = -10$

$A_{21} = -5,$ $A_{22} = -6,$ $A_{23} = 1$

$A_{31} = -1,$ $A_{32} = 9,$ $A_{33} = 7$

Therefore $$A^{-1} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

So $$X = A^{-1}B = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence $x = 1$, $y = 2$ and $z = 3$.

Example 29 The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Solution Let first, second and third numbers be denoted by x, y and z, respectively. Then, according to given conditions, we have

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x + z = 2y \text{ or } x - 2y + z = 0$$

This system can be written as A X = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Here $|A| = 1(1+6) - (0-3) + (0-1) = 9 \neq 0$. Now we find *adj* A

$A_{11} = 1\,(1 + 6) = 7,$ $A_{12} = -(0 - 3) = 3,$ $A_{13} = -1$

$A_{21} = -(1 + 2) = -3,$ $A_{22} = 0,$ $A_{23} = -(-2 - 1) = 3$

$A_{31} = (3 - 1) = 2,$ $A_{32} = -(3 - 0) = -3,$ $A_{33} = (1 - 0) = 1$

Hence $$adj\ \mathrm{A} = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Thus $$\mathrm{A}^{-1} = \frac{1}{|\mathrm{A}|}\ adj\ (\mathrm{A}) = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Since $$\mathrm{X} = \mathrm{A}^{-1}\,\mathrm{B}$$

$$\mathrm{X} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42-33+0 \\ 18+\ 0+0 \\ -6+33+0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus $$x = 1,\ y = 2,\ z = 3$$

EXERCISE 4.6

Examine the consistency of the system of equations in Exercises 1 to 6.

1. $x + 2y = 2$, $2x + 3y = 3$

2. $2x - y = 5$, $x + y = 4$

3. $x + 3y = 5$, $2x + 6y = 8$

4. $x + y + z = 1$, $2x + 3y + 2z = 2$, $ax + ay + 2az = 4$

5. $3x - y - 2z = 2$, $2y - z = -1$, $3x - 5y = 3$

6. $5x - y + 4z = 5$, $2x + 3y + 5z = 2$, $5x - 2y + 6z = -1$

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

7. $5x + 2y = 4$, $7x + 3y = 5$

8. $2x - y = -2$, $3x + 4y = 3$

9. $4x - 3y = 3$, $3x - 5y = 7$

10. $5x + 2y = 3$, $3x + 2y = 5$

11. $2x + y + z = 1$, $x - 2y - z = \frac{3}{2}$, $3y - 5z = 9$

12. $x - y + z = 4$, $2x + y - 3z = 0$, $x + y + z = 2$

13. $2x + 3y + 3z = 5$, $x - 2y + z = -4$, $3x - y - 2z = 3$

14. $x - y + 2z = 7$, $3x + 4y - 5z = -5$, $2x - y + 3z = 12$

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x + 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70. Find cost of each item per kg by matrix method.

Miscellaneous Examples

Example 30 If a, b, c are positive and unequal, show that value of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ is negative.}$$

Solution Applying $C_1 \to C_1 + C_2 + C_3$ to the given determinant, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \text{(Applying } R_2 \to R_2 - R_1, \text{and } R_3 \to R_3 - R_1\text{)}$$

$$= (a+b+c)\,[(c-b)(b-c) - (a-c)(a-b)] \quad \text{(Expanding along } C_1\text{)}$$

$$= (a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= \frac{-1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= \frac{-1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

which is negative (since $a + b + c > 0$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$)

Example 31 If a, b, c, are in A.P, find value of

$$\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

Solution Applying $R_1 \to R_1 + R_3 - 2R_2$ to the given determinant, we obtain

$$\begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} = 0 \qquad \text{(Since } 2b = a + c\text{)}$$

Example 32 Show that

$$\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz\,(x+y+z)^3$$

Solution Applying $R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3$ to Δ and dividing by xyz, we get

$$\Delta = \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common factors x, y, z from C_1 C_2 and C_3, respectively, we get

$$\Delta = \frac{xyz}{xyz}\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have

$$\Delta = \begin{vmatrix} (y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2 \\ y^2 & (x+z)^2-y^2 & 0 \\ z^2 & 0 & (x+y)^2-z^2 \end{vmatrix}$$

Taking common factor $(x + y + z)$ from C_2 and C_3, we have

$$\Delta = (x+y+z)^2 \begin{vmatrix} (y+z)^2 & x-(y+z) & x-(y+z) \\ y^2 & (x+z)-y & 0 \\ z^2 & 0 & (x+y)-z \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$, we have

$$\Delta = (x+y+z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x-y+z & 0 \\ z^2 & 0 & x+y-z \end{vmatrix}$$

Applying $C_2 \to (C_2 + \frac{1}{y} C_1)$ and $C_3 \to \left(C_3 + \frac{1}{z} C_1\right)$, we get

$$\Delta = (x+y+z)^2 \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & x+z & \dfrac{y^2}{z} \\ z^2 & \dfrac{z^2}{y} & x+y \end{vmatrix}$$

Finally expanding along R_1, we have

$$\Delta = (x+y+z)^2 (2yz) [(x+z)(x+y) - yz] = (x+y+z)^2 (2yz) (x^2 + xy + xz)$$
$$= (x+y+z)^3 (2xyz)$$

Example 33 Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations

$$x - y + 2z = 1$$
$$2y - 3z = 1$$
$$3x - 2y + 4z = 2$$

Solution Consider the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence $x = 0$, $y = 5$ and $z = 3$

Example 34 Prove that

$$\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

Solution Applying $R_1 \to R_1 - x\,R_2$ to Δ, we get

$$\Delta = \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

$$= (1-x^2)\begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

Applying $R_2 \to R_2 - x\,R_1$, we get

$$\Delta = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

Miscellaneous Exercises on Chapter 4

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

4. If a, b and c are real numbers, and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$$

Show that either $a + b + c = 0$ or $a = b = c$.

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0,\ a \neq 0$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

8. Let A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $[adj\ A]^{-1} = adj\ (A^{-1})$ (ii) $(A^{-1})^{-1} = A$

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Using properties of determinants in Exercises 11 to 15, prove that:

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta - \gamma)\ (\gamma - \alpha)\ (\alpha - \beta)\ (\alpha + \beta + \gamma)$

12. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1 + pxyz)\ (x - y)\ (y - z)\ (z - x)$, where p is any scalar.

13. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a + b + c)\ (ab + bc + ca)$

14. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

15. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$

16. Solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Choose the correct answer in Exercise 17 to 19.

17. If a, b, c, are in A.P, then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \text{ is}$$

(A) 0 (B) 1 (C) x (D) $2x$

18. If x, y, z are nonzero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then

(A) $\text{Det}(A) = 0$ (B) $\text{Det}(A) \in (2, \infty)$

(C) $\text{Det}(A) \in (2, 4)$ (D) $\text{Det}(A) \in [2, 4]$

Summary

- Determinant of a matrix $A = [a_{11}]_{1\times1}$ is given by $|a_{11}| = a_{11}$
- Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

- Determinant of a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by (expanding along R_1)

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

For any square matrix A, the |A| satisfy following properties.

- $|A'| = |A|$, where A' = transpose of A.
- If we interchange any two rows (or columns), then sign of determinant changes.
- If any two rows or any two columns are identical or proportional, then value of determinant is zero.
- If we multiply each element of a row or a column of a determinant by constant k, then value of determinant is multiplied by k.
- Multiplying a determinant by k means multiply elements of only one row (or one column) by k.
- If $A = [a_{ij}]_{3\times3}$, then $|k.A| = k^3\,|A|$
- If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.
- If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added, then value of determinant remains same.

- Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and denoted by M_{ij}.
- Cofactor of a_{ij} of given by $A_{ij} = (-1)^{i+j} M_{ij}$
- Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example,

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example, $a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0$
- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is cofactor of a_{ij}
- $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$, where A is square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.
- If AB = BA = I, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B$ or $B^{-1} = A$ and hence $(A^{-1})^{-1} = A$.
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1} = \frac{1}{|A|}(adj\ A)$
- If $a_1 x + b_1 y + c_1 z = d_1$
  $a_2 x + b_2 y + c_2 z = d_2$
  $a_3 x + b_3 y + c_3 z = d_3$,
  then these equations can be written as A X = B, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- Unique solution of equation AX = B is given by $X = A^{-1} B$, where $|A| \neq 0$.
- A system of equation is consistent or inconsistent according as its solution exists or not.
- For a square matrix A in matrix equation AX = B
  - (i) $|A| \neq 0$, there exists unique solution
  - (ii) $|A| = 0$ and $(adj\ A)\ B \neq 0$, then there exists no solution
  - (iii) $|A| = 0$ and $(adj\ A)\ B = 0$, then system may or may not be consistent.

Historical Note

The Chinese method of representing the coefficients of the unknowns of several linear equations by using rods on a calculating board naturally led to the discovery of simple method of elimination. The arrangement of rods was precisely that of the numbers in a determinant. The Chinese, therefore, early developed the idea of subtracting columns and rows as in simplification of a determinant '*Mikami, China*, *pp* 30, 93.

Seki Kowa, the greatest of the Japanese Mathematicians of seventeenth century in his work '*Kai Fukudai no Ho*' in 1683 showed that he had the idea of determinants and of their expansion. But he used this device only in eliminating a quantity from two equations and not directly in the solution of a set of simultaneous linear equations. 'T. Hayashi, "*The Fakudoi and Determinants in Japanese Mathematics*," in the proc. of the Tokyo Math. Soc., V.

Vendermonde was the first to recognise determinants as independent functions. He may be called the formal founder. Laplace (1772), gave general method of expanding a determinant in terms of its complementary minors. In 1773 Lagrange treated determinants of the second and third orders and used them for purpose other than the solution of equations. In 1801, Gauss used determinants in his theory of numbers.

The next great contributor was Jacques - Philippe - Marie Binet, (1812) who stated the theorem relating to the product of two matrices of m-columns and n-rows, which for the special case of $m = n$ reduces to the multiplication theorem.

Also on the same day, Cauchy (1812) presented one on the same subject. He used the word 'determinant' in its present sense. He gave the proof of multiplication theorem more satisfactory than Binet's.

The greatest contributor to the theory was Carl Gustav Jacob Jacobi, after this the word determinant received its final acceptance.

Chapter 5

CONTINUITY AND DIFFERENTIABILITY

❖ *The whole of science is nothing more than a refinement of everyday thinking." — ALBERT EINSTEIN* ❖

5.1 Introduction

This chapter is essentially a continuation of our study of differentiation of functions in Class XI. We had learnt to differentiate certain functions like polynomial functions and trigonometric functions. In this chapter, we introduce the very important concepts of continuity, differentiability and relations between them. We will also learn differentiation of inverse trigonometric functions. Further, we introduce a new class of functions called exponential and logarithmic functions. These functions lead to powerful techniques of differentiation. We illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.

Sir Issac Newton (1642-1727)

5.2 Continuity

We start the section with two informal examples to get a feel of continuity. Consider the function

$$f(x) = \begin{cases} 1, \text{ if } x \le 0 \\ 2, \text{ if } x > 0 \end{cases}$$

This function is of course defined at every point of the real line. Graph of this function is given in the Fig 5.1. One can deduce from the graph that the value of the function at *nearby* points on x-axis remain *close* to each other except at $x = 0$. At the points near and to the left of 0, i.e., at points like – 0.1, – 0.01, – 0.001, the value of the function is 1. At the points near and to the right of 0, i.e., at points like 0.1, 0.01,

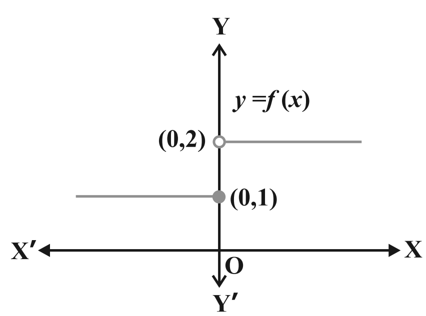


Fig 5.1

0.001, the value of the function is 2. Using the language of left and right hand limits, we may say that the left (respectively right) hand limit of f at 0 is 1 (respectively 2). In particular the left and right hand limits do not coincide. We also observe that the value of the function at $x = 0$ concides with the left hand limit. Note that when we try to draw the graph, we cannot draw it in one stroke, i.e., without lifting pen from the plane of the paper, we can not draw the graph of this function. In fact, we need to lift the pen when we come to 0 from left. This is one instance of function being not continuous at $x = 0$.

Now, consider the function defined as

$$f(x) = \begin{cases} 1, \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases}$$

This function is also defined at every point. Left and the right hand limits at $x = 0$ are both equal to 1. But the value of the function at $x = 0$ equals 2 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is yet another instance of a function being not continuous at $x = 0$.

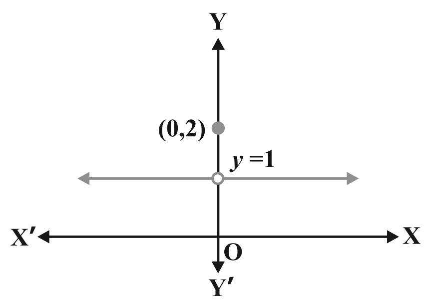


Fig 5.2

Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function *around* that point without lifting the pen from the plane of the paper.

Mathematically, it may be phrased precisely as follows:

Definition 1 Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other, then f is said to be continuous at $x = c$. Recall that if the right hand and left hand limits at $x = c$ coincide, then we say that the common value is the limit of the function at $x = c$. Hence we may also rephrase the definition of continuity as follows: *a function is continuous at x = c if the function is defined at x = c and if the value of the function at x = c equals the limit of the function at x = c.* If f is not continuous at c, we say f is *discontinuous* at c and c is called a *point of discontinuity* of f.

Example 1 Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.

Solution First note that the function is defined at the given point $x = 1$ and its value is 5. Then find the limit of the function at $x = 1$. Clearly

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(2x+3) = 2(1)+3 = 5$$

Thus $$\lim_{x\to 1} f(x) = 5 = f(1)$$

Hence, f is continuous at $x = 1$.

Example 2 Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.

Solution First note that the function is defined at the given point $x = 0$ and its value is 0. Then find the limit of the function at $x = 0$. Clearly

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0^2 = 0$$

Thus $$\lim_{x\to 0} f(x) = 0 = f(0)$$

Hence, f is continuous at $x = 0$.

Example 3 Discuss the continuity of the function f given by $f(x) = |\, x \,|$ at $x = 0$.

Solution By definition

$$f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \quad \text{if } x \geq 0 \end{cases}$$

Clearly the function is defined at 0 and $f(0) = 0$. Left hand limit of f at 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$$

Similarly, the right hand limit of f at 0 is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

Thus, the left hand limit, right hand limit and the value of the function coincide at $x = 0$. Hence, f is continuous at $x = 0$.

Example 4 Show that the function f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Solution The function is defined at $x = 0$ and its value at $x = 0$ is 1. When $x \neq 0$, the function is given by a polynomial. Hence,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^3 + 3) = 0^3 + 3 = 3$$

Since the limit of f at $x = 0$ does not coincide with $f(0)$, the function is not continuous at $x = 0$. It may be noted that $x = 0$ is the only point of discontinuity for this function.

Example 5 Check the points where the constant function $f(x) = k$ is continuous.

Solution The function is defined at all real numbers and by definition, its value at any real number equals k. Let c be any real number. Then

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$$

Since $f(c) = k = \lim_{x \to c} f(x)$ for any real number c, the function f is continuous at every real number.

Example 6 Prove that the identity function on real numbers given by $f(x) = x$ is continuous at every real number.

Solution The function is clearly defined at every point and $f(c) = c$ for every real number c. Also,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

Thus, $\lim_{x \to c} f(x) = c = f(c)$ and hence the function is continuous at every real number.

Having defined continuity of a function at a given point, now we make a natural extension of this definition to discuss continuity of a function.

Definition 2 A real function f is said to be continuous if it is continuous at every point in the domain of f.

This definition requires a bit of elaboration. Suppose f is a function defined on a closed interval $[a, b]$, then for f to be continuous, it needs to be continuous at every point in $[a, b]$ including the end points a and b. Continuity of f at a means

$$\lim_{x \to a^+} f(x) = f(a)$$

and continuity of f at b means

$$\lim_{x \to b^-} f(x) = f(b)$$

Observe that $\lim_{x \to a^-} f(x)$ *and* $\lim_{x \to b^+} f(x)$ *do not make sense. As a consequence of this definition, if f is defined only at one point, it is continuous there, i.e., if the domain of f is a singleton, f is a continuous function.*

Example 7 Is the function defined by $f(x) = |x|$, a continuous function?

Solution We may rewrite f as

$$f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \quad \text{if } x \geq 0 \end{cases}$$

By Example 3, we know that f is continuous at $x = 0$.

Let c be a real number such that $c < 0$. Then $f(c) = -c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x) = -c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all negative real numbers.

Now, let c be a real number such that $c > 0$. Then $f(c) = c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all positive real numbers. Hence, f is continuous at all points.

Example 8 Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$.

Solution Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$. We also know that

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus $\lim_{x \to c} f(x) = f(c)$, and hence f is continuous at every real number. This means f is a continuous function.

Example 9 Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}$, $x \neq 0$.

Solution Fix any non zero real number c, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

Also, since for $c \neq 0$, $f(c) = \frac{1}{c}$, we have $\lim_{x \to c} f(x) = f(c)$ and hence, f is continuous at every point in the domain of f. Thus f is a continuous function.

We take this opportunity to explain the concept of *infinity*. This we do by analysing the function $f(x) = \dfrac{1}{x}$ near $x = 0$. To carry out this analysis we follow the usual trick of finding the value of the function at real numbers *close* to 0. Essentially we are trying to find the right hand limit of f at 0. We tabulate this in the following (Table 5.1).

Table 5.1

x	1	0.3	0.2	$0.1 = 10^{-1}$	$0.01 = 10^{-2}$	$0.001 = 10^{-3}$	10^{-n}
$f(x)$	1	3.333...	5	10	$100 = 10^2$	$1000 = 10^3$	10^n

We observe that as x gets closer to 0 from the right, the value of $f(x)$ shoots up higher. This may be rephrased as: the value of $f(x)$ may be made larger than any given number by choosing a positive real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^+} f(x) = +\infty$$

(to be read as: the right hand limit of $f(x)$ at 0 is plus infinity). We wish to emphasise that $+\infty$ is NOT a real number and hence the right hand limit of f at 0 does not exist (as a real number).

Similarly, the left hand limit of f at 0 may be found. The following table is self explanatory.

Table 5.2

x	-1	-0.3	-0.2	-10^{-1}	-10^{-2}	-10^{-3}	-10^{-n}
$f(x)$	-1	$-3.333...$	-5	-10	-10^2	-10^3	-10^n

From the Table 5.2, we deduce that the value of $f(x)$ may be made smaller than any given number by choosing a negative real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^-} f(x) = -\infty$$

(to be read as: the left hand limit of $f(x)$ at 0 is minus infinity). Again, we wish to emphasise that $-\infty$ is NOT a real number and hence the left hand limit of f at 0 does not exist (as a real number). The graph of the reciprocal function given in Fig 5.3 is a geometric representation of the above mentioned facts.

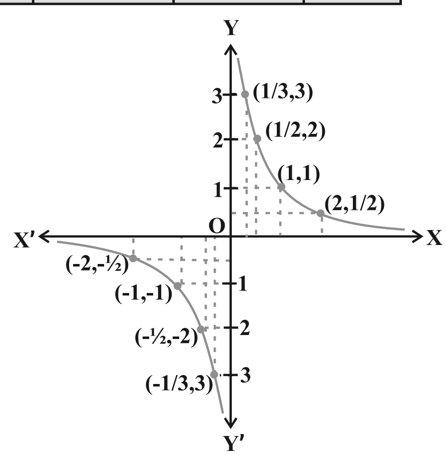


Fig 5.3

Example 10 Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x \le 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution The function f is defined at all points of the real line.

Case 1 If $c < 1$, then $f(c) = c + 2$. Therefore, $\lim_{x \to c} f(x) = \lim_{x \to c} f(x+2) = c+2$

Thus, f is continuous at all real numbers less than 1.

Case 2 If $c > 1$, then $f(c) = c - 2$. Therefore,

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x - 2) = c - 2 = f(c)$$

Thus, f is continuous at all points $x > 1$.

Case 3 If $c = 1$, then the left hand limit of f at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

Since the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in Fig 5.4.

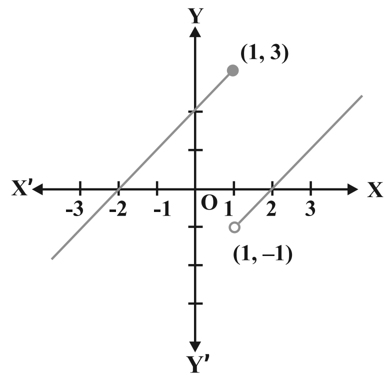


Fig 5.4

Example 11 Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x < 1 \\ 0, \quad \text{if } x = 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution As in the previous example we find that f is continuous at all real numbers $x \neq 1$. The left hand limit of f at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

Since, the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in the Fig 5.5.

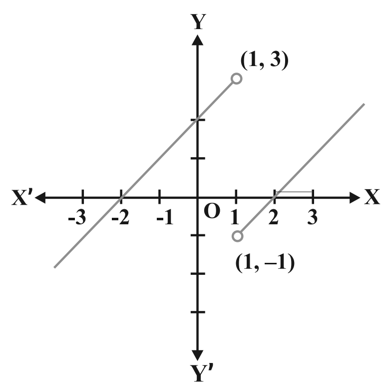


Fig 5.5

Example 12 Discuss the continuity of the function defined by

$$f(x) = \begin{cases} x+2, \text{ if } x<0 \\ -x+2, \text{ if } x>0 \end{cases}$$

Solution Observe that the function is defined at all real numbers except at 0. Domain of definition of this function is

$D_1 \cup D_2$ where $D_1 = \{x \in \mathbf{R} : x < 0\}$ and
$D_2 = \{x \in \mathbf{R} : x > 0\}$

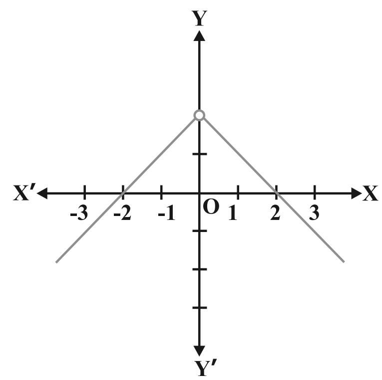


Fig 5.6

Case 1 If $c \in D_1$, then $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x + 2)$ $= c + 2 = f(c)$ and hence f is continuous in D_1.

Case 2 If $c \in D_2$, then $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (-x + 2)$ $= -c + 2 = f(c)$ and hence f is continuous in D_2. Since f is continuous at all points in the domain of f, we deduce that f is continuous. Graph of this function is given in the Fig 5.6. Note that to graph this function we need to lift the pen from the plane of the paper, but we need to do that only for those points where the function is not defined.

Example 13 Discuss the continuity of the function f given by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$$

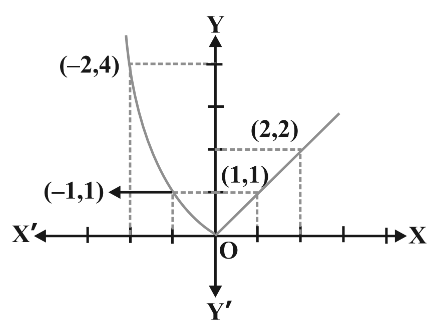


Fig 5.7

Solution Clearly the function is defined at every real number. Graph of the function is given in Fig 5.7. By inspection, it seems prudent to partition the domain of definition of f into three disjoint subsets of the real line.

Let $D_1 = \{x \in \mathbf{R} : x < 0\}$, $D_2 = \{0\}$ and
$D_3 = \{x \in \mathbf{R} : x > 0\}$

Case 1 At any point in D_1, we have $f(x) = x^2$ and it is easy to see that it is continuous there (see Example 2).

Case 2 At any point in D_3, we have $f(x) = x$ and it is easy to see that it is continuous there (see Example 6).

Case 3 Now we analyse the function at $x = 0$. The value of the function at 0 is $f(0) = 0$. The left hand limit of f at 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

The right hand limit of f at 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0. This means that f is continuous at every point in its domain and hence, f is a continuous function.

Example 14 Show that every polynomial function is continuous.

Solution Recall that a function p is a polynomial function if it is defined by $p(x) = a_0 + a_1 x + ... + a_n x^n$ for some natural number n, $a_n \neq 0$ and $a_i \in \mathbf{R}$. Clearly this function is defined for every real number. For a fixed real number c, we have

$$\lim_{x \to c} p(x) = p(c)$$

By definition, p is continuous at c. Since c is any real number, p is continuous at every real number and hence p is a continuous function.

Example 15 Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x.

Solution First observe that f is defined for all real numbers. Graph of the function is given in Fig 5.8. From the graph it looks like that f is discontinuous at every integral point. Below we explore, if this is true.

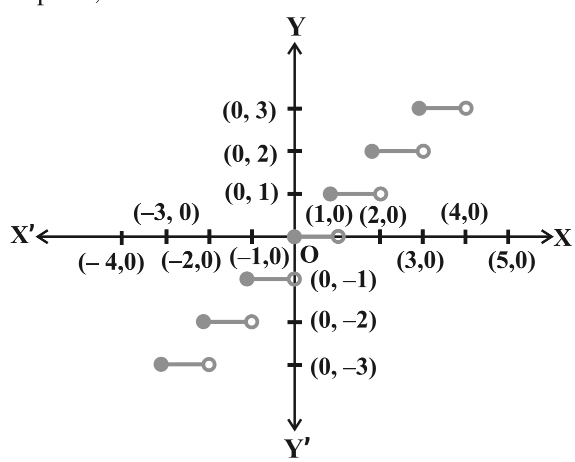


Fig 5.8

Case 1 Let c be a real number which is not equal to any integer. It is evident from the graph that for all real numbers *close* to c the value of the function is equal to $[c]$; i.e., $\lim_{x \to c} f(x) = \lim_{x \to c} [x] = [c]$. Also $f(c) = [c]$ and hence the function is continuous at all real numbers not equal to integers.

Case 2 Let c be an integer. Then we can find a sufficiently small real number $r > 0$ such that $[c - r] = c - 1$ whereas $[c + r] = c$.

This, in terms of limits mean that

$$\lim_{x \to c^-} f(x) = c - 1, \quad \lim_{x \to c^+} f(x) = c$$

Since these limits cannot be equal to each other for any c, the function is discontinuous at every integral point.

5.2.1 Algebra of continuous functions

In the previous class, after having understood the concept of limits, we learnt some algebra of limits. Analogously, now we will study some algebra of continuous functions. Since continuity of a function at a point is entirely dictated by the limit of the function at that point, it is reasonable to expect results analogous to the case of limits.

Theorem 1 Suppose f and g be two real functions continuous at a real number c. Then

(1) $f + g$ is continuous at $x = c$.

(2) $f - g$ is continuous at $x = c$.

(3) $f \,.\, g$ is continuous at $x = c$.

(4) $\left(\dfrac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).

Proof We are investigating continuity of $(f + g)$ at $x = c$. Clearly it is defined at $x = c$. We have

$$\begin{aligned}
\lim_{x \to c} (f + g)(x) &= \lim_{x \to c} [f(x) + g(x)] && \text{(by definition of } f + g) \\
&= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) && \text{(by the theorem on limits)} \\
&= f(c) + g(c) && \text{(as } f \text{ and } g \text{ are continuous)} \\
&= (f + g)(c) && \text{(by definition of } f + g)
\end{aligned}$$

Hence, $f + g$ is continuous at $x = c$.

Proofs for the remaining parts are similar and left as an exercise to the reader.

Remarks

(i) As a special case of (3) above, if f is a constant function, i.e., $f(x) = \lambda$ for some real number λ, then the function $(\lambda . g)$ defined by $(\lambda . g)(x) = \lambda . g(x)$ is also continuous. In particular if $\lambda = -1$, the continuity of f implies continuity of $-f$.

(ii) As a special case of (4) above, if f is the constant function $f(x) = \lambda$, then the function $\frac{\lambda}{g}$ defined by $\frac{\lambda}{g}(x) = \frac{\lambda}{g(x)}$ is also continuous wherever $g(x) \neq 0$. In particular, the continuity of g implies continuity of $\frac{1}{g}$.

The above theorem can be exploited to generate many continuous functions. They also aid in deciding if certain functions are continuous or not. The following examples illustrate this:

Example 16 Prove that every rational function is continuous.

Solution Recall that every rational function f is given by

$$f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$$

where p and q are polynomial functions. The domain of f is all real numbers except points at which q is zero. Since polynomial functions are continuous (Example 14), f is continuous by (4) of Theorem 1.

Example 17 Discuss the continuity of sine function.

Solution To see this we use the following facts

$$\lim_{x \to 0} \sin x = 0$$

We have not proved it, but is intuitively clear from the graph of $\sin x$ near 0.

Now, observe that $f(x) = \sin x$ is defined for every real number. Let c be a real number. Put $x = c + h$. If $x \to c$ we know that $h \to 0$. Therefore

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c} \sin x \\
&= \lim_{h \to 0} \sin(c + h) \\
&= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\
&= \lim_{h \to 0} [\sin c \cos h] + \lim_{h \to 0} [\cos c \sin h] \\
&= \sin c + 0 = \sin c = f(c)
\end{aligned}$$

Thus $\lim_{x \to c} f(x) = f(c)$ and hence f is a continuous function.

Remark A similar proof may be given for the continuity of cosine function.

Example 18 Prove that the function defined by $f(x) = \tan x$ is a continuous function.

Solution The function $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. This is defined for all real numbers such that $\cos x \neq 0$, i.e., $x \neq (2n+1)\dfrac{\pi}{2}$. We have just proved that both sine and cosine functions are continuous. Thus $\tan x$ being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if f and g are two real functions, then

$$(f \circ g)\ (x) = f(g(x))$$

is defined whenever the range of g is a subset of domain of f. The following theorem (stated without proof) captures the continuity of composite functions.

Theorem 2 Suppose f and g are real valued functions such that $(f \circ g)$ is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

The following examples illustrate this theorem.

Example 19 Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

Solution Observe that the function is defined for every real number. The function f may be thought of as a composition $g \circ h$ of the two functions g and h, where $g(x) = \sin x$ and $h(x) = x^2$. Since both g and h are continuous functions, by Theorem 2, it can be deduced that f is a continuous function.

Example 20 Show that the function f defined by

$$f(x) = |1 - x + |x||,$$

where x is any real number, is a continuous function.

Solution Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$ for all real x. Then

$$\begin{aligned}(h \circ g)\ (x) &= h(g(x)) \\ &= h(1 - x + |x|) \\ &= |1 - x + |x|| = f(x)\end{aligned}$$

In Example 7, we have seen that h is a continuous function. Hence g being a sum of a polynomial function and the modulus function is continuous. But then f being a composite of two continuous functions is continuous.

EXERCISE 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.
2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.
3. Examine the following functions for continuity.

   (a) $f(x) = x - 5$ (b) $f(x) = \dfrac{1}{x-5}$

   (c) $f(x) = \dfrac{x^2 - 25}{x+5}$ (d) $f(x) = |x - 5|$
4. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.
5. Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Find all points of discontinuity of f, where f is defined by

6. $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

7. $f(x) = \begin{cases} |x|+3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \ge 3 \end{cases}$

8. $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \ge 0 \end{cases}$

10. $f(x) = \begin{cases} x+1, & \text{if } x \ge 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

11. $f(x) = \begin{cases} x^3-3, & \text{if } x \le 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$

12. $f(x) = \begin{cases} x^{10}-1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$

13. Is the function defined by

$$f(x) = \begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Discuss the continuity of the function f, where f is defined by

14. $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

15. $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

16. $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

17. Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

18. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \,.\, \cos x$

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

23. Find all points of discontinuity of f, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \ge 0 \end{cases}$$

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

26. $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ at $x = \dfrac{\pi}{2}$

27. $f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$ at $x = 2$

28. $f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ at $x = \pi$

29. $f(x) = \begin{cases} kx + 1, & \text{if } x \leq 5 \\ 3x - 5, & \text{if } x > 5 \end{cases}$ at $x = 5$

30. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

is a continuous function.

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

33. Examine that $\sin |x|$ is a continuous function.

34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

5.3. Differentiability

Recall the following facts from previous class. We had defined the derivative of a real function as follows:

Suppose f is a real function and c is a point in its domain. The derivative of f at c is defined by

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))|_c$. The function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f. The derivative of f is denoted by $f'(x)$ or $\frac{d}{dx}(f(x))$ or if $y = f(x)$ by $\frac{dy}{dx}$ or y'. The process of finding derivative of a function is called differentiation. We also use the phrase *differentiate $f(x)$ with respect to x* to mean *find $f'(x)$*.

The following rules were established as a part of algebra of derivatives:

(1) $(u \pm v)' = u' \pm v'$

(2) $(uv)' = u'v + uv'$ (Leibnitz or product rule)

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, wherever $v \neq 0$ (Quotient rule).

The following table gives a list of derivatives of certain standard functions:

Table 5.3

$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$

Whenever we defined derivative, we had put a caution *provided the limit exists*. Now the natural question is; what if it doesn't? The question is quite pertinent and so is its answer. If $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ does not exist, we say that f is not differentiable at c. In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$ and $\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal. A function is said to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. As in case of continuity, at the end points a and b, we take the right hand limit and left hand limit, which are nothing but left hand derivative and right hand derivative of the function at a and b respectively. Similarly, a function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

Theorem 3 If a function f is differentiable at a point c, then it is also continuous at that point.

Proof Since f is differentiable at c, we have

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x-c)$$

Therefore $$\lim_{x\to c} [f(x)-f(c)] = \lim_{x\to c} \left[\frac{f(x)-f(c)}{x-c} \cdot (x-c)\right]$$

or $$\lim_{x\to c} [f(x)] - \lim_{x\to c} [f(c)] = \lim_{x\to c} \left[\frac{f(x)-f(c)}{x-c}\right] \cdot \lim_{x\to c} [(x-c)]$$

$$= f'(c) \cdot 0 = 0$$

or $$\lim_{x\to c} f(x) = f(c)$$

Hence f is continuous at $x = c$.

Corollary 1 Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x) = |x|$ is a continuous function. Consider the left hand limit

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \frac{-h}{h} = -1$$

The right hand limit

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \frac{h}{h} = 1$$

Since the above left and right hand limits at 0 are not equal, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

5.3.1 *Derivatives of composite functions*

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of f, where

$$f(x) = (2x + 1)^3$$

One way is to expand $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= \frac{d}{dx}(8x^3 + 12x^2 + 6x + 1)$$

$$= 24x^2 + 24x + 6$$

$$= 6\,(2x + 1)^2$$

Now, observe that $\qquad f(x) = (h \circ g)(x)$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6\,(2x + 1)^2 = 3(2x + 1)^2 \,.\, 2 = 3t^2 \,.\, 2 = \frac{dh}{dt}\cdot\frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say, $(2x + 1)^{100}$. We may formalise this observation in the following theorem called the chain rule.

Theorem 4 (Chain Rule) Let f be a real valued function which is a composite of two functions u and v; i.e., $f = v \circ u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have

$$\frac{df}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose f is a real valued function which is a composite of three functions u, v and w; i.e.,

$f = (w \circ u) \circ v$. If $t = v(x)$ and $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \circ u)}{dt}\cdot\frac{dt}{dx} = \frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

Example 21 Find the derivative of the function given by $f(x) = \sin(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \circ u)(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x. Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

Alternatively, We can also directly proceed as follows:

$$y = \sin (x^2) \Rightarrow \frac{dy}{dx} = \frac{d}{dx} (\sin x^2)$$

$$= \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2$$

Example 22 Find the derivative of tan $(2x + 3)$.

Solution Let $f(x) = \tan (2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \text{ o } u)(x) = v(u(x)) = v(2x + 3) = \tan (2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2 (2x+3)$$

Example 23 Differentiate $\sin (\cos (x^2))$ with respect to x.

Solution The function $f(x) = \sin (\cos (x^2))$ is a composition $f(x) = (w \text{ o } v \text{ o } u)(x)$ of the three functions u, v and w, where $u(x) = x^2$, $v(t) = \cos t$ and $w(s) = \sin s$. Put $t = u(x) = x^2$ and $s = v(t) = \cos t$. Observe that $\frac{dw}{ds} = \cos s$, $\frac{ds}{dt} = -\sin t$ and $\frac{dt}{dx} = 2x$ exist for all real x. Hence by a generalisation of chain rule, we have

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} = (\cos s) . (-\sin t) . (2x) = -2x \sin x^2 . \cos (\cos x^2)$$

Alternatively, we can proceed as follows:

$$y = \sin(\cos x^2)$$

Therefore $$\frac{dy}{dx} = \frac{d}{dx}\sin(\cos x^2) = \cos(\cos x^2)\frac{d}{dx}(\cos x^2)$$

$$= \cos(\cos x^2)(-\sin x^2)\frac{d}{dx}(x^2)$$

$$= -\sin x^2 \cos(\cos x^2)(2x)$$
$$= -2x \sin x^2 \cos(\cos x^2)$$

EXERCISE 5.2

Differentiate the functions with respect to x in Exercises 1 to 8.

1. $\sin(x^2 + 5)$
2. $\cos(\sin x)$
3. $\sin(ax + b)$
4. $\sec(\tan(\sqrt{x}))$
5. $\dfrac{\sin(ax+b)}{\cos(cx+d)}$
6. $\cos x^3 . \sin^2(x^5)$
7. $2\sqrt{\cot(x^2)}$
8. $\cos(\sqrt{x})$
9. Prove that the function f given by
$$f(x) = |x - 1|,\ x \in \mathbf{R}$$
is not differentiable at $x = 1$.
10. Prove that the greatest integer function defined by
$$f(x) = [x],\ 0 < x < 3$$
is not differentiable at $x = 1$ and $x = 2$.

5.3.2 *Derivatives of implicit functions*

Until now we have been differentiating various functions given in the form $y = f(x)$. But it is not necessary that functions are always expressed in this form. For example, consider one of the following relationships between x and y:

$$x - y - \pi = 0$$
$$x + \sin xy - y = 0$$

In the first case, we can *solve for y* and rewrite the relationship as $y = x - \pi$. In the second case, it does not seem that there is an easy way to *solve for y*. Nevertheless, there is no doubt about the dependence of y on x in either of the cases. When a relationship between x and y is expressed in a way that it is easy to *solve for y* and write $y = f(x)$, we say that y is given as an *explicit function* of x. In the latter case it is implicit that y is a function of x and we say that the relationship of the second type, above, gives function *implicitly*. In this subsection, we learn to differentiate implicit functions.

Example 24 Find $\frac{dy}{dx}$ if $x - y = \pi$.

Solution One way is to solve for y and rewrite the above as

$$y = x - \pi$$

But then
$$\frac{dy}{dx} = 1$$

Alternatively, *directly* differentiating the relationship w.r.t., x, we have

$$\frac{d}{dx}(x - y) = \frac{d\pi}{dx}$$

Recall that $\frac{d\pi}{dx}$ means to differentiate the constant function taking value π everywhere w.r.t., x. Thus

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

which implies that

$$\frac{dy}{dx} = \frac{dx}{dx} = 1$$

Example 25 Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$.

Solution We differentiate the relationship directly with respect to x, i.e.,

$$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$

which implies using chain rule

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = -\sin x$$

This gives
$$\frac{dy}{dx} = -\frac{\sin x}{1 + \cos y}$$

where
$$y \neq (2n + 1)\pi$$

5.3.3 *Derivatives of inverse trigonometric functions*

We remark that inverse trigonometric functions are continuous functions, but we will not prove this. Now we use chain rule to find derivatives of these functions.

Example 26 Find the derivative of f given by $f(x) = \sin^{-1} x$ assuming it exists.

Solution Let $y = \sin^{-1} x$. Then, $x = \sin y$.

Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$.

To make this result a bit more attractive, we carry out the following manipulation. Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is positive and hence $\cos y = \sqrt{1 - x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Example 27 Find the derivative of f given by $f(x) = \tan^{-1} x$ assuming it exists.

Solution Let $y = \tan^{-1} x$. Then, $x = \tan y$.

Differentiating both sides w.r.t. x, we get

$$1 = \sec^2 y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + (\tan(\tan^{-1} x))^2} = \frac{1}{1 + x^2}$$

Finding of the derivatives of other inverse trigonometric functions is left as exercise. The following table gives the derivatives of the remaining inverse trigonometric functions (Table 5.4):

Table 5.4

$f(x)$	$\cos^{-1}x$	$\cot^{-1}x$	$\sec^{-1}x$	$\text{cosec}^{-1}x$
$f'(x)$	$\dfrac{-1}{\sqrt{1-x^2}}$	$\dfrac{-1}{1+x^2}$	$\dfrac{1}{x\sqrt{x^2-1}}$	$\dfrac{-1}{x\sqrt{x^2-1}}$
Domain of f'	$(-1, 1)$	**R**	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1) \cup (1, \infty)$

EXERCISE 5.3

Find $\dfrac{dy}{dx}$ in the following:

1. $2x + 3y = \sin x$
2. $2x + 3y = \sin y$
3. $ax + by^2 = \cos y$
4. $xy + y^2 = \tan x + y$
5. $x^2 + xy + y^2 = 100$
6. $x^3 + x^2y + xy^2 + y^3 = 81$
7. $\sin^2 y + \cos xy = \pi$
8. $\sin^2 x + \cos^2 y = 1$
9. $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$
10. $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), \quad -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$
11. $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$
12. $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$
13. $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right), -1 < x < 1$
14. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$
15. $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right), 0 < x < \dfrac{1}{\sqrt{2}}$

5.4 Exponential and Logarithmic Functions

Till now we have learnt some aspects of different classes of functions like polynomial functions, rational functions and trigonometric functions. In this section, we shall learn about a new class of (related) functions called exponential functions and logarithmic functions. It needs to be emphasized that many statements made in this section are motivational and precise proofs of these are well beyond the scope of this text.

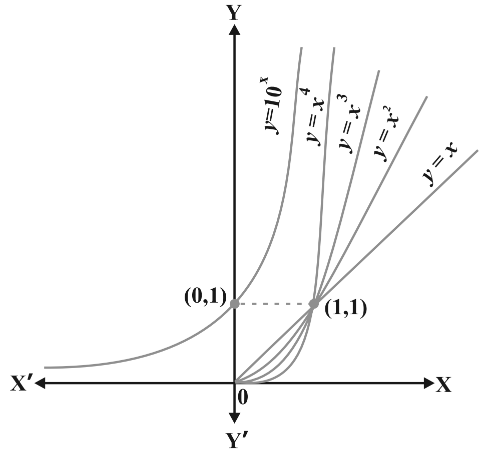


Fig 5.9

The Fig 5.9 gives a sketch of $y = f_1(x) = x$, $y = f_2(x) = x^2$, $y = f_3(x) = x^3$ and $y = f_4(x) = x^4$. Observe that the curves get steeper as the power of x increases. Steeper the curve, faster is the rate of growth. What this means is that for a fixed increment in the value of $x\,(> 1)$, the increment in the value of $y = f_n(x)$ increases as n increases for $n = 1, 2, 3, 4$. It is conceivable that such a statement is true for all positive values of n, where $f_n(x) = x^n$. Essentially, this means that the graph of $y = f_n(x)$ *leans* more towards the y-axis as n increases. For example, consider $f_{10}(x) = x^{10}$ and $f_{15}(x) = x^{15}$. If x increases from 1 to 2, f_{10} increases from 1 to 2^{10} whereas f_{15} increases from 1 to 2^{15}. Thus, for the same increment in x, f_{15} grow faster than f_{10}.

Upshot of the above discussion is that the growth of polynomial functions is dependent on the degree of the polynomial function – higher the degree, greater is the growth. The next natural question is: Is there a function which grows faster than any polynomial function. The answer is in affirmative and an example of such a function is

$$y = f(x) = 10^x.$$

Our claim is that this function f grows faster than $f_n(x) = x^n$ for any positive integer n. For example, we can prove that 10^x grows faster than $f_{100}(x) = x^{100}$. For large values of x like $x = 10^3$, note that $f_{100}(x) = (10^3)^{100} = 10^{300}$ whereas $f(10^3) = 10^{10^3} = 10^{1000}$. Clearly $f(x)$ is much greater than $f_{100}(x)$. It is not difficult to prove that for all $x > 10^3$, $f(x) > f_{100}(x)$. But we will not attempt to give a proof of this here. Similarly, by choosing large values of x, one can verify that $f(x)$ grows faster than $f_n(x)$ for any positive integer n.

Definition 3 The exponential function with positive base $b > 1$ is the function

$$y = f(x) = b^x$$

The graph of $y = 10^x$ is given in the Fig 5.9.

It is advised that the reader plots this graph for particular values of b like 2, 3 and 4. Following are some of the salient features of the exponential functions:

(1) Domain of the exponential function is **R**, the set of all real numbers.

(2) Range of the exponential function is the set of all positive real numbers.

(3) The point (0, 1) is always on the graph of the exponential function (this is a restatement of the fact that $b^0 = 1$ for any real $b > 1$).

(4) Exponential function is ever increasing; i.e., as we move from left to right, the graph rises above.

(5) For very large negative values of x, the exponential function is very close to 0. In other words, in the second quadrant, the graph approaches x-axis (but never meets it).

Exponential function with base 10 is called the *common exponential function*. In the Appendix A.1.4 of Class XI, it was observed that the sum of the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + ...$$

is a number between 2 and 3 and is denoted by e. Using this e as the base we obtain an extremely important exponential function $y = e^x$.

This is called *natural exponential function*.

It would be interesting to know if the inverse of the exponential function exists and has *nice* interpretation. This search motivates the following definition.

Definition 4 Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$.

Logarithm of a to base b is denoted by $\log_b a$. Thus $\log_b a = x$ if $b^x = a$. Let us work with a few explicit examples to get a feel for this. We know $2^3 = 8$. In terms of logarithms, we may rewrite this as $\log_2 8 = 3$. Similarly, $10^4 = 10000$ is equivalent to saying $\log_{10} 10000 = 4$. Also, $625 = 5^4 = 25^2$ is equivalent to saying $\log_5 625 = 4$ or $\log_{25} 625 = 2$.

On a slightly more mature note, fixing a base $b > 1$, we may look at logarithm as a function from positive real numbers to all real numbers. This function, called the *logarithmic function*, is defined by

$$\begin{aligned} \log_b : \mathbf{R}^+ &\to \mathbf{R} \\ x &\to \log_b x = y \quad \text{if } b^y = x \end{aligned}$$

As before if the base $b = 10$, we say it is *common logarithms* and if $b = e$, then we say it is *natural logarithms*. Often natural logarithm is denoted by *ln*. *In this chapter, log x denotes the logarithm function to base e*, i.e., *ln x* will be written as simply log *x*. The Fig 5.10 gives the plots of logarithm function to base 2, *e* and 10.

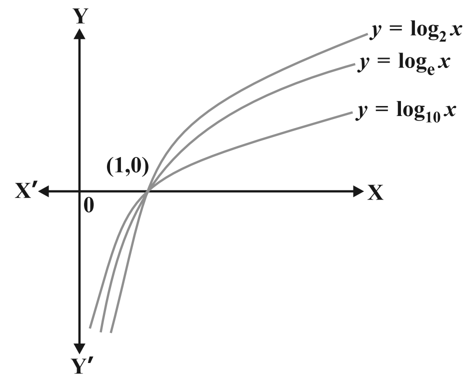


Fig 5.10

Some of the important observations about the logarithm function to any base $b > 1$ are listed below:

(1) We cannot make a meaningful definition of logarithm of non-positive numbers and hence the domain of log function is $\mathbf{R}^+$.

(2) The range of log function is the set of all real numbers.

(3) The point (1, 0) is always on the graph of the log function.

(4) The log function is ever increasing, i.e., as we move from left to right the graph rises above.

(5) For x very near to zero, the value of log x can be made lesser than any given real number. In other words in the fourth quadrant the graph approaches y-axis (but never meets it).

(6) Fig 5.11 gives the plot of $y = e^x$ and $y = ln\ x$. It is of interest to observe that the two curves are the mirror images of each other reflected in the line $y = x$.

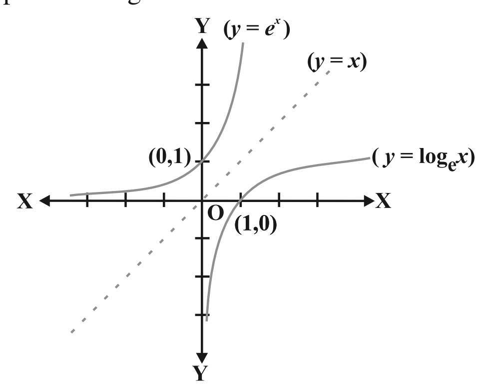


Fig 5.11

Two properties of 'log' functions are proved below:

(1) There is a standard change of base rule to obtain $\log_a p$ in terms of $\log_b p$. Let $\log_a p = \alpha$, $\log_b p = \beta$ and $\log_b a = \gamma$. This means $a^\alpha = p$, $b^\beta = p$ and $b^\gamma = a$.

Substituting the third equation in the first one, we have

$$(b^\gamma)^\alpha = b^{\gamma\alpha} = p$$

Using this in the second equation, we get

$$b^\beta = p = b^{\gamma\alpha}$$

which implies $\beta = \alpha\gamma$ or $\alpha = \dfrac{\beta}{\gamma}$. But then

$$\mathbf{\log_a p = \frac{\log_b p}{\log_b a}}$$

(2) Another interesting property of the log function is its effect on products. Let $\log_b pq = \alpha$. Then $b^\alpha = pq$. If $\log_b p = \beta$ and $\log_b q = \gamma$, then $b^\beta = p$ and $b^\gamma = q$. But then $b^\alpha = pq = b^\beta b^\gamma = b^{\beta + \gamma}$

which implies $\alpha = \beta + \gamma$, i.e.,

$$\mathbf{\log_b pq = \log_b p + \log_b q}$$

A particularly interesting and important consequence of this is when $p = q$. In this case the above may be rewritten as

$$\mathbf{\log_b p^2 = \log_b p + \log_b p = 2 \log p}$$

An easy generalisation of this (left as an exercise!) is

$$\mathbf{\log_b p^n = n \log p}$$

for any positive integer n. In fact this is true for any real number n, but we will not attempt to prove this. On the similar lines the reader is invited to verify

$$\mathbf{\log_b \frac{x}{y} = \log_b x - \log_b y}$$

Example 28 Is it true that $x = e^{\log x}$ for all real x?

Solution First, observe that the domain of log function is set of all positive real numbers. So the above equation is not true for non-positive real numbers. Now, let $y = e^{\log x}$. If $y > 0$, we may take logarithm which gives us $\log y = \log (e^{\log x}) = \log x \,.\, \log e = \log x$. Thus $y = x$. Hence $x = e^{\log x}$ is true only for positive values of x.

One of the striking properties of the natural exponential function in differential calculus is that it doesn't change during the process of differentiation. This is captured in the following theorem whose proof we skip.

Theorem 5

(1) The derivative of e^x w.r.t., x is e^x; i.e., $\dfrac{d}{dx}(e^x) = e^x$.

(2) The derivative of $\log x$ w.r.t., x is $\dfrac{1}{x}$; i.e., $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$.

Example 29 Differentiate the following w.r.t. x:

(i) e^{-x} (ii) $\sin(\log x), x > 0$ (iii) $\cos^{-1}(e^x)$ (iv) $e^{\cos x}$

Solution

(i) Let $y = e^{-x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

(ii) Let $y = \sin(\log x)$. Using chain rule, we have

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

(iii) Let $y = \cos^{-1}(e^x)$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x) = \frac{-e^x}{\sqrt{1-e^{2x}}}$$

(iv) Let $y = e^{\cos x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -(\sin x)\, e^{\cos x}$$

EXERCISE 5.4

Differentiate the following w.r.t. x:

1. $\dfrac{e^x}{\sin x}$ **2.** $e^{\sin^{-1} x}$ **3.** e^{x^3}

4. $\sin(\tan^{-1} e^{-x})$ **5.** $\log(\cos e^x)$ **6.** $e^x + e^{x^2} + \ldots + e^{x^5}$

7. $\sqrt{e^{\sqrt{x}}},\ x > 0$ **8.** $\log(\log x), x > 1$ **9.** $\dfrac{\cos x}{\log x},\ x > 0$

10. $\cos(\log x + e^x), x > 0$

5.5. Logarithmic Differentiation

In this section, we will learn to differentiate certain special class of functions given in the form

$$y = f(x) = [u(x)]^{v(x)}$$

By taking logarithm (to base e) the above may be rewritten as

$$\log y = v(x) \log [u(x)]$$

Using chain rule we may differentiate this to get

$$\frac{1}{y}\cdot\frac{dy}{dx} = v(x)\cdot\frac{1}{u(x)}\cdot u'(x) + v'(x)\cdot\log\,[u(x)]$$

which implies that

$$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}\cdot u'(x) + v'(x)\cdot\log\left[u(x)\right]\right]$$

The main point to be noted in this method is that $f(x)$ and $u(x)$ must always be positive as otherwise their logarithms are not defined. This process of differentiation is known as *logarithms differentiation* and is illustrated by the following examples:

Example 30 Differentiate $\sqrt{\frac{(x-3)\,(x^2+4)}{3x^2+4x+5}}$ w.r.t. x.

Solution Let $y = \sqrt{\frac{(x-3)\,(x^2+4)}{(3x^2+4x+5)}}$

Taking logarithm on both sides, we have

$$\log y = \frac{1}{2}\,[\log\,(x-3) + \log\,(x^2+4) - \log\,(3x^2+4x+5)]$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

or
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$= \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

Example 31 Differentiate a^x w.r.t. x, where a is a positive constant.

Solution Let $y = a^x$. Then

$$\log y = x \log a$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \log a$$

or $$\frac{dy}{dx} = y \log a$$

Thus $$\frac{d}{dx}(a^x) = a^x \log a$$

Alternatively $$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log a}) = e^{x\log a}\,\frac{d}{dx}(x\log a)$$

$$= e^{x \log a} \,.\, \log a = a^x \log a.$$

Example 32 Differentiate $x^{\sin x}$, $x > 0$ w.r.t. x.

Solution Let $y = x^{\sin x}$. Taking logarithm on both sides, we have

$$\log y = \sin x \log x$$

Therefore $$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sin x)$$

or $$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x$$

or $$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x - 1}\cdot \sin x + x^{\sin x}\cdot \cos x \log x$$

Example 33 Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$.

Solution Given that $y^x + x^y + x^x = a^b$.

Putting $u = y^x$, $v = x^y$ and $w = x^x$, we get $u + v + w = a^b$

Therefore $$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \qquad \text{... (1)}$$

Now, $u = y^x$. Taking logarithm on both sides, we have

$$\log u = x \log y$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)$$

$$= x\frac{1}{y}\cdot\frac{dy}{dx} + \log y\cdot 1$$

So $$\frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \quad \ldots (2)$$

Also $v = x^y$

Taking logarithm on both sides, we have

$$\log v = y \log x$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\cdot\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

So $$\frac{dv}{dx} = v\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] \quad \ldots (3)$$

Again $w = x^x$

Taking logarithm on both sides, we have

$$\log w = x \log x.$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{w}\cdot\frac{dw}{dx} = x\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{x} + \log x\cdot 1$$

i.e. $$\frac{dw}{dx} = w\,(1 + \log x)$$

$$= x^x\,(1 + \log x) \quad \ldots (4)$$

From (1), (2), (3), (4), we have

$$y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)+x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right) + x^x\,(1 + \log x) = 0$$

or $\qquad (x \,.\, y^{x-1} + x^y \,.\, \log x)\,\dfrac{dy}{dx} = -\,x^x\,(1 + \log x) - y \,.\, x^{y-1} - y^x \log y$

Therefore $\qquad \dfrac{dy}{dx} = \dfrac{-\,[\,y^x \log y + y \,.\, x^{y-1} + x^x\,(1+\log x)]}{x \,.\, y^{x-1} + x^y \log x}$

EXERCISE 5.5

Differentiate the functions given in Exercises 1 to 11 w.r.t. x.

1. $\cos x \,.\, \cos 2x \,.\, \cos 3x$

2. $\sqrt{\dfrac{(x-1)\,(x-2)}{(x-3)\,(x-4)\,(x-5)}}$

3. $(\log x)^{\cos x}$

4. $x^x - 2^{\sin x}$

5. $(x + 3)^2 \,.\, (x + 4)^3 \,.\, (x + 5)^4$

6. $\left(x+\dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

7. $(\log x)^x + x^{\log x}$

8. $(\sin x)^x + \sin^{-1}\sqrt{x}$

9. $x^{\sin x} + (\sin x)^{\cos x}$

10. $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

11. $(x \cos x)^x + (x\sin x)^{\frac{1}{x}}$

Find $\dfrac{dy}{dx}$ of the functions given in Exercises 12 to 15.

12. $x^y + y^x = 1$

13. $y^x = x^y$

14. $(\cos x)^y = (\cos y)^x$

15. $xy = e^{(x-y)}$

16. Find the derivative of the function given by $f(x) = (1 + x)\,(1 + x^2)\,(1 + x^4)\,(1 + x^8)$ and hence find $f'(1)$.

17. Differentiate $(x^2 - 5x + 8)\,(x^3 + 7x + 9)$ in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

18. If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.\ v.\ w) = \frac{du}{dx}\ v.\ w + u\ .\ \frac{dv}{dx}\ .\ w + u\ .\ v\ \frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

5.6 Derivatives of Functions in Parametric Forms

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed via a third variable. The third variable is called the parameter. More precisely, a relation expressed between two variables x and y in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with t as a parameter.

In order to find derivative of function in such form, we have by chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\text{whenever } \frac{dx}{dt} \neq 0\right)$$

Thus
$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \quad \left(\text{as } \frac{dy}{dt} = g'(t) \text{ and } \frac{dx}{dt} = f'(t)\right) \quad [\text{provided } f'(t) \neq 0]$$

Example 34 Find $\frac{dy}{dx}$, if $x = a \cos \theta$, $y = a \sin \theta$.

Solution Given that

$$x = a \cos \theta,\ y = a \sin \theta$$

Therefore
$$\frac{dx}{d\theta} = -a \sin \theta,\ \frac{dy}{d\theta} = a \cos \theta$$

Hence
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

Example 35 Find $\frac{dy}{dx}$, if $x = at^2$, $y = 2at$.

Solution Given that $x = at^2$, $y = 2at$

So $$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

Example 36 Find $\frac{dy}{dx}$, if $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$.

Solution We have $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\,(\sin\theta)$

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a\,(1+\cos\theta)} = \tan\frac{\theta}{2}$$

> **Note** It may be noted here that $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables x and y.

Example 37 Find $\frac{dy}{dx}$, if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution Let $x = a\cos^3\theta$, $y = a\sin^3\theta$. Then

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a\cos^3\theta)^{\frac{2}{3}} + (a\sin^3\theta)^{\frac{2}{3}}$$
$$= a^{\frac{2}{3}}(\cos^2\theta + (\sin^2\theta) = a^{\frac{2}{3}}$$

Hence, $x = a\cos^3\theta$, $y = a\sin^3\theta$ is parametric equation of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Now $$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta = -\sqrt[3]{\frac{y}{x}}$$

Note Had we proceeded in implicit way, it would have been quite tedious.

EXERCISE 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

1. $x = 2at^2$, $y = at^4$
2. $x = a\cos\theta$, $y = b\cos\theta$
3. $x = \sin t$, $y = \cos 2t$
4. $x = 4t$, $y = \frac{4}{t}$
5. $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$
6. $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$
7. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
8. $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$ $y = a\sin t$
9. $x = a\sec\theta$, $y = b\tan\theta$
10. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$
11. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

5.7 Second Order Derivative

Let $y = f(x)$. Then

$$\frac{dy}{dx} = f'(x) \qquad \text{... (1)}$$

If $f'(x)$ is differentiable, we may differentiate (1) again w.r.t. x. Then, the left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is called the *second order derivative* of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also denoted by $D^2 y$ or y'' or y_2 if $y = f(x)$. We remark that higher order derivatives may be defined similarly.

Example 38 Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

Solution Given that $y = x^3 + \tan x$. Then

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

Therefore
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 + \sec^2 x\right)$$
$$= 6x + 2 \sec x \,.\, \sec x \tan x = 6x + 2 \sec^2 x \tan x$$

Example 39 If $y = A \sin x + B \cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.

Solution We have

$$\frac{dy}{dx} = A \cos x - B \sin x$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}(A \cos x - B \sin x)$$
$$= - A \sin x - B \cos x = - y$$

Hence
$$\frac{d^2y}{dx^2} + y = 0$$

Example 40 If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Solution Given that $y = 3e^{2x} + 2e^{3x}$. Then

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6\,(e^{2x} + e^{3x})$$

Therefore
$$\frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x} = 6\,(2e^{2x} + 3e^{3x})$$

Hence
$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6\,(2e^{2x} + 3e^{3x})$$
$$- 30\,(e^{2x} + e^{3x}) + 6\,(3e^{2x} + 2e^{3x}) = 0$$

Example 41 If $y = \sin^{-1} x$, show that $(1 - x^2)\ \dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} = 0$.

Solution We have $y = \sin^{-1}x$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

or
$$\sqrt{(1-x^2)}\ \frac{dy}{dx} = 1$$

So
$$\frac{d}{dx}\left(\sqrt{(1-x^2)}\ .\ \frac{dy}{dx}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2 y}{dx^2} + \frac{dy}{dx}\cdot\frac{d}{dx}\left(\sqrt{(1-x^2)}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2 y}{dx^2} - \frac{dy}{dx}\cdot\frac{2x}{2\sqrt{1-x^2}} = 0$$

Hence
$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} = 0$$

Alternatively, Given that $y = \sin^{-1} x$, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}\text{, i.e., } \left(1-x^2\right) y_1^2 = 1$$

So
$$(1-x^2)\ .\ 2y_1 y_2 + y_1^2(0-2x) = 0$$

Hence
$$(1 - x^2)\ y_2 - xy_1 = 0$$

EXERCISE 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

1. $x^2 + 3x + 2$ **2.** x^{20} **3.** $x\ .\ \cos x$

4. $\log x$ **5.** $x^3 \log x$ **6.** $e^x \sin 5x$

7. $e^{6x} \cos 3x$ **8.** $\tan^{-1} x$ **9.** $\log (\log x)$

10. $\sin (\log x)$

11. If $y = 5 \cos x - 3 \sin x$, prove that $\dfrac{d^2 y}{dx^2} + y = 0$

12. If $y = \cos^{-1} x$, Find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

13. If $y = 3 \cos (\log x) + 4 \sin (\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

14. If $y = Ae^{mx} + Be^{nx}$, show that $\dfrac{d^2y}{dx^2} - (m+n)\dfrac{dy}{dx} + mny = 0$

15. If $y = 500e^{7x} + 600e^{-7x}$, show that $\dfrac{d^2y}{dx^2} = 49y$

16. If $e^y (x + 1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

5.8 Mean Value Theorem

In this section, we will state two fundamental results in Calculus without proof. We shall also learn the geometric interpretation of these theorems.

Theorem 6 (Rolle's Theorem) Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b), such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists some c in (a, b) such that $f'(c) = 0$.

In Fig 5.12 and 5.13, graphs of a few typical differentiable functions satisfying the hypothesis of Rolle's theorem are given.

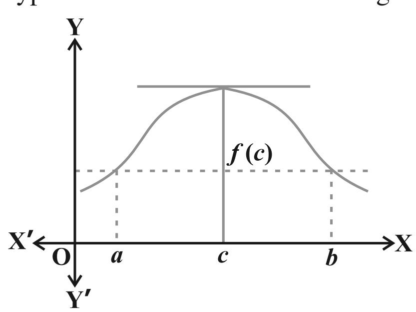


Fig 5.12

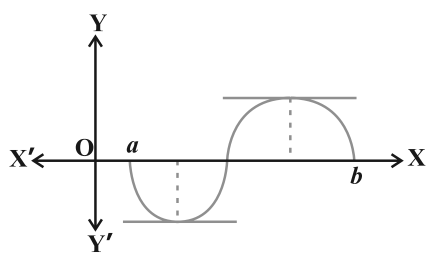


Fig 5.13

Observe what happens to the slope of the tangent to the curve at various points between a and b. In each of the graphs, the slope becomes zero at least at one point. That is precisely the claim of the Rolle's theorem as the slope of the tangent at any point on the graph of $y = f(x)$ is nothing but the derivative of $f(x)$ at that point.

Theorem 7 (Mean Value Theorem) Let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Observe that the Mean Value Theorem (MVT) is an extension of Rolle's theorem. Let us now understand a geometric interpretation of the MVT. The graph of a function $y = f(x)$ is given in the Fig 5.14. We have already interpreted $f'(c)$ as the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. From the Fig 5.14 it is clear that $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant drawn between $(a, f(a))$ and $(b, f(b))$. The MVT states that there is a point c in (a, b) such that the slope of the tangent at $(c, f(c))$ is same as the slope of the secant between $(a, f(a))$ and $(b, f(b))$. In other words, there is a point c in (a, b) such that the tangent at $(c, f(c))$ is parallel to the secant between $(a, f(a))$ and $(b, f(b))$.

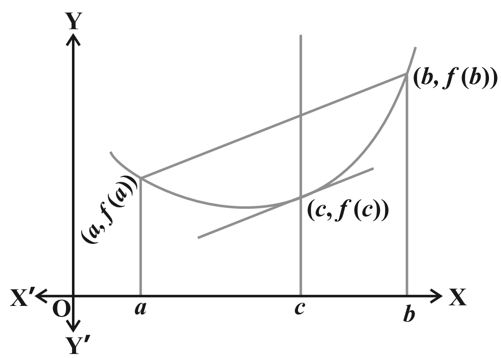


Fig 5.14

Example 42 Verify Rolle's theorem for the function $y = x^2 + 2$, $a = -2$ and $b = 2$.

Solution The function $y = x^2 + 2$ is continuous in $[-2, 2]$ and differentiable in $(-2, 2)$. Also $f(-2) = f(2) = 6$ and hence the value of $f(x)$ at -2 and 2 coincide. Rolle's theorem states that there is a point $c \in (-2, 2)$, where $f'(c) = 0$. Since $f'(x) = 2x$, we get $c = 0$. Thus at $c = 0$, we have $f'(c) = 0$ and $c = 0 \in (-2, 2)$.

Example 43 Verify Mean Value Theorem for the function $f(x) = x^2$ in the interval $[2, 4]$.

Solution The function $f(x) = x^2$ is continuous in $[2, 4]$ and differentiable in $(2, 4)$ as its derivative $f'(x) = 2x$ is defined in $(2, 4)$.

Now, $f(2) = 4$ and $f(4) = 16$. Hence

$$\frac{f(b)-f(a)}{b-a} = \frac{16-4}{4-2} = 6$$

MVT states that there is a point $c \in (2, 4)$ such that $f'(c) = 6$. But $f'(x) = 2x$ which implies $c = 3$. Thus at $c = 3 \in (2, 4)$, we have $f'(c) = 6$.

EXERCISE 5.8

1. Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.
2. Examine if Rolle's theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's theorem from these example?
   (i) $f(x) = [x]$ for $x \in [5, 9]$ (ii) $f(x) = [x]$ for $x \in [-2, 2]$
   (iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$
3. If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.
4. Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.
5. Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.
6. Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Miscellaneous Examples

Example 44 Differentiate w.r.t. x, the following function:

(i) $\sqrt{3x+2} + \dfrac{1}{\sqrt{2x^2+4}}$ (ii) $e^{\sec^2 x} + 3\cos^{-1} x$ (iii) $\log_7 (\log x)$

Solution

(i) Let $y = \sqrt{3x+2} + \dfrac{1}{\sqrt{2x^2+4}} = (3x+2)^{\frac{1}{2}} + (2x^2+4)^{-\frac{1}{2}}$

Note that this function is defined at all real numbers $x > -\dfrac{2}{3}$. Therefore

$$\frac{dy}{dx} = \frac{1}{2}(3x+2)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(3x+2) + \left(-\frac{1}{2}\right)(2x^2+4)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(2x^2+4)$$

$$= \frac{1}{2}(3x+2)^{-\frac{1}{2}} \cdot (3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}} \cdot 4x$$

$$= \frac{3}{2\sqrt{3x+2}} - \frac{2x}{\left(2x^2+4\right)^{\frac{3}{2}}}$$

This is defined for all real numbers $x > -\frac{2}{3}$.

(ii) Let $y = e^{\sec^2 x} + 3\cos^{-1} x$

This is defined at every real number in $[-1, 1] - \{0\}$. Therefore

$$\frac{dy}{dx} = e^{\sec^2 x} \cdot \frac{d}{dx}(\sec^2 x) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= e^{\sec^2 x} \cdot \left(2\sec x \frac{d}{dx}(\sec x)\right) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= 2\sec x\,(\sec x \tan x)\, e^{\sec^2 x} + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= 2\sec^2 x \tan x\, e^{\sec^2 x} + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

Observe that the derivative of the given function is valid only in $[-1, 1] - \{0\}$ as the derivative of $\cos^{-1} x$ exists only in $(-1, 1)$ and the function itself is not defined at 0.

(iii) Let $y = \log_7 (\log x) = \frac{\log(\log x)}{\log 7}$ (by change of base formula).

The function is defined for all real numbers $x > 1$. Therefore

$$\frac{dy}{dx} = \frac{1}{\log 7}\frac{d}{dx}(\log(\log x))$$

$$= \frac{1}{\log 7}\frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{x \log 7 \log x}$$

Example 45 Differentiate the following w.r.t. x.

(i) $\cos^{-1}(\sin x)$ (ii) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ (iii) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

Solution

(i) Let $f(x) = \cos^{-1}(\sin x)$. Observe that this function is defined for all real numbers. We may rewrite this function as

$$f(x) = \cos^{-1}(\sin x)$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= \frac{\pi}{2} - x$$

Thus $f'(x) = -1$.

(ii) Let $f(x) = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$. Observe that this function is defined for all real numbers, where $\cos x \neq -1$; i.e., at all odd multiples of π. We may rewrite this function as

$$f(x) = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{x}{2}\right)\right] = \frac{x}{2}$$

Observe that we could cancel $\cos\left(\frac{x}{2}\right)$ in both numerator and denominator as it is not equal to zero. Thus $f'(x) = \frac{1}{2}$.

(iii) Let $f(x) = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$. To find the domain of this function we need to find all x such that $-1 \leq \frac{2^{x+1}}{1+4^x} \leq 1$. Since the quantity in the middle is always positive, we need to find all x such that $\frac{2^{x+1}}{1+4^x} \leq 1$, i.e., all x such that $2^{x+1} \leq 1 + 4^x$. We may rewrite this as $2 \leq \frac{1}{2^x} + 2^x$ which is true for all x. Hence the function is defined at every real number. By putting $2^x = \tan\theta$, this function may be rewritten as

$$f(x) = \sin^{-1}\left[\frac{2^{x+1}}{1+4^x}\right]$$

$$= \sin^{-1}\left[\frac{2^x \cdot 2}{1+(2^x)^2}\right]$$

$$= \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= \sin^{-1}[\sin 2\theta]$$

$$= 2\theta = 2\tan^{-1}(2^x)$$

Thus $$f'(x) = 2\cdot\frac{1}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x)$$

$$= \frac{2}{1+4^x}\cdot(2^x)\log 2$$

$$= \frac{2^{x+1}\log 2}{1+4^x}$$

Example 46 Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $0 < x < \pi$.

Solution The function $y = (\sin x)^{\sin x}$ is defined for all positive real numbers. Taking logarithms, we have

$$\log y = \log(\sin x)^{\sin x} = \sin x \log(\sin x)$$

Then $$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\sin x \log(\sin x))$$

$$= \cos x \log(\sin x) + \sin x \,.\, \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)$$

$$= \cos x \log(\sin x) + \cos x$$

$$= (1 + \log(\sin x))\cos x$$

Thus $$\frac{dy}{dx} = y((1 + \log (\sin x)) \cos x) = (1 + \log (\sin x)) (\sin x)^{\sin x} \cos x$$

Example 47 For a positive constant a find $\frac{dy}{dx}$, where

$$y = a^{t+\frac{1}{t}}, \text{ and } x = \left(t + \frac{1}{t}\right)^a$$

Solution Observe that both y and x are defined for all real $t \neq 0$. Clearly

$$\frac{dy}{dt} = \frac{d}{dt}\left(a^{t+\frac{1}{t}}\right) = a^{t+\frac{1}{t}} \frac{d}{dt}\left(t + \frac{1}{t}\right) \cdot \log a$$

$$= a^{t+\frac{1}{t}}\left(1 - \frac{1}{t^2}\right) \log a$$

Similarly $$\frac{dx}{dt} = a\left[t + \frac{1}{t}\right]^{a-1} \cdot \frac{d}{dt}\left(t + \frac{1}{t}\right)$$

$$= a\left[t + \frac{1}{t}\right]^{a-1} \cdot \left(1 - \frac{1}{t^2}\right)$$

$\frac{dx}{dt} \neq 0$ only if $t \neq \pm 1$. Thus for $t \neq \pm 1$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a^{t+\frac{1}{t}}\left(1 - \frac{1}{t^2}\right) \log a}{a\left[t + \frac{1}{t}\right]^{a-1} \cdot \left(1 - \frac{1}{t^2}\right)}$$

$$= \frac{a^{t+\frac{1}{t}} \log a}{a\left(t + \frac{1}{t}\right)^{a-1}}$$

Example 48 Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Solution Let $u(x) = \sin^2 x$ and $v(x) = e^{\cos x}$. We want to find $\frac{du}{dv} = \frac{du/dx}{dv/dx}$. Clearly

$$\frac{du}{dx} = 2 \sin x \cos x \text{ and } \frac{dv}{dx} = e^{\cos x} (-\sin x) = -(\sin x)\, e^{\cos x}$$

Thus $$\frac{du}{dv} = \frac{2\sin x\cos x}{-\sin x\, e^{\cos x}} = -\frac{2\cos x}{e^{\cos x}}$$

Miscellaneous Exercise on Chapter 5

Differentiate w.r.t. x the function in Exercises 1 to 11.

1. $(3x^2 - 9x + 5)^9$

2. $\sin^3 x + \cos^6 x$

3. $(5x)^{3\cos 2x}$

4. $\sin^{-1}(x\sqrt{x}),\ 0 \le x \le 1$

5. $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},\ -2 < x < 2$

6. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],\ 0 < x < \dfrac{\pi}{2}$

7. $(\log x)^{\log x},\ x > 1$

8. $\cos(a\cos x + b\sin x)$, for some constant a and b.

9. $(\sin x - \cos x)^{(\sin x - \cos x)},\ \dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$

10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

11. $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

12. Find $\dfrac{dy}{dx}$, if $y = 12(1 - \cos t),\ x = 10(t - \sin t),\ -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

13. Find $\dfrac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2},\ -1 \le x \le 1$

14. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for , $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

15. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

16. If $\cos y = x \cos (a + y)$, with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$.

17. If $x = a (\cos t + t \sin t)$ and $y = a (\sin t - t \cos t)$, find $\dfrac{d^2y}{dx^2}$.

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

19. Using mathematical induction prove that $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ for all positive integers n.

20. Using the fact that $\sin (A + B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

22. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

23. If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$, show that $\left(1 - x^2\right)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2 y = 0$.

Summary

- A real valued function is ***continuous*** at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.
- Sum, difference, product and quotient of continuous functions are continuous. i.e., if f and g are continuous functions, then

  $(f \pm g)(x) = f(x) \pm g(x)$ is continuous.

  $(f \,.\, g)(x) = f(x) \,.\, g(x)$ is continuous.

  $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ (wherever $g(x) \neq 0$) is continuous.
- Every differentiable function is continuous, but the converse is not true.

- Chain rule is rule to differentiate composites of functions. If $f = v \circ u$, $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

- Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \frac{d}{dx}\left(\log x\right) = \frac{1}{x}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both $f(x)$ and $u(x)$ need to be positive for this technique to make sense.
- ***Rolle's Theorem***: If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there exists some c in (a, b) such that $f'(c) = 0$.
- ***Mean Value Theorem***: If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Chapter 6

APPLICATION OF DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature." — WHITEHEAD* ❖

6.1 Introduction

In Chapter 5, we have learnt how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions. In this chapter, we will study applications of the derivative in various disciplines, e.g., in engineering, science, social science, and many other fields. For instance, we will learn how the derivative can be used (i) to determine rate of change of quantities, (ii) to find the equations of tangent and normal to a curve at a point, (iii) to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs. We will also use derivative to find intervals on which a function is increasing or decreasing. Finally, we use the derivative to find approximate value of certain quantities.

6.2 Rate of Change of Quantities

Recall that by the derivative $\frac{ds}{dt}$, we mean the rate of change of distance s with respect to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.

Let us consider some examples.

Example 1 Find the rate of change of the area of a circle per second with respect to its radius r when $r = 5$ cm.

Solution The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of the area A with respect to its radius r is given by $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$.

When $r = 5$ cm, $\frac{dA}{dr} = 10\pi$. Thus, the area of the circle is changing at the rate of 10π cm^2/s.

Example 2 The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

Solution Let x be the length of a side, V be the volume and S be the surface area of the cube. Then, $V = x^3$ and $S = 6x^2$, where x is a function of time t.

Now $$\frac{dV}{dt} = 9\text{cm}^3/\text{s (Given)}$$

Therefore $$9 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 3x^2\cdot\frac{dx}{dt}$$

or $$\frac{dx}{dt} = \frac{3}{x^2} \quad \text{... (1)}$$

Now $$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 12x\cdot\left(\frac{3}{x^2}\right) = \frac{36}{x} \quad \text{(Using (1))}$$

Hence, when $x = 10$ cm, $\frac{dS}{dt} = 3.6\,\text{cm}^2/\text{s}$

Example 3 A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution The area A of a circle with radius r is given by A = πr^2. Therefore, the rate of change of area A with respect to time t is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{(By Chain Rule)}$$

It is given that $\frac{dr}{dt} = 4\text{cm/s}$

Therefore, when r = 10 cm, $\frac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Thus, the enclosed area is increasing at the rate of 80π cm²/s, when r = 10 cm.

> **Note** $\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

Example 4 The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Solution Since the length x is decreasing and the width y is increasing with respect to time, we have

$$\frac{dx}{dt} = -3 \text{ cm/min} \quad \text{and} \quad \frac{dy}{dt} = 2 \text{ cm/min}$$

(a) The perimeter P of a rectangle is given by

$$P = 2(x + y)$$

Therefore $$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-3 + 2) = -2 \text{ cm/min}$$

(b) The area A of the rectangle is given by

$$A = x \cdot y$$

Therefore $$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$$= -3(6) + 10(2) \quad (\text{as } x = 10 \text{ cm and } y = 6 \text{ cm})$$

$$= 2 \text{ cm}^2/\text{min}$$

Example 5 The total cost C(x) in Rupees, associated with the production of x units of an item is given by

$$C(x) = 0.005\, x^3 - 0.02\, x^2 + 30x + 5000$$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution Since marginal cost is the rate of change of total cost with respect to the output, we have

Marginal cost (MC) = $\frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$

When $x = 3$, MC = $0.015(3^2) - 0.04(3) + 30$

$= 0.135 - 0.12 + 30 = 30.015$

Hence, the required marginal cost is Rs 30.02 (nearly).

Example 6 The total revenue in Rupees received from the sale of x units of a product is given by R(x) = $3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Solution Since marginal revenue is the rate of change of total revenue with respect to the number of units sold, we have

Marginal Revenue (MR) = $\frac{dR}{dx} = 6x + 36$

When $x = 5$, MR = $6(5) + 36 = 66$

Hence, the required marginal revenue is Rs 66.

EXERCISE 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when
   (a) $r = 3$ cm (b) $r = 4$ cm
2. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?
3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.
4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$cm and $y = 6$cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.
8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.
10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?
11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.
12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.
14. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
15. The total cost C (x) in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

Find the marginal cost when 17 units are produced.
16. The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

Choose the correct answer in the Exercises 17 and 18.

17. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is
(A) 10π (B) 12π (C) 8π (D) 11π

18. The total revenue in Rupees received from the sale of x units of a product is given by
$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is
(A) 116 (B) 96 (C) 90 (D) 126

6.3 Increasing and Decreasing Functions

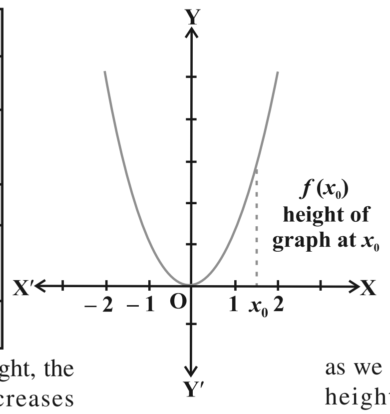
In this section, we will use differentiation to find out whether a function is increasing or decreasing or none.

Consider the function f given by $f(x) = x^2$, $x \in \mathbf{R}$. The graph of this function is a parabola as given in Fig 6.1.

Values left to origin

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0

as we move from left to right, the height of the graph decreases



Values right to origin

x	$f(x) = x^2$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

as we move from left to right, the height of the graph increases

Fig 6.1

First consider the graph (Fig 6.1) to the right of the origin. Observe that as we move from left to right along the graph, the height of the graph continuously increases. For this reason, the function is said to be increasing for the real numbers $x > 0$.

Now consider the graph to the left of the origin and observe here that as we move from left to right along the graph, the height of the graph continuously decreases. Consequently, the function is said to be decreasing for the real numbers $x < 0$.

We shall now give the following analytical definitions for a function which is increasing or decreasing on an interval.

Definition 1 Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in$ I.

(ii) strictly increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(iv) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

For graphical representation of such functions see Fig 6.2.

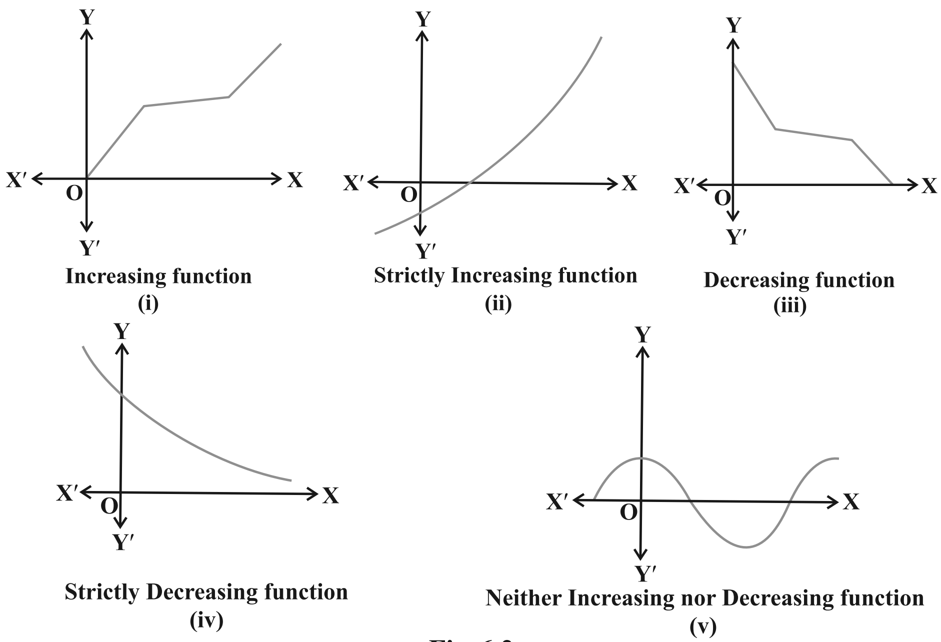


Fig 6.2

We shall now define when a function is increasing or decreasing at a point.

Definition 2 Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing, strictly increasing, decreasing or strictly decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, strictly increasing, decreasing or strictly decreasing, respectively, in I.

Let us clarify this definition for the case of increasing function.

A function f is said to be increasing at x_0 if there exists an interval I = $(x_0 - h, x_0 + h)$, $h > 0$ such that for $x_1, x_2 \in$ I

$$x_1 < x_2 \text{ in I} \Rightarrow f(x_1) \leq f(x_2)$$

Similarly, the other cases can be clarified.

Example 7 Show that the function given by $f(x) = 7x - 3$ is strictly increasing on **R**.

Solution Let x_1 and x_2 be any two numbers in **R**. Then

$$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, by Definition 1, it follows that f is strictly increasing on **R**.

We shall now give the first derivative test for increasing and decreasing functions. The proof of this test requires the Mean Value Theorem studied in Chapter 5.

Theorem 1 Let f be continuous on $[a, b]$ and differentiable on the open interval (a,b). Then

(a) f is increasing in $[a,b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in $[a,b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in $[a,b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Proof (a) Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$.

Then, by Mean Value Theorem (Theorem 8 in Chapter 5), there exists a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1)$$

i.e. $$f(x_2) - f(x_1) > 0 \qquad \text{(as } f'(c) > 0 \text{ (given))}$$

i.e. $$f(x_2) > f(x_1)$$

Thus, we have

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \text{for all } x_1, x_2 \in [a,b]$$

Hence, f is an increasing function in $[a,b]$.

The proofs of part (b) and (c) are similar. It is left as an exercise to the reader.

Remarks

(i) f is strictly increasing in (a, b) if $f'(x) > 0$ for each $x \in (a, b)$

(ii) f is strictly decreasing in (a, b) if $f'(x) < 0$ for each $x \in (a, b)$

(iii) A function will be increasing (decreasing) in **R** if it is so in every interval of **R**.

Example 8 Show that the function f given by

$$f(x) = x^3 - 3x^2 + 4x,\ x \in \mathbf{R}$$

is strictly increasing on **R**.

Solution Note that

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 4 \\ &= 3(x^2 - 2x + 1) + 1 \\ &= 3(x-1)^2 + 1 > 0, \text{ in every interval of } \mathbf{R} \end{aligned}$$

Therefore, the function f is strictly increasing on **R**.

Example 9 Prove that the function given by $f(x) = \cos x$ is

(a) strictly decreasing in $(0, \pi)$

(b) strictly increasing in $(\pi, 2\pi)$, and

(c) neither increasing nor decreasing in $(0, 2\pi)$.

Solution Note that $f'(x) = -\sin x$

(a) Since for each $x \in (0, \pi)$, $\sin x > 0$, we have $f'(x) < 0$ and so f is strictly decreasing in $(0, \pi)$.

(b) Since for each $x \in (\pi, 2\pi)$, $\sin x < 0$, we have $f'(x) > 0$ and so f is strictly increasing in $(\pi, 2\pi)$.

(c) Clearly by (a) and (b) above, f is neither increasing nor decreasing in $(0, 2\pi)$.

> **Note** One may note that the function in Example 9 is neither strictly increasing in $[\pi, 2\pi]$ nor strictly decreasing in $[0, \pi]$. However, since the function is continuous at the end points 0 and π, by Theorem 1, f is increasing in $[\pi, 2\pi]$ and decreasing in $[0, \pi]$.

Example 10 Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is

(a) strictly increasing (b) strictly decreasing

Solution We have

$$f(x) = x^2 - 4x + 6$$

or

$$f'(x) = 2x - 4$$

Therefore, $f'(x) = 0$ gives $x = 2$. Now the point $x = 2$ divides the real line into two disjoint intervals namely, $(-\infty, 2)$ and $(2, \infty)$ (Fig 6.3). In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$.

Therefore, f is strictly decreasing in this interval. Also, in the interval $(2, \infty)$, $f'(x) > 0$ and so the function f is strictly increasing in this interval.

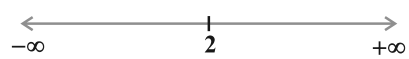


Fig 6.3

> **Note** Note that the given function is continuous at 2 which is the point joining the two intervals. So, by Theorem 1, we conclude that the given function is decreasing in $(-\infty, 2]$ and increasing in $[2, \infty)$.

Example 11 Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) strictly increasing (b) strictly decreasing.

Solution We have

$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

or $$f'(x) = 12x^2 - 12x - 72$$
$$= 12(x^2 - x - 6)$$
$$= 12(x - 3)(x + 2)$$

Therefore, $f'(x) = 0$ gives $x = -2, 3$. The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Fig 6.4

In the intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in the interval $(-2, 3)$, $f'(x)$ is negative. Consequently, the function f is strictly increasing in the intervals $(-\infty, -2)$ and $(3, \infty)$ while the function is strictly decreasing in the interval $(-2, 3)$. However, f is neither increasing nor decreasing in **R**.

Interval	Sign of $f'(x)$	Nature of function f
$(-\infty, -2)$	$(-)(-) > 0$	f is strictly increasing
$(-2, 3)$	$(-)(+) < 0$	f is strictly decreasing
$(3, \infty)$	$(+)(+) > 0$	f is strictly increasing

Example 12 Find intervals in which the function given by $f(x) = \sin 3x$, $x \in \left[0, \frac{\pi}{2}\right]$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \sin 3x$$

or $$f'(x) = 3\cos 3x$$

Therefore, $f'(x) = 0$ gives $\cos 3x = 0$ which in turn gives $3x = \frac{\pi}{2}, \frac{3\pi}{2}$ (as $x \in \left[0, \frac{\pi}{2}\right]$ implies $3x \in \left[0, \frac{3\pi}{2}\right]$). So $x = \frac{\pi}{6}$ and $\frac{\pi}{2}$. The point $x = \frac{\pi}{6}$ divides the interval $\left[0, \frac{\pi}{2}\right]$ into two disjoint intervals $\left[0, \frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{2}\right]$.

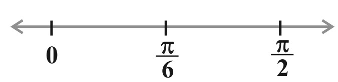


Fig 6.5

Now, $f'(x) > 0$ for all $x \in \left[0, \frac{\pi}{6}\right)$ as $0 \le x < \frac{\pi}{6} \Rightarrow 0 \le 3x < \frac{\pi}{2}$ and $f'(x) < 0$ for all $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ as $\frac{\pi}{6} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3x < \frac{3\pi}{2}$.

Therefore, f is strictly increasing in $\left[0, \frac{\pi}{6}\right)$ and strictly decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

Also, the given function is continuous at $x = 0$ and $x = \frac{\pi}{6}$. Therefore, by Theorem 1, f is increasing on $\left[0, \frac{\pi}{6}\right]$ and decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Example 13 Find the intervals in which the function f given by

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

is strictly increasing or strictly decreasing.

Solution We have

$$f(x) = \sin x + \cos x,$$

or $$f'(x) = \cos x - \sin x$$

Now $f'(x) = 0$ gives $\sin x = \cos x$ which gives that $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2\pi$

The points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ divide the interval $[0, 2\pi]$ into three disjoint intervals, namely, $\left[0, \frac{\pi}{4}\right)$, $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

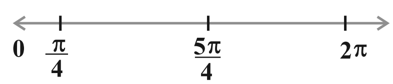


Fig 6.6

Note that $f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

or f is strictly increasing in the intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$

Also $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

or f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right)$	> 0	f is strictly increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	< 0	f is strictly decreasing
$\left(\frac{5\pi}{4}, 2\pi\right]$	> 0	f is strictly increasing

EXERCISE 6.2

1. Show that the function given by $f(x) = 3x + 17$ is strictly increasing on **R**.
2. Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.
3. Show that the function given by $f(x) = \sin x$ is

   (a) strictly increasing in $\left(0, \frac{\pi}{2}\right)$ (b) strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$

   (c) neither increasing nor decreasing in $(0, \pi)$
4. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

   (a) strictly increasing (b) strictly decreasing
5. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

   (a) strictly increasing (b) strictly decreasing
6. Find the intervals in which the following functions are strictly increasing or decreasing:

   (a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$

   (c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$

   (e) $(x + 1)^3 (x - 3)^3$
7. Show that $y = \log(1 + x) - \frac{2x}{2 + x}$, $x > -1$, is an increasing function of x throughout its domain.
8. Find the values of x for which $y = [x(x - 2)]^2$ is an increasing function.
9. Prove that $y = \frac{4\sin\theta}{(2 + \cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

10. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.

12. Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing ?

(A) $(0,1)$ (B) $\left(\frac{\pi}{2}, \pi\right)$ (C) $\left(0, \frac{\pi}{2}\right)$ (D) None of these

14. Find the least value of a such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on $(1, 2)$.

15. Let I be any interval disjoint from $(-1, 1)$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

16. Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

19. The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

6.4 Tangents and Normals

In this section, we shall use differentiation to find the equation of the tangent line and the normal line to a curve at a given point.

Recall that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by

$$y - y_0 = m(x - x_0)$$

Note that the slope of the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} (= f'(x_0))$. So the equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by

$$y - y_0 = f'(x_0)(x - x_0)$$

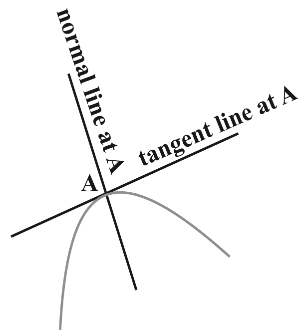


Fig 6.7

Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_0, y_0) is $\frac{-1}{f'(x_0)}$, if $f'(x_0) \neq 0$. Therefore, the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is given by

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$$

i.e. $\quad (y - y_0) f'(x_0) + (x - x_0) = 0$

Note If a tangent line to the curve $y = f(x)$ makes an angle θ with x-axis in the positive direction, then $\frac{dy}{dx}$ = slope of the tangent = $\tan\theta$.

Particular cases

(i) If slope of the tangent line is zero, then $\tan \theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point (x_0, y_0) is given by $y = y_0$.

(ii) If $\theta \to \frac{\pi}{2}$, then $\tan \theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $x = x_0$ (Why?).

Example 14 Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

Solution The slope of the tangent at $x = 2$ is given by

$$\left.\frac{dy}{dx}\right]_{x=2} = \left.3x^2 - 1\right]_{x=2} = 11.$$

Example 15 Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

Solution Slope of tangent to the given curve at (x, y) is

$$\frac{dy}{dx} = \frac{1}{2}(4x-3)^{\frac{-1}{2}} 4 = \frac{2}{\sqrt{4x-3}}$$

The slope is given to be $\frac{2}{3}$.

So $$\frac{2}{\sqrt{4x-3}} = \frac{2}{3}$$

or $$4x - 3 = 9$$

or $$x = 3$$

Now $y = \sqrt{4x-3} - 1$. So when $x = 3$, $y = \sqrt{4(3)-3} - 1 = 2$.

Therefore, the required point is (3, 2).

Example 16 Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

Solution Slope of the tangent to the given curve at any point (x,y) is given by

$$\frac{dy}{dx} = \frac{2}{(x-3)^2}$$

But the slope is given to be 2. Therefore

$$\frac{2}{(x-3)^2} = 2$$

or $$(x-3)^2 = 1$$

or $$x - 3 = \pm 1$$

or $$x = 2, 4$$

Now $x = 2$ gives $y = 2$ and $x = 4$ gives $y = -2$. Thus, there are two tangents to the given curve with slope 2 and passing through the points (2, 2) and (4, – 2). The equation of tangent through (2, 2) is given by

$$y - 2 = 2(x - 2)$$

or $$y - 2x + 2 = 0$$

and the equation of the tangent through (4, – 2) is given by

$$y - (-2) = 2(x - 4)$$

or $$y - 2x + 10 = 0$$

Example 17 Find points on the curve $\frac{x^2}{4}+\frac{y^2}{25}=1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Solution Differentiating $\frac{x^2}{4}+\frac{y^2}{25}=1$ with respect to x, we get

$$\frac{x}{2}+\frac{2y}{25}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = \frac{-25}{4}\frac{x}{y}$$

(i) Now, the tangent is parallel to the x-axis if the slope of the tangent is zero which gives $\frac{-25}{4}\frac{x}{y}=0$. This is possible if $x = 0$. Then $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $x = 0$ gives $y^2 = 25$, i.e., $y = \pm 5$.

Thus, the points at which the tangents are parallel to the x-axis are (0, 5) and (0, – 5).

(ii) The tangent line is parallel to y-axis if the slope of the normal is 0 which gives $\frac{4y}{25x}=0$, i.e., $y = 0$. Therefore, $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $y = 0$ gives $x = \pm 2$. Hence, the points at which the tangents are parallel to the y-axis are (2, 0) and (–2, 0).

Example 18 Find the equation of the tangent to the curve $y=\frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Solution Note that on x-axis, $y = 0$. So the equation of the curve, when $y = 0$, gives $x = 7$. Thus, the curve cuts the x-axis at (7, 0). Now differentiating the equation of the curve with respect to x, we obtain

$$\frac{dy}{dx} = \frac{1-y(2x-5)}{(x-2)(x-3)} \qquad \text{(Why?)}$$

or
$$\left.\frac{dy}{dx}\right]_{(7,0)} = \frac{1-0}{(5)(4)}=\frac{1}{20}$$

Therefore, the slope of the tangent at (7, 0) is $\frac{1}{20}$. Hence, the equation of the tangent at (7, 0) is

$$y - 0 = \frac{1}{20}(x - 7) \qquad \text{or} \qquad 20y - x + 7 = 0$$

Example 19 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Solution Differentiating $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ with respect to x, we get

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Therefore, the slope of the tangent at (1, 1) is $\left.\frac{dy}{dx}\right]_{(1,1)} = -1$.

So the equation of the tangent at (1, 1) is

$$y - 1 = -1\,(x - 1) \qquad \text{or} \qquad y + x - 2 = 0$$

Also, the slope of the normal at (1, 1) is given by

$$\frac{-1}{\text{slope of the tangent at }(1,1)} = 1$$

Therefore, the equation of the normal at (1, 1) is

$$y - 1 = 1\,(x - 1) \qquad \text{or} \qquad y - x = 0$$

Example 20 Find the equation of tangent to the curve given by

$$x = a\sin^3 t\,, \qquad y = b\cos^3 t \qquad \text{... (1)}$$

at a point where $t = \frac{\pi}{2}$.

Solution Differentiating (1) with respect to t, we get

$$\frac{dx}{dt} = 3a\sin^2 t\cos t \qquad \text{and} \qquad \frac{dy}{dt} = -3b\cos^2 t\sin t$$

or
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t} = \frac{-b}{a}\frac{\cos t}{\sin t}$$

Therefore, slope of the tangent at $t = \frac{\pi}{2}$ is

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{2}} = \frac{-b\cos\frac{\pi}{2}}{a\sin\frac{\pi}{2}} = 0$$

Also, when $t = \frac{\pi}{2}$, $x = a$ and $y = 0$. Hence, the equation of tangent to the given curve at $t = \frac{\pi}{2}$, i.e., at $(a, 0)$ is

$$y - 0 = 0\,(x - a), \text{ i.e., } y = 0.$$

EXERCISE 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.
4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
5. Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.
6. Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.
7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.
8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

10. Find the equation of all lines having slope -1 that are tangents to the curve $y = \dfrac{1}{x-1}$, $x \neq 1$.

11. Find the equation of all lines having slope 2 which are tangents to the curve $y = \dfrac{1}{x-3}$, $x \neq 3$.

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \dfrac{1}{x^2 - 2x + 3}$.

13. Find points on the curve $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ at which the tangents are

(i) parallel to x-axis (ii) parallel to y-axis.

14. Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(iii) $y = x^3$ at (1, 1)

(iv) $y = x^2$ at (0, 0)

(v) $x = \cos t$, $y = \sin t$ at $t = \dfrac{\pi}{4}$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles* if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

25. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Choose the correct answer in Exercises 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is

(A) 3 (B) $\frac{1}{3}$ (C) –3 (D) $-\frac{1}{3}$

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, – 2) (D) (– 1, 2)

6.5 Approximations

In this section, we will use differentials to approximate values of certain quantities.

Let $f: D \to \mathbf{R}$, $D \subset \mathbf{R}$, be a given function and let $y = f(x)$. Let Δx denote a small increment in x. Recall that the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the following

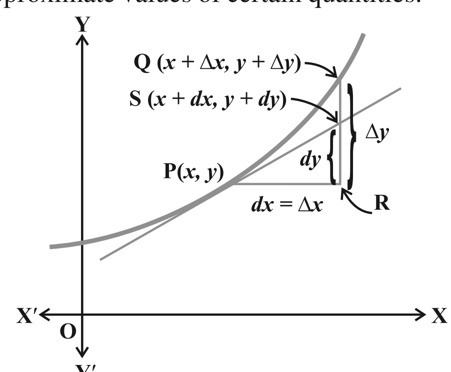


Fig 6.8

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)\, dx$ or
$$dy = \left(\frac{dy}{dx}\right)\Delta x.$$

* Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.

In case $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

For geometrical meaning of Δx, Δy, dx and dy, one may refer to Fig 6.8.

> **Note** In view of the above discussion and Fig 6.8, we may note that the differential of the dependent variable is not equal to the increment of the variable where as the differential of independent variable is equal to the increment of the variable.

Example 21 Use differential to approximate $\sqrt{36.6}$.

Solution Take $y = \sqrt{x}$. Let $x = 36$ and let $\Delta x = 0.6$. Then

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6$$

or $\qquad \sqrt{36.6} = 6 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05 \qquad (\text{as } y = \sqrt{x})$$

Thus, the approximate value of $\sqrt{36.6}$ is $6 + 0.05 = 6.05$.

Example 22 Use differential to approximate $(25)^{\frac{1}{3}}$.

Solution Let $y = x^{\frac{1}{3}}$. Let $x = 27$ and let $\Delta x = -2$. Then

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (25)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (25)^{\frac{1}{3}} - 3$$

or $\qquad (25)^{\frac{1}{3}} = 3 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{\frac{2}{3}}}(-2) \qquad (\text{as } y = x^{\frac{1}{3}})$$

$$= \frac{1}{3((27)^{\frac{1}{3}})^2}(-2) = \frac{-2}{27} = -0.074$$

Thus, the approximate value of $(25)^{\frac{1}{3}}$ is given by

$$3 + (-0.074) = 2.926$$

Example 23 Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.

Solution Let $x = 3$ and $\Delta x = 0.02$. Then

$$f(3.\ 02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 5(x + \Delta x) + 3$$

Note that $\Delta y = f(x + \Delta x) - f(x)$. Therefore

$$f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x)\ \Delta x \qquad \text{(as } dx = \Delta x\text{)}$$

or

$$f(3.02) \approx (3x^2 + 5x + 3) + (6x + 5)\ \Delta x$$

$$= (3(3)^2 + 5(3) + 3) + (6(3) + 5)\ (0.02) \qquad \text{(as } x = 3, \Delta x = 0.02\text{)}$$

$$= (27 + 15 + 3) + (18 + 5)\ (0.02)$$

$$= 45 + 0.46 = 45.46$$

Hence, approximate value of $f(3.02)$ is 45.46.

Example 24 Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

Solution Note that

$$\mathrm{V} = x^3$$

or

$$d\mathrm{V} = \left(\frac{d\mathrm{V}}{dx}\right)\Delta x = (3x^2)\ \Delta x$$

$$= (3x^2)\ (0.02x) = 0.06x^3\ \mathrm{m}^3 \qquad \text{(as 2\% of } x \text{ is } 0.02x\text{)}$$

Thus, the approximate change in volume is $0.06\ x^3\ \mathrm{m}^3$.

Example 25 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Solution Let r be the radius of the sphere and Δr be the error in measuring the radius. Then $r = 9$ cm and $\Delta r = 0.03$ cm. Now, the volume V of the sphere is given by

$$\mathrm{V} = \frac{4}{3}\pi r^3$$

or

$$\frac{d\mathrm{V}}{dr} = 4\pi r^2$$

Therefore

$$d\mathrm{V} = \left(\frac{d\mathrm{V}}{dr}\right)\Delta r = (4\pi r^2)\Delta r$$

$$= 4\pi(9)^2\ (0.03) = 9.72\pi\ \mathrm{cm}^3$$

Thus, the approximate error in calculating the volume is $9.72\pi\ \mathrm{cm}^3$.

EXERCISE 6.4

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal.

(i) $\sqrt{25.3}$ (ii) $\sqrt{49.5}$ (iii) $\sqrt{0.6}$

(iv) $(0.009)^{\frac{1}{3}}$ (v) $(0.999)^{\frac{1}{10}}$ (vi) $(15)^{\frac{1}{4}}$

(vii) $(26)^{\frac{1}{3}}$ (viii) $(255)^{\frac{1}{4}}$ (ix) $(82)^{\frac{1}{4}}$

(x) $(401)^{\frac{1}{2}}$ (xi) $(0.0037)^{\frac{1}{2}}$ (xii) $(26.57)^{\frac{1}{3}}$

(xiii) $(81.5)^{\frac{1}{4}}$ (xiv) $(3.968)^{\frac{3}{2}}$ (xv) $(32.15)^{\frac{1}{5}}$

2. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.
3. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.
4. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 1%.
5. Find the approximate change in the surface area of a cube of side x metres caused by decreasing the side by 1%.
6. If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.
7. If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating its surface area.
8. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is

   (A) 47.66 (B) 57.66 (C) 67.66 (D) 77.66
9. The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is

   (A) $0.06\,x^3$ m^3 (B) $0.6\,x^3$ m^3 (C) $0.09\,x^3$ m^3 (D) $0.9\,x^3$ m^3

6.6 Maxima and Minima

In this section, we will use the concept of derivatives to calculate the maximum or minimum values of various functions. In fact, we will find the 'turning points' of the graph of a function and thus find points at which the graph reaches its highest (or lowest) *locally*. The knowledge of such points is very useful in sketching the graph of a given function. Further, we will also find the absolute maximum and absolute minimum of a function that are necessary for the solution of many applied problems.

Let us consider the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is given by $P(x) = ax + bx^2$, where a,b are constants and x is the number of orange trees per acre. How many trees per acre will maximise the profit?

(ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by $h(x) = 60 + x - \dfrac{x^2}{60}$, where x is the horizontal distance from the building and $h(x)$ is the height of the ball . What is the maximum height the ball will reach?

(iii) An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

Definition 3 Let f be a function defined on an interval I. Then

(a) f is said to have a *maximum value* in I, if there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in$ I.

The number $f(c)$ is called the maximum value of f in I and the point c is called a *point of maximum value* of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in$ I.

The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a *point of minimum value* of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The number $f(c)$, in this case, is called an *extreme value* of f in I and the point c is called an *extreme point*.

Remark In Fig 6.9(a), (b) and (c), we have exhibited that graphs of certain particular functions help us to find maximum value and minimum value at a point. Infact, through graphs, we can even find maximum/minimum value of a function at a point at which it is not even differentiable (Example 27).

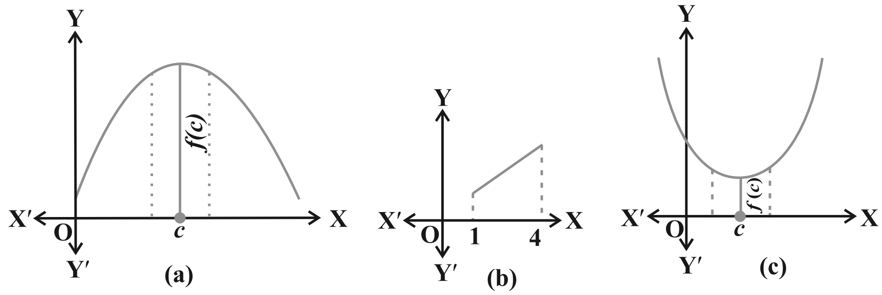


Fig 6.9

Example 26 Find the maximum and the minimum values, if any, of the function f given by

$$f(x) = x^2,\ x \in \mathbf{R}.$$

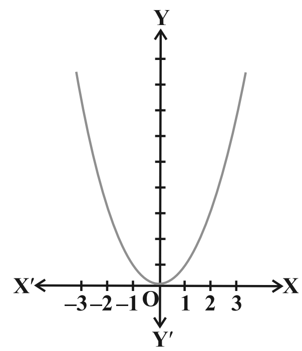


Fig 6.10

Solution From the graph of the given function (Fig 6.10), we have $f(x) = 0$ if $x = 0$. Also

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R}.$$

Therefore, the minimum value of f is 0 and the point of minimum value of f is $x = 0$. Further, it may be observed from the graph of the function that f has no maximum value and hence no point of maximum value of f in $\mathbf{R}$.

Note If we restrict the domain of f to $[-2, 1]$ only, then f will have maximum value $(-2)^2 = 4$ at $x = -2$.

Example 27 Find the maximum and minimum values of f, if any, of the function given by $f(x) = |x|$, $x \in \mathbf{R}$.

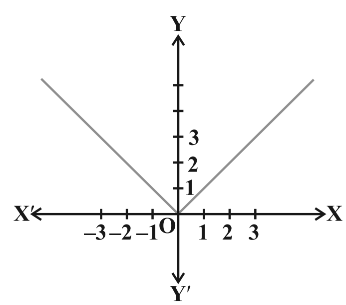


Fig 6.11

Solution From the graph of the given function (Fig 6.11) , note that

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R} \text{ and } f(x) = 0 \text{ if } x = 0.$$

Therefore, the function f has a minimum value 0 and the point of minimum value of f is $x = 0$. Also, the graph clearly shows that f has no maximum value in $\mathbf{R}$ and hence no point of maximum value in $\mathbf{R}$.

Note

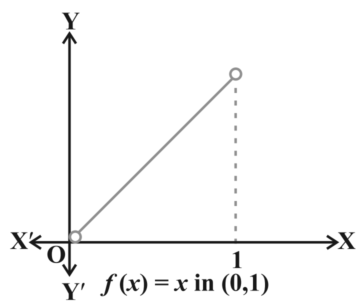
(i) If we restrict the domain of f to $[-2, 1]$ only, then f will have maximum value $|-2| = 2$.

(ii) One may note that the function f in Example 27 is not differentiable at $x = 0$.

Example 28 Find the maximum and the minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1).$$

Solution The given function is an increasing (strictly) function in the given interval (0, 1). From the graph (Fig 6.12) of the function f, it seems that, it should have the minimum value at a point closest to 0 on its right and the maximum value at a point closest to 1 on its left. Are such points available? Of course, not. It is not possible to locate such points. Infact, if a point x_0 is closest to 0, then we find $\frac{x_0}{2} < x_0$ for all $x_0 \in (0,1)$. Also, if x_1 is closest to 1, then $\frac{x_1+1}{2} > x_1$ for all $x_1 \in (0,1)$.



$f(x) = x$ in (0,1)

Fig 6.12

Therefore, the given function has neither the maximum value nor the minimum value in the interval (0,1).

Remark The reader may observe that in Example 28, if we include the points 0 and 1 in the domain of f, i.e., if we extend the domain of f to [0,1], then the function f has minimum value 0 at $x = 0$ and maximum value 1 at $x = 1$. Infact, we have the following results (The proof of these results are beyond the scope of the present text)

Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.

A more general result is

Every continuous function on a closed interval has a maximum and a minimum value.

Note By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.

Maximum and minimum values of a function defined on a closed interval will be discussed later in this section.

Let us now examine the graph of a function as shown in Fig 6.13. Observe that at points A, B, C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called *turning points* of the given function. Further, observe that at turning points, the graph has either a little hill or a little valley. Roughly speaking, the function has minimum value in some neighbourhood (interval) of each of the points A and C which are at the bottom of their respective

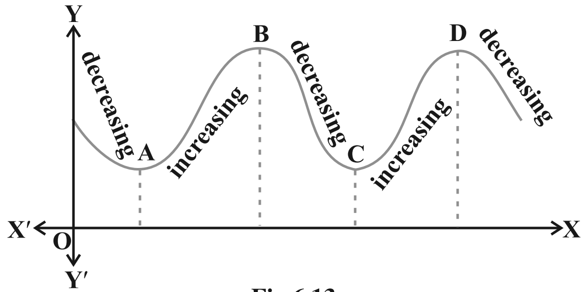


Fig 6.13

valleys. Similarly, the function has maximum value in some neighbourhood of points B and D which are at the top of their respective hills. For this reason, the points A and C may be regarded as points of *local minimum value* (or *relative minimum value*) and points B and D may be regarded as points of *local maximum value* (or *relative maximum value*) for the function. The *local maximum value* and *local minimum value* of the function are referred to as *local maxima* and *local minima*, respectively, of the function.

We now formally give the following definition

Definition 4 Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of *local maxima* if there is an $h > 0$ such that

$$f(c) \geq f(x), \text{ for all } x \text{ in } (c - h, c + h)$$

The value $f(c)$ is called the *local maximum value* of f.

(b) c is called a point of *local minima* if there is an $h > 0$ such that

$$f(c) \leq f(x), \text{ for all } x \text{ in } (c - h, c + h)$$

The value $f(c)$ is called the *local minimum value* of f.

Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of f around c will be as shown in Fig 6.14(a). Note that the function f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$.

This suggests that $f'(c)$ must be zero.

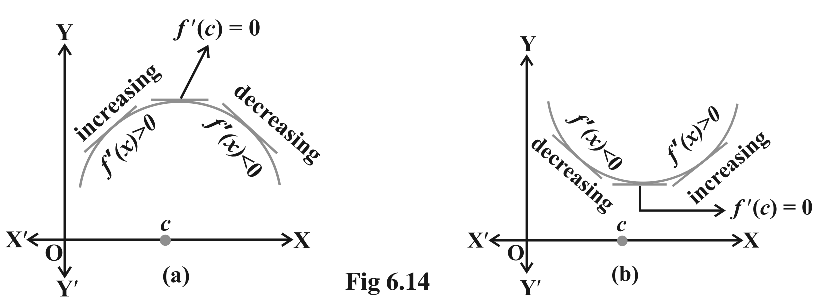


Fig 6.14

Similarly, if c is a point of local minima of f, then the graph of f around c will be as shown in Fig 6.14(b). Here f is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c + h)$. This again suggest that $f'(c)$ must be zero.

The above discussion lead us to the following theorem (without proof).

Theorem 2 Let f be a function defined on an open interval I. Suppose $c \in$ I be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

Remark The converse of above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima or local minima. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and so $f'(0) = 0$. But 0 is neither a point of local maxima nor a point of local minima (Fig 6.15).

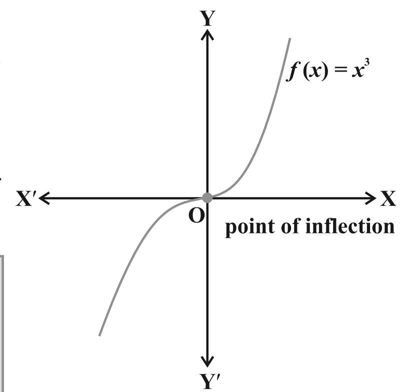


Fig 6.15

Note A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a *critical point* of f. Note that if f is continuous at c and $f'(c) = 0$, then there exists an $h > 0$ such that f is differentiable in the interval $(c - h, c + h)$.

We shall now give a working rule for finding points of local maxima or points of local minima using only the first order derivatives.

Theorem 3 (First Derivative Test) Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.15).

Note If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then $f(c)$ is a local minimum value of f.

Figures 6.15 and 6.16, geometrically explain Theorem 3.

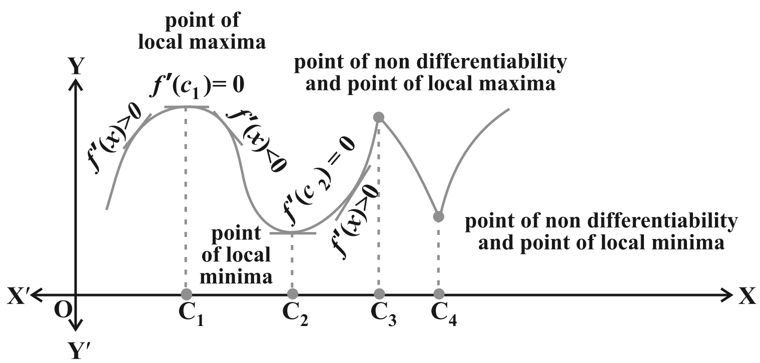


Fig 6.16

Example 29 Find all points of local maxima and local minima of the function f given by

$$f(x) = x^3 - 3x + 3.$$

Solution We have

$$f(x) = x^3 - 3x + 3$$

or $$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$$

or $$f'(x) = 0 \text{ at } x = 1 \text{ and } x = -1$$

Thus, $x = \pm 1$ are the only critical points which could possibly be the points of local maxima and/or local minima of f. Let us first examine the point $x = 1$.

Note that for values close to 1 and to the right of 1, $f'(x) > 0$ and for values close to 1 and to the left of 1, $f'(x) < 0$. Therefore, by first derivative test, $x = 1$ is a point of local minima and local minimum value is $f(1) = 1$. In the case of $x = -1$, note that $f'(x) > 0$, for values close to and to the left of -1 and $f'(x) < 0$, for values close to and to the right of -1. Therefore, by first derivative test, $x = -1$ is a point of local maxima and local maximum value is $f(-1) = 5$.

Values of x		Sign of $f'(x) = 3(x - 1)(x + 1)$
Close to 1	to the right (say 1.1 etc.)	>0
	to the left (say 0.9 etc.)	<0
Close to -1	to the right (say -0.9 etc.)	<0
	to the left (say -1.1 etc.)	>0

Example 30 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or $$f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$$

or $$f'(x) = 0 \quad \text{at} \quad x = 1$$

Thus, $x = 1$ is the only critical point of f. We shall now examine this point for local maxima and/or local minima of f. Observe that $f'(x) \geq 0$, for all $x \in \mathbf{R}$ and in particular $f'(x) > 0$, for values close to 1 and to the left and to the right of 1. Therefore, by first derivative test, the point $x = 1$ is neither a point of local maxima nor a point of local minima. Hence $x = 1$ is a point of inflexion.

Remark One may note that since $f'(x)$, in Example 30, never changes its sign on $\mathbf{R}$, graph of f has no turning points and hence no point of local maxima or local minima.

We shall now give another test to examine local maxima and local minima of a given function. This test is often easier to apply than the first derivative test.

Theorem 4 (Second Derivative Test) Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Note As f is twice differentiable at c, we mean second order derivative of f exists at c.

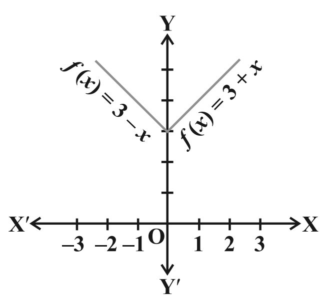


Fig 6.17

Example 31 Find local minimum value of the function f given by $f(x) = 3 + |x|, x \in \mathbf{R}$.

Solution Note that the given function is not differentiable at $x = 0$. So, second derivative test fails. Let us try first derivative test. Note that 0 is a critical point of f. Now to the left of 0, $f(x) = 3 - x$ and so $f'(x) = -1 < 0$. Also to the right of 0, $f(x) = 3 + x$ and so $f'(x) = 1 > 0$. Therefore, by first derivative test, $x = 0$ is a point of local minima of f and local minimum value of f is $f(0) = 3$.

Example 32 Find local maximum and local minimum values of the function f given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Solution We have

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

or $$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$$

or $$f'(x) = 0 \text{ at } x = 0,\ x = 1 \text{ and } x = -2.$$

Now $$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 1)$$

or $$\begin{cases} f''(0) & = -12 < 0 \\ f''(1) & = 48 > 0 \\ f''(-2) & = 84 > 0 \end{cases}$$

Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of f at $x = 0$ is $f(0) = 12$ while $x = 1$ and $x = -2$ are the points of local minima and local minimum values of f at $x = -1$ and -2 are $f(1) = 7$ and $f(-2) = -20$, respectively.

Example 33 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or $$\begin{cases} f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2 \\ f''(x) = 12(x-1) \end{cases}$$

Now $f'(x) = 0$ gives $x = 1$. Also $f''(1) = 0$. Therefore, the second derivative test fails in this case. So, we shall go back to the first derivative test.

We have already seen (Example 30) that, using first derivative test, $x = 1$ is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

Example 34 Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Solution Let one of the numbers be x. Then the other number is $(15 - x)$. Let $S(x)$ denote the sum of the squares of these numbers. Then

$$S(x) = x^2 + (15 - x)^2 = 2x^2 - 30x + 225$$

or
$$\begin{cases} S'(x) = 4x - 30 \\ S''(x) = 4 \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{15}{2}$. Also $S''\left(\frac{15}{2}\right) = 4 > 0$. Therefore, by second derivative test, $x = \frac{15}{2}$ is the point of local minima of S. Hence the sum of squares of numbers is minimum when the numbers are $\frac{15}{2}$ and $15 - \frac{15}{2} = \frac{15}{2}$.

Remark Proceeding as in Example 34 one may prove that the two positive numbers, whose sum is k and the sum of whose squares is minimum, are $\frac{k}{2}$ and $\frac{k}{2}$.

Example 35 Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 \le c \le 5$.

Solution Let (h, k) be any point on the parabola $y = x^2$. Let D be the required distance between (h, k) and $(0, c)$. Then

$$D = \sqrt{(h-0)^2 + (k-c)^2} = \sqrt{h^2 + (k-c)^2} \quad \text{... (1)}$$

Since (h, k) lies on the parabola $y = x^2$, we have $k = h^2$. So (1) gives

$$D \equiv D(k) = \sqrt{k + (k-c)^2}$$

or
$$D'(k) = \frac{1 + 2(k-c)}{2\sqrt{k + (k-c)^2}}$$

Now
$$D'(k) = 0 \text{ gives } k = \frac{2c-1}{2}$$

Observe that when $k < \frac{2c-1}{2}$, then $2(k-c) + 1 < 0$, i.e., $D'(k) < 0$. Also when $k > \frac{2c-1}{2}$, then $D'(k) > 0$. So, by first derivative test, D (k) is minimum at $k = \frac{2c-1}{2}$.

Hence, the required shortest distance is given by

$$D\left(\frac{2c-1}{2}\right)=\sqrt{\frac{2c-1}{2}+\left(\frac{2c-1}{2}-c\right)^2}=\frac{\sqrt{4c-1}}{2}$$

Note The reader may note that in Example 35, we have used first derivative test instead of the second derivative test as the former is easy and short.

Example 36 Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

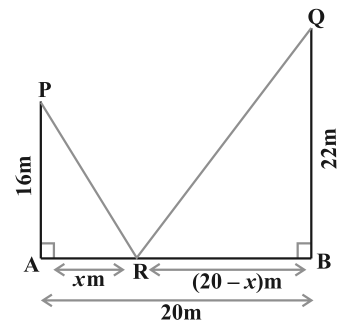


Fig 6.18

Solution Let R be a point on AB such that AR = x m. Then RB = $(20 - x)$ m (as AB = 20 m). From Fig 6.18, we have

$$RP^2 = AR^2 + AP^2$$

and $$RQ^2 = RB^2 + BQ^2$$

Therefore
$$RP^2 + RQ^2 = AR^2 + AP^2 + RB^2 + BQ^2$$
$$= x^2 + (16)^2 + (20 - x)^2 + (22)^2$$
$$= 2x^2 - 40x + 1140$$

Let $$S \equiv S(x) = RP^2 + RQ^2 = 2x^2 - 40x + 1140.$$

Therefore $$S'(x) = 4x - 40.$$

Now $S'(x) = 0$ gives $x = 10$. Also $S''(x) = 4 > 0$, for all x and so $S''(10) > 0$. Therefore, by second derivative test, $x = 10$ is the point of local minima of S. Thus, the distance of R from A on AB is AR = x = 10 m.

Example 37 If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

Solution The required trapezium is as given in Fig 6.19. Draw perpendiculars DP and

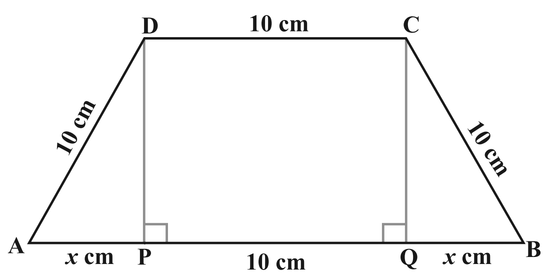


Fig 6.19

CQ on AB. Let AP = x cm. Note that ΔAPD ~ ΔBQC. Therefore, QB = x cm. Also, by Pythagoras theorem, DP = QC = $\sqrt{100-x^2}$. Let A be the area of the trapezium. Then

$$A \equiv A(x) = \frac{1}{2}\text{(sum of parallel sides) (height)}$$

$$= \frac{1}{2}(2x+10+10)\left(\sqrt{100-x^2}\right)$$

$$= (x+10)\left(\sqrt{100-x^2}\right)$$

or
$$A'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}}+\left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

Now $A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, i.e., $x = 5$ and $x = -10$.

Since x represents distance, it can not be negative.

So, $x = 5$. Now

$$A''(x) = \frac{\sqrt{100-x^2}(-4x-10)-(-2x^2-10x+100)\frac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \text{ (on simplification)}$$

or
$$A''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and the area is given by

$$A(5) = (5+10)\sqrt{100-(5)^2} = 15\sqrt{75} = 75\sqrt{3}\text{ cm}^2$$

Example 38 Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Solution Let OC = r be the radius of the cone and OA = h be its height. Let a cylinder with radius OE = x inscribed in the given cone (Fig 6.20). The height QE of the cylinder is given by

$$\frac{QE}{OA} = \frac{EC}{OC} \quad \text{(since } \Delta QEC \sim \Delta AOC)$$

or $$\frac{QE}{h} = \frac{r-x}{r}$$

or $$QE = \frac{h(r-x)}{r}$$

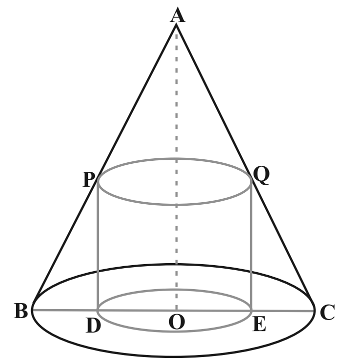


Fig 6.20

Let S be the curved surface area of the given cylinder. Then

$$S \equiv S(x) = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

or $$\begin{cases} S'(x) = \dfrac{2\pi h}{r}(r-2x) \\ S''(x) = \dfrac{-4\pi h}{r} \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{r}{2}$. Since $S''(x) < 0$ for all x, $S''\left(\frac{r}{2}\right) < 0$. So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

6.6.1 Maximum and Minimum Values of a Function in a Closed Interval

Let us consider a function f given by

$$f(x) = x + 2, x \in (0, 1)$$

Observe that the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of f to the closed interval [0, 1], then f still may not have a local maximum (minimum) values but it certainly does have maximum value $3 = f(1)$ and minimum value $2 = f(0)$. The maximum value 3 of f at $x = 1$ is called *absolute maximum* value (*global maximum* or *greatest value*) of f on the interval [0, 1]. Similarly, the minimum value 2 of f at $x = 0$ is called the *absolute minimum* value (*global minimum* or *least value*) of f on [0, 1].

Consider the graph given in Fig 6.21 of a continuous function defined on a closed interval [a, d]. Observe that the function f has a local minima at $x = b$ and local

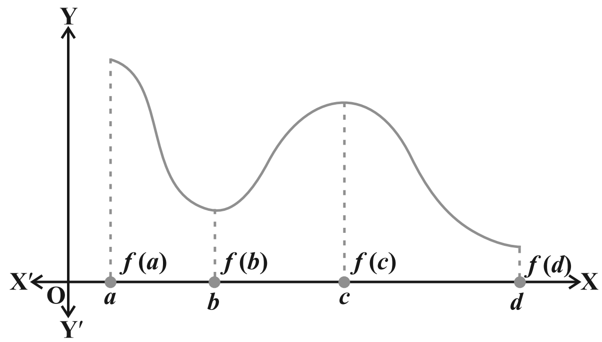


Fig 6.21

minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$.

Also from the graph, it is evident that f has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f.

We will now state two results (without proof) regarding absolute maximum and absolute minimum values of a function on a closed interval I.

Theorem 5 Let f be a continuous function on an interval I = $[a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem 6 Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) $f'(c) = 0$ if f attains its absolute maximum value at c.

(ii) $f'(c) = 0$ if f attains its absolute minimum value at c.

In view of the above results, we have the following working rule for finding absolute maximum and/or absolute minimum values of a function in a given closed interval $[a, b]$.

Working Rule

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.

Example 39 Find the absolute maximum and minimum values of a function f given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5].$$

Solution We have

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

or
$$f'(x) = 6x^2 - 30x + 36 = 6(x-3)(x-2)$$

Note that $f'(x) = 0$ gives $x = 2$ and $x = 3$.

We shall now evaluate the value of f at these points and at the end points of the interval [1, 5], i.e., at $x = 1$, $x = 2$, $x = 3$ and at $x = 5$. So

$$f(1) = 2(1^3) - 15(1^2) + 36(1) + 1 = 24$$
$$f(2) = 2(2^3) - 15(2^2) + 36(2) + 1 = 29$$
$$f(3) = 2(3^3) - 15(3^2) + 36(3) + 1 = 28$$
$$f(5) = 2(5^3) - 15(5^2) + 36(5) + 1 = 56$$

Thus, we conclude that absolute maximum value of f on [1, 5] is 56, occurring at $x = 5$, and absolute minimum value of f on [1, 5] is 24 which occurs at $x = 1$.

Example 40 Find absolute maximum and minimum values of a function f given by

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}},\ x \in [-1, 1]$$

Solution We have

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$$

or
$$f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}} = \frac{2(8x-1)}{x^{\frac{2}{3}}}$$

Thus, $f'(x) = 0$ gives $x = \frac{1}{8}$. Further note that $f'(x)$ is not defined at $x = 0$. So the critical points are $x = 0$ and $x = \frac{1}{8}$. Now evaluating the value of f at critical points $x = 0, \frac{1}{8}$ and at end points of the interval $x = -1$ and $x = 1$, we have

$$f(-1) = 12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}} = 18$$
$$f(0) = 12(0) - 6(0) = 0$$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{-9}{4}$$

$$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}} = 6$$

Hence, we conclude that absolute maximum value of f is 18 that occurs at $x = -1$ and absolute minimum value of f is $\frac{-9}{4}$ that occurs at $x = \frac{1}{8}$.

Example 41 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at (3,7) is

$$\sqrt{(x-3)^2 + (x^2+7-7)^2}, \text{ i.e., } \sqrt{(x-3)^2 + x^4}.$$

Let $\quad f(x) = (x-3)^2 + x^4$

or $\quad f'(x) = 2(x-3) + 4x^3 = 2(x-1)(2x^2 + 2x + 3)$

Thus, $f'(x) = 0$ gives $x = 1$ or $2x^2 + 2x + 3 = 0$ for which there are no real roots. Also, there are no end points of the interval to be added to the set for which f' is zero, i.e., there is only one point, namely, $x = 1$. The value of f at this point is given by $f(1) = (1-3)^2 + (1)^4 = 5$. Thus, the distance between the solider and the helicopter is $\sqrt{f(1)} = \sqrt{5}$.

Note that $\sqrt{5}$ is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5},$$

it follows that $\sqrt{5}$ is the minimum value of $\sqrt{f(x)}$. Hence, $\sqrt{5}$ is the minimum distance between the soldier and the helicopter.

EXERCISE 6.5

1. Find the maximum and minimum values, if any, of the following functions given by

   (i) $f(x) = (2x-1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

   (iii) $f(x) = -(x-1)^2 + 10$ (iv) $g(x) = x^3 + 1$

2. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$ (ii) $g(x) = -|x + 1| + 3$

(iii) $h(x) = \sin(2x) + 5$ (iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$ (ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$ (vi) $g(x) = \frac{x}{2} + \frac{2}{x},\quad x > 0$

(vii) $g(x) = \frac{1}{x^2 + 2}$ (viii) $f(x) = x\sqrt{1-x},\quad x > 0$

4. Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ (ii) $g(x) = \log x$

(iii) $h(x) = x^3 + x^2 + x + 1$

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, x \in [-2, 2]$ (ii) $f(x) = \sin x + \cos x,\ x \in [0, \pi]$

(iii) $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$ (iv) $f(x) = (x-1)^2 + 3,\ x \in [-3, 1]$

6. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 24x - 18x^2$$

7. Find both the maximum value and the minimum value of

$3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

8. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

9. What is the maximum value of the function $\sin x + \cos x$?

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

13. Find two numbers whose sum is 24 and whose product is as large as possible.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

15. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Choose the correct answer in the Exercises 27 and 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2}, 4)$ (B) $(2\sqrt{2}, 0)$ (C) (0, 0) (D) (2, 2)

28. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1 (C) 3 (D) $\frac{1}{3}$

29. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}$, $0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ (B) $\frac{1}{2}$ (C) 1 (D) 0

Miscellaneous Examples

Example 42 A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by

$$x = t^2\left(2 - \frac{t}{3}\right)$$

Find the time taken by it to reach Q and also find distance between P and Q.

Solution Let v be the velocity of the car at t seconds.

Now $$x = t^2\left(2 - \frac{t}{3}\right)$$

Therefore $$v = \frac{dx}{dt} = 4t - t^2 = t(4 - t)$$

Thus, $v = 0$ gives $t = 0$ and/or $t = 4$.

Now $v = 0$ at P as well as at Q and at P, $t = 0$. So, at Q, $t = 4$. Thus, the car will reach the point Q after 4 seconds. Also the distance travelled in 4 seconds is given by

$$x]_{t=4} = 4^2\left(2 - \frac{4}{3}\right) = 16\left(\frac{2}{3}\right) = \frac{32}{3}\text{ m}$$

Example 43 A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

Solution Let r, h and α be as in Fig 6.22. Then $\tan\alpha = \frac{r}{h}\cdot$

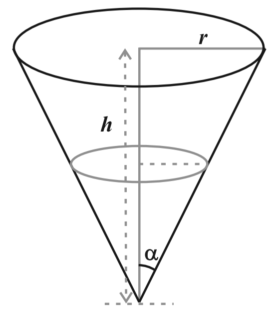


Fig 6.22

So $\qquad \alpha = \tan^{-1}\left(\frac{r}{h}\right).$

But $\qquad \alpha = \tan^{-1}(0.5) \quad$ (given)

or $\qquad \frac{r}{h} = 0.5$

or $\qquad r = \frac{h}{2}$

Let V be the volume of the cone. Then

$$\text{V} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Therefore $\qquad \frac{d\text{V}}{dt} = \frac{d}{dh}\left(\frac{\pi h^3}{12}\right)\cdot\frac{dh}{dt} \qquad$ (by Chain Rule)

$$= \frac{\pi}{4}h^2\frac{dh}{dt}$$

Now rate of change of volume, i.e., $\frac{d\text{V}}{dt} = 5\ \text{m}^3/\text{h}$ and $h = 4$ m.

Therefore $\qquad 5 = \frac{\pi}{4}(4)^2\cdot\frac{dh}{dt}$

or $\qquad \frac{dh}{dt} = \frac{5}{4\pi} = \frac{35}{88}\ \text{m/h} \quad \left(\pi = \frac{22}{7}\right)$

Thus, the rate of change of water level is $\frac{35}{88}$ m/h .

Example 44 A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.

Solution In Fig 6.23, Let AB be the lamp-post, the lamp being at the position B and let MN be the man at a particular time t and let AM = l metres. Then, MS is the shadow of the man. Let MS = s metres.

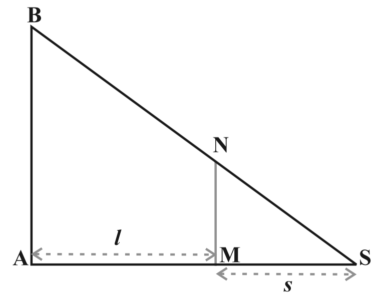


Fig 6.23

Note that $\Delta MSN \sim \Delta ASB$

or $$\frac{MS}{AS} = \frac{MN}{AB}$$

or $AS = 3s$ (as MN = 2 and AB = 6 (given))

Thus $AM = 3s - s = 2s$. But $AM = l$

So $l = 2s$

Therefore $$\frac{dl}{dt} = 2\frac{ds}{dt}$$

Since $\frac{dl}{dt} = 5$ km/h. Hence, the length of the shadow increases at the rate $\frac{5}{2}$ km/h.

Example 45 Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).

Solution Differentiating $x^2 = 4y$ with respect to x, we get

$$\frac{dy}{dx} = \frac{x}{2}$$

Let (h, k) be the coordinates of the point of contact of the normal to the curve $x^2 = 4y$. Now, slope of the tangent at (h, k) is given by

$$\left.\frac{dy}{dx}\right]_{(h,k)} = \frac{h}{2}$$

Hence, slope of the normal at $(h, k) = \frac{-2}{h}$

Therefore, the equation of normal at (h, k) is

$$y - k = \frac{-2}{h}(x - h) \qquad \text{... (1)}$$

Since it passes through the point (1, 2), we have

$$2 - k = \frac{-2}{h}(1 - h) \quad \text{or} \quad k = 2 + \frac{2}{h}(1 - h) \qquad \text{... (2)}$$

Since (h, k) lies on the curve $x^2 = 4y$, we have

$$h^2 = 4k \qquad \ldots (3)$$

From (2) and (3), we have $h = 2$ and $k = 1$. Substituting the values of h and k in (1), we get the required equation of normal as

$$y - 1 = \frac{-2}{2}(x - 2) \text{ or } x + y = 3$$

Example 46 Find the equation of tangents to the curve

$$y = \cos(x + y), \ -2\pi \le x \le 2\pi$$

that are parallel to the line $x + 2y = 0$.

Solution Differentiating $y = \cos(x + y)$ with respect to x, we have

$$\frac{dy}{dx} = \frac{-\sin(x+y)}{1+\sin(x+y)}$$

or $\qquad$ slope of tangent at $(x, y) = \dfrac{-\sin(x+y)}{1+\sin(x+y)}$

Since the tangents to the given curve are parallel to the line $x + 2y = 0$, whose slope is $\dfrac{-1}{2}$, we have

$$\frac{-\sin(x+y)}{1+\sin(x+y)} = \frac{-1}{2}$$

or $\qquad \sin(x + y) = 1$

or $\qquad x + y = n\pi + (-1)^n \dfrac{\pi}{2}, \ n \in \mathbf{Z}$

Then $\qquad y = \cos(x + y) = \cos\left(n\pi + (-1)^n \dfrac{\pi}{2}\right), \ n \in \mathbf{Z}$

$\qquad = 0$, for all $n \in \mathbf{Z}$

Also, since $-2\pi \le x \le 2\pi$, we get $x = \dfrac{-3\pi}{2}$ and $x = \dfrac{\pi}{2}$. Thus, tangents to the given curve are parallel to the line $x + 2y = 0$ only at points $\left(\dfrac{-3\pi}{2}, 0\right)$ and $\left(\dfrac{\pi}{2}, 0\right)$. Therefore, the required equation of tangents are

$$y - 0 = \frac{-1}{2}\left(x + \frac{3\pi}{2}\right) \quad \text{or} \quad 2x + 4y + 3\pi = 0$$

and
$$y - 0 = \frac{-1}{2}\left(x - \frac{\pi}{2}\right) \quad \text{or} \quad 2x + 4y - \pi = 0$$

Example 47 Find intervals in which the function given by

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

is (a) strictly increasing (b) strictly decreasing.

Solution We have

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

Therefore
$$f'(x) = \frac{3}{10}(4x^3) - \frac{4}{5}(3x^2) - 3(2x) + \frac{36}{5}$$

$$= \frac{6}{5}(x-1)(x+2)(x-3) \qquad \text{(on simplification)}$$

Now $f'(x) = 0$ gives $x = 1$, $x = -2$, or $x = 3$. The points $x = 1, -2$, and 3 divide the real line into four disjoint intervals namely, $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$ (Fig 6.24).

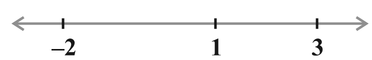


Fig 6.24

Consider the interval $(-\infty, -2)$, i.e., when $-\infty < x < -2$.

In this case, we have $x - 1 < 0$, $x + 2 < 0$ and $x - 3 < 0$.

(In particular, observe that for $x = -3$, $f'(x) = (x - 1)(x + 2)(x - 3) = (-4)(-1)(-6) < 0$)

Therefore, $\quad f'(x) < 0$ when $-\infty < x < -2$.

Thus, the function f is strictly decreasing in $(-\infty, -2)$.

Consider the interval $(-2, 1)$, i.e., when $-2 < x < 1$.

In this case, we have $x - 1 < 0$, $x + 2 > 0$ and $x - 3 < 0$

(In particular, observe that for $x = 0$, $f'(x) = (x - 1)(x + 2)(x - 3) = (-1)(2)(-3) = 6 > 0$)

So $\quad f'(x) > 0$ when $-2 < x < 1$.

Thus, $\quad f$ is strictly increasing in $(-2, 1)$.

Now consider the interval (1, 3), i.e., when $1 < x < 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 < 0$.

So, $f'(x) < 0$ when $1 < x < 3$.

Thus, f is strictly decreasing in (1, 3).

Finally, consider the interval $(3, \infty)$, i.e., when $x > 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 > 0$. So $f'(x) > 0$ when $x > 3$.

Thus, f is strictly increasing in the interval $(3, \infty)$.

Example 48 Show that the function f given by

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

is always an strictly increasing function in $\left(0, \frac{\pi}{4}\right)$.

Solution We have

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

Therefore $$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}(\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{2 + \sin 2x} \qquad \text{(on simplification)}$$

Note that $2 + \sin 2x > 0$ for all x in $\left(0, \frac{\pi}{4}\right)$.

Therefore $f'(x) > 0$ if $\cos x - \sin x > 0$

or $f'(x) > 0$ if $\cos x > \sin x$ or $\cot x > 1$

Now $\cot x > 1$ if $\tan x < 1$, i.e., if $0 < x < \frac{\pi}{4}$

Thus $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$

Hence f is strictly increasing function in $\left(0, \frac{\pi}{4}\right)$.

Example 49 A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.

Solution Let r be the radius of the given disc and A be its area. Then

$$A = \pi r^2$$

or $$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{(by Chain Rule)}$$

Now approximate rate of increase of radius $= dr = \frac{dr}{dt}\Delta t = 0.05$ cm/s.

Therefore, the approximate rate of increase in area is given by

$$dA = \frac{dA}{dt}(\Delta t) = 2\pi r\left(\frac{dr}{dt}\Delta t\right)$$

$$= 2\pi\,(3.2)\,(0.05) = 0.320\pi \text{ cm}^2/\text{s} \quad (r = 3.2 \text{ cm})$$

Example 50 An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

Solution Let x metre be the length of a side of the removed squares. Then, the height of the box is x, length is $8 - 2x$ and breadth is $3 - 2x$ (Fig 6.25). If $V(x)$ is the volume of the box, then

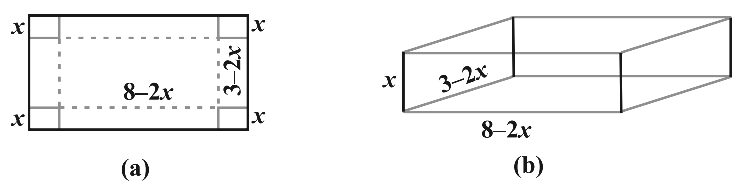


Fig 6.25

$$V(x) = x(3 - 2x)(8 - 2x)$$
$$= 4x^3 - 22x^2 + 24x$$

Therefore $$\begin{cases} V'(x) = 12x^2 - 44x + 24 = 4(x-3)(3x-2) \\ V''(x) = 24x - 44 \end{cases}$$

Now $\quad V'(x) = 0$ gives $x = 3, \frac{2}{3}$. But $x \neq 3$ (Why?)

Thus, we have $x = \frac{2}{3}$. Now $V''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 44 = -28 < 0$.

Therefore, $x = \frac{2}{3}$ is the point of maxima, i.e., if we remove a square of side $\frac{2}{3}$ metre from each corner of the sheet and make a box from the remaining sheet, then the volume of the box such obtained will be the largest and it is given by

$$V\left(\frac{2}{3}\right) = 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right)$$

$$= \frac{200}{27} \text{ m}^3$$

Example 51 Manufacturer can sell x items at a price of rupees $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is Rs $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution Let $S(x)$ be the selling price of x items and let $C(x)$ be the cost price of x items. Then, we have

$$S(x) = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and

$$C(x) = \frac{x}{5} + 500$$

Thus, the profit function $P(x)$ is given by

$$P(x) = S(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

i.e.

$$P(x) = \frac{24}{5}x - \frac{x^2}{100} - 500$$

or

$$P'(x) = \frac{24}{5} - \frac{x}{50}$$

Now $P'(x) = 0$ gives $x = 240$. Also $P''(x) = \frac{-1}{50}$. So $P''(240) = \frac{-1}{50} < 0$

Thus, $x = 240$ is a point of maxima. Hence, the manufacturer can earn maximum profit, if he sells 240 items.

Miscellaneous Exercise on Chapter 6

1. Using differentials, find the approximate value of each of the following:

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$

2. Show that the function given by $f(x)=\frac{\log x}{x}$ has maximum at $x = e$.

3. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

4. Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

5. Show that the normal at any point θ to the curve

$x = a \cos\theta + a\,\theta \sin\theta,\ y = a \sin\theta - a\theta \cos\theta$

is at a constant distance from the origin.

6. Find the intervals in which the function f given by

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

is (i) increasing (ii) decreasing.

7. Find the intervals in which the function f given by $f(x)=x^3+\frac{1}{x^3},\ x\neq 0$ is

(i) increasing (ii) decreasing.

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

10. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

12. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

Show that the maximum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

13. Find the points at which the function f given by $f(x) = (x-2)^4 (x+1)^3$ has

(i) local maxima (ii) local minima

(iii) point of inflexion

14. Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$$

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

16. Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

18. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Choose the correct answer in the Exercises from 19 to 24.

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m^3/h (B) 0.1 m^3/h

(C) 1.1 m^3/h (D) 0.5 m^3/h

20. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2,– 1) is

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$

21. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

(A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

22. The normal at the point (1,1) on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ (B) $x - y = 0$

(C) $x + y + 1 = 0$ (D) $x - y = 0$

23. The normal to the curve $x^2 = 4y$ passing (1,2) is

(A) $x + y = 3$ (B) $x - y = 3$

(C) $x + y = 1$ (D) $x - y = 1$

24. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, \frac{-8}{3}\right)$

(C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$

Summary

- If a quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.
- If two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0.$$

- A function f is said to be

  (a) increasing on an interval (a, b) if
  $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in (a, b)$.

Alternatively, if $f'(x) \geq 0$ for each x in (a, b)

(b) decreasing on (a,b) if

$$x_1 < x_2 \text{ in } (a, b) \Rightarrow f(x_1) \geq f(x_2) \text{ for all } x_1, x_2 \in (a, b).$$

Alternatively, if $f'(x) \leq 0$ for each x in (a, b)

- The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by

$$y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0)$$

- If $\frac{dy}{dx}$ does not exist at the point (x_0, y_0), then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$.
- If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x-axis, then $\left.\frac{dy}{dx}\right]_{x=x_0} = 0$.
- Equation of the normal to the curve $y = f(x)$ at a point (x_0, y_0) is given by

$$y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}} (x - x_0)$$

- If $\frac{dy}{dx}$ at the point (x_0, y_0) is zero, then equation of the normal is $x = x_0$.
- If $\frac{dy}{dx}$ at the point (x_0, y_0) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$.
- Let $y = f(x)$, Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then dy given by

$$dy = f'(x)dx \text{ or } dy = \left(\frac{dy}{dx}\right)\Delta x.$$

is a good approximation of Δy when $dx = \Delta x$ is relatively small and we denote it by $dy \approx \Delta y$.

- A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a *critical point* of f.

- ***First Derivative Test*** Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then
  - (i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.
  - (ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.
  - (iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflexion*.
- ***Second Derivative Test*** Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then
  - (i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

    The values $f(c)$ is local maximum value of f.
  - (ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

    In this case, $f(c)$ is local minimum value of f.
  - (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

    In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.
- Working rule for finding absolute maxima and/or absolute minima

  Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

  Step 2: Take the end points of the interval.

  Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

  Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

ANSWERS

EXERCISE 1.1

1. (i) Neither reflexive nor symmetric nor transitive.

(ii) Neither reflexive nor symmetric nor transitive.

(iii) Reflexive and transitive but not symmetric.

(iv) Reflexive, symmetric and transitive.

(v) (a) Reflexive, symmetric and transitive.

(b) Reflexive, symmetric and transitive.

(c) Neither reflexive nor symmetric nor transitive.

(d) Neither reflexive nor symmetric nor transitive.

(e) Neither reflexive nor symmetric nor transitive.

3. Neither reflexive nor symmetric nor transitive.

5. Neither reflexive nor symmetric nor transitive.

9. (i) $\{1, 5, 9\}$, (ii) $\{1\}$

12. T_1 is related to T_3.

13. The set of all triangles

14. The set of all lines $y = 2x + c$, $c \in \mathbf{R}$

15. B

16. C

EXERCISE 1.2

1. No

2. (i) Injective but not surjective

(ii) Neither injective nor surjective

(iii) Neither injective nor surjective

(iv) Injective but not surjective

(v) Injective but not surjective

7. (i) One-one and onto

(ii) Neither one-one nor onto.

9. No

10. Yes

11. D

12. A

EXERCISE 1.3

1. $gof = \{(1, 3), (3,1), (4,3)\}$

3. (i) $(gof)(x) = |5|x| - 2|$, $(fog)(x) = |5x - 2|$

(ii) $(gof)(x) = 2x$, $(fog)(x) = 8x$

4. Inverse of f is f itself

5. (i) No, since f is many-one (ii) No, since g is many-one.
(iii) Yes, since h is one-one-onto.

6. f^{-1} is given by $f^{-1}(y) = \dfrac{2y}{1-y}$, $y \neq 1$ 7. f^{-1} is given by $f^{-1}(y) = \dfrac{y-3}{4}$

11. f^{-1} is given by $f^{-1}(a) = 1$, $f^{-1}(b) = 2$ and $f^{-1}(c) = 3$.

13. (C) 14. (B)

EXERCISE 1.4

1. (i) No (ii) Yes (iii) Yes (iv) Yes (v) Yes
2. (i) $*$ is neither commutative nor associative
   (ii) $*$ is commutative but not associative
   (iii) $*$ is both commutative and associative
   (iv) $*$ is commutative but not associative
   (v) $*$ is neither commutative nor associative
   (vi) $*$ is neither commutative nor associative

3.

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

4. (i) $(2 * 3) * 4 = 1$ and $2 * (3 * 4) = 1$ (ii) Yes (iii) 1
5. Yes
6. (i) $5 * 7 = 35$, $20 * 16 = 80$ (ii) Yes (iii) Yes (iv) 1 (v) 1
7. No 8. $*$ is both commutative and associative; $*$ does not have any identity in **N**
9. (ii) , (iv), (v) are commutative; (v) is associative.
11. Identity element does not exist.
12. (ii) False (ii) True 13. B

Miscellaneous Exercise on Chapter 1

1. $g(y) = \dfrac{y-7}{10}$
2. The inverse of f is f itself
3. $x^4 - 6x^3 + 10x^2 - 3x$
8. No
10. $n!$
11. (i) $F^{-1} = \{(3, a), (2, b), (1, c)\}$, (ii) F^{-1} does not exist
12. No
15. Yes
16. A
17. B
18. No
19. B

EXERCISE 2.1

1. $\dfrac{-\pi}{6}$
2. $\dfrac{\pi}{6}$
3. $\dfrac{\pi}{6}$
4. $\dfrac{-\pi}{3}$
5. $\dfrac{2\pi}{3}$
6. $-\dfrac{\pi}{4}$
7. $\dfrac{\pi}{6}$
8. $\dfrac{\pi}{6}$
9. $\dfrac{3\pi}{4}$
10. $\dfrac{-\pi}{4}$
11. $\dfrac{3\pi}{4}$
12. $\dfrac{2\pi}{3}$
13. B
14. B

EXERCISE 2.2

5. $\dfrac{1}{2}\tan^{-1} x$
6. $\dfrac{\pi}{2} - \sec^{-1} x$
7. $\dfrac{x}{2}$
8. $\dfrac{\pi}{4} - x$
9. $\sin^{-1}\dfrac{x}{a}$
10. $3\tan^{-1}\dfrac{x}{a}$
11. $\dfrac{\pi}{4}$
12. 0
13. $\dfrac{x+y}{1-xy}$
14. $\dfrac{1}{5}$
15. $\pm\dfrac{1}{\sqrt{2}}$
16. $\dfrac{\pi}{3}$
17. $\dfrac{-\pi}{4}$
18. $\dfrac{17}{6}$
19. B
20. D
21. B

Miscellaneous Exercise on Chapter 2

1. $\dfrac{\pi}{6}$
2. $\dfrac{\pi}{6}$
13. $x = \dfrac{\pi}{4}$
14. $x = \dfrac{1}{\sqrt{3}}$
15. D
16. C
17. C

EXERCISE 3.1

1. (i) 3×4 (ii) 12 (iii) $19, 35, -5, 12, \frac{5}{2}$

2. $1 \times 24, 2 \times 12, 3 \times 8, 4 \times 6, 6 \times 4, 8 \times 3, 12 \times 2, 24 \times 1; 1 \times 13, 13 \times 1$

3. $1 \times 18, 2 \times 9, 3 \times 6, 6 \times 3, 9 \times 2, 18 \times 1; 1 \times 5, 5 \times 1$

4. (i) $\begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$

5. (i) $\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$

6. (i) $x = 1, \quad y = 4, \quad z = 3$

(ii) $x = 4, \quad y = 2, \quad z = 0$ or $x = 2, \quad y = 4, \quad z = 0$

(iii) $x = 2, \quad y = 4, \quad z = 3$

7. $a = 1, b = 2, c = 3, d = 4$

8. C **9.** B **10.** D

EXERCISE 3.2

1. (i) $A + B = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$ (ii) $A - B = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$

(iii) $3A - C = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$ (iv) $AB = \begin{bmatrix} -6 & 26 \\ 1 & 19 \end{bmatrix}$ (v) $BA = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

2. (i) $\begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$ (ii) $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$

(iii) $\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. (i) $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$ (iii) $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

(iv) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

4. $A+B=\begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}, B-C=\begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7. (i) $X=\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}, Y=\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (ii) $X=\begin{bmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{bmatrix}, Y=\begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

8. $X=\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$ **9.** $x = 3, y = 3$ **10.** $x = 3, y = 6, z = 9, t = 6$

11. $x = 3, y = -4$ **12.** $x = 2, y = 4, w = 3, z = 1$

15. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ **17.** $k = 1$

19. (a) Rs 15000, Rs 15000 (b) Rs 5000, Rs 25000

20. Rs 20160 **21.** A **22.** B

EXERCISE 3.3

1. (i) $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$ **9.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. (i) $A = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & -2 & -2 \\ \frac{-5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ \frac{-5}{2} & 0 & 3 \\ \frac{-3}{2} & -3 & 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

11. A **12.** B

EXERCISE 3.4

1. $\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ **5.** $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$ **6.** $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ **8.** $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$ **11.** $\begin{bmatrix} -1 & 3 \\ \frac{-1}{2} & 1 \end{bmatrix}$ **12.** Inverse does not exist.

13. $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ **14.** Inverse does not exist.

15. $\begin{bmatrix} \frac{-2}{5} & 0 & \frac{3}{5} \\ \frac{-1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix}$ **16.** $\begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$ **17.** $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

18. D

Miscellaneous Exercise on Chapter 3

6. $x = \pm \frac{1}{\sqrt{2}}, y = \pm \frac{1}{\sqrt{6}}, z = \pm \frac{1}{\sqrt{3}}$

7. $x = -1$ **9.** $x = \pm 4\sqrt{3}$

10. (a) Total revenue in the market - I = Rs 46000

Total revenue in the market - II = Rs 53000

(b) Rs 15000, Rs 17000

11. $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$ **13.** C **14.** B **15.** C

EXERCISE 4.1

1. (i) 18 **2.** (i) 1, (ii) $x^3 - x^2 + 2$

5. (i) – 12, (ii) 46, (iii) 0, (iv) 5 **6.** 0

7. (i) $x = \pm\sqrt{3}$, (ii) $x = 2$ **8.** (B)

EXERCISE 4.2

15. C **16.** C

EXERCISE 4.3

1. (i) $\frac{15}{2}$, (ii) $\frac{47}{2}$, (iii) 15

3. (i) 0, 8, (ii) 0, 8 **4.** (i) $y = 2x$, (ii) $x - 3y = 0$ **5.** (D)

EXERCISE 4.4

1. (i) $M_{11} = 3, M_{12} = 0, M_{21} = -4, M_{22} = 2, A_{11} = 3, A_{12} = 0, A_{21} = 4, A_{22} = 2$

(ii) $M_{11} = d, M_{12} = b, M_{21} = c, M_{22} = a$

$A_{11} = d, A_{12} = -b, A_{21} = -c, A_{22} = a$

2. (i) $M_{11} = 1, M_{12} = 0, M_{13} = 0, M_{21} = 0, M_{22} = 1, M_{23} = 0, M_{31} = 0, M_{32} = 0, M_{33} = 1,$

$A_{11} = 1, A_{12} = 0, A_{13} = 0, A_{21} = 0, A_{22} = 1, A_{23} = 0, A_{31} = 0, A_{32} = 0, A_{33} = 1$

(ii) $M_{11} = 11, M_{12} = 6, M_{13} = 3, M_{21} = -4, M_{22} = 2, M_{23} = 1, M_{31} = -20, M_{32} = -13, M_{33} = 5$

$A_{11} = 11, A_{12} = -6, A_{13} = 3, A_{21} = 4, A_{22} = 2, A_{23} = -1, A_{31} = -20, A_{32} = 13, A_{33} = 5$

3. 7 **4.** $(x - y)(y - z)(z - x)$ **5.** (D)

EXERCISE 4.5

1. $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$ **5.** $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

6. $\frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$ **7.** $\frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ **8.** $\frac{-1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

9. $\frac{-1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$ **10.** $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

13. $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ **14.** $a = -4, b = 1$ **15.** $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

16. $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ **17.** B **18.** B

EXERCISE 4.6

1. Consistent **2.** Consistent **3.** Inconsistent

4. Consistent **5.** Inconsistent **6.** Consistent

7. $x = 2, y = -3$ **8.** $x = \frac{-5}{11},\ y = \frac{12}{11}$ **9.** $x = \frac{-6}{11},\ y = \frac{-19}{11}$

10. $x = -1, y = 4$ **11.** $x = 1,\ y = \frac{1}{2},\ z = \frac{-3}{2}$

12. $x = 2, y = -1, z = 1$ **13.** $x = 1, y = 2, z = -1$

14. $x = 2, y = 1, z = 3$

15. $\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$, $x = 1, y = 2, z = 3$

16. cost of onions per kg = Rs 5
cost of wheat per kg = Rs 8
cost of rice per kg = Rs 8

Miscellaneous Exercise on Chapter 4

3. 1 **5.** $x = \frac{-a}{3}$ **7.** $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

9. $-2(x^3 + y^3)$ **10.** xy **16.** $x = 2, y = 3, z = 5$

17. A **18.** A **19.** D

EXERCISE 5.1

2. f is continuous at $x = 3$
3. (a), (b), (c) and (d) are all continuous functions
5. f is continuous at $x = 0$ and $x = 2$; Not continuous at $x = 1$
6. Discontinuous at $x = 2$
7. Discontinuous at $x = 3$
8. Discontinuous at $x = 0$
9. No point of discontinuity
10. No point of discontinuity
11. No point of discontinuity
12. f is continuous at $x = 1$
13. f is not continuous at $x = 1$
14. f is not continuous at $x = 1$ and $x = 3$
15. $x = 1$ is the only point of discontinuity
16. Continuous
17. $a = b + \frac{2}{3}$
18. For no value of λ, f is continuous at $x = 0$ but f is continuous at $x = 1$ for any value of λ.
20. f is continuous at $x = \pi$
21. (a), (b) and (c) are all continuous
22. Cosine function is continuous for all $x \in \mathbf{R}$; cosecant is continuous except for $x = n\pi$, $n \in \mathbf{Z}$; secant is continuous except for $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbf{Z}$ and cotangent function is continuous except for $x = n\pi$, $n \in \mathbf{Z}$
23. There is no point of discontinuity.
24. Yes, f is continuous for all $x \in \mathbf{R}$
25. f is continuous for all $x \in \mathbf{R}$
26. $k = 6$
27. $k = \frac{3}{4}$
28. $k = \frac{-2}{\pi}$
29. $k = \frac{9}{5}$
30. $a = 2$, $b = 1$
34. There is no point of discontinuity.

EXERCISE 5.2

1. $2x\cos(x^2 + 5)$
2. $-\cos x \sin(\sin x)$
3. $a\cos(ax + b)$
4. $\dfrac{\sec(\tan\sqrt{x}).\tan(\tan\sqrt{x}).\sec^2\sqrt{x}}{2\sqrt{x}}$
5. $a\cos(ax + b)\sec(cx + d) + c\sin(ax + b)\tan(cx + d)\sec(cx + d)$
6. $10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2 x^5$

7. $\dfrac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$ 8. $-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$

EXERCISE 5.3

1. $\dfrac{\cos x-2}{3}$ 2. $\dfrac{2}{\cos y-3}$ 3. $-\dfrac{a}{2by+\sin y}$

4. $\dfrac{\sec^2 x-y}{x+2y-1}$ 5. $-\dfrac{(2x+y)}{(x+2y)}$ 6. $-\dfrac{(3x^2+2xy+y^2)}{(x^2+2xy+3y^2)}$

7. $\dfrac{y\sin xy}{\sin 2y-x\sin xy}$ 8. $\dfrac{\sin 2x}{\sin 2y}$ 9. $\dfrac{2}{1+x^2}$ 10. $\dfrac{3}{1+x^2}$

11. $\dfrac{2}{1+x^2}$ 12. $\dfrac{-2}{1+x^2}$ 13. $\dfrac{-2}{1+x^2}$ 14. $\dfrac{2}{\sqrt{1-x^2}}$

15. $-\dfrac{2}{\sqrt{1-x^2}}$

EXERCISE 5.4

1. $\dfrac{e^x(\sin x-\cos x)}{\sin^2 x}$, $x \neq n\pi,\ n \in \mathbf{Z}$ 2. $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x\in(-1,1)$

3. $3x^2e^{x^3}$ 4. $-\dfrac{e^{-x}\cos(\tan^{-1}e^{-x})}{1+e^{-2x}}$

5. $-e^x\tan e^x,\ e^x \neq (2n+1)\dfrac{\pi}{2}, n\in\mathbf{N}$ 6. $e^x+2xe^{x^2}+3x^2e^{x^3}+4x^3e^{x^4}+5x^4e^{x^5}$

7. $\dfrac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x>0$ 8. $\dfrac{1}{x\log x}, x>1$

9. $-\dfrac{(x\sin x\cdot\log x+\cos x)}{x(\log x)^2}, x>0$ 10. $-\left(\dfrac{1}{x}+e^x\right)\sin(\log x+e^x), x>0$

EXERCISE 5.5

1. $-\cos x \cos 2x \cos 3x\,[\tan x + 2\tan 2x + 3\tan 3x]$

2. $\dfrac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x-3}-\dfrac{1}{x-4}-\dfrac{1}{x-5}\right]$

3. $(\log x)^{\cos x}\left[\dfrac{\cos x}{x\log x}-\sin x\log(\log x)\right]$

4. $x^x(1+\log x)-2^{\sin x}\cos x\log 2$

5. $(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$

6. $\left(x+\dfrac{1}{x}\right)^x\left[\dfrac{x^2-1}{x^2+1}+\log\left(x+\dfrac{1}{x}\right)\right]+x^{1+\frac{1}{x}}\left(\dfrac{x+1-\log x}{x^2}\right)$

7. $(\log x)^{x-1}[1+\log x \,.\, \log(\log x)]+2x^{\log x-1}\,.\,\log x$

8. $(\sin x)^x(x\cot x+\log\sin x)+\dfrac{1}{2}\dfrac{1}{\sqrt{x-x^2}}$

9. $x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x\log x\right]+(\sin x)^{\cos x}[\cos x\cot x-\sin x\log\sin x]$

10. $x^{x\cos x}[\cos x \,.\, (1+\log x)-x\sin x\log x]-\dfrac{4x}{(x^2-1)^2}$

11. $(x\cos x)^x[1-x\tan x+\log(x\cos x)]+(x\sin x)^{\frac{1}{x}}\left[\dfrac{x\cot x+1-\log(x\sin x)}{x^2}\right]$

12. $-\dfrac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

13. $\dfrac{y}{x}\left(\dfrac{y-x\log y}{x-y\log x}\right)$

14. $\dfrac{y\tan x+\log\cos y}{x\tan y+\log\cos x}$

15. $\dfrac{y(x-1)}{x(y+1)}$

16. $(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}+\dfrac{4x^3}{1+x^4}+\dfrac{8x^7}{1+x^8}\right]$; $f'(1)=120$

17. $5x^4-20x^3+45x^2-52x+11$

EXERCISE 5.6

1. $2t^2$ 2. $\dfrac{b}{a}$ 3. $-4\sin t$ 4. $-\dfrac{1}{t^2}$

5. $\dfrac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$ **6.** $-\cot\dfrac{\theta}{2}$ **7.** $-\cot 3t$ **8.** $\tan t$

9. $\dfrac{b}{a}\operatorname{cosec}\theta$ **10.** $\tan\theta$

EXERCISE 5.7

1. 2 **2.** $380\, x^{18}$ **3.** $-x\cos x - 2\sin x$

4. $-\dfrac{1}{x^2}$ **5.** $x(5 + 6\log x)$ **6.** $2e^x(5\cos 5x - 12\sin 5x)$

7. $9\, e^{6x}\,(3\cos 3x - 4\sin 3x)$ **8.** $-\dfrac{2x}{(1+x^2)^2}$

9. $-\dfrac{(1+\log x)}{(x\log x)^2}$ **10.** $-\dfrac{\sin(\log x) + \cos(\log x)}{x^2}$

12. $-\cot y\,\operatorname{cosec}^2 y$

Miscellaneous Exercise on Chapter 5

1. $27\,(3x^2 - 9x + 5)^8\,(2x - 3)$ **2.** $3\sin x\cos x\,(\sin x - 2\cos^4 x)$

3. $(5x)^{3\cos 2x}\left[\dfrac{3\cos 2x}{x} - 6\sin 2x\log 5x\right]$

4. $\dfrac{3}{2}\sqrt{\dfrac{x}{1-x^3}}$ **5.** $-\left[\dfrac{1}{\sqrt{4-x^2}\,\sqrt{2x+7}} + \dfrac{\cos^{-1}\dfrac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$

6. $\dfrac{1}{2}$ **7.** $(\log x)^{\log x}\left[\dfrac{1}{x} + \dfrac{\log(\log x)}{x}\right],\ x > 1$

8. $(a\sin x - b\cos x)\sin(a\cos x + b\sin x)$

9. $(\sin x - \cos x)^{\sin x - \cos x}\,(\cos x + \sin x)\,(1 + \log(\sin x - \cos x)),\ \sin x > \cos x$

10. $x^x\,(1 + \log x) + ax^{a-1} + a^x\log a$

11. $x^{x^2-3}\left[\dfrac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\dfrac{x^2}{x-3} + 2x\log(x-3)\right]$

12. $\frac{6}{5}\cot\frac{t}{2}$ **13.** 0 **17.** $\frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$

EXERCISE 6.1

1. (a) 6π cm^2/s (b) 8π cm^2/s

2. $\frac{8}{3}$ cm^2/s **3.** 60π cm^2/s **4.** 900 cm^3/s

5. 80π cm^2/s **6.** 1.4π cm/s

7. (a) –2 cm/min (b) 2 cm^2/min

8. $\frac{1}{\pi}$ cm/s **9.** 400π cm^3/s **10.** $\frac{8}{3}$ cm/s

11. (4, 11) and $\left(-4, \frac{-31}{3}\right)$ **12.** 2π cm^3/s

13. $\frac{27}{8}\pi\,(2x+1)^2$ **14.** $\frac{1}{48\pi}$ cm/s **15.** Rs 20.967

16. Rs 208 **17.** B **18.** D

EXERCISE 6.2

4. (a) $\left(\frac{3}{4}, \infty\right)$ (b) $\left(-\infty, \frac{3}{4}\right)$

5. (a) $(-\infty, -2)$ and $(3, \infty)$ (b) $(-2, 3)$

6. (a) Strictly decreasing for $x < -1$ and strictly increasing for $x > -1$

(b) Strictly decreasing for $x > -\frac{3}{2}$ and strictly increasing for $x < -\frac{3}{2}$

(c) Strictly increasing for $-2 < x < -1$ and strictly decreasing for $x < -2$ and $x > -1$

(d) Strictly increasing for $x < -\frac{9}{2}$ and strictly decreasing for $x > -\frac{9}{2}$

(e) Strictly increasing in (1, 3) and (3, ∞), strictly decreasing in ($-\infty$, -1) and (-1, 1).

8. $0 < x < 1$ and $x > 2$ **12.** A, B

13. D **14.** $a = -2$ **19.** D

EXERCISE 6.3

1. 764 **2.** $\frac{-1}{64}$ **3.** 11 **4.** 24

5. 1 **6.** $\frac{-a}{2b}$ **7.** (3, -20) and (-1, 12)

8. (3, 1) **9.** (2, -9)

10. (i) $y + x + 1 = 0$ and $y + x - 3 = 0$

11. No tangent to the curve which has slope 2.

12. $y = \frac{1}{2}$ **13.** (i) (0, ± 4) (ii) (± 3, 0)

14. (i) Tangent: $10x + y = 5$; Normal: $x - 10y + 50 = 0$

(ii) Tangent: $y = 2x + 1$; Normal: $x + 2y - 7 = 0$

(iii) Tangent: $y = 3x - 2$; Normal: $x + 3y - 4 = 0$

(iv) Tangent: $y = 0$; Normal: $x = 0$

(v) Tangent: $x + y - \sqrt{2} = 0$; Normal $x = y$

15. (a) $y - 2x - 3 = 0$ (b) $36y + 12x - 227 = 0$

17. (0, 0), (3, 27) **18.** (0, 0), (1, 2), (-1, -2)

19. (1, ± 2) **20.** $2x + 3my - am^2(2 + 3m^2) = 0$

21. $x + 14y - 254 = 0$, $x + 14y + 86 = 0$

22. $ty = x + at^2$, $y = -tx + 2at + at^3$

24. $\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$, $\frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0$

25. $48x - 24y = 23$ **26.** D **27.** A

EXERCISE 6.4

1. (i) 5.03 (ii) 7.035 (iii) 0.8

(iv) 0.208 (v) 0.9999 (vi) 1.96875

(vii) 2.9629	(viii) 3.9961	(ix) 3.009
(x) 20.025	(xi) 0.06083	(xii) 2.948
(xiii) 3.0046	(xiv) 7.904	(xv) 2.00187

2. 28.21 **3.** – 34.995 **4.** $0.03\ x^3\ \text{m}^3$

5. $0.12\ x^2\ \text{m}^2$ **6.** $3.92\ \pi\ \text{m}^3$ **7.** $2.16\ \pi\ \text{m}^3$

8. D **9.** C

EXERCISE 6.5

1. (i) Minimum Value = 3 (ii) Minimum Value = – 2

(iii) Maximum Value = 10 (iv) Neither minimum nor maximum value

2. (i) Minimum Value = – 1; No maximum value

(ii) Maximum Value = 3; No minimum value

(iii) Minimum Value = 4; Maximum Value = 6

(iv) Minimum Value = 2; Maximum Value = 4

(v) Neither minimum nor Maximum Value

3. (i) local minimum at $x = 0$, local minimum value = 0

(ii) local minimum at $x = 1$, local minimum value = – 2

local maximum at $x = -1$, local maximum value = 2

(iii) local maximum at $x = \frac{\pi}{4}$, local maximum value = $\sqrt{2}$

(iv) local maximum at $x = \frac{3\pi}{4}$, local maximum value = $\sqrt{2}$

local minimum at $x = \frac{7\pi}{4}$, local minimum value = $-\sqrt{2}$

(v) local maximum at $x = 1$, local maximum value = 19

local minimum at $x = 3$, local minimum value = 15

(vi) local minimum at $x = 2$, local minimum value = 2

(vii) local maximum at $x = 0$, local maximum value $= \frac{1}{2}$

(viii) local maximum at $x = \frac{2}{3}$, local maximum value $= \frac{2\sqrt{3}}{9}$

5. (i) Absolute minimum value $= -8$, absolute maximum value $= 8$

(ii) Absolute minimum value $= -1$, absolute maximum value $= \sqrt{2}$

(iii) Absolute minimum value $= -10$, absolute maximum value $= 8$

(iv) Absolute minimum value $= 19$, absolute maximum value $= 3$

6. Maximum profit = 49 unit.

7. Minima at $x = 2$, minimum value $= -39$, Maxima at $x = 0$, maximum value $= 25$.

8. At $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

9. Maximum value $= \sqrt{2}$

10. Maximum at $x = 3$, maximum value 89; maximum at $x = -2$, maximum value $= 139$

11. $a = 120$

12. Maximum at $x = 2\pi$, maximum value $= 2\pi$; Minimum at $x = 0$, minimum value $= 0$

13. 12, 12 **14.** 45, 15 **15.** 25, 10 **16.** 8, 8

17. 3 cm **18.** $x = 5$ cm

21. radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

22. $\frac{112}{\pi + 4}$ cm, $\frac{28\pi}{\pi + 4}$ cm **27.** A **28.** D **29.** C

Miscellaneous Exercise on Chapter 6

1. (a) 0.677 (b) 0.497

3. $b\sqrt{3}$ cm^2/s **4.** $x + y - 3 = 0$

6. (i) $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$ (ii) $\frac{\pi}{2} < x < \frac{3\pi}{2}$

7. (i) $x < -1$ and $x > 1$ (ii) $-1 < x < 1$

8. $\frac{3\sqrt{3}}{4}ab$ **9.** Rs 1000

11. length $= \frac{20}{\pi + 4}$ m, breadth $= \frac{10}{\pi + 4}$ m

13. (i) local maxima at $x = 2$ (ii) local minima at $x = \frac{2}{7}$

(iii) point of inflection at $x = -1$

14. Absolute maximum $= \frac{5}{4}$, Absolute minimum $= 1$

17. $\frac{4\pi R^3}{3\sqrt{3}}$ **19.** A **20.** B **21.** A

22. B **23.** A **24.** A

Appendix 1

PROOFS IN MATHEMATICS

❖*Proofs are to Mathematics what calligraphy is to poetry.*
Mathematical works do consist of proofs just as
poems do consist of characters.
— VLADIMIR ARNOLD ❖

A.1.1 Introduction

In Classes IX, X and XI, we have learnt about the concepts of a statement, compound statement, negation, converse and contrapositive of a statement; axioms, conjectures, theorems and deductive reasoning.

Here, we will discuss various methods of proving mathematical propositions.

A.1.2 What is a Proof?

Proof of a mathematical statement consists of sequence of statements, each statement being justified with a definition or an axiom or a proposition that is previously established by the method of deduction using only the allowed logical rules.

Thus, each proof is a chain of deductive arguments each of which has its premises and conclusions. Many a times, we prove a proposition directly from what is given in the proposition. But some times it is easier to prove an equivalent proposition rather than proving the proposition itself. This leads to, two ways of proving a proposition directly or indirectly and the proofs obtained are called direct proof and indirect proof and further each has three different ways of proving which is discussed below.

Direct Proof It is the proof of a proposition in which we directly start the proof with what is given in the proposition.

(i) **Straight forward approach** It is a chain of arguments which leads directly from what is given or assumed, with the help of axioms, definitions or already proved theorems, to what is to be proved using rules of logic.

Consider the following example:

Example 1 Show that if $x^2 - 5x + 6 = 0$, then $x = 3$ or $x = 2$.

Solution $x^2 - 5x + 6 = 0$ (given)

$\Rightarrow$ $(x-3)(x-2) = 0$ (replacing an expression by an equal/equivalent expression)

$\Rightarrow$ $x-3 = 0$ or $x-2 = 0$ (from the established theorem $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$, for a, b in $\mathbf{R}$)

$\Rightarrow$ $x-3+3 = 0+3$ or $x-2+2 = 0+2$ (adding equal quantities on either side of the equation does not alter the nature of the equation)

$\Rightarrow$ $x+0 = 3$ or $x+0 = 2$ (using the identity property of integers under addition)

$\Rightarrow$ $x = 3$ or $x = 2$ (using the identity property of integers under addition)

Hence, $x^2 - 5x + 6 = 0$ implies $x = 3$ or $x = 2$.

Explanation Let p be the given statement "$x^2 - 5x + 6 = 0$" and q be the conclusion statement "$x = 3$ or $x = 2$".

From the statement p, we deduced the statement r: "$(x-3)(x-2) = 0$" by replacing the expression $x^2 - 5x + 6$ in the statement p by another expression $(x-3)(x-2)$ which is equal to $x^2 - 5x + 6$.

There arise two questions:

(i) How does the expression $(x-3)(x-2)$ is equal to the expression $x^2 - 5x + 6$?

(ii) How can we replace an expression with another expression which is equal to the former?

The first one is proved in earlier classes by factorization, i.e.,

$x^2 - 5x + 6 = x^2 - 3x - 2x + 6 = x(x-3) - 2(x-3) = (x-3)(x-2)$.

The second one is by valid form of argumentation (rules of logic)

Next this statement r becomes premises or given and deduce the statement s "$x-3 = 0$ or $x-2 = 0$" and the reasons are given in the brackets.

This process continues till we reach the conclusion.

The symbolic equivalent of the argument is to prove by deduction that $p \Rightarrow q$ is true.

Starting with p, we deduce $p \Rightarrow r \Rightarrow s \Rightarrow \ldots \Rightarrow q$. This implies that "$p \Rightarrow q$" is true.

Example 2 Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 2x + 5$ is one-one.

Solution Note that a function f is one-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \text{(definition of one-one function)}$$

Now, given that $\quad f(x_1) = f(x_2)$, i.e., $2x_1 + 5 = 2x_2 + 5$

$\Rightarrow \quad 2x_1 + 5 - 5 = 2x_2 + 5 - 5$ (adding the same quantity on both sides)

$$\Rightarrow \qquad 2x_1 + 0 = 2x_2 + 0$$

$$\Rightarrow \qquad 2x_1 = 2x_2 \text{ (using additive identity of real number)}$$

$$\Rightarrow \qquad \frac{2}{2}x_1 = \frac{2}{2}x_2 \text{ (dividing by the same non zero quantity)}$$

$$\Rightarrow \qquad x_1 = x_2$$

Hence, the given function is one-one.

(ii) Mathematical Induction

Mathematical induction, is a strategy, of proving a proposition which is deductive in nature. The whole basis of proof of this method depends on the following axiom:

For a given subset S of **N**, if

(i) the natural number $1 \in$ S and

(ii) the natural number $k + 1 \in$ S whenever $k \in$ S, then S = **N**.

According to the principle of mathematical induction, if a statement "S(n) is true for $n = 1$" (or for some starting point j), and if "S(n) is true for $n = k$" implies that "S(n) is true for $n = k + 1$" (whatever integer $k \geq j$ may be), then the statement is true for any positive integer n, for all $n \geq j$.

We now consider some examples.

Example 3 Show that if

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

Solution We have

$$P(n) : A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

We note that
$$P(1) : A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, P(1) is true.

Assume that P(k) is true, i.e.,

$$P(k) : A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

We want to prove that P(k + 1) is true whenever P(k) is true, i.e.,

$$P(k+1) : A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Now $A^{k+1} = A^k . A$

Since P(k) is true, we have

$$A^{k+1} = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

(by matrix multiplication)

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, P(n) is true for all $n \geq 1$ (by the principle of mathematical induction).

(iii) Proof by cases or by exhaustion

This method of proving a statement $p \Rightarrow q$ is possible only when p can be split into several cases, r, s, t (say) so that $p = r \vee s \vee t$ (where "$\vee$" is the symbol for "OR").

If the conditionals $r \Rightarrow q$;

$s \Rightarrow q$;

and $t \Rightarrow q$

are proved, then $(r \vee s \vee t) \Rightarrow q$, is proved and so $p \Rightarrow q$ is proved.

The method consists of examining every possible case of the hypothesis. It is practically convenient only when the number of possible cases are few.

Example 4 Show that in any triangle ABC,

$$a = b\cos C + c\cos B$$

Solution Let p be the statement "ABC is any triangle" and q be the statement "$a = b\cos C + c\cos B$"

Let ABC be a triangle. From A draw AD a perpendicular to BC (BC produced if necessary).

As we know that any triangle has to be either acute or obtuse or right angled, we can split p into three statements r, s and t, where *r* : ABC is an acute angled triangle with $\angle$ C is acute.

s : ABC is an obtuse angled triangle with $\angle$ C is obtuse.

t : ABC is a right angled triangle with $\angle$ C is right angle.

Hence, we prove the theorem by three cases.

Case (i) When $\angle$ C is acute (Fig. A1.1).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $\quad BD = AB \cos B$

$$= c \cos B$$

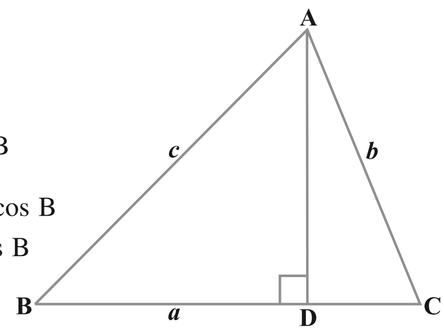


Fig A1.1

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos C$$

i.e. $\quad CD = AC \cos C$

$$= b \cos C$$

Now $\quad a = BD + CD$

$$= c \cos B + b \cos C \qquad \text{... (1)}$$

Case (ii) When $\angle$ C is obtuse (Fig A1.2).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $\quad BD = AB \cos B$

$$= c \cos B$$

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos \angle ACD$$

$$= \cos (180° - C)$$

$$= - \cos C$$

i.e. $\quad CD = - AC \cos C$

$$= - b \cos C$$

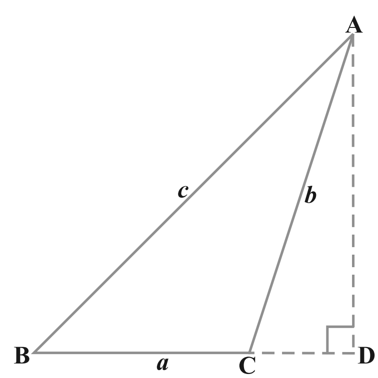


Fig A1.2

Now $a = BC = BD - CD$

i.e. $a = c \cos B - (-b \cos C)$

$a = c \cos B + b \cos C$ ... (2)

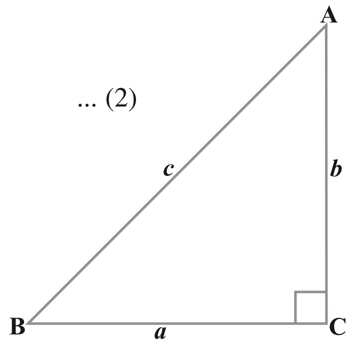


Fig A1.3

Case (iii) When $\angle$ C is a right angle (Fig A1.3).

From the right angled triangle ACB,

$$\frac{BC}{AB} = \cos B$$

i.e. $BC = AB \cos B$

$a = c \cos B,$

and $b \cos C = b \cos 90^0 = 0.$

Thus, we may write $a = 0 + c \cos B$

$= b \cos C + c \cos B$ ... (3)

From (1), (2) and (3). We assert that for any triangle ABC,

$$a = b \cos C + c \cos B$$

By case (i), $r \Rightarrow q$ is proved.

By case (ii), $s \Rightarrow q$ is proved.

By case (iii), $t \Rightarrow q$ is proved.

Hence, from the proof by cases, $(r \vee s \vee t) \Rightarrow q$ is proved, i.e., $p \Rightarrow q$ is proved.

Indirect Proof Instead of proving the given proposition directly, we establish the proof of the proposition through proving a proposition which is equivalent to the given proposition.

(i) **Proof by contradiction** (*Reductio Ad Absurdum*) : Here, we start with the assumption that the given statement is false. By rules of logic, we arrive at a conclusion contradicting the assumption and hence it is inferred that the assumption is wrong and hence the given statement is true.

Let us illustrate this method by an example.

Example 5 Show that the set of all prime numbers is infinite.

Solution Let P be the set of all prime numbers. We take the negation of the statement "the set of all prime numbers is infinite", i.e., we assume the set of all prime numbers to be finite. Hence, we can list all the prime numbers as $P_1, P_2, P_3, ..., P_k$ (say). Note that we have assumed that there is no prime number other than $P_1, P_2, P_3, ..., P_k$.

Now consider $N = (P_1 P_2 P_3 \ldots P_k) + 1$ ... (1)

N is not in the list as N is larger than any of the numbers in the list.

N is either prime or composite.

If N is a prime, then by (1), there exists a prime number which is not listed.

On the other hand, if N is composite, it should have a prime divisor. But none of the numbers in the list can divide N, because they all leave the remainder 1. Hence, the prime divisor should be other than the one in the list.

Thus, in both the cases whether N is a prime or a composite, we ended up with contradiction to the fact that we have listed all the prime numbers.

Hence, our assumption that set of all prime numbers is finite is false.

Thus, the set of all prime numbers is infinite.

Note Observe that the above proof also uses the method of proof by cases.

(ii) Proof by using contrapositive statement of the given statement

Instead of proving the conditional $p \Rightarrow q$, we prove its equivalent, i.e., $\sim q \Rightarrow \sim p$. (students can verify).

The contrapositive of a conditional can be formed by interchanging the conclusion and the hypothesis and negating both.

Example 6 Prove that the function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = 2x + 5$ is one-one.

Solution A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Using this we have to show that "$2x_1 + 5 = 2x_2 + 5$" $\Rightarrow$ "$x_1 = x_2$". This is of the form $p \Rightarrow q$, where, p is $2x_1 + 5 = 2x_2 + 5$ and $q : x_1 = x_2$. We have proved this in Example 2 of "direct method".

We can also prove the same by using contrapositive of the statement. Now contrapositive of this statement is $\sim q \Rightarrow \sim p$, i.e., contrapositive of "if $f(x_1) = f(x_2)$, then $x_1 = x_2$" is "if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$".

$$
\begin{aligned}
\text{Now} \quad & x_1 \neq x_2 \\
\Rightarrow \quad & 2x_1 \neq 2x_2 \\
\Rightarrow \quad & 2x_1 + 5 \neq 2x_2 + 5 \\
\Rightarrow \quad & f(x_1) \neq f(x_2).
\end{aligned}
$$

Since "$\sim q \Rightarrow \sim p$", is equivalent to "$p \Rightarrow q$" the proof is complete.

Example 7 Show that "if a matrix A is invertible, then A is non singular".

Solution Writing the above statement in symbolic form, we have

$p \Rightarrow q$, where, p is "matrix A is invertible" and q is "A is non singular"

Instead of proving the given statement, we prove its contrapositive statement, i.e., if A is not a non singular matrix, then the matrix A is not invertible.

If A is not a non singular matrix, then it means the matrix A is singular, i.e.,

$$|A| = 0$$

Then $$A^{-1} = \frac{adj\ A}{|A|} \text{ does not exist as } |A| = 0$$

Hence, A is not invertible.

Thus, we have proved that if A is not a non singular matrix, then A is not invertible. i.e., $\sim q \Rightarrow \sim p$.

Hence, if a matrix A is invertible, then A is non singular.

(iii) Proof by a counter example

In the history of Mathematics, there are occasions when all attempts to find a valid proof of a statement fail and the uncertainty of the truth value of the statement remains unresolved.

In such a situation, it is beneficial, if we find an example to falsify the statement. The example to disprove the statement is called a *counter example*. Since the disproof of a proposition $p \Rightarrow q$ is merely a proof of the proposition $\sim (p \Rightarrow q)$. Hence, this is also a method of proof.

Example 8 For each $n, 2^{2^n} + 1$ is a prime ($n \in$ **N**).
This was once thought to be true on the basis that

$$2^{2^1} + 1 = 2^2 + 1 = 5 \text{ is a prime.}$$

$$2^{2^2} + 1 = 2^4 + 1 = 17 \text{ is a prime.}$$

$$2^{2^3} + 1 = 2^8 + 1 = 257 \text{ is a prime.}$$

However, at first sight the generalisation looks to be correct. But, eventually it was shown that $$2^{2^5} + 1 = 2^{32} + 1 = 4294967297$$
which is not a prime since $4294967297 = 641 \times 6700417$ (a product of two numbers).

So the generalisation "For each n, $2^{2^n} + 1$ is a prime ($n \in$ **N**)" is false.

Just this one example $2^{2^5} + 1$ is sufficient to disprove the generalisation. This is the counter example.

Thus, we have proved that the generalisation "For each $n, 2^{2^n} + 1$ is a prime ($n \in$ **N**)" is not true in general.

Example 9 Every continuous function is differentiable.

Proof We consider some functions given by

(i) $f(x) = x^2$

(ii) $g(x) = e^x$

(iii) $h(x) = \sin x$

These functions are continuous for all values of x. If we check for their differentiability, we find that they are all differentiable for all the values of x. This makes us to believe that the generalisation "Every continuous function is differentiable" may be true. But if we check the differentiability of the function given by "$\phi(x) = |x|$" which is continuous, we find that it is not differentiable at $x = 0$. This means that the statement **"**Every continuous function is differentiable" is false, in general. Just this one function "$\phi(x) = |x|$" is sufficient to disprove the statement. Hence, "$\phi(x) = |x|$" is called a counter example to disprove **"**Every continuous function is differentiable".

Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

In class XI, we have learnt about mathematical modelling as an attempt to study some part (or form) of some real-life problems in mathematical terms, i.e., the conversion of a physical situation into mathematics using some suitable conditions. Roughly speaking mathematical modelling is an activity in which we make models to describe the behaviour of various phenomenal activities of our interest in many ways using words, drawings or sketches, computer programs, mathematical formulae etc.

In earlier classes, we have observed that solutions to many problems, involving applications of various mathematical concepts, involve mathematical modelling in one way or the other. Therefore, it is important to study mathematical modelling as a separate topic.

In this chapter, we shall further study mathematical modelling of some real-life problems using techniques/results from matrix, calculus and linear programming.

A.2.2 Why Mathematical Modelling?

Students are aware of the solution of word problems in arithmetic, algebra, trigonometry and linear programming etc. Sometimes we solve the problems without going into the physical insight of the situational problems. Situational problems need physical insight that is **introduction** of physical laws and some symbols to compare the mathematical results obtained with practical values. To solve many problems faced by us, we need a technique and this is what is known as *mathematical modelling*. Let us consider the following problems:

(i) To find the width of a river (particularly, when it is difficult to cross the river).

(ii) To find the optimal angle in case of shot-put (by considering the variables such as : the height of the thrower, resistance of the media, acceleration due to gravity etc.).

(iii) To find the height of a tower (particularly, when it is not possible to reach the top of the tower).

(iv) To find the temperature at the surface of the Sun.

(v) Why heart patients are not allowed to use lift? (without knowing the physiology of a human being).

(vi) To find the mass of the Earth.

(vii) Estimate the yield of pulses in India from the standing crops (a person is not allowed to cut all of it).

(viii) Find the volume of blood inside the body of a person (a person is not allowed to bleed completely).

(ix) Estimate the population of India in the year 2020 (a person is not allowed to wait till then).

All of these problems can be solved and infact have been solved with the help of Mathematics using mathematical modelling. In fact, you might have studied the methods for solving some of them in the present textbook itself. However, it will be instructive if you first try to solve them yourself and that too without the help of Mathematics, if possible, you will then appreciate the power of Mathematics and the need for mathematical modelling.

A.2.3 Principles of Mathematical Modelling

Mathematical modelling is a principled activity and so it has some principles behind it. These principles are almost philosophical in nature. Some of the basic principles of mathematical modelling are listed below in terms of instructions:

(i) Identify the need for the model. (for what we are looking for)

(ii) List the parameters/variables which are required for the model.

(iii) Identify the available relevent data. (what is given?)

(iv) Identify the circumstances that can be applied (assumptions)

(v) Identify the governing physical principles.

(vi) Identify

- (a) the equations that will be used.
- (b) the calculations that will be made.
- (c) the solution which will follow.

(vii) Identify tests that can check the

- (a) consistency of the model.
- (b) utility of the model.

(viii) Identify the parameter values that can improve the model.

The above principles of mathematical modelling lead to the following: steps for mathematical modelling.

Step 1: Identify the physical situation.

Step 2: Convert the physical situation into a mathematical model by introducing parameters / variables and using various known physical laws and symbols.

Step 3: Find the solution of the mathematical problem.

Step 4: Interpret the result in terms of the original problem and compare the result with observations or experiments.

Step 5: If the result is in good agreement, then accept the model. Otherwise modify the hypotheses / assumptions according to the physical situation and go to Step 2.

The above steps can also be viewed through the following diagram:

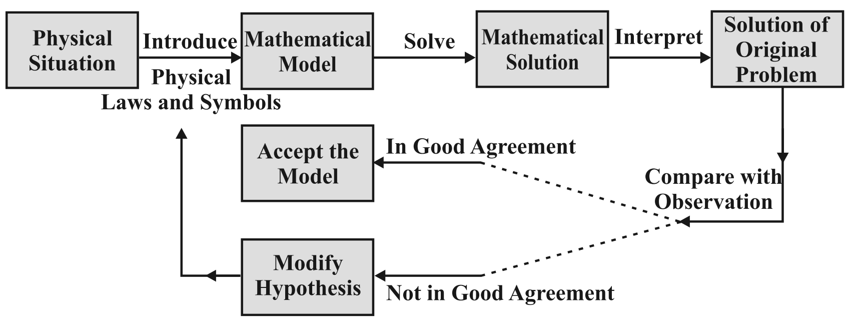


Fig A.2.1

Example 1 Find the height of a given tower using mathematical modelling.

Solution Step 1 Given physical situation is "to find the height of a given tower".

Step 2 Let AB be the given tower (Fig A.2.2). Let PQ be an observer measuring the height of the tower with his eye at P. Let PQ = h and let height of tower be H. Let α be the angle of elevation from the eye of the observer to the top of the tower.

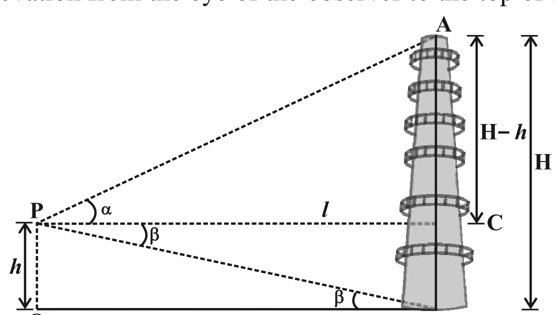


Fig A.2.2

Let $$l = \text{PC} = \text{QB}$$

Now $$\tan\alpha = \frac{\text{AC}}{\text{PC}} = \frac{\text{H} - h}{l}$$

or $$\text{H} = h + l\tan\alpha \qquad \ldots (1)$$

Step 3 Note that the values of the parameters h, l and α (using sextant) are known to the observer and so (1) gives the solution of the problem.

Step 4 In case, if the foot of the tower is not accessible, i.e., when l is not known to the observer, let β be the angle of depression from P to the foot B of the tower. So from ΔPQB, we have

$$\tan\beta = \frac{\text{PQ}}{\text{QB}} = \frac{h}{l} \text{ or } l = h\cot\beta$$

Step 5 is not required in this situation as exact values of the parameters h, l, α and β are known.

Example 2 Let a business firm produces three types of products P_1, P_2 and P_3 that uses three types of raw materials R_1, R_2 and R_3. Let the firm has purchase orders from two clients F_1 and F_2. Considering the situation that the firm has a limited quantity of R_1, R_2 and R_3, respectively, prepare a model to determine the quantities of the raw material R_1, R_2 and R_3 required to meet the purchase orders.

Solution Step 1 The physical situation is well identified in the problem.

Step 2 Let A be a matrix that represents purchase orders from the two clients F_1 and F_2. Then, A is of the form

$$A = \begin{array}{c} \\ F_1 \\ F_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Let B be the matrix that represents the amount of raw materials R_1, R_2 and R_3, required to manufacture each unit of the products P_1, P_2 and P_3. Then, B is of the form

$$B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Step 3 Note that the product (which in this case is well defined) of matrices A and B is given by the following matrix

$$\mathrm{AB} = \begin{array}{c} \\ \mathrm{F}_1 \\ \mathrm{F}_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \mathrm{R}_1 & \mathrm{R}_2 & \mathrm{R}_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

which in fact gives the desired quantities of the raw materials R_1, R_2 and R_3 to fulfill the purchase orders of the two clients F_1 and F_2.

Example 3 Interpret the model in Example 2, in case

$$A = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

and the available raw materials are 330 units of R_1, 455 units of R_2 and 140 units of R_3.

Solution Note that

$$AB = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

$$= \begin{array}{c} \\ \mathrm{F}_1 \\ \mathrm{F}_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \mathrm{R}_1 & \mathrm{R}_2 & \mathrm{R}_3 \end{array} \\ \begin{bmatrix} 165 & 247 & 87 \\ 170 & 220 & 60 \end{bmatrix} \end{array}$$

This clearly shows that to meet the purchase order of F_1 and F_2, the raw material required is 335 units of R_1, 467 units of R_2 and 147 units of R_3 which is much more than the available raw material. Since the amount of raw material required to manufacture each unit of the three products is fixed, we can either ask for an increase in the available raw material or we may ask the clients to reduce their orders.

Remark If we replace A in Example 3 by A_1 given by

$$A_1 = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}$$

i.e., if the clients agree to reduce their purchase orders, then

$$A_1 B = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} = \begin{bmatrix} 141 & 216 & 78 \\ 170 & 220 & 60 \end{bmatrix}$$

This requires 311 units of R_1, 436 units of R_2 and 138 units of R_3 which are well below the available raw materials, i.e., 330 units of R_1, 455 units of R_2 and 140 units of R_3. Thus, if the revised purchase orders of the clients are given by A_1, then the firm can easily supply the purchase orders of the two clients.

Note One may further modify A so as to make full use of the available raw material.

Query Can we make a mathematical model with a given B and with fixed quantities of the available raw material that can help the firm owner to ask the clients to modify their orders in such a way that the firm makes the full use of its available raw material?

The answer to this query is given in the following example:

Example 4 Suppose P_1, P_2, P_3 and R_1, R_2, R_3 are as in Example 2. Let the firm has 330 units of R_1, 455 units of R_2 and 140 units of R_3 available with it and let the amount of raw materials R_1, R_2 and R_3 required to manufacture each unit of the three products is given by

$$B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} \end{array}$$

How many units of each product is to be made so as to utilise the full available raw material?

Solution Step 1 The situation is easily identifiable.

Step 2 Suppose the firm produces x units of P_1, y units of P_2 and z units of P_3. Since product P_1 requires 3 units of R_1, P_2 requires 7 units of R_1 and P_3 requires 5 units of R_1 (observe matrix B) and the total number of units, of R_1, available is 330, we have

$$3x + 7y + 5z = 330 \text{ (for raw material } R_1)$$

Similarly, we have

$$4x + 9y + 12z = 455 \text{ (for raw material } R_2)$$

and

$$3y + 7z = 140 \text{ (for raw material } R_3)$$

This system of equations can be expressed in matrix form as

$$\begin{bmatrix} 3 & 7 & 5 \\ 4 & 9 & 12 \\ 0 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 330 \\ 455 \\ 140 \end{bmatrix}$$

Step 3 Using elementary row operations, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \\ 5 \end{bmatrix}$$

This gives $x = 20$, $y = 35$ and $z = 5$. Thus, the firm can produce 20 units of P_1, 35 units of P_2 and 5 units of P_3 to make full use of its available raw material.

Remark One may observe that if the manufacturer decides to manufacture according to the available raw material and not according to the purchase orders of the two clients F_1 and F_2 (as in Example 3), he/she is unable to meet these purchase orders as F_1 demanded 6 units of P_3 where as the manufacturer can make only 5 units of P_3.

Example 5 A manufacturer of medicines is preparing a production plan of medicines M_1 and M_2. There are sufficient raw materials available to make 20000 bottles of M_1 and 40000 bottles of M_2, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of M_1, it takes 1 hour to prepare enough material to fill 1000 bottles of M_2 and there are 66 hours available for this operation. The profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2. How should the manufacturer schedule his/her production in order to maximise profit?

Solution Step 1 To find the number of bottles of M_1 and M_2 in order to maximise the profit under the given hypotheses.

Step 2 Let x be the number of bottles of type M_1 medicine and y be the number of bottles of type M_2 medicine. Since profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2, therefore the objective function (which is to be maximised) is given by

$$Z \equiv Z(x, y) = 8x + 7y$$

The objective function is to be maximised subject to the constraints (Refer Chapter 12 on Linear Programming)

$$\left.\begin{aligned} & x \leq 20000 \\ & y \leq 40000 \\ & x + y \leq 45000 \\ & 3x + y \leq 66000 \\ & x \geq 0, y \geq 0 \end{aligned}\right\} \quad \text{... (1)}$$

Step 3 The shaded region OPQRST is the feasible region for the constraints (1) (Fig A.2.3). The co-ordinates of vertices O, P, Q, R, S and T are (0, 0), (20000, 0), (20000, 6000), (10500, 34500), (5000, 40000) and (0, 40000), respectively.

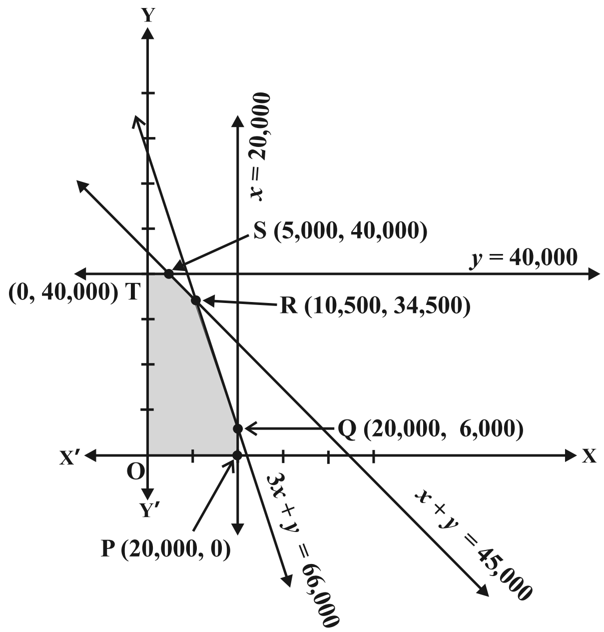


Fig A.2.3

Note that

Z at P (0, 0) = 0

Z at P (20000, 0) = 8 × 20000 = 160000

Z at Q (20000, 6000) = 8 × 20000 + 7 × 6000 = 202000

Z at R (10500, 34500) = 8 × 10500 + 7 × 34500 = 325500

Z at S = (5000, 40000) = 8 × 5000 + 7 × 40000 = 320000

Z at T = (0, 40000) = 7 × 40000 = 280000

Now observe that the profit is maximum at x = 10500 and y = 34500 and the maximum profit is Rs 325500. Hence, the manufacturer should produce 10500 bottles of M_1 medicine and 34500 bottles of M_2 medicine in order to get maximum profit of Rs 325500.

Example 6 Suppose a company plans to produce a new product that incur some costs (fixed and variable) and let the company plans to sell the product at a fixed price. Prepare a mathematical model to examine the profitability.

Solution Step 1 Situation is clearly identifiable.

Step 2 Formulation: We are given that the costs are of two types: fixed and variable. The fixed costs are independent of the number of units produced (e.g., rent and rates), while the variable costs increase with the number of units produced (e.g., material). Initially, we assume that the variable costs are directly proportional to the number of units produced — this should simplify our model. The company earn a certain amount of money by selling its products and wants to ensure that it is maximum. For convenience, we assume that all units produced are sold immediately.

The mathematical model

Let x = number of units produced and sold

C = total cost of production (in rupees)

I = income from sales (in rupees)

P = profit (in rupees)

Our assumptions above state that C consists of two parts:

(i) fixed cost = a (in rupees),

(ii) variable cost = b (rupees/unit produced).

Then $$C = a + bx \qquad \text{... (1)}$$

Also, income I depends on selling price s (rupees/unit)

Thus $$I = sx \qquad \text{... (2)}$$

The profit P is then the difference between income and costs. So

$$\begin{aligned} P &= I - C \\ &= sx - (a + bx) \\ &= (s - b)\,x - a \qquad \text{... (3)} \end{aligned}$$

We now have a mathematical model of the relationships (1) to (3) *between the variables x,* C, I, P*, a, b, s.* These variables may be classified as:

independent	x
dependent	C, I, P
parameters	a, b, s

The manufacturer, knowing x, a, b, s can determine P.

Step 3 From (3), we can observe that for the break even point (i.e., make neither profit nor loss), he must have P = 0, i.e., $x = \frac{a}{s-b}$ units.

Steps 4 and 5 In view of the break even point, one may conclude that if the company produces few units, i.e., less than $x = \frac{a}{s-b}$ units, then the company will suffer loss and if it produces large number of units, i.e., much more than $\frac{a}{s-b}$ units, then it can make huge profit. Further, if the break even point proves to be unrealistic, then another model could be tried or the assumptions regarding cash flow may be modified.

Remark From (3), we also have

$$\frac{d\,\mathrm{P}}{d\,x} = s - b$$

This means that rate of change of P with respect to x depends on the quantity $s - b$, which is the difference of selling price and the variable cost of each product. Thus, in order to gain profit, this should be positive and to get large gains, we need to produce large quantity of the product and at the same time try to reduce the variable cost.

Example 7 Let a tank contains 1000 litres of brine which contains 250 g of salt per litre. Brine containing 200 g of salt per litre flows into the tank at the rate of 25 litres per minute and the mixture flows out at the same rate. Assume that the mixture is kept uniform all the time by stirring. What would be the amount of salt in the tank at any time t?

Solution Step 1 The situation is easily identifiable.

Step 2 Let $y = y\,(t)$ denote the amount of salt (in kg) in the tank at time t (in minutes) after the inflow, outflow starts. Further assume that y is a differentiable function.

When $t = 0$, i.e., before the inflow–outflow of the brine starts,

$$y = 250 \text{ g} \times 1000 = 250 \text{ kg}$$

Note that the change in y occurs due to the inflow, outflow of the mixture.

Now the inflow of brine brings salt into the tank at the rate of 5 kg per minute (as 25×200 g = 5 kg) and the outflow of brine takes salt out of the tank at the rate of $25\left(\frac{y}{1000}\right) = \frac{y}{40}$ kg per minute (as at time t, the salt in the tank is $\frac{y}{1000}$ kg).

Thus, the rate of change of salt with respect to t is given by

$$\frac{dy}{dt} = 5 - \frac{y}{40} \qquad \text{(Why?)}$$

or

$$\frac{dy}{dt} + \frac{1}{40}y = 5 \qquad \text{... (1)}$$

This gives a mathematical model for the given problem.

Step 3 Equation (1) is a linear equation and can be easily solved. The solution of (1) is given by

$$y e^{\frac{t}{40}} = 200 e^{\frac{t}{40}} + \mathrm{C} \text{ or } y\,(t) = 200 + \mathrm{C}\ e^{-\frac{t}{40}} \qquad \text{... (2)}$$

where, c is the constant of integration.

Note that when $t = 0$, $y = 250$. Therefore, $250 = 200 + \mathrm{C}$

or $\mathrm{C} = 50$

Then (2) reduces to

$$y = 200 + 50\ e^{-\frac{t}{40}} \qquad \text{... (3)}$$

or $$\frac{y-200}{50} = e^{-\frac{t}{40}}$$

or $$e^{\frac{t}{40}} = \frac{50}{y-200}$$

Therefore $$t = 40 \log_e \left(\frac{50}{y-200}\right) \qquad \text{... (4)}$$

Here, the equation (4) gives the time t at which the salt in tank is y kg.

Step 4 Since $e^{-\frac{t}{40}}$ is always positive, from (3), we conclude that $y > 200$ at all times Thus, the minimum amount of salt content in the tank is 200 kg.

Also, from (4), we conclude that $t > 0$ if and only if $0 < y - 200 < 50$ i.e., if and only if $200 < y < 250$ i.e., the amount of salt content in the tank after the start of inflow and outflow of the brine is between 200 kg and 250 kg.

Limitations of Mathematical Modelling

Till today many mathematical models have been developed and applied successfully to understand and get an insight into thousands of situations. Some of the subjects like mathematical physics, mathematical economics, operations research, bio-mathematics etc. are almost synonymous with mathematical modelling.

But there are still a large number of situations which are yet to be modelled. The reason behind this is that either the situation are found to be very complex or the mathematical models formed are mathematically intractable.

The development of the powerful computers and super computers has enabled us to mathematically model a large number of situations (even complex situations). Due to these fast and advanced computers, it has been possible to prepare more realistic models which can obtain better agreements with observations.

However, we do not have good guidelines for choosing various parameters / variables and also for estimating the values of these parameters / variables used in a mathematical model. Infact, we can prepare reasonably accurate models to fit any data by choosing five or six parameters / variables. We require a minimal number of parameters / variables to be able to estimate them accurately.

Mathematical modelling of large or complex situations has its own special problems. These type of situations usually occur in the study of world models of environment, oceanography, pollution control etc. Mathematical modellers from all disciplines — mathematics, computer science, physics, engineering, social sciences, etc., are involved in meeting these challenges with courage.

CONSTITUTION OF INDIA

Part III (Articles 12 – 35)

(Subject to certain conditions, some exceptions and reasonable restrictions)

guarantees these

Fundamental Rights

Right to Equality

- before law and equal protection of laws;
- irrespective of religion, race, caste, sex or place of birth;
- of opportunity in public employment;
- by abolition of untouchability and titles.

Right to Freedom

- of expression, assembly, association, movement, residence and profession;
- of certain protections in respect of conviction for offences;
- of protection of life and personal liberty;
- of free and compulsory education for children between the age of six and fourteen years;
- of protection against arrest and detention in certain cases.

Right against Exploitation

- for prohibition of traffic in human beings and forced labour;
- for prohibition of employment of children in hazardous jobs.

Right to Freedom of Religion

- freedom of conscience and free profession, practice and propagation of religion;
- freedom to manage religious affairs;
- freedom as to payment of taxes for promotion of any particular religion;
- freedom as to attendance at religious instruction or religious worship in educational institutions wholly maintained by the State.

Cultural and Educational Rights

- for protection of interests of minorities to conserve their language, script and culture;
- for minorities to establish and administer educational institutions of their choice.

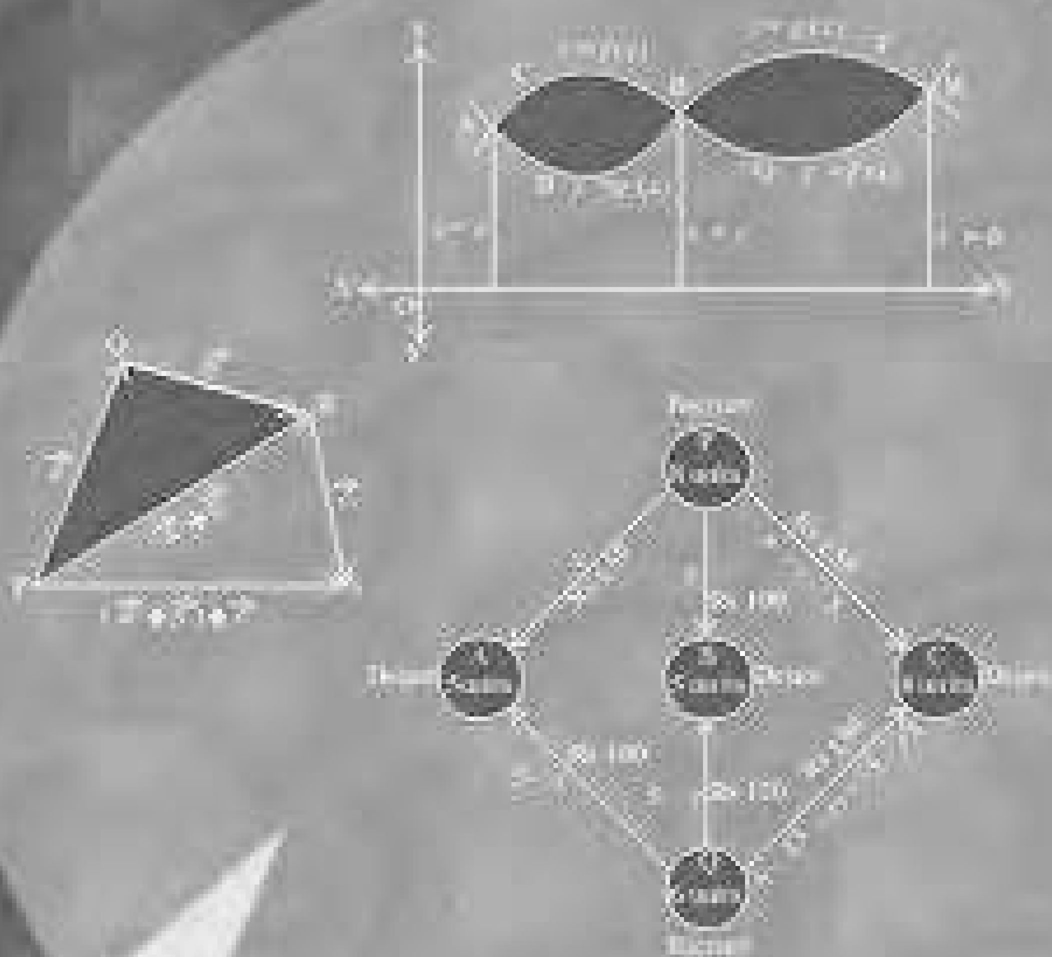
Right to Constitutional Remedies

- by issuance of directions or orders or writs by the Supreme Court and High Courts for enforcement of these Fundamental Rights.

MATHEMATICS

Textbook for Class XII

Part II



MATHEMATICS Part II Class XII NCERT

Contents

Part II

Foreword *v*
Preface *vii*

7. Integrals **287**
7.1 Introduction 288
7.2 Integration as an Inverse Process of Differentiation 288
7.3 Methods of Integration 300
7.4 Integrals of some Particular Functions 307
7.5 Integration by Partial Fractions 316
7.6 Integration by Parts 323
7.7 Definite Integral 331
7.8 Fundamental Theorem of Calculus 334
7.9 Evaluation of Definite Integrals by Substitution 338
7.10 Some Properties of Definite Integrals 341

8. Application of Integrals **359**
8.1 Introduction 359
8.2 Area under Simple Curves 359
8.3 Area between Two Curves 366

9. Differential Equations **379**
9.1 Introduction 379
9.2 Basic Concepts 379
9.3 General and Particular Solutions of a Differential Equation 383
9.4 Formation of a Differential Equation whose General Solution is given 385
9.5 Methods of Solving First order, First Degree Differential Equations 391

10. Vector Algebra **424**
10.1 Introduction 424
10.2 Some Basic Concepts 424
10.3 Types of Vectors 427
10.4 Addition of Vectors 429

	10.5	Multiplication of a Vector by a Scalar	432
	10.6	Product of Two Vectors	441
11.	**Three Dimensional Geometry**		**463**
	11.1	Introduction	463
	11.2	Direction Cosines and Direction Ratios of a Line	463
	11.3	Equation of a Line in Space	468
	11.4	Angle between Two Lines	471
	11.5	Shortest Distance between Two Lines	473
	11.6	Plane	479
	11.7	Coplanarity of Two Lines	487
	11.8	Angle between Two Planes	488
	11.9	Distance of a Point from a Plane	490
	11.10	Angle between a Line and a Plane	492
12.	**Linear Programming**		**504**
	12.1	Introduction	504
	12.2	Linear Programming Problem and its Mathematical Formulation	505
	12.3	Different Types of Linear Programming Problems	514
13.	**Probability**		**531**
	13.1	Introduction	531
	13.2	Conditional Probability	531
	13.3	Multiplication Theorem on Probability	540
	13.4	Independent Events	542
	13.5	Bayes' Theorem	548
	13.6	Random Variables and its Probability Distributions	557
	13.7	Bernoulli Trials and Binomial Distribution	572
	Answers		**588**

Chapter 7

INTEGRALS

❖ *Just as a mountaineer climbs a mountain – because it is there, so a good mathematics student studies new material because it is there.* — *JAMES B. BRISTOL* ❖

7.1 Introduction

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.

G .W. Leibnitz (1646 -1716)

If a function f is differentiable in an interval I, i.e., its derivative f' exists at each point of I, then a natural question arises that given f' at each point of I, can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the ***indefinite integral*** of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types:

(a) the problem of finding a function whenever its derivative is given,

(b) the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the ***Integral Calculus***.

There is a connection, known as the ***Fundamental Theorem of Calculus***, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering. The definite integral is also used to solve many interesting problems from various disciplines like economics, finance and probability.

In this Chapter, we shall confine ourselves to the study of indefinite and definite integrals and their elementary properties including some techniques of integration.

7.2 Integration as an Inverse Process of Differentiation

Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*.
Let us consider the following examples:

We know that $$\frac{d}{dx}(\sin x) = \cos x \qquad \text{... (1)}$$

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \qquad \text{... (2)}$$

and $$\frac{d}{dx}(e^x) = e^x \qquad \text{... (3)}$$

We observe that in (1), the function $\cos x$ is the derived function of $\sin x$. We say that $\sin x$ is an anti derivative (or an integral) of $\cos x$. Similarly, in (2) and (3), $\frac{x^3}{3}$ and e^x are the anti derivatives (or integrals) of x^2 and e^x, respectively. Again, we note that for any real number C, treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows :

$$\frac{d}{dx}(\sin x + \text{C}) = \cos x\,,\ \frac{d}{dx}\left(\frac{x^3}{3} + \text{C}\right) = x^2 \text{ and } \frac{d}{dx}(e^x + \text{C}) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason C is customarily referred to as ***arbitrary constant***. In fact, C is the ***parameter*** by varying which one gets different anti derivatives (or integrals) of the given function.

More generally, if there is a function F such that $\frac{d}{dx}\text{F}(x) = f(x)$, $\forall x \in \text{I}$ (interval), then for any arbitrary real number C, (also called *constant of integration*)

$$\frac{d}{dx}\left[\text{F}(x) + \text{C}\right] = f(x),\ x \in \text{I}$$

Thus, $\{F + C, C \in \mathbf{R}\}$ denotes a family of anti derivatives of f.

Remark Functions with same derivatives differ by a constant. To show this, let g and h be two functions having the same derivatives on an interval I.

Consider the function $f = g - h$ defined by $f(x) = g(x) - h(x)$, $\forall x \in$ I

Then $\frac{df}{dx} = f' = g' - h'$ giving $f'(x) = g'(x) - h'(x)$ $\forall x \in$ I

or $f'(x) = 0$, $\forall x \in$ I by hypothesis,

i.e., the rate of change of f with respect to x is zero on I and hence f is constant.

In view of the above remark, it is justified to infer that the family $\{F + C, C \in \mathbf{R}\}$ provides all possible anti derivatives of f.

We introduce a new symbol, namely, $\int f(x)\, dx$ which will represent the entire class of anti derivatives read as the indefinite integral of f with respect to x.

Symbolically, we write $\int f(x)\, dx = F(x) + C$.

Notation Given that $\frac{dy}{dx} = f(x)$, we write $y = \int f(x)\, dx$.

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the Table (7.1).

Table 7.1

Symbols/Terms/Phrases	**Meaning**
$\int f(x)\, dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\, dx$	Integrand
x in $\int f(x)\, dx$	Variable of integration
Integrate	Find the integral
An integral of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding the integral
Constant of Integration	Any real number C, considered as constant function

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down immediately the corresponding formulae (referred to as standard formulae) for the integrals of these functions, as listed below which will be used to find integrals of other functions.

Derivatives	**Integrals (Anti derivatives)**
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
Particularly, we note that	
$\frac{d}{dx}(x) = 1$;	$\int dx = x + C$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x \, dx = \sin x + C$
(iii) $\frac{d}{dx}(-\cos x) = \sin x$;	$\int \sin x \, dx = -\cos x + C$
(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$;	$\int \sec^2 x \, dx = \tan x + C$
(v) $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$;	$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$;	$\int \sec x \tan x \, dx = \sec x + C$
(vii) $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$;	$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
(viii) $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
(ix) $\frac{d}{dx}\left(-\cos^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
(x) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
(xi) $\frac{d}{dx}\left(-\cot^{-1} x\right) = \frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xiv) $\frac{d}{dx}(e^x) = e^x$; $\int e^x\, dx = e^x + C$

(xv) $\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}$; $\int \frac{1}{x}\, dx = \log|x| + C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$; $\int a^x\, dx = \frac{a^x}{\log a} + C$

Note In practice, we normally do not mention the interval over which the various functions are defined. However, in any specific problem one has to keep it in mind.

7.2.1 *Geometrical interpretation of indefinite integral*

Let $f(x) = 2x$. Then $\int f(x)\, dx = x^2 + C$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

Clearly, for C = 0, we obtain $y = x^2$, a parabola with its vertex on the origin. The curve $y = x^2 + 1$ for C = 1 is obtained by shifting the parabola $y = x^2$ one unit along y-axis in positive direction. For C = – 1, $y = x^2 - 1$ is obtained by shifting the parabola $y = x^2$ one unit along y-axis in the negative direction. Thus, for each positive value of C, each parabola of the family has its vertex on the positive side of the y-axis and for negative values of C, each has its vertex along the negative side of the y-axis. Some of these have been shown in the Fig 7.1.

Let us consider the intersection of all these parabolas by a line $x = a$. In the Fig 7.1, we have taken $a > 0$. The same is true when $a < 0$. If the line $x = a$ intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\frac{dy}{dx}$ at these points equals $2a$. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x\, dx = x^2 + C = F_C(x)$ (say), implies that

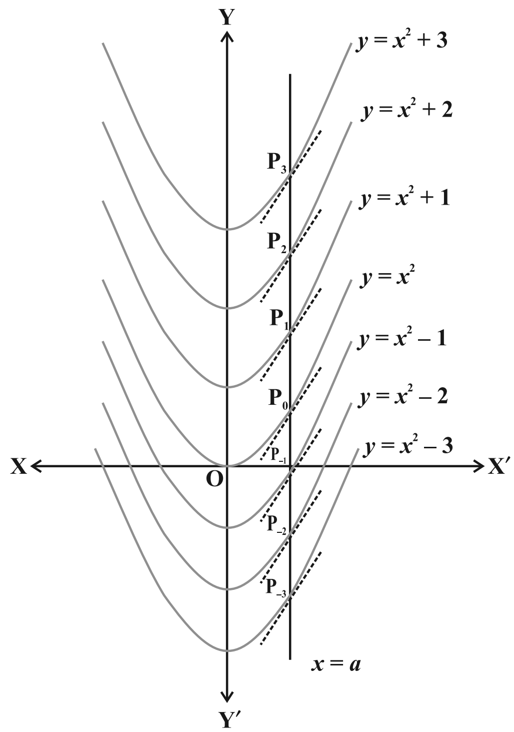


Fig 7.1

the tangents to all the curves $y = F_C(x)$, $C \in \mathbf{R}$, at the points of intersection of the curves by the line $x = a$, ($a \in \mathbf{R}$), are parallel.

Further, the following equation (statement) $\int f(x)\, dx = F(x) + C = y$ (say), represents a family of curves. The different values of C will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

7.2.2 *Some properties of indefinite integral*

In this sub section, we shall derive some properties of indefinite integrals.

(I) The process of differentiation and integration are inverses of each other in the sense of the following results :

$$\frac{d}{dx}\int f(x)\, dx = f(x)$$

and $$\int f'(x)\, dx = f(x) + C,$$ where C is any arbitrary constant.

Proof Let F be any anti derivative of f, i.e.,

$$\frac{d}{dx}\mathrm{F}(x) = f(x)$$

Then $\int f(x)\,dx = \mathrm{F}(x) + \mathrm{C}$

Therefore $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\left(\mathrm{F}(x) + \mathrm{C}\right)$

$$= \frac{d}{dx}\mathrm{F}(x) = f(x)$$

Similarly, we note that

$$f'(x) = \frac{d}{dx}f(x)$$

and hence $\int f'(x)\,dx = f(x) + \mathrm{C}$

where C is arbitrary constant called constant of integration.

(II) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

Proof Let f and g be two functions such that

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$$

or $\frac{d}{dx}\left[\int f(x)\,dx - \int g(x)\,dx\right] = 0$

Hence $\int f(x)\,dx - \int g(x)\,dx =$ C, where C is any real number (Why?)

or $\int f(x)\,dx = \int g(x)\,dx + \mathrm{C}$

So the families of curves $\left\{\int f(x)\,dx + \mathrm{C}_1, \mathrm{C}_1 \in \mathbf{R}\right\}$

and $\left\{\int g(x)\,dx + \mathrm{C}_2, \mathrm{C}_2 \in \mathbf{R}\right\}$ are identical.

Hence, in this sense, $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

Note The equivalence of the families $\left\{\int f(x)\,dx + C_1, C_1 \in \mathbf{R}\right\}$ and $\left\{\int g(x)\,dx + C_2, C_2 \in \mathbf{R}\right\}$ is customarily expressed by writing $\int f(x)\,dx = \int g(x)\,dx$, without mentioning the parameter.

(III) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Proof By Property (I), we have

$$\frac{d}{dx}\left[\int [f(x) + g(x)]\,dx\right] = f(x) + g(x) \qquad \text{... (1)}$$

On the otherhand, we find that

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$$

$$= f(x) + g(x) \qquad \text{... (2)}$$

Thus, in view of Property (II), it follows by (1) and (2) that

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

(IV) For any real number k, $\int k\ f(x)\,dx = k\int f(x)\,dx$

Proof By the Property (I), $\frac{d}{dx}\int k\ f(x)\,dx = k\ f(x)$.

Also $\quad \frac{d}{dx}\left[k\int f(x)\,dx\right] = k\frac{d}{dx}\int f(x)\,dx = k\ f(x)$

Therefore, using the Property (II), we have $\int k\ f(x)\,dx = k\int f(x)\,dx$.

(V) Properties (III) and (IV) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int [k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)]\,dx$$

$$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + ... + k_n\int f_n(x)\,dx.$$

To find an anti derivative of a given function, we search intuitively for a function whose derivative is the given function. The search for the requisite function for finding an anti derivative is known as integration by the method of inspection. We illustrate it through some examples.

Example 1 Write an anti derivative for each of the following functions using the method of inspection:

(i) $\cos 2x$ (ii) $3x^2 + 4x^3$ (iii) $\frac{1}{x}, x \neq 0$

Solution

(i) We look for a function whose derivative is $\cos 2x$. Recall that

$$\frac{d}{dx} \sin 2x = 2 \cos 2x$$

or $$\cos 2x = \frac{1}{2}\frac{d}{dx} (\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

Therefore, an anti derivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$.

(ii) We look for a function whose derivative is $3x^2 + 4x^3$. Note that

$$\frac{d}{dx}\left(x^3 + x^4\right) = 3x^2 + 4x^3.$$

Therefore, an anti derivative of $3x^2 + 4x^3$ is $x^3 + x^4$.

(iii) We know that

$$\frac{d}{dx}(\log x) = \frac{1}{x}, x > 0 \text{ and } \frac{d}{dx}[\log(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}, x < 0$$

Combining above, we get $\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}, x \neq 0$

Therefore, $\int \frac{1}{x}\, dx = \log|x|$ is one of the anti derivatives of $\frac{1}{x}$.

Example 2 Find the following integrals:

(i) $\int \frac{x^3 - 1}{x^2} dx$ (ii) $\int (x^{\frac{2}{3}} + 1)\, dx$ (iii) $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})\, dx$

Solution

(i) We have

$$\int \frac{x^3 - 1}{x^2}\, dx = \int x\, dx - \int x^{-2}\, dx \qquad \text{(by Property V)}$$

$$= \left(\frac{x^{1+1}}{1+1} + C_1\right) - \left(\frac{x^{-2+1}}{-2+1} + C_2\right); \; C_1, C_2 \text{ are constants of integration}$$

$$= \frac{x^2}{2} + C_1 - \frac{x^{-1}}{-1} - C_2 = \frac{x^2}{2} + \frac{1}{x} + C_1 - C_2$$

$$= \frac{x^2}{2} + \frac{1}{x} + C \text{, where } C = C_1 - C_2 \text{ is another constant of integration.}$$

Note From now onwards, we shall write only one constant of integration in the final answer.

(ii) We have

$$\int (x^{\frac{2}{3}} + 1)\, dx = \int x^{\frac{2}{3}}\, dx + \int dx$$

$$= \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + x + C = \frac{3}{5} x^{\frac{5}{3}} + x + C$$

(iii) We have $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})\, dx = \int x^{\frac{3}{2}}\, dx + \int 2e^x\, dx - \int \frac{1}{x}\, dx$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2e^x - \log|x| + C$$

$$= \frac{2}{5} x^{\frac{5}{2}} + 2e^x - \log|x| + C$$

Example 3 Find the following integrals:

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \operatorname{cosec} x (\operatorname{cosec} x + \cot x)\, dx$

(iii) $\int \frac{1 - \sin x}{\cos^2 x}\, dx$

Solution

(i) We have

$$\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx$$

$$= -\cos x + \sin x + C$$

(ii) We have

$$\int (\text{cosec } x (\text{cosec } x + \cot x)\, dx = \int \text{cosec}^2 x\, dx + \int \text{cosec } x \cot x\, dx$$

$$= -\cot x - \text{cosec } x + \text{C}$$

(iii) We have

$$\int \frac{1-\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx - \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 x\, dx - \int \tan x \sec x\, dx$$

$$= \tan x - \sec x + \text{C}$$

Example 4 Find the anti derivative F of f defined by $f(x) = 4x^3 - 6$, where F (0) = 3

Solution One anti derivative of $f(x)$ is $x^4 - 6x$ since

$$\frac{d}{dx}(x^4 - 6x) = 4x^3 - 6$$

Therefore, the anti derivative F is given by

$$\text{F}(x) = x^4 - 6x + \text{C}, \text{ where C is constant.}$$

Given that $\quad \text{F}(0) = 3$, which gives,

$$3 = 0 - 6 \times 0 + \text{C} \quad \text{or} \quad \text{C} = 3$$

Hence, the required anti derivative is the unique function F defined by

$$\text{F}(x) = x^4 - 6x + 3.$$

Remarks

(i) We see that if F is an anti derivative of f, then so is F + C, where C is any constant. Thus, if we know one anti derivative F of a function f, we can write down an infinite number of anti derivatives of f by adding any constant to F expressed by $\text{F}(x) + \text{C}, \text{C} \in \mathbf{R}$. In applications, it is often necessary to satisfy an additional condition which then determines a specific value of C giving unique anti derivative of the given function.

(ii) Sometimes, F is not expressible in terms of elementary functions viz., polynomial, logarithmic, exponential, trigonometric functions and their inverses etc. We are therefore blocked for finding $\int f(x)\, dx$. For example, it is not possible to find $\int e^{-x^2} dx$ by inspection since we can not find a function whose derivative is e^{-x^2}

(iii) When the variable of integration is denoted by a variable other than x, the integral formulae are modified accordingly. For instance

$$\int y^4 \, dy = \frac{y^{4+1}}{4+1} + C = \frac{1}{5} y^5 + C$$

7.2.3 *Comparison between differentiation and integration*

1. Both are operations on functions.
2. Both satisfy the property of linearity, i.e.,

   (i) $\dfrac{d}{dx}\left[k_1 \, f_1(x) + k_2 \, f_2(x)\right] = k_1 \dfrac{d}{dx} f_1(x) + k_2 \dfrac{d}{dx} f_2(x)$

   (ii) $\int \left[k_1 \, f_1(x) + k_2 \, f_2(x)\right] dx = k_1 \int f_1(x) \, dx + k_2 \int f_2(x) \, dx$

   Here k_1 and k_2 are constants.
3. We have already seen that all functions are not differentiable. Similarly, all functions are not integrable. We will learn more about nondifferentiable functions and nonintegrable functions in higher classes.
4. The derivative of a function, when it exists, is a unique function. The integral of a function is not so. However, they are unique upto an additive constant, i.e., any two integrals of a function differ by a constant.
5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.
6. We can speak of the derivative at a point. We never speak of the integral at a point, we speak of the integral of a function over an interval on which the integral is defined as will be seen in Section 7.7.
7. The derivative of a function has a geometrical meaning, namely, the slope of the tangent to the corresponding curve at a point. Similarly, the indefinite integral of a function represents geometrically, a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines orthogonal (perpendicular) to the axis representing the variable of integration.
8. The derivative is used for finding some physical quantities like the velocity of a moving particle, when the distance traversed at any time t is known. Similarly, the integral is used in calculating the distance traversed when the velocity at time t is known.
9. Differentiation is a process involving limits. **So is integration,** as will be seen in Section 7.7.

10. The process of differentiation and integration are inverses of each other as discussed in Section 7.2.2 (i).

EXERCISE 7.1

Find an anti derivative (or integral) of the following functions by the method of inspection.

1. $\sin 2x$ **2.** $\cos 3x$ **3.** e^{2x}

4. $(ax + b)^2$ **5.** $\sin 2x - 4\, e^{3x}$

Find the following integrals in Exercises 6 to 20:

6. $\int (4\, e^{3x} + 1)\, dx$ **7.** $\int x^2 \left(1 - \frac{1}{x^2}\right) dx$ **8.** $\int (ax^2 + bx + c)\, dx$

9. $\int (2x^2 + e^x)\, dx$ **10.** $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ **11.** $\int \frac{x^3 + 5x^2 - 4}{x^2}\, dx$

12. $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx$ **13.** $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$ **14.** $\int (1 - x)\sqrt{x}\, dx$

15. $\int \sqrt{x}\,(3x^2 + 2x + 3)\, dx$ **16.** $\int (2x - 3\cos x + e^x)\, dx$

17. $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$ **18.** $\int \sec x\, (\sec x + \tan x)\, dx$

19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\, dx$ **20.** $\int \frac{2 - 3\sin x}{\cos^2 x}\, dx.$

Choose the correct answer in Exercises 21 and 22.

21. The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

22. If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

7.3 Methods of Integration

In previous section, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function F whose derivative is f which led us to the integral of f. However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on:

1. Integration by Substitution
2. Integration using Partial Fractions
3. Integration by Parts

7.3.1 *Integration by substitution*

In this section, we consider the method of integration by substitution.

The given integral $\int f(x)\, dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider $\qquad I = \int f(x)\, dx$

Put $x = g(t)$ so that $\dfrac{dx}{dt} = g'(t)$.

We write $\qquad dx = g'(t)\, dt$

Thus $\qquad I = \int f(x)\, dx = \int f(g(t))\, g'(t)\, dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

Example 5 Integrate the following functions w.r.t. x:

(i) $\sin mx$ $\qquad$ (ii) $2x \sin (x^2 + 1)$

(iii) $\dfrac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$ $\qquad$ (iv) $\dfrac{\sin (\tan^{-1} x)}{1 + x^2}$

Solution

(i) We know that derivative of mx is m. Thus, we make the substitution $mx = t$ so that $mdx = dt$.

Therefore, $\int \sin mx\, dx = \dfrac{1}{m}\int \sin t\, dt = -\dfrac{1}{m}\cos t + C = -\dfrac{1}{m}\cos mx + C$

(ii) Derivative of $x^2 + 1$ is $2x$. Thus, we use the substitution $x^2 + 1 = t$ so that $2x\, dx = dt$.

Therefore, $\int 2x \sin (x^2+1)\, dx = \int \sin t\, dt = -\cos t + C = -\cos (x^2 + 1) + C$

(iii) Derivative of $\sqrt{x}$ is $\frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Thus, we use the substitution

$\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}} dx = dt$ giving $dx = 2t\, dt$.

Thus, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \int \frac{2t \tan^4 t \sec^2 t\, dt}{t} = 2 \int \tan^4 t \sec^2 t\, dt$$

Again, we make another substitution $\tan t = u$ so that $\sec^2 t\, dt = du$

Therefore, $$2 \int \tan^4 t \sec^2 t\, dt = 2 \int u^4\, du = 2 \frac{u^5}{5} + C$$

$$= \frac{2}{5} \tan^5 t + C \text{ (since } u = \tan t)$$

$$= \frac{2}{5} \tan^5 \sqrt{x} + C \text{ (since } t = \sqrt{x})$$

Hence, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \frac{2}{5} \tan^5 \sqrt{x} + C$$

Alternatively, make the substitution $\tan \sqrt{x} = t$

(iv) Derivative of $\tan^{-1} x = \frac{1}{1+x^2}$. Thus, we use the substitution

$$\tan^{-1} x = t \text{ so that } \frac{dx}{1+x^2} = dt.$$

Therefore, $$\int \frac{\sin (\tan^{-1} x)}{1+x^2} dx = \int \sin t\, dt = -\cos t + C = -\cos (\tan^{-1} x) + C$$

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\boldsymbol{\int \tan x\, dx = \log |\sec x| + C}$

We have

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$$

Put $\cos x = t$ so that $\sin x\, dx = -\, dt$

Then $$\int \tan x\, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$$

or $$\int \tan x\, dx = \log|\sec x| + C$$

(ii) $\displaystyle\int \cot x\, dx = \log|\sin x| + C$

We have $\displaystyle\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$

Put $\sin x = t$ so that $\cos x\, dx = dt$

Then $$\int \cot x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$$

(iii) $\displaystyle\int \sec x\, dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x\, dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

Put $\sec x + \tan x = t$ so that $\sec x\,(\tan x + \sec x)\, dx = dt$

Therefore, $\displaystyle\int \sec x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv) $\displaystyle\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

We have

$$\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)}\, dx$$

Put $\operatorname{cosec} x + \cot x = t$ so that $-\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\, dx = dt$

So $$\int \operatorname{cosec} x\, dx = -\int \frac{dt}{t} = -\log|t| = -\log|\operatorname{cosec} x + \cot x| + C$$

$$= -\log\left|\frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x - \cot x}\right| + C$$

$$= \log|\operatorname{cosec} x - \cot x| + C$$

Example 6 Find the following integrals:

(i) $\displaystyle\int \sin^3 x \cos^2 x\, dx$ (ii) $\displaystyle\int \frac{\sin x}{\sin(x + a)}\, dx$ (iii) $\displaystyle\int \frac{1}{1 + \tan x}\, dx$

Solution

(i) We have

$$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x \,(\sin x)\, dx$$

$$= \int (1-\cos^2 x)\cos^2 x\,(\sin x)\, dx$$

Put $t = \cos x$ so that $dt = -\sin x\, dx$

Therefore, $\int \sin^2 x \cos^2 x\,(\sin x)\, dx = -\int (1-t^2)\, t^2\, dt$

$$= -\int (t^2 - t^4)\, dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + \text{C}$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + \text{C}$$

(ii) Put $x + a = t$. Then $dx = dt$. Therefore

$$\int \frac{\sin x}{\sin (x+a)}\, dx = \int \frac{\sin (t-a)}{\sin t}\, dt$$

$$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t}\, dt$$

$$= \cos a \int dt - \sin a \int \cot t\, dt$$

$$= (\cos a)\, t - (\sin a)\left[\log |\sin t| + \text{C}_1\right]$$

$$= (\cos a)\,(x+a) - (\sin a)\left[\log |\sin (x+a)| + \text{C}_1\right]$$

$$= x \cos a + a \cos a - (\sin a) \log |\sin (x+a)| - \text{C}_1 \sin a$$

Hence, $\int \frac{\sin x}{\sin (x+a)}\, dx = x \cos a - \sin a \log |\sin (x+a)| + \text{C}$,

where, $\text{C} = -\text{C}_1 \sin a + a \cos a$, is another arbitrary constant.

(iii) $\int \frac{dx}{1+\tan x} = \int \frac{\cos x\, dx}{\cos x + \sin x}$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x + \cos x - \sin x)\, dx}{\cos x + \sin x}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx \qquad \text{... (1)}$$

Now, consider $I = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

Put $\cos x + \sin x = t$ so that $(\cos x - \sin x)\,dx = dt$

Therefore $\quad I = \int \frac{dt}{t} = \log|t| + C_2 = \log|\cos x + \sin x| + C_2$

Putting it in (1), we get

$$\int \frac{dx}{1+\tan x} = \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_1}{2} + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + C, \left(C = \frac{C_1}{2} + \frac{C_2}{2}\right)$$

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

1. $\frac{2x}{1+x^2}$ **2.** $\frac{(\log x)^2}{x}$ **3.** $\frac{1}{x + x\log x}$

4. $\sin x \sin(\cos x)$ **5.** $\sin(ax+b)\cos(ax+b)$

6. $\sqrt{ax+b}$ **7.** $x\sqrt{x+2}$ **8.** $x\sqrt{1+2x^2}$

9. $(4x+2)\sqrt{x^2+x+1}$ **10.** $\frac{1}{x-\sqrt{x}}$ **11.** $\frac{x}{\sqrt{x+4}}, x > 0$

12. $(x^3-1)^{\frac{1}{3}}\,x^5$ **13.** $\frac{x^2}{(2+3x^3)^3}$ **14.** $\frac{1}{x(\log x)^m}, x > 0$

15. $\frac{x}{9-4x^2}$ **16.** e^{2x+3} **17.** $\frac{x}{e^{x^2}}$

18. $\dfrac{e^{\tan^{-1} x}}{1+x^2}$

19. $\dfrac{e^{2x}-1}{e^{2x}+1}$

20. $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$

22. $\sec^2(7-4x)$

23. $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

24. $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

25. $\dfrac{1}{\cos^2 x\,(1-\tan x)^2}$

26. $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

27. $\sqrt{\sin 2x}\cos 2x$

28. $\dfrac{\cos x}{\sqrt{1+\sin x}}$

29. $\cot x \log \sin x$

30. $\dfrac{\sin x}{1+\cos x}$

31. $\dfrac{\sin x}{(1+\cos x)^2}$

32. $\dfrac{1}{1+\cot x}$

33. $\dfrac{1}{1-\tan x}$

34. $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

35. $\dfrac{(1+\log x)^2}{x}$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$

37. $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38. $\displaystyle\int \frac{10x^9 + 10^x \log_{e^{10}} dx}{x^{10}+10^x}$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

39. $\displaystyle\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

7.3.2 *Integration using trigonometric identities*

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

Example 7 Find (i) $\int \cos^2 x\, dx$ (ii) $\int \sin 2x \cos 3x\, dx$ (iii) $\int \sin^3 x\, dx$

Solution

(i) Recall the identity $\cos 2x = 2\cos^2 x - 1$, which gives

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Therefore, $$\int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + \text{C}$$

(ii) Recall the identity $\sin x \cos y = \frac{1}{2}[\sin (x+y) + \sin (x-y)]$ (Why?)

Then $$\int \sin 2x \cos 3x\, dx = \frac{1}{2}\left[\int \sin 5x\, dx \bullet \int \sin x\, dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right] + \text{C}$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + \text{C}$$

(iii) From the identity $\sin 3x = 3\sin x - 4\sin^3 x$, we find that

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Therefore, $$\int \sin^3 x\, dx = \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + \text{C}$$

Alternatively, $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1-\cos^2 x)\sin x\, dx$

Put $\cos x = t$ so that $-\sin x\, dx = dt$

Therefore, $$\int \sin^3 x\, dx = -\int \left(1-t^2\right) dt = -\int dt + \int t^2\, dt = -t + \frac{t^3}{3} + \text{C}$$

$$= -\cos x + \frac{1}{3}\cos^3 x + \text{C}$$

Remark It can be shown using trigonometric identities that both answers are equivalent.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

1. $\sin^2 (2x + 5)$
2. $\sin 3x \cos 4x$
3. $\cos 2x \cos 4x \cos 6x$
4. $\sin^3 (2x + 1)$
5. $\sin^3 x \cos^3 x$
6. $\sin x \sin 2x \sin 3x$
7. $\sin 4x \sin 8x$
8. $\dfrac{1-\cos x}{1+\cos x}$
9. $\dfrac{\cos x}{1+\cos x}$
10. $\sin^4 x$
11. $\cos^4 2x$
12. $\dfrac{\sin^2 x}{1+\cos x}$
13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$
14. $\dfrac{\cos x - \sin x}{1+\sin 2x}$
15. $\tan^3 2x \sec 2x$
16. $\tan^4 x$
17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
18. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
19. $\dfrac{1}{\sin x \cos^3 x}$
20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$
21. $\sin^{-1} (\cos x)$
22. $\dfrac{1}{\cos (x-a) \cos (x-b)}$

Choose the correct answer in Exercises 23 and 24.

23. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

(A) $\tan x + \cot x + C$ (B) $\tan x + \text{cosec } x + C$

(C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

24. $\displaystyle\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

(A) $-\cot (ex^x) + C$ (B) $\tan (xe^x) + C$

(C) $\tan (e^x) + C$ (D) $\cot (e^x) + C$

7.4 Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1) $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$

(2) $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + C$$

(3) $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

(4) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

(5) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

(6) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

We now prove the above results:

(1) We have $\dfrac{1}{x^2 - a^2} = \dfrac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right] = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

Therefore, $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$$

$$= \frac{1}{2a}\left[\log|(x-a)| - \log|(x+a)|\right] + C$$

$$= \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left[\frac{(a+x)+(a-x)}{(a+x)(a-x)}\right] = \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$$

Therefore,
$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\left[\int \frac{dx}{a-x} + \int \frac{dx}{a+x}\right]$$
$$= \frac{1}{2a}[-\log|a-x| + \log|a+x|] + C$$
$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Note The technique used in (1) will be explained in Section 7.5.

(3) Put $x = a \tan \theta$. Then $dx = a \sec^2 \theta \, d\theta$.

Therefore,
$$\int \frac{dx}{x^2 + a^2} = \int \frac{a \sec^2 \theta \, d\theta}{a^2 \tan^2\theta + a^2}$$
$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) Let $x = a \sec\theta$. Then $dx = a \sec\theta \tan\theta \, d\theta$.

Therefore,
$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec\theta \tan\theta \, d\theta}{\sqrt{a^2 \sec^2\theta - a^2}}$$
$$= \int \sec\theta \, d\theta = \log|\sec\theta + \tan\theta| + C_1$$
$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C_1$$
$$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log|a| + C_1$$
$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C, \text{ where } C = C_1 - \log|a|$$

(5) Let $x = a \sin\theta$. Then $dx = a \cos\theta \, d\theta$.

Therefore,
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos\theta \, d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}}$$
$$= \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C$$

(6) Let $x = a \tan\theta$. Then $dx = a \sec^2\theta \, d\theta$.

Therefore,
$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \sec^2\theta \, d\theta}{\sqrt{a^2 \tan^2\theta + a^2}}$$
$$= \int \sec\theta \, d\theta = \log|(\sec\theta + \tan\theta)| + C_1$$

$$= \log \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1} \right| + C_1$$

$$= \log \left| x + \sqrt{x^2 + a^2} \right| - \log |a| + C_1$$

$$= \log \left| x + \sqrt{x^2 + a^2} \right| + C, \text{ where } C = C_1 - \log |a|$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

(7) To find the integral $\int \frac{dx}{ax^2 + bx + c}$, we write

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Now, put $x + \frac{b}{2a} = t$ so that $dx = dt$ and writing $\frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$. We find the integral reduced to the form $\frac{1}{a}\int \frac{dt}{t^2 \pm k^2}$ depending upon the sign of $\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$ and hence can be evaluated.

(8) To find the integral of the type $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, proceeding as in (7), we obtain the integral using the standard formulae.

(9) To find the integral of the type $\int \frac{px + q}{ax^2 + bx + c}\, dx$, where p, q, a, b, c are constants, we are to find real numbers A, B such that

$$px + q = \mathrm{A}\frac{d}{dx}(ax^2 + bx + c) + \mathrm{B} = \mathrm{A}\,(2ax + b) + \mathrm{B}$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type $\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$, we proceed as in (9) and transform the integral into known standard forms.

Let us illustrate the above methods by some examples.

Example 8 Find the following integrals:

(i) $\int \frac{dx}{x^2-16}$ (ii) $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution

(i) We have $\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + \mathrm{C}$ [by 7.4 (1)]

(ii) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

Put $x - 1 = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + \mathrm{C}$ [by 7.4 (5)]

$$= \sin^{-1}(x-1) + \mathrm{C}$$

Example 9 Find the following integrals :

(i) $\int \frac{dx}{x^2-6x+13}$ (ii) $\int \frac{dx}{3x^2+13x-10}$ (iii) $\int \frac{dx}{\sqrt{5x^2-2x}}$

Solution

(i) We have $x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$

So, $\int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+2^2}\,dx$

Let $x - 3 = t$. Then $dx = dt$

Therefore, $\int \frac{dx}{x^2-6x+13} = \int \frac{dt}{t^2+2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + \mathrm{C}$ [by 7.4 (3)]

$$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + \mathrm{C}$$

(ii) The given integral is of the form 7.4 (7). We write the denominator of the integrand,

$$3x^2+13x-10 = 3\left(x^2+\frac{13x}{3}-\frac{10}{3}\right)$$

$$= 3\left[\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2\right] \text{ (completing the square)}$$

Thus $$\int\frac{dx}{3x^2+13x-10} = \frac{1}{3}\int\frac{dx}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}$$

Put $x+\frac{13}{6}=t$. Then $dx = dt$.

Therefore, $$\int\frac{dx}{3x^2+13x-10} = \frac{1}{3}\int\frac{dt}{t^2-\left(\frac{17}{6}\right)^2}$$

$$= \frac{1}{3\times 2\times\frac{17}{6}}\log\left|\frac{t-\frac{17}{6}}{t+\frac{17}{6}}\right|+C_1 \qquad \text{[by 7.4 (i)]}$$

$$= \frac{1}{17}\log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right|+C_1$$

$$= \frac{1}{17}\log\left|\frac{6x-4}{6x+30}\right|+C_1$$

$$= \frac{1}{17}\log\left|\frac{3x-2}{x+5}\right|+C_1+\frac{1}{17}\log\frac{1}{3}$$

$$= \frac{1}{17}\log\left|\frac{3x-2}{x+5}\right|+C, \text{ where } C = C_1+\frac{1}{17}\log\frac{1}{3}$$

(iii) We have $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \int \frac{dx}{\sqrt{5\left(x^2 - \frac{2x}{5}\right)}}$

$$= \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} \quad \text{(completing the square)}$$

Put $x - \frac{1}{5} = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}}$

$$= \frac{1}{\sqrt{5}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C \qquad \text{[by 7.4 (4)]}$$

$$= \frac{1}{\sqrt{5}} \log \left| x - \frac{1}{5} + \sqrt{x^2 - \frac{2x}{5}} \right| + C$$

Example 10 Find the following integrals:

(i) $\int \frac{x+2}{2x^2 + 6x + 5} dx$ (ii) $\int \frac{x+3}{\sqrt{5 - 4x + x^2}} dx$

Solution

(i) Using the formula 7.4 (9), we express

$$x + 2 = A \frac{d}{dx}\left(2x^2 + 6x + 5\right) + B = A(4x + 6) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$4A = 1$ and $6A + B = 2$ or $A = \frac{1}{4}$ and $B = \frac{1}{2}$.

Therefore, $\int \frac{x+2}{2x^2 + 6x + 5} = \frac{1}{4} \int \frac{4x+6}{2x^2 + 6x + 5} dx + \frac{1}{2} \int \frac{dx}{2x^2 + 6x + 5}$

$$= \frac{1}{4} I_1 + \frac{1}{2} I_2 \quad \text{(say)} \qquad \text{... (1)}$$

In I_1, put $2x^2 + 6x + 5 = t$, so that $(4x + 6)\, dx = dt$

Therefore, $$I_1 = \int \frac{dt}{t} = \log|t| + C_1$$

$$= \log|2x^2 + 6x + 5| + C_1 \quad \ldots (2)$$

and $$I_2 = \int \frac{dx}{2x^2 + 6x + 5} = \frac{1}{2}\int \frac{dx}{x^2 + 3x + \frac{5}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

Put $x + \frac{3}{2} = t$, so that $dx = dt$, we get

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2 \times \frac{1}{2}} \tan^{-1} 2t + C_2 \quad \text{[by 7.4 (3)]}$$

$$= \tan^{-1} 2\left(x + \frac{3}{2}\right) + C_2 = \tan^{-1}(2x + 3) + C_2 \quad \ldots (3)$$

Using (2) and (3) in (1), we get

$$\int \frac{x+2}{2x^2 + 6x + 5}\, dx = \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{2}\tan^{-1}(2x + 3) + C$$

where, $$C = \frac{C_1}{4} + \frac{C_2}{2}$$

(ii) This integral is of the form given in 7.4 (10). Let us express

$$x + 3 = A\frac{d}{dx}(5 - 4x - x^2) + B = A(-4 - 2x) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$-2A = 1$ and $-4A + B = 3$, i.e., $A = -\frac{1}{2}$ and $B = 1$

Therefore, $\displaystyle\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} + \int \frac{dx}{\sqrt{5-4x-x^2}}$

$$= -\frac{1}{2}\ \mathrm{I}_1 + \mathrm{I}_2 \qquad \text{... (1)}$$

In I_1, put $5 - 4x - x^2 = t$, so that $(-4 - 2x)\,dx = dt$.

Therefore, $\displaystyle\mathrm{I}_1 = \int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + \mathrm{C}_1$

$$= 2\sqrt{5-4x-x^2} + \mathrm{C}_1 \qquad \text{... (2)}$$

Now consider $\displaystyle\mathrm{I}_2 = \int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}}$

Put $x + 2 = t$, so that $dx = dt$.

Therefore, $\displaystyle\mathrm{I}_2 = \int \frac{dt}{\sqrt{3^2-t^2}} = \sin^{-1}\frac{t}{3} + \mathrm{C}_2$ [by 7.4 (5)]

$$= \sin^{-1}\frac{x+2}{3} + \mathrm{C}_2 \qquad \text{... (3)}$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{\sqrt{5-4x-x^2}} = -\sqrt{5-4x-x^2} + \sin^{-1}\frac{x+2}{3} + \mathrm{C}, \text{ where } \mathrm{C} = \mathrm{C}_2 - \frac{\mathrm{C}_1}{2}$$

EXERCISE 7.4

Integrate the functions in Exercises 1 to 23.

1. $\dfrac{3x^2}{x^6+1}$ **2.** $\dfrac{1}{\sqrt{1+4x^2}}$ **3.** $\dfrac{1}{\sqrt{(2-x)^2+1}}$

4. $\dfrac{1}{\sqrt{9-25x^2}}$ **5.** $\dfrac{3x}{1+2x^4}$ **6.** $\dfrac{x^2}{1-x^6}$

7. $\dfrac{x-1}{\sqrt{x^2-1}}$ **8.** $\dfrac{x^2}{\sqrt{x^6+a^6}}$ **9.** $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

10. $\dfrac{1}{\sqrt{x^2+2x+2}}$ **11.** $\dfrac{1}{9x^2+6x+5}$ **12.** $\dfrac{1}{\sqrt{7-6x-x^2}}$

13. $\dfrac{1}{\sqrt{(x-1)(x-2)}}$ **14.** $\dfrac{1}{\sqrt{8+3x-x^2}}$ **15.** $\dfrac{1}{\sqrt{(x-a)(x-b)}}$

16. $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$ **17.** $\dfrac{x+2}{\sqrt{x^2-1}}$ **18.** $\dfrac{5x-2}{1+2x+3x^2}$

19. $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$ **20.** $\dfrac{x+2}{\sqrt{4x-x^2}}$ **21.** $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

22. $\dfrac{x+3}{x^2-2x-5}$ **23.** $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$.

Choose the correct answer in Exercises 24 and 25.

24. $\int \dfrac{dx}{x^2+2x+2}$ equals

(A) $x \tan^{-1}(x+1) + C$ (B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1}x + C$ (D) $\tan^{-1}x + C$

25. $\int \dfrac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\dfrac{1}{9}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (B) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{8x-9}{9}\right)+C$

(C) $\dfrac{1}{3}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (D) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{9x-8}{9}\right)+C$

7.5 Integration by Partial Fractions

Recall that a rational function is defined as the ratio of two polynomials in the form $\dfrac{P(x)}{Q(x)}$, where P (x) and Q(x) are polynomials in x and Q$(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T($x$) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function. As we know how to integrate polynomials, the integration of any rational function is reduced to the integration of a proper rational function. The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors. Assume that we want to evaluate $\int \frac{P(x)}{Q(x)}\, dx$, where $\frac{P(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods. The following Table 7.2 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.2

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$,
	where $x^2 + bx + c$ cannot be factorised further	

In the above table, A, B and C are real numbers to be determined suitably.

Example 11 Find $\int \frac{dx}{(x+1)(x+2)}$

Solution The integrand is a proper rational function. Therefore, by using the form of partial fraction [Table 7.2 (i)], we write

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad \text{... (1)}$$

where, real numbers A and B are to be determined suitably. This gives

$$1 = A(x+2) + B(x+1).$$

Equating the coefficients of x and the constant term, we get

$$A + B = 0$$

and $$2A + B = 1$$

Solving these equations, we get A =1 and B = – 1.

Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

Therefore,
$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log|x+1| - \log|x+2| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

Remark The equation (1) above is an identity, i.e. a statement true for all (permissible) values of x. Some authors use the symbol '≡' to indicate that the statement is an identity and use the symbol '=' to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of x.

Example 12 Find $\int \frac{x^2+1}{x^2-5x+6}\,dx$

Solution Here the integrand $\frac{x^2+1}{x^2-5x+6}$ is not proper rational function, so we divide $x^2 + 1$ by $x^2 - 5x + 6$ and find that

$$\frac{x^2+1}{x^2-5x+6} = 1+\frac{5x-5}{x^2-5x+6}=1+\frac{5x-5}{(x-2)(x-3)}$$

Let $$\frac{5x-5}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3}$$

So that $$5x - 5 = A\,(x-3) + B\,(x-2)$$

Equating the coefficients of x and constant terms on both sides, we get A + B = 5 and 3A + 2B = 5. Solving these equations, we get A = – 5 and B = 10

Thus, $$\frac{x^2+1}{x^2-5x+6} = 1-\frac{5}{x-2}+\frac{10}{x-3}$$

Therefore, $$\int\frac{x^2+1}{x^2-5x+6}\,dx = \int dx-5\int\frac{1}{x-2}\,dx+10\int\frac{dx}{x-3}$$

$$= x - 5\log|x-2| + 10\log|x-3| + C.$$

Example 13 Find $\int\frac{3x-2}{(x+1)^2(x+3)}dx$

Solution The integrand is of the type as given in Table 7.2 (4). We write

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$$

So that $$3x-2 = A\,(x+1)(x+3) + B\,(x+3) + C\,(x+1)^2$$

$$= A\,(x^2+4x+3) + B\,(x+3) + C\,(x^2+2x+1)$$

Comparing coefficient of x^2, x and constant term on both sides, we get A + C = 0, 4A + B + 2C = 3 and 3A + 3B + C = – 2. Solving these equations, we get $A=\frac{11}{4}, B=\frac{-5}{2}$ and $C=\frac{-11}{4}$. Thus the integrand is given by

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4(x+1)}-\frac{5}{2(x+1)^2}-\frac{11}{4(x+3)}$$

Therefore, $$\int\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4}\int\frac{dx}{x+1}-\frac{5}{2}\int\frac{dx}{(x+1)^2}-\frac{11}{4}\int\frac{dx}{x+3}$$

$$= \frac{11}{4}\log|x+1|+\frac{5}{2(x+1)}-\frac{11}{4}\log|x+3|+C$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2(x+1)}+C$$

Example 14 Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Solution Consider $\frac{x^2}{(x^2+1)(x^2+4)}$ and put $x^2 = y$.

Then $$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$

Write $$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$$

So that $$y = A(y+4) + B(y+1)$$

Comparing coefficients of y and constant terms on both sides, we get A + B = 1 and 4A + B = 0, which give

$$A = -\frac{1}{3} \quad \text{and} \quad B = \frac{4}{3}$$

Thus, $$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

Therefore, $$\int \frac{x^2 dx}{(x^2+1)(x^2+4)} = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{4}{3} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

In the above example, the substitution was made only for the partial fraction part and not for the integration part. Now, we consider an example, where the integration involves a combination of the substitution method and the partial fraction method.

Example 15 Find $\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi} d\phi$

Solution Let $y = \sin\phi$

Then $$dy = \cos\phi \, d\phi$$

Therefore, $\displaystyle\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi}\,d\phi = \int \frac{(3y-2)\,dy}{5-(1-y^2)-4y}$

$$= \int \frac{3y-2}{y^2-4y+4}\,dy$$

$$= \int \frac{3y-2}{(y-2)^2} = \text{I (say)}$$

Now, we write $\displaystyle\frac{3y-2}{(y-2)^2} = \frac{A}{y-2} + \frac{B}{(y-2)^2}$ [by Table 7.2 (2)]

Therefore, $3y - 2 = A\,(y - 2) + B$

Comparing the coefficients of y and constant term, we get A = 3 and B – 2A = – 2, which gives A = 3 and B = 4.

Therefore, the required integral is given by

$$\text{I} = \int \left[\frac{3}{y-2} + \frac{4}{(y-2)^2}\right] dy = 3\int \frac{dy}{y-2} + 4\int \frac{dy}{(y-2)^2}$$

$$= 3\log|\,y-2\,| + 4\left(-\frac{1}{y-2}\right) + C$$

$$= 3\log|\,\sin\phi - 2\,| + \frac{4}{2-\sin\phi} + C$$

$$= 3\log(2-\sin\phi) + \frac{4}{2-\sin\phi} + C \quad \text{(since, } 2 - \sin\phi \text{ is always positive)}$$

Example 16 Find $\displaystyle\int \frac{x^2+x+1\,dx}{(x+2)\,(x^2+1)}$

Solution The integrand is a proper rational function. Decompose the rational function into partial fraction [Table 2.2(5)]. Write

$$\frac{x^2+x+1}{(x^2+1)\,(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{(x^2+1)}$$

Therefore, $x^2 + x + 1 = A\,(x^2 + 1) + (Bx + C)\,(x + 2)$

Equating the coefficients of x^2, x and of constant term of both sides, we get A + B =1, 2B + C = 1 and A + 2C = 1. Solving these equations, we get

$A = \frac{3}{5}$, $B = \frac{2}{5}$ and $C = \frac{1}{5}$

Thus, the integrand is given by

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{3}{5(x+2)} + \frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1} = \frac{3}{5(x+2)} + \frac{1}{5}\left(\frac{2x+1}{x^2+1}\right)$$

Therefore, $$\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx = \frac{3}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + C$$

EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

1. $\frac{x}{(x+1)(x+2)}$
2. $\frac{1}{x^2-9}$
3. $\frac{3x-1}{(x-1)(x-2)(x-3)}$
4. $\frac{x}{(x-1)(x-2)(x-3)}$
5. $\frac{2x}{x^2+3x+2}$
6. $\frac{1-x^2}{x(1-2x)}$
7. $\frac{x}{(x^2+1)(x-1)}$
8. $\frac{x}{(x-1)^2(x+2)}$
9. $\frac{3x+5}{x^3-x^2-x+1}$
10. $\frac{2x-3}{(x^2-1)(2x+3)}$
11. $\frac{5x}{(x+1)(x^2-4)}$
12. $\frac{x^3+x+1}{x^2-1}$
13. $\frac{2}{(1-x)(1+x^2)}$
14. $\frac{3x-1}{(x+2)^2}$
15. $\frac{1}{x^4-1}$
16. $\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]
17. $\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint : Put $\sin x = t$]

18. $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ **19.** $\dfrac{2x}{(x^2+1)(x^2+3)}$ **20.** $\dfrac{1}{x(x^4-1)}$

21. $\dfrac{1}{(e^x-1)}$ [Hint : Put $e^x = t$]

Choose the correct answer in each of the Exercises 22 and 23.

22. $\displaystyle\int \frac{x\,dx}{(x-1)(x-2)}$ equals

(A) $\log\left|\dfrac{(x-1)^2}{x-2}\right| + \text{C}$ (B) $\log\left|\dfrac{(x-2)^2}{x-1}\right| + \text{C}$

(C) $\log\left|\left(\dfrac{x-1}{x-2}\right)^2\right| + \text{C}$ (D) $\log\left|(x-1)(x-2)\right| + \text{C}$

23. $\displaystyle\int \frac{dx}{x(x^2+1)}$ equals

(A) $\log|x| - \dfrac{1}{2}\log(x^2+1) + \text{C}$ (B) $\log|x| + \dfrac{1}{2}\log(x^2+1) + \text{C}$

(C) $-\log|x| + \dfrac{1}{2}\log(x^2+1) + \text{C}$ (D) $\dfrac{1}{2}\log|x| + \log(x^2+1) + \text{C}$

7.6 Integration by Parts

In this section, we describe one more method of integration, that is found quite useful in integrating products of functions.

If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides, we get

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

or
$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \qquad \text{... (1)}$$

Let
$$u = f(x) \text{ and } \frac{dv}{dx} = g(x). \text{ Then}$$

$$\frac{du}{dx} = f'(x) \text{ and } v = \int g(x)\,dx$$

Therefore, expression (1) can be rewritten as

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [\int g(x)\, dx]\, f'(x)\, dx$$

i.e.,
$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [f'(x) \int g(x)\, dx]\, dx$$

If we take f as the first function and g as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Example 17 Find $\int x \cos x\, dx$

Solution Put $f(x) = x$ (first function) and $g(x) = \cos x$ (second function).
Then, integration by parts gives

$$\int x \cos x\, dx = x \int \cos x\, dx - \int [\frac{d}{dx}(x) \int \cos x\, dx]\, dx$$

$$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + C$$

Suppose, we take $f(x) = \cos x$ and $g(x) = x$. Then

$$\int x \cos x\, dx = \cos x \int x\, dx - \int [\frac{d}{dx}(\cos x) \int x\, dx]\, dx$$

$$= (\cos x)\frac{x^2}{2} + \int \sin x \frac{x^2}{2}\, dx$$

Thus, it shows that the integral $\int x \cos x\, dx$ is reduced to the comparatively more complicated integral having more power of x. Therefore, the proper choice of the first function and the second function is significant.

Remarks

(i) It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int \sqrt{x} \sin x\, dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}\, \sin x$.

(ii) Observe that while finding the integral of the second function, we did not add any constant of integration. If we write the integral of the second function $\cos x$

as $\sin x + k$, where k is any constant, then

$$\int x \cos x \, dx = x(\sin x + k) - \int (\sin x + k)\, dx$$

$$= x(\sin x + k) - \int (\sin x \, dx - \int k \, dx$$

$$= x(\sin x + k) - \cos x - kx + C = x \sin x + \cos x + C$$

This shows that adding a constant to the integral of the second function is superfluous so far as the final result is concerned while applying the method of integration by parts.

(iii) Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, in cases where other function is inverse trigonometric function or logarithmic function, then we take them as first function.

Example 18 Find $\int \log x \, dx$

Solution To start with, we are unable to guess a function whose derivative is $\log x$. We take $\log x$ as the first function and the constant function 1 as the second function. Then, the integral of the second function is x.

Hence,
$$\int (\log x.1)\, dx = \log x \int 1\, dx - \int [\frac{d}{dx}(\log x) \int 1\, dx]\, dx$$

$$= (\log x) \cdot x - \int \frac{1}{x} x \, dx = x \log x - x + C.$$

Example 19 Find $\int x e^x dx$

Solution Take first function as x and second function as e^x. The integral of the second function is e^x.

Therefore,
$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx = xe^x - e^x + C.$$

Example 20 Find $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution Let first function be $\sin^{-1} x$ and second function be $\frac{x}{\sqrt{1-x^2}}$.

First we find the integral of the second function, i.e., $\int \frac{x\, dx}{\sqrt{1-x^2}}$.

Put $t = 1 - x^2$. Then $dt = -2x \, dx$

Therefore, $$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

Hence, $$\int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\,dx = (\sin^{-1}x)\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\,(-\sqrt{1-x^2})\,dx$$
$$= -\sqrt{1-x^2}\,\sin^{-1}x + x + \text{C} = x - \sqrt{1-x^2}\,\sin^{-1}x + \text{C}$$

Alternatively, this integral can also be worked out by making substitution $\sin^{-1}x = \theta$ and then integrating by parts.

Example 21 Find $\int e^x \sin x\,dx$

Solution Take e^x as the first function and $\sin x$ as second function. Then, integrating by parts, we have

$$\text{I} = \int e^x \sin x\,dx = e^x(-\cos x) + \int e^x \cos x\,dx$$
$$= -e^x \cos x + \text{I}_1 \text{ (say)} \quad \text{... (1)}$$

Taking e^x and $\cos x$ as the first and second functions, respectively, in I_1, we get

$$\text{I}_1 = e^x \sin x - \int e^x \sin x\,dx$$

Substituting the value of I_1 in (1), we get

$$\text{I} = -e^x \cos x + e^x \sin x - \text{I} \quad \text{or} \quad 2\text{I} = e^x(\sin x - \cos x)$$

Hence, $$\text{I} = \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + \text{C}$$

Alternatively, above integral can also be determined by taking $\sin x$ as the first function and e^x the second function.

7.6.1 *Integral of the type* $\int e^x\,[\,f(x) + f'(x)]\,dx$

We have $$\text{I} = \int e^x\,[\,f(x) + f'(x)]\,dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$
$$= \text{I}_1 + \int e^x f'(x)\,dx, \text{ where } \text{I}_1 = \int e^x f(x)\,dx \quad \text{... (1)}$$

Taking $f(x)$ and e^x as the first function and second function, respectively, in I_1 and integrating it by parts, we have $\text{I}_1 = f(x)\,e^x - \int f'(x)\,e^x dx + \text{C}$

Substituting I_1 in (1), we get

$$\text{I} = e^x f(x) - \int f'(x)\,e^x dx + \int e^x f'(x)\,dx + \text{C} = e^x f(x) + \text{C}$$

Thus, $$\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$$

Example 22 Find (i) $\int e^x (\tan^{-1}x + \frac{1}{1+x^2})\,dx$ (ii) $\int \frac{(x^2+1)\,e^x}{(x+1)^2}\,dx$

Solution

(i) We have $I = \int e^x (\tan^{-1}x + \frac{1}{1+x^2})\,dx$

Consider $f(x) = \tan^{-1}x$, then $f'(x) = \frac{1}{1+x^2}$

Thus, the given integrand is of the form $e^x\,[f(x) + f'(x)]$.

Therefore, $I = \int e^x (\tan^{-1}x + \frac{1}{1+x^2})\,dx = e^x \tan^{-1}x + C$

(ii) We have $I = \int \frac{(x^2+1)\,e^x}{(x+1)^2}\,dx = \int e^x\,[\frac{x^2-1+1+1)}{(x+1)^2}]\,dx$

$$= \int e^x\,[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]\,dx = \int e^x\,[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}]\,dx$$

Consider $f(x) = \frac{x-1}{x+1}$, then $f'(x) = \frac{2}{(x+1)^2}$

Thus, the given integrand is of the form $e^x\,[f(x) + f'(x)]$.

Therefore, $$\int \frac{x^2+1}{(x+1)^2}\,e^x\,dx = \frac{x-1}{x+1}\,e^x + C$$

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. $x \sin x$	**2.** $x \sin 3x$	**3.** $x^2\,e^x$	**4.** $x \log x$
5. $x \log 2x$	**6.** $x^2 \log x$	**7.** $x \sin^{-1}x$	**8.** $x \tan^{-1} x$
9. $x \cos^{-1} x$	**10.** $(\sin^{-1}x)^2$	**11.** $\frac{x \cos^{-1}x}{\sqrt{1-x^2}}$	**12.** $x \sec^2 x$
13. $\tan^{-1}x$	**14.** $x (\log x)^2$	**15.** $(x^2 + 1) \log x$	

16. $e^x (\sin x + \cos x)$ **17.** $\dfrac{x\, e^x}{(1+x)^2}$ **18.** $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$

19. $e^x \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$ **20.** $\dfrac{(x-3)\, e^x}{(x-1)^3}$ **21.** $e^{2x} \sin x$

22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3} dx$ equals

(A) $\dfrac{1}{3} e^{x^3} + \text{C}$ (B) $\dfrac{1}{3} e^{x^2} + \text{C}$

(C) $\dfrac{1}{2} e^{x^3} + \text{C}$ (D) $\dfrac{1}{2} e^{x^2} + \text{C}$

24. $\int e^x \sec x\, (1+\tan x)\, dx$ equals

(A) $e^x \cos x + \text{C}$ (B) $e^x \sec x + \text{C}$

(C) $e^x \sin x + \text{C}$ (D) $e^x \tan x + \text{C}$

7.6.2 ***Integrals of some more types***

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i) $\int \sqrt{x^2 - a^2}\; dx$ (ii) $\int \sqrt{x^2 + a^2}\; dx$ (iii) $\int \sqrt{a^2 - x^2}\; dx$

(i) Let $\text{I} = \int \sqrt{x^2 - a^2}\; dx$

Taking constant function 1 as the second function and integrating by parts, we have

$$\text{I} = x\sqrt{x^2 - a^2} - \int \frac{1}{2} \frac{2x}{\sqrt{x^2 - a^2}}\, x\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}}\, dx = x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= x\sqrt{x^2 - a^2} - \mathrm{I} - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

or $$2\mathrm{I} = x\sqrt{x^2 - a^2} - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

or $$\mathbf{I} = \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + \mathbf{C}$$

Similarly, integrating other two integrals by parts, taking constant function 1 as the second function, we get

(ii) $$\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + \mathbf{C}$$

(iii) $$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \mathbf{C}$$

Alternatively, integrals (i), (ii) and (iii) can also be found by making trigonometric substitution $x = a \sec\theta$ in (i), $x = a \tan\theta$ in (ii) and $x = a \sin\theta$ in (iii) respectively.

Example 23 Find $\int \sqrt{x^2 + 2x + 5}\, dx$

Solution Note that

$$\int \sqrt{x^2 + 2x + 5}\, dx = \int \sqrt{(x+1)^2 + 4}\, dx$$

Put $x + 1 = y$, so that $dx = dy$. Then

$$\int \sqrt{x^2 + 2x + 5}\, dx = \int \sqrt{y^2 + 2^2}\, dy$$

$$= \frac{1}{2} y \sqrt{y^2 + 4} + \frac{4}{2} \log \left| y + \sqrt{y^2 + 4} \right| + \mathrm{C} \qquad \text{[using 7.6.2 (ii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{x^2 + 2x + 5} + 2 \log \left| x + 1 + \sqrt{x^2 + 2x + 5} \right| + \mathrm{C}$$

Example 24 Find $\int \sqrt{3 - 2x - x^2}\, dx$

Solution Note that $\int \sqrt{3 - 2x - x^2}\, dx = \int \sqrt{4 - (x+1)^2}\, dx$

Put $x + 1 = y$ so that $dx = dy$.

Thus $$\int \sqrt{3-2x-x^2}\, dx = \int \sqrt{4-y^2}\, dy$$

$$= \frac{1}{2} y \sqrt{4-y^2} + \frac{4}{2} \sin^{-1} \frac{y}{2} + C \qquad \text{[using 7.6.2 (iii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C$$

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

1. $\sqrt{4-x^2}$ **2.** $\sqrt{1-4x^2}$ **3.** $\sqrt{x^2+4x+6}$

4. $\sqrt{x^2+4x+1}$ **5.** $\sqrt{1-4x-x^2}$ **6.** $\sqrt{x^2+4x-5}$

7. $\sqrt{1+3x-x^2}$ **8.** $\sqrt{x^2+3x}$ **9.** $\sqrt{1+\frac{x^2}{9}}$

Choose the correct answer in Exercises 10 to 11.

10. $\int \sqrt{1+x^2}\, dx$ is equal to

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right| + C$

(B) $\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$ (C) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}} + C$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C$

11. $\int \sqrt{x^2-8x+7}\, dx$ is equal to

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

7.7 Definite Integral

In the previous sections, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)\,dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$. Here, we shall consider these two cases separately as discussed below:

7.7.1 *Definite integral as the limit of a sum*

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a$ and $x = b$ (Fig 7.2).

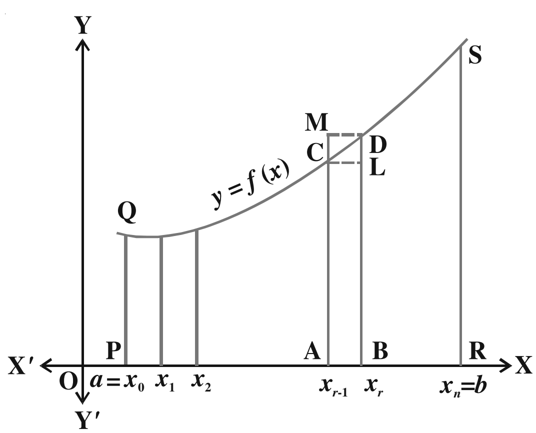


Fig 7.2

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,...., $[x_{r-1}, x_r]$, ..., $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... , $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \frac{b-a}{h}\cdot$ We note that as $n \to \infty$, $h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From Fig 7.2, we have

area of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \qquad \ldots (2)$$

and
$$S_n = h\,[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \qquad \ldots (3)$$

Here, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region PRSQP} < S_n \qquad \ldots (4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region PRSQP} = \int_a^b f(x)dx \qquad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h]$$

or
$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h] \qquad \ldots (6)$$

where
$$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression (6) is known as the definition of definite integral as the *limit of sum*.

Remark The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)\, dt$ or $\int_a^b f(u)\, du$ instead of $\int_a^b f(x)\, dx$. Hence, the variable of integration is called a *dummy variable*.

Example 25 Find $\int_0^2 (x^2+1)\, dx$ as the limit of a sum.

Solution By definition

$$\int_a^b f(x)\, dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+...+f(a+(n-1)h],$$

where, $\quad h = \dfrac{b-a}{n}$

In this example, $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

Therefore,

$$\int_0^2 (x^2+1)\, dx = 2\lim_{n\to\infty}\frac{1}{n}[f(0)+f(\frac{2}{n})+f(\frac{4}{n})+...+f(\frac{2(n-1)}{n})]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[1+(\frac{2^2}{n^2}+1)+(\frac{4^2}{n^2}+1)+...+\left(\frac{(2n-2)^2}{n^2}+1\right)]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[\underbrace{(1+1+...+1)}_{n\text{-terms}}+\frac{1}{n^2}(2^2+4^2+...+(2n-2)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2^2}{n^2}(1^2+2^2+...+(n-1)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{4}{n^2}\frac{(n-1)\,n\,(2n-1)}{6}]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2}{3}\frac{(n-1)\ (2n-1)}{n}]$$

$$= 2\lim_{n\to\infty}\ [1+\frac{2}{3}(1-\frac{1}{n})\ (2-\frac{1}{n})] = 2\ [1+\frac{4}{3}] = \frac{14}{3}$$

Example 26 Evaluate $\int_0^2 e^x\, dx$ as the limit of a sum.

Solution By definition

$$\int_0^2 e^x\, dx = (2-0)\lim_{n\to\infty}\frac{1}{n}\left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}}\right]$$

Using the sum to n terms of a G.P., where $a = 1$, $r = e^{\frac{2}{n}}$, we have

$$\int_0^2 e^x\, dx = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{2\,(e^2-1)}{\lim\limits_{n\to\infty}\left[\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right]\cdot 2} = e^2 - 1 \qquad \left[\text{using } \lim_{h\to 0}\frac{(e^h-1)}{h} = 1\right]$$

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x\, dx$ **2.** $\int_0^5 (x+1)\, dx$ **3.** $\int_2^3 x^2\, dx$

4. $\int_1^4 (x^2 - x)\, dx$ **5.** $\int_{-1}^1 e^x\, dx$ **6.** $\int_0^4 (x + e^{2x})\, dx$

7.8 Fundamental Theorem of Calculus

7.8.1 *Area function*

We have defined $\int_a^b f(x)\, dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\, dx$ represents the area of the shaded region

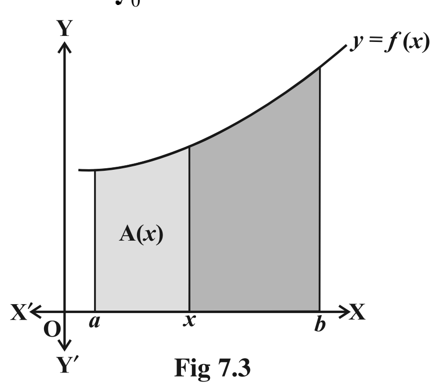


Fig 7.3

in Fig 7.3 [Here it is assumed that $f(x) > 0$ for $x \in [a, b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x.

In other words, the area of this shaded region is a function of x. We denote this function of x by $A(x)$. We call the function $A(x)$ as *Area function* and is given by

$$\mathbf{A}(x) = \int_a^x f(x)\,dx \quad \text{... (1)}$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

7.8.2 *First fundamental theorem of integral calculus*

Theorem 1 Let f be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $\mathbf{A}'(x) = f(x)$, for all $x \in [a, b]$.

7.8.3 *Second fundamental theorem of integral calculus*

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2 Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [\mathbf{F}(x)]_a^b = \mathbf{F}(b) - \mathbf{F}(a)$.

Remarks

(i) In words, the Theorem 2 tells us that $\int_a^b f(x)\,dx$ = (value of the anti derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

(ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

(iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(iv) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in $[a, b]$. For instance, the consideration of definite integral $\int_{-2}^{3} x(x^2-1)^{\frac{1}{2}}\,dx$ is erroneous since the function f expressed by $f(x) = x(x^2-1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

Steps for calculating $\int_a^b f(x)\,dx$.

(i) Find the indefinite integral $\int f(x)\,dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get

$\int_a^b f(x)\,dx = [\mathrm{F}(x) + \mathrm{C}]_a^b = [\mathrm{F}(b) + \mathrm{C}] - [\mathrm{F}(a) + \mathrm{C}] = \mathrm{F}(b) - \mathrm{F}(a)$.

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate F(b) – F(a) = $[\mathrm{F}(x)]_a^b$, which is the value of $\int_a^b f(x)\,dx$.

We now consider some examples

Example 27 Evaluate the following integrals:

(i) $\int_2^3 x^2\,dx$ (ii) $\int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$

(iii) $\int_1^2 \frac{x\,dx}{(x+1)(x+2)}$ (iv) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$

Solution

(i) Let $\mathrm{I} = \int_2^3 x^2\,dx$. Since $\int x^2\,dx = \frac{x^3}{3} = \mathrm{F}(x)$,

Therefore, by the second fundamental theorem, we get

$$\mathrm{I} = \mathrm{F}(3) - \mathrm{F}(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(ii) Let $\mathrm{I} = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$. We first find the anti derivative of the integrand.

Put $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\,dx = dt$ or $\sqrt{x}\,dx = -\frac{2}{3}dt$

Thus, $\int \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right] = \mathrm{F}(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$\text{I} = \text{F}(9) - \text{F}(4) = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-27)} - \frac{1}{30-8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

(iii) Let $\text{I} = \int_1^2 \frac{x\,dx}{(x+1)\,(x+2)}$

Using partial fraction, we get $\frac{x}{(x+1)\,(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So $$\int \frac{x\,dx}{(x+1)\,(x+2)} = -\log\left|\,x+1\,\right| + 2\log\left|\,x+2\,\right| = \text{F}(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$\text{I} = \text{F}(2) - \text{F}(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3]$$

$$= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right)$$

(iv) Let $\text{I} = \int_0^{\frac{\pi}{4}} \sin^3 2t\cos 2t\,dt$. Consider $\int \sin^3 2t\cos 2t\,dt$

Put $\sin 2t = u$ so that $2\cos 2t\,dt = du$ or $\cos 2t\,dt = \frac{1}{2}\,du$

So $$\int \sin^3 2t\cos 2t\,dt = \frac{1}{2}\int u^3 du$$

$$= \frac{1}{8}[u^4] = \frac{1}{8}\sin^4 2t = \text{F}(t) \text{ say}$$

Therefore, by the second fundamental theorem of integral calculus

$$\text{I} = \text{F}\left(\frac{\pi}{4}\right) - \text{F}(0) = \frac{1}{8}\left[\sin^4\frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

1. $\int_{-1}^{1}(x+1)\,dx$
2. $\int_{2}^{3}\frac{1}{x}\,dx$
3. $\int_{1}^{2}(4x^3-5x^2+6x+9)\,dx$
4. $\int_{0}^{\frac{\pi}{4}}\sin 2x\,dx$
5. $\int_{0}^{\frac{\pi}{2}}\cos 2x\,dx$
6. $\int_{4}^{5}e^x\,dx$
7. $\int_{0}^{\frac{\pi}{4}}\tan x\,dx$
8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\operatorname{cosec} x\,dx$
9. $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$
10. $\int_{0}^{1}\frac{dx}{1+x^2}$
11. $\int_{2}^{3}\frac{dx}{x^2-1}$
12. $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx$
13. $\int_{2}^{3}\frac{x\,dx}{x^2+1}$
14. $\int_{0}^{1}\frac{2x+3}{5x^2+1}\,dx$
15. $\int_{0}^{1}x\,e^{x^2}\,dx$
16. $\int_{1}^{2}\frac{5x^2}{x^2+4x+3}$
17. $\int_{0}^{\frac{\pi}{4}}(2\sec^2 x+x^3+2)\,dx$
18. $\int_{0}^{\pi}(\sin^2\frac{x}{2}-\cos^2\frac{x}{2})\,dx$
19. $\int_{0}^{2}\frac{6x+3}{x^2+4}\,dx$
20. $\int_{0}^{1}(x\,e^x+\sin\frac{\pi x}{4})\,dx$

Choose the correct answer in Exercises 21 and 22.

21. $\int_{1}^{\sqrt{3}}\frac{dx}{1+x^2}$ equals

    (A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

22. $\int_{0}^{\frac{2}{3}}\frac{dx}{4+9x^2}$ equals

    (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate $\int_a^b f(x)\,dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

Note In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

Example 28 Evaluate $\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\,dx$.

Solution Put $t = x^5 + 1$, then $dt = 5x^4\,dx$.

Therefore, $$\int 5x^4\sqrt{x^5+1}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$$

Hence, $$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Alternatively, first we transform the integral and then evaluate the transformed integral with new limits.

Let $t = x^5 + 1$. Then $dt = 5\,x^4\,dx$.

Note that, when $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

Thus, as x varies from – 1 to 1, t varies from 0 to 2

Therefore
$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_0^2 \sqrt{t}\,dt$$
$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Example 29 Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx$

Solution Let $t = \tan^{-1}x$, then $dt = \dfrac{1}{1+x^2}\,dx$. The new limits are, when $x = 0$, $t = 0$ and when $x = 1$, $t = \dfrac{\pi}{4}$. Thus, as x varies from 0 to 1, t varies from 0 to $\dfrac{\pi}{4}$.

Therefore
$$\int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} t\,dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

1. $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$ **2.** $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$ **3.** $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

4. $\displaystyle\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$) **5.** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

6. $\displaystyle\int_0^2 \frac{dx}{x+4-x^2}$ **7.** $\displaystyle\int_{-1}^1 \frac{dx}{x^2+2x+5}$ **8.** $\displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}\,dx$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\displaystyle\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\,dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

10. If $f(x) = \displaystyle\int_0^x t\sin t\,dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$ (B) $x\sin x$

(C) $x\cos x$ (D) $\sin x + x\cos x$

7.10 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

P$_0$: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P$_1$: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

P$_2$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P$_3$: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P$_4$: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(Note that P$_4$ is a particular case of P$_3$)

P$_5$: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P$_6$: $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and

0 if $f(2a-x) = -f(x)$

P$_7$: (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.

We give the proofs of these properties one by one.

Proof of P$_0$ It follows directly by making the substitution $x = t$.

Proof of P$_1$ Let F be anti derivative of f. Then, by the second fundamental theorem of calculus, we have $\int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) = -[\mathrm{F}(a) - \mathrm{F}(b)] = -\int_b^a f(x)\,dx$

Here, we observe that, if $a = b$, then $\int_a^a f(x)\,dx = 0$.

Proof of P$_2$ Let F be anti derivative of f. Then

$$\int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) \quad \text{... (1)}$$

$$\int_a^c f(x)\,dx = \mathrm{F}(c) - \mathrm{F}(a) \quad \text{... (2)}$$

and

$$\int_c^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(c) \quad \text{... (3)}$$

Adding (2) and (3), we get $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) = \int_a^b f(x)\,dx$

This proves the property P_2.

Proof of P_3 Let $t = a + b - x$. Then $dt = -\,dx$. When $x = a$, $t = b$ and when $x = b$, $t = a$. Therefore

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \quad \text{(by } P_1\text{)}$$

$$= \int_a^b f(a+b-x)\,dx \text{ by } P_0$$

Proof of P_4 Put $t = a - x$. Then $dt = -\,dx$. When $x = 0$, $t = a$ and when $x = a$, $t = 0$. Now proceed as in P_3.

Proof of P_5 Using P_2, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$.

Let $t = 2a - x$ in the second integral on the right hand side. Then $dt = -\,dx$. When $x = a$, $t = a$ and when $x = 2a$, $t = 0$. Also $x = 2a - t$.

Therefore, the second integral becomes

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

Hence $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Proof of P_6 Using P_5, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ ... (1)

Now, if $f(2a - x) = f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx,$$

and if $f(2a - x) = -f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

Proof of P_7 Using P_2, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx. \text{ Then}$$

Let $t = -x$ in the first integral on the right hand side. $dt = -\,dx$. When $x = -a$, $t = a$ and when $x = 0$, $t = 0$. Also $x = -t$.

Therefore $$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{(by } P_0) \quad ... (1)$$

(i) Now, if f is an even function, then $f(-x) = f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If f is an odd function, then $f(-x) = -f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Example 30 Evaluate $\int_{-1}^{2} \left| x^3 - x \right| dx$

Solution We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$. So by P_2 we write

$$\int_{-1}^{2} \left| x^3 - x \right| dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

Example 31 Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$

Solution We observe that $\sin^2 x$ is an even function. Therefore, by P_7 (i), we get

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{(1-\cos 2x)}{2}\,dx = \int_0^{\frac{\pi}{4}} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{\pi}{4} - \frac{1}{2}$$

Example 32 Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

Solution Let I = $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$. Then, by P_4, we have

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\,dx}{1+\cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I$$

or $$2\,I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

or $$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$. Therefore, (by P_1) we get

$$I = \frac{-\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi\int_0^{1} \frac{dt}{1+t^2} \text{ (by } P_7\text{, since } \frac{1}{1+t^2} \text{ is even function)}$$

$$= \pi\left[\tan^{-1} t\right]_0^1 = \pi\left[\tan^{-1} 1 - \tan^{-1} 0\right] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

Example 33 Evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx$

Solution Let I = $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx$. Let $f(x) = \sin^5 x \cos^4 x$. Then

$f(-x) = \sin^5(-x)\cos^4(-x) = -\sin^5 x \cos^4 x = -f(x)$, i.e., f is an odd function. Therefore, by P_7 (ii), I = 0

Example 34 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

Solution Let I = $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ ... (1)

Then, by P_4

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 (\frac{\pi}{2} - x)}{\sin^4 (\frac{\pi}{2} - x) + \cos^4 (\frac{\pi}{2} - x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence $\qquad I = \frac{\pi}{4}$

Example 35 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}}$

Solution Let I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}}$ ... (1)

Then, by P_3 $\qquad I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} dx}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \text{ Hence } I = \frac{\pi}{12}$$

Example 36 Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Solution Let $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Then, by P_4

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

Adding the two values of I, we get

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x + \log 2 - \log 2) \, dx \text{ (by adding and subtracting } \log 2)$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx \quad \text{(Why?)}$$

Put $2x = t$ in the first integral. Then $2\,dx = dt$, when $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$.

Therefore $$2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2} \log 2 \quad [\text{by } P_6 \text{ as } \sin(\pi - t) = \sin t]$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2 \quad \text{(by changing variable } t \text{ to } x)$$

$$= I - \frac{\pi}{2} \log 2$$

Hence $$\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{-\pi}{2} \log 2.$$

EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$ **2.** $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ **3.** $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x}$ **5.** $\int_{-5}^{5} |x+2| \, dx$ **6.** $\int_2^8 |x-5| \, dx$

7. $\int_0^1 x(1-x)^n dx$ **8.** $\int_0^{\frac{\pi}{4}} \log(1+\tan x) \, dx$ **9.** $\int_0^2 x\sqrt{2-x} \, dx$

10. $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x) \, dx$ **11.** $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$

12. $\int_0^{\pi} \frac{x \, dx}{1+\sin x}$ **13.** $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$ **14.** $\int_0^{2\pi} \cos^5 x \, dx$

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$ **16.** $\int_0^{\pi} \log(1+\cos x) \, dx$ **17.** $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$

18. $\int_0^4 |x-1| \, dx$

19. Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Choose the correct answer in Exercises 20 and 21.

20. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1) \, dx$ is

(A) 0 (B) 2 (C) π (D) 1

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(A) 2 (B) $\frac{3}{4}$ (C) 0 (D) –2

Miscellaneous Examples

Example 37 Find $\int \cos 6x \sqrt{1+\sin 6x}\, dx$

Solution Put $t = 1 + \sin 6x$, so that $dt = 6 \cos 6x\, dx$

Therefore $\int \cos 6x\sqrt{1+\sin 6x}\, dx = \frac{1}{6}\int t^{\frac{1}{2}} dt$

$$= \frac{1}{6}\times\frac{2}{3}(t)^{\frac{3}{2}} + \text{C} = \frac{1}{9}(1+\sin 6x)^{\frac{3}{2}} + \text{C}$$

Example 38 Find $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5}\, dx$

Solution We have $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5}\, dx = \int \frac{(1-\frac{1}{x^3})^{\frac{1}{4}}}{x^4}\, dx$

Put $1-\frac{1}{x^3} = 1-x^{-3} = t$, so that $\frac{3}{x^4}dx = dt$

Therefore $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5}\, dx = \frac{1}{3}\int t^{\frac{1}{4}}\, dt = \frac{1}{3}\times\frac{4}{5}t^{\frac{5}{4}} + \text{C} = \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}} + \text{C}$

Example 39 Find $\int \frac{x^4\, dx}{(x-1)\,(x^2+1)}$

Solution We have

$$\frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{x^3 - x^2 + x - 1}$$

$$= (x+1) + \frac{1}{(x-1)\,(x^2+1)} \qquad \text{... (1)}$$

Now express $$\frac{1}{(x-1)(x^2+1)} = \frac{\text{A}}{(x-1)} + \frac{\text{B}x+\text{C}}{(x^2+1)} \qquad \text{... (2)}$$

So $$1 = \mathrm{A}\,(x^2 + 1) + (\mathrm{B}x + \mathrm{C})\,(x - 1)$$
$$= (\mathrm{A} + \mathrm{B})\,x^2 + (\mathrm{C} - \mathrm{B})\,x + \mathrm{A} - \mathrm{C}$$

Equating coefficients on both sides, we get A + B = 0, C – B = 0 and A – C = 1, which give $\mathrm{A} = \frac{1}{2}, \mathrm{B} = \mathrm{C} = -\frac{1}{2}$. Substituting values of A, B and C in (2), we get

$$\frac{1}{(x-1)\,(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)} \qquad \text{... (3)}$$

Again, substituting (3) in (1), we have

$$\frac{x^4}{(x-1)\,(x^2+x+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)}$$

Therefore

$$\int \frac{x^4}{(x-1)\,(x^2+x+1)}dx = \frac{x^2}{2} + x + \frac{1}{2}\log|\,x-1\,| - \frac{1}{4}\log\,(x^2+1) - \frac{1}{2}\tan^{-1}x + \mathrm{C}$$

Example 40 Find $\int \left[\log\,(\log x) + \frac{1}{(\log x)^2}\right]dx$

Solution Let $\mathrm{I} = \int \left[\log\,(\log x) + \frac{1}{(\log x)^2}\right]dx$

$$= \int \log\,(\log x)\,dx + \int \frac{1}{(\log x)^2}\,dx$$

In the first integral, let us take 1 as the second function. Then integrating it by parts, we get

$$\mathrm{I} = x\log\,(\log x) - \int \frac{1}{x\log x}\,x\,dx + \int \frac{dx}{(\log x)^2}$$

$$= x\log\,(\log x) - \int \frac{dx}{\log x} + \int \frac{dx}{(\log x)^2} \qquad \text{... (1)}$$

Again, consider $\int \frac{dx}{\log x}$, take 1 as the second function and integrate it by parts, we have $\int \frac{dx}{\log x} = \left[\frac{x}{\log x} - \int x\left\{-\frac{1}{(\log x)^2}\left(\frac{1}{x}\right)\right\}dx\right]$ ... (2)

Putting (2) in (1), we get

$$\mathrm{I} = x \log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2} + \int \frac{dx}{(\log x)^2} = x \log(\log x) - \frac{x}{\log x} + \mathrm{C}$$

Example 41 Find $\int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx$

Solution We have

$$\mathrm{I} = \int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx = \int \sqrt{\tan x}\,(1 + \cot x)\, dx$$

Put $\tan x = t^2$, so that $\sec^2 x\, dx = 2t\, dt$

or $\qquad dx = \dfrac{2t\, dt}{1+t^4}$

Then $\qquad \mathrm{I} = \displaystyle\int t\left(1+\frac{1}{t^2}\right)\frac{2t}{(1+t^4)}\, dt$

$$= 2\int \frac{(t^2+1)}{t^4+1}\, dt = 2\int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t^2+\frac{1}{t^2}\right)} = 2\int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t-\frac{1}{t}\right)^2 + 2}$$

Put $t - \dfrac{1}{t} = y$, so that $\left(1+\dfrac{1}{t^2}\right) dt = dy$. Then

$$\mathrm{I} = 2\int \frac{dy}{y^2 + \left(\sqrt{2}\right)^2} = \sqrt{2}\, \tan^{-1} \frac{y}{\sqrt{2}} + \mathrm{C} = \sqrt{2}\, \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + \mathrm{C}$$

$$= \sqrt{2}\, \tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\, t}\right) + \mathrm{C} = \sqrt{2}\, \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \mathrm{C}$$

Example 42 Find $\displaystyle\int \frac{\sin 2x \cos 2x\, dx}{\sqrt{9 - \cos^4(2x)}}$

Solution Let $\mathrm{I} = \displaystyle\int \frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}}\, dx$

Put $\cos^2(2x) = t$ so that $4 \sin 2x \cos 2x \, dx = -dt$

Therefore $\quad \mathrm{I} = -\frac{1}{4}\int \frac{dt}{\sqrt{9-t^2}} = -\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right) + \mathrm{C} = -\frac{1}{4}\sin^{-1}\left[\frac{1}{3}\cos^2 2x\right] + \mathrm{C}$

Example 43 Evaluate $\int_{-1}^{\frac{3}{2}} \left| x \sin(\pi x) \right| dx$

Solution Here $f(x) = |x \sin \pi x| = \begin{cases} x \sin \pi x \text{ for } -1 \le x \le 1 \\ -x \sin \pi x \text{ for } 1 \le x \le \frac{3}{2} \end{cases}$

Therefore
$$\int_{-1}^{\frac{3}{2}} |x \sin \pi x| \, dx = \int_{-1}^{1} x \sin \pi x \, dx + \int_{1}^{\frac{3}{2}} -x \sin \pi x \, dx$$
$$= \int_{-1}^{1} x \sin \pi x \, dx - \int_{1}^{\frac{3}{2}} x \sin \pi x \, dx$$

Integrating both integrals on righthand side, we get

$$\int_{-1}^{\frac{3}{2}} |x \sin \pi x| \, dx = \left[\frac{-x\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{-1}^{1} - \left[\frac{-x\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{1}^{\frac{3}{2}}$$
$$= \frac{2}{\pi} - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3}{\pi} + \frac{1}{\pi^2}$$

Example 44 Evaluate $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Solution Let $\mathrm{I} = \int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi} \frac{(\pi - x) \, dx}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)}$ (using P_4)

$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - \int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$
$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - \mathrm{I}$$

Thus $\quad 2\mathrm{I} = \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

or $$\text{I} = \frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$
(using P_6)

$$= \pi\int_0^{\frac{\pi}{2}}\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x} \quad \text{(dividing numerator and denominator by } \cos^2 x\text{).}$$

Put $b \tan x = t$, so that $b \sec^2 x\, dx = dt$. Also, when $x = 0$, $t = 0$, and when $x = \frac{\pi}{2}$, $t \to \infty$.

Therefore, $$\text{I} = \frac{\pi}{b}\int_0^{\infty}\frac{dt}{a^2 + t^2} = \frac{\pi}{b}\cdot\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^{\infty} = \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2\,ab}.$$

Miscellaneous Exercise on Chapter 7

Integrate the functions in Exercises 1 to 24.

1. $\frac{1}{x - x^3}$ **2.** $\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$ **3.** $\frac{1}{x\sqrt{ax - x^2}}$ [Hint: Put $x = \frac{a}{t}$]

4. $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$ **5.** $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ [Hint: $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1 + x^{\frac{1}{6}}\right)}$, put $x = t^6$]

6. $\frac{5x}{(x+1)(x^2+9)}$ **7.** $\frac{\sin x}{\sin(x-a)}$ **8.** $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

9. $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$ **10.** $\frac{\sin^8 - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}$ **11.** $\frac{1}{\cos(x+a)\cos(x+b)}$

12. $\frac{x^3}{\sqrt{1 - x^8}}$ **13.** $\frac{e^x}{(1+e^x)(2+e^x)}$ **14.** $\frac{1}{(x^2+1)(x^2+4)}$

15. $\cos^3 x\, e^{\log \sin x}$ **16.** $e^{3\log x}(x^4 + 1)^{-1}$ **17.** $f'(ax + b)\,[f(ax + b)]^n$

18. $\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$ **19.** $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$, $x \in [0, 1]$

20. $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$ **21.** $\frac{2 + \sin 2x}{1 + \cos 2x}e^x$ **22.** $\frac{x^2 + x + 1}{(x+1)^2(x+2)}$

23. $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$ **24.** $\dfrac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$

Evaluate the definite integrals in Exercises 25 to 33.

25. $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\dfrac{1-\sin x}{1+\cos x}\right)dx$ **26.** $\int_0^{\frac{\pi}{4}}\dfrac{\sin x\,\cos x}{\cos^4 x+\sin^4 x}\,dx$ **27.** $\int_0^{\frac{\pi}{2}}\dfrac{\cos^2 x\,dx}{\cos^2 x+4\sin^2 x}$

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\dfrac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$ **29.** $\int_0^1\dfrac{dx}{\sqrt{1+x}-\sqrt{x}}$ **30.** $\int_0^{\frac{\pi}{4}}\dfrac{\sin x+\cos x}{9+16\sin 2x}\,dx$

31. $\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)\,dx$ **32.** $\int_0^{\pi}\dfrac{x\tan x}{\sec x+\tan x}\,dx$

33. $\int_1^4[|x-1|+|x-2|+|x-3|]\,dx$

Prove the following (Exercises 34 to 39)

34. $\int_1^3\dfrac{dx}{x^2(x+1)}=\dfrac{2}{3}+\log\dfrac{2}{3}$ **35.** $\int_0^1 x\,e^x dx=1$

36. $\int_{-1}^1 x^{17}\cos^4 x\,dx=0$ **37.** $\int_0^{\frac{\pi}{2}}\sin^3 x\,dx=\dfrac{2}{3}$

38. $\int_0^{\frac{\pi}{4}}2\tan^3 x\,dx=1-\log 2$ **39.** $\int_0^1\sin^{-1}x\,dx=\dfrac{\pi}{2}-1$

40. Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Choose the correct answers in Exercises 41 to 44.

41. $\int\dfrac{dx}{e^x+e^{-x}}$ is equal to

(A) $\tan^{-1}(e^x)+C$ (B) $\tan^{-1}(e^{-x})+C$

(C) $\log(e^x-e^{-x})+C$ (D) $\log(e^x+e^{-x})+C$

42. $\int\dfrac{\cos 2x}{(\sin x+\cos x)^2}\,dx$ is equal to

(A) $\dfrac{-1}{\sin x+\cos x}+C$ (B) $\log|\sin x+\cos x|+C$

(C) $\log|\sin x-\cos x|+C$ (D) $\dfrac{1}{(\sin x+\cos x)^2}$

43. If $f(a + b - x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to

(A) $\frac{a+b}{2}\int_a^b f(b-x)\, dx$ (B) $\frac{a+b}{2}\int_a^b f(b+x)\, dx$

(C) $\frac{b-a}{2}\int_a^b f(x)\, dx$ (D) $\frac{a+b}{2}\int_a^b f(x)\, dx$

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ is

(A) 1 (B) 0 (C) –1 (D) $\frac{\pi}{4}$

Summary

- Integration is the inverse process of differentiation. In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation.

  Let $\frac{d}{dx}\mathrm{F}(x) = f(x)$. Then we write $\int f(x)\, dx = \mathrm{F}(x) + \mathrm{C}$. These integrals are called indefinite integrals or general integrals, C is called constant of integration. All these integrals differ by a constant.
- From the geometric point of view, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along the y-axis.
- Some properties of indefinite integrals are as follows:

  1. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

  2. For any real number k, $\int k\, f(x)\, dx = k\int f(x)\, dx$

  More generally, if $f_1, f_2, f_3, \ldots, f_n$ are functions and $k_1, k_2, \ldots, k_n$ are real numbers. Then

  $$\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\, dx$$
  $$= k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx + \ldots + k_n \int f_n(x)\, dx$$

Some standard integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$. Particularly, $\int dx = x + C$

(ii) $\int \cos x\, dx = \sin x + C$

(iii) $\int \sin x\, dx = -\cos x + C$

(iv) $\int \sec^2 x\, dx = \tan x + C$

(v) $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

(vi) $\int \sec x \tan x\, dx = \sec x + C$

(vii) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(xi) $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\int e^x dx = e^x + C$

(xiii) $\int a^x dx = \frac{a^x}{\log a} + C$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xvi) $\int \frac{1}{x}\, dx = \log |x| + C$

Integration by partial fractions

Recall that a rational function is ratio of two polynomials of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If degree of the polynomial $P(x)$ is greater than the degree of the polynomial $Q(x)$, then we may divide $P(x)$ by $Q(x)$ so that $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where $T(x)$ is a polynomial in x and degree of $P_1(x)$ is less than the degree of $Q(x)$. $T(x)$ being polynomial can be easily integrated. $\frac{P_1(x)}{Q(x)}$ can be integrated by expressing $\frac{P_1(x)}{Q(x)}$ as the sum of partial fractions of the following type:

1. $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b$

2. $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

4. $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

5. $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ can not be factorised further.

◆ **Integration by substitution**

A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method in which we change the variable to some other variable is called the method of substitution. When the integrand involves some trigonometric functions, we use some well known identities to find the integrals. Using substitution technique, we obtain the following standard integrals.

(i) $\int \tan x\, dx = \log|\sec x| + C$ (ii) $\int \cot x\, dx = \log|\sin x| + C$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

◆ **Integrals of some special functions**

(i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(ii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$ (iii) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$ (v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

(vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}| + C$

- **Integration by parts**

  For given functions f_1 and f_2, we have

  $\int f_1(x) \cdot f_2(x)\, dx = f_1(x) \int f_2(x)\, dx - \int \left[\frac{d}{dx} f_1(x) \cdot \int f_2(x)\, dx\right] dx$, i.e., the integral of the product of two functions = first function × integral of the second function – integral of {differential coefficient of the first function × integral of the second function}. Care must be taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

- $\int e^x [f(x) + f'(x)]\, dx = \int e^x f(x)\, dx + C$

- **Some special types of integrals**

  (i) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

  (ii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

  (iii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

  (iv) Integrals of the types $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing

  $$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

  (v) Integrals of the types $\int \frac{px + q\, dx}{ax^2 + bx + c}$ or $\int \frac{px + q\, dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing

$px + q = \text{A}\frac{d}{dx}(ax^2 + bx + c) + \text{B} = \text{A}\,(2ax + b) + \text{B}$, where A and B are determined by comparing coefficients on both sides.

- We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, $a \le x \le b$, the x-axis and the ordinates $x = a$ and $x = b$. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the **Area function** A (x). This concept of area function leads to the Fundamental Theorems of Integral Calculus.
- **First fundamental theorem of integral calculus**

  Let the area function be defined by A$(x) = \int_a^x f(x)\,dx$ for all $x \ge a$, where the function f is assumed to be continuous on $[a, b]$. Then A′ $(x) = f(x)$ for all $x \in [a, b]$.
- **Second fundamental theorem of integral calculus**

  Let f be a continuous function of x defined on the closed interval $[a, b]$ and let F be another function such that $\frac{d}{dx}\text{F}(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)\,dx = \left[\text{F}(x) + \text{C}\right]_a^b = \text{F}\,(b) - \text{F}\,(a)$.

  This is called the definite integral of f over the range $[a, b]$, where a and b are called the limits of integration, a being the lower limit and b the upper limit.

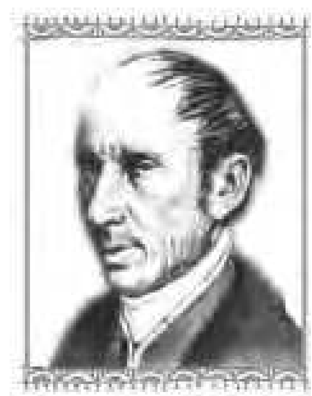
Chapter 8

APPLICATION OF INTEGRALS

❖ *One should study Mathematics because it is only through Mathematics that nature can be conceived in harmonious form. – BIRKHOFF* ❖

8.1 Introduction

In geometry, we have learnt formulae to calculate areas of various geometrical figures including triangles, rectangles, trapezias and circles. Such formulae are fundamental in the applications of mathematics to many real life problems. The formulae of elementary geometry allow us to calculate areas of many simple figures. However, they are inadequate for calculating the areas enclosed by curves. For that we shall need some concepts of Integral Calculus.



A.L. Cauchy
(1789-1857)

In the previous chapter, we have studied to find the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and x-axis, while calculating definite integral as the limit of a sum. Here, in this chapter, we shall study a specific application of integrals to find the area under simple curves, area between lines and arcs of circles, parabolas and ellipses (standard forms only). We shall also deal with finding the area bounded by the above said curves.

8.2 Area under Simple Curves

In the previous chapter, we have studied definite integral as the limit of a sum and how to evaluate definite integral using Fundamental Theorem of Calculus. Now, we consider the easy and intuitive way of finding the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. From Fig 8.1, we can think of area under the curve as composed of large number of very thin vertical strips. Consider an arbitrary strip of height y and width dx, then dA (area of the elementary strip) $= y dx$, where, $y = f(x)$.

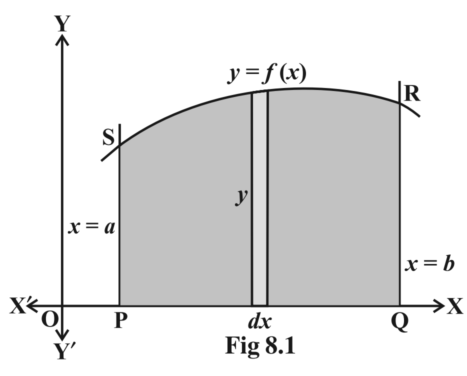


Fig 8.1

This area is called the *elementary area* which is located at an arbitrary position within the region which is specified by some value of x between a and b. We can think of the total area A of the region between x-axis, ordinates $x = a$, $x = b$ and the curve $y = f(x)$ as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically, we express

$$A = \int_a^b dA = \int_a^b y dx = \int_a^b f(x)\, dx$$

The area A of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$, $y = d$ is given by

$$A = \int_c^d x dy = \int_c^d g(y)\, dy$$

Here, we consider horizontal strips as shown in the Fig 8.2

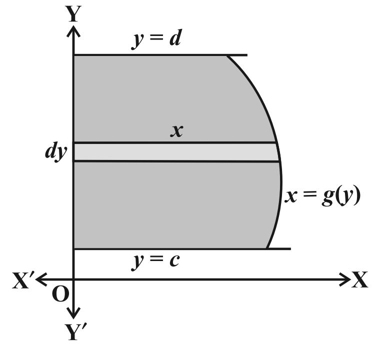


Fig 8.2

Remark If the position of the curve under consideration is below the x-axis, then since $f(x) < 0$ from $x = a$ to $x = b$, as shown in Fig 8.3, the area bounded by the curve, x-axis and the ordinates $x = a$, $x = b$ come out to be negative. But, it is only the numerical value of the area which is taken into consideration. Thus, if the area is negative, we take its absolute value, i.e., $\left|\int_a^b f(x)\, dx\right|$.

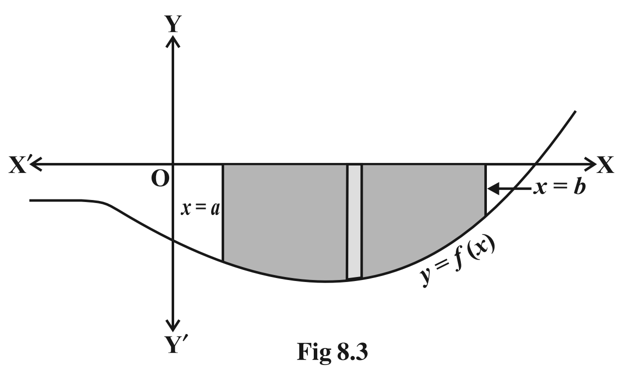


Fig 8.3

Generally, it may happen that some portion of the curve is above x-axis and some is below the x-axis as shown in the Fig 8.4. Here, $A_1 < 0$ and $A_2 > 0$. Therefore, the area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by $A = |A_1| + A_2$.

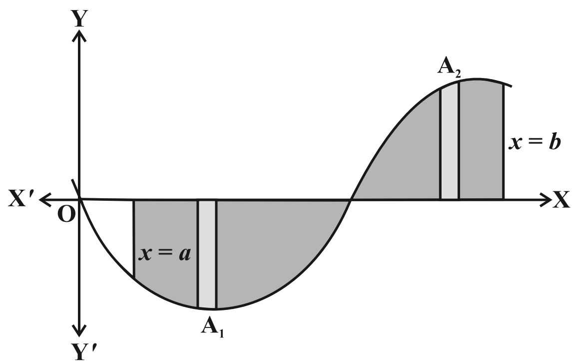


Fig 8.4

Example 1 Find the area enclosed by the circle $x^2 + y^2 = a^2$.

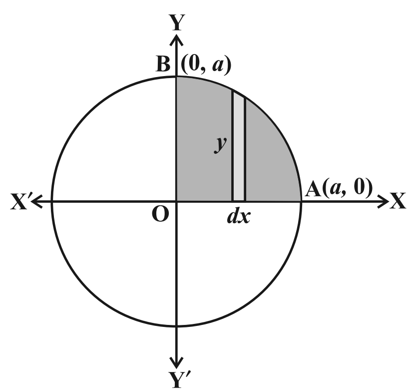


Fig 8.5

Solution From Fig 8.5, the whole area enclosed by the given circle

= 4 (area of the region AOBA bounded by the curve, x-axis and the ordinates $x = 0$ and $x = a$) [as the circle is symmetrical about both x-axis and y-axis]

$$= 4\int_0^a y\,dx \text{ (taking vertical strips)}$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

Since $x^2 + y^2 = a^2$ gives $\quad y = \pm\sqrt{a^2 - x^2}$

As the region AOBA lies in the first quadrant, y is taken as positive. Integrating, we get the whole area enclosed by the given circle

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right] = 4\left(\frac{a^2}{2}\right)\left(\frac{\pi}{2}\right) = \pi a^2$$

Alternatively, considering horizontal strips as shown in Fig 8.6, the whole area of the region enclosed by circle

$$= 4\int_0^a x\,dy = 4\int_0^a \sqrt{a^2 - y^2}\,dy \qquad \text{(Why?)}$$

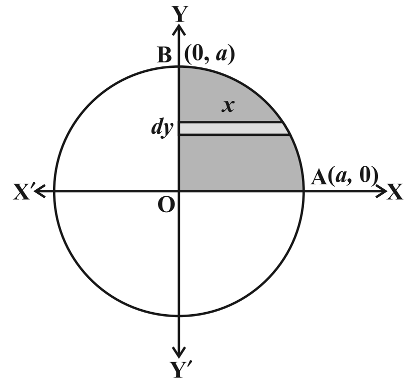


Fig 8.6

$$= 4\left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= 4\frac{a^2}{2}\frac{\pi}{2} = \pi a^2$$

Example 2 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution From Fig 8.7, the area of the region ABA′B′A bounded by the ellipse

$$= 4\begin{pmatrix}\textit{area of the region AOBA in the first quadrant bounded}\\ \textit{by the curve, } x - \textit{axis and the ordinates } x = 0, x = a\end{pmatrix}$$

(as the ellipse is symmetrical about both x-axis and y-axis)

$$= 4\int_0^a y\,dx \quad \text{(taking vertical strips)}$$

Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$, but as the region AOBA lies in the first quadrant, y is taken as positive. So, the required area is

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a \quad \text{(Why?)}$$

$$= \frac{4b}{a}\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= \frac{4b}{a}\frac{a^2}{2}\frac{\pi}{2} = \pi ab$$

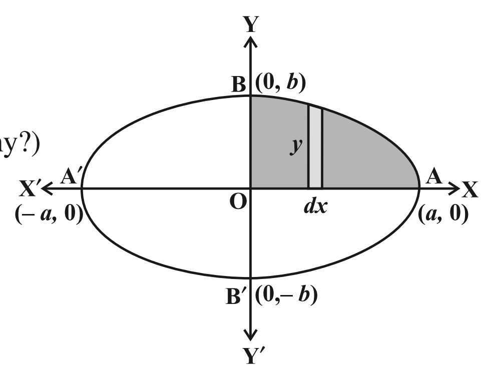


Fig 8.7

Alternatively, considering horizontal strips as shown in the Fig 8.8, the area of the ellipse is

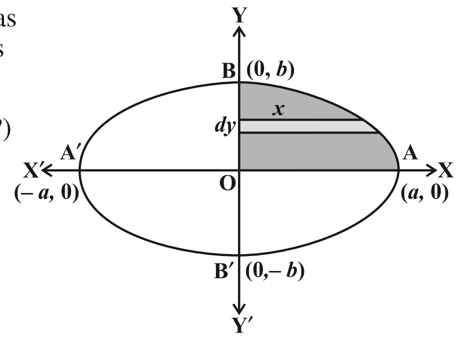


Fig 8.8

$$= 4\int_0^b x\,dy = 4\frac{a}{b}\int_0^b \sqrt{b^2 - y^2}\,dy \quad \text{(Why?)}$$

$$= \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2 - y^2} + \frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$= \frac{4a}{b}\left[\left(\frac{b}{2}\times 0 + \frac{b^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= \frac{4a}{b}\frac{b^2}{2}\frac{\pi}{2} = \pi ab$$

8.2.1 *The area of the region bounded by a curve and a line*

In this subsection, we will find the area of the region bounded by a line and a circle, a line and a parabola, a line and an ellipse. Equations of above mentioned curves will be in their standard forms only as the cases in other forms go beyond the scope of this textbook.

Example 3 Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution Since the given curve represented by the equation $y = x^2$ is a parabola symmetrical about y-axis only, therefore, from Fig 8.9, the required area of the region AOBA is given by

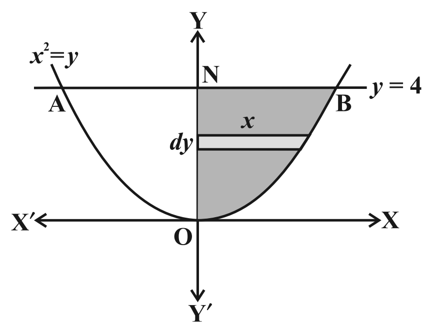


Fig 8.9

$$2\int_0^4 x\,dy =$$

$$2\begin{pmatrix}\text{area of the region BONB bounded by curve, } y-\text{axis} \\ \text{and the lines } y=0 \text{ and } y=4\end{pmatrix}$$

$$= 2\int_0^4 \sqrt{y}\,dy = 2\times\frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^4 = \frac{4}{3}\times 8 = \frac{32}{3} \quad \text{(Why?)}$$

Here, we have taken horizontal strips as indicated in the Fig 8.9.

Alternatively, we may consider the vertical strips like PQ as shown in the Fig 8.10 to obtain the area of the region AOBA. To this end, we solve the equations $x^2 = y$ and $y = 4$ which gives $x = -2$ and $x = 2$.

Thus, the region AOBA may be stated as the region bounded by the curve $y = x^2$, $y = 4$ and the ordinates $x = -2$ and $x = 2$.

Therefore, the area of the region AOBA

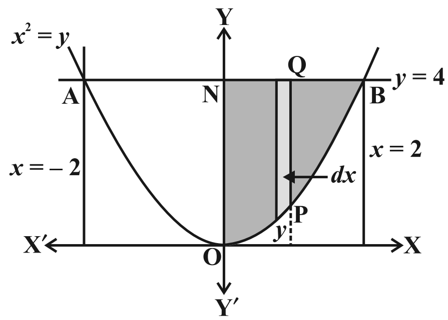


Fig 8.10

$$= \int_{-2}^{2} y\,dx$$

$[y = (y\text{-coordinate of Q}) - (y\text{-coordinate of P}) = 4 - x^2]$

$$= 2\int_{0}^{2}\left(4 - x^2\right)dx \qquad \text{(Why?)}$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2\left[4\times 2 - \frac{8}{3}\right] = \frac{32}{3}$$

Remark From the above examples, it is inferred that we can consider either vertical strips or horizontal strips for calculating the area of the region. Henceforth, we shall consider either of these two, most preferably vertical strips.

Example 4 Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$, and the circle $x^2 + y^2 = 32$.

Solution The given equations are

$$y = x \qquad \text{... (1)}$$

and $\qquad x^2 + y^2 = 32 \qquad \text{... (2)}$

Solving (1) and (2), we find that the line and the circle meet at B(4, 4) in the first quadrant (Fig 8.11). Draw perpendicular BM to the x-axis.

Therefore, the required area = area of the region OBMO + area of the region BMAB.

Now, the area of the region OBMO

$$= \int_0^4 y\,dx = \int_0^4 x\,dx \qquad \text{... (3)}$$

$$= \frac{1}{2}\left[x^2\right]_0^4 = 8$$

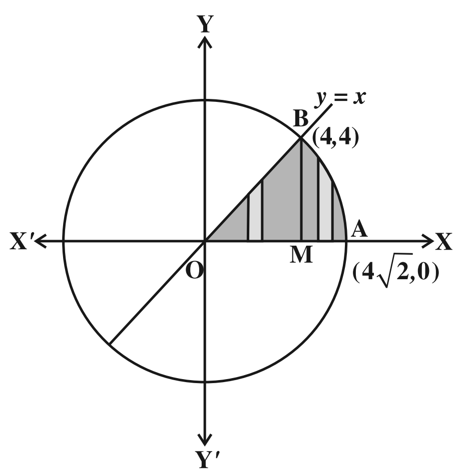


Fig 8.11

Again, the area of the region BMAB

$$= \int_4^{4\sqrt{2}} y\,dx = \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$= \left[\frac{1}{2} x\sqrt{32 - x^2} + \frac{1}{2} \times 32 \times \sin^{-1} \frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{1}{2} 4\sqrt{2} \times 0 + \frac{1}{2} \times 32 \times \sin^{-1} 1\right) - \left(\frac{4}{2}\sqrt{32-16} + \frac{1}{2} \times 32 \times \sin^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= 8\,\pi - (8 + 4\pi) = 4\pi - 8 \qquad \ldots (4)$$

Adding (3) and (4), we get, the required area = 4π.

Example 5 Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$, where, $b^2 = a^2\,(1 - e^2)$ and $e < 1$.

Solution The required area (Fig 8.12) of the region BOB′RFSB is enclosed by the ellipse and the lines $x = 0$ and $x = ae$.

Note that the area of the region BOB′RFSB

$$= 2\int_0^{ae} y\,dx = 2\frac{b}{a}\int_0^{ae} \sqrt{a^2 - x^2}\,dx$$

$$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{2b}{2a}\left[ae\sqrt{a^2 - a^2e^2} + a^2 \sin^{-1} e\right]$$

$$= ab\left[e\sqrt{1 - e^2} + \sin^{-1} e\right]$$

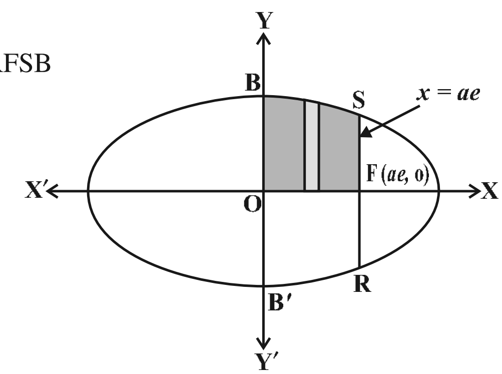


Fig 8.12

EXERCISE 8.1

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.
2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

4. Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

5. Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}\,y$ and the circle $x^2 + y^2 = 4$.

7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$.

8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Choose the correct answer in the following Exercises 12 and 13.

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

13. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(A) 2 (B) $\frac{9}{4}$ (C) $\frac{9}{3}$ (D) $\frac{9}{2}$

8.3 Area between Two Curves

Intuitively, true in the sense of Leibnitz, integration is the act of calculating the area by cutting the region into a large number of small strips of elementary area and then adding up these elementary areas. Suppose we are given two curves represented by $y = f(x)$, $y = g(x)$, where $f(x) \geq g(x)$ in $[a, b]$ as shown in Fig 8.13. Here the points of intersection of these two curves are given by $x = a$ and $x = b$ obtained by taking common values of y from the given equation of two curves.

For setting up a formula for the integral, it is convenient to take elementary area in the form of vertical strips. As indicated in the Fig 8.13, elementary strip has height

$f(x) - g(x)$ and width dx so that the elementary area

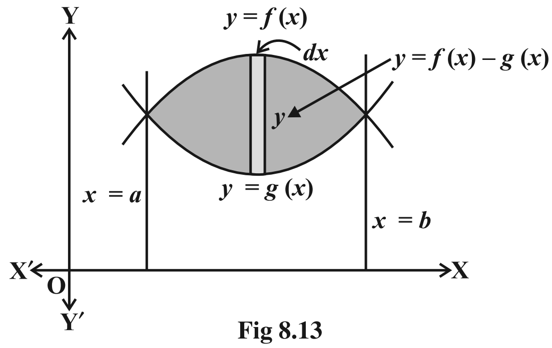


Fig 8.13

$dA = [f(x) - g(x)]\, dx$, and the total area A can be taken as

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Alternatively,

A = [area bounded by $y = f(x)$, x-axis and the lines $x = a$, $x = b$]
– [area bounded by $y = g(x)$, x-axis and the lines $x = a$, $x = b$]

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \left[f(x) - g(x)\right]dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b]$$

If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, where $a < c < b$ as shown in the Fig 8.14, then the area of the regions bounded by curves can be written as

Total Area = Area of the region ACBDA + Area of the region BPRQB

$$= \int_a^c \left[f(x) - g(x)\right]dx + \int_c^b \left[g(x) - f(x)\right]dx$$

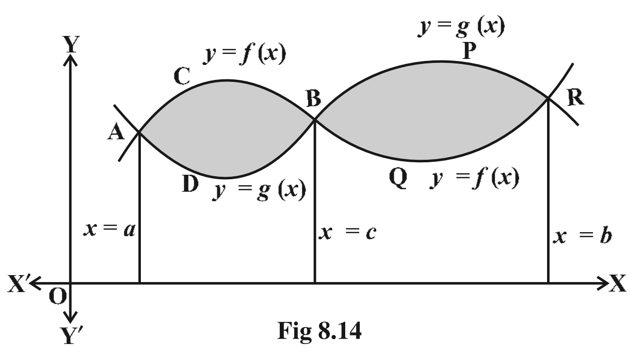


Fig 8.14

Example 6 Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Solution The point of intersection of these two parabolas are O (0, 0) and A (1, 1) as shown in the Fig 8.15.

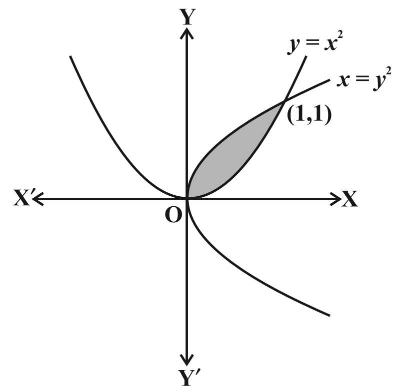


Fig 8.15

Here, we can set $y^2 = x$ or $y = \sqrt{x} = f(x)$ and $y = x^2 = g(x)$, where, $f(x) \geq g(x)$ in [0, 1].

Therefore, the required area of the shaded region

$$= \int_0^1 [f(x) - g(x)]\, dx$$

$$= \int_0^1 \left[\sqrt{x} - x^2\right] dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Example 7 Find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Solution The given equation of the circle $x^2 + y^2 = 8x$ can be expressed as $(x - 4)^2 + y^2 = 16$. Thus, the centre of the circle is (4, 0) and radius is 4. Its intersection with the parabola $y^2 = 4x$ gives

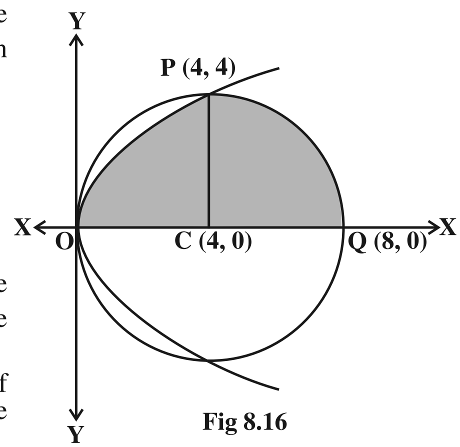


Fig 8.16

$$x^2 + 4x = 8x$$

or $$x^2 - 4x = 0$$

or $$x(x - 4) = 0$$

or $$x = 0, x = 4$$

Thus, the points of intersection of these two curves are O(0, 0) and P(4,4) above the x-axis.

From the Fig 8.16, the required area of the region OPQCO included between these two curves above x-axis is

= (area of the region OCPO) + (area of the region PCQP)

$$= \int_0^4 y\,dx + \int_4^8 y\,dx$$

$$= 2\int_0^4 \sqrt{x}\,dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\,dx \quad \text{(Why?)}$$

$$= 2 \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4 + \int_0^4 \sqrt{4^2 - t^2}\,dt, \text{ where, } x - 4 = t \quad \text{(Why?)}$$

$$= \frac{32}{3} + \left[\frac{t}{2}\sqrt{4^2 - t^2} + \frac{1}{2} \times 4^2 \times \sin^{-1}\frac{t}{4}\right]_0^4$$

$$= \frac{32}{3} + \left[\frac{4}{2} \times 0 + \frac{1}{2} \times 4^2 \times \sin^{-1} 1\right] = \frac{32}{3} + \left[0 + 8 \times \frac{\pi}{2}\right] = \frac{32}{3} + 4\pi = \frac{4}{3}(8 + 3\pi)$$

Example 8 In Fig 8.17, AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that OA = 2 and OB = 6. Find the area between the arc AB and the chord AB.

Solution Given equation of the ellipse $9x^2 + y^2 = 36$ can be expressed as $\frac{x^2}{4} + \frac{y^2}{36} = 1$ or $\frac{x^2}{2^2} + \frac{y^2}{6^2} = 1$ and hence, its shape is as given in Fig 8.17.

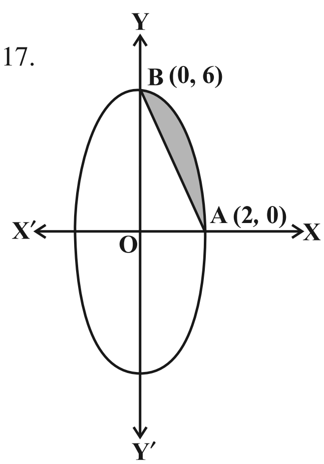


Fig 8.17

Accordingly, the equation of the chord AB is

$$y - 0 = \frac{6-0}{0-2}(x-2)$$

or $\quad y = -3(x - 2)$

or $\quad y = -3x + 6$

Area of the shaded region as shown in the Fig 8.17.

$$= 3\int_0^2 \sqrt{4 - x^2}\,dx - \int_0^2 (6 - 3x)\,dx \quad \text{(Why?)}$$

$$= 3\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[6x - \frac{3x^2}{2}\right]_0^2$$

$$= 3\left[\frac{2}{2} \times 0 + 2\sin^{-1}(1)\right] - \left[12 - \frac{12}{2}\right] = 3 \times 2 \times \frac{\pi}{2} - 6 = 3\pi - 6$$

Example 9 Using integration find the area of region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1).

Solution Let A (1, 0), B (2, 2) and C (3, 1) be the vertices of a triangle ABC (Fig 8.18).

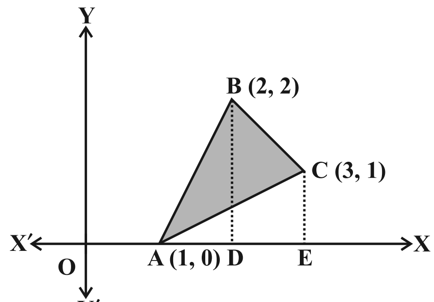


Fig 8.18

Area of ΔABC

= Area of ΔABD + Area of trapezium BDEC – Area of ΔAEC

Now equation of the sides AB, BC and CA are given by

$$y = 2(x-1),\ y = 4 - x,\ y = \frac{1}{2}(x-1), \text{ respectively.}$$

Hence, area of Δ ABC $= \int_1^2 2(x-1)\,dx + \int_2^3 (4-x)\,dx - \int_1^3 \frac{x-1}{2}\,dx$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right] + \left[\left(4\times 3 - \frac{3^2}{2}\right) - \left(4\times 2 - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{3}{2}$$

Example 10 Find the area of the region enclosed between the two circles: $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Solution Equations of the given circles are

$$x^2 + y^2 = 4 \quad \text{... (1)}$$

and $$(x-2)^2 + y^2 = 4 \quad \text{... (2)}$$

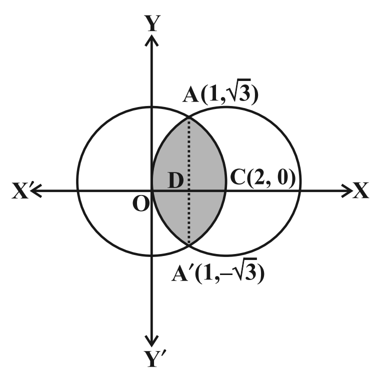


Fig 8.19

Equation (1) is a circle with centre O at the origin and radius 2. Equation (2) is a circle with centre C (2, 0) and radius 2. Solving equations (1) and (2), we have

$$(x-2)^2 + y^2 = x^2 + y^2$$

or $$x^2 - 4x + 4 + y^2 = x^2 + y^2$$

or $x = 1$ which gives $y = \pm\sqrt{3}$

Thus, the points of intersection of the given circles are A(1, $\sqrt{3}$) and A′(1, $-\sqrt{3}$) as shown in the Fig 8.19.

Required area of the enclosed region OACA′O between circles

$= 2$ [area of the region ODCAO] (Why?)

$= 2$ [area of the region ODAO + area of the region DCAD]

$$= 2\left[\int_0^1 y\,dx + \int_1^2 y\,dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right] \qquad \text{(Why?)}$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+\ 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] + \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

EXERCISE 8.2

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.
2. Find the area bounded by curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.
3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.
4. Using integration find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$.
5. Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Choose the correct answer in the following exercises 6 and 7.

6. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(A) $2(\pi - 2)$ (B) $\pi - 2$ (C) $2\pi - 1$ (D) $2(\pi + 2)$

7. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

Miscellaneous Examples

Example 11 Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.

Solution From Fig 8.20, the vertex of the parabola $y^2 = 4ax$ is at origin (0, 0). The equation of the latus rectum LSL′ is $x = a$. Also, parabola is symmetrical about the x-axis.

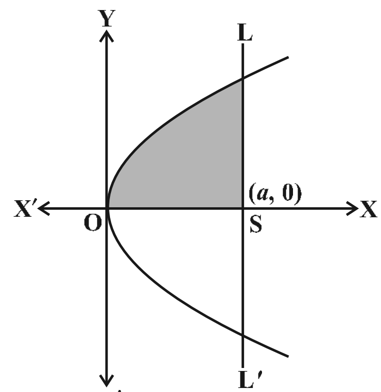


Fig 8.20

The required area of the region OLL′O

$= 2$ (area of the region OLSO)

$$= 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2\times 2\sqrt{a}\int_0^a \sqrt{x}\,dx$$

$$= 4\sqrt{a}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\left[a^{\frac{3}{2}}\right] = \frac{8}{3}a^2$$

Example 12 Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.

Solution As shown in the Fig 8.21, the line $y = 3x + 2$ meets x-axis at $x = \frac{-2}{3}$ and its graph lies below x-axis for $x \in \left(-1, \frac{-2}{3}\right)$ and above x-axis for $x \in \left(\frac{-2}{3}, 1\right)$.

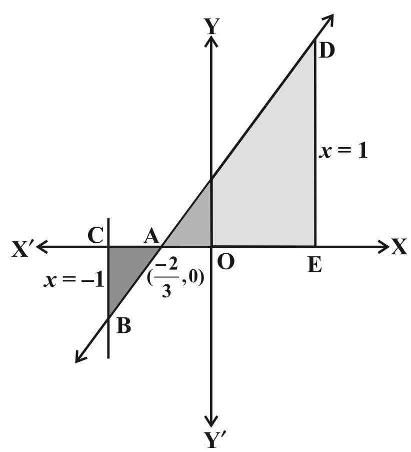


Fig 8.21

The required area = Area of the region ACBA + Area of the region ADEA

$$= \left|\int_{-1}^{\frac{-2}{3}} (3x+2)dx\right| + \int_{\frac{-2}{3}}^{1} (3x+2)dx$$

$$= \left|\left[\frac{3x^2}{2}+2x\right]_{-1}^{\frac{-2}{3}}\right| + \left[\frac{3x^2}{2}+2x\right]_{\frac{-2}{3}}^{1} = \frac{1}{6}+\frac{25}{6}=\frac{13}{3}$$

Example 13 Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.

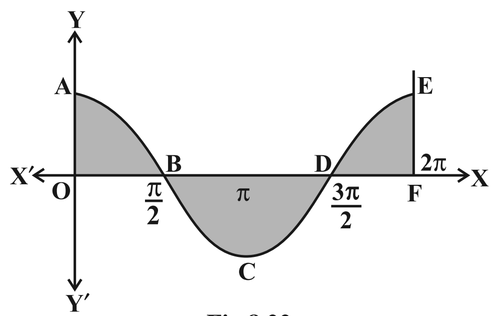


Fig 8.22

Solution From the Fig 8.22, the required area = area of the region OABO + area of the region BCDB + area of the region DEFD.

Thus, we have the required area

$$= \int_0^{\frac{\pi}{2}} \cos x\, dx + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx\right| + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\, dx$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}} + \left|\left[\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + \left[\sin x\right]_{\frac{3\pi}{2}}^{2\pi}$$

$$= 1 + 2 + 1 = 4$$

Example 13 Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.

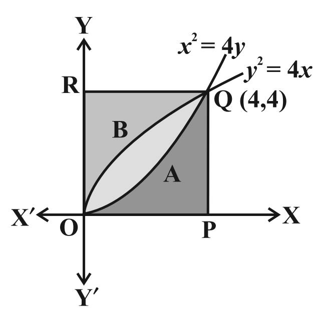


Fig 8.23

Solution Note that the point of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ are (0, 0) and (4, 4) as shown in the Fig 8.23.

Now, the area of the region OAQBO bounded by curves $y^2 = 4x$ and $x^2 = 4y$.

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2 \times \frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \qquad \text{... (1)}$$

Again, the area of the region OPQAO bounded by the curves $x^2 = 4y$, $x = 0$, $x = 4$ and x-axis

$$= \int_0^4 \frac{x^2}{4} dx = \frac{1}{12}\left[x^3\right]_0^4 = \frac{16}{3} \qquad \text{... (2)}$$

Similarly, the area of the region OBQRO bounded by the curve $y^2 = 4x$, y-axis, $y = 0$ and $y = 4$

$$= \int_0^4 x dy = \int_0^4 \frac{y^2}{4} dy = \frac{1}{12}\left[y^3\right]_0^4 = \frac{16}{3} \qquad \text{... (3)}$$

From (1), (2) and (3), it is concluded that the area of the region OAQBO = area of the region OPQAO = area of the region OBQRO, i.e., area bounded by parabolas $y^2 = 4x$ and $x^2 = 4y$ divides the area of the square in three equal parts.

Example 14 Find the area of the region

$\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

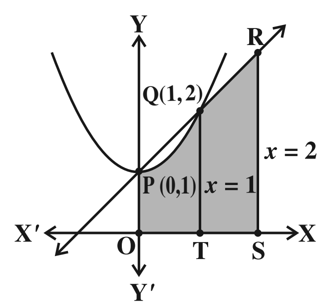


Fig 8.24

Solution Let us first sketch the region whose area is to be found out. This region is the intersection of the following regions.

$A_1 = \{(x, y) : 0 \le y \le x^2 + 1\}$,

$A_2 = \{(x, y) : 0 \le y \le x + 1\}$

and $A_3 = \{(x, y) : 0 \le x \le 2\}$

The points of intersection of $y = x^2 + 1$ and $y = x + 1$ are points P(0, 1) and Q(1, 2). From the Fig 8.24, the required region is the shaded region OPQRSTO whose area

= area of the region OTQPO + area of the region TSRQT

$$= \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx \qquad \text{(Why?)}$$

$$= \left[\left(\frac{x^3}{3} + x\right)\right]_0^1 + \left[\left(\frac{x^2}{2} + x\right)\right]_1^2$$

$$= \left[\left(\frac{1}{3} + 1\right) - 0\right] + \left[(2+2) - \left(\frac{1}{2} + 1\right)\right] = \frac{23}{6}$$

Miscellaneous Exercise on Chapter 8

1. Find the area under the given curves and given lines:

(i) $y = x^2$, $x = 1$, $x = 2$ and x-axis

(ii) $y = x^4$, $x = 1$, $x = 5$ and x-axis

2. Find the area between the curves $y = x$ and $y = x^2$.

3. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

4. Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3|\, dx$.

5. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

6. Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

7. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

8. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

9. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.

10. Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and the x-axis.

11. Using the method of integration find the area bounded by the curve $|x| + |y| = 1$.

[Hint: The required region is bounded by lines $x + y = 1$, $x - y = 1$, $-x + y = 1$ and $-x - y = 1$].

12. Find the area bounded by curves $\{(x, y) : y \geq x^2 \text{ and } y = |x|\}$.

13. Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are A(2, 0), B (4, 5) and C (6, 3).

14. Using the method of integration find the area of the region bounded by lines:

$2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$

15. Find the area of the region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$

Choose the correct answer in the following Exercises from 16 to 20.

16. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(A) -9 (B) $\frac{-15}{4}$ (C) $\frac{15}{4}$ (D) $\frac{17}{4}$

17. The area bounded by the curve $y = x\,|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

[Hint : $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$].

18. The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

(A) $\frac{4}{3}(4\pi - \sqrt{3})$ (B) $\frac{4}{3}(4\pi + \sqrt{3})$ (C) $\frac{4}{3}(8\pi - \sqrt{3})$ (D) $\frac{4}{3}(8\pi + \sqrt{3})$

19. The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \leq x \leq \frac{\pi}{2}$ is

(A) $2(\sqrt{2} - 1)$ (B) $\sqrt{2} - 1$ (C) $\sqrt{2} + 1$ (D) $\sqrt{2}$

Summary

- The area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by the formula: $\text{Area} = \int_a^b y\,dx = \int_a^b f(x)\,dx$.
- The area of the region bounded by the curve $x = \phi(y)$, y-axis and the lines $y = c$, $y = d$ is given by the formula: $\text{Area} = \int_c^d x\,dy = \int_c^d \phi(y)\,dy$.

- The area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by the formula,

$$\text{Area} = \int_a^b \left[f(x) - g(x)\right] dx \text{, where, } f(x) \geq g(x) \text{ in } [a, b]$$

- If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then

$$\text{Area} = \int_a^c \left[f(x) - g(x)\right] dx + \int_c^b \left[g(x) - f(x)\right] dx .$$

Historical Note

The origin of the Integral Calculus goes back to the early period of development of Mathematics and it is related to the method of exhaustion developed by the mathematicians of ancient Greece. This method arose in the solution of problems on calculating areas of plane figures, surface areas and volumes of solid bodies etc. In this sense, the method of exhaustion can be regarded as an early method of integration. The greatest development of method of exhaustion in the early period was obtained in the works of Eudoxus (440 B.C.) and Archimedes (300 B.C.)

Systematic approach to the theory of Calculus began in the 17th century. In 1665, Newton began his work on the Calculus described by him as the theory of fluxions and used his theory in finding the tangent and radius of curvature at any point on a curve. Newton introduced the basic notion of inverse function called the anti derivative (indefinite integral) or the inverse method of tangents.

During 1684-86, Leibnitz published an article in the *Acta Eruditorum* which he called *Calculas summatorius*, since it was connected with the summation of a number of infinitely small areas, whose sum, he indicated by the symbol '$\int$'. In 1696, he followed a suggestion made by J. Bernoulli and changed this article to Calculus integrali. This corresponded to Newton's inverse method of tangents.

Both Newton and Leibnitz adopted quite independent lines of approach which was radically different. However, respective theories accomplished results that were practically identical. Leibnitz used the notion of definite integral and what is quite certain is that he first clearly appreciated tie up between the antiderivative and the definite integral.

Conclusively, the fundamental concepts and theory of Integral Calculus and primarily its relationships with Differential Calculus were developed in the work of P.de Fermat, I. Newton and G. Leibnitz at the end of 17th century.

However, this justification by the concept of limit was only developed in the works of A.L. Cauchy in the early 19th century. Lastly, it is worth mentioning the following quotation by Lie Sophie's:

"It may be said that the conceptions of differential quotient and integral which in their origin certainly go back to Archimedes were introduced in Science by the investigations of Kepler, Descartes, Cavalieri, Fermat and Wallis .... The discovery that differentiation and integration are inverse operations belongs to Newton and Leibnitz".

Chapter 13

PROBABILITY

❖ *The theory of probabilities is simply the Science of logic quantitatively treated. – C.S. PEIRCE* ❖

13.1 Introduction

Pierre de Fermat
(1601-1665)

In earlier Classes, we have studied the probability as a measure of uncertainty of events in a random experiment. We discussed the axiomatic approach formulated by Russian Mathematician, A.N. Kolmogorov (1903-1987) and treated probability as a function of outcomes of the experiment. We have also established equivalence between the axiomatic theory and the classical theory of probability in case of equally likely outcomes. On the basis of this relationship, we obtained probabilities of events associated with discrete sample spaces. We have also studied the addition rule of probability. In this chapter, we shall discuss the important concept of conditional probability of an event given that another event has occurred, which will be helpful in understanding the Bayes' theorem, multiplication rule of probability and independence of events. We shall also learn an important concept of random variable and its probability distribution and also the mean and variance of a probability distribution. In the last section of the chapter, we shall study an important discrete probability distribution called Binomial distribution. Throughout this chapter, we shall take up the experiments having equally likely outcomes, unless stated otherwise.

13.2 Conditional Probability

Uptill now in probability, we have discussed the methods of finding the probability of events. If we have two events from the same sample space, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur.

Consider the experiment of tossing three fair coins. The sample space of the experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Since the coins are fair, we can assign the probability $\frac{1}{8}$ to each sample point. Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'. Then

$$E = \{HHH, HHT, HTH, THH\}$$

and $$F = \{THH, THT, TTH, TTT\}$$

Therefore $$P(E) = P(\{HHH\}) + P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2} \text{ (Why ?)}$$

and $$P(F) = P(\{THH\}) + P(\{THT\}) + P(\{TTH\}) + P(\{TTT\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

Also $$E \cap F = \{THH\}$$

with $$P(E \cap F) = P(\{THH\}) = \frac{1}{8}$$

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E? With the information of occurrence of F, we are sure that the cases in which first coin does not result into a tail should not be considered while finding the probability of E. This information reduces our sample space from the set S to its subset F for the event E. In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favourable to the occurrence of the event F.

Now, the sample point of F which is favourable to event E is THH.

Thus, Probability of E considering F as the sample space = $\frac{1}{4}$,

or Probability of E given that the event F has occurred = $\frac{1}{4}$

This probability of the event E is called the *conditional probability of E given that F has already occurred*, and is denoted by P (E|F).

Thus $$P(E|F) = \frac{1}{4}$$

Note that the elements of F which favour the event E are the common elements of E and F, i.e. the sample points of $E \cap F$.

Thus, we can also write the conditional probability of E given that F has occurred as

$$P(E|F) = \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$$

$$= \frac{n(E \cap F)}{n(F)}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that P(E|F) can also be written as

$$P(E|F) = \frac{\frac{n(E \cap F)}{n(S)}}{\frac{n(F)}{n(S)}} = \frac{P(E \cap F)}{P(F)} \qquad \text{... (1)}$$

Note that (1) is valid only when $P(F) \neq 0$ i.e., $F \neq \phi$ (Why?)

Thus, we can define the conditional probability as follows :

Definition 1 If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e. P (E|F) is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \text{ provided } P(F) \neq 0$$

13.2.1 *Properties of conditional probability*

Let E and F be events of a sample space S of an experiment, then we have

Property 1 $P(S|F) = P(F|F) = 1$

We know that

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Also $$P(F|F) = \frac{P(F \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Thus $$P(S|F) = P(F|F) = 1$$

Property 2 *If* A *and* B *are any two events of a sample space* S *and* F *is an event of* S *such that* $P(F) \neq 0$*, then*

$$P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$$

In particular, if A *and* B *are disjoint events, then*

$$P((A \cup B)|F) = P(A|F) + P(B|F)$$

We have

$$P((A \cup B)|F) = \frac{P[(A \cup B) \cap F]}{P(F)}$$

$$= \frac{P[(A \cap F) \cup (B \cap F)]}{P(F)}$$

(by distributive law of union of sets over intersection)

$$= \frac{P(A \cap F) + P(B \cap F) - P(A \cap B \cap F)}{P(F)}$$

$$= \frac{P(A \cap F)}{P(F)} + \frac{P(B \cap F)}{P(F)} - \frac{P[(A \cap B) \cap F]}{P(F)}$$

$$= P(A|F) + P(B|F) - P((A \cap B)|F)$$

When A and B are disjoint events, then

$$P((A \cap B)|F) = 0$$

$$\Rightarrow \qquad P((A \cup B)|F) = P(A|F) + P(B|F)$$

Property 3 $P(E'|F) = 1 - P(E|F)$

From Property 1, we know that $P(S|F) = 1$

$\Rightarrow \qquad P(E \cup E'|F) = 1 \qquad$ since $S = E \cup E'$

$\Rightarrow \qquad P(E|F) + P(E'|F) = 1 \qquad$ since E and E′ are disjoint events

Thus, $\qquad P(E'|F) = 1 - P(E|F)$

Let us now take up some examples.

Example 1 If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, evaluate $P(A|B)$.

Solution We have $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{13}}{\frac{9}{13}} = \dfrac{4}{9}$

Example 2 A family has two children. What is the probability that both the children are boys given that at least one of them is a boy ?

Solution Let b stand for boy and g for girl. The sample space of the experiment is

$$S = \{(b, b), (g, b), (b, g), (g, g)\}$$

Let E and F denote the following events :

E : 'both the children are boys'

F : 'at least one of the child is a boy'

Then $\qquad E = \{(b,b)\}$ and $F = \{(b,b), (g,b), (b,g)\}$

Now $\qquad E \cap F = \{(b,b)\}$

Thus $\qquad P(F) = \frac{3}{4}$ and $P(E \cap F) = \frac{1}{4}$

Therefore $\qquad P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Example 3 Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

Solution Let A be the event 'the number on the card drawn is even' and B be the event 'the number on the card drawn is greater than 3'. We have to find P(A|B).

Now, the sample space of the experiment is $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Then $\qquad A = \{2, 4, 6, 8, 10\}, \; B = \{4, 5, 6, 7, 8, 9, 10\}$

and $\qquad A \cap B = \{4, 6, 8, 10\}$

Also $\qquad P(A) = \frac{5}{10}, P(B) = \frac{7}{10}$ and $P(A \cap B) = \frac{4}{10}$

Then $\qquad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{10}}{\frac{7}{10}} = \frac{4}{7}$

Example 4 In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in Class XII given that the chosen student is a girl?

Solution Let E denote the event that a student chosen randomly studies in Class XII and F be the event that the randomly chosen student is a girl. We have to find P (E|F).

Now $\quad P(F) = \frac{430}{1000} = 0.43$ and $P(E \cap F) = \frac{43}{1000} = 0.043$ (Why?)

Then $\quad P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.043}{0.43} = 0.1$

Example 5 A die is thrown three times. Events A and B are defined as below:

A : 4 on the third throw

B : 6 on the first and 5 on the second throw

Find the probability of A given that B has already occurred.

Solution The sample space has 216 outcomes.

Now
$$A = \begin{Bmatrix} (1,1,4) & (1,2,4) & ... & (1,6,4) & (2,1,4) & (2,2,4) & ... & (2,6,4) \\ (3,1,4) & (3,2,4) & ... & (3,6,4) & (4,1,4) & (4,2,4) & ... & (4,6,4) \\ (5,1,4) & (5,2,4) & ... & (5,6,4) & (6,1,4) & (6,2,4) & ... & (6,6,4) \end{Bmatrix}$$

$$B = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$$

and $\quad A \cap B = \{(6,5,4)\}$.

Now $\quad P(B) = \frac{6}{216}$ and $P(A \cap B) = \frac{1}{216}$

Then $\quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{216}}{\frac{6}{216}} = \frac{1}{6}$

Example 6 A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Solution Let E be the event that 'number 4 appears at least once' and F be the event that 'the sum of the numbers appearing is 6'.

Then, $\quad E = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (5,4), (6,4)\}$

and $\quad F = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

We have $\quad P(E) = \frac{11}{36}$ and $P(F) = \frac{5}{36}$

Also $\quad E \cap F = \{(2,4), (4,2)\}$

Therefore $\quad P(E \cap F) = \dfrac{2}{36}$

Hence, the required probability

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

For the conditional probability discussed above, we have considered the elementary events of the experiment to be equally likely and the corresponding definition of the probability of an event was used. However, the same definition can also be used in the general case where the elementary events of the sample space are not equally likely, the probabilities $P(E \cap F)$ and $P(F)$ being calculated accordingly. Let us take up the following example.

Example 7 Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Find the conditional probability of the event that 'the die shows a number greater than 4' given that 'there is at least one tail'.

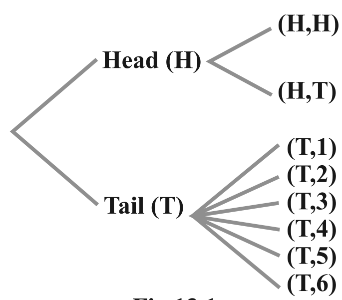


Fig 13.1

Solution The outcomes of the experiment can be represented in following diagrammatic manner called the 'tree diagram'.

The sample space of the experiment may be described as

$$S = \{(H,H), (H,T), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

where (H, H) denotes that both the tosses result into head and (T, i) denote the first toss result into a tail and the number i appeared on the die for i = 1,2,3,4,5,6.

Thus, the probabilities assigned to the 8 elementary events

(H, H), (H, T), (T, 1), (T, 2), (T, 3) (T, 4), (T, 5), (T, 6)

are $\dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{12}, \dfrac{1}{12}, \dfrac{1}{12}, \dfrac{1}{12}, \dfrac{1}{12}, \dfrac{1}{12}$ respectively which is clear from the Fig 13.2.

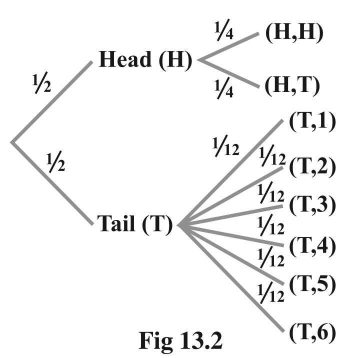


Fig 13.2

Let F be the event that 'there is at least one tail' and E be the event 'the die shows a number greater than 4'. Then

$$F = \{(H,T), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

$$E = \{(T,5), (T,6)\} \text{ and } E \cap F = \{(T,5), (T,6)\}$$

Now
$$P(F) = P(\{(H,T)\}) + P(\{(T,1)\}) + P(\{(T,2)\}) + P(\{(T,3)\}) + P(\{(T,4)\}) + P(\{(T,5)\}) + P(\{(T,6)\})$$

$$= \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4}$$

and
$$P(E \cap F) = P(\{(T,5)\}) + P(\{(T,6)\}) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

Hence
$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{2}{9}$$

EXERCISE 13.1

1. Given that E and F are events such that P(E) = 0.6, P(F) = 0.3 and $P(E \cap F) = 0.2$, find P(E|F) and P(F|E)
2. Compute P(A|B), if P(B) = 0.5 and $P(A \cap B) = 0.32$
3. If P(A) = 0.8, P(B) = 0.5 and P(B|A) = 0.4, find
   (i) $P(A \cap B)$ (ii) P(A|B) (iii) $P(A \cup B)$
4. Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$
5. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find
   (i) $P(A \cap B)$ (ii) P(A|B) (iii) P(B|A)

Determine P(E|F) in Exercises 6 to 9.

6. A coin is tossed three times, where
   (i) E : head on third toss , F : heads on first two tosses
   (ii) E : at least two heads , F : at most two heads
   (iii) E : at most two tails , F : at least one tail

7. Two coins are tossed once, where

(i) E : tail appears on one coin, F : one coin shows head

(ii) E : no tail appears, F : no head appears

8. A die is thrown three times,

E : 4 appears on the third toss, F : 6 and 5 appears respectively on first two tosses

9. Mother, father and son line up at random for a family picture

E : son on one end, F : father in middle

10. A black and a red dice are rolled.

(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

11. A fair die is rolled. Consider events $E = \{1,3,5\}$, $F = \{2,3\}$ and $G = \{2,3,4,5\}$ Find

(i) $P(E|F)$ and $P(F|E)$ (ii) $P(E|G)$ and $P(G|E)$

(iii) $P((E \cup F)|G)$ and $P((E \cap F)|G)$

12. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

13. An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

14. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

15. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

In each of the Exercises 16 and 17 choose the correct answer:

16. If $P(A) = \frac{1}{2}$, $P(B) = 0$, then $P(A|B)$ is

(A) 0 (B) $\frac{1}{2}$

(C) not defined (D) 1

17. If A and B are events such that $P(A|B) = P(B|A)$, then

(A) $A \subset B$ but $A \neq B$ (B) $A = B$

(C) $A \cap B = \phi$ (D) $P(A) = P(B)$

13.3 Multiplication Theorem on Probability

Let E and F be two events associated with a sample space S. Clearly, the set $E \cap F$ denotes the event that both E and F have occurred. In other words, $E \cap F$ denotes the simultaneous occurrence of the events E and F. The event $E \cap F$ is also written as EF.

Very often we need to find the probability of the event EF. For example, in the experiment of drawing two cards one after the other, we may be interested in finding the probability of the event 'a king and a queen'. The probability of event EF is obtained by using the conditional probability as obtained below :

We know that the conditional probability of event E given that F has occurred is denoted by $P(E|F)$ and is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$$

From this result, we can write

$$P(E \cap F) = P(F) \ . \ P(E|F) \qquad \ldots (1)$$

Also, we know that

$$P(F|E) = \frac{P(F \cap E)}{P(E)}, P(E) \neq 0$$

or

$$P(F|E) = \frac{P(E \cap F)}{P(E)} \quad (\text{since } E \cap F = F \cap E)$$

Thus,

$$P(E \cap F) = P(E).\ P(F|E) \qquad \ldots. (2)$$

Combining (1) and (2), we find that

$$P(E \cap F) = P(E)\ P(F|E)$$
$$= P(F)\ P(E|F) \text{ provided } P(E) \neq 0 \text{ and } P(F) \neq 0.$$

The above result is known as the *multiplication rule of probability*.

Let us now take up an example.

Example 8 An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

Solution Let E and F denote respectively the events that first and second ball drawn are black. We have to find $P(E \cap F)$ or $P(EF)$.

Now $\quad P(E) = P\text{ (black ball in first draw)} = \frac{10}{15}$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. $\quad P(F|E) = \frac{9}{14}$

By multiplication rule of probability, we have

$$P(E \cap F) = P(E)\ P(F|E)$$

$$= \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$$

Multiplication rule of probability for more than two events If E, F and G are three events of sample space, we have

$$P(E \cap F \cap G) = P(E)\ P(F|E)\ P(G|(E \cap F)) = P(E)\ P(F|E)\ P(G|EF)$$

Similarly, the multiplication rule of probability can be extended for four or more events.

The following example illustrates the extension of multiplication rule of probability for three events.

Example 9 Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

Solution Let K denote the event that the card drawn is king and A be the event that the card drawn is an ace. Clearly, we have to find P (KKA)

Now $\quad P(K) = \frac{4}{52}$

Also, P (K|K) is the probability of second king with the condition that one king has already been drawn. Now there are three kings in $(52 - 1) = 51$ cards.

Therefore $\quad P(K|K) = \frac{3}{51}$

Lastly, P(A|KK) is the probability of third drawn card to be an ace, with the condition that two kings have already been drawn. Now there are four aces in left 50 cards.

Therefore $$\text{P}(\text{A}|\text{KK}) = \frac{4}{50}$$

By multiplication law of probability, we have

$$\text{P}(\text{KKA}) = \text{P}(\text{K}) \quad \text{P}(\text{K}|\text{K}) \quad \text{P}(\text{A}|\text{KK})$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$$

13.4 Independent Events

Consider the experiment of drawing a card from a deck of 52 playing cards, in which the elementary events are assumed to be equally likely. If E and F denote the events 'the card drawn is a spade' and 'the card drawn is an ace' respectively, then

$$\text{P(E)} = \frac{13}{52} = \frac{1}{4} \text{ and P(F)} = \frac{4}{52} = \frac{1}{13}$$

Also E and F is the event ' the card drawn is the ace of spades' so that

$$\text{P}(\text{E} \cap \text{F}) = \frac{1}{52}$$

Hence $$\text{P(E|F)} = \frac{\text{P}(\text{E} \cap \text{F})}{\text{P(F)}} = \frac{\frac{1}{52}}{\frac{1}{13}} = \frac{1}{4}$$

Since P(E) = $\frac{1}{4}$ = P (E|F), we can say that the occurrence of event F has not affected the probability of occurrence of the event E.

We also have

$$\text{P(F|E)} = \frac{\text{P}(\text{E} \cap \text{F})}{\text{P(E)}} = \frac{\frac{1}{52}}{\frac{1}{4}} = \frac{1}{13} = \text{P(F)}$$

Again, $\text{P(F)} = \frac{1}{13}$ = P(F|E) shows that occurrence of event E has not affected the probability of occurrence of the event F.

Thus, E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other.

Such events are called *independent events*.

Definition 2 Two events E and F are said to be independent, if

$$P(F|E) = P(F) \text{ provided } P(E) \neq 0$$

and $$P(E|F) = P(E) \text{ provided } P(F) \neq 0$$

Thus, in this definition we need to have P (E) $\neq$ 0 and P(F) $\neq$ 0

Now, by the multiplication rule of probability, we have

$$P(E \cap F) = P(E) \,.\, P(F|E) \qquad \ldots (1)$$

If E and F are independent, then (1) becomes

$$P(E \cap F) = P(E) \,.\, P(F) \qquad \ldots (2)$$

Thus, using (2), the independence of two events is also defined as follows:

Definition 3 Let E and F be two events associated with the same random experiment, then E and F are said to be independent if

$$P(E \cap F) = P(E) \,.\, P(F)$$

Remarks

(i) Two events E and F are said to be dependent if they are not independent, i.e. if

$$P(E \cap F) \neq P(E) \,.\, P(F)$$

(ii) Sometimes there is a confusion between independent events and mutually exclusive events. Term 'independent' is defined in terms of 'probability of events' whereas mutually exclusive is defined in term of events (subset of sample space). Moreover, mutually exclusive events never have an outcome common, but independent events, may have common outcome. Clearly, 'independent' and 'mutually exclusive' do not have the same meaning.

In other words, two independent events having nonzero probabilities of occurrence can not be mutually exclusive, and conversely, i.e. two mutually exclusive events having nonzero probabilities of occurrence can not be independent.

(iii) Two experiments are said to be independent if for every pair of events E and F, where E is associated with the first experiment and F with the second experiment, the probability of the simultaneous occurrence of the events E and F when the two experiments are performed is the product of P(E) and P(F) calculated separately on the basis of two experiments, i.e., P (E $\cap$ F) = P (E) . P(F)

(iv) Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A)\, P(B)$$

$$P(A \cap C) = P(A)\, P(C)$$

$$P(B \cap C) = P(B)\, P(C)$$

and $$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

If at least one of the above is not true for three given events, we say that the events are not independent.

Example 10 A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent ?

Solution We know that the sample space is S = {1, 2, 3, 4, 5, 6}

Now $\quad E = \{3, 6\}, F = \{2, 4, 6\}$ and $E \cap F = \{6\}$

Then $\quad P(E) = \frac{2}{6} = \frac{1}{3},\ P(F) = \frac{3}{6} = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{6}$

Clearly $\quad P(E \cap F) = P(E).\ P(F)$

Hence $\quad$ E and F are independent events.

Example 11 An unbiased die is thrown twice. Let the event A be 'odd number on the first throw' and B the event 'odd number on the second throw'. Check the independence of the events A and B.

Solution If all the 36 elementary events of the experiment are considered to be equally likely, we have

$$P(A) = \frac{18}{36} = \frac{1}{2} \text{ and } P(B) = \frac{18}{36} = \frac{1}{2}$$

Also $\quad P(A \cap B) = P$ (odd number on both throws)

$$= \frac{9}{36} = \frac{1}{4}$$

Now $\quad P(A)\ P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Clearly $\quad P(A \cap B) = P(A) \times P(B)$

Thus, $\quad$ A and B are independent events

Example 12 Three coins are tossed simultaneously. Consider the event E 'three heads or three tails', F 'at least two heads' and G 'at most two heads'. Of the pairs (E,F), (E,G) and (F,G), which are independent? which are dependent?

Solution The sample space of the experiment is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Clearly $\quad E = \{HHH, TTT\}, F = \{HHH, HHT, HTH, THH\}$

and $\quad G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also $\quad E \cap F = \{HHH\}, E \cap G = \{TTT\}, F \cap G = \{HHT, HTH, THH\}$

Therefore $\quad P(E) = \frac{2}{8} = \frac{1}{4}, P(F) = \frac{4}{8} = \frac{1}{2}, P(G) = \frac{7}{8}$

and $\quad P(E \cap F) = \frac{1}{8}, P(E \cap G) = \frac{1}{8}, P(F \cap G) = \frac{3}{8}$

Also $\quad P(E) \,.\, P(F) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}, P(E) \cdot P(G) = \frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$

and $\quad P(F) \,.\, P(G) = \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$

Thus $\quad P(E \cap F) = P(E) \,.\, P(F)$

$P(E \cap G) \neq P(E) \,.\, P(G)$

and $\quad P(F \cap G) \neq P(F) \,.\, P(G)$

Hence, the events (E and F) are independent, and the events (E and G) and (F and G) are dependent.

Example 13 Prove that if E and F are independent events, then so are the events E and F′.

Solution Since E and F are independent, we have

$$P(E \cap F) = P(E) \,.\, P(F) \qquad ....(1)$$

From the venn diagram in Fig 13.3, it is clear that $E \cap F$ and $E \cap F'$ are mutually exclusive events and also $E = (E \cap F) \cup (E \cap F')$.

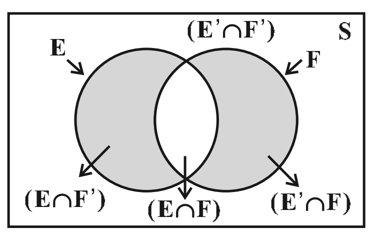


Fig 13.3

Therefore $\quad P(E) = P(E \cap F) + P(E \cap F')$

or $\quad P(E \cap F') = P(E) - P(E \cap F)$

$= P(E) - P(E) \,.\, P(F)$ (by (1))

$= P(E)\,(1 - P(F))$

$= P(E).\; P(F')$

Hence, E and F′ are independent

Note In a similar manner, it can be shown that if the events E and F are independent, then

(a) E' and F are independent,

(b) E' and F' are independent

Example 14 If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by $1 - P(A')\,P(B')$

Solution We have

$$
\begin{aligned}
P(\text{at least one of A and B}) &= P(A \cup B)\\
&= P(A) + P(B) - P(A \cap B)\\
&= P(A) + P(B) - P(A)\,P(B)\\
&= P(A) + P(B)\,[1 - P(A)]\\
&= P(A) + P(B).\,P(A')\\
&= 1 - P(A') + P(B)\,P(A')\\
&= 1 - P(A')\,[1 - P(B)]\\
&= 1 - P(A')\,P(B')
\end{aligned}
$$

EXERCISE 13.2

1. If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.
2. Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.
3. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.
4. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.
5. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even,' and B be the event, 'the number is red'. Are A and B independent?
6. Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

7. Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are (i) mutually exclusive (ii) independent.

8. Let A and B be independent events with P(A) = 0.3 and P(B) = 0.4. Find
   (i) $P(A \cap B)$ (ii) $P(A \cup B)$
   (iii) $P(A|B)$ (iv) $P(B|A)$

9. If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find P (not A and not B).

10. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and P(not A or not B) $= \frac{1}{4}$. State whether A and B are independent ?

11. Given two independent events A and B such that P(A) = 0.3, P(B) = 0.6. Find
    (i) P(A and B) (ii) P(A and not B)
    (iii) P(A or B) (iv) P(neither A nor B)

12. A die is tossed thrice. Find the probability of getting an odd number at least once.

13. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that
    (i) both balls are red.
    (ii) first ball is black and second is red.
    (iii) one of them is black and other is red.

14. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that
    (i) the problem is solved (ii) exactly one of them solves the problem.

15. One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?
    (i) E : 'the card drawn is a spade'
    F : 'the card drawn is an ace'
    (ii) E : 'the card drawn is black'
    F : 'the card drawn is a king'
    (iii) E : 'the card drawn is a king or queen'
    F : 'the card drawn is a queen or jack'.

16. In a hostel, 60% of the students read Hindi news paper, 40% read English news paper and 20% read both Hindi and English news papers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English news papers.

(b) If she reads Hindi news paper, find the probability that she reads English news paper.

(c) If she reads English news paper, find the probability that she reads Hindi news paper.

Choose the correct answer in Exercises 17 and 18.

17. The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{12}$ (D) $\frac{1}{36}$

18. Two events A and B will be independent, if

(A) A and B are mutually exclusive

(B) $P(A'B') = [1 - P(A)]\ [1 - P(B)]$

(C) $P(A) = P(B)$

(D) $P(A) + P(B) = 1$

13.5 Bayes' Theorem

Consider that there are two bags I and II. Bag I contains 2 white and 3 red balls and Bag II contains 4 white and 5 red balls. One ball is drawn at random from one of the bags. We can find the probability of selecting any of the bags (i.e. $\frac{1}{2}$) or probability of drawing a ball of a particular colour (say white) from a particular bag (say Bag I). In other words, we can find the probability that the ball drawn is of a particular colour, if we are given the bag from which the ball is drawn. But, can we find the probability that the ball drawn is from a particular bag (say Bag II), if the colour of the ball drawn is given? Here, we have to find the reverse probability of Bag II to be selected when an event occurred after it is known. Famous mathematician, John Bayes' solved the problem of finding reverse probability by using conditional probability. The formula developed by him is known as *'Bayes theorem'* which was published posthumously in 1763. Before stating and proving the Bayes' theorem, let us first take up a definition and some preliminary results.

13.5.1 *Partition of a sample space*

A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of the sample space S if

(a) $E_i \cap E_j = \phi,\ i \neq j,\ i, j = 1, 2, 3, ..., n$

(b) $E_1 \cup E_2 \cup ... \cup E_n = S$ and

(c) $P(E_i) > 0$ for all $i = 1, 2, ..., n$.

In other words, the events $E_1, E_2, ..., E_n$ represent a partition of the sample space S if they are pairwise disjoint, exhaustive and have nonzero probabilities.

As an example, we see that any nonempty event E and its complement E′ form a partition of the sample space S since they satisfy $E \cap E' = \phi$ and $E \cup E' = S$.

From the Venn diagram in Fig 13.3, one can easily observe that if E and F are any two events associated with a sample space S, then the set $\{E \cap F', E \cap F, E' \cap F, E' \cap F'\}$ is a partition of the sample space S. It may be mentioned that the partition of a sample space is not unique. There can be several partitions of the same sample space.

We shall now prove a theorem known as *Theorem of total probability*.

13.5.2 *Theorem of total probability*

Let $\{E_1, E_2,...,E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2,..., E_n$ has nonzero probability of occurrence. Let A be any event associated with S, then

$$P(A) = P(E_1)\ P(A|E_1) + P(E_2)\ P(A|E_2) + ... + P(E_n)\ P(A|E_n)$$

$$= \sum_{j=1}^{n} P(E_j)\, P(A|E_j)$$

Proof Given that $E_1, E_2,..., E_n$ is a partition of the sample space S (Fig 13.4). Therefore,

$$S = E_1 \cup E_2 \cup ... \cup E_n \quad ... (1)$$

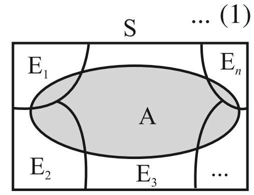


Fig 13.4

and $\quad E_i \cap E_j = \phi,\ i \neq j,\ i, j = 1, 2, ..., n$

Now, we know that for any event A,

$$A = A \cap S$$
$$= A \cap (E_1 \cup E_2 \cup ... \cup E_n)$$
$$= (A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)$$

Also $A \cap E_i$ and $A \cap E_j$ are respectively the subsets of E_i and E_j. We know that E_i and E_j are disjoint, for $i \neq j$, therefore, $A \cap E_i$ and $A \cap E_j$ are also disjoint for all $i \neq j,\ i, j = 1, 2, ..., n$.

Thus, $\quad P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup ..... \cup (A \cap E_n)]$

$$= P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_n)$$

Now, by multiplication rule of probability, we have

$$P(A \cap E_i) = P(E_i)\ P(A|E_i) \text{ as } P(E_i) \neq 0\ \forall i = 1,2,..., n$$

Therefore, $\quad P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + ... + P(E_n)P(A|E_n)$

or $$P(A) = \sum_{j=1}^{n} P(E_j)\,P(A|E_j)$$

Example 15 A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

Solution Let A be the event that the construction job will be completed on time, and B be the event that there will be a strike. We have to find P(A).
We have

$$P(B) = 0.65,\ P(\text{no strike}) = P(B') = 1 - P(B) = 1 - 0.65 = 0.35$$
$$P(A|B) = 0.32,\ P(A|B') = 0.80$$

Since events B and B′ form a partition of the sample space S, therefore, by theorem on total probability, we have

$$\begin{aligned} P(A) &= P(B)\ P(A|B) + P(B')\ P(A|B') \\ &= 0.65 \times 0.32 + 0.35 \times 0.8 \\ &= 0.208 + 0.28 = 0.488 \end{aligned}$$

Thus, the probability that the construction job will be completed in time is 0.488.

We shall now state and prove the Bayes' theorem.

Bayes' Theorem If $E_1, E_2, ..., E_n$ are n non empty events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup ... \cup E_n = S$ and A is any event of nonzero probability, then

$$P(E_i|A) = \frac{P(E_i)\,P(A|E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A|E_j)} \quad \text{for any } i = 1, 2, 3, ..., n$$

Proof By formula of conditional probability, we know that

$$\begin{aligned} P(E_i|A) &= \frac{P(A \cap E_i)}{P(A)} \\ &= \frac{P(E_i)\,P(A|E_i)}{P(A)} \quad \text{(by multiplication rule of probability)} \\ &= \frac{P(E_i)\,P(A|E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A|E_j)} \quad \text{(by the result of theorem of total probability)} \end{aligned}$$

Remark The following terminology is generally used when Bayes' theorem is applied.

The events $E_1, E_2, ..., E_n$ are called *hypotheses.*

The probability $P(E_i)$ is called the *priori probability* of the hypothesis E_i

The conditional probability $P(E_i|A)$ is called *a posteriori probability* of the hypothesis E_i.

Bayes' theorem is also called the formula for the probability of "causes". Since the E_i's are a partition of the sample space S, one and only one of the events E_i occurs (i.e. one of the events E_i must occur and only one can occur). Hence, the above formula gives us the probability of a particular E_i (i.e. a "Cause"), given that the event A has occurred.

The Bayes' theorem has its applications in variety of situations, few of which are illustrated in following examples.

Example 16 Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

Solution Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a red ball.

Then $\quad P(E_1) = P(E_2) = \frac{1}{2}$

Also $\quad P(A|E_1) = P(\text{drawing a red ball from Bag I}) = \frac{3}{7}$

and $\quad P(A|E_2) = P(\text{drawing a red ball from Bag II}) = \frac{5}{11}$

Now, the probability of drawing a ball from Bag II, being given that it is red, is $P(E_2|A)$

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)} = \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{35}{68}$$

Example 17 Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Solution Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Also, let A be the event that 'the coin drawn is of gold'

Then $$P(A|E_1) = P(\text{a gold coin from bag I}) = \frac{2}{2} = 1$$

$$P(A|E_2) = P(\text{a gold coin from bag II}) = 0$$

$$P(A|E_3) = P(\text{a gold coin from bag III}) = \frac{1}{2}$$

Now, the probability that the other coin in the box is of gold

= the probability that gold coin is drawn from the box I.

= $P(E_1|A)$

By Bayes' theorem, we know that

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times 0+\frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

Example 18 Suppose that the reliability of a HIV test is specified as follows:

Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV–ive but 1% are diagnosed as showing HIV+ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV+ive. What is the probability that the person actually has HIV?

Solution Let E denote the event that the person selected is actually having HIV and A the event that the person's HIV test is diagnosed as +ive. We need to find $P(E|A)$.

Also E' denotes the event that the person selected is actually not having HIV.

Clearly, $\{E, E'\}$ is a partition of the sample space of all people in the population. We are given that

$$P(E) = 0.1\% = \frac{0.1}{100} = 0.001$$

$$P(E') = 1 - P(E) = 0.999$$

$P(A|E)$ = P(Person tested as HIV+ive given that he/she is actually having HIV)

$$= 90\% = \frac{90}{100} = 0.9$$

and $P(A|E')$ = P(Person tested as HIV +ive given that he/she is actually not having HIV)

$$= 1\% = \frac{1}{100} = 0.01$$

Now, by Bayes' theorem

$$P(E|A) = \frac{P(E)\,P(A|E)}{P(E)\,P(A|E) + P(E')\,P(A|E')}$$

$$= \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089}$$

$$= 0.083 \text{ approx.}$$

Thus, the probability that a person selected at random is actually having HIV given that he/she is tested HIV+ive is 0.083.

Example 19 In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

Solution Let events B_1, B_2, B_3 be the following :

B_1 : the bolt is manufactured by machine A

B_2 : the bolt is manufactured by machine B

B_3 : the bolt is manufactured by machine C

Clearly, B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space.

Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that,

$$P(B_1) = 25\% = 0.25, \quad P(B_2) = 0.35 \text{ and } P(B_3) = 0.40$$

Again $P(E|B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5% = 0.05

Similarly, $P(E|B_2) = 0.04$, $P(E|B_3) = 0.02$.

Hence, by Bayes' Theorem, we have

$$P(B_2|E) = \frac{P(B_2)\,P(E|B_2)}{P(B_1)\,P(E|B_1) + P(B_2)\,P(E|B_2) + P(B_3)\,P(E|B_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02}$$

$$= \frac{0.0140}{0.0345} = \frac{28}{69}$$

Example 20 A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$, and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Solution Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus, scooter, and other means of transport respectively.

Then $\quad P(T_1) = \frac{3}{10}, P(T_2) = \frac{1}{5}, P(T_3) = \frac{1}{10}$ and $P(T_4) = \frac{2}{5}$ (given)

$P(E|T_1)$ = Probability that the doctor arriving late comes by train = $\frac{1}{4}$

Similarly, $P(E|T_2) = \frac{1}{3}$, $P(E|T_3) = \frac{1}{12}$ and $P(E|T_4) = 0$, since he is not late if he comes by other means of transport.

Therefore, by Bayes' Theorem, we have

$P(T_1|E)$ = Probability that the doctor arriving late comes by train

$$= \frac{P(T_1)\,P(E|T_1)}{P(T_1)\,P(E|T_1) + P(T_2)\,P(E|T_2) + P(T_3)\,P(E|T_3) + P(T_4)\,P(E|T_4)}$$

$$= \frac{\frac{3}{10} \times \frac{1}{4}}{\frac{3}{10} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{10} \times \frac{1}{12} + \frac{2}{5} \times 0} = \frac{3}{40} \times \frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

Example 21 A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Solution Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur.

Then $\quad P(S_1)$ = Probability that *six* occurs = $\frac{1}{6}$

$P(S_2)$ = Probability that *six* does not occur = $\frac{5}{6}$

$P(E|S_1)$ = Probability that the man reports that *six* occurs when *six* has actually occurred on the die

= Probability that the man speaks the truth = $\frac{3}{4}$

$P(E|S_2)$ = Probability that the man reports that *six* occurs when *six* has not actually occurred on the die

= Probability that the man does not speak the truth $= 1 - \frac{3}{4} = \frac{1}{4}$

Thus, by Bayes' theorem, we get

$P(S_1|E)$ = Probability that the report of the man that *six* has occurred is actually a *six*

$$= \frac{P(S_1)\,P(E|S_1)}{P(S_1)\,P(E|S_1) + P(S_2)\,P(E|S_2)}$$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{1}{8} \times \frac{24}{8} = \frac{3}{8}$$

Hence, the required probability is $\frac{3}{8}$.

EXERCISE 13.3

1. An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

2. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

3. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

4. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

5. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

6. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

7. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

8. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

9. Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are

0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

10. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

11. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

12. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

13. Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(A) $\frac{4}{5}$ (B) $\frac{1}{2}$ (C) $\frac{1}{5}$ (D) $\frac{2}{5}$

14. If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct?

(A) $P(A|B) = \frac{P(B)}{P(A)}$ (B) $P(A|B) < P(A)$

(C) $P(A|B) \geq P(A)$ (D) None of these

13.6 Random Variables and its Probability Distributions

We have already learnt about random experiments and formation of sample spaces. In most of these experiments, we were not only interested in the particular outcome that occurs but rather in some number associated with that outcomes as shown in following examples/experiments.

(i) In tossing two dice, we may be interested in the sum of the numbers on the two dice.

(ii) In tossing a coin 50 times, we may want the number of heads obtained.

(iii) In the experiment of taking out four articles (one after the other) at random from a lot of 20 articles in which 6 are defective, we want to know the number of defectives in the sample of four and not in the particular sequence of defective and nondefective articles.

In all of the above experiments, we have a rule which assigns to each outcome of the experiment a single real number. This single real number may vary with different outcomes of the experiment. Hence, it is a variable. Also its value depends upon the outcome of a random experiment and, hence, is called random variable. A random variable is usually denoted by X.

If you recall the definition of a function, you will realise that the random variable X is really speaking a function whose domain is the set of outcomes (or sample space) of a random experiment. A random variable can take any real value, therefore, its co-domain is the set of real numbers. Hence, a random variable can be defined as follows :

Definition 4 A random variable is a real valued function whose domain is the sample space of a random experiment.

For example, let us consider the experiment of tossing a coin two times in succession.

The sample space of the experiment is S = {HH, HT, TH, TT}.

If X denotes the number of heads obtained, then X is a random variable and for each outcome, its value is as given below :

$$X(HH) = 2,\ X(HT) = 1,\ X(TH) = 1,\ X(TT) = 0.$$

More than one random variables can be defined on the same sample space. For example, let Y denote the number of heads minus the number of tails for each outcome of the above sample space S.

Then $\quad Y(HH) = 2,\ Y(HT) = 0,\ Y(TH) = 0,\ Y(TT) = -2.$

Thus, X and Y are two different random variables defined on the same sample space S.

Example 22 A person plays a game of tossing a coin thrice. For each head, he is given Rs 2 by the organiser of the game and for each tail, he has to give Rs 1.50 to the organiser. Let X denote the amount gained or lost by the person. Show that X is a random variable and exhibit it as a function on the sample space of the experiment.

Solution X is a number whose values are defined on the outcomes of a random experiment. Therefore, X is a random variable.

Now, sample space of the experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Then $\quad X(HHH) = \text{Rs } (2 \times 3) = \text{Rs } 6$

$X(HHT) = X(HTH) = X(THH) = \text{Rs } (2 \times 2 - 1 \times 1.50) = \text{Rs } 2.50$

$X(HTT) = X(THT) = (TTH) = \text{Rs } (1 \times 2) - (2 \times 1.50) = -\text{ Re } 1$

and $\quad X(TTT) = -\text{ Rs } (3 \times 1.50) = -\text{ Rs } 4.50$

where, minus sign shows the loss to the player. Thus, for each element of the sample space, X takes a unique value, hence, X is a function on the sample space whose range is

$$\{-1,\ 2.50,\ -4.50,\ 6\}$$

Example 23 A bag contains 2 white and 1 red balls. One ball is drawn at random and then put back in the box after noting its colour. The process is repeated again. If X denotes the number of red balls recorded in the two draws, describe X.

Solution Let the balls in the bag be denoted by w_1, w_2, r. Then the sample space is

$$S = \{w_1\, w_1,\ w_1\, w_2,\ w_2\, w_2,\ w_2\, w_1,\ w_1\, r,\ w_2\, r,\ r\, w_1,\ r\, w_2,\ r\, r\}$$

Now, for $\quad \omega \in S$

$X(\omega) = \text{number of red balls}$

Therefore

$$X(\{w_1\, w_1\}) = X(\{w_1\, w_2\}) = X(\{w_2\, w_2\}) = X(\{w_2\, w_1\}) = 0$$

$$X(\{w_1\, r\}) = X(\{w_2\, r\}) = X(\{r\, w_1\}) = X(\{r\, w_2\}) = 1 \text{ and } X(\{r\, r\}) = 2$$

Thus, X is a random variable which can take values 0, 1 or 2.

13.6.1 *Probability distribution of a random variable*

Let us look at the experiment of selecting one family out of ten families $f_1, f_2, ..., f_{10}$ in such a manner that each family is equally likely to be selected. Let the families $f_1, f_2, ..., f_{10}$ have 3, 4, 3, 2, 5, 4, 3, 6, 4, 5 members, respectively.

Let us select a family and note down the number of members in the family denoting X. Clearly, X is a random variable defined as below :

$$X(f_1) = 3,\ X(f_2) = 4,\ X(f_3) = 3,\ X(f_4) = 2,\ X(f_5) = 5,$$

$$X(f_6) = 4,\ X(f_7) = 3,\ X(f_8) = 6,\ X(f_9) = 4,\ X(f_{10}) = 5$$

Thus, X can take any value 2,3,4,5 or 6 depending upon which family is selected.

Now, X will take the value 2 when the family f_4 is selected. X can take the value 3 when any one of the families f_1, f_3, f_7 is selected.

Similarly, $\quad X = 4$, when family f_2, f_6 or f_9 is selected,

$X = 5$, when family f_5 or f_{10} is selected

and $\quad X = 6$, when family f_8 is selected.

Since we had assumed that each family is equally likely to be selected, the probability that family f_4 is selected is $\frac{1}{10}$.

Thus, the probability that X can take the value 2 is $\frac{1}{10}$. We write $P(X = 2) = \frac{1}{10}$

Also, the probability that any one of the families f_1, f_3 or f_7 is selected is

$$P(\{f_1, f_3, f_7\}) = \frac{3}{10}$$

Thus, the probability that X can take the value 3 = $\frac{3}{10}$

We write $$P(X = 3) = \frac{3}{10}$$

Similarly, we obtain

$$P(X = 4) = P(\{f_2, f_6, f_9\}) = \frac{3}{10}$$

$$P(X = 5) = P(\{f_5, f_{10}\}) = \frac{2}{10}$$

and $$P(X = 6) = P(\{f_8\}) = \frac{1}{10}$$

Such a description giving the values of the random variable along with the corresponding probabilities is called the *probability distribution of the random variable X*.

In general, the probability distribution of a random variable X is defined as follows:

Definition 5 The probability distribution of a random variable X is the system of numbers

X	:	x_1	x_2	...	x_n
P(X)	:	p_1	p_2	...	p_n

where, $$p_i > 0, \quad \sum_{i=1}^{n} p_i = 1, \; i = 1, 2, ..., n$$

The real numbers $x_1, x_2, ..., x_n$ are the possible values of the random variable X and p_i (i = 1,2,..., n) is the probability of the random variable X taking the value x_i i.e., $P(X = x_i) = p_i$

Note If x_i is one of the possible values of a random variable X, the statement $X = x_i$ is true only at some point (s) of the sample space. Hence, the probability that X takes value x_i is always nonzero, i.e. $P(X = x_i) \neq 0$.

Also for all possible values of the random variable X, all elements of the sample space are covered. Hence, the sum of all the probabilities in a probability distribution must be one.

Example 24 Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Solution The number of aces is a random variable. Let it be denoted by X. Clearly, X can take the values 0, 1, or 2.

Now, since the draws are done with replacement, therefore, the two draws form independent experiments.

Therefore,
$$\begin{aligned} P(X = 0) &= P(\text{non-ace and non-ace}) \\ &= P(\text{non-ace}) \times P(\text{non-ace}) \\ &= \frac{48}{52} \times \frac{48}{52} = \frac{144}{169} \end{aligned}$$

$$\begin{aligned} P(X = 1) &= P(\text{ace and non-ace or non-ace and ace}) \\ &= P(\text{ace and non-ace}) + P(\text{non-ace and ace}) \\ &= P(\text{ace}).\ P(\text{non-ace}) + P\ (\text{non-ace})\ .\ P(\text{ace}) \\ &= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} = \frac{24}{169} \end{aligned}$$

and
$$\begin{aligned} P(X = 2) &= P\ (\text{ace and ace}) \\ &= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169} \end{aligned}$$

Thus, the required probability distribution is

X	0	1	2
P(X)	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Example 25 Find the probability distribution of number of doublets in three throws of a pair of dice.

Solution Let X denote the number of doublets. Possible doublets are

$$(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)$$

Clearly, X can take the value 0, 1, 2, or 3.

Probability of getting a doublet $= \frac{6}{36} = \frac{1}{6}$

Probability of not getting a doublet $= 1 - \frac{1}{6} = \frac{5}{6}$

Now $\quad P(X = 0) = P\text{ (no doublet)} = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

$P(X = 1) = P\text{ (one doublet and two non-doublets)}$

$$= \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$= 3\left(\frac{1}{6} \times \frac{5^2}{6^2}\right) = \frac{75}{216}$$

$P(X = 2) = P\text{ (two doublets and one non-doublet)}$

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} = 3\left(\frac{1}{6^2} \times \frac{5}{6}\right) = \frac{15}{216}$$

and $\quad P(X = 3) = P\text{ (three doublets)}$

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$$

Thus, the required probability distribution is

X	0	1	2	3
P(X)	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Verification Sum of the probabilities

$$\sum_{i=1}^{n} p_i = \frac{125}{216} + \frac{75}{216} + \frac{15}{216} + \frac{1}{216}$$

$$= \frac{125 + 75 + 15 + 1}{216} = \frac{216}{216} = 1$$

Example 26 Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the values x, has the following form, where k is some unknown constant.

$$P(X = x) = \begin{cases} 0.1, \text{ if } x = 0 \\ kx, \text{ if } x = 1 \text{ or } 2 \\ k(5-x), \text{ if } x = 3 \text{ or } 4 \\ 0, \text{ otherwise} \end{cases}$$

(a) Find the value of k.

(b) What is the probability that you study at least two hours ? Exactly two hours? At most two hours?

Solution The probability distribution of X is

X	0	1	2	3	4
P(X)	0.1	k	$2k$	$2k$	k

(a) We know that $\sum_{i=1}^{n} p_i = 1$

Therefore $0.1 + k + 2k + 2k + k = 1$

i.e. $k = 0.15$

(b) P(you study at least two hours) $= P(X \geq 2)$

$= P(X = 2) + P(X = 3) + P(X = 4)$

$= 2k + 2k + k = 5k = 5 \times 0.15 = 0.75$

P(you study exactly two hours) $= P(X = 2)$

$= 2k = 2 \times 0.15 = 0.3$

P(you study at most two hours) $= P(X \leq 2)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$= 0.1 + k + 2k = 0.1 + 3k = 0.1 + 3 \times 0.15$

$= 0.55$

13.6.2 *Mean of a random variable*

In many problems, it is desirable to describe some feature of the random variable by means of a single number that can be computed from its probability distribution. Few such numbers are mean, median and mode. In this section, we shall discuss mean only. Mean is a measure of location or central tendency in the sense that it roughly locates a *middle* or *average value* of the random variable.

Definition 6 Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, p_3,..., p_n$, respectively. The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$ i.e. the mean of X is the weighted average of the possible values of X, each value being weighted by its probability with which it occurs.

The mean of a random variable X is also called the expectation of X, denoted by E(X).

Thus,
$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + ... + x_n p_n.$$

In other words, the mean or expectation of a random variable X is the sum of the products of all possible values of X by their respective probabilities.

Example 27 Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X.

Solution The sample space of the experiment consists of 36 elementary events in the form of ordered pairs (x_i, y_i), where $x_i = 1, 2, 3, 4, 5, 6$ and $y_i = 1, 2, 3, 4, 5, 6$.

The random variable X i.e. the sum of the numbers on the two dice takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

Now
$$P(X = 2) = P(\{(1,1)\}) = \frac{1}{36}$$

$$P(X = 3) = P(\{(1,2), (2,1)\}) = \frac{2}{36}$$

$$P(X = 4) = P(\{(1,3), (2,2), (3,1)\}) = \frac{3}{36}$$

$$P(X = 5) = P(\{(1,4), (2,3), (3,2), (4,1)\}) = \frac{4}{36}$$

$$P(X = 6) = P(\{(1,5), (2,4), (3,3), (4,2), (5,1)\}) = \frac{5}{36}$$

$$P(X = 7) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}) = \frac{6}{36}$$

$$P(X = 8) = P(\{(2,6), (3,5), (4,4), (5,3), (6,2)\}) = \frac{5}{36}$$

$$P(X = 9) = P(\{(3,6), (4,5), (5,4), (6,3)\}) = \frac{4}{36}$$

$$P(X = 10) = P(\{(4,6), (5,5), (6,4)\}) = \frac{3}{36}$$

$$P(X = 11) = P(\{(5,6), (6,5)\}) = \frac{2}{36}$$

$$P(X = 12) = P(\{(6,6)\}) = \frac{1}{36}$$

The probability distribution of X is

X or x_i	2	3	4	5	6	7	8	9	10	11	12
P(X) or p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Therefore,

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i = 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36}$$

$$+ 6\times\frac{5}{36} + 7\times\frac{6}{36} + 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = 7$$

Thus, the mean of the sum of the numbers that appear on throwing two fair dice is 7.

13.6.3 *Variance of a random variable*

The mean of a random variable does not give us information about the variability in the values of the random variable. In fact, if the variance is small, then the values of the random variable are close to the mean. Also random variables with different probability distributions can have equal means, as shown in the following distributions of X and Y.

X	1	2	3	4
P(X)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{2}{8}$

Y	–1	0	4	5	6
P(Y)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Clearly $\quad E(X) = 1\times\frac{1}{8}+2\times\frac{2}{8}+3\times\frac{3}{8}+4\times\frac{2}{8}=\frac{22}{8}=2.75$

and $\quad E(Y) = -1\times\frac{1}{8}+0\times\frac{2}{8}+4\times\frac{3}{8}+5\times\frac{1}{8}= 6\times\frac{1}{8}=\frac{22}{8}=2.75$

The variables X and Y are different, however their means are same. It is also easily observable from the diagramatic representation of these distributions (Fig 13.5).

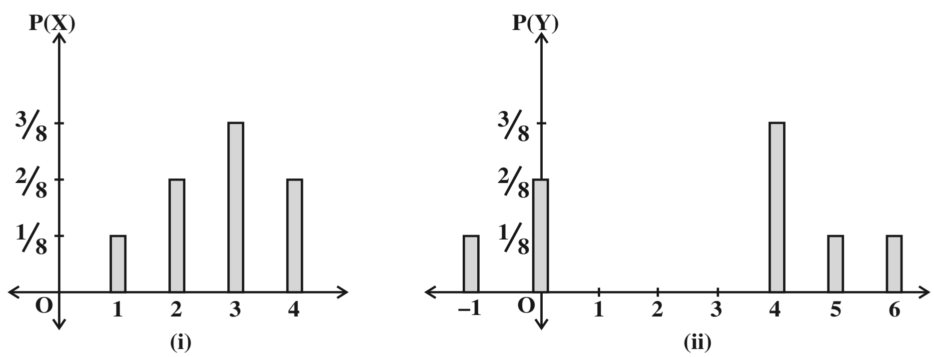


Fig 13.5

To distinguish X from Y, we require a measure of the extent to which the values of the random variables spread out. In Statistics, we have studied that the variance is a measure of the spread or scatter in data. Likewise, the variability or spread in the values of a random variable may be measured by variance.

Definition 7 Let X be a random variable whose possible values $x_1, x_2,...,x_n$ occur with probabilities $p(x_1), p(x_2),..., p(x_n)$ respectively.

Let $\mu = E(X)$ be the mean of X. The variance of X, denoted by Var (X) or σ_x^2 is defined as

$$\sigma_x^2 = \text{Var}(X)=\sum_{i=1}^{n}(x_i-\mu)^2\, p(x_i)$$

or equivalently $\quad \sigma_x^2 = E(X-\mu)^2$

The non-negative number

$$\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 \, p(x_i)}$$

is called the *standard deviation* of the random variable X.

Another formula to find the variance of a random variable. We know that,

$$\begin{aligned}
\text{Var (X)} &= \sum_{i=1}^{n} (x_i - \mu)^2 \, p(x_i) \\
&= \sum_{i=1}^{n} (x_i^2 + \mu^2 - 2\mu\, x_i)\, p(x_i) \\
&= \sum_{i=1}^{n} x_i^2 \, p(x_i) + \sum_{i=1}^{n} \mu^2 \, p(x_i) - \sum_{i=1}^{n} 2\mu x_i \, p(x_i) \\
&= \sum_{i=1}^{n} x_i^2 \, p(x_i) + \mu^2 \sum_{i=1}^{n} p(x_i) - 2\mu \sum_{i=1}^{n} x_i \, p(x_i) \\
&= \sum_{i=1}^{n} x_i^2 \, p(x_i) + \mu^2 - 2\mu^2 \left[\text{since} \sum_{i=1}^{n} p\ (x_i) = 1 \text{ and } \mu = \sum_{i=1}^{n} x_i \, p(x_i)\right] \\
&= \sum_{i=1}^{n} x_i^2 \, p(x_i) - \mu^2
\end{aligned}$$

or $$\text{Var (X)} = \sum_{i=1}^{n} x_i^2 \, p(x_i) - \left(\sum_{i=1}^{n} x_i \, p(x_i)\right)^2$$

or $$\text{Var (X)} = E(X^2) - [E(X)]^2, \text{ where } E(X^2) = \sum_{i=1}^{n} x_i^2 \, p(x_i)$$

Example 28 Find the variance of the number obtained on a throw of an unbiased die.

Solution The sample space of the experiment is S = {1, 2, 3, 4, 5, 6}.

Let X denote the number obtained on the throw. Then X is a random variable which can take values 1, 2, 3, 4, 5, or 6.

Also $$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

Therefore, the Probability distribution of X is

X	1	2	3	4	5	6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now $$E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$$

$$= 1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=\frac{21}{6}$$

Also $$E(X^2) = 1^2\times\frac{1}{6}+2^2\times\frac{1}{6}+3^2\times\frac{1}{6}+4^2\times\frac{1}{6}+5^2\times\frac{1}{6}+6^2\times\frac{1}{6}=\frac{91}{6}$$

Thus, $$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{91}{6}-\left(\frac{21}{6}\right)^2 = \frac{91}{6}-\frac{441}{36} = \frac{35}{12}$$

Example 29 Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

Solution Let X denote the number of kings in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

Now $$P(X = 0) = P(\text{no king}) = \frac{{}^{48}C_2}{{}^{52}C_2} = \frac{\frac{48!}{2!(48-2)!}}{\frac{52!}{2!(52-2)!}} = \frac{48\times 47}{52\times 51} = \frac{188}{221}$$

$$P(X = 1) = P(\text{one king and one non-king}) = \frac{{}^{4}C_1\ {}^{48}C_1}{{}^{52}C_2}$$

$$= \frac{4\times 48\times 2}{52\times 51} = \frac{32}{221}$$

and $\qquad P(X = 2) = P \text{ (two kings)} = \dfrac{{}^4C_2}{{}^{52}C_2} = \dfrac{4\times3}{52\times51} = \dfrac{1}{221}$

Thus, the probability distribution of X is

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Now $\qquad$ Mean of $\quad X = E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$

$$= 0\times\frac{188}{221} + 1\times\frac{32}{221} + 2\times\frac{1}{221} = \frac{34}{221}$$

Also $\qquad E(X^2) = \sum_{i=1}^{n} x_i^2\, p(x_i)$

$$= 0^2\times\frac{188}{221} + 1^2\times\frac{32}{221} + 2^2\times\frac{1}{221} = \frac{36}{221}$$

Now $\qquad \text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{36}{221} - \left(\frac{34}{221}\right)^2 = \frac{6800}{(221)^2}$$

Therefore $\qquad \sigma_x = \sqrt{\text{Var}(X)} = \dfrac{\sqrt{6800}}{221} = 0.37$

EXERCISE 13.4

1. State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

(i)

X	0	1	2
P(X)	0.4	0.4	0.2

(ii)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	– 0.1	0.3

(iii)

Y	– 1	0	1
P(Y)	0.6	0.1	0.2

(iv)

Z	3	2	1	0	–1
P(Z)	0.3	0.2	0.4	0.1	0.05

2. An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable ?

3. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are possible values of X?

4. Find the probability distribution of
   (i) number of heads in two tosses of a coin.
   (ii) number of tails in the simultaneous tosses of three coins.
   (iii) number of heads in four tosses of a coin.

5. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as
   (i) number greater than 4
   (ii) six appears on at least one die

6. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

7. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

8. A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
P(X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine

(i) k (ii) P(X < 3)

(iii) P(X > 6) (iv) P(0 < X < 3)

9. The random variable X has a probability distribution P(X) of the following form, where k is some number :

$$P(X) = \begin{cases} k, & if \quad x=0 \\ 2k, & if \quad x=1 \\ 3k, & if \quad x=2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the value of k.

(b) Find $P(X < 2)$, $P(X \leq 2)$, $P(X \geq 2)$.

10. Find the mean number of heads in three tosses of a fair coin.

11. Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

12. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E(X).

13. Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

14. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

15. In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).

Choose the correct answer in each of the following:

16. The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

(A) 1 (B) 2 (C) 5 (D) $\frac{8}{3}$

17. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is

(A) $\frac{37}{221}$ (B) $\frac{5}{13}$ (C) $\frac{1}{13}$ (D) $\frac{2}{13}$

13.7 Bernoulli Trials and Binomial Distribution

13.7.1 *Bernoulli trials*

Many experiments are dichotomous in nature. For example, a tossed coin shows a 'head' or 'tail', a manufactured item can be 'defective' or 'non-defective', the response to a question might be 'yes' or 'no', an egg has 'hatched' or 'not hatched', the decision is 'yes' or 'no' etc. In such cases, it is customary to call one of the outcomes a 'success' and the other 'not success' or 'failure'. For example, in tossing a coin, if the occurrence of the head is considered a success, then occurrence of tail is a failure.

Each time we toss a coin or roll a die or perform any other experiment, we call it a trial. If a coin is tossed, say, 4 times, the number of trials is 4, each having exactly two outcomes, namely, success or failure. The outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred as 'success' or 'failure' are called *Bernoulli trials*.

Definition 8 Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes : success or failure.

(iv) The probability of success remains the same in each trial.

For example, throwing a die 50 times is a case of 50 Bernoulli trials, in which each trial results in success (say an even number) or failure (an odd number) and the probability of success (p) is same for all 50 throws. Obviously, the successive throws of the die are independent experiments. If the die is fair and have six numbers 1 to 6 written on six faces, then $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$ = probability of failure.

Example 30 Six balls are drawn successively from an urn containing 7 red and 9 black balls. Tell whether or not the trials of drawing balls are Bernoulli trials when after each draw the ball drawn is

(i) replaced (ii) not replaced in the urn.

Solution

(i) The number of trials is finite. When the drawing is done with replacement, the probability of success (say, red ball) is $p = \frac{7}{16}$ which is same for all six trials (draws). Hence, the drawing of balls with replacements are Bernoulli trials.

(ii) When the drawing is done without replacement, the probability of success (i.e., red ball) in first trial is $\frac{7}{16}$, in 2nd trial is $\frac{6}{15}$ if the first ball drawn is red or $\frac{7}{15}$ if the first ball drawn is black and so on. Clearly, the probability of success is not same for all trials, hence the trials are not Bernoulli trials.

13.7.2 *Binomial distribution*

Consider the experiment of tossing a coin in which each trial results in success (say, heads) or failure (tails). Let S and F denote respectively success and failure in each trial. Suppose we are interested in finding the ways in which we have one success in six trials.

Clearly, six different cases are there as listed below:

SFFFFF, FSFFFF, FFSFFF, FFFSFF, FFFFSF, FFFFFS.

Similarly, two successes and four failures can have $\frac{6!}{4! \times 2!}$ combinations. It will be lengthy job to list all of these ways. Therefore, calculation of probabilities of 0, 1, 2,..., n number of successes may be lengthy and time consuming. To avoid the lengthy calculations and listing of all the possible cases, for the probabilities of number of successes in n-Bernoulli trials, a formula is derived. For this purpose, let us take the experiment made up of three Bernoulli trials with probabilities p and $q = 1 - p$ for success and failure respectively in each trial. The sample space of the experiment is the set

$$S = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

The number of successes is a random variable X and can take values 0, 1, 2, or 3. The probability distribution of the number of successes is as below :

$$\begin{aligned}
P(X = 0) &= P(\text{no success}) \\
&= P(\{FFF\}) = P(F)\ P(F)\ P(F) \\
&= q\ .\ q\ .\ q = q^3 \text{ since the trials are independent} \\
P(X = 1) &= P(\text{one successes}) \\
&= P(\{SFF, FSF, FFS\}) \\
&= P(\{SFF\}) + P(\{FSF\}) + P(\{FFS\}) \\
&= P(S)\ P(F)\ P(F) + P(F)\ P(S)\ P(F) + P(F)\ P(F)\ P(S) \\
&= p.q.q + q.p.q + q.q.p = 3pq^2 \\
P(X = 2) &= P\ (\text{two successes}) \\
&= P(\{SSF, SFS, FSS\}) \\
&= P(\{SSF\}) + P\ (\{SFS\}) + P(\{FSS\})
\end{aligned}$$

$= P(S)\ P(S)\ P(F) + P(S)\ P(F)\ P(S) + P(F)\ P(S)\ P(S)$

$= p.p.q. + p.q.p + q.p.p = 3p^2q$

and $\quad P(X = 3) = P(\text{three success}) = P(\{SSS\})$

$= P(S)\ .\ P(S)\ .\ P(S) = p^3$

Thus, the probability distribution of X is

X	0	1	2	3
P(X)	q^3	$3q^2p$	$3qp^2$	p^3

Also, the binominal expansion of $(q + p)^3$ is

$$q^3 + 3q^2p + 3qp^2 + p^3$$

Note that the probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q + p)^3$.

Also, since $q + p = 1$, it follows that the sum of these probabilities, as expected, is 1.

Thus, we may conclude that in an experiment of n-Bernoulli trials, the probabilities of 0, 1, 2,..., n successes can be obtained as 1st, 2nd,...,$(n + 1)^{\text{th}}$ terms in the expansion of $(q + p)^n$. To prove this assertion (result), let us find the probability of x-successes in an experiment of n-Bernoulli trials.

Clearly, in case of x successes (S), there will be $(n - x)$ failures (F).

Now, x successes (S) and $(n - x)$ failures (F) can be obtained in $\dfrac{n!}{x!(n-x)!}$ ways.

In each of these ways, the probability of x successes and $(n - x)$ failures is

$= P(x \text{ successes})\ .\ P(n–x) \text{ failures is}$

$$= \underbrace{P(S).P(S)...P(S)}_{x\,\text{times}} \cdot \underbrace{P(F).P(F)...P(F)}_{(n-x)\text{ times}} = p^x\, q^{n-x}$$

Thus, the probability of x successes in n-Bernoulli trials is $\dfrac{n!}{x!(n-x)!} p^x\, q^{n-x}$

or ${}^nC_x\, p^x\, q^{n-x}$

Thus $\quad P(x \text{ successes}) = {}^nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2,...,n.\ (q = 1 - p)$

Clearly, P(x successes), i.e. ${}^nC_x\, p^x\, q^{n-x}$ is the $(x + 1)^{\text{th}}$ term in the binomial expansion of $(q + p)^n$.

Thus, the probability distribution of number of successes in an experiment consisting of n Bernoulli trials may be obtained by the binomial expansion of $(q + p)^n$. Hence, this distribution of number of successes X can be written as

X	0	1	2	...	x	...	n
P(X)	${}^nC_0\, q^n$	${}^nC_1\, q^{n-1}p^1$	${}^nC_2\, q^{n-2}p^2$		${}^nC_x\, q^{n-x}p^x$		${}^nC_n\, p^n$

The above probability distribution is known as *binomial distribution* with parameters n and p, because for given values of n and p, we can find the complete probability distribution.

The probability of x successes $P(X = x)$ is also denoted by $P(x)$ and is given by

$$P(x) = {}^nC_x\, q^{n-x}p^x, \quad x = 0, 1, ..., n.\ (q = 1 - p)$$

This $P(x)$ is called the *probability function* of the binomial distribution.

A binomial distribution with n-Bernoulli trials and probability of success in each trial as p, is denoted by $B(n, p)$.

Let us now take up some examples.

Example 31 If a fair coin is tossed 10 times, find the probability of

(i) exactly six heads

(ii) at least six heads

(iii) at most six heads

Solution The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{1}{2}$

Therefore $\quad P(X = x) = {}^nC_x q^{n-x}p^x,\ x = 0, 1, 2, ..., n$

Here $\quad n = 10,\ p = \frac{1}{2},\ q = 1 - p = \frac{1}{2}$

Therefore $\quad P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^x = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

Now (i) $P(X = 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4!}\frac{1}{2^{10}} = \frac{105}{512}$

(ii) $P(\text{at least six heads}) = P(X \geq 6)$

$= P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\left(\frac{10!}{6!\times 4!}\right) + \left(\frac{10!}{7!\times 3!}\right) + \left(\frac{10!}{8!\times 2!}\right) + \left(\frac{10!}{9!\times 1!}\right) + \left(\frac{10!}{10!}\right)\right]\frac{1}{2^{10}} = \frac{193}{512}$$

(iii) P(at most six heads) = $P(X \le 6)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$+ P(X=4) + P(X=5) + P(X=6)$$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10}$$

$$+ {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}$$

Example 32 Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Solution Let X denote the number of defective eggs in the 10 eggs drawn. Since the drawing is done with replacement, the trials are Bernoulli trials. Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{10}{100} = \frac{1}{10}$.

Therefore $$q = 1 - p = \frac{9}{10}$$

Now P(at least one defective egg) = $P(X \ge 1) = 1 - P(X = 0)$

$$= 1 - {}^{10}C_0\left(\frac{9}{10}\right)^{10} = 1 - \frac{9^{10}}{10^{10}}$$

EXERCISE 13.5

1. A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of
   (i) 5 successes?
   (ii) at least 5 successes?
   (iii) at most 5 successes?

2. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.
3. There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?
4. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that
   (i) all the five cards are spades?
   (ii) only 3 cards are spades?
   (iii) none is a spade?
5. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs
   (i) none
   (ii) not more than one
   (iii) more than one
   (iv) at least one

   will fuse after 150 days of use.
6. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?
7. In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.
8. Suppose X has a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome.

   (Hint : $P(X = 3)$ is the maximum among all $P(x_i)$, $x_i = 0,1,2,3,4,5,6$)
9. On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing ?
10. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize (a) at least once (b) exactly once (c) at least twice?

11. Find the probability of getting 5 exactly twice in 7 throws of a die.

12. Find the probability of throwing at most 2 sixes in 6 throws of a single die.

13. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

In each of the following, choose the correct answer:

14. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is

(A) 10^{-1} (B) $\left(\frac{1}{2}\right)^5$ (C) $\left(\frac{9}{10}\right)^5$ (D) $\frac{9}{10}$

15. The probability that a student is not a swimmer is $\frac{1}{5}\cdot$ Then the probability that out of five students, four are swimmers is

(A) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$ (B) $\left(\frac{4}{5}\right)^4\frac{1}{5}$

(C) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4$ (D) None of these

Miscellaneous Examples

Example 33 Coloured balls are distributed in four boxes as shown in the following table:

Box	Colour			
	Black	White	Red	Blue
I	3	4	5	6
II	2	2	2	2
III	1	2	3	1
IV	4	3	1	5

A box is selected at random and then a ball is randomly drawn from the selected box. The colour of the ball is black, what is the probability that ball drawn is from the box III?

Solution Let A, E_1, E_2, E_3 and E_4 be the events as defined below :

A : a black ball is selected $\quad$ E_1 : box I is selected

E_2 : box II is selected $\quad$ E_3 : box III is selected

E_4 : box IV is selected

Since the boxes are chosen at random,

Therefore $\quad P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

Also $\quad P(A|E_1) = \frac{3}{18}$, $P(A|E_2) = \frac{2}{8}$, $P(A|E_3) = \frac{1}{7}$ and $P(A|E_4) = \frac{4}{13}$

P(box III is selected, given that the drawn ball is black) = $P(E_3|A)$. By Bayes' theorem,

$$P(E_3|A) = \frac{P(E_3)\cdot P(A|E_3)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)+P(E_4)P(A|E_4)}$$

$$= \frac{\frac{1}{4}\times\frac{1}{7}}{\frac{1}{4}\times\frac{3}{18}+\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{7}+\frac{1}{4}\times\frac{4}{13}} = 0.165$$

Example 34 Find the mean of the Binomial distribution $B\left(4, \frac{1}{3}\right)$.

Solution Let X be the random variable whose probability distribution is $B\left(4, \frac{1}{3}\right)$.

Here $\quad n = 4,\ p = \frac{1}{3}$ and $q = 1 - \frac{1}{3} = \frac{2}{3}$

We know that $\quad P(X = x) = {}^4C_x\left(\frac{2}{3}\right)^{4-x}\left(\frac{1}{3}\right)^x,\ x = 0, 1, 2, 3, 4.$

i.e. the distribution of X is

x_i	$P(x_i)$	$x_i\ P(x_i)$
0	${}^4C_0\left(\frac{2}{3}\right)^4$	0
1	${}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$	${}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$

2	${}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$	$2\left({}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2\right)$
3	${}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3$	$3\left({}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3\right)$
4	${}^4C_4\left(\frac{1}{3}\right)^4$	$4\left({}^4C_4\left(\frac{1}{3}\right)^4\right)$

Now Mean (μ) = $\sum_{i=1}^{4} x_i\, p(x_i)$

$$= 0 + {}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 2\cdot{}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 3\cdot{}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + 4\cdot{}^4C_4\left(\frac{1}{3}\right)^4$$

$$= 4\times\frac{2^3}{3^4} + 2\times 6\times\frac{2^2}{3^4} + 3\times 4\times\frac{2}{3^4} + 4\times 1\times\frac{1}{3^4}$$

$$= \frac{32+48+24+4}{3^4} = \frac{108}{81} = \frac{4}{3}$$

Example 35 The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?

Solution Let the shooter fire n times. Obviously, n fires are n Bernoulli trials. In each trial, p = probability of hitting the target = $\frac{3}{4}$ and q = probability of not hitting the target = $\frac{1}{4}$. Then $P(X = x) = {}^nC_x\, q^{n-x}\, p^x = {}^nC_x\left(\frac{1}{4}\right)^{n-x}\left(\frac{3}{4}\right)^x = {}^nC_x\,\frac{3^x}{4^n}$.

Now, given that,

P(hitting the target at least once) > 0.99

i.e. $\qquad P(x \geq 1) > 0.99$

Therefore, $$1 - P(x = 0) > 0.99$$

or $$1 - {}^nC_0 \frac{1}{4^n} > 0.99$$

or $${}^nC_0 \frac{1}{4^n} < 0.01 \quad \text{i.e.} \quad \frac{1}{4^n} < 0.01$$

or $$4^n > \frac{1}{0.01} = 100 \qquad \ldots (1)$$

The minimum value of n to satisfy the inequality (1) is 4.

Thus, the shooter must fire 4 times.

Example 36 A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

Solution Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus, $$P(S) = \frac{1}{6},\ P(F) = \frac{5}{6}$$

$$P(\text{A wins in the first throw}) = P(S) = \frac{1}{6}$$

A gets the third throw, when the first throw by A and second throw by B result into failures.

Therefore, $$P(\text{A wins in the 3rd throw}) = P(FFS) = P(F)P(F)P(S) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$= \left(\frac{5}{6}\right)^2 \times \frac{1}{6}$$

$$P(\text{A wins in the 5th throw}) = P\ (FFFFS) = \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) \text{ and so on.}$$

Hence, $$P(\text{A wins}) = \frac{1}{6} + \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) + \ldots$$

$$= \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$$

$$P(\text{B wins}) = 1 - P(\text{A wins}) = 1-\frac{6}{11}=\frac{5}{11}$$

Remark If $a + ar + ar^2 + ... + ar^{n-1} + ...$, where $|r| < 1$, then sum of this infinite G.P. is given by $\frac{a}{1-r}\cdot$ (Refer A.1.3 of Class XI Text book).

Example 37 If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly setup.

Solution Let A be the event that the machine produces 2 acceptable items.

Also let B_1 represent the event of correct set up and B_2 represent the event of incorrect setup.

Now $\quad P(B_1) = 0.8,\ P(B_2) = 0.2$

$$P(A|B_1) = 0.9 \times 0.9 \text{ and } P(A|B_2) = 0.4 \times 0.4$$

Therefore
$$P(B_1|A) = \frac{P(B_1)\,P(A|B_1)}{P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2)}$$

$$= \frac{0.8\times0.9\times0.9}{0.8\times0.9\times0.9+0.2\times0.4\times0.4} = \frac{648}{680} = 0.95$$

Miscellaneous Exercise on Chapter 13

1. A and B are two events such that $P(A) \neq 0$. Find $P(B|A)$, if
   (i) A is a subset of B   (ii) $A \cap B = \phi$

2. A couple has two children,
   (i) Find the probability that both children are males, if it is known that at least one of the children is male.
   (ii) Find the probability that both children are females, if it is known that the elder child is a female.

3. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

4. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?

5. An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that
   (i) all will bear 'X' mark.
   (ii) not more than 2 will bear 'Y' mark.
   (iii) at least one ball will bear 'Y' mark.
   (iv) the number of balls with 'X' mark and 'Y' mark will be equal.
6. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?
7. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.
8. If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?
9. An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be atleast 4 successes.
10. How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?
11. In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins / loses.
12. Suppose we have four boxes A,B,C and D containing coloured marbles as given below:

Box	**Marble colour**		
	Red	**White**	**Black**
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A?, box B?, box C?

13. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

14. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$).

15. An electronic assembly consists of two subsystems, say, A and B. From previous testing procedures, the following probabilities are assumed to be known:

$$P(\text{A fails}) = 0.2$$

$$P(\text{B fails alone}) = 0.15$$

$$P(\text{A and B fail}) = 0.15$$

Evaluate the following probabilities

(i) $P(\text{A fails}|\text{B has failed})$ (ii) $P(\text{A fails alone})$

16. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Choose the correct answer in each of the following:

17. If A and B are two events such that $P(A) \neq 0$ and $P(B \mid A) = 1$, then

(A) $A \subset B$ (B) $B \subset A$ (C) $B = \phi$ (D) $A = \phi$

18. If $P(A|B) > P(A)$, then which of the following is correct :

(A) $P(B|A) < P(B)$ (B) $P(A \cap B) < P(A) \,.\, P(B)$

(C) $P(B|A) > P(B)$ (D) $P(B|A) = P(B)$

19. If A and B are any two events such that $P(A) + P(B) - P(A \text{ and } B) = P(A)$, then

(A) $P(B|A) = 1$ (B) $P(A|B) = 1$

(C) $P(B|A) = 0$ (D) $P(A|B) = 0$

Summary

The salient features of the chapter are –

- The conditional probability of an event E, given the occurrence of the event F is given by $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$, $P(F) \neq 0$
- $0 \leq P(E|F) \leq 1$, $P(E'|F) = 1 - P(E|F)$
  $P((E \cup F)|G) = P(E|G) + P(F|G) - P((E \cap F)|G)$
- $P(E \cap F) = P(E)\, P(F|E)$, $P(E) \neq 0$
  $P(E \cap F) = P(F)\, P(E|F)$, $P(F) \neq 0$
- If E and F are independent, then
  $P(E \cap F) = P(E)\, P(F)$
  $P(E|F) = P(E)$, $P(F) \neq 0$
  $P(F|E) = P(F)$, $P(E) \neq 0$
- **Theorem of total probability**
  Let $\{E_1, E_2, ..., E_n)$ be a partition of a sample space and suppose that each of $E_1, E_2, ..., E_n$ has nonzero probability. Let A be any event associated with S, then
  $P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + ... + P(E_n)\, P(A|E_n)$
- **Bayes' theorem** If $E_1, E_2, ..., E_n$ are events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup ... \cup E_n = S$ and A be any event with nonzero probability, then

$$P(E_i \mid A) = \frac{P(E_i)\,P(A|E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A|E_j)}$$

- A random variable is a real valued function whose domain is the sample space of a random experiment.
- The probability distribution of a random variable X is the system of numbers

X	:	x_1	x_2	...	x_n
P(X)	:	p_1	p_2	...	p_n

where, $p_i > 0,\ \sum_{i=1}^{n} p_i = 1,\ i = 1, 2, ..., n$

- Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, p_3, ... p_n$ respectively. The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$.

  The mean of a random variable X is also called the expectation of X, denoted by E (X).
- Let X be a random variable whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p(x_1), p(x_2), ..., p(x_n)$ respectively.

  Let $\mu = E(X)$ be the mean of X. The variance of X, denoted by Var (X) or σ_x^2, is defined as $\sigma_x^2 = \mathrm{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$

  or equivalently $\sigma_x^2 = E(X - \mu)^2$

  The non-negative number

  $$\sigma_x = \sqrt{\mathrm{Var}(X)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)}$$

  is called the standard deviation of the random variable X.
- $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$
- Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :
  (i) There should be a finite number of trials.
  (ii) The trials should be independent.
  (iii) Each trial has exactly two outcomes : success or failure.
  (iv) The probability of success remains the same in each trial.

  For Binomial distribution B (n, p), $P(X = x) = {}^nC_x\, q^{n-x} p^x$, $x = 0, 1, ..., n$ $(q = 1 - p)$

Historical Note

The earliest indication on measurement of chances in game of dice appeared in 1477 in a commentary on Dante's Divine Comedy. A treatise on gambling named *liber de Ludo Alcae*, by Geronimo Carden (1501-1576) was published posthumously in 1663. In this treatise, he gives the number of favourable cases for each event when two dice are thrown.

Galileo (1564-1642) gave casual remarks concerning the correct evaluation of chance in a game of three dice. Galileo analysed that when three dice are thrown, the sum of the number that appear is more likely to be 10 than the sum 9, because the number of cases favourable to 10 are more than the number of cases for the appearance of number 9.

Apart from these early contributions, it is generally acknowledged that the true origin of the science of probability lies in the correspondence between two great men of the seventeenth century, Pascal (1623-1662) and Pierre de Fermat (1601-1665). A French gambler, Chevalier de Metre asked Pascal to explain some seeming contradiction between his theoretical reasoning and the observation gathered from gambling. In a series of letters written around 1654, Pascal and Fermat laid the first foundation of science of probability. Pascal solved the problem in algebraic manner while Fermat used the method of combinations.

Great Dutch Scientist, Huygens (1629-1695), became acquainted with the content of the correspondence between Pascal and Fermat and published a first book on probability, "*De Ratiociniis in Ludo Aleae*" containing solution of many interesting rather than difficult problems on probability in games of chances.

The next great work on probability theory is by Jacob Bernoulli (1654-1705), in the form of a great book, "*Ars Conjectendi*" published posthumously in 1713 by his nephew, Nicholes Bernoulli. To him is due the discovery of one of the most important probability distribution known as Binomial distribution. The next remarkable work on probability lies in 1993. A. N. Kolmogorov (1903-1987) is credited with the axiomatic theory of probability. His book, 'Foundations of probability' published in 1933, introduces probability as a set function and is considered a 'classic!'.

Chapter 9

DIFFERENTIAL EQUATIONS

❖ *He who seeks for methods without having a definite problem in mind seeks for the most part in vain.* – D. HILBERT ❖

9.1 Introduction

In Class XI and in Chapter 5 of the present book, we discussed how to differentiate a given function f with respect to an independent variable, i.e., how to find $f'(x)$ for a given function f at each x in its domain of definition. Further, in the chapter on Integral Calculus, we discussed how to find a function f whose derivative is the function g, which may also be formulated as follows:

For a given function g, find a function f such that

$$\frac{dy}{dx} = g(x), \text{ where } y = f(x) \quad \ldots (1)$$

Henri Poincare (1854-1912)

An equation of the form (1) is known as a *differential equation*. A formal definition will be given later.

These equations arise in a variety of applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics etc. Hence, an indepth study of differential equations has assumed prime importance in all modern scientific investigations.

In this chapter, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation, formation of differential equations, some methods to solve a first order - first degree differential equation and some applications of differential equations in different areas.

9.2 Basic Concepts

We are already familiar with the equations of the type:

$$x^2 - 3x + 3 = 0 \quad \ldots (1)$$

$$\sin x + \cos x = 0 \quad \ldots (2)$$

$$x + y = 7 \quad \ldots (3)$$

Let us consider the equation:

$$x\frac{dy}{dx} + y = 0 \qquad \text{... (4)}$$

We see that equations (1), (2) and (3) involve independent and/or dependent variable (variables) only but equation (4) involves variables as well as derivative of the dependent variable y with respect to the independent variable x. Such an equation is called a *differential equation*.

In general, an equation involving derivative (derivatives) of the dependent variable with respect to independent variable (variables) is called a differential equation.

A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation, e.g.,

$$2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0 \text{ is an ordinary differential equation} \qquad \text{.... (5)}$$

Of course, there are differential equations involving derivatives with respect to more than one independent variables, called partial differential equations but at this stage we shall confine ourselves to the study of ordinary differential equations only. Now onward, we will use the term 'differential equation' for 'ordinary differential equation'.

Note

1. We shall prefer to use the following notations for derivatives:

   $$\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'''$$

2. For derivatives of higher order, it will be inconvenient to use so many dashes as supersuffix therefore, we use the notation y_n for nth order derivative $\frac{d^ny}{dx^n}$.

9.2.1. *Order of a differential equation*

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Consider the following differential equations:

$$\frac{dy}{dx} = e^x \qquad \text{... (6)}$$

$$\frac{d^2y}{dx^2} + y = 0 \qquad \text{... (7)}$$

$$\left(\frac{d^3y}{dx^3}\right) + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0 \qquad \text{... (8)}$$

The equations (6), (7) and (8) involve the highest derivative of first, second and third order respectively. Therefore, the order of these equations are 1, 2 and 3 respectively.

9.2.2 *Degree of a differential equation*

To study the degree of a differential equation, the key point is that the differential equation must be a polynomial equation in derivatives, i.e., y', y'', y''' etc. Consider the following differential equations:

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0 \qquad \text{... (9)}$$

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin^2 y = 0 \qquad \text{... (10)}$$

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0 \qquad \text{... (11)}$$

We observe that equation (9) is a polynomial equation in y''', y'' and y', equation (10) is a polynomial equation in y' (not a polynomial in y though). Degree of such differential equations can be defined. But equation (11) is not a polynomial equation in y' and degree of such a differential equation can not be defined.

By the degree of a differential equation, when it is a polynomial equation in derivatives, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

In view of the above definition, one may observe that differential equations (6), (7), (8) and (9) each are of degree one, equation (10) is of degree two while the degree of differential equation (11) is not defined.

Note Order and degree (if defined) of a differential equation are always positive integers.

Example 1 Find the order and degree, if defined, of each of the following differential equations:

(i) $\frac{dy}{dx} - \cos x = 0$ (ii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

(iii) $y''' + y^2 + e^{y'} = 0$

Solution

(i) The highest order derivative present in the differential equation is $\frac{dy}{dx}$, so its order is one. It is a polynomial equation in y' and the highest power raised to $\frac{dy}{dx}$ is one, so its degree is one.

(ii) The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$, so its order is two. It is a polynomial equation in $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and the highest power raised to $\frac{d^2y}{dx^2}$ is one, so its degree is one.

(iii) The highest order derivative present in the differential equation is y''', so its order is three. The given differential equation is not a polynomial equation in its derivatives and so its degree is not defined.

EXERCISE 9.1

Determine order and degree (if defined) of differential equations given in Exercises 1 to 10.

1. $\frac{d^4y}{dx^4} + \sin(y''') = 0$ **2.** $y' + 5y = 0$ **3.** $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

4. $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$ **5.** $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$

6. $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ **7.** $y''' + 2y'' + y' = 0$

8. $y' + y = e^x$ **9.** $y'' + (y')^2 + 2y = 0$ **10.** $y'' + 2y' + \sin y = 0$

11. The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is}$$

(A) 3 (B) 2 (C) 1 (D) not defined

12. The order of the differential equation

$$2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0 \text{ is}$$

(A) 2 (B) 1 (C) 0 (D) not defined

9.3. General and Particular Solutions of a Differential Equation

In earlier Classes, we have solved the equations of the type:

$$x^2 + 1 = 0 \quad \text{... (1)}$$

$$\sin^2 x - \cos x = 0 \quad \text{... (2)}$$

Solution of equations (1) and (2) are numbers, real or complex, that will satisfy the given equation i.e., when that number is substituted for the unknown x in the given equation, L.H.S. becomes equal to the R.H.S..

Now consider the differential equation $\frac{d^2y}{dx^2} + y = 0$ ... (3)

In contrast to the first two equations, the solution of this differential equation is a function ϕ that will satisfy it i.e., when the function ϕ is substituted for the unknown y (dependent variable) in the given differential equation, L.H.S. becomes equal to R.H.S..

The curve $y = \phi(x)$ is called the solution curve (integral curve) of the given differential equation. Consider the function given by

$$y = \phi(x) = a \sin(x + b), \quad \text{... (4)}$$

where $a, b \in \mathbf{R}$. When this function and its derivative are substituted in equation (3), L.H.S. = R.H.S.. So it is a solution of the differential equation (3).

Let a and b be given some particular values say $a = 2$ and $b = \frac{\pi}{4}$, then we get a

function $y = \phi_1(x) = 2\sin\left(x + \frac{\pi}{4}\right)$ ... (5)

When this function and its derivative are substituted in equation (3) again L.H.S. = R.H.S.. Therefore ϕ_1 is also a solution of equation (3).

Function ϕ consists of two arbitrary constants (parameters) a, b and it is called *general solution* of the given differential equation. Whereas function ϕ_1 contains no arbitrary constants but only the particular values of the parameters a and b and hence is called a *particular solution* of the given differential equation.

The solution which contains arbitrary constants is called the *general solution* (*primitive*) of the differential equation.

The solution free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a *particular solution* of the differential equation.

Example 2 Verify that the function $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

Solution Given function is $y = e^{-3x}$. Differentiating both sides of equation with respect to x, we get

$$\frac{dy}{dx} = -3e^{-3x} \qquad \text{... (1)}$$

Now, differentiating (1) with respect to x, we have

$$\frac{d^2y}{dx^2} = 9\,e^{-3x}$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get

L.H.S. $= 9\ e^{-3x} + (-3e^{-3x}) - 6.e^{-3x} = 9\ e^{-3x} - 9\ e^{-3x} = 0 =$ R.H.S..

Therefore, the given function is a solution of the given differential equation.

Example 3 Verify that the function $y = a \cos x + b \sin x$, where, $a, b \in \mathbf{R}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

Solution The given function is

$$y = a \cos x + b \sin x \qquad \text{... (1)}$$

Differentiating both sides of equation (1) with respect to x, successively, we get

$$\frac{dy}{dx} = -\,a \sin x + b \cos x$$

$$\frac{d^2y}{dx^2} = -\,a \cos x - b \sin x$$

Substituting the values of $\frac{d^2 y}{dx^2}$ and y in the given differential equation, we get

L.H.S. = $(-a \cos x - b \sin x) + (a \cos x + b \sin x) = 0$ = R.H.S..

Therefore, the given function is a solution of the given differential equation.

EXERCISE 9.2

In each of the Exercises 1 to 10 verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

1. $y = e^x + 1$: $y'' - y' = 0$

2. $y = x^2 + 2x + C$: $y' - 2x - 2 = 0$

3. $y = \cos x + C$: $y' + \sin x = 0$

4. $y = \sqrt{1+x^2}$: $y' = \frac{xy}{1+x^2}$

5. $y = Ax$: $xy' = y \ (x \neq 0)$

6. $y = x \sin x$: $xy' = y + x\sqrt{x^2 - y^2} \ (x \neq 0$ and $x > y$ or $x < -y)$

7. $xy = \log y + C$: $y' = \frac{y^2}{1-xy} \ (xy \neq 1)$

8. $y - \cos y = x$: $(y \sin y + \cos y + x)\, y' = y$

9. $x + y = \tan^{-1}y$: $y^2 y' + y^2 + 1 = 0$

10. $y = \sqrt{a^2 - x^2} \ x \in (-a, a)$: $x + y\frac{dy}{dx} = 0 \ (y \neq 0)$

11. The number of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0 (B) 2 (C) 3 (D) 4

12. The number of arbitrary constants in the particular solution of a differential equation of third order are:

(A) 3 (B) 2 (C) 1 (D) 0

9.4 Formation of a Differential Equation whose General Solution is given

We know that the equation

$$x^2 + y^2 + 2x - 4y + 4 = 0 \qquad \ldots (1)$$

represents a circle having centre at $(-1, 2)$ and radius 1 unit.

Differentiating equation (1) with respect to x, we get

$$\frac{dy}{dx} = \frac{x+1}{2-y} \quad (y \neq 2) \qquad \text{... (2)}$$

which is a differential equation. You will find later on [See (example 9 section 9.5.1.)] that this equation represents the family of circles and one member of the family is the circle given in equation (1).

Let us consider the equation

$$x^2 + y^2 = r^2 \qquad \text{... (3)}$$

By giving different values to r, we get different members of the family e.g. $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$ etc. (see Fig 9.1). Thus, equation (3) represents a family of concentric circles centered at the origin and having different radii.

We are interested in finding a differential equation that is satisfied by each member of the family. The differential equation must be free from r because r is different for different members of the family. This equation is obtained by differentiating equation (3) with respect to x, i.e.,

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad x + y\frac{dy}{dx} = 0 \qquad \text{... (4)}$$

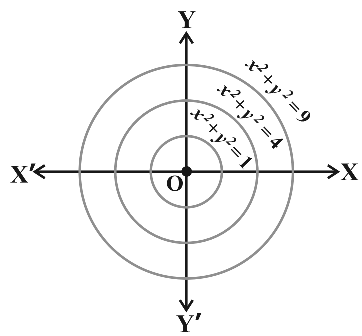


Fig 9.1

which represents the family of concentric circles given by equation (3).

Again, let us consider the equation

$$y = mx + c \qquad \text{... (5)}$$

By giving different values to the parameters m and c, we get different members of the family, e.g.,

$y = x$ $\quad (m = 1, \ c = 0)$

$y = \sqrt{3}\,x$ $\quad (m = \sqrt{3}, \ c = 0)$

$y = x + 1$ $\quad (m = 1, \ c = 1)$

$y = -x$ $\quad (m = -1, \ c = 0)$

$y = -x - 1$ $\quad (m = -1, \ c = -1)$ etc. (see Fig 9.2).

Thus, equation (5) represents the family of straight lines, where m, c are parameters.

We are now interested in finding a differential equation that is satisfied by each member of the family. Further, the equation must be free from m and c because m and

c are different for different members of the family. This is obtained by differentiating equation (5) with respect to x, successively we get

$$\frac{dy}{dx} = m \text{, and } \frac{d^2y}{dx^2} = 0 \qquad \dots (6)$$

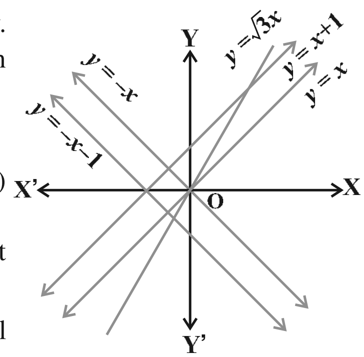


Fig 9.2

The equation (6) represents the family of straight lines given by equation (5).

Note that equations (3) and (5) are the general solutions of equations (4) and (6) respectively.

9.4.1 *Procedure to form a differential equation that will represent a given family of curves*

(a) If the given family F_1 of curves depends on only one parameter then it is represented by an equation of the form

$$F_1 (x, y, a) = 0 \qquad \dots (1)$$

For example, the family of parabolas $y^2 = ax$ can be represented by an equation of the form $f(x, y, a) : y^2 = ax$.

Differentiating equation (1) with respect to x, we get an equation involving y', y, x, and a, i.e.,

$$g (x, y, y', a) = 0 \qquad \dots (2)$$

The required differential equation is then obtained by eliminating a from equations (1) and (2) as

$$F(x, y, y') = 0 \qquad \dots (3)$$

(b) If the given family F_2 of curves depends on the parameters a, b (say) then it is represented by an equation of the from

$$F_2 (x, y, a, b) = 0 \qquad \dots (4)$$

Differentiating equation (4) with respect to x, we get an equation involving y', x, y, a, b, i.e.,

$$g (x, y, y', a, b) = 0 \qquad \dots (5)$$

But it is not possible to eliminate two parameters a and b from the two equations and so, we need a third equation. This equation is obtained by differentiating equation (5), with respect to x, to obtain a relation of the form

$$h (x, y, y', y'', a, b) = 0 \qquad \dots (6)$$

The required differential equation is then obtained by eliminating a and b from equations (4), (5) and (6) as

$$F(x, y, y', y'') = 0 \qquad \text{... (7)}$$

Note The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

Example 4 Form the differential equation representing the family of curves $y = mx$, where, m is arbitrary constant.

Solution We have

$$y = mx \qquad \text{... (1)}$$

Differentiating both sides of equation (1) with respect to x, we get

$$\frac{dy}{dx} = m$$

Substituting the value of m in equation (1) we get $y = \frac{dy}{dx} \cdot x$

or $$x\frac{dy}{dx} - y = 0$$

which is free from the parameter m and hence this is the required differential equation.

Example 5 Form the differential equation representing the family of curves $y = a \sin (x + b)$, where a, b are arbitrary constants.

Solution We have

$$y = a \sin (x + b) \qquad \text{... (1)}$$

Differentiating both sides of equation (1) with respect to x, successively we get

$$\frac{dy}{dx} = a \cos (x + b) \qquad \text{... (2)}$$

$$\frac{d^2 y}{dx^2} = -a \sin (x + b) \qquad \text{... (3)}$$

Eliminating a and b from equations (1), (2) and (3), we get

$$\frac{d^2 y}{dx^2} + y = 0 \qquad \text{... (4)}$$

which is free from the arbitrary constants a and b and hence this the required differential equation.

Example 6 Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.

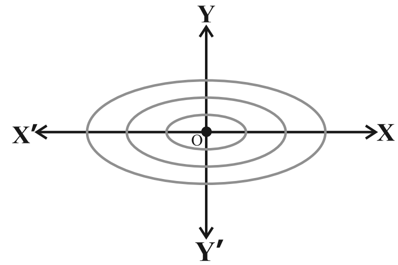


Fig 9.3

Solution We know that the equation of said family of ellipses (see Fig 9.3) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad \text{... (1)}$$

Differentiating equation (1) with respect to x, we get $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

or $$\frac{y}{x}\left(\frac{dy}{dx}\right) = \frac{-b^2}{a^2} \qquad \text{... (2)}$$

Differentiating both sides of equation (2) with respect to x, we get

$$\left(\frac{y}{x}\right)\left(\frac{d^2y}{dx^2}\right)+\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)\frac{dy}{dx} = 0$$

or $$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx} = 0 \qquad \text{... (3)}$$

which is the required differential equation.

Example 7 Form the differential equation of the family of circles touching the x-axis at origin.

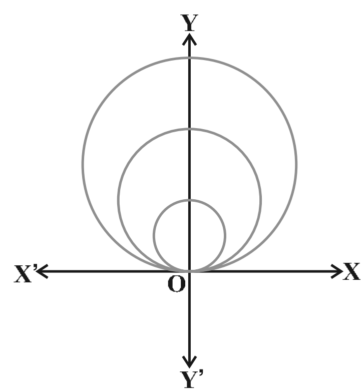


Fig 9.4

Solution Let C denote the family of circles touching x-axis at origin. Let $(0, a)$ be the coordinates of the centre of any member of the family (see Fig 9.4). Therefore, equation of family C is

$$x^2 + (y - a)^2 = a^2 \text{ or } x^2 + y^2 = 2ay \qquad \text{... (1)}$$

where, a is an arbitrary constant. Differentiating both sides of equation (1) with respect to x, we get

$$2x+2y\frac{dy}{dx} = 2a\frac{dy}{dx}$$

or $$x + y\frac{dy}{dx} = a\frac{dy}{dx} \text{ or } a = \frac{x + y\dfrac{dy}{dx}}{\dfrac{dy}{dx}} \qquad \ldots (2)$$

Substituting the value of a from equation (2) in equation (1), we get

$$x^2 + y^2 = 2y\frac{\left[x + y\dfrac{dy}{dx}\right]}{\dfrac{dy}{dx}}$$

or $$\frac{dy}{dx}(x^2 + y^2) = 2xy + 2y^2\frac{dy}{dx}$$

or $$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

This is the required differential equation of the given family of circles.

Example 8 Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

Solution Let P denote the family of above said parabolas (see Fig 9.5) and let $(a, 0)$ be the focus of a member of the given family, where a is an arbitrary constant. Therefore, equation of family P is

$$y^2 = 4ax \qquad \ldots (1)$$

Differentiating both sides of equation (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4a \qquad \ldots (2)$$

Substituting the value of $4a$ from equation (2) in equation (1), we get

$$y^2 = \left(2y\frac{dy}{dx}\right)(x)$$

or $$y^2 - 2xy\frac{dy}{dx} = 0$$

which is the differential equation of the given family of parabolas.

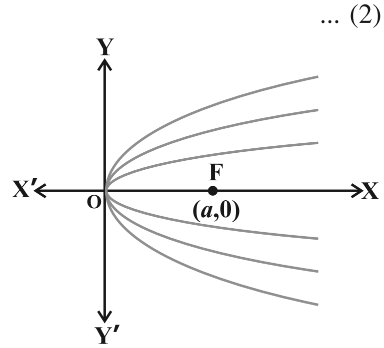


Fig 9.5

EXERCISE 9.3

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

1. $\frac{x}{a} + \frac{y}{b} = 1$
2. $y^2 = a\,(b^2 - x^2)$
3. $y = a\,e^{3x} + b\,e^{-2x}$
4. $y = e^{2x}\,(a + bx)$
5. $y = e^x\,(a \cos x + b \sin x)$
6. Form the differential equation of the family of circles touching the y-axis at origin.
7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.
8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.
9. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.
10. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.
11. Which of the following differential equations has $y = c_1\, e^x + c_2\, e^{-x}$ as the general solution?

    (A) $\frac{d^2y}{dx^2} + y = 0$ (B) $\frac{d^2y}{dx^2} - y = 0$ (C) $\frac{d^2y}{dx^2} + 1 = 0$ (D) $\frac{d^2y}{dx^2} - 1 = 0$
12. Which of the following differential equations has $y = x$ as one of its particular solution?

    (A) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$ (B) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$

    (C) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$ (D) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

9.5. Methods of Solving First Order, First Degree Differential Equations

In this section we shall discuss three methods of solving first order first degree differential equations.

9.5.1 *Differential equations with variables separable*

A first order-first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y) \qquad \text{... (1)}$$

If F(x, y) can be expressed as a product g (x) $h(y)$, where, $g(x)$ is a function of x and $h(y)$ is a function of y, then the differential equation (1) is said to be of variable separable type. The differential equation (1) then has the form

$$\frac{dy}{dx} = h(y) \cdot g(x) \qquad \text{... (2)}$$

If $h(y) \neq 0$, separating the variables, (2) can be rewritten as

$$\frac{1}{h(y)}\, dy = g(x)\, dx \qquad \text{... (3)}$$

Integrating both sides of (3), we get

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx \qquad \text{... (4)}$$

Thus, (4) provides the solutions of given differential equation in the form

$$\mathrm{H}(y) = \mathrm{G}(x) + \mathrm{C}$$

Here, H (y) and G (x) are the anti derivatives of $\frac{1}{h(y)}$ and $g(x)$ respectively and C is the arbitrary constant.

Example 9 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}$, $(y \neq 2)$

Solution We have

$$\frac{dy}{dx} = \frac{x+1}{2-y} \qquad \text{... (1)}$$

Separating the variables in equation (1), we get

$$(2-y)\, dy = (x+1)\, dx \qquad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\int (2-y)\, dy = \int (x+1)\, dx$$

or $$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + \mathrm{C}_1$$

or $$x^2 + y^2 + 2x - 4y + 2\,\mathrm{C}_1 = 0$$

or $$x^2 + y^2 + 2x - 4y + \mathrm{C} = 0, \text{ where } \mathrm{C} = 2\mathrm{C}_1$$

which is the general solution of equation (1).

Example 10 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Solution Since $1 + y^2 \neq 0$, therefore separating the variables, the given differential equation can be written as

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \qquad \ldots (1)$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

which is the general solution of equation (1).

Example 11 Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

Solution If $y \neq 0$, the given differential equation can be written as

$$\frac{dy}{y^2} = -4x\, dx \qquad \ldots (1)$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{y^2} = -4\int x\, dx$$

or $$-\frac{1}{y} = -2x^2 + C$$

or $$y = \frac{1}{2x^2 - C} \qquad \ldots (2)$$

Substituting $y = 1$ and $x = 0$ in equation (2), we get, $C = -1$.

Now substituting the value of C in equation (2), we get the particular solution of the given differential equation as $y = \frac{1}{2x^2+1}$.

Example 12 Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\, dy = (2x^2 + 1)\, dx$ $(x \neq 0)$.

Solution The given differential equation can be expressed as

$$dy^* = \left(\frac{2x^2+1}{x}\right) dx^*$$

or
$$dy = \left(2x + \frac{1}{x}\right) dx \qquad \ldots (1)$$

Integrating both sides of equation (1), we get

$$\int dy = \int \left(2x + \frac{1}{x}\right) dx$$

or
$$y = x^2 + \log |x| + C \qquad \ldots (2)$$

Equation (2) represents the family of solution curves of the given differential equation but we are interested in finding the equation of a particular member of the family which passes through the point (1, 1). Therefore substituting $x = 1$, $y = 1$ in equation (2), we get C = 0.

Now substituting the value of C in equation (2) we get the equation of the required curve as $y = x^2 + \log |x|$.

Example 13 Find the equation of a curve passing through the point (–2, 3), given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$.

Solution We know that the slope of the tangent to a curve is given by $\frac{dy}{dx}$.

so,
$$\frac{dy}{dx} = \frac{2x}{y^2} \qquad \ldots (1)$$

Separating the variables, equation (1) can be written as

$$y^2\, dy = 2x\, dx \qquad \ldots (2)$$

Integrating both sides of equation (2), we get

$$\int y^2 dy = \int 2x\, dx$$

or
$$\frac{y^3}{3} = x^2 + C \qquad \ldots (3)$$

---

* The notation $\frac{dy}{dx}$ due to Leibnitz is extremely flexible and useful in many calculation and formal transformations, where, we can deal with symbols dy and dx exactly as if they were ordinary numbers. By treating dx and dy like separate entities, we can give neater expressions to many calculations.

Refer: Introduction to Calculus and Analysis, volume-I page 172, By Richard Courant, Fritz John Spinger – Verlog New York.

Substituting $x = -2$, $y = 3$ in equation (3), we get C = 5.

Substituting the value of C in equation (3), we get the equation of the required curve as

$$\frac{y^3}{3} = x^2 + 5 \quad \text{or} \quad y = (3x^2 + 15)^{\frac{1}{3}}$$

Example 14 In a bank, principal increases continuously at the rate of 5% per year. In how many years Rs 1000 double itself?

Solution Let P be the principal at any time t. According to the given problem,

$$\frac{dp}{dt} = \left(\frac{5}{100}\right) \times \text{P}$$

or $$\frac{dp}{dt} = \frac{\text{P}}{20} \qquad \text{... (1)}$$

separating the variables in equation (1), we get

$$\frac{dp}{\text{P}} = \frac{dt}{20} \qquad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\log \text{P} = \frac{t}{20} + \text{C}_1$$

or $$\text{P} = e^{\frac{t}{20}} \cdot e^{\text{C}_1}$$

or $$\text{P} = \text{C}\, e^{\frac{t}{20}} \quad (\text{where } e^{\text{C}_1} = \text{C}) \qquad \text{... (3)}$$

Now $\text{P} = 1000$, when $t = 0$

Substituting the values of P and t in (3), we get C = 1000. Therefore, equation (3), gives

$$\text{P} = 1000\, e^{\frac{t}{20}}$$

Let t years be the time required to double the principal. Then

$$2000 = 1000\, e^{\frac{t}{20}} \Rightarrow t = 20 \log_e 2$$

EXERCISE 9.4

For each of the differential equations in Exercises 1 to 10, find the general solution:

1. $\dfrac{dy}{dx} = \dfrac{1 - \cos x}{1 + \cos x}$

2. $\dfrac{dy}{dx} = \sqrt{4 - y^2} \ (-2 < y < 2)$

3. $\dfrac{dy}{dx} + y = 1 \; (y \neq 1)$

4. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

5. $(e^x + e^{-x}) \, dy - (e^x - e^{-x}) \, dx = 0$

6. $\dfrac{dy}{dx} = (1 + x^2)(1 + y^2)$

7. $y \log y \, dx - x \, dy = 0$

8. $x^5 \dfrac{dy}{dx} = -\, y^5$

9. $\dfrac{dy}{dx} = \sin^{-1} x$

10. $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

For each of the differential equations in Exercises 11 to 14, find a particular solution satisfying the given condition:

11. $(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x$; $y = 1$ when $x = 0$

12. $x(x^2 - 1)\dfrac{dy}{dx} = 1$; $y = 0$ when $x = 2$

13. $\cos\left(\dfrac{dy}{dx}\right) = a \;\; (a \in \mathbf{R})$; $y = 1$ when $x = 0$

14. $\dfrac{dy}{dx} = y \tan x$; $y = 1$ when $x = 0$

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

16. For the differential equation $xy\dfrac{dy}{dx} = (x + 2)(y + 2)$, find the solution curve passing through the point (1, –1).

17. Find the equation of a curve passing through the point (0, –2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

18. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, –3). Find the equation of the curve given that it passes through (–2, 1).

19. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

20. In a bank, principal increases continuously at the rate of $r\%$ per year. Find the value of r if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

9.5.2 *Homogeneous differential equations*

Consider the following functions in x and y

$F_1(x, y) = y^2 + 2xy$, $F_2(x, y) = 2x - 3y$,

$F_3(x, y) = \cos\left(\frac{y}{x}\right)$, $F_4(x, y) = \sin x + \cos y$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$$

$$F_2(\lambda x, \lambda y) = \lambda (2x - 3y) = \lambda F_2(x, y)$$

$$F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right) = \lambda^0 F_3(x, y)$$

$$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n F_4(x, y), \text{ for any } n \in \mathbf{N}$$

Here, we observe that the functions F_1, F_2, F_3 can be written in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ but F_4 can not be written in this form. This leads to the following definition:

A function $F(x, y)$ is said to be *homogeneous function of degree n* if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

We note that in the above examples, F_1, F_2, F_3 are homogeneous functions of degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

We also observe that

$$F_1(x, y) = x^2\left(\frac{y^2}{x^2}+\frac{2y}{x}\right) = x^2 h_1\left(\frac{y}{x}\right)$$

or
$$F_1(x, y) = y^2\left(1+\frac{2x}{y}\right) = y^2 h_2\left(\frac{x}{y}\right)$$

$$F_2(x, y) = x^1\left(2-\frac{3y}{x}\right) = x^1 h_3\left(\frac{y}{x}\right)$$

or
$$F_2(x, y) = y^1\left(2\frac{x}{y}-3\right) = y^1 h_4\left(\frac{x}{y}\right)$$

$$F_3(x, y) = x^0 \cos\left(\frac{y}{x}\right) = x^0\ h_5\left(\frac{y}{x}\right)$$

$$F_4(x, y) \neq x^n\ h_6\left(\frac{y}{x}\right)\ \text{, for any } n \in \mathbf{N}$$

or
$$F_4\ (x, y) \neq y^n\ h_7\left(\frac{x}{y}\right)\text{, for any } n \in \mathbf{N}$$

Therefore, a function F (x, y) is a homogeneous function of degree n if

$$F(x, y) = x^n g\left(\frac{y}{x}\right) \quad \text{or} \quad y^n h\left(\frac{x}{y}\right)$$

A differential equation of the form $\frac{dy}{dx}$= F (x, y) is said to be *homogenous* if F(x, y) is a homogenous function of degree zero.

To solve a homogeneous differential equation of the type

$$\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right) \qquad \text{... (1)}$$

We make the substitution $\qquad y = v \,.\, x \qquad$ ... (2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get

$$v + x\frac{dv}{dx} = g(v)$$

or $$x\frac{dv}{dx} = g(v) - v \quad \ldots (4)$$

Separating the variables in equation (4), we get

$$\frac{dv}{g(v) - v} = \frac{dx}{x} \quad \ldots (5)$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{g(v) - v} = \int \frac{1}{x}dx + C \quad \ldots (6)$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by $\frac{y}{x}$.

Note If the homogeneous differential equation is in the form $\frac{dx}{dy} = F(x, y)$ where, $F(x, y)$ is homogenous function of degree zero, then we make substitution $\frac{x}{y} = v$ i.e., $x = vy$ and we proceed further to find the general solution as discussed above by writing $\frac{dx}{dy} = F(x, y) = h\left(\frac{x}{y}\right)$.

Example 15 Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y} \quad \ldots (1)$$

Let $$F(x, y) = \frac{x + 2y}{x - y}$$

Now $$F(\lambda x, \lambda y) = \frac{\lambda(x + 2y)}{\lambda(x - y)} = \lambda^0 \cdot f(x, y)$$

Therefore, $F(x, y)$ is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \qquad \text{... (2)}$$

R.H.S. of differential equation (2) is of the form $g\left(\frac{y}{x}\right)$ and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (3)}$$

Differentiating equation (3) with respect to, x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (4)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1) we get

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

or
$$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

or
$$x\frac{dv}{dx} = \frac{v^2+v+1}{1-v}$$

or
$$\frac{v-1}{v^2+v+1}dv = \frac{-dx}{x}$$

Integrating both sides of equation (5), we get

$$\int\frac{v-1}{v^2+v+1}dv = -\int\frac{dx}{x}$$

or
$$\frac{1}{2}\int\frac{2v+1-3}{v^2+v+1}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv-\frac{3}{2}\int\frac{1}{v^2+v+1}dv=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\int\frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|-\frac{3}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)+C_1 \qquad \text{(Why?)}$$

Replacing v by $\frac{y}{x}$, we get

or $$\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right|=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\log|(y^2+xy+x^2)|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+2C_1$$

or $$\log|(x^2+xy+y^2)|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)+C$$

which is the general solution of the differential equation (1)

Example 16 Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$ is homogeneous and solve it.

Solution The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

It is a differential equation of the form $\frac{dy}{dx} = \mathrm{F}(x, y)$.

Here $$\mathrm{F}(x, y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Replacing x by λx and y by λy, we get

$$\mathrm{F}(\lambda x, \lambda y) = \frac{\lambda\left[y\cos\left(\frac{y}{x}\right) + x\right]}{\lambda\left(x\cos\frac{y}{x}\right)} = \lambda^0[\mathrm{F}(x, y)]$$

Thus, $\mathrm{F}(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

or $$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

or $$x\frac{dv}{dx} = \frac{1}{\cos v}$$

or $$\cos v\ dv = \frac{dx}{x}$$

Therefore $$\int \cos v\ dv = \int \frac{1}{x}\, dx$$

or $\sin v = \log |x| + \log |C|$

or $\sin v = \log |Cx|$

Replacing v by $\frac{y}{x}$, we get

$$\sin\left(\frac{y}{x}\right) = \log |Cx|$$

which is the general solution of the differential equation (1).

Example 17 Show that the differential equation $2y\, e^{\frac{x}{y}} dx + \left(y - 2x\, e^{\frac{x}{y}}\right) dy = 0$ is homogeneous and find its particular solution, given that, $x = 0$ when $y = 1$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x\, e^{\frac{x}{y}} - y}{2y\, e^{\frac{x}{y}}} \qquad \text{... (1)}$$

Let $$F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then $$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0 [F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \qquad \text{... (2)}$$

Differentiating equation (2) with respect to y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get

$$v + y\frac{dv}{dy} = \frac{2v\,e^{v} - 1}{2e^{v}}$$

or $$y\frac{dv}{dy} = \frac{2v\,e^{v} - 1}{2e^{v}} - v$$

or $$y\frac{dv}{dy} = -\frac{1}{2e^{v}}$$

or $$2e^{v}\,dv = \frac{-dy}{y}$$

or $$\int 2e^{v}\cdot dv = -\int\frac{dy}{y}$$

or $$2\,e^{v} = -\log|y| + C$$

and replacing v by $\frac{x}{y}$, we get

$$2\,e^{\frac{x}{y}} + \log|y| = C \qquad \text{... (3)}$$

Substituting $x = 0$ and $y = 1$ in equation (3), we get

$$2\,e^{0} + \log|1| = C \Rightarrow C = 2$$

Substituting the value of C in equation (3), we get

$$2\,e^{\frac{x}{y}} + \log|y| = 2$$

which is the particular solution of the given differential equation.

Example 18 Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Solution We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \qquad \text{... (1)}$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

or $$\frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

Therefore $$\int\frac{2v}{v^2-1}dv = -\int\frac{1}{x}dx$$

or $$\log|v^2 - 1| = -\log|x| + \log|C_1|$$

or $$\log|(v^2 - 1)(x)| = \log|C_1|$$

or $$(v^2 - 1)\,x = \pm\, C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2}-1\right)x = \pm\, C_1$$

or $$(y^2 - x^2) = \pm\, C_1\, x \text{ or } x^2 - y^2 = Cx$$

EXERCISE 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

2. $y' = \dfrac{x+y}{x}$

3. $(x - y)\, dy - (x + y)\, dx = 0$

4. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

5. $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

6. $x\, dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

7. $\left\{x\cos\left(\dfrac{y}{x}\right) + y\sin\left(\dfrac{y}{x}\right)\right\} y\, dx = \left\{y\sin\left(\dfrac{y}{x}\right) - x\cos\left(\dfrac{y}{x}\right)\right\} x\, dy$

8. $x\dfrac{dy}{dx} - y + x\sin\left(\dfrac{y}{x}\right) = 0$

9. $y\, dx + x\log\left(\dfrac{y}{x}\right) dy - 2x\, dy = 0$

10. $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right) dy = 0$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)\, dy + (x - y)\, dx = 0$; $y = 1$ when $x = 1$

12. $x^2\, dy + (xy + y^2)\, dx = 0$; $y = 1$ when $x = 1$

13. $\left[x\sin^2\left(\dfrac{y}{x}\right) - y\right] dx + x\, dy = 0$; $y = \dfrac{\pi}{4}$ when $x = 1$

14. $\dfrac{dy}{dx} - \dfrac{y}{x} + \operatorname{cosec}\left(\dfrac{y}{x}\right) = 0$; $y = 0$ when $x = 1$

15. $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0$; $y = 2$ when $x = 1$

16. A homogeneous differential equation of the from $\dfrac{dx}{dy} = h\left(\dfrac{x}{y}\right)$ can be solved by making the substitution.

(A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$

17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

9.5.3 *Linear differential equations*

A differential equation of the from

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$$

where, P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q} \qquad \text{... (1)}$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x) \frac{dy}{dx} + \mathrm{P}.(g(x))\, y = \mathrm{Q}\,.\, g(x) \qquad \text{... (2)}$$

Choose $g(x)$ in such a way that R.H.S. becomes a derivative of $y \cdot g(x)$.

i.e. $$g(x)\frac{dy}{dx} + \text{P}.\, g(x)\, y = \frac{d}{dx}[y \cdot g(x)]$$

or $$g(x)\frac{dy}{dx} + \text{P}.\, g(x)\, y = g(x)\frac{dy}{dx} + y\, g'(x)$$

$\Rightarrow$ $$\text{P}.\, g(x) = g'(x)$$

or $$\text{P} = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to x, we get

$$\int \text{P}dx = \int \frac{g'(x)}{g(x)}dx$$

or $$\int \text{P}\cdot dx = \log(g(x))$$

or $$g(x) = e^{\int \text{P}\, dx}$$

On multiplying the equation (1) by $g(x) = e^{\int \text{P}\, dx}$, the L.H.S. becomes the derivative of some function of x and y. This function $g(x) = e^{\int \text{P}\, dx}$ is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of $g(x)$ in equation (2), we get

$$e^{\int \text{P}\, dx}\frac{dy}{dx} + \text{P}e^{\int \text{P}\, dx}y = \text{Q}\cdot e^{\int \text{P}\, dx}$$

or $$\frac{d}{dx}\left(y\, e^{\int \text{P}dx}\right) = \text{Q}e^{\int \text{P}dx}$$

Integrating both sides with respect to x, we get

$$y \cdot e^{\int \text{P}dx} = \int\left(\text{Q}\cdot e^{\int \text{P}dx}\right)dx$$

or $$y = e^{-\int \text{P}dx}\cdot\int\left(\text{Q}\cdot e^{\int \text{P}dx}\right)dx + \text{C}$$

which is the general solution of the differential equation.

Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + Py = Q$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = $e^{\int P\,dx}$.

(iii) Write the solution of the given differential equation as

$$y\ (\text{I.F}) = \int (Q \times \text{I.F})\, dx + C$$

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + P_1 x = Q_1$, where, P_1 and Q_1 are constants or functions of y only. Then I.F = $e^{\int P_1\,dy}$ and the solution of the differential equation is given by

$$x\ .\ (\text{I.F}) = \int (Q_1 \times \text{I.F})\, dy + C$$

Example 19 Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$.

Solution Given differential equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -1 \text{ and } Q = \cos x$$

Therefore $\qquad \text{I.F} = e^{\int -1\,dx} = e^{-x}$

Multiplying both sides of equation by I.F, we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

or $\qquad \frac{dy}{dx}\left(ye^{-x}\right) = e^{-x}\cos x$

On integrating both sides with respect to x, we get

$$ye^{-x} = \int e^{-x}\cos x\, dx + C \qquad \text{... (1)}$$

Let $\qquad I = \int e^{-x}\cos x\, dx$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int (-\sin x)\,(-e^{-x})\, dx$$

$$= -\cos x\, e^{-x} - \int \sin x\, e^{-x}\, dx$$

$$= -\cos x\, e^{-x} - \left[\sin x(-e^{-x}) - \int \cos x\, (-e^{-x})\, dx\right]$$

$$= -\cos x\, e^{-x} + \sin x\, e^{-x} - \int \cos x\, e^{-x}\, dx$$

or $$\mathrm{I} = -\, e^{-x} \cos x + \sin x\, e^{-x} - \mathrm{I}$$

or $$2\mathrm{I} = (\sin x - \cos x)\, e^{-x}$$

or $$\mathrm{I} = \frac{(\sin x - \cos x)e^{-x}}{2}$$

Substituting the value of I in equation (1), we get

$$ye^{-x} = \left(\frac{\sin x - \cos x}{2}\right)e^{-x} + \mathrm{C}$$

or $$y = \left(\frac{\sin x - \cos x}{2}\right) + \mathrm{C}e^{x}$$

which is the general solution of the given differential equation.

Example 20 Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2 \ (x \neq 0)$.

Solution The given differential equation is

$$x\frac{dy}{dx} + 2y = x^2 \qquad \text{... (1)}$$

Dividing both sides of equation (1) by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is a linear differential equation of the type $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$, where $\mathrm{P} = \frac{2}{x}$ and $\mathrm{Q} = x$.

So $$\text{I.F} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2 \ [as\ e^{\log f(x)} = f(x)]$$

Therefore, solution of the given equation is given by

$$y \,.\, x^2 = \int (x)\,(x^2)\,dx + \mathrm{C} = \int x^3 dx + \mathrm{C}$$

or $$y = \frac{x^2}{4} + \mathrm{C}\,x^{-2}$$

which is the general solution of the given differential equation.

Example 21 Find the general solution of the differential equation $y\,dx - (x + 2y^2)\,dy = 0$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a linear differential equation of the type $\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$, where $\mathrm{P}_1 = -\frac{1}{y}$ and $\mathrm{Q}_1 = 2y$. Therefore $\text{I.F} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log (y)^{-1}} = \frac{1}{y}$

Hence, the solution of the given differential equation is

$$x\frac{1}{y} = \int (2y)\left(\frac{1}{y}\right) dy + \mathrm{C}$$

or
$$\frac{x}{y} = \int (2dy) + \mathrm{C}$$

or
$$\frac{x}{y} = 2y + \mathrm{C}$$

or
$$x = 2y^2 + \mathrm{C}y$$

which is a general solution of the given differential equation.

Example 22 Find the particular solution of the differential equation

$$\frac{dx}{dy} + y\cot x = 2x + x^2 \cot x \ (x \neq 0)$$

given that $y = 0$ when $x = \frac{\pi}{2}$.

Solution The given equation is a linear differential equation of the type $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$, where $\mathrm{P} = \cot x$ and $\mathrm{Q} = 2x + x^2 \cot x$. Therefore

$$\text{I.F} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Hence, the solution of the differential equation is given by

$$y \,.\, \sin x = \int (2x + x^2 \cot x) \sin x\, dx + \mathrm{C}$$

or $$y \sin x = \int 2x \sin x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = \sin x\left(\frac{2x^2}{2}\right) - \int \cos x\left(\frac{2x^2}{2}\right)dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x - \int x^2 \cos x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x + C \qquad \text{... (1)}$$

Substituting $y = 0$ and $x = \frac{\pi}{2}$ in equation (1), we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

or $$C = \frac{-\pi^2}{4}$$

Substituting the value of C in equation (1), we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

or $$y = x^2 - \frac{\pi^2}{4 \sin x} \; (\sin x \neq 0)$$

which is the particular solution of the given differential equation.

Example 23 Find the equation of a curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.

Solution We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = x + xy$$

or $$\frac{dy}{dx} - xy = x \qquad \text{... (1)}$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = -x$ and $Q = x$.

Therefore, $$\text{I.F} = e^{\int -x \, dx} = e^{\frac{-x^2}{2}}$$

Hence, the solution of equation is given by

$$y \cdot e^{\frac{-x^2}{2}} = \int (x)\left(e^{\frac{-x^2}{2}}\right) dx + C \qquad \text{... (2)}$$

Let $$I = \int (x)\, e^{\frac{-x^2}{2}}\, dx$$

Let $\dfrac{-x^2}{2} = t$, then $-\,x\,dx = dt$ or $x\,dx = -\,dt$.

Therefore, $I = -\int e^t\,dt = -e^t = -\,e^{\frac{-x^2}{2}}$

Substituting the value of I in equation (2), we get

$$y\, e^{\frac{-x^2}{2}} = -e^{\frac{-x^2}{2}} + C$$

or $$y = -1 + C\, e^{\frac{x^2}{2}} \qquad \text{... (3)}$$

Now (3) represents the equation of family of curves. But we are interested in finding a particular member of the family passing through (0, 1). Substituting $x = 0$ and $y = 1$ in equation (3) we get

$$1 = -\,1 + C\,.\,e^0 \quad \text{or} \quad C = 2$$

Substituting the value of C in equation (3), we get

$$y = -1 + 2\, e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

EXERCISE 9.6

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1. $\dfrac{dy}{dx} + 2y = \sin x$ 2. $\dfrac{dy}{dx} + 3y = e^{-2x}$ 3. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

4. $\dfrac{dy}{dx} + \sec x\, y = \tan x \left(0 \le x < \dfrac{\pi}{2}\right)$ 5. $\cos^2 x \dfrac{dy}{dx} + y = \tan x \left(0 \le x < \dfrac{\pi}{2}\right)$

6. $x\dfrac{dy}{dx} + 2y = x^2 \log x$ 7. $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x}\log x$

8. $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx\ (x \ne 0)$

9. $x\frac{dy}{dx}+y-x+xy\cot x=0\ (x\neq 0)$ 10. $(x+y)\frac{dy}{dx}=1$

11. $y\,dx + (x - y^2)\,dy = 0$ 12. $(x+3y^2)\frac{dy}{dx}=y\ (y>0)$.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx}+2y\tan x=\sin x;\ y=0$ when $x=\frac{\pi}{3}$

14. $(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2};\ y=0$ when $x=1$

15. $\frac{dy}{dx}-3y\cot x=\sin 2x;\ y=2$ when $x=\frac{\pi}{2}$

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

17. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

18. The Integrating Factor of the differential equation $x\frac{dy}{dx}-y=2x^2$ is

(A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

19. The Integrating Factor of the differential equation

$(1-y^2)\frac{dx}{dy}+yx = ay\,(-1<y<1)$ is

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Miscellaneous Examples

Example 24 Verify that the function $y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation

$$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)y=0$$

Solution The given function is

$$y = e^{ax}\,[c_1 \cos bx + c_2 \sin bx] \qquad \text{... (1)}$$

Differentiating both sides of equation (1) with respect to x, we get

$$\frac{dy}{dx} = e^{ax}\left[-bc_1 \sin bx + bc_2 \cos bx\right] + \left[c_1 \cos bx + c_2 \sin bx\right] e^{ax} \cdot a$$

or

$$\frac{dy}{dx} = e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx] \qquad \text{... (2)}$$

Differentiating both sides of equation (2) with respect to x, we get

$$\frac{d^2y}{dx^2} = e^{ax}[(bc_2 + ac_1)(-b\sin bx) + (ac_2 - bc_1)(b\cos bx)]$$

$$+ [(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx]\, e^{ax}.a$$

$$= e^{ax}[(a^2 c_2 - 2abc_1 - b^2c_2)\sin bx + (a^2 c_1 + 2abc_2 - b^2c_1)\cos bx]$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get

$$\text{L.H.S.} = e^{ax}[a^2c_2 - 2abc_1 - b^2c_2)\sin bx + (a^2c_1 + 2abc_2 - b^2c_1)\cos bx]$$

$$-2ae^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx]$$

$$+(a^2 + b^2)\,e^{ax}[c_1 \cos bx + c_2 \sin bx]$$

$$= e^{ax}\begin{bmatrix} \left(a^2c_2 - 2abc_1 - b^2c_2 - 2a^2c_2 + 2abc_1 + a^2c_2 + b^2c_2\right)\sin bx \\ +(a^2c_1 + 2abc_2 - b^2c_1 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1)\cos bx \end{bmatrix}$$

$$= e^{ax}[0 \times \sin bx + 0\cos bx] = e^{ax} \times 0 = 0 = \text{R.H.S.}$$

Hence, the given function is a solution of the given differential equation.

Example 25 Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Solution Let C denote the family of circles in the second quadrant and touching the coordinate axes. Let $(-a, a)$ be the coordinate of the centre of any member of this family (see Fig 9.6).

Equation representing the family C is

$$(x + a)^2 + (y - a)^2 = a^2 \quad \dots (1)$$

or $$x^2 + y^2 + 2ax - 2ay + a^2 = 0 \quad \dots (2)$$

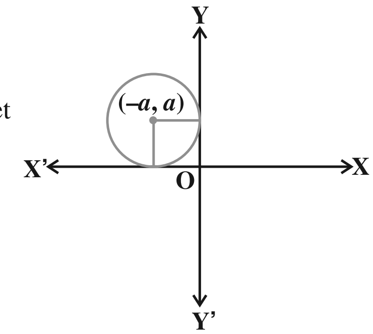


Fig 9.6

Differentiating equation (2) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

or $$x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right)$$

or $$a = \frac{x + y\,y'}{y' - 1}$$

Substituting the value of a in equation (1), we get

$$\left[x + \frac{x + y\,y'}{y' - 1}\right]^2 + \left[y - \frac{x + y\,y'}{y' - 1}\right]^2 = \left[\frac{x + y\,y'}{y' - 1}\right]^2$$

or $$[xy' - x + x + y\,y']^2 + [y\,y' - y - x - y\,y']^2 = [x + y\,y']^2$$

or $$(x + y)^2\,y'^2 + [x + y]^2 = [x + y\,y']^2$$

or $$(x + y)^2\,[(y')^2 + 1] = [x + y\,y']^2$$

which is the differential equation representing the given family of circles.

Example 26 Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$ when $x = 0$.

Solution The given differential equation can be written as

$$\frac{dy}{dx} = e^{(3x + 4y)}$$

or $$\frac{dy}{dx} = e^{3x} \cdot e^{4y} \quad \dots (1)$$

Separating the variables, we get

$$\frac{dy}{e^{4y}} = e^{3x}\,dx$$

Therefore $$\int e^{-4y}\,dy = \int e^{3x}\,dx$$

or $$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

or $$4\,e^{3x} + 3\,e^{-4y} + 12\,C = 0 \qquad \text{... (2)}$$

Substituting $x = 0$ and $y = 0$ in (2), we get

$$4 + 3 + 12\,C = 0 \text{ or } C = \frac{-7}{12}$$

Substituting the value of C in equation (2), we get

$$4\,e^{3x} + 3\,e^{-4y} - 7 = 0,$$

which is a particular solution of the given differential equation.

Example 27 Solve the differential equation

$$(x\,dy - y\,dx)\,y\,\sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x\,\cos\left(\frac{y}{x}\right).$$

Solution The given differential equation can be written as

$$\left[x\,y\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right]dy = \left[xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right]dx$$

or $$\frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)}$$

Dividing numerator and denominator on RHS by x^2, we get

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right) + \left(\frac{y^2}{x^2}\right)\sin\left(\frac{y}{x}\right)}{\frac{y}{x}\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

Clearly, equation (1) is a homogeneous differential equation of the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

To solve it, we make the substitution

$$y = vx \qquad \text{... (2)}$$

or $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} \qquad \text{(using (1) and (2))}$$

or $$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

or $$\left(\frac{v\sin v - \cos v}{v\cos v}\right)dv = \frac{2\,dx}{x}$$

Therefore $$\int\left(\frac{v\sin v - \cos v}{v\cos v}\right)dv = 2\int\frac{1}{x}\,dx$$

or $$\int \tan v\,dv - \int\frac{1}{v}\,dv = 2\int\frac{1}{x}\,dx$$

or $$\log|\sec v| - \log|v| = 2\log|x| + \log|C_1|$$

or $$\log\left|\frac{\sec v}{v\,x^2}\right| = \log|C_1|$$

or $$\frac{\sec v}{v\,x^2} = \pm C_1 \qquad \text{... (3)}$$

Replacing v by $\frac{y}{x}$ in equation (3), we get

$$\frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)(x^2)} = C \text{ where, } C = \pm C_1$$

or $$\sec\left(\frac{y}{x}\right) = C\,xy$$

which is the general solution of the given differential equation.

Example 28 Solve the differential equation

$$(\tan^{-1}y - x)\,dy = (1 + y^2)\,dx.$$

Solution The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \qquad \text{... (1)}$$

Now (1) is a linear differential equation of the form $\frac{dx}{dy} + P_1\, x = Q_1$,

where, $P_1 = \frac{1}{1+y^2}$ and $Q_1 = \frac{\tan^{-1} y}{1+y^2}$.

Therefore, $I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Thus, the solution of the given differential equation is

$$x e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y}\, dy + C \qquad \text{... (2)}$$

Let $$I = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y}\, dy$$

Substituting $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2}\right) dy = dt$, we get

$$I = \int t\, e^t\, dt = t\, e^t - \int 1\,.\, e^t\, dt = t\, e^t - e^t = e^t\,(t-1)$$

or $$I = e^{\tan^{-1} y}\,(\tan^{-1} y - 1)$$

Substituting the value of I in equation (2), we get

$$x\,.\,e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

or $$x = (\tan^{-1} y - 1) + C\, e^{-\tan^{-1} y}$$

which is the general solution of the given differential equation.

Miscellaneous Exercise on Chapter 9

1. For each of the differential equations given below, indicate its order and degree (if defined).

(i) $\frac{d^2 y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$ (ii) $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$

(iii) $\frac{d^4 y}{dx^4} - \sin\left(\frac{d^3 y}{dx^3}\right) = 0$

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(i) $y = a\, e^{x} + b\, e^{-x} + x^2$: $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$

(ii) $y = e^{x}\,(a \cos x + b \sin x)$: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

(iii) $y = x \sin 3x$: $\dfrac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2)\dfrac{dy}{dx} - xy = 0$

3. Form the differential equation representing the family of curves given by $(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

4. Prove that $x^2 - y^2 = c\,(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3x\,y^2)\,dx = (y^3 - 3x^2y)\,dy$, where c is a parameter.

5. Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

6. Find the general solution of the differential equation $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$.

7. Show that the general solution of the differential equation $\dfrac{dy}{dx} + \dfrac{y^2+y+1}{x^2+x+1} = 0$ is given by $(x + y + 1) = \text{A}\,(1 - x - y - 2xy)$, where A is parameter.

8. Find the equation of the curve passing through the point $\left(0, \dfrac{\pi}{4}\right)$ whose differential equation is $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$.

9. Find the particular solution of the differential equation
$(1 + e^{2x})\, dy + (1 + y^2)\, e^{x}\, dx = 0$, given that $y = 1$ when $x = 0$.

10. Solve the differential equation $y\, e^{\frac{x}{y}} dx = \left(x\, e^{\frac{x}{y}} + y^2\right) dy\ (y \neq 0)$.

11. Find a particular solution of the differential equation $(x - y)\,(dx + dy) = dx - dy$, given that $y = -1$, when $x = 0$. (Hint: put $x - y = t$)

12. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1\ (x \neq 0)$.

13. Find a particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$ $(x \neq 0)$, given that $y = 0$ when $x = \frac{\pi}{2}$.

14. Find a particular solution of the differential equation $(x+1)\frac{dy}{dx} = 2\,e^{-y} - 1$, given that $y = 0$ when $x = 0$.

15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20, 000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

16. The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$ is

(A) $xy = C$ (B) $x = Cy^2$ (C) $y = Cx$ (D) $y = Cx^2$

17. The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is

(A) $y\,e^{\int P_1\,dy} = \int\left(Q_1 e^{\int P_1\,dy}\right)dy + C$

(B) $y\,.\,e^{\int P_1\,dx} = \int\left(Q_1 e^{\int P_1\,dx}\right)dx + C$

(C) $x\,e^{\int P_1\,dy} = \int\left(Q_1 e^{\int P_1\,dy}\right)dy + C$

(D) $x\,e^{\int P_1\,dx} = \int\left(Q_1 e^{\int P_1\,dx}\right)dx + C$

18. The general solution of the differential equation $e^x\,dy + (y\,e^x + 2x)\,dx = 0$ is

(A) $x\,e^y + x^2 = C$ (B) $x\,e^y + y^2 = C$

(C) $y\,e^x + x^2 = C$ (D) $y\,e^y + x^2 = C$

Summary

- An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.
- Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
- Degree of a differential equation is defined if it is a polynomial equation in its derivatives.
- Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.
- To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.
- Variable separable method is used to solve such an equation in which variables can be separated completely i.e. terms containing y should remain with dy and terms containing x should remain with dx.
- A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$ where, $f(x, y)$ and $g(x, y)$ are homogenous functions of degree zero is called a homogeneous differential equation.
- A differential equation of the form $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$, where P and Q are constants or functions of x only is called a first order linear differential equation.

Historical Note

One of the principal languages of Science is that of differential equations. Interestingly, the date of birth of differential equations is taken to be November, 11,1675, when Gottfried Wilthelm Freiherr Leibnitz (1646 - 1716) first put in black and white the identity $\int y\, dy = \frac{1}{2} y^2$, thereby introducing both the symbols $\int$ and dy.

Leibnitz was actually interested in the problem of finding a curve whose tangents were prescribed. This led him to discover the '*method of separation of variables*' 1691. A year later he formulated the '*method of solving the homogeneous differential equations of the first order*'. He went further in a very short time to the discovery of the '*method of solving a linear differential equation of the first-order*'. How surprising is it that all these methods came from a single man and that too within 25 years of the birth of differential equations!

In the old days, what we now call the 'solution' of a differential equation, was used to be referred to as 'integral' of the differential equation, the word being coined by James Bernoulli (1654 - 1705) in 1690. The word 'solution was first used by Joseph Louis Lagrange (1736 - 1813) in 1774, which was almost hundred years since the birth of differential equations. It was Jules Henri Poincare (1854 - 1912) who strongly advocated the use of the word 'solution' and thus the word 'solution' has found its deserved place in modern terminology. The name of the '*method of separation of variables*' is due to John Bernoulli (1667 - 1748), a younger brother of James Bernoulli.

Application to geometric problems were also considered. It was again John Bernoulli who first brought into light the intricate nature of differential equations. In a letter to Leibnitz, dated May 20, 1715, he revealed the solutions of the differential equation

$$x^2 y'' = 2y,$$

which led to three types of curves, viz., parabolas, hyperbolas and a class of cubic curves. This shows how varied the solutions of such innocent looking differential equation can be. From the second half of the twentieth century attention has been drawn to the investigation of this complicated nature of the solutions of differential equations, under the heading '*qualitative analysis of differential equations*'. Now-a-days, this has acquired prime importance being absolutely necessary in almost all investigations.

Chapter 11

THREE DIMENSIONAL GEOMETRY

❖ *The moving power of mathematical invention is not reasoning but imagination. – A. DEMORGAN* ❖

11.1 Introduction

Leonhard Euler
(1707-1783)

In Class XI, while studying Analytical Geometry in two dimensions, and the introduction to three dimensional geometry, we confined to the Cartesian methods only. In the previous chapter of this book, we have studied some basic concepts of vectors. We will now use vector algebra to three dimensional geometry. The purpose of this approach to 3-dimensional geometry is that it makes the study simple and elegant*.

In this chapter, we shall study the direction cosines and direction ratios of a line joining two points and also discuss about the equations of lines and planes in space under different conditions, angle between two lines, two planes, a line and a plane, shortest distance between two skew lines and distance of a point from a plane. Most of the above results are obtained in vector form. Nevertheless, we shall also translate these results in the Cartesian form which, at times, presents a more clear geometric and analytic picture of the situation.

11.2 Direction Cosines and Direction Ratios of a Line

From Chapter 10, recall that if a directed line L passing through the origin makes angles α, β and γ with *x*, *y* and *z*-axes, respectively, called direction angles, then cosine of these angles, namely, cos α, cos β and cos γ are called direction cosines of the directed line L.

If we reverse the direction of L, then the direction angles are replaced by their supplements, i.e., $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Thus, the signs of the direction cosines are reversed.

* For various activities in three dimensional geometry, one may refer to the Book ***"A Hand Book for designing Mathematics Laboratory in Schools"***, NCERT, 2005

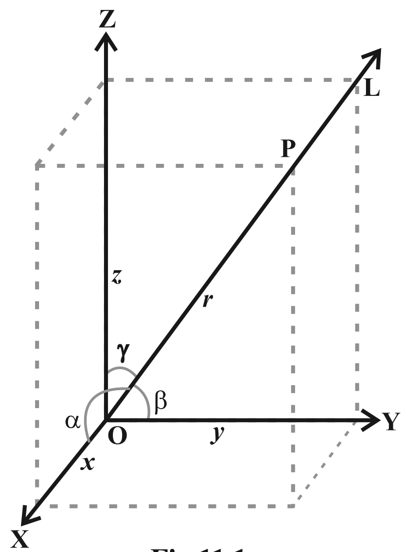


Fig 11.1

Note that a given line in space can be extended in two opposite directions and so it has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by l, m and n.

Remark If the given line in space does not pass through the origin, then, in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now take one of the directed lines from the origin and find its direction cosines as two parallel line have same set of direction cosines.

Any three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line. If l, m, n are direction cosines and a, b, c are direction ratios of a line, then $a = \lambda l$, $b=\lambda m$ and $c = \lambda n$, for any nonzero $\lambda \in \mathbf{R}$.

Note Some authors also call direction ratios as direction numbers.

Let a, b, c be direction ratios of a line and let l, m and n be the direction cosines ($d.c$'s) of the line. Then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ (say), } k \text{ being a constant.}$$

Therefore $\qquad l = ak,\ m = bk,\ n = ck \qquad \text{... (1)}$

But $\qquad l^2 + m^2 + n^2 = 1$

Therefore $\qquad k^2 (a^2 + b^2 + c^2) = 1$

or $\qquad k = \pm \dfrac{1}{\sqrt{a^2+b^2+c^2}}$

Hence, from (1), the *d.c.'s* of the line are

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

where, depending on the desired sign of k, either a positive or a negative sign is to be taken for l, m and n.

For any line, if a, b, c are direction ratios of a line, then ka, kb, kc; $k \neq 0$ is also a set of direction ratios. So, any two sets of direction ratios of a line are also proportional. Also, for any line there are infinitely many sets of direction ratios.

11.2.1 *Relation between the direction cosines of a line*

Consider a line RS with direction cosines l, m, n. Through the origin draw a line parallel to the given line and take a point P(x, y, z) on this line. From P draw a perpendicular PA on the x-axis (Fig. 11.2).

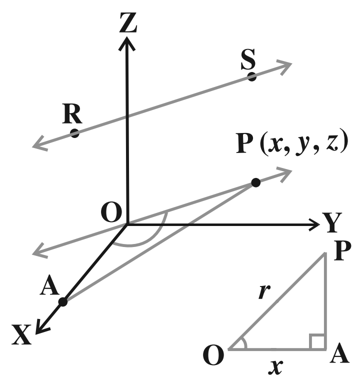


Fig 11.2

Let OP = r. Then $\cos\alpha = \frac{\text{OA}}{\text{OP}} = \frac{x}{r}$. This gives $x = lr$.

Similarly, $\quad y = mr$ and $z = nr$

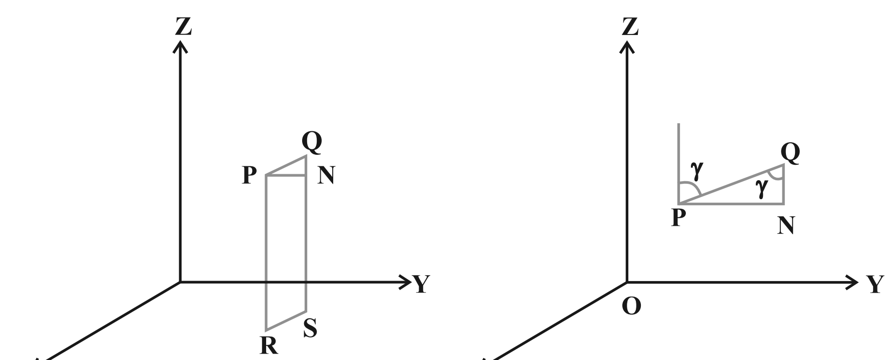
Thus $\quad x^2 + y^2 + z^2 = r^2 (l^2 + m^2 + n^2)$

But $\quad x^2 + y^2 + z^2 = r^2$

Hence $\quad \boldsymbol{l^2 + m^2 + n^2 = 1}$

11.2.2 *Direction cosines of a line passing through two points*

Since one and only one line passes through two given points, we can determine the direction cosines of a line passing through the given points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) as follows (Fig 11.3 (a)).



(a) (b)

Fig 11.3

Let l, m, n be the direction cosines of the line PQ and let it makes angles α, β and γ with the x, y and z-axis, respectively.

Draw perpendiculars from P and Q to XY-plane to meet at R and S. Draw a perpendicular from P to QS to meet at N. Now, in right angle triangle PNQ, $\angle$PQN= γ (Fig 11.3 (b).

Therefore, $$\cos\gamma = \frac{NQ}{PQ} = \frac{z_2 - z_1}{PQ}$$

Similarly $$\cos\alpha = \frac{x_2 - x_1}{PQ} \text{ and } \cos\beta = \frac{y_2 - y_1}{PQ}$$

Hence, the direction cosines of the line segment joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + \left(z_2 - z_1\right)^2}$$

Note The direction ratios of the line segment joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) may be taken as

$$x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \text{ or } x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2$$

Example 1 If a line makes angle 90°, 60° and 30° with the positive direction of x, y and z-axis respectively, find its direction cosines.

Solution Let the $d.c.$'s of the lines be l , m, n. Then $l = \cos 90^0 = 0$, $m = \cos 60^0 = \frac{1}{2}$,

$n = \cos 30^0 = \frac{\sqrt{3}}{2}$.

Example 2 If a line has direction ratios 2, – 1, – 2, determine its direction cosines.

Solution Direction cosines are

$$\frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-2}{\sqrt{2^2 + \left(-1\right)^2 + (-2)^2}}$$

or $$\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$$

Example 3 Find the direction cosines of the line passing through the two points (– 2, 4, – 5) and (1, 2, 3).

Solution We know the direction cosines of the line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Here P is $(-2, 4, -5)$ and Q is $(1, 2, 3)$.

So $$PQ = \sqrt{(1-(-2))^2 + (2-4)^2 + (3-(-5))^2} = \sqrt{77}$$

Thus, the direction cosines of the line joining two points is

$$\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

Example 4 Find the direction cosines of x, y and z-axis.

Solution The x-axis makes angles 0°, 90° and 90° respectively with x, y and z-axis. Therefore, the direction cosines of x-axis are cos 0°, cos 90°, cos 90° i.e., 1,0,0. Similarly, direction cosines of y-axis and z-axis are 0, 1, 0 and 0, 0, 1 respectively.

Example 5 Show that the points A (2, 3, – 4), B (1, – 2, 3) and C (3, 8, – 11) are collinear.

Solution Direction ratios of line joining A and B are

$1 - 2, -2 - 3, 3 + 4$ i.e., $-1, -5, 7$.

The direction ratios of line joining B and C are

$3 - 1, 8 + 2, -11 - 3$, i.e., $2, 10, -14$.

It is clear that direction ratios of AB and BC are proportional, hence, AB is parallel to BC. But point B is common to both AB and BC. Therefore, A, B, C are collinear points.

EXERCISE 11.1

1. If a line makes angles 90°, 135°, 45° with the x, y and z-axes respectively, find its direction cosines.
2. Find the direction cosines of a line which makes equal angles with the coordinate axes.
3. If a line has the direction ratios –18, 12, – 4, then what are its direction cosines ?
4. Show that the points (2, 3, 4), (– 1, – 2, 1), (5, 8, 7) are collinear.
5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2).

11.3 Equation of a Line in Space

We have studied equation of lines in two dimensions in Class XI, we shall now study the vector and cartesian equations of a line in space.

A line is uniquely determined if

(i) it passes through a given point and has given direction, or

(ii) it passes through two given points.

11.3.1 *Equation of a line through a given point and parallel to a given vector* $\vec{b}$

Let $\vec{a}$ be the position vector of the given point A with respect to the origin O of the rectangular coordinate system. Let l be the line which passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the line (Fig 11.4).

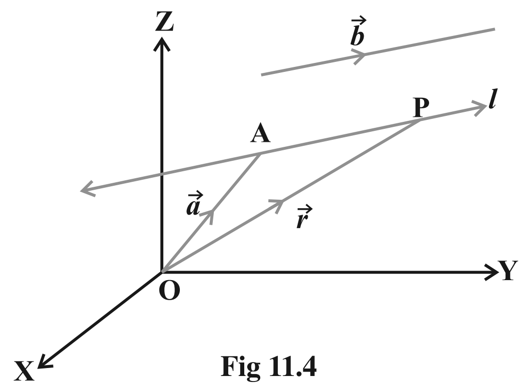


Fig 11.4

Then $\overrightarrow{AP}$ is parallel to the vector $\vec{b}$, i.e., $\overrightarrow{AP} = \lambda\vec{b}$, where λ is some real number.

But $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

i.e. $\lambda\vec{b} = \vec{r} - \vec{a}$

Conversely, for each value of the parameter λ, this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda\vec{b} \quad \text{... (1)}$$

Remark If $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$, then a, b, c are direction ratios of the line and conversely, if a, b, c are direction ratios of a line, then $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ will be the parallel to the line. Here, b should not be confused with $|\vec{b}|$.

Derivation of cartesian form from vector form

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b, c. Consider the coordinates of any point P be (x, y, z). Then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k};\ \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

and $$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$$

Substituting these values in (1) and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a;\ y = y_1 + \lambda b;\ z = z_1 + \lambda c \quad \text{... (2)}$$

These are parametric equations of the line. Eliminating the parameter λ from (2), we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{... (3)}$$

This is the Cartesian equation of the line.

Note If l, m, n are the direction cosines of the line, the equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Example 6 Find the vector and the Cartesian equations of the line through the point (5, 2, – 4) and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Solution We have

$$\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$$

Therefore, the vector equation of the line is

$$\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$$

Now, $\vec{r}$ is the position vector of any point P(x, y, z) on the line.

Therefore, $\quad x\hat{i} + y\hat{j} + z\hat{k} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$

$$= (5 + 3\lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (-4 - 8\lambda)\hat{k}$$

Eliminating λ, we get

$$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$$

which is the equation of the line in Cartesian form.

11.3.2 *Equation of a line passing through two given points*

Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2), respectively that are lying on a line (Fig 11.5).

Let $\vec{r}$ be the position vector of an arbitrary point P(x, y, z), then P is a point on the line if and only if $\overrightarrow{AP} = \vec{r} - \vec{a}$ and $\overrightarrow{AB} = \vec{b} - \vec{a}$ are collinear vectors. Therefore, P is on the line if and only if

$$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

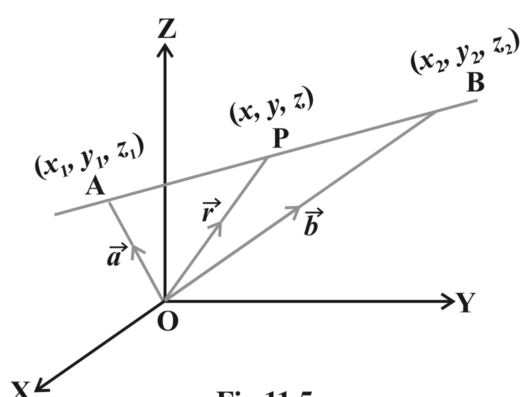


Fig 11.5

or $\qquad \boldsymbol{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$, $\lambda \in$ **R**. ... (1)

This is the vector equation of the line.

Derivation of cartesian form from vector form

We have

$$\vec{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k},\ \vec{a} = x_1\hat{i} + y_1\,\hat{j} + z_1\,\hat{k} \text{ and } \vec{b} = x_2\hat{i} + y_2\,\hat{j} + z_2\,\hat{k},$$

Substituting these values in (1), we get

$$x\hat{i} + y\,\hat{j} + z\,\hat{k} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\hat{k} + \lambda\,[(x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}]$$

Equating the like coefficients of $\hat{i},\ \hat{j},\ \hat{k}$, we get

$$x = x_1 + \lambda\ (x_2 - x_1);\ y = y_1 + \lambda\ (y_2 - y_1);\ z = z_1 + \lambda\ (z_2 - z_1)$$

On eliminating λ, we obtain

$$\boldsymbol{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

which is the equation of the line in Cartesian form.

Example 7 Find the vector equation for the line passing through the points (–1, 0, 2) and (3, 4, 6).

Solution Let $\vec{a}$ and $\vec{b}$ be the position vectors of the point A (– 1, 0, 2) and B (3, 4, 6).

Then $\qquad \vec{a} = -\,\hat{i} + 2\hat{k}$

and $\qquad \vec{b} = 3\hat{i} + 4\,\hat{j} + 6\hat{k}$

Therefore $\qquad \vec{b} - \vec{a} = 4\hat{i} + 4\,\hat{j} + 4\hat{k}$

Let $\vec{r}$ be the position vector of any point on the line. Then the vector equation of the line is

$$\vec{r} = -\,\hat{i} + 2\hat{k} + \lambda\,(4\hat{i} + 4\,\hat{j} + 4\hat{k})$$

Example 8 The Cartesian equation of a line is

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$$

Find the vector equation for the line.

Solution Comparing the given equation with the standard form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

We observe that $\qquad x_1 = -\,3,\ y_1 = 5,\ z_1 = -\,6;\ a = 2,\ b = 4,\ c = 2.$

Thus, the required line passes through the point (– 3, 5, – 6) and is parallel to the vector $2\hat{i}+4\hat{j}+2\hat{k}$. Let $\vec{r}$ be the position vector of any point on the line, then the vector equation of the line is given by

$$\vec{r}=(-3\hat{i}+5\hat{j}-6\hat{k})+\lambda\ (2\hat{i}+4\hat{j}+2\hat{k})$$

11.4 Angle between Two Lines

Let L_1 and L_2 be two lines passing through the origin and with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, respectively. Let P be a point on L_1 and Q be a point on L_2. Consider the directed lines OP and OQ as given in Fig 11.6. Let θ be the acute angle between OP and OQ. Now recall that the directed line segments OP and OQ are vectors with components a_1, b_1, c_1 and a_2, b_2, c_2, respectively. Therefore, the angle θ between them is given by

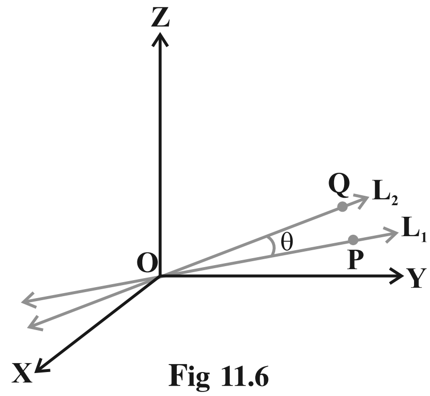


Fig 11.6

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right| \qquad \text{... (1)}$$

The angle between the lines in terms of sin θ is given by

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{(a_1a_2+b_1b_2+c_1c_2)^2}{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)}}$$

$$= \frac{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)-(a_1a_2+b_1b_2+c_1c_2)^2}}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$$

$$= \frac{\sqrt{(a_1\,b_2-a_2\,b_1)^2+(b_1\,c_2-b_2\,c_1)^2+(c_1\,a_2-c_2\,a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}} \qquad \text{... (2)}$$

Note In case the lines L_1 and L_2 do not pass through the origin, we may take lines L_1' and L_2' which are parallel to L_1 and L_2 respectively and pass through the origin.

If instead of direction ratios for the lines L_1 and L_2, direction cosines, namely, l_1, m_1, n_1 for L_1 and l_2, m_2, n_2 for L_2 are given, then (1) and (2) takes the following form:

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| \quad (\text{as } l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2) \qquad \ldots (3)$$

and $$\sin\theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 - (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2} \qquad \ldots (4)$$

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are

(i) perpendicular i.e. if $\theta = 90°$ by (1)

$$\boldsymbol{a_1 a_2 + b_1 b_2 + c_1 c_2 = 0}$$

(ii) parallel i.e. if $\theta = 0$ by (2)

$$\boldsymbol{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$$

Now, we find the angle between two lines when their equations are given. If θ is acute the angle between the lines

$$\vec{r} = \bar{a}_1 + \lambda \vec{b}_1 \text{ and } \vec{r} = \vec{a}_2 + \mu \vec{b}_2$$

then $$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \right|$$

In Cartesian form, if θ is the angle between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \qquad \ldots (1)$$

and $$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \qquad \ldots (2)$$

where, a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of the lines (1) and (2), respectively, then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

Example 9 Find the angle between the pair of lines given by

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

and $$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

Solution Here $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

The angle θ between the two lines is given by

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| = \left|\frac{(\hat{i}+2\hat{j}+2\hat{k})\cdot(3\hat{i}+2\hat{j}+6\hat{k})}{\sqrt{1+4+4}\,\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21}$$

Hence $\theta = \cos^{-1}\left(\frac{19}{21}\right)$

Example 10 Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$$

and

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

Solution The direction ratios of the first line are 3, 5, 4 and the direction ratios of the second line are 1, 1, 2. If θ is the angle between them, then

$$\cos\theta = \left|\frac{3.1+5.1+4.2}{\sqrt{3^2+5^2+4^2}\,\sqrt{1^2+1^2+2^2}}\right| = \frac{16}{\sqrt{50}\,\sqrt{6}} = \frac{16}{5\sqrt{2}\,\sqrt{6}} = \frac{8\sqrt{3}}{15}$$

Hence, the required angle is $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$.

11.5 Shortest Distance between Two Lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line.

Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are *non coplanar* and are called *skew lines*. For example, let us consider a room of size 1, 3, 2 units along *x*, *y* and *z*-axes respectively Fig 11.7.

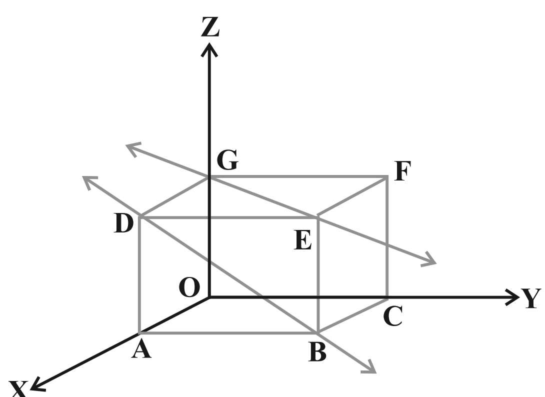


Fig 11.7

The line GE that goes diagonally across the ceiling and the line DB passes through one corner of the ceiling directly above A and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

11.5.1 *Distance between two skew lines*

We now determine the shortest distance between two skew lines in the following way: Let l_1 and l_2 be two skew lines with equations (Fig. 11.8)

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \qquad \text{... (1)}$$

and

$$\vec{r} = \vec{a}_2 + \mu \vec{b}_2 \qquad \text{... (2)}$$

Take any point S on l_1 with position vector $\vec{a}_1$ and T on l_2, with position vector $\vec{a}_2$. Then the magnitude of the shortest distance vector will be equal to that of the projection of ST along the direction of the line of shortest distance (See 10.6.2).

If $\overrightarrow{PQ}$ is the shortest distance vector between l_1 and l_2, then it being perpendicular to both $\vec{b}_1$ and $\vec{b}_2$, the unit vector $\hat{n}$ along $\overrightarrow{PQ}$ would therefore be

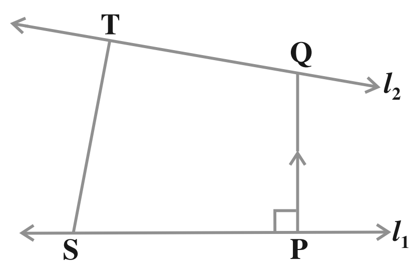


Fig 11.8

$$\hat{n} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|} \qquad \text{... (3)}$$

Then

$$\overrightarrow{PQ} = d\,\hat{n}$$

where, d is the magnitude of the shortest distance vector. Let θ be the angle between $\overrightarrow{ST}$ and $\overrightarrow{PQ}$. Then

$$PQ = ST\,|\cos\theta|$$

But

$$\cos\theta = \left|\frac{\overrightarrow{PQ}\cdot\overrightarrow{ST}}{|\overrightarrow{PQ}|\,|\overrightarrow{ST}|}\right|$$

$$= \left|\frac{d\,\hat{n}\cdot(\vec{a}_2 - \vec{a}_1)}{d\ ST}\right| \qquad (\text{since } \overrightarrow{ST} = \vec{a}_2 - \vec{a}_1)$$

$$= \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{ST\,|\vec{b}_1 \times \vec{b}_2|}\right| \qquad \text{[From (3)]}$$

Hence, the required shortest distance is

$$d = \text{PQ} = \text{ST}\ |\cos\theta|$$

or

$$\boldsymbol{d} = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Cartesian form

The shortest distance between the lines

$$l_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and

$$l_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

11.5.2 *Distance between parallel lines*

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be given by

$$\vec{r} = \vec{a}_1 + \lambda \vec{b} \qquad \text{... (1)}$$

and

$$\vec{r} = \vec{a}_2 + \mu \vec{b} \qquad \text{... (2)}$$

where, $\vec{a}_1$ is the position vector of a point S on l_1 and $\vec{a}_2$ is the position vector of a point T on l_2 Fig 11.9.

As l_1, l_2 are coplanar, if the foot of the perpendicular from T on the line l_1 is P, then the distance between the lines l_1 and l_2 = |TP|.

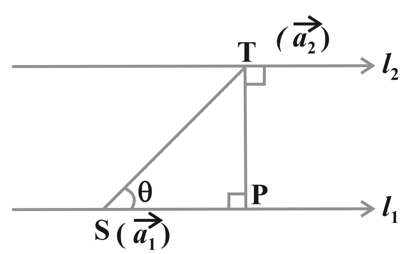


Fig 11.9

Let θ be the angle between the vectors $\overrightarrow{\text{ST}}$ and $\vec{b}$. Then

$$\vec{b} \times \overrightarrow{\text{ST}} = (|\vec{b}||\overrightarrow{\text{ST}}|\sin\theta)\hat{n} \qquad \text{... (3)}$$

where $\hat{n}$ is the unit vector perpendicular to the plane of the lines l_1 and l_2

But $\qquad \overrightarrow{\text{ST}} = \vec{a}_2 - \vec{a}_1$

Therefore, from (3), we get

$$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|\,\text{PT}\,\hat{n} \qquad \text{(since PT = ST sin } \theta)$$

i.e.,
$$|\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|\,\text{PT}\cdot 1 \qquad (\text{as } |\hat{n}| = 1)$$

Hence, the distance between the given parallel lines is

$$\boldsymbol{d = |\overrightarrow{\mathbf{PT}}| = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|}$$

Example 11 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} - \hat{j} + \hat{k}) \qquad \text{... (1)}$$

and
$$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu\,(3\hat{i} - 5\hat{j} + 2\hat{k}) \qquad \text{... (2)}$$

Solution Comparing (1) and (2) with $\vec{r} = \vec{a}_1 + \lambda\,\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\,\vec{b}_2$ respectively, we get

$$\vec{a}_1 = \hat{i} + \hat{j},\ \vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k} \text{ and } \vec{b}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}$$

Therefore
$$\vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$$

and
$$\vec{b}_1 \times \vec{b}_2 = (2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} - 5\hat{j} + 2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

So
$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9 + 1 + 49} = \sqrt{59}$$

Hence, the shortest distance between the given lines is given by

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\,.\,(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \frac{|3 - 0 + 7|}{\sqrt{59}} = \frac{10}{\sqrt{59}}$$

Example 12 Find the distance between the lines l_1 and l_2 given by

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and
$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Solution The two lines are parallel (Why?) We have

$$\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k},\ \vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k} \text{ and } \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Therefore, the distance between the lines is given by

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right| = \left| \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4+9+36}} \right|$$

or $$= \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{\sqrt{49}} = \frac{\sqrt{293}}{\sqrt{49}} = \frac{\sqrt{293}}{7}$$

EXERCISE 11.2

1. Show that the three lines with direction cosines

   $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13};\ \frac{4}{13}, \frac{12}{13}, \frac{3}{13};\ \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

2. Show that the line through the points (1, – 1, 2), (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).
3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1), (1, 2, 5).
4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.
5. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i} - j + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.
6. Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.
7. The cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.
8. Find the vector and the cartesian equations of the lines that passes through the origin and (5, – 2, 3).

9. Find the vector and the cartesian equations of the line that passes through the points (3, – 2, – 5), (3, – 2, 6).

10. Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and
$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and
$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

11. Find the angle between the following pair of lines:

(i) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$ and $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(ii) $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ and $\dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$

12. Find the values of p so that the lines $\dfrac{1-x}{3} = \dfrac{7y-14}{2p} = \dfrac{z-3}{2}$

and $\dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5}$ are at right angles.

13. Show that the lines $\dfrac{x-5}{7} = \dfrac{y+2}{-5} = \dfrac{z}{1}$ and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$ are perpendicular to each other.

14. Find the shortest distance between the lines
$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and
$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

15. Find the shortest distance between the lines
$\dfrac{x+1}{7} = \dfrac{y+1}{-6} = \dfrac{z+1}{1}$ and $\dfrac{x-3}{1} = \dfrac{y-5}{-2} = \dfrac{z-7}{1}$

16. Find the shortest distance between the lines whose vector equations are
$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$
and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

17. Find the shortest distance between the lines whose vector equations are
$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and
$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

11.6 Plane

A plane is determined uniquely if any one of the following is known:

(i) the normal to the plane and its distance from the origin is given, i.e., equation of a plane in normal form.

(ii) it passes through a point and is perpendicular to a given direction.

(iii) it passes through three given non collinear points.

Now we shall find vector and Cartesian equations of the planes.

11.6.1 *Equation of a plane in normal form*

Consider a plane whose perpendicular distance from the origin is d $(d \neq 0)$. Fig 11.10.

If $\overrightarrow{\mathrm{ON}}$ is the normal from the origin to the plane, and $\hat{n}$ is the unit normal vector along $\overrightarrow{\mathrm{ON}}$. Then $\overrightarrow{\mathrm{ON}} = d\ \hat{n}$. Let P be any point on the plane. Therefore, $\overrightarrow{\mathrm{NP}}$ is perpendicular to $\overrightarrow{\mathrm{ON}}$.

Therefore, $\overrightarrow{\mathrm{NP}} \cdot \overrightarrow{\mathrm{ON}} = 0$ ... (1)

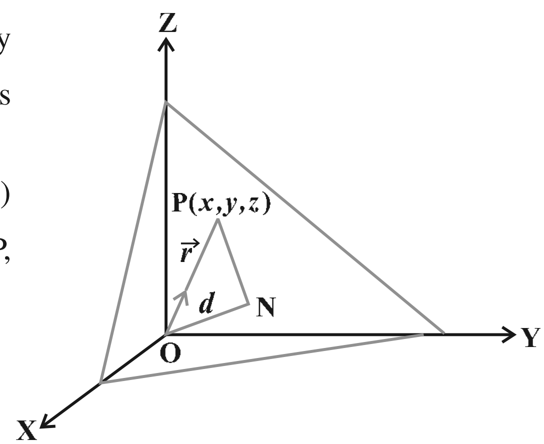


Fig 11.10

Let $\vec{r}$ be the position vector of the point P, then $\overrightarrow{\mathrm{NP}} = \vec{r} - d\ \hat{n}$ (as $\overrightarrow{\mathrm{ON}} + \overrightarrow{\mathrm{NP}} = \overrightarrow{\mathrm{OP}}$)

Therefore, (1) becomes

$$(\vec{r} - d\,\hat{n}) \cdot d\,\hat{n} = 0$$

or $$(\vec{r} - d\,\hat{n}) \cdot \hat{n} = 0 \qquad (d \neq 0)$$

or $$\vec{r} \cdot \hat{n} - d\,\hat{n} \cdot \hat{n} = 0$$

i.e., $$\boldsymbol{\vec{r} \cdot \hat{n} = d} \qquad (\text{as } \hat{n} \cdot \hat{n} = 1) \qquad \ldots (2)$$

This is the vector form of the equation of the plane.

Cartesian form

Equation (2) gives the vector equation of a plane, where $\hat{n}$ is the unit vector normal to the plane. Let P(x, y, z) be any point on the plane. Then

$$\overrightarrow{\mathrm{OP}} = \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Let l, m, n be the direction cosines of $\hat{n}$. Then

$$\hat{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

Therefore, (2) gives

$$(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})\cdot(l\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = d$$

i.e., $\boldsymbol{lx + my + nz = d}$ ... (3)

This is the cartesian equation of the plane in the normal form.

Note Equation (3) shows that if $\vec{r}\cdot(a\,\hat{i} + b\,\hat{j} + c\,\hat{k}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the Cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.

Example 13 Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also find its cartesian form.

Solution Let $\vec{n} = 2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}$. Then

$$\hat{n} = \frac{\vec{n}}{|\,\vec{n}\,|} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{4+9+16}} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{29}}$$

Hence, the required equation of the plane is

$$\vec{r}\cdot\left(\frac{2}{\sqrt{29}}\,\hat{i} + \frac{-3}{\sqrt{29}}\,\hat{j} + \frac{4}{\sqrt{29}}\,\hat{k}\right) = \frac{6}{\sqrt{29}}$$

Example 14 Find the direction cosines of the unit vector perpendicular to the plane $\vec{r}\cdot(6\,\hat{i} - 3\,\hat{j} - 2\,\hat{k}) + 1 = 0$ passing through the origin.

Solution The given equation can be written as

$$\vec{r}\cdot(-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}) = 1 \qquad \text{... (1)}$$

Now $|-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}| = \sqrt{36+9+4} = 7$

Therefore, dividing both sides of (1) by 7, we get

$$\vec{r}\cdot\left(-\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k}\right) = \frac{1}{7}$$

which is the equation of the plane in the form $\vec{r}\cdot\hat{n} = d$.

This shows that $\hat{n} = -\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k}$ is a unit vector perpendicular to the plane through the origin. Hence, the direction cosines of $\hat{n}$ are $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$.

Example 15 Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin.

Solution Since the direction ratios of the normal to the plane are 2, –3, 4; the direction cosines of it are

$$\frac{2}{\sqrt{2^2+(-3)^2+4^2}}, \frac{-3}{\sqrt{2^2+(-3)^2+4^2}}, \frac{4}{\sqrt{2^2+(-3)^2+4^2}}, \text{ i.e., } \frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

Hence, dividing the equation $2x-3y+4z-6=0$ i.e., $2x-3y+4z=6$ throughout by $\sqrt{29}$, we get

$$\frac{2}{\sqrt{29}}x + \frac{-3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

This is of the form $lx + my + nz = d$, where d is the distance of the plane from the origin. So, the distance of the plane from the origin is $\frac{6}{\sqrt{29}}$.

Example 16 Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$.

Solution Let the coordinates of the foot of the perpendicular P from the origin to the plane is (x_1, y_1, z_1) (Fig 11.11).

Then, the direction ratios of the line OP are x_1, y_1, z_1.

Writing the equation of the plane in the normal form, we have

$$\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

where, $\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$ are the direction cosines of the OP.

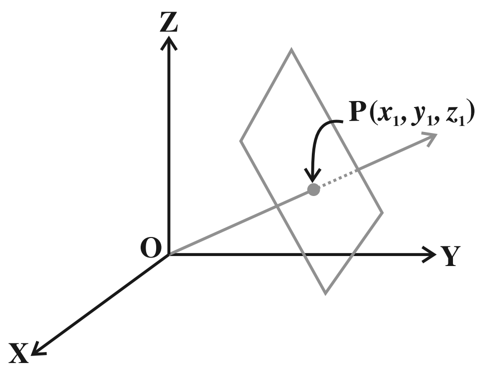


Fig 11.11

Since *d.c.'s* and direction ratios of a line are proportional, we have

$$\frac{x_1}{\frac{2}{\sqrt{29}}} = \frac{y_1}{\frac{-3}{\sqrt{29}}} = \frac{z_1}{\frac{4}{\sqrt{29}}} = k$$

i.e.,
$$x_1 = \frac{2k}{\sqrt{29}}, y_1 = \frac{-3k}{\sqrt{29}}, z_1 = \frac{4k}{\sqrt{29}}$$

Substituting these in the equation of the plane, we get $k = \dfrac{6}{\sqrt{29}}$.

Hence, the foot of the perpendicular is $\left(\dfrac{12}{29}, \dfrac{-18}{29}, \dfrac{24}{29}\right)$.

Note If d is the distance from the origin and l, m, n are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

11.6.2 ***Equation of a plane perpendicular to a given vector and passing through a given point***

In the space, there can be many planes that are perpendicular to the given vector, but through a given point P(x_1, y_1, z_1), only one such plane exists (see Fig 11.12).

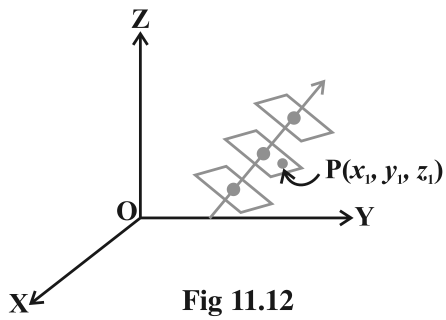


Fig 11.12

Let a plane pass through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$.

Let $\vec{r}$ be the position vector of any point P(x, y, z) in the plane. (Fig 11.13).

Then the point P lies in the plane if and only if $\overrightarrow{AP}$ is perpendicular to $\vec{N}$. i.e., $\overrightarrow{AP} . \vec{N} = 0$. But $\overrightarrow{AP} = \vec{r} - \vec{a}$. Therefore, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$ ... (1)

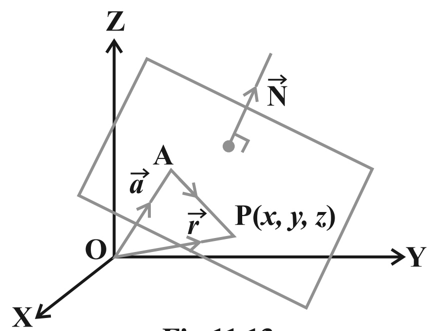


Fig 11.13

This is the vector equation of the plane.

Cartesian form

Let the given point A be (x_1, y_1, z_1), P be (x, y, z) and direction ratios of $\vec{N}$ are A, B and C. Then,

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ and } \vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Now $\qquad (\vec{r} - \vec{a}) \cdot \vec{N} = 0$

So $\qquad \left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] \cdot (A\hat{i} + B\hat{j} + C\hat{k}) = 0$

i.e. $\qquad \mathbf{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$

Example 17 Find the vector and cartesian equations of the plane which passes through the point (5, 2, – 4) and perpendicular to the line with direction ratios 2, 3, – 1.

Solution We have the position vector of point (5, 2, – 4) as $\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and the normal vector $\vec{N}$ perpendicular to the plane as $\vec{N} = 2\hat{i} + 3\hat{j} - \hat{k}$

Therefore, the vector equation of the plane is given by $(\vec{r} - \vec{a}).\vec{N} = 0$

or $$[\vec{r} - (5\hat{i} + 2\hat{j} - 4\hat{k})] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0 \qquad \ldots (1)$$

Transforming (1) into Cartesian form, we have

$$[(x-5)\hat{i} + (y-2)\hat{j} + (z+4)\hat{k}] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$$

or $$2(x-5) + 3(y-2) - 1(z+4) = 0$$

i.e. $$2x + 3y - z = 20$$

which is the cartesian equation of the plane.

11.6.3 *Equation of a plane passing through three non collinear points*

Let R, S and T be three non collinear points on the plane with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively (Fig 11.14).

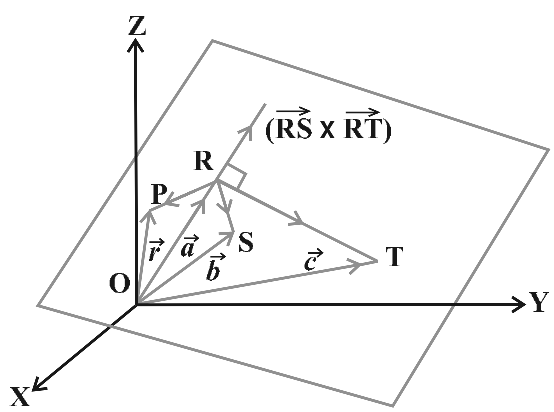


Fig 11.14

The vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are in the given plane. Therefore, the vector $\overrightarrow{RS} \times \overrightarrow{RT}$ is perpendicular to the plane containing points R, S and T. Let $\vec{r}$ be the position vector of any point P in the plane. Therefore, the equation of the plane passing through R and perpendicular to the vector $\overrightarrow{RS} \times \overrightarrow{RT}$ is

$$(\vec{r} - \vec{a}) \cdot (\overrightarrow{RS} \times \overrightarrow{RT}) = 0$$

or $$(\vec{r} - \vec{a}) \times [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0 \qquad \ldots (1)$$

This is the equation of the plane in vector form passing through three noncollinear points.

Note Why was it necessary to say that the three points had to be non collinear? If the three points were on the same line, then there will be many planes that will contain them (Fig 11.15).

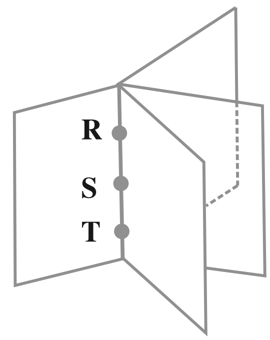


Fig 11.15

These planes will resemble the pages of a book where the line containing the points R, S and T are members in the binding of the book.

Cartesian form

Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the points R, S and T respectively. Let (x, y, z) be the coordinates of any point P on the plane with position vector $\vec{r}$. Then

$$\overrightarrow{RP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

$$\overrightarrow{RS} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\overrightarrow{RT} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$

Substituting these values in equation (1) of the vector form and expressing it in the form of a determinant, we have

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

which is the equation of the plane in Cartesian form passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Example 18 Find the vector equations of the plane passing through the points R(2, 5, – 3), S(– 2, – 3, 5) and T(5, 3,– 3).

Solution Let $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{b} = -2\hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 5\hat{i} + 3\hat{j} - 3\hat{k}$

Then the vector equation of the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is given by

$$(\vec{r} - \vec{a}) \cdot (\overrightarrow{RS} \times \overrightarrow{RT}) = 0 \qquad \text{(Why?)}$$

or $$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

i.e. $$[\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})] \cdot [(-4\hat{i} - 8\hat{j} + 8\hat{k}) \times (3\hat{i} - 2\hat{j})] = 0$$

11.6.4 *Intercept form of the equation of a plane*

In this section, we shall deduce the equation of a plane in terms of the intercepts made by the plane on the coordinate axes. Let the equation of the plane be

$$Ax + By + Cz + D = 0 \quad (D \neq 0) \qquad \text{... (1)}$$

Let the plane make intercepts a, b, c on x, y and z axes, respectively (Fig 11.16).

Hence, the plane meets x, y and z-axes at $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, respectively.

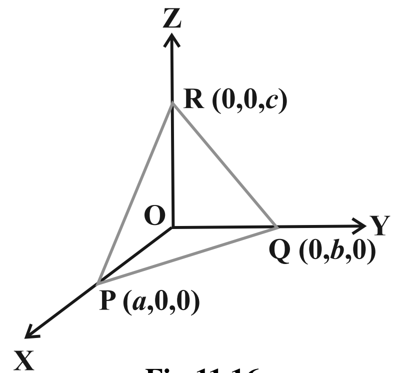


Fig 11.16

Therefore
$$Aa + D = 0 \text{ or } A = \frac{-D}{a}$$
$$Bb + D = 0 \text{ or } B = \frac{-D}{b}$$
$$Cc + D = 0 \text{ or } C = \frac{-D}{c}$$

Substituting these values in the equation (1) of the plane and simplifying, we get

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

which is the required equation of the plane in the intercept form.

Example 19 Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z-axis respectively.

Solution Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

Here $\qquad a = 2, b = 3, c = 4.$

Substituting the values of a, b and c in (1), we get the required equation of the plane as $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ or $6x + 4y + 3z = 12$.

11.6.5 *Plane passing through the intersection of two given planes*

Let π_1 and π_2 be two planes with equations $\vec{r} \cdot \hat{n}_1 = d_1$ and $\vec{r} \cdot \hat{n}_2 = d_2$ respectively. The position vector of any point on the line of intersection must satisfy both the equations (Fig 11.17).

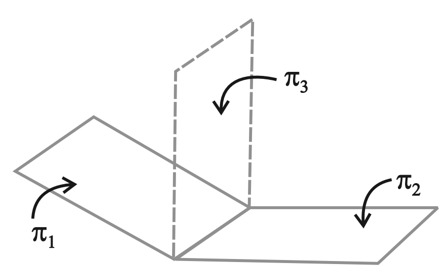


Fig 11.17

If $\vec{t}$ is the position vector of a point on the line, then

$$\vec{t}\cdot\hat{n}_1 = d_1 \text{ and } \vec{t}\cdot\hat{n}_2 = d_2$$

Therefore, for all real values of λ, we have

$$\vec{t}\cdot(\hat{n}_1+\lambda\hat{n}_2) = d_1+\lambda d_2$$

Since $\vec{t}$ is arbitrary, it satisfies for any point on the line.

Hence, the equation $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ satisfies both the equations π_1 and π_2, it also satisfies the equation π_3 i.e., any plane passing through the intersection of the planes

$$\vec{r}\cdot\vec{n}_1 = d_1 \text{ and } \vec{r}\cdot\vec{n}_2 = d_2$$

has the equation $$\boldsymbol{\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1 + \lambda d_2} \qquad \text{... (1)}$$

Cartesian form

In Cartesian system, let

$$\vec{n}_1 = \mathrm{A}_1\hat{i}+\mathrm{B}_2\hat{j}+\mathrm{C}_1\hat{k}$$

$$\vec{n}_2 = \mathrm{A}_2\hat{i}+\mathrm{B}_2\hat{j}+\mathrm{C}_2\hat{k}$$

and $$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

Then (1) becomes

$$x(\mathrm{A}_1 + \lambda\mathrm{A}_2) + y(\mathrm{B}_1 + \lambda\mathrm{B}_2) + z(\mathrm{C}_1 + \lambda\mathrm{C}_2) = d_1 + \lambda d_2$$

or $$\boldsymbol{(\mathrm{A}_1x + \mathrm{B}_1y + \mathrm{C}_1z - d_1) + \lambda(\mathrm{A}_2x + \mathrm{B}_2y + \mathrm{C}_2z - d_2) = 0} \qquad \text{... (2)}$$

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

Example 20 Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=6$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}+4\hat{k})=-5$, and the point (1, 1, 1).

Solution Here, $\vec{n}_1=\hat{i}+\hat{j}+\hat{k}$ and $\vec{n}_2 = 2\hat{i}+3\hat{j}+4\hat{k}$;

and $\quad d_1 = 6$ and $d_2 = -5$

Hence, using the relation $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$, we get

$$\vec{r}\cdot[\hat{i}+\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}+4\hat{k})] = 6-5\lambda$$

or $$\vec{r}\cdot[(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1+4\lambda)\hat{k}] = 6-5\lambda \qquad \text{... (1)}$$

where, λ is some real number.

Taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we get

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot[(1+2\lambda)\hat{i} + (1+3\lambda)\hat{j} + (1+4\lambda)\hat{k}] = 6 - 5\lambda$$

or $$(1 + 2\lambda)\,x + (1 + 3\lambda)\,y + (1 + 4\lambda)\,z = 6 - 5\lambda$$

or $$(x + y + z - 6) + \lambda\ (2x + 3y + 4z + 5) = 0 \qquad \ldots (2)$$

Given that the plane passes through the point (1,1,1), it must satisfy (2), i.e.

$$(1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0$$

or $$\lambda = \frac{3}{14}$$

Putting the values of λ in (1), we get

$$\vec{r}\left[\left(1+\frac{3}{7}\right)\hat{i} + \left(1+\frac{9}{14}\right)\hat{j} + \left(1+\frac{6}{7}\right)\hat{k}\right] = 6 - \frac{15}{14}$$

or $$\vec{r}\left(\frac{10}{7}\hat{i} + \frac{23}{14}\hat{j} + \frac{13}{7}\hat{k}\right) = \frac{69}{14}$$

or $$\vec{r}\cdot(20\hat{i} + 23\hat{j} + 26\hat{k}) = 69$$

which is the required vector equation of the plane.

11.7 Coplanarity of Two Lines

Let the given lines be

$$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1 \qquad \ldots (1)$$

and $$\vec{r} = \vec{a}_2 + \mu\vec{b}_2 \qquad \ldots (2)$$

The line (1) passes through the point, say A, with position vector $\vec{a}_1$ and is parallel to $\vec{b}_1$. The line (2) passes through the point, say B with position vector $\vec{a}_2$ and is parallel to $\vec{b}_2$.

Thus, $$\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$$

The given lines are coplanar if and only if $\overrightarrow{AB}$ is perpendicular to $\vec{b}_1 \times \vec{b}_2$.

i.e. $$\overrightarrow{AB}.(\vec{b}_1 \times \vec{b}_2) = 0 \text{ or } (\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$$

Cartesian form

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points A and B respectively.

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of $\vec{b}_1$ and $\vec{b}_2$, respectively. Then

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \text{ and } \vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

The given lines are coplanar if and only if $\overrightarrow{AB} \cdot \left(\vec{b}_1 \times \vec{b}_2\right) = 0$. In the cartesian form, it can be expressed as

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \qquad \text{... (4)}$$

Example 21 Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \text{ are coplanar.}$$

Solution Here, $x_1 = -3$, $y_1 = 1$, $z_1 = 5$, $a_1 = -3$, $b_1 = 1$, $c_1 = 5$

$x_2 = -1$, $y_2 = 2$, $z_2 = 5$, $a_2 = -1$, $b_2 = 2$, $c_2 = 5$

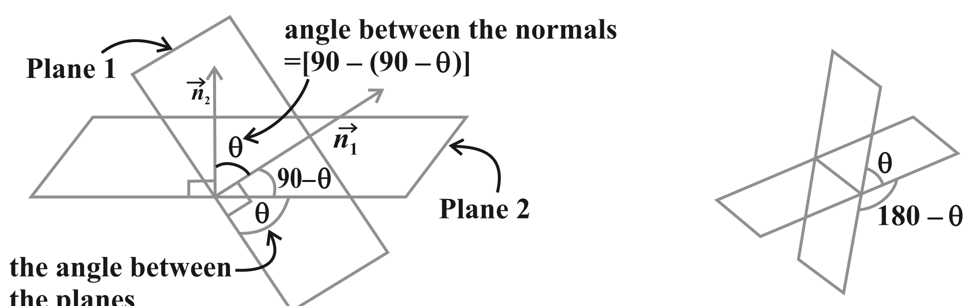
Now, consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

Therefore, lines are coplanar.

11.8 Angle between Two Planes

Definition 2 The angle between two planes is defined as the angle between their normals (Fig 11.18 (a)). Observe that if θ is an angle between the two planes, then so is 180 – θ (Fig 11.18 (b)). We shall take the acute angle as the angles between two planes.



(a) (b)

Fig 11.18

If $\vec{n}_1$ and $\vec{n}_2$ are normals to the planes and θ be the angle between the planes

$$\vec{r} \cdot \vec{n}_1 = d_1 \text{ and } \vec{r} \,.\, \vec{n}_2 = d_2 .$$

Then θ is the angle between the normals to the planes drawn from some common point.

We have, $$\cos\theta = \left|\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right|$$

Note The planes are perpendicular to each other if $\vec{n}_1 . \vec{n}_2 = 0$ and parallel if $\vec{n}_1$ is parallel to $\vec{n}_2$.

Cartesian form Let θ be the angle between the planes,

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

The direction ratios of the normal to the planes are A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

Therefore, $$\cos\theta = \left|\frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

Note

1. If the planes are at right angles, then θ = 90° and so cos θ = 0. Hence, $\cos\theta = A_1A_2 + B_1B_2 + C_1C_2 = 0$.
2. If the planes are parallel, then $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}$.

Example 22 Find the angle between the two planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ using vector method.

Solution The angle between two planes is the angle between their normals. From the equation of the planes, the normal vectors are

$$\vec{N}_1 = 2\hat{i} + \hat{j} - 2\hat{k} \text{ and } \vec{N}_2 = 3\hat{i} - 6\hat{j} - 2\hat{k}$$

Therefore $$\cos\theta = \left|\frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|}\right| = \left|\frac{(2\hat{i} + \hat{j} - 2\hat{k})\cdot(3\hat{i} - 6\hat{j} - 2\hat{k})}{\sqrt{4+1+4}\ \sqrt{9+36+4}}\right| = \left(\frac{4}{21}\right)$$

Hence $$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Example 23 Find the angle between the two planes $3x - 6y + 2z = 7$ and $2x + 2y - 2z = 5$.

Solution Comparing the given equations of the planes with the equations

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

We get
$$A_1 = 3, B_1 = -6, C_1 = 2$$
$$A_2 = 2, B_2 = 2, C_2 = -2$$

$$\cos\theta = \left| \frac{3 \times 2 + (-6)\ (2) + (2)\ (-2)}{\sqrt{\left(3^2 + (-6)^2 + (-2)^2\right)}\ \sqrt{\left(2^2 + 2^2 + (-2)^2\right)}} \right|$$

$$= \left| \frac{-10}{7 \times 2\sqrt{3}} \right| = \frac{5}{7\sqrt{3}} = \frac{5\sqrt{3}}{21}$$

Therefore, $$\theta = \cos^{-1}\left(\frac{5\sqrt{3}}{21}\right)$$

11.9 Distance of a Point from a Plane

Vector form

Consider a point P with position vector $\vec{a}$ and a plane π_1 whose equation is $\vec{r} \cdot \hat{n} = d$ (Fig 11.19).

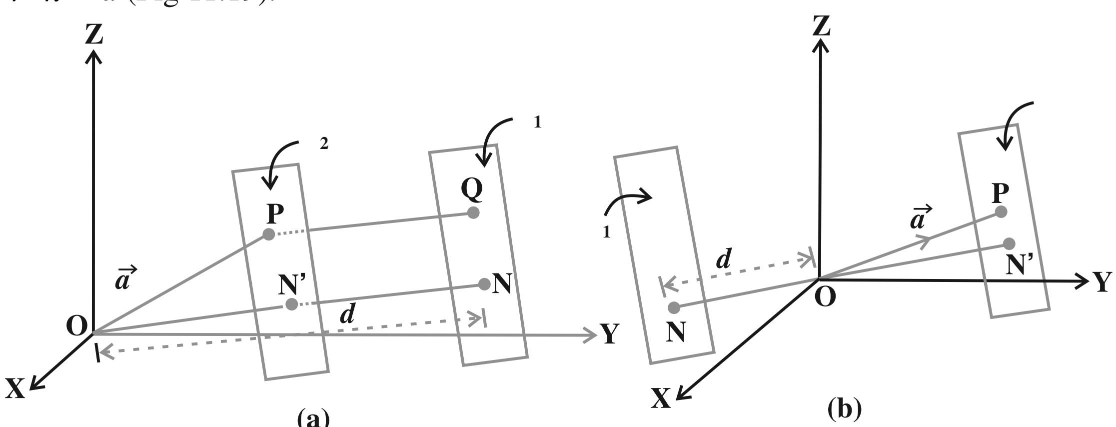


Fig 11.19

Consider a plane π_2 through P parallel to the plane π_1. The unit vector normal to π_2 is $\hat{n}$. Hence, its equation is $(\vec{r} - \vec{a}) \cdot \hat{n} = 0$

i.e., $$\vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n}$$

Thus, the distance ON′ of this plane from the origin is $|\vec{a} \cdot \hat{n}|$. Therefore, the distance PQ from the plane π_1 is (Fig. 11.21 (a))

i.e., $$\text{ON} - \text{ON}' = |d - \vec{a} \cdot \hat{n}|$$

which is the length of the perpendicular from a point to the given plane.

We may establish the similar results for (Fig 11.19 (b)).

Note

1. If the equation of the plane π_2 is in the form $\vec{r} \cdot \vec{N} = d$, where $\vec{N}$ is normal to the plane, then the perpendicular distance is $\dfrac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$.

2. The length of the perpendicular from origin O to the plane $\vec{r} \cdot \vec{N} = d$ is $\dfrac{|d|}{|\vec{N}|}$ (since $\vec{a} = 0$).

Cartesian form

Let P(x_1, y_1, z_1) be the given point with position vector $\vec{a}$ and

$$Ax + By + Cz = D$$

be the Cartesian equation of the given plane. Then

$$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

$$\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Hence, from Note 1, the perpendicular from P to the plane is

$$\left| \frac{(x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) \cdot (A\hat{i} + B\hat{j} + C\hat{k}) - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \left| \frac{A x_1 + B y_1 + C z_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Example 24 Find the distance of a point (2, 5, – 3) from the plane

$$\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$$

Solution Here, $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{N} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ and $d = 4$.

Therefore, the distance of the point (2, 5, – 3) from the given plane is

$$\frac{|(2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) - 4|}{|6\hat{i} - 3\hat{j} + 2\hat{k}|} = \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}} = \frac{13}{7}$$

11.10 Angle between a Line and a Plane

Definition 3 The angle between a line and a plane is the complement of the angle between the line and normal to the plane (Fig 11.20).

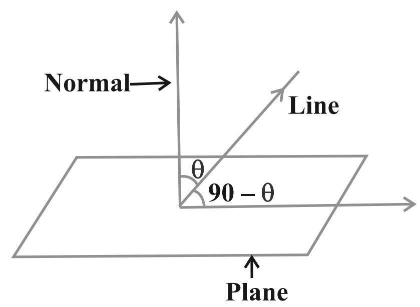


Fig 11.20

Vector form If the equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and the equation of the plane is $\vec{r} \cdot \vec{n} = d$. Then the angle θ between the line and the normal to the plane is

$$\cos\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|$$

and so the angle ϕ between the line and the plane is given by 90 – θ, i.e.,

$$\sin(90-\theta) = \cos\theta$$

i.e. $$\sin\phi = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\ |\vec{n}|}\right| \text{ or } \phi = \sin^{-1}\left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}||\bar{n}|}\right|$$

Example 25 Find the angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$$

and the plane $10\,x + 2y - 11\,z = 3$.

Solution Let θ be the angle between the line and the normal to the plane. Converting the given equations into vector form, we have

$$\vec{r} = (-\hat{i} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $$\vec{r}\cdot(10\hat{i} + 2\hat{j} - 11\hat{k}) = 3$$

Here $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{n} = 10\hat{i} + 2\hat{j} - 11\hat{k}$

$$\sin\phi = \left|\frac{(2\hat{i} + 3\hat{j} + 6\hat{k})\cdot(10\hat{i} + 2\hat{j} - 11\hat{k})}{\sqrt{2^2+3^2+6^2}\ \sqrt{10^2+2^2+11^2}}\right|$$

$$= \left|\frac{-40}{7\times 15}\right| = \left|\frac{-8}{21}\right| = \frac{8}{21} \text{ or } \phi = \sin^{-1}\left(\frac{8}{21}\right)$$

EXERCISE 11.3

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.
   (a) $z = 2$ (b) $x + y + z = 1$
   (c) $2x + 3y - z = 5$ (d) $5y + 8 = 0$
2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.
3. Find the Cartesian equation of the following planes:
   (a) $\vec{r}\cdot(\hat{i} + \hat{j} - \hat{k}) = 2$ (b) $\vec{r}\cdot(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$
   (c) $\vec{r}\cdot[(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$
4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.
   (a) $2x + 3y + 4z - 12 = 0$ (b) $3y + 4z - 6 = 0$
   (c) $x + y + z = 1$ (d) $5y + 8 = 0$
5. Find the vector and cartesian equations of the planes
   (a) that passes through the point (1, 0, – 2) and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.
   (b) that passes through the point (1,4, 6) and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.
6. Find the equations of the planes that passes through three points.
   (a) (1, 1, – 1), (6, 4, – 5), (– 4, – 2, 3)
   (b) (1, 1, 0), (1, 2, 1), (– 2, 2, – 1)
7. Find the intercepts cut off by the plane $2x + y - z = 5$.
8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.
9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1).
10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point (2, 1, 3).
11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

12. Find the angle between the planes whose vector equations are

$\vec{r}\cdot(2\,\hat{i} + 2\,\hat{j} - 3\,\hat{k}) = 5$ and $\vec{r}\cdot(3\,\hat{i} - 3\,\hat{j} + 5\,\hat{k}) = 3$.

13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

	Point	**Plane**
(a)	(0, 0, 0)	$3x - 4y + 12\,z = 3$
(b)	(3, – 2, 1)	$2x - y + 2z + 3 = 0$
(c)	(2, 3, – 5)	$x + 2y - 2z = 9$
(d)	(– 6, 0, 0)	$2x - 3y + 6z - 2 = 0$

Miscellaneous Examples

Example 26 A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Solution A cube is a rectangular parallelopiped having equal length, breadth and height. Let OADBFEGC be the cube with each side of length a units. (Fig 11.21)

The four diagonals are OE, AF, BG and CD.

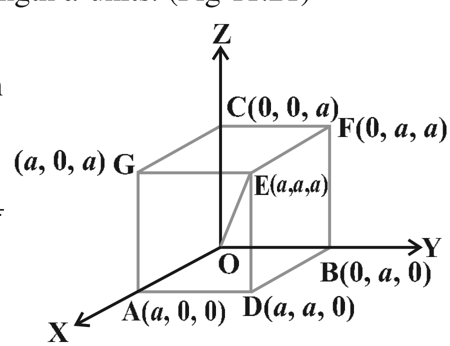


Fig 11.21

The direction cosines of the diagonal OE which is the line joining two points O and E are

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e., $$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Similarly, the direction cosines of AF, BG and CD are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$, respectively.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OE, AF, BG, CD, respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l + m + n); \cos\beta = \frac{1}{\sqrt{3}}(-l + m + n);$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l - m + n); \cos\delta = \frac{1}{\sqrt{3}}(l + m - n) \qquad \text{(Why?)}$$

Squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\ [\,(l + m + n\,)^2 + (-l + m + n)^2\,]\ + (l - m + n)^2 + (l + m - n)^2]$$

$$= \frac{1}{3}\ [\,4\,(l^2 + m^2 + n^2\,)\,]\ = \frac{4}{3} \qquad (\text{as } l^2 + m^2 + n^2 = 1)$$

Example 27 Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.

Solution The equation of the plane containing the given point is

$$A\,(x - 1) + B(y + 1) + C\,(z - 2) = 0 \qquad \text{... (1)}$$

Applying the condition of perpendicularly to the plane given in (1) with the planes

$$2x + 3y - 2z = 5 \text{ and } x + 2y - 3z = 8, \text{ we have}$$

$$2A + 3B - 2C = 0 \text{ and } A + 2B - 3C = 0$$

Solving these equations, we find A = – 5C and B = 4C. Hence, the required equation is

$$-5C\,(x - 1) + 4\,C\,(y + 1) + C(z - 2) = 0$$

i.e. $\qquad 5x - 4y - z = 7$

Example 28 Find the distance between the point P(6, 5, 9) and the plane determined by the points A (3, – 1, 2), B (5, 2, 4) and C(– 1, – 1, 6).

Solution Let A, B, C be the three points in the plane. D is the foot of the perpendicular drawn from a point P to the plane. PD is the required distance to be determined, which is the projection of $\overrightarrow{AP}$ on $\overrightarrow{AB} \times \overrightarrow{AC}$.

Hence, PD = the dot product of $\overrightarrow{AP}$ with the unit vector along $\overrightarrow{AB} \times \overrightarrow{AC}$.

So $$\overrightarrow{AP} = 3\hat{i} + 6\hat{j} + 7\hat{k}$$

and $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 12\hat{i} - 16\hat{j} + 12\hat{k}$$

Unit vector along $$\overrightarrow{AB} \times \overrightarrow{AC} = \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

Hence $$PD = (3\hat{i} + 6\hat{j} + 7\hat{k}) \cdot \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

$$= \frac{3\sqrt{34}}{17}$$

Alternatively, find the equation of the plane passing through A, B and C and then compute the distance of the point P from the plane.

Example 29 Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$$

and $$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$$ are coplanar.

Solution

Here
$$\begin{aligned} x_1 &= a-d & x_2 &= b-c \\ y_1 &= a & y_2 &= b \\ z_1 &= a+d & z_2 &= b+c \\ a_1 &= \alpha-\delta & a_2 &= \beta-\gamma \\ b_1 &= \alpha & b_2 &= \beta \\ c_1 &= \alpha+\delta & c_2 &= \beta+\gamma \end{aligned}$$

Now consider the determinant

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b-c-a+d & b-a & b+c-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$$

Adding third column to the first column, we get

$$2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

Since the first and second columns are identical. Hence, the given two lines are coplanar.

Example 30 Find the coordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the XY-plane.

Solution The vector equation of the line through the points A and B is

$$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda\,[\,(5-3)\hat{i} + (1-4)\hat{j} + (6-1)\hat{k}\,]$$

i.e. $$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda\,(2\hat{i} - 3\hat{j} + 5\hat{k}) \qquad \text{... (1)}$$

Let P be the point where the line AB crosses the XY-plane. Then the position vector of the point P is of the form $x\hat{i} + y\hat{j}$.

This point must satisfy the equation (1). (Why ?)

i.e. $$x\hat{i} + y\hat{j} = (3+2\lambda)\hat{i} + (4-3\lambda)\hat{j} + (1+5\lambda)\hat{k}$$

Equating the like coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$x = 3 + 2\lambda$$
$$y = 4 - 3\lambda$$
$$0 = 1 + 5\lambda$$

Solving the above equations, we get

$$x = \frac{13}{5} \text{ and } y = \frac{23}{5}$$

Hence, the coordinates of the required point are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Miscellaneous Exercise on Chapter 11

1. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, – 1), (4, 3, – 1).
2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1 n_2 - m_2 n_1,\ n_1 l_2 - n_2 l_1,\ l_1 m_2 - l_2 m_1$

3. Find the angle between the lines whose direction ratios are a, b, c and $b - c$, $c - a$, $a - b$.
4. Find the equation of a line parallel to x-axis and passing through the origin.
5. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.
6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.
7. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.
8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.
9. Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.
10. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4,1) crosses the YZ-plane.
11. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX-plane.
12. Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.
13. Find the equation of the plane passing through the point (– 1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.
14. If the points (1, 1, p) and (– 3, 0, 1) be equidistant from the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, then find the value of p.
15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis.
16. If O be the origin and the coordinates of P be (1, 2, – 3), then find the equation of the plane passing through P and perpendicular to OP.
17. Find the equation of the plane which contains the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$, $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$.

18. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6$.

20. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

21. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Choose the correct answer in Exercises 22 and 23.

22. Distance between the two planes: $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is

(A) 2 units (B) 4 units (C) 8 units (D) $\frac{2}{\sqrt{29}}$ units

23. The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular (B) Parallel

(C) intersect y-axis (D) passes through $\left(0, 0, \frac{5}{4}\right)$

Summary

- **Direction cosines of a line** are the cosines of the angles made by the line with the positive directions of the coordinate axes.
- If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- **Direction ratios of a line** are the numbers which are proportional to the direction cosines of a line.
- If l, m, n are the direction cosines and a, b, c are the direction ratios of a line

then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \; ; m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \; ; n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- **Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.
- **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between the two lines; then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$.
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- The vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$.
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.
- If θ is the acute angle between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, then

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \, |\vec{b}_2|} \right|$$

- If $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.

- Shortest distance between two skew lines is the line segment perpendicular to both the lines.
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

- Shortest distance between the lines: $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and

$\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

- Distance between parallel lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is

$$\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

- In the vector form, equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$.
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}) . \vec{N} = 0$.
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is

$$A (x - x_1) + B (y - y_1) + C (z - z_1) = 0$$

- Equation of a plane passing through three non collinear points (x_1, y_1, z_1),

(x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

◆ Vector equation of a plane that contains three non collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

◆ Equation of a plane that cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

◆ Vector equation of a plane that passes through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.

◆ Vector equation of a plane that passes through the intersection of two given planes $A_1 x + B_1 y + C_1 z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is $(A_1 x + B_1 y + C_1 z + D_1) + \lambda(A_2 x + B_2 y + C_2 z + D_2) = 0$.

◆ Two planes $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are coplanar if

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$$

◆ Two planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$.

◆ In the vector form, if θ is the angle between the two planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then $\theta = \cos^{-1} \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$.

◆ The angle ϕ between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \hat{n} = d$ is

$$\sin \phi = \left| \frac{\vec{b} \cdot \hat{n}}{|\vec{b}||\hat{n}|} \right|$$

- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is given by

$$\cos\theta = \left|\frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

- The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r}\cdot\hat{n} = d$ is $|d - \vec{a}\cdot\hat{n}|$
- The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|.$$

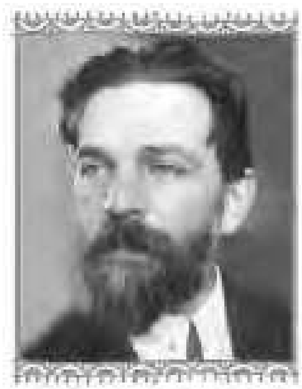
Chapter 12

LINEAR PROGRAMMING

❖ *The mathematical experience of the student is incomplete if he never had the opportunity to solve a problem invented by himself. – G. POLYA* ❖

12.1 Introduction

In earlier classes, we have discussed systems of linear equations and their applications in day to day problems. In Class XI, we have studied linear inequalities and systems of linear inequalities in two variables and their solutions by graphical method. Many applications in mathematics involve systems of inequalities/equations. In this chapter, we shall apply the systems of linear inequalities/equations to solve some real life problems of the type as given below:



L. Kantorovich

A furniture dealer deals in only two items–tables and chairs. He has Rs 50,000 to invest and has storage space of at most 60 pieces. A table costs Rs 2500 and a chair Rs 500. He estimates that from the sale of one table, he can make a profit of Rs 250 and that from the sale of one chair a profit of Rs 75. He wants to know how many tables and chairs he should buy from the available money so as to maximise his total profit, assuming that he can sell all the items which he buys.

Such type of problems which seek to maximise (or, minimise) profit (or, cost) form a general class of problems called **optimisation problems**. Thus, an optimisation problem may involve finding maximum profit, minimum cost, or minimum use of resources etc.

A special but a very important class of optimisation problems is **linear programming problem.** The above stated optimisation problem is an example of linear programming problem. Linear programming problems are of much interest because of their wide applicability in industry, commerce, management science etc.

In this chapter, we shall study some linear programming problems and their solutions by graphical method only, though there are many other methods also to solve such problems.

12.2 Linear Programming Problem and its Mathematical Formulation

We begin our discussion with the above example of furniture dealer which will further lead to a mathematical formulation of the problem in two variables. In this example, we observe

(i) The dealer can invest his money in buying tables or chairs or combination thereof. Further he would earn different profits by following different investment strategies.

(ii) There are certain **overriding conditions** or **constraints** viz., his investment is limited to a **maximum** of Rs 50,000 and so is his storage space which is for a maximum of 60 pieces.

Suppose he decides to buy tables only and no chairs, so he can buy 50000 ÷ 2500, i.e., 20 tables. His profit in this case will be Rs (250 × 20), i.e., **Rs 5000.**

Suppose he chooses to buy chairs only and no tables. With his capital of Rs 50,000, he can buy 50000 ÷ 500, i.e. 100 chairs. But he can store only 60 pieces. Therefore, he is forced to buy only 60 chairs which will give him a total profit of Rs (60 × 75), i.e., **Rs 4500**.

There are many other possibilities, for instance, he may choose to buy 10 tables and 50 chairs, as he can store only 60 pieces. Total profit in this case would be Rs (10 × 250 + 50 × 75), i.e., **Rs 6250** and so on.

We, thus, find that the dealer can invest his money in different ways and he would earn different profits by following different investment strategies.

Now the problem is : How should he invest his money in order to get maximum profit? To answer this question, let us try to formulate the problem mathematically.

12.2.1 *Mathematical formulation of the problem*

Let x be the number of tables and y be the number of chairs that the dealer buys. Obviously, x and y must be non-negative, i.e.,

$$\left.\begin{array}{l} x \geq 0 \quad \text{... (1)} \\ y \geq 0 \quad \text{... (2)} \end{array}\right\} \text{(Non-negative constraints)}$$

The dealer is constrained by the maximum amount he can invest (Here it is Rs 50,000) and by the maximum number of items he can store (Here it is 60).

Stated mathematically,

$$2500x + 500y \leq 50000 \text{ (investment constraint)}$$

or $$5x + y \leq 100 \qquad \text{... (3)}$$

and $$x + y \leq 60 \text{ (storage constraint)} \qquad \text{... (4)}$$

The dealer wants to invest in such a way so as to maximise his profit, say, Z which stated as a function of x and y is given by

$Z = 250x + 75y$ (called *objective function*) ... (5)

Mathematically, the given problems now reduces to:

Maximise $Z = 250x + 75y$

subject to the constraints:

$$5x + y \leq 100$$

$$x + y \leq 60$$

$$x \geq 0, \quad y \geq 0$$

So, we have to maximise the linear function Z subject to certain conditions determined by a set of linear inequalities with variables as non-negative. There are also some other problems where we have to minimise a linear function subject to certain conditions determined by a set of linear inequalities with variables as non-negative. Such problems are called **Linear Programming Problems.**

Thus, a Linear Programming Problem is one that is concerned with finding the **optimal value** (maximum or minimum value) of a linear function (called **objective function**) of several variables (say x and y), subject to the conditions that the variables are **non-negative** and satisfy a set of linear inequalities (called **linear constraints).** The term **linear** implies that all the mathematical relations used in the problem are **linear relations** while the term programming refers to the method of determining a particular **programme** or plan of action.

Before we proceed further, we now formally define some terms (which have been used above) which we shall be using in the linear programming problems:

Objective function Linear function $Z = ax + by$, where a, b are constants, which has to be maximised or minimized is called a linear **objective function.**

In the above example, $Z = 250x + 75y$ is a linear objective function. Variables x and y are called **decision variables**.

Constraints The linear inequalities or equations or restrictions on the variables of a linear programming problem are called **constraints**. The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions. In the above example, the set of inequalities (1) to (4) are **constraints**.

Optimisation problem A problem which seeks to maximise or minimise a linear function (say of two variables x and y) subject to certain constraints as determined by a set of linear inequalities is called an **optimisation problem**. Linear programming problems are special type of optimisation problems. The above problem of investing a given sum by the dealer in purchasing chairs and tables is an example of an optimisation problem as well as of a linear programming problem.

We will now discuss how to find solutions to a linear programming problem. In this chapter, we will be concerned only with the graphical method.

12.2.2 *Graphical method of solving linear programming problems*

In Class XI, we have learnt how to graph a system of linear inequalities involving two variables x and y and to find its solutions graphically. Let us refer to the problem of investment in tables and chairs discussed in Section 12.2. We will now solve this problem graphically. Let us graph the constraints stated as linear inequalities:

$$5x + y \leq 100 \quad \ldots (1)$$

$$x + y \leq 60 \quad \ldots (2)$$

$$x \geq 0 \quad \ldots (3)$$

$$y \geq 0 \quad \ldots (4)$$

The graph of this system (shaded region) consists of the points common to all half planes determined by the inequalities (1) to (4) (Fig 12.1). Each point in this region represents a **feasible choice** open to the dealer for investing in tables and chairs. The region, therefore, is called the **feasible region** for the problem. Every point of this region is called a **feasible solution** to the problem. Thus, we have,

Feasible region The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem is called the **feasible region** (or solution region) for the problem. In Fig 12.1, the region OABC (shaded) is the feasible region for the problem. The region other than feasible region is called an **infeasible region**.

Feasible solutions Points within and on the boundary of the feasible region represent feasible solutions of the constraints. In Fig 12.1, every point within and on the boundary of the feasible region OABC represents feasible solution to the problem. For example, the point (10, 50) is a feasible solution of the problem and so are the points (0, 60), (20, 0) etc.

Any point outside the feasible region is called an **infeasible solution.** For example, the point (25, 40) is an infeasible solution of the problem.

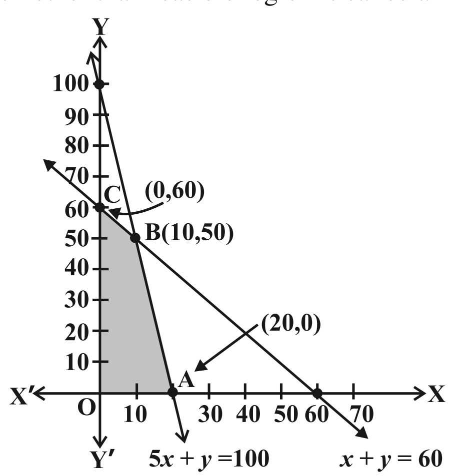


Fig 12.1

Optimal (feasible) solution: Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**

Now, we see that every point in the feasible region OABC satisfies all the constraints as given in (1) to (4), and since there are **infinitely many points**, it is not evident how we should go about finding a point that gives a maximum value of the objective function Z = 250x + 75y. To handle this situation, we use the following theorems which are fundamental in solving linear programming problems. The proofs of these theorems are beyond the scope of the book.

Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and let Z = $ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point* (vertex) of the feasible region.

Theorem 2 Let R be the feasible region for a linear programming problem, and let Z = $ax + by$ be the objective function. If R is **bounded****, then the objective function Z has both a **maximum** and a **minimum** value on R and each of these occurs at a corner point (vertex) of R.

Remark If R is **unbounded**, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R. (By Theorem 1).

In the above example, the corner points (vertices) of the bounded (feasible) region are: O, A, B and C and it is easy to find their coordinates as (0, 0), (20, 0), (10, 50) and (0, 60) respectively. Let us now compute the values of Z at these points.

We have

Vertex of the Feasible Region	Corresponding value of Z (in Rs)	
O (0,0)	0	
A (0,60)	4500	
B (10,50)	**6250** ⟵	Maximum
C (20,0)	5000	

* A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.

** A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a circle. Otherwise, it is called unbounded. Unbounded means that the feasible region does extend indefinitely in any direction.

We observe that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs.

This method of solving linear programming problem is referred as **Corner Point Method**. The method comprises of the following steps:

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Evaluate the objective function Z = $ax + by$ at each corner point. Let M and m, respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is **bounded**, M and m are the maximum and minimum values of Z.

   (ii) In case, the feasible region is **unbounded**, we have:
4. (a) M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

   (b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

We will now illustrate these steps of Corner Point Method by considering some examples:

Example 1 Solve the following linear programming problem graphically:

Maximise $Z = 4x + y$ ... (1)

subject to the constraints:

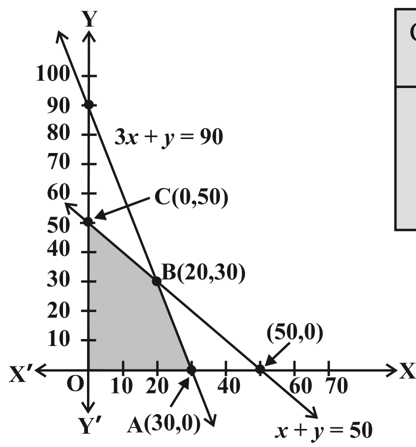
$$x + y \leq 50 \quad \text{... (2)}$$

$$3x + y \leq 90 \quad \text{... (3)}$$

$$x \geq 0, y \geq 0 \quad \text{... (4)}$$

Solution The shaded region in Fig 12.2 is the feasible region determined by the system of constraints (2) to (4). We observe that the feasible region OABC is **bounded.** So, we now use Corner Point Method to determine the maximum value of Z.

The coordinates of the corner points O, A, B and C are (0, 0), (30, 0), (20, 30) and (0, 50) respectively. Now we evaluate Z at each corner point.



Corner Point	Corresponding value of Z	
(0, 0)	0	
(30, 0)	**120** ←	Maximum
(20, 30)	110	
(0, 50)	50	

Fig 12.2

Hence, maximum value of Z is 120 at the point (30, 0).

Example 2 Solve the following linear programming problem graphically:

Minimise Z = 200 x + 500 y ... (1)

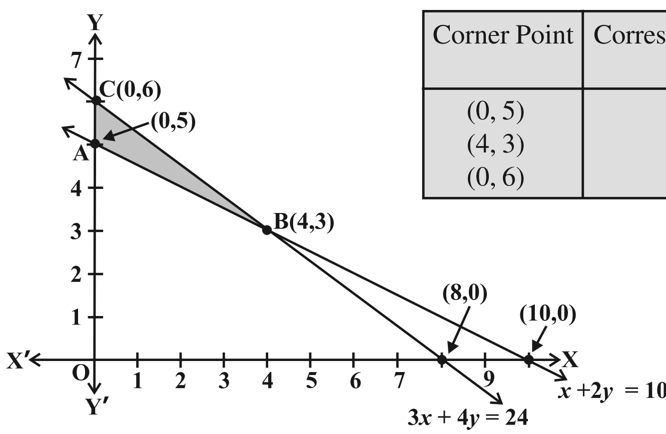
subject to the constraints:

$$x + 2y \geq 10 \quad \text{... (2)}$$

$$3x + 4y \leq 24 \quad \text{... (3)}$$

$$x \geq 0,\ y \geq 0 \quad \text{... (4)}$$

Solution The shaded region in Fig 12.3 is the feasible region ABC determined by the system of constraints (2) to (4), which is **bounded**. The coordinates of corner points



Corner Point	Corresponding value of Z	
(0, 5)	2500	
(4, 3)	**2300** ←	Minimum
(0, 6)	3000	

Fig 12.3

A, B and C are (0,5), (4,3) and (0,6) respectively. Now we evaluate Z $= 200x + 500y$ at these points.

Hence, minimum value of Z is 2300 attained at the point (4, 3)

Example 3 Solve the following problem graphically:

Minimise and Maximise $Z = 3x + 9y$ ... (1)

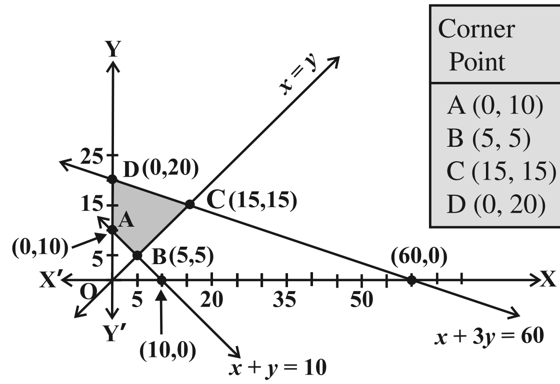
subject to the constraints: $x + 3y \le 60$ ... (2)

$x + y \ge 10$ ... (3)

$x \le y$ ... (4)

$x \ge 0, y \ge 0$ ... (5)

Solution First of all, let us graph the feasible region of the system of linear inequalities (2) to (5). The feasible region ABCD is shown in the Fig 12.4. Note that the region is bounded. The coordinates of the corner points A, B, C and D are (0, 10), (5, 5), (15,15) and (0, 20) respectively.



Corner Point	Corresponding value of $Z = 3x + 9y$	
A (0, 10)	90	
B (5, 5)	**60** ←	Minimum
C (15, 15)	**180** ←	Maximum (Multiple optimal solutions)
D (0, 20)	**180**	

Fig 12.4

We now find the minimum and maximum value of Z. From the table, we find that the minimum value of Z is 60 at the point B (5, 5) of the feasible region.

The maximum value of Z on the feasible region occurs at the two corner points C (15, 15) and D (0, 20) and it is 180 in each case.

Remark Observe that in the above example, the problem has multiple optimal solutions at the corner points C and D, i.e. the both points produce same maximum value 180. In such cases, you can see that every point on the line segment CD joining the two corner points C and D also give the same maximum value. Same is also true in the case if the two points produce same minimum value.

Example 4 Determine graphically the minimum value of the objective function

$$Z = -50x + 20y \qquad \text{... (1)}$$

subject to the constraints:

$$2x - y \geq -5 \qquad \text{... (2)}$$

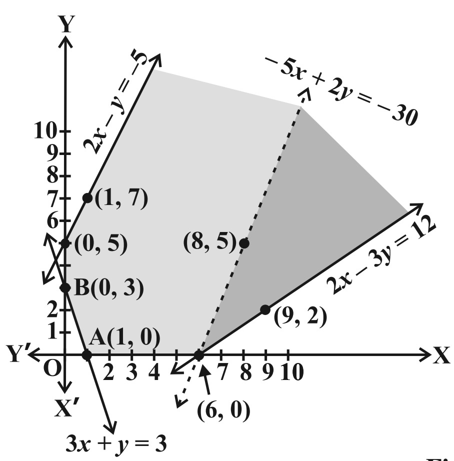
$$3x + y \geq 3 \qquad \text{... (3)}$$

$$2x - 3y \leq 12 \qquad \text{... (4)}$$

$$x \geq 0, y \geq 0 \qquad \text{... (5)}$$

Solution First of all, let us graph the feasible region of the system of inequalities (2) to (5). The feasible region (shaded) is shown in the Fig 12.5. Observe that the feasible region is **unbounded**.

We now evaluate Z at the corner points.



Corner Point	$Z = -50x + 20y$	
(0, 5)	100	
(0, 3)	**60**	
(1, 0)	**–50**	
(6, 0)	**–300** ⟵	smallest

Fig 12.5

From this table, we find that – 300 is the smallest value of Z at the corner point (6, 0). Can we say that minimum value of Z is – 300? Note that if the region would have been bounded, this smallest value of Z is the minimum value of Z (Theorem 2). But here we see that the feasible region is unbounded. Therefore, – 300 may or may not be the minimum value of Z. To decide this issue, we graph the inequality

$$-50x + 20y < -300 \text{ (see Step 3(ii) of corner Point Method.)}$$

i.e., $\qquad -5x + 2y < -30$

and check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then –300 will not be the minimum value of *Z*. Otherwise, –300 will be the minimum value of Z.

As shown in the Fig 12.5, it has common points. Therefore, Z = –50 x + 20 y has no minimum value subject to the given constraints.

In the above example, can you say whether $z = -50x + 20y$ has the maximum value 100 at (0,5)? For this, check whether the graph of $-50x + 20y > 100$ has points in common with the feasible region. (Why?)

Example 5 Minimise Z = $3x + 2y$

subject to the constraints:

$$x + y \geq 8 \quad \text{... (1)}$$

$$3x + 5y \leq 15 \quad \text{... (2)}$$

$$x \geq 0, y \geq 0 \quad \text{... (3)}$$

Solution Let us graph the inequalities (1) to (3) (Fig 12.6). Is there any feasible region? Why is so?

From Fig 12.6, you can see that there is no point satisfying all the constraints simultaneously. Thus, the problem is having no feasible region and hence no feasible solution.

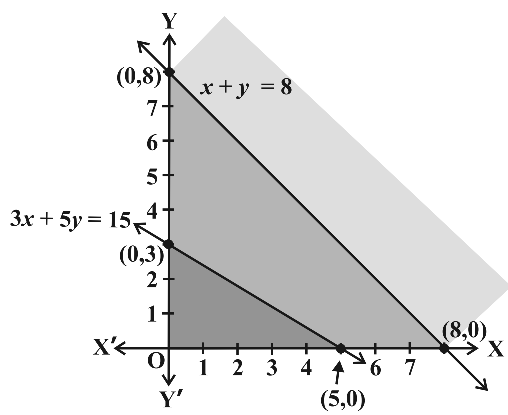


Fig 12.6

Remarks From the examples which we have discussed so far, we notice some general features of linear programming problems:

(i) The feasible region is always a convex region.

(ii) The maximum (or minimum) solution of the objective function occurs at the vertex (corner) of the feasible region. If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

EXERCISE 12.1

Solve the following Linear Programming Problems graphically:

1. Maximise Z = $3x + 4y$

   subject to the constraints : $x + y \leq 4$, $x \geq 0$, $y \geq 0$.

2. Minimise $Z = -3x + 4y$
subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$.
3. Maximise $Z = 5x + 3y$
subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.
4. Minimise $Z = 3x + 5y$
such that $x + 3y \geq 3$, $x + y \geq 2$, $x, y \geq 0$.
5. Maximise $Z = 3x + 2y$
subject to $x + 2y \leq 10$, $3x + y \leq 15$, $x, y \geq 0$.
6. Minimise $Z = x + 2y$
subject to $2x + y \geq 3$, $x + 2y \geq 6$, $x, y \geq 0$.

Show that the minimum of Z occurs at more than two points.

7. Minimise and Maximise $Z = 5x + 10y$
subject to $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$.
8. Minimise and Maximise $Z = x + 2y$
subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$; $x, y \geq 0$.
9. Maximise $Z = -x + 2y$, subject to the constraints:
$x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$.
10. Maximise $Z = x + y$, subject to $x - y \leq -1$, $-x + y \leq 0$, $x, y \geq 0$.

12.3 Different Types of Linear Programming Problems

A few important linear programming problems are listed below:

1. **Manufacturing problems** In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hour per unit of product, warehouse space per unit of the output etc., in order to make maximum profit.
2. **Diet problems** In these problems, we determine the amount of different kinds of constituents/nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrients.
3. **Transportation problems** In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Let us now solve some of these types of linear programming problems:

Example 6 (Diet problem): A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.

Solution Let the mixture contain x kg of Food 'I' and y kg of Food 'II'. Clearly, $x \geq 0$, $y \geq 0$. We make the following table from the given data:

Resources	Food I (x)	Food II (y)	Requirement
Vitamin A (units/kg)	2	1	8
Vitamin C (units/kg)	1	2	10
Cost (Rs/kg)	50	70	

Since the mixture must contain at least 8 units of vitamin A and 10 units of vitamin C, we have the constraints:

$$2x + y \geq 8$$

$$x + 2y \geq 10$$

Total cost Z of purchasing x kg of food 'I' and y kg of Food 'II' is

$$Z = 50x + 70y$$

Hence, the mathematical formulation of the problem is:

Minimise $$Z = 50x + 70y \quad \text{... (1)}$$

subject to the constraints:

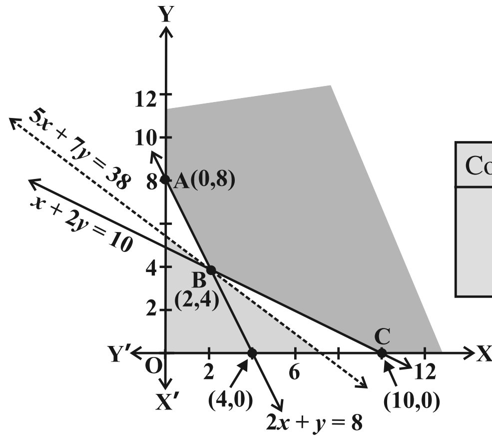
$$2x + y \geq 8 \quad \text{... (2)}$$

$$x + 2y \geq 10 \quad \text{... (3)}$$

$$x, y \geq 0 \quad \text{... (4)}$$

Let us graph the inequalities (2) to (4). The feasible region determined by the system is shown in the Fig 12.7. Here again, observe that the feasible region is **unbounded**.

Let us evaluate Z at the corner points A(0,8), B(2,4) and C(10,0).



Corner Point	Z = 50x + 70y	
(0,8)	560	
(2,4)	**380** ⟵	Minimum
(10,0)	500	

Fig 12.7

In the table, we find that smallest value of Z is 380 at the point (2,4). Can we say that the minimum value of Z is 380? Remember that the feasible region is unbounded. Therefore, we have to draw the graph of the inequality

$$50x + 70y < 380 \text{ i.e., } \quad 5x + 7y < 38$$

to check whether the resulting open half plane has any point common with the feasible region. From the Fig 12.7, we see that it has no points in common.

Thus, the minimum value of Z is 380 attained at the point (2, 4). Hence, the optimal mixing strategy for the dietician would be to mix 2 kg of Food 'I' and 4 kg of Food 'II', and with this strategy, the minimum cost of the mixture will be Rs 380.

Example 7 **(Allocation problem)** A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society?

Solution Let x hectare of land be allocated to crop X and y hectare to crop Y. Obviously, $x \geq 0$, $y \geq 0$.

Profit per hectare on crop X = Rs 10500

Profit per hectare on crop Y = Rs 9000

Therefore, total profit = Rs $(10500x + 9000y)$

The mathematical formulation of the problem is as follows:

Maximise $Z = 10500\,x + 9000\,y$

subject to the constraints:

$$x + y \le 50 \quad \text{(constraint related to land)} \qquad \dots (1)$$

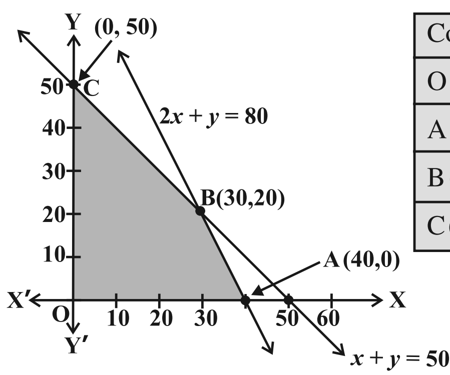
$$20x + 10y \le 800 \text{ (constraint related to use of herbicide)}$$

i.e. $$2x + y \le 80 \qquad \dots (2)$$

$$x \ge 0,\ y \ge 0 \qquad \text{(non negative constraint)} \qquad \dots (3)$$

Let us draw the graph of the system of inequalities (1) to (3). The feasible region OABC is shown (shaded) in the Fig 12.8. Observe that the feasible region is **bounded**.

The coordinates of the corner points O, A, B and C are (0, 0), (40, 0), (30, 20) and (0, 50) respectively. Let us evaluate the objective function $Z = 10500\,x + 9000y$ at these vertices to find which one gives the maximum profit.



Corner Point	$Z = 10500x + 9000y$	
O(0, 0)	0	
A(40, 0)	420000	
B(30, 20)	**495000** ←	Maximum
C(0,50)	450000	

Fig 12.8

Hence, the society will get the maximum profit of Rs 4,95,000 by allocating 30 hectares for crop X and 20 hectares for crop Y.

Example 8 (Manufacturing problem) A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?

Solution Suppose x is the number of pieces of Model A and y is the number of pieces of Model B. Then

$$\text{Total profit (in Rs)} = 8000\,x + 12000\,y$$

Let $$Z = 8000\,x + 12000\,y$$

We now have the following mathematical model for the given problem.

Maximise $Z = 8000\,x + 12000\,y$ ... (1)

subject to the constraints:

$$9x + 12y \leq 180 \quad \text{(Fabricating constraint)}$$

i.e. $$3x + 4y \leq 60 \quad \text{... (2)}$$

$$x + 3y \leq 30 \quad \text{(Finishing constraint)} \quad \text{... (3)}$$

$$x \geq 0,\ y \geq 0 \quad \text{(non-negative constraint)} \quad \text{... (4)}$$

The feasible region (shaded) OABC determined by the linear inequalities (2) to (4) is shown in the Fig 12.9. Note that the feasible region is bounded.

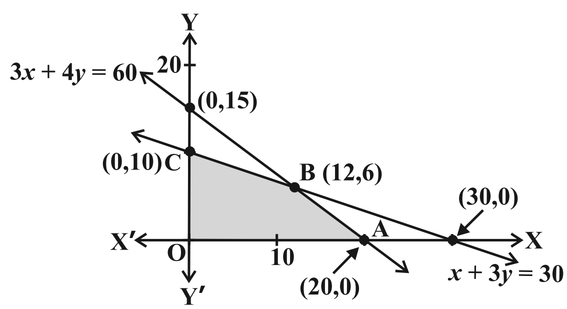


Fig 12.9

Let us evaluate the objective function Z at each corner point as shown below:

Corner Point	$Z = 8000\,x + 12000\,y$	
0 (0, 0)	0	
A (20, 0)	160000	
B (12, 6)	**168000** ←	Maximum
C (0, 10)	120000	

We find that maximum value of Z is 1,68,000 at B (12, 6). Hence, the company should produce 12 pieces of Model A and 6 pieces of Model B to realise maximum profit and maximum profit then will be Rs 1,68,000.

EXERCISE 12.2

1. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units / kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.
2. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
3. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
   (i) What number of rackets and bats must be made if the factory is to work at full capacity?
   (ii) If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.
4. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?
5. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

6. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?
7. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?
8. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.
9. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.
10. There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?
11. The corner points of the feasible region determined by the following system of linear inequalities:

    $2x + y \leq 10$, $x + 3y \leq 15$, $x, y \geq 0$ are (0, 0), (5, 0), (3, 4) and (0, 5). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both (3, 4) and (0, 5) is

    (A) $p = q$ (B) $p = 2q$ (C) $p = 3q$ (D) $q = 3p$

Miscellaneous Examples

Example 9 **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

Solution Let x and y be the number of packets of food P and Q respectively. Obviously $x \geq 0$, $y \geq 0$. Mathematical formulation of the given problem is as follows:
Minimise $Z = 6x + 3y$ (vitamin A)

subject to the constraints

$12x + 3y \geq 240$ (constraint on calcium), i.e. $4x + y \geq 80$ ... (1)

$4x + 20y \geq 460$ (constraint on iron), i.e. $x + 5y \geq 115$ ... (2)

$6x + 4y \leq 300$ (constraint on cholesterol), i.e. $3x + 2y \leq 150$ ... (3)

$x \geq 0$, $y \geq 0$ ... (4)

Let us graph the inequalities (1) to (4).

The feasible region (shaded) determined by the constraints (1) to (4) is shown in Fig 12.10 and note that it is bounded.

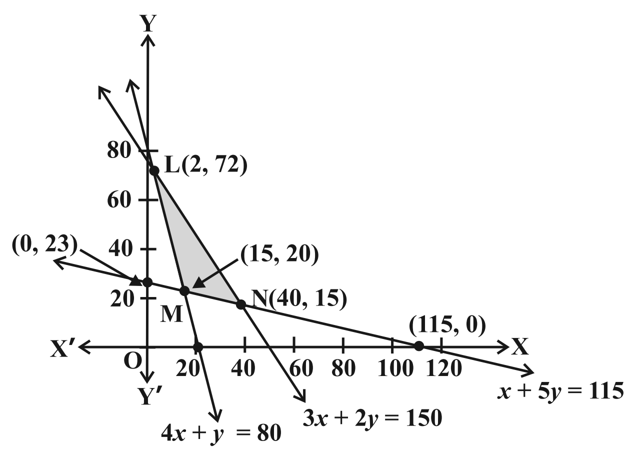


Fig 12.10

The coordinates of the corner points L, M and N are (2, 72), (15, 20) and (40, 15) respectively. Let us evaluate Z at these points:

Corner Point	$Z = 6x + 3y$	
(2, 72)	228	
(15, 20)	**150** $\leftarrow$	Minimum
(40, 15)	285	

From the table, we find that Z is minimum at the point (15, 20). Hence, the amount of vitamin A under the constraints given in the problem will be minimum, if 15 packets of food P and 20 packets of food Q are used in the special diet. The minimum amount of vitamin A will be 150 units.

Example 10 **(Manufacturing problem)** A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines.

The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Items	**Number of hours required on machines**		
	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

Solution Let x and y be the number of items M and N respectively.

Total profit on the production = Rs $(600x + 400y)$

Mathematical formulation of the given problem is as follows:

Maximise $Z = 600x + 400y$

subject to the constraints:

$$x + 2y \leq 12 \text{ (constraint on Machine I)} \quad \dots (1)$$

$$2x + y \leq 12 \text{ (constraint on Machine II)} \quad \dots (2)$$

$$x + \frac{5}{4}y \geq 5 \text{ (constraint on Machine III)} \quad \dots (3)$$

$$x \geq 0, \ y \geq 0 \quad \dots (4)$$

Let us draw the graph of constraints (1) to (4). ABCDE is the feasible region (shaded) as shown in Fig 12.11 determined by the constraints (1) to (4). Observe that the feasible region is bounded, coordinates of the corner points A, B, C, D and E are (5, 0) (6, 0), (4, 4), (0, 6) and (0, 4) respectively.

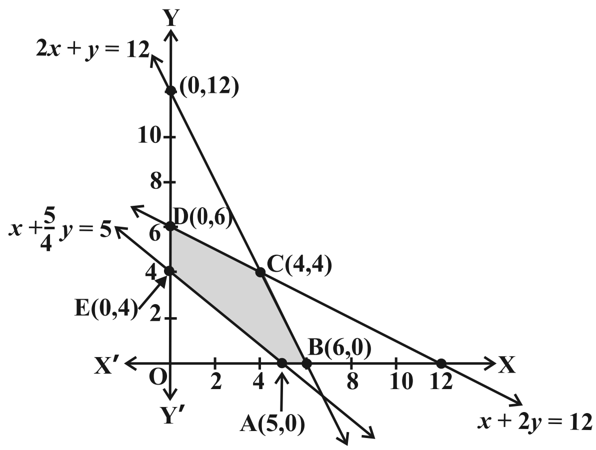


Fig 12.11

Let us evaluate Z = 600 x + 400 y at these corner points.

Corner point	Z = 600 x + 400 y	
(5, 0)	3000	
(6, 0)	3600	
(4, 4)	**4000** ←	Maximum
(0, 6)	2400	
(0, 4)	1600	

We see that the point (4, 4) is giving the maximum value of Z. Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of Rs 4000.

Example 11 (Transportation problem) There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

From/To	Cost (in Rs)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Solution The problem can be explained diagrammatically as follows (Fig 12.12):

Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then $(8 - x - y)$ units will be transported to depot at C (Why?)

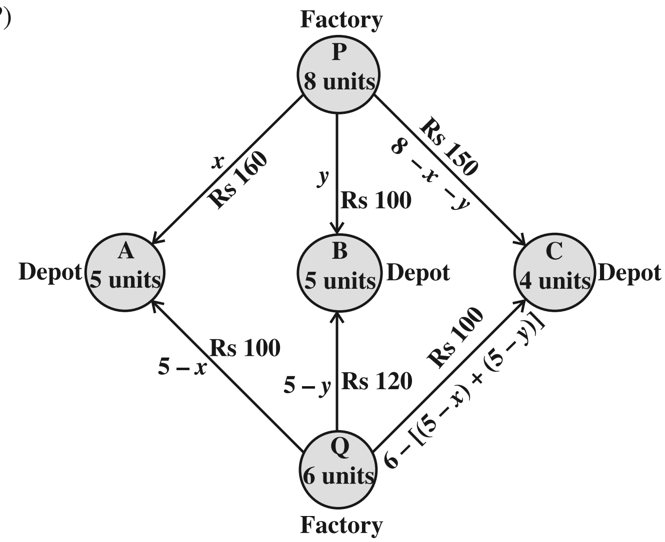


Fig 12.12

Hence, we have $\quad x \geq 0, y \geq 0 \quad$ and $\quad 8 - x - y \geq 0$

i.e. $\quad x \geq 0, y \geq 0 \quad$ and $\quad x + y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5 - x)$ units need to be transported from the factory at Q. Obviously, $5 - x \geq 0$, i.e. $x \leq 5$.

Similarly, $(5 - y)$ and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the factory at Q to the depots at B and C respectively.

Thus, $\quad 5 - y \geq 0 \;,\; x + y - 4 \geq 0$

i.e. $\quad y \leq 5 \;,\; x + y \geq 4$

Total transportation cost Z is given by

$$Z = 160\,x + 100\,y + 100\,(5 - x) + 120\,(5 - y) + 100\,(x + y - 4) + 150\,(8 - x - y)$$
$$= 10\,(x - 7\,y + 190)$$

Therefore, the problem reduces to

Minimise $Z = 10\,(x - 7y + 190)$

subject to the constraints:

$$x \geq 0,\ y \geq 0 \quad \ldots (1)$$
$$x + y \leq 8 \quad \ldots (2)$$
$$x \leq 5 \quad \ldots (3)$$
$$y \leq 5 \quad \ldots (4)$$
and $$x + y \geq 4 \quad \ldots (5)$$

The shaded region ABCDEF represented by the constraints (1) to (5) is the feasible region (Fig 12.13).

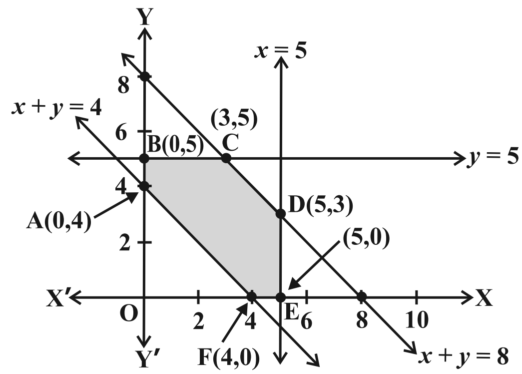


Fig 12.13

Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0). Let us evaluate Z at these points.

Corner Point	$Z = 10\,(x - 7\,y + 190)$	
(0, 4)	1620	
(0, 5)	1550 ←	Minimum
(3, 5)	1580	
(5, 3)	1740	
(5, 0)	1950	
(4, 0)	1940	

From the table, we see that the minimum value of Z is 1550 at the point (0, 5).

Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., Rs 1550.

Miscellaneous Exercise on Chapter 12

1. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

2. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

3. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

4. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Types of Toys	Machines		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

5. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

6. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per quintal (in Rs)		
From/To	**A**	**B**
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

7. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From / To	**A**	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

8. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine.

If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	**Brand Q**
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

9. Refer to Question 8. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

10. A toy company manufactures two types of dolls, A and B. Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?

Summary

- A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum) of a linear function of several variables (called **objective function)** subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear **constraints**). Variables are sometimes called **decision variables** and are **non-negative.**
- A few important linear programming problems are:
  - (i) Diet problems
  - (ii) Manufacturing problems
  - (iii) Transportation problems
- The common region determined by all the constraints including the non-negative constraints $x \geq 0$, $y \geq 0$ of a linear programming problem is called the **feasible region** (or **solution region**) for the problem.
- Points within and on the boundary of the feasible region represent **feasible solutions** of the constraints.

  Any point outside the feasible region is an **infeasible solution.**

- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**
- The following Theorems are fundamental in solving linear programming problems:

  Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

  Theorem 2 Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is **bounded**, then the objective function Z has both a **maximum** and a **minimum** value on R and each of these occurs at a corner point (vertex) of R.
- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.
- **Corner point method** for solving a linear programming problem. The method comprises of the following steps:
  (i) Find the feasible region of the linear programming problem and determine its corner points (vertices).
  (ii) Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.
  (iii) If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective function.

  If the feasible region is unbounded, then
  (i) M is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.
  (ii) m is the minimum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.
- If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

Historical Note

In the World War II, when the war operations had to be planned to economise expenditure, maximise damage to the enemy, linear programming problems came to the forefront.

The first problem in linear programming was formulated in 1941 by the Russian mathematician, L. Kantorovich and the American economist, F. L. Hitchcock, both of whom worked at it independently of each other. This was the well known *transportation problem*. In 1945, an English economist, G.Stigler, described yet another linear programming problem – that of determining an *optimal diet*.

In 1947, the American economist, G. B. Dantzig suggested an efficient method known as the simplex method which is an iterative procedure to solve any linear programming problem in a finite number of steps.

L. Katorovich and American mathematical economist, T. C. Koopmans were awarded the nobel prize in the year 1975 in economics for their pioneering work in linear programming. With the advent of computers and the necessary softwares, it has become possible to apply linear programming model to increasingly complex problems in many areas.

Chapter 10

VECTOR ALGEBRA

❖ *In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure. – HERMAN HANKEL* ❖

10.1 Introduction

W.R. Hamilton (1805-1865)

In our day to day life, we come across many queries such as – What is your height? How should a football player hit the ball to give a pass to another player of his team? Observe that a possible answer to the first query may be 1.6 meters, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors.* In mathematics, physics and engineering, we frequently come across with both types of quantities, namely, scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc.

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors, and their algebraic and geometric properties. These two type of properties, when considered together give a full realisation to the concept of vectors, and lead to their vital applicability in various areas as mentioned above.

10.2 Some Basic Concepts

Let '*l*' be any straight line in plane or three dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a *directed line* (Fig 10.1 (i), (ii)).

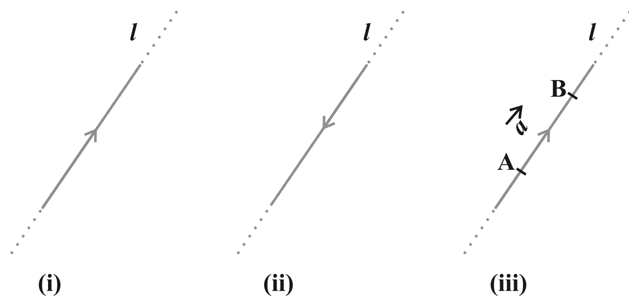


Fig 10.1

Now observe that if we restrict the line l to the line segment AB, then a magnitude is prescribed on the line l with one of the two directions, so that we obtain a *directed line segment* (Fig 10.1(iii)). Thus, a directed line segment has magnitude as well as direction.

Definition 1 A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig 10.1(iii)), denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

The point A from where the vector $\overrightarrow{AB}$ starts is called its *initial point*, and the point B where it ends is called its *terminal point*. The distance between initial and terminal points of a vector is called the *magnitude* (or length) of the vector, denoted as $|\overrightarrow{AB}|$, or $|\vec{a}|$, or a. The arrow indicates the direction of the vector.

Note Since the length is never negative, the notation $|\vec{a}| < 0$ has no meaning.

Position Vector

From Class XI, recall the three dimensional right handed rectangular coordinate system (Fig 10.2(i)). Consider a point P in space, having coordinates (x, y, z) with respect to the origin O(0, 0, 0). Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the *position vector* of the point P with respect to O. Using distance formula (from Class XI), the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

In practice, the position vectors of points A, B, C, etc., with respect to the origin O are denoted by $\vec{a}$, $\vec{b}, \vec{c}$, etc., respectively (Fig 10.2 (ii)).

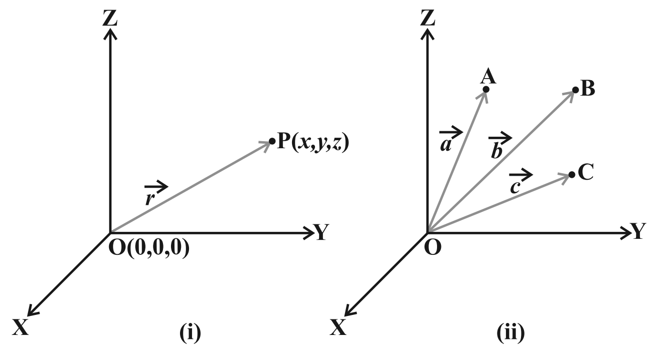


Fig 10.2

Direction Cosines

Consider the position vector $\overrightarrow{\mathrm{OP}}\left(\text{or } \vec{r}\right)$ of a point P(x, y, z) as in Fig 10.3. The angles α, β, γ made by the vector $\vec{r}$ with the positive directions of x, y and z-axes respectively, are called its *direction angles*. The cosine values of these angles, i.e., $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called *direction cosines* of the vector $\vec{r}$, and usually denoted by l, m and n, respectively.

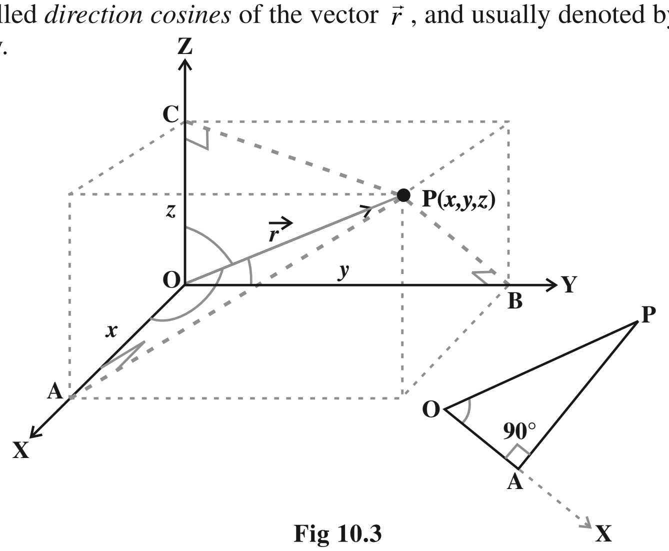


Fig 10.3

From Fig 10.3, one may note that the triangle OAP is right angled, and in it, we have $\cos\alpha = \dfrac{x}{r}$ $\left(r \text{ stands for } |\vec{r}|\right)$. Similarly, from the right angled triangles OBP and OCP, we may write $\cos\beta = \dfrac{y}{r}$ and $\cos\gamma = \dfrac{z}{r}$. Thus, the coordinates of the point P may also be expressed as (lr, mr,nr). The numbers lr, mr and nr, proportional to the direction cosines are called as *direction ratios* of vector $\vec{r}$, and denoted as a, b and c, respectively.

Note One may note that $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.

10.3 Types of Vectors

Zero Vector A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{0}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors $\overrightarrow{AA}, \overrightarrow{BB}$ represent the zero vector,

Unit Vector A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$.

Coinitial Vectors Two or more vectors having the same initial point are called coinitial vectors.

Collinear Vectors Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

Equal Vectors Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

Negative of a Vector A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but direction is opposite to that of it, is called *negative* of the given vector. For example, vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Remark The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called *free vectors*. Throughout this chapter, we will be dealing with free vectors only.

Example 1 Represent graphically a displacement of 40 km, 30° west of south.

Solution The vector $\overrightarrow{OP}$ represents the required displacement (Fig 10.4).

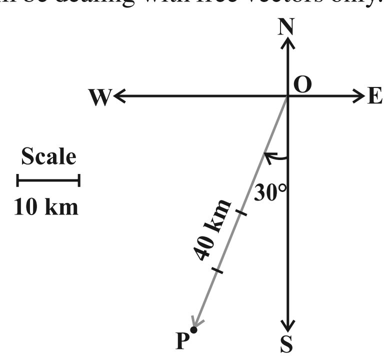


Fig 10.4

Example 2 Classify the following measures as scalars and vectors.

(i) 5 seconds

(ii) 1000 cm^3

(iii) 10 Newton (iv) 30 km/hr (v) 10 g/cm^3

(vi) 20 m/s towards north

Solution

(i) Time-scalar (ii) Volume-scalar (iii) Force-vector

(iv) Speed-scalar (v) Density-scalar (vi) Velocity-vector

Example 3 In Fig 10.5, which of the vectors are:

(i) Collinear (ii) Equal (iii) Coinitial

Solution

(i) Collinear vectors : $\vec{a}$, $\vec{c}$ and $\vec{d}$.

(ii) Equal vectors : $\vec{a}$ and $\vec{c}$.

(iii) Coinitial vectors : $\vec{b}$, $\vec{c}$ and $\vec{d}$.

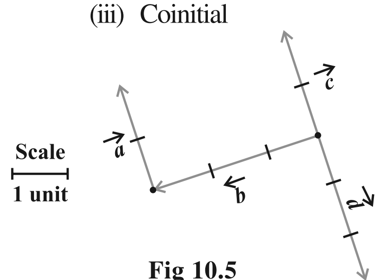


Fig 10.5

EXERCISE 10.1

1. Represent graphically a displacement of 40 km, 30° east of north.
2. Classify the following measures as scalars and vectors.

   (i) 10 kg (ii) 2 meters north-west (iii) 40°

   (iv) 40 watt (v) 10^{-19} coulomb (vi) 20 m/s^2
3. Classify the following as scalar and vector quantities.

   (i) time period (ii) distance (iii) force

   (iv) velocity (v) work done
4. In Fig 10.6 (a square), identify the following vectors.

   (i) Coinitial (ii) Equal

   (iii) Collinear but not equal

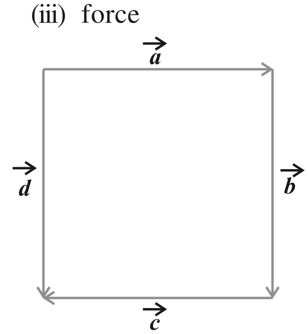


Fig 10.6

5. Answer the following as true or false.

   (i) $\vec{a}$ and $-\vec{a}$ are collinear.

   (ii) Two collinear vectors are always equal in magnitude.

   (iii) Two vectors having same magnitude are collinear.

   (iv) Two collinear vectors having the same magnitude are equal.

10.4 Addition of Vectors

A vector $\overrightarrow{AB}$ simply means the displacement from a point A to the point B. Now consider a situation that a girl moves from A to B and then from B to C (Fig 10.7). The net displacement made by the girl from point A to the point C, is given by the vector $\overrightarrow{AC}$ and expressed as

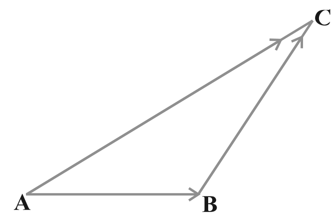


Fig 10.7

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

This is known as the *triangle law of vector addition.*

In general, if we have two vectors $\vec{a}$ and $\vec{b}$ (Fig 10.8 (i)), then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other (Fig 10.8(ii)).

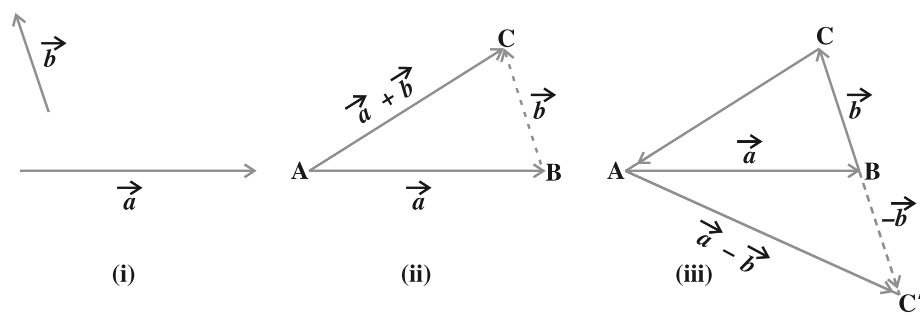


Fig 10.8

For example, in Fig 10.8 (ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction, so that it's initial point coincides with the terminal point of $\vec{a}$. Then, the vector $\vec{a} + \vec{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ i.e., in triangle ABC (Fig 10.8 (ii)), we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Now again, since $\overrightarrow{AC} = -\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig 10.8(iii)).

Now, construct a vector $\overrightarrow{BC'}$ so that its magnitude is same as the vector $\overrightarrow{BC}$, but the direction opposite to that of it (Fig 10.8 (iii)), i.e.,

$$\overrightarrow{BC'} = -\overrightarrow{BC}$$

Then, on applying triangle law from the Fig 10.8 (iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

The vector $\overrightarrow{AC'}$ is said to represent the *difference of* $\vec{a}$ and $\vec{b}$.

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors–one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat in actual starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig 10.9), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the *parallelogram law of vector addition.*

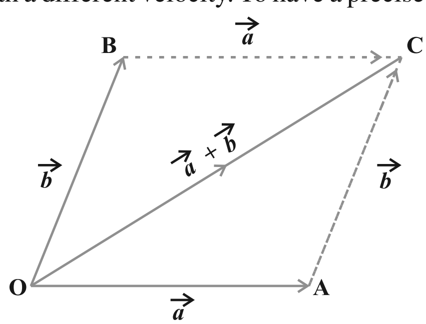


Fig 10.9

> **Note** From Fig 10.9, using the triangle law, one may note that
>
> $$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$
>
> or $$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \qquad (\text{since } \overrightarrow{AC} = \overrightarrow{OB})$$
>
> which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

Properties of vector addition

Property 1 For any two vectors $\vec{a}$ and $\vec{b}$,

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(Commutative property)}$$

Proof Consider the parallelogram ABCD (Fig 10.10). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, then using the triangle law, from triangle ABC, we have

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

Now, since the opposite sides of a parallelogram are equal and parallel, from Fig 10.10, we have, $\overrightarrow{AD} = \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$. Again using triangle law, from triangle ADC, we have

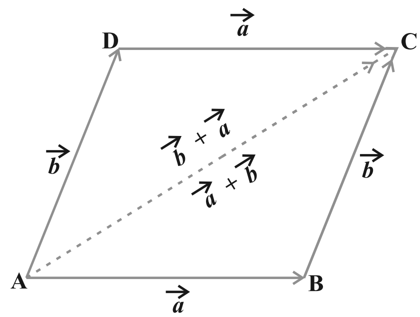


Fig 10.10

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$$

Hence $\quad \vec{a} + \vec{b} = \vec{b} + \vec{a}$

Property 2 For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(Associative property)}$$

Proof Let the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$, respectively, as shown in Fig 10.11(i) and (ii).

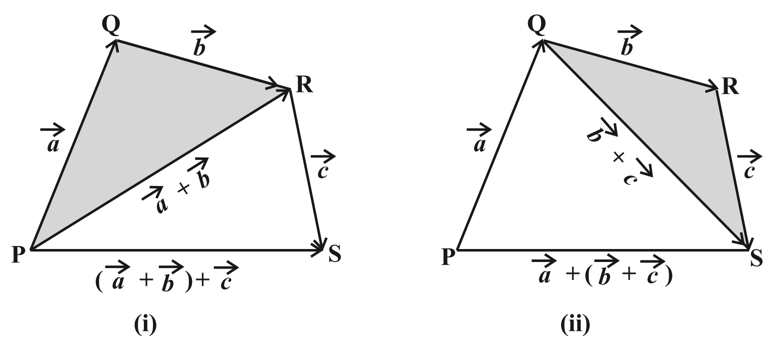


Fig 10.11

Then $\quad \vec{a} + \vec{b} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

and $\quad \vec{b} + \vec{c} = \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$

So $\quad (\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$

and $\quad \vec{a}+(\vec{b}+\vec{c}) = \overrightarrow{PQ}+\overrightarrow{QS}=\overrightarrow{PS}$

Hence $\quad (\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

Remark The associative property of vector addition enables us to write the sum of three vectors $\vec{a}$,$\vec{b}$, $\vec{c}$ as $\vec{a}+\vec{b}+\vec{c}$ without using brackets.

Note that for any vector $\vec{a}$, we have

$$\vec{a}+\vec{0} = \vec{0}+\vec{a}=\vec{a}$$

Here, the zero vector $\vec{0}$ is called the *additive identity* for the vector addition.

10.5 Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ, denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that, $\lambda\vec{a}$ is also a vector, collinear to the vector $\vec{a}$. The vector $\lambda\vec{a}$ has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of the vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualisation of multiplication of a vector by a scalar is given in Fig 10.12.

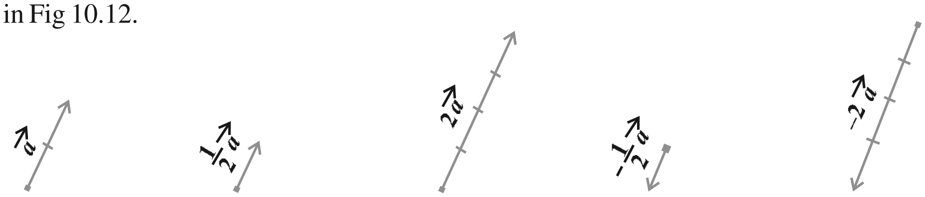


Fig 10.12

When λ = –1, then $\lambda\vec{a}=-\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$. The vector $-\vec{a}$ is called the *negative* (or *additive inverse*) *of vector* $\vec{a}$ and we always have

$$\vec{a}+(-\vec{a}) = (-\vec{a})+\vec{a}=\vec{0}$$

Also, if $\lambda=\frac{1}{|\vec{a}|}$, provided $\vec{a}\neq 0$, i.e. $\vec{a}$ is not a null vector, then

$$|\lambda\vec{a}|=|\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}|=1$$

So, $\lambda\vec{a}$ *represents the unit vector in the direction of* $\vec{a}$. We write it as

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

Note For any scalar k, $k\vec{0} = \vec{0}$.

10.5.1 *Components of a vector*

Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the x-axis, y-axis and z-axis, respectively. Then, clearly

$$|\overrightarrow{OA}| = 1, |\overrightarrow{OB}| = 1 \text{ and } |\overrightarrow{OC}| = 1$$

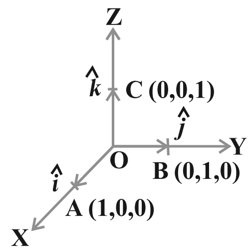


Fig 10.13

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called *unit vectors along the axes* OX, OY and OZ, respectively, and denoted by $\hat{i}, \hat{j}$ and $\hat{k}$, respectively (Fig 10.13).

Now, consider the position vector $\overrightarrow{OP}$ of a point P(x, y, z) as in Fig 10.14. Let P_1 be the foot of the perpendicular from P on the plane XOY. We, thus, see that P_1P is

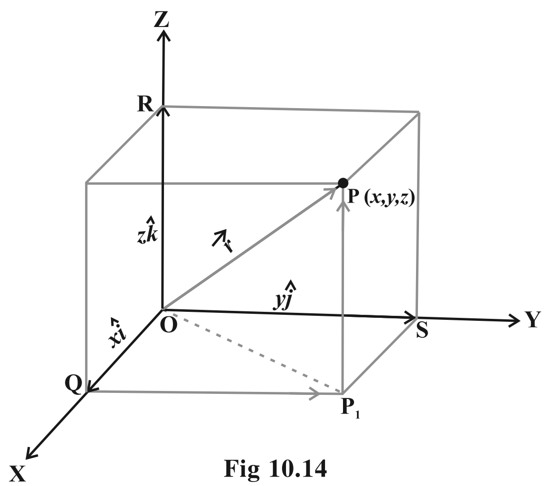


Fig 10.14

parallel to z-axis. As $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the x, y and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$

and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}\text{)} = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its *component form.* Here, x, y and z are called as the *scalar components* of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the *vector components* of $\vec{r}$ along the respective axes. Sometimes x, y and z are also termed as *rectangular components*.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, is readily determined by applying the Pythagoras theorem twice. We note that in the right angle triangle OQP_1 (Fig 10.14)

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2},$$

and in the right angle triangle OP_1P, we have

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then

(i) the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) the difference of the vector $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) the vectors $\vec{a}$ and $\vec{b}$ are equal if and only if

$$a_1 = b_1,\ a_2 = b_2 \quad \text{and} \quad a_3 = b_3$$

(iv) the multiplication of vector $\vec{a}$ by any scalar λ is given by

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ and $\vec{b}$ be any two vectors, and k and m be any scalars. Then

(i) $k\vec{a} + m\vec{a} = (k+m)\vec{a}$

(ii) $k(m\vec{a}) = (km)\vec{a}$

(iii) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

Remarks

(i) One may observe that whatever be the value of λ, the vector $\lambda\vec{a}$ is always collinear to the vector $\vec{a}$. In fact, two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a nonzero scalar λ such that $\vec{b} = \lambda\vec{a}$. If the vectors $\vec{a}$ and $\vec{b}$ are given in the component form, i.e. $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then the two vectors are collinear if and only if

$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\Leftrightarrow \quad b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Leftrightarrow \quad b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ b_3 = \lambda a_3$$

$$\Leftrightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2, a_3 are also called direction ratios of $\vec{a}$.

(iii) In case if it is given that l, m, n are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k}$ $= (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector, where α, β and γ are the angles which the vector makes with x, y and z axes respectively.

Example 4 Find the values of x, y and z so that the vectors $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal.

Solution Note that two vectors are equal if and only if their corresponding components are equal. Thus, the given vectors $\vec{a}$ and $\vec{b}$ will be equal if and only if

$$x = 2,\ y = 2,\ z = 1$$

Example 5 Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal?

Solution We have $|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $|\vec{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

So, $|\vec{a}| = |\vec{b}|$. But, the two vectors are not equal since their corresponding components are distinct.

Example 6 Find unit vector in the direction of vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

Solution The unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$.

Now $$|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

Therefore $$\hat{a} = \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k}) = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$$

Example 7 Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

Solution The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j}) = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\vec{a}$ is

$$7\hat{a} = 7\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

Example 8 Find the unit vector in the direction of the sum of the vectors, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

Solution The sum of the given vectors is

$$\vec{a} + \vec{b}\ (= \vec{c}, \text{say}) = 4\hat{i} + 3\hat{j} - 2\hat{k}$$

and $$|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

Thus, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Example 9 Write the direction ratio's of the vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and hence calculate its direction cosines.

Solution Note that the direction ratio's a, b, c of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are just the respective components x, y and z of the vector. So, for the given vector, we have $a = 1$, $b = 1$ and $c = -2$. Further, if l, m and n are the direction cosines of the given vector, then

$$l = \frac{a}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad m = \frac{b}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad n = \frac{c}{|\vec{r}|} = \frac{-2}{\sqrt{6}} \text{ as } |\vec{r}| = \sqrt{6}$$

Thus, the direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

10.5.2 *Vector joining two points*

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$ (Fig 10.15).

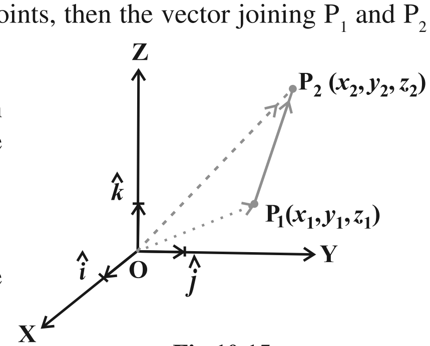


Fig 10.15

Joining the points P_1 and P_2 with the origin O, and applying triangle law, from the triangle OP_1P_2, we have

$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}.$$

Using the properties of vector addition, the above equation becomes

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

i.e.
$$\overrightarrow{P_1P_2} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 10 Find the vector joining the points P(2, 3, 0) and Q(– 1, – 2, – 4) directed from P to Q.

Solution Since the vector is to be directed from P to Q, clearly P is the initial point and Q is the terminal point. So, the required vector joining P and Q is the vector $\overrightarrow{PQ}$, given by

$$\overrightarrow{PQ} = (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{k}$$

i.e.
$$\overrightarrow{PQ} = -3\hat{i} - 5\hat{j} - 4\hat{k}.$$

10.5.3 *Section formula*

Let P and Q be two points represented by the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, respectively, with respect to the origin O. Then the line segment joining the points P and Q may be divided by a third point, say R, in two ways – internally (Fig 10.16) and externally (Fig 10.17). Here, we intend to find the position vector $\overrightarrow{OR}$ for the point R with respect to the origin O. We take the two cases one by one.

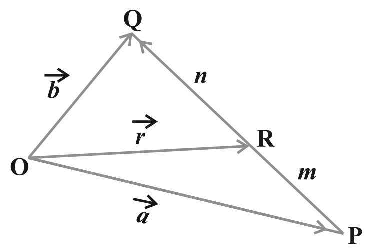


Fig 10.16

Case I When R divides PQ internally (Fig 10.16).

If R divides $\overrightarrow{PQ}$ such that $m\,\overrightarrow{RQ} = n\,\overrightarrow{PR}$,

where m and n are positive scalars, we say that the point R divides $\overrightarrow{PQ}$ internally in the ratio of $m : n$. Now from triangles ORQ and OPR, we have

$$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR} = \vec{b} - \vec{r}$$

and
$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \vec{r} - \vec{a},$$

Therefore, we have
$$m\,(\vec{b} - \vec{r}) = n\,(\vec{r} - \vec{a}) \quad \text{(Why?)}$$

or
$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n} \quad \text{(on simplification)}$$

Hence, the position vector of the point R which divides P and Q internally in the ratio of $m : n$ is given by

$$\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Case II When R divides PQ externally (Fig 10.17). We leave it to the reader as an exercise to verify that the position vector of the point R which divides the line segment PQ externally in the ratio $m : n \left(\text{i.e. } \frac{\text{PR}}{\text{QR}} = \frac{m}{n}\right)$ is given by

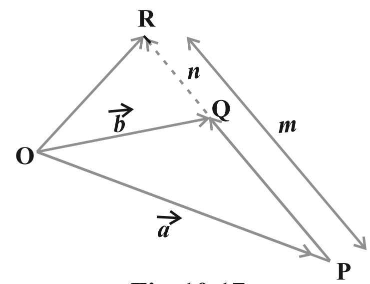


Fig 10.17

$$\overrightarrow{\text{OR}} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

Remark If R is the midpoint of PQ , then $m = n$. And therefore, from Case I, the midpoint R of $\overrightarrow{\text{PQ}}$, will have its position vector as

$$\overrightarrow{\text{OR}} = \frac{\vec{a} + \vec{b}}{2}$$

Example 11 Consider two points P and Q with position vectors $\overrightarrow{\text{OP}} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{\text{OQ}} = \vec{a} + \vec{b}$. Find the position vector of a point R which divides the line joining P and Q in the ratio 2:1, (i) internally, and (ii) externally.

Solution

(i) The position vector of the point R dividing the join of P and Q internally in the ratio 2:1 is

$$\overrightarrow{\text{OR}} = \frac{2(\vec{a} + \vec{b}) + (3\vec{a} - 2\vec{b})}{2 + 1} = \frac{5\vec{a}}{3}$$

(ii) The position vector of the point R dividing the join of P and Q externally in the ratio 2:1 is

$$\overrightarrow{\text{OR}} = \frac{2(\vec{a} + \vec{b}) - (3\vec{a} - 2\vec{b})}{2 - 1} = 4\vec{b} - \vec{a}$$

Example 12 Show that the points $\text{A}(2\hat{i} - \hat{j} + \hat{k}),\ \text{B}(\hat{i} - 3\hat{j} - 5\hat{k}),\ \text{C}(3\hat{i} - 4j - 4\hat{k})$ are the vertices of a right angled triangle.

Solution We have

$$\overrightarrow{\text{AB}} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{\text{BC}} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

and $$\overrightarrow{\text{CA}} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

Further, note that

$$|\overrightarrow{AB}|^2 = 41 = 6 + 35 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$$

Hence, the triangle is a right angled triangle.

EXERCISE 10.2

1. Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + k; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \quad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

2. Write two different vectors having same magnitude.
3. Write two different vectors having same direction.
4. Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.
5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (– 5, 7).
6. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.
7. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.
8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.
9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.
10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.
11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.
12. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.
13. Find the direction cosines of the vector joining the points A (1, 2, –3) and B(–1, –2, 1), directed from A to B.
14. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.
15. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1

    (i) internally (ii) externally

16. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

17. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right angled triangle.

18. In triangle ABC (Fig 10.18), which of the following is not true:

(A) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(B) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(C) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

(D) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

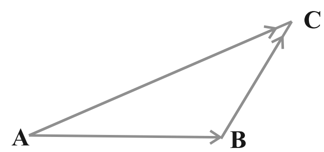


Fig 10.18

19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

(A) $\vec{b} = \lambda\vec{a}$, for some scalar λ

(B) $\vec{a} = \pm\vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are proportional

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes.

10.6 Product of Two Vectors

So far we have studied about addition and subtraction of vectors. An other algebraic operation which we intend to discuss regarding vectors is their product. We may recall that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiply them in two ways, namely, multiplication of two functions pointwise and composition of two functions. Similarly, multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector. Based upon these two types of products for vectors, they have found various applications in geometry, mechanics and engineering. In this section, we will discuss these two types of products.

10.6.1 *Scalar (or dot) product of two vectors*

Definition 2 The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta,$

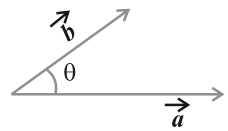


Fig 10.19

where, θ is the angle between $\vec{a}$ and $\vec{b}, 0 \le \theta \le \pi$ (Fig 10.19).

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined, and in this case, we define $\vec{a} \cdot \vec{b} = 0$

Observations

1. $\vec{a} \cdot \vec{b}$ is a real number.
2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors, then $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other. i.e.

   $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
3. If θ = 0, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

   In particular, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, as θ in this case is 0.
4. If θ = π, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

   In particular, $\vec{a} \cdot (-\vec{a}) = -|\vec{a}|^2$, as θ in this case is π.
5. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1,$$
$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

6. The angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}, \text{ or } \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

7. The scalar product is commutative. i.e.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad \text{(Why?)}$$

Two important properties of scalar product

Property 1 (Distributivity of scalar product over addition) Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Property 2 Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then

$$(\lambda\vec{a})\cdot\vec{b} = (\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$$

If two vectors $\vec{a}$ and $\vec{b}$ are given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then their scalar product is given as

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\ &= a_1\hat{i}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})+a_2\hat{j}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})+a_3\hat{k}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\ &= a_1b_1(\hat{i}\cdot\hat{i})+a_1b_2(\hat{i}\cdot\hat{j})+a_1b_3(\hat{i}\cdot\hat{k})+a_2b_1(\hat{j}\cdot\hat{i})+a_2b_2(\hat{j}\cdot\hat{j})+a_2b_3(\hat{j}\cdot\hat{k})\\ &\quad + a_3b_1(\hat{k}\cdot\hat{i})+a_3b_2(\hat{k}\cdot\hat{j})+a_3b_3(\hat{k}\cdot\hat{k}) \text{ (Using the above Properties 1 and 2)}\\ &= a_1b_1 + a_2b_2 + a_3b_3 \quad \text{(Using Observation 5)}\end{aligned}$$

Thus $\quad \vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$

10.6.2 *Projection of a vector on a line*

Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a given directed line l (say), in the *anticlockwise direction* (Fig 10.20). Then the projection of $\overrightarrow{AB}$ on l is a vector $\vec{p}$ (say) with magnitude $|\overrightarrow{AB}|\cos\theta$, and the direction of $\vec{p}$ being the same (or opposite) to that of the line l, depending upon whether $\cos\theta$ is positive or negative. The vector $\vec{p}$

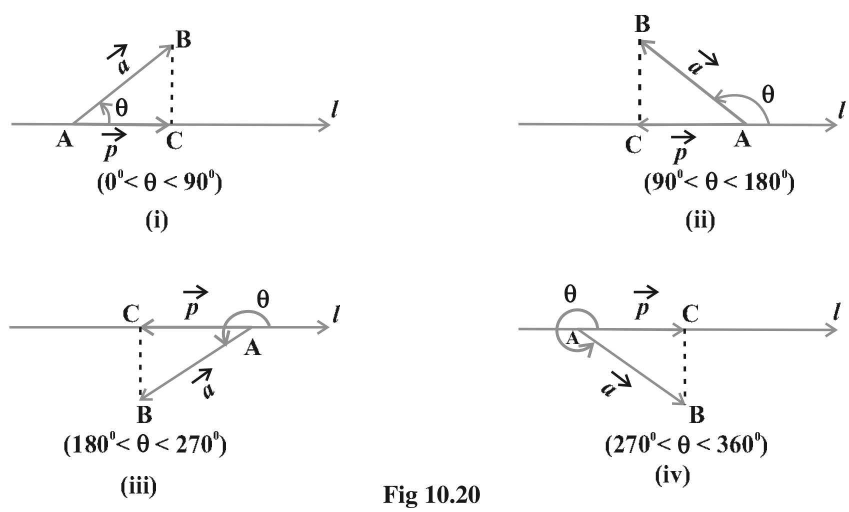


Fig 10.20

is called the *projection vector*, and its magnitude $|\vec{p}|$ is simply called as the *projection* of the vector $\overrightarrow{AB}$ on the directed line l.

For example, in each of the following figures (Fig 10.20 (i) to (iv)), projection vector of $\overrightarrow{AB}$ along the line l is vector $\overrightarrow{AC}$.

Observations

1. If $\hat{p}$ is the unit vector along a line l, then the projection of a vector $\vec{a}$ on the line l is given by $\vec{a} \cdot \hat{p}$.
2. Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a} \cdot \hat{b}, \quad \text{or} \quad \vec{a} \cdot \left(\frac{\vec{b}}{|\vec{b}|}\right), \quad \text{or} \quad \frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b})$$

3. If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.
4. If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

Remark If α, β and γ are the direction angles of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines may be given as

$$\cos\alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Also, note that $|\vec{a}|\cos\alpha$, $|\vec{a}|\cos\beta$ and $|\vec{a}|\cos\gamma$ are respectively the projections of $\vec{a}$ along OX, OY and OZ. i.e., the scalar components a_1, a_2 and a_3 of the vector $\vec{a}$, are precisely the projections of $\vec{a}$ along x-axis, y-axis and z-axis, respectively. Further, if $\vec{a}$ is a unit vector, then it may be expressed in terms of its direction cosines as

$$\vec{a} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$$

Example 13 Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $\vec{a} \cdot \vec{b} = 1$.

Solution Given $\vec{a} \cdot \vec{b} = 1, |\vec{a}| = 1$ and $|\vec{b}| = 2$. We have

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Example 14 Find angle 'θ' between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Solution The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Now $\vec{a}\cdot\vec{b} = (\hat{i} + \hat{j} - \hat{k})\cdot(\hat{i} - \hat{j} + \hat{k}) = 1 - 1 - 1 = -1$.

Therefore, we have $\cos\theta = \frac{-1}{3}$

hence the required angle is $\theta = \cos^{-1}\left(-\frac{1}{3}\right)$

Example 15 If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

Solution We know that two nonzero vectors are perpendicular if their scalar product is zero.

Here $\vec{a} + \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k}) = 6\hat{i} + 2\hat{j} - 8\hat{k}$

and $\vec{a} - \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k}) = 4\hat{i} - 4\hat{j} + 2\hat{k}$

So $(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k})\cdot(4\hat{i} - 4\hat{j} + 2\hat{k}) = 24 - 8 - 16 = 0.$

Hence $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors.

Example 16 Find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Solution The projection of vector $\vec{a}$ on the vector $\vec{b}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{(2\times 1 + 3\times 2 + 2\times 1)}{\sqrt{(1)^2 + (2)^2 + (1)^2}} = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}$$

Example 17 Find $|\vec{a} - \vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 4$.

Solution We have

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$$

$$= \vec{a}.\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 - 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

$$= (2)^2 - 2(4) + (3)^2$$

Therefore $\qquad |\vec{a}-\vec{b}| = \sqrt{5}$

Example 18 If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$, then find $|\vec{x}|$.

Solution Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$. Also,

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$$

or $\qquad \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{a} = 8$

or $\qquad |\vec{x}|^2 - 1 = 8 \text{ i.e. } |\vec{x}|^2 = 9$

Therefore $\qquad |\vec{x}| = 3$ (as magnitude of a vector is non negative).

Example 19 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$ (Cauchy-Schwartz inequality).

Solution The inequality holds trivially when either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. Actually, in such a situation we have $|\vec{a}\cdot\vec{b}| = 0 = |\vec{a}||\vec{b}|$. So, let us assume that $|\vec{a}| \ne 0 \ne |\vec{b}|$. Then, we have

$$\frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \le 1$$

Therefore $\qquad |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$

Example 20 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ (triangle inequality).

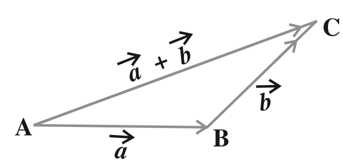


Fig 10.21

Solution The inequality holds trivially in case either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ (How?). So, let $|\vec{a}| \ne \vec{0} \ne |\vec{b}|$. Then,

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \qquad \text{(scalar product is commutative)}$$

$$\le |\vec{a}|^2 + 2|\vec{a}\cdot\vec{b}| + |\vec{b}|^2 \qquad (\text{since } x \le |x|\ \forall x \in \mathbf{R})$$

$$\le |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 \qquad \text{(from Example 19)}$$

$$= (|\vec{a}| + |\vec{b}|)^2$$

Hence $$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$$

Remark If the equality holds in triangle inequality (in the above Example 20), i.e.

$$|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|,$$

then $$|\overrightarrow{AC}| = |\overrightarrow{AB}|+|\overrightarrow{BC}|$$

showing that the points A, B and C are collinear.

Example 21 Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k})$, $B(\hat{i}+2\hat{j}+3\hat{k})$ and $C(7\hat{i}-\hat{k})$ are collinear.

Solution We have

$$\overrightarrow{AB} = (1+2)\hat{i}+(2-3)\hat{j}+(3-5)\hat{k} = 3\hat{i}-\hat{j}-2\hat{k},$$

$$\overrightarrow{BC} = (7-1)\hat{i}+(0-2)\hat{j}+(-1-3)\hat{k} = 6\hat{i}-2\hat{j}-4\hat{k},$$

$$\overrightarrow{AC} = (7+2)\hat{i}+(0-3)\hat{j}+(-1-5)\hat{k} = 9\hat{i}-3\hat{j}-6\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{14},\ |\overrightarrow{BC}| = 2\sqrt{14} \text{ and } |\overrightarrow{AC}| = 3\sqrt{14}$$

Therefore $$\left|\overrightarrow{AC}\right| = |\overrightarrow{AB}|+|\overrightarrow{BC}|$$

Hence the points A, B and C are collinear.

Note In Example 21, one may note that although $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \vec{0}$ but the points A, B and C do not form the vertices of a triangle.

EXERCISE 10.3

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}\cdot\vec{b} = \sqrt{6}$.
2. Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$
3. Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.
4. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.
5. Show that each of the given three vectors is a unit vector:

   $\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}),\ \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}),\ \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$

   Also, show that they are mutually perpendicular to each other.

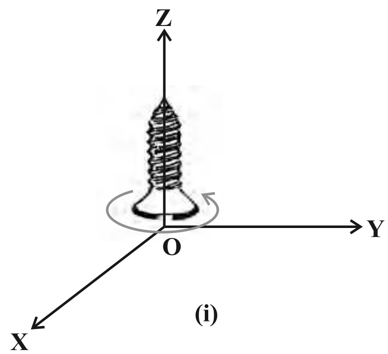
6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.
7. Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.
8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.
9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$.
10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.
11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.
12. If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?
13. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.
14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.
15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$].
16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, –1) are collinear.
17. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.
18. If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = 1/|\lambda|$

10.6.3 *Vector (or cross) product of two vectors*

In Section 10.2, we have discussed on the three dimensional right handed rectangular coordinate system. In this system, when the positive x-axis is rotated counterclockwise into the positive y-axis, a right handed (standard) screw would advance in the direction of the positive z-axis (Fig 10.22(i)).

In a right handed coordinate system, the thumb of the right hand points in the direction of the positive z-axis when the fingers are curled in the direction away from the positive x-axis toward the positive y-axis (Fig 10.22(ii)).



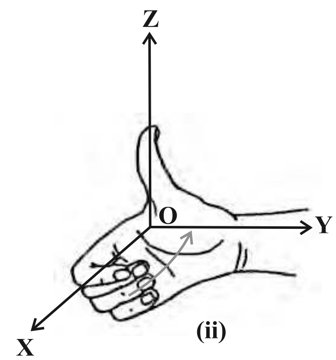


Fig 10.22 (i), (ii)

Definition 3 The vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$, is denoted by $\vec{a} \times \vec{b}$ and defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n},$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (Fig 10.23). i.e., the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.

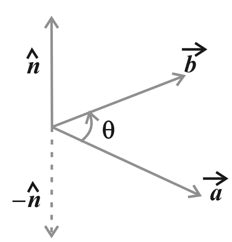


Fig 10.23

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case, we define $\vec{a} \times \vec{b} = \vec{0}$.

Observations

1. $\vec{a} \times \vec{b}$ is a vector.
2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other, i.e.,

$$\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$$

In particular, $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times (-\vec{a}) = \vec{0}$, since in the first situation, $\theta = 0$ and in the second one, $\theta = \pi$, making the value of $\sin\theta$ to be 0.

3. If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$.

4. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ (Fig 10.24), we have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

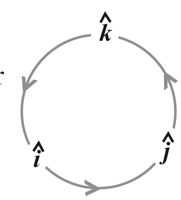
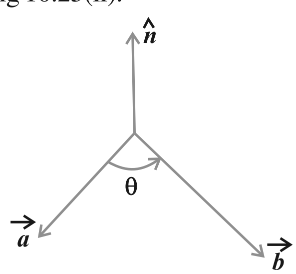


Fig 10.24

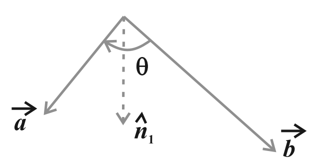
5. In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

6. It is always true that the vector product is not commutative, as $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Indeed, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system, i.e., θ is traversed from $\vec{a}$ to $\vec{b}$, Fig 10.25 (i). While, $\vec{b} \times \vec{a} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}_1$, where $\vec{b}, \vec{a}$ and $\hat{n}_1$ form a right handed system i.e. θ is traversed from $\vec{b}$ to $\vec{a}$, Fig 10.25(ii).



(i)



(ii)

Fig 10.25 (i), (ii)

Thus, if we assume $\vec{a}$ and $\vec{b}$ to lie in the plane of the paper, then $\hat{n}$ and $\hat{n}_1$ both will be perpendicular to the plane of the paper. But, $\hat{n}$ being directed above the paper while $\hat{n}_1$ directed below the paper. i.e. $\hat{n}_1 = -\hat{n}$.

Hence $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$= -|\vec{a}||\vec{b}|\sin\theta\,\hat{n}_1 = -\vec{b}\times\vec{a}$

7. In view of the Observations 4 and 6, we have

$$\hat{j}\times\hat{i} = -\hat{k},\quad \hat{k}\times\hat{j} = -\hat{i} \text{ and } \hat{i}\times\hat{k} = -\hat{j}.$$

8. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as $\frac{1}{2}|\vec{a}\times\vec{b}|$.

By definition of the area of a triangle, we have from Fig 10.26,

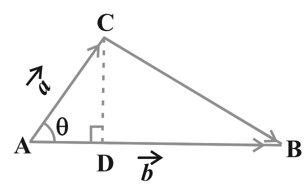


Fig 10.26

Area of triangle ABC $= \frac{1}{2}\text{AB}\cdot\text{CD}.$

But $\text{AB} = |\vec{b}|$ (as given), and CD $= |\vec{a}|\sin\theta$.

Thus, Area of triangle ABC $= \frac{1}{2}|\vec{b}||\vec{a}|\sin\theta = \frac{1}{2}|\vec{a}\times\vec{b}|.$

9. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a}\times\vec{b}|$.

From Fig 10.27, we have

Area of parallelogram ABCD = AB. DE.

But $\text{AB} = |\vec{b}|$ (as given), and

$\text{DE} = |\vec{a}|\sin\theta$.

Thus,

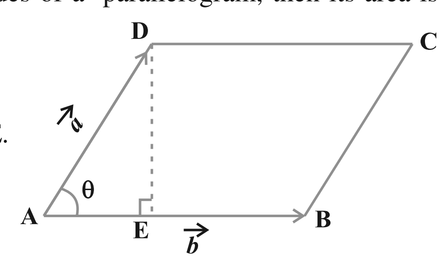


Fig 10.27

Area of parallelogram ABCD $= |\vec{b}||\vec{a}|\sin\theta = |\vec{a}\times\vec{b}|.$

We now state two important properties of vector product.

Property 3 (Distributivity of vector product over addition): If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors and λ be a scalar, then

(i) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

(ii) $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$

Let $\vec{a}$ and $\vec{b}$ be two vectors given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, respectively. Then their cross product may be given by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Explanation We have

$$\begin{aligned}
\vec{a}\times\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \\
&= a_1b_1(\hat{i}\times\hat{i})+a_1b_2(\hat{i}\times\hat{j})+a_1b_3(\hat{i}\times\hat{k})+a_2b_1(\hat{j}\times\hat{i}) \\
&\quad + a_2b_2(\hat{j}\times\hat{j})+a_2b_3(\hat{j}\times\hat{k}) \\
&\quad + a_3b_1(\hat{k}\times\hat{i})+a_3b_2(\hat{k}\times\hat{j})+a_3b_3(\hat{k}\times\hat{k}) \qquad \text{(by Property 1)} \\
&= a_1b_2(\hat{i}\times\hat{j})-a_1b_3(\hat{k}\times\hat{i})-a_2b_1(\hat{i}\times\hat{j}) \\
&\quad + a_2b_3(\hat{j}\times\hat{k})+a_3b_1(\hat{k}\times\hat{i})-a_3b_2(\hat{j}\times\hat{k})
\end{aligned}$$

(as $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ and $\hat{i}\times\hat{k}=-\hat{k}\times\hat{i}$, $\hat{j}\times\hat{i}=-\hat{i}\times\hat{j}$ and $\hat{k}\times\hat{j}=-\hat{j}\times\hat{k}$)

$$= a_1b_2\hat{k}-a_1b_3\hat{j}-a_2b_1\hat{k}+a_2b_3\hat{i}+a_3b_1\hat{j}-a_3b_2\hat{i}$$

(as $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$)

$$= (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 22 Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$

Solution We have

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$$= \hat{i}(-2-15)-(-4-9)\hat{j}+(10-3)\hat{k} = -17\hat{i}+13\hat{j}+7\hat{k}$$

Hence $\quad |\vec{a}\times\vec{b}| = \sqrt{(-17)^2+(13)^2+(7)^2} = \sqrt{507}$

Example 23 Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.

Solution We have $\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$

A vector which is perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\hat{j}-2\hat{k} \quad (=\vec{c},\ \text{say})$$

Now $$|\vec{c}| = \sqrt{4+16+4}=\sqrt{24}=2\sqrt{6}$$

Therefore, the required unit vector is

$$\frac{\vec{c}}{|\vec{c}|} = \frac{-1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$$

Note There are two perpendicular directions to any plane. Thus, another unit vector perpendicular to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ will be $\frac{1}{\sqrt{6}}\hat{i}-\frac{2}{\sqrt{6}}\hat{j}+\frac{1}{\sqrt{6}}\hat{k}$. But that will be a consequence of $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$.

Example 24 Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Solution We have $\overrightarrow{AB}=\hat{j}+2\hat{k}$ and $\overrightarrow{AC}=\hat{i}+2\hat{j}$. The area of the given triangle is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

Now, $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\hat{i}+2\hat{j}-\hat{k}$$

Therefore $$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{16+4+1}=\sqrt{21}$$

Thus, the required area is $\frac{1}{2}\sqrt{21}$

Example 25 Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Solution The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a} \times \vec{b}|$.

Now $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{i} + \hat{j} - 4\hat{k}$$

Therefore $$|\vec{a} \times \vec{b}| = \sqrt{25+1+16} = \sqrt{42}$$

and hence, the required area is $\sqrt{42}$.

EXERCISE 10.4

1. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.
2. Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.
3. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.
4. Show that
$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$$
5. Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.
6. Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?
7. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.
9. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Miscellaneous Examples

Example 26 Write all the unit vectors in XY-plane.

Solution Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in XY-plane (Fig 10.28). Then, from the figure, we have $x = \cos\theta$ and $y = \sin\theta$ (since $|\vec{r}| = 1$). So, we may write the vector $\vec{r}$ as

$$\vec{r}\left(=\overrightarrow{OP}\right) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \text{... (1)}$$

Clearly, $$|\vec{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

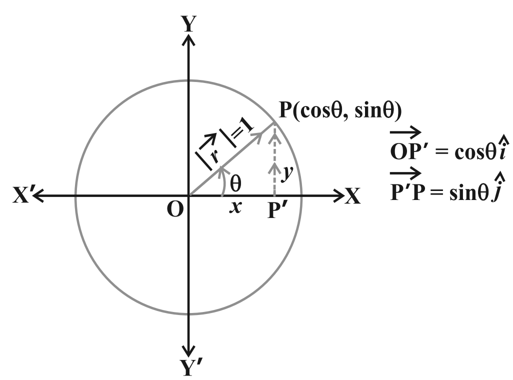


Fig 10.28

Also, as θ varies from 0 to 2π, the point P (Fig 10.28) traces the circle $x^2 + y^2 = 1$ counterclockwise, and this covers all possible directions. So, (1) gives every unit vector in the XY-plane.

Example 27 If $\hat{i}+\hat{j}+\hat{k},\ 2\hat{i}+5\hat{j},\ 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Solution Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\quad \overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$$

Therefore $\quad |\overrightarrow{AB}| = \sqrt{(1)^2+(4)^2+(-1)^2} = 3\sqrt{2}$

Similarly $\quad \overrightarrow{CD} = -2\hat{i}-8\hat{j}+2\hat{k}$ and $|\overrightarrow{CD}| = 6\sqrt{2}$

Thus $\quad \cos\theta = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}$

$$= \frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1$$

Since $0 \le \theta \le \pi$, it follows that $\theta = \pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Alternatively, $\overrightarrow{AB} = -\dfrac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

Example 28 Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Solution Given $\vec{a}\cdot(\vec{b}+\vec{c}) = 0,\ \vec{b}\cdot(\vec{c}+\vec{a}) = 0,\ \vec{c}\cdot(\vec{a}+\vec{b}) = 0.$

Now $\quad |\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot\vec{b}+\vec{b}\cdot(\vec{a}+\vec{c})$$

$$+\ \vec{c}.(\vec{a}+\vec{b})+\vec{c}.\vec{c}$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$$

$$= 9 + 16 + 25 = 50$$

Therefore $\quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$

Example 29 Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=1$, $|\vec{b}|=4$ and $|\vec{c}|=2$.

Solution Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have

$$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$$

or $$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = 0$$

Therefore $$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = -\left|\vec{a}\right|^2=-1 \qquad \text{... (1)}$$

Again, $$\vec{b}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right) = 0$$

or $$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c} = -\left|\vec{b}\right|^2=-16 \qquad \text{... (2)}$$

Similarly $$\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} = -4. \qquad \text{... (3)}$$

Adding (1), (2) and (3), we have

$$2\ (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c}) = -21$$

or $$2\mu=-21,\ \text{i.e., } \mu = \frac{-21}{2}$$

Example 30 If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, $\vec{\alpha}=3\hat{i}-\hat{j}$, $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Solution Let $\vec{\beta}_1=\lambda\vec{\alpha}$, λ is a scalar, i.e., $\vec{\beta}_1=3\lambda\hat{i}-\lambda\hat{j}$.

Now $$\vec{\beta}_2=\vec{\beta}-\vec{\beta}_1 = (2-3\lambda)\hat{i}+(1+\lambda)\hat{j}-3\hat{k}.$$

Now, since $\vec{\beta}_2$ is to be perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha}\cdot\vec{\beta}_2=0$. i.e.,

$$3(2-3\lambda)-(1+\lambda) = 0$$

or $$\lambda = \frac{1}{2}$$

Therefore $$\vec{\beta}_1 = \frac{3}{2}\hat{i}-\frac{1}{2}\hat{j} \quad \text{and} \quad \vec{\beta}_2=\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$$

Miscellaneous Exercise on Chapter 10

1. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.
2. Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.
3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
4. If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.
5. Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.
6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.
7. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.
8. Show that the points A (1, – 2, – 8), B (5, 0, –2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.
9. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.
10. The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.
11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.
12. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 15$.
13. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.
14. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

15. Prove that $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Choose the correct answer in Exercises 16 to 19.

16. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$ (D) $0 \leq \theta \leq \pi$

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

18. The value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$ is

(A) 0 (B) –1 (C) 1 (D) 3

19. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Summary

- Position vector of a point P(*x*, *y*, *z*) is given as $\overrightarrow{OP}(=\vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$, and its magnitude by $\sqrt{x^2+y^2+z^2}$.
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude (*r*), direction ratios (*a*, *b*, *c*) and direction cosines (*l*, *m*, *n*) of any vector are related as:

$$l = \frac{a}{r},\quad m = \frac{b}{r},\quad n = \frac{c}{r}$$

- The vector sum of the three sides of a triangle taken in order is $\vec{0}$.

- The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- The multiplication of a given vector by a scalar λ, changes the magnitude of the vector by the multiple $|\lambda|$, and keeps the direction same (or makes it opposite) according as the value of λ is positive (or negative).
- For a given vector $\vec{a}$, the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$.
- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m : n$

  (i) internally, is given by $\frac{n\vec{a} + m\vec{b}}{m+n}$.

  (ii) externally, is given by $\frac{m\vec{b} - n\vec{a}}{m-n}$.

- The scalar product of two given vectors $\vec{a}$ and $\vec{b}$ having angle θ between them is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta .$$

  Also, when $\vec{a} \cdot \vec{b}$ is given, the angle 'θ' between the vectors $\vec{a}$ and $\vec{b}$ may be determined by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

- If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then their cross product is given as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$$

  where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Such that $\vec{a}, \vec{b}, \hat{n}$ form right handed system of coordinate axes.

- If we have two vectors $\vec{a}$ and $\vec{b}$, given in component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ any scalar, then $\vec{a}+\vec{b} = (a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$;

$$\lambda\vec{a} = (\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}\ ;$$

$$\vec{a}.\vec{b} = a_1b_1+a_2b_2+a_3b_3\ ;$$

and $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$.

Historical Note

The word *vector* has been derived from a Latin word *vectus*, which means "to carry". The germinal ideas of modern vector theory date from around 1800 when Caspar Wessel (1745-1818) and Jean Robert Argand (1768-1822) described that how a complex number $a + ib$ could be given a geometric interpretation with the help of a directed line segment in a coordinate plane. William Rowen Hamilton (1805-1865) an Irish mathematician was the first to use the term vector for a directed line segment in his book *Lectures on Quaternions* (1853). Hamilton's method of quaternions (an ordered set of four real numbers given as: $a+b\hat{i}+c\hat{j}+d\hat{k}$, $\hat{i}$, $\hat{j}$, $\hat{k}$ following certain algebraic rules) was a solution to the problem of multiplying vectors in three dimensional space. Though, we must mention here that in practice, the idea of vector concept and their addition was known much earlier ever since the time of Aristotle (384-322 B.C.), a Greek philosopher, and pupil of Plato (427-348 B.C.). That time it was supposed to be known that the combined action of two or more forces could be seen by adding them according to parallelogram law. The correct law for the composition of forces, that forces add vectorially, had been discovered in the case of perpendicular forces by Stevin-Simon (1548-1620). In 1586 A.D., he analysed the principle of geometric addition of forces in his treatise *DeBeghinselen der Weeghconst* ("Principles of the Art of Weighing"), which caused a major breakthrough in the development of mechanics. But it took another 200 years for the general concept of vectors to form.

In the 1880, Josaih Willard Gibbs (1839-1903), an American physicist and mathematician, and Oliver Heaviside (1850-1925), an English engineer, created what we now know as *vector analysis*, essentially by separating the real (*scalar*) part of quaternion from its imaginary (*vector*) part. In 1881 and 1884, Gibbs printed a treatise entitled *Element of Vector Analysis*. This book gave a systematic and concise account of vectors. However, much of the credit for demonstrating the applications of vectors is due to the D. Heaviside and P.G. Tait (1831-1901) who contributed significantly to this subject.

ANSWERS

EXERCISE 7.1

1. $-\frac{1}{2}\cos 2x$
2. $\frac{1}{3}\sin 3x$
3. $\frac{1}{2}e^{2x}$
4. $\frac{1}{3a}(ax+b)^3$
5. $-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}$
6. $\frac{4}{3}e^{3x}+x+C$
7. $\frac{x^3}{3}-x+C$
8. $\frac{ax^3}{3}+\frac{bx^2}{2}+cx+C$
9. $\frac{2}{3}x^3+e^x+C$
10. $\frac{x^2}{2}+\log|x|-2x+C$
11. $\frac{x^2}{2}+5x+\frac{4}{x}+C$
12. $\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$
13. $\frac{x^3}{3}+x+C$
14. $\frac{2}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}+C$
15. $\frac{6}{7}x^{\frac{7}{2}}+\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+C$
16. $x^2-3\sin x+e^x+C$
17. $\frac{2}{3}x^3+3\cos x+\frac{10}{3}x^{\frac{3}{2}}+C$
18. $\tan x + \sec x + C$
19. $\tan x - x + C$
20. $2\tan x - 3\sec x + C$
21. C
22. A

EXERCISE 7.2

1. $\log(1+x^2)+C$
2. $\frac{1}{3}(\log|x|)^3+C$
3. $\log|1+\log x|+C$
4. $\cos(\cos x)+C$
5. $-\frac{1}{4a}\cos 2(ax+b)+C$
6. $\frac{2}{3a}(ax+b)^{\frac{3}{2}}+C$
7. $\frac{2}{5}(x+2)^{\frac{5}{2}}-\frac{4}{3}(x+2)^{\frac{3}{2}}+C$

8. $\frac{1}{6}(1+2x^2)^{\frac{3}{2}}+C$ **9.** $\frac{4}{3}(x^2+x+1)^{\frac{3}{2}}+C$ **10.** $2\log\left|\sqrt{x}-1\right|+C$

11. $\frac{2}{3}\sqrt{x+4}(x-8)+C$

12. $\frac{1}{7}(x^3-1)^{\frac{7}{3}}+\frac{1}{4}(x^3-1)^{\frac{4}{3}}+C$ **13.** $-\frac{1}{18(2+3x^3)^2}+C$

14. $\frac{(\log x)^{1-m}}{1-m}+C$ **15.** $-\frac{1}{8}\log|9-4x^2|$ **16.** $\frac{1}{2}e^{2x+3}+C$

17. $-\frac{1}{2e^{x^2}}+C$ **18.** $e^{\tan^{-1}x}+C$ **19.** $\log(e^x+e^{-x})+C$

20. $\frac{1}{2}\log(e^{2x}+e^{-2x})+C$ **21.** $\frac{1}{2}\tan(2x-3)-x+C$

22. $-\frac{1}{4}\tan(7-4x)+C$ **23.** $\frac{1}{2}(\sin^{-1}x)^2+C$

24. $\frac{1}{2}\log\left|2\sin x+3\cos x\right|+C$ **25.** $\frac{1}{(1-\tan x)}+C$

26. $2\sin\sqrt{x}+C$ **27.** $\frac{1}{3}(\sin 2x)^{\frac{3}{2}}+C$ **28.** $2\sqrt{1+\sin x}+C$

29. $\frac{1}{2}(\log\sin x)^2+C$ **30.** $-\log(1+\cos x)$ **31.** $\frac{1}{1+\cos x}+C$

32. $\frac{x}{2}-\frac{1}{2}\log\left|\cos x+\sin x\right|+C$ **33.** $\frac{x}{2}-\frac{1}{2}\log\left|\cos x-\sin x\right|+C$

34. $2\sqrt{\tan x}+C$ **35.** $\frac{1}{3}(1+\log x)^3+C$ **36.** $\frac{1}{3}(x+\log x)^3+C$

37. $-\frac{1}{4}\cos(\tan^{-1}x^4)+C$ **38.** D

39. B

EXERCISE 7.3

1. $\frac{x}{2}-\frac{1}{8}\sin(4x+10)+C$

2. $-\frac{1}{14}\cos 7x+\frac{1}{2}\cos x+C$

3. $\frac{1}{4}\left[\frac{1}{12}\sin 12x+x+\frac{1}{8}\sin 8x+\frac{1}{4}\sin 4x\right]+C$

4. $-\frac{1}{2}\cos(2x+1)+\frac{1}{6}\cos^3(2x+1)+C$

5. $\frac{1}{6}\cos^6 x-\frac{1}{4}\cos^4 x+C$

6. $\frac{1}{4}\left[\frac{1}{6}\cos 6x-\frac{1}{4}\cos 4x-\frac{1}{2}\cos 2x\right]+C$

7. $\frac{1}{2}\left[\frac{1}{4}\sin 4x-\frac{1}{12}\sin 12x\right]+C$

8. $2\tan\frac{x}{2}-x+C$

9. $x-\tan\frac{x}{2}+C$

10. $\frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$

11. $\frac{3x}{8}+\frac{1}{8}\sin 4x+\frac{1}{64}\sin 8x+C$

12. $x - \sin x + C$

13. $2(\sin x + x\cos\alpha) + C$

14. $-\frac{1}{\cos x+\sin x}+C$

15. $\frac{1}{6}\sec^3 2x-\frac{1}{2}\sec 2x+C$

16. $\frac{1}{3}\tan^3 x-\tan x+x+C$

17. $\sec x - \operatorname{cosec} x + C$

18. $\tan x + C$

19. $\log|\tan x|+\frac{1}{2}\tan^2 x+C$

20. $\log|\cos x+\sin x|+C$

21. $\frac{\pi x}{2}-\frac{x^2}{2}+C$

22. $\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|+C$

23. A

24. B

EXERCISE 7.4

1. $\tan^{-1}x^3+C$

2. $\frac{1}{2}\log\left|2x+\sqrt{1+4x^2}\right|+C$

3. $\log\left|\dfrac{1}{2-x+\sqrt{x^2-4x+5}}\right|+C$

4. $\dfrac{1}{5}\sin^{-1}\dfrac{5x}{3}+C$

5. $\dfrac{3}{2\sqrt{2}}\tan^{-1}\sqrt{2}\,x^2+C$

6. $\dfrac{1}{6}\log\left|\dfrac{1+x^3}{1-x^3}\right|+C$

7. $\sqrt{x^2-1}-\log\left|x+\sqrt{x^2-1}\right|+C$

8. $\dfrac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right|+C$

9. $\log\left|\tan x+\sqrt{\tan^2 x+4}\right|+C$

10. $\log\left|x+1+\sqrt{x^2+2x+2}\right|+C$

11. $\dfrac{1}{6}\tan^{-1}\left(\dfrac{3x+1}{2}\right)+C$

12. $\sin^{-1}\left(\dfrac{x+3}{4}\right)+C$

13. $\log\left|x-\dfrac{3}{2}+\sqrt{x^2-3x+2}\right|+C$

14. $\sin^{-1}\left(\dfrac{2x-3}{\sqrt{41}}\right)+C$

15. $\log\left|x-\dfrac{a+b}{2}+\sqrt{(x-a)(x-b)}\right|+C$

16. $2\sqrt{2x^2+x-3}+C$

17. $\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+C$

18. $\dfrac{5}{6}\log\left|3x^2+2x+1\right|-\dfrac{11}{3\sqrt{2}}\tan^{-1}\left(\dfrac{3x+1}{\sqrt{2}}\right)+C$

19. $6\sqrt{x^2-9x+20}+34\log\left|x-\dfrac{9}{2}+\sqrt{x^2-9x+20}\right|+C$

20. $-\sqrt{4x-x^2}+4\sin^{-1}\left(\dfrac{x-2}{2}\right)+C$

21. $\sqrt{x^2+2x+3}+\log\left|x+1+\sqrt{x^2+2x+3}\right|+C$

22. $\dfrac{1}{2}\log\left|x^2-2x-5\right|+\dfrac{2}{\sqrt{6}}\log\left|\dfrac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$

23. $5\sqrt{x^2+4x+10}-7\log\left|x+2+\sqrt{x^2+4x+10}\right|+C$

24. B

25. B

EXERCISE 7.5

1. $\log\frac{(x+2)^2}{|x+1|}+C$

2. $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right|+C$

3. $\log|x-1|-5\log|x-2|+4\log|x-3|+C$

4. $\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+C$

5. $4\log|x+2|-2\log|x+1|+C$

6. $\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$

7. $\frac{1}{2}\log|x-1|-\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+C$

8. $\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+C$

9. $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+C$

10. $\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$

11. $\frac{5}{3}\log|x+1|-\frac{5}{2}\log|x+2|+\frac{5}{6}\log|x-2|+C$

12. $\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+C$

13. $-\log|x-1|+\frac{1}{2}\log(1+x^2)+\tan^{-1}x+C$

14. $3\log|x-2|-\frac{5}{x-2}+C$

15. $\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$

16. $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$

17. $\log\left|\frac{2-\sin x}{1-\sin x}\right|+C$

18. $x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$

19. $\frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right)+C$

20. $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$

21. $\log\left(\frac{e^x-1}{e^x}\right)+C$

22. B

23. A

EXERCISE 7.6

1. $-x\cos x+\sin x+C$

2. $-\frac{x}{3}\cos 3x+\frac{1}{9}\sin 3x+C$

3. $e^x(x^2-2x+2)+C$

4. $\frac{x^2}{2}\log x-\frac{x^2}{4}+C$

5. $\frac{x^2}{2}\log 2x-\frac{x^2}{4}+C$

6. $\frac{x^3}{3}\log x-\frac{x^3}{9}+C$

7. $\frac{1}{4}(2x^2-1)\sin^{-1}x+\frac{x\sqrt{1-x^2}}{4}+C$

8. $\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C$

9. $(2x^2-1)\frac{\cos^{-1}x}{4}-\frac{x}{4}\sqrt{1-x^2}+C$

10. $\left(\sin^{-1}x\right)^2 x+2\sqrt{1-x^2}\sin^{-1}x-2x+C$

11. $-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$

12. $x\tan x+\log|\cos x|+C$

13. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)+C$

14. $\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{x^2}{4}+C$

15. $\left(\frac{x^3}{3}+x\right)\log x-\frac{x^3}{9}-x+C$

16. $e^x\sin x+C$

17. $\frac{e^x}{1+x}+C$

18. $e^x\tan\frac{x}{2}+C$

19. $\frac{e^x}{x}+C$

20. $\frac{e^x}{(x-1)^2}+C$

21. $\frac{e^{2x}}{5}(2\sin x-\cos x)+C$

22. $2x\tan^{-1}x-\log(1+x^2)+C$

23. A

24. B

EXERCISE 7.7

1. $\frac{1}{2}x\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

2. $\frac{1}{4}\sin^{-1}2x+\frac{1}{2}x\sqrt{1-4x^2}+C$

3. $\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|x+2+\sqrt{x^2+4x+6}\right|+C$

4. $\frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|x+2+\sqrt{x^2+4x+1}\right|+C$

5. $\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+\frac{x+2}{2}\sqrt{1-4x-x^2}+C$

6. $\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|x+2+\sqrt{x^2+4x-5}\right|+C$

7. $\frac{(2x-3)}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$

8. $\frac{2x+3}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right|+C$

9. $\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$

10. A

11. D

EXERCISE 7.8

1. $\frac{1}{2}(b^2-a^2)$

2. $\frac{35}{2}$

3. $\frac{19}{3}$

4. $\frac{27}{2}$

5. $e-\frac{1}{e}$

6. $\frac{15+e^8}{2}$

EXERCISE 7.9

1. 2

2. $\log\frac{3}{2}$

3. $\frac{64}{3}$

4. $\frac{1}{2}$

5. 0

6. $e^4(e-1)$

7. $\frac{1}{2}\log 2$ **8.** $\log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$ **9.** $\frac{\pi}{2}$

10. $\frac{\pi}{4}$ **11.** $\frac{1}{2}\log\frac{3}{2}$ **12.** $\frac{\pi}{4}$

13. $\frac{1}{2}\log 2$ **14.** $\frac{1}{5}\log 6+\frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

15. $\frac{1}{2}(e-1)$ **16.** $5-\frac{5}{2}\left(9\log\frac{5}{4}-\log\frac{3}{2}\right)$

17. $\frac{\pi^4}{1024}+\frac{\pi}{2}+2$ **18.** 0 **19.** $3\log 2+\frac{3\pi}{8}$

20. $1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$ **21.** D **22.** C

EXERCISE 7.10

1. $\frac{1}{2}\log 2$ **2.** $\frac{64}{231}$ **3.** $\frac{\pi}{2}-\log 2$

4. $\frac{16\sqrt{2}}{15}(\sqrt{2}+1)$ **5.** $\frac{\pi}{4}$ **6.** $\frac{1}{\sqrt{17}}\log\frac{21+5\sqrt{17}}{4}$

7. $\frac{\pi}{8}$ **8.** $\frac{e^2(e^2-2)}{4}$ **9.** D

10. B

EXERCISE 7.11

1. $\frac{\pi}{4}$ **2.** $\frac{\pi}{4}$ **3.** $\frac{\pi}{4}$ **4.** $\frac{\pi}{4}$

5. 29 **6.** 9 **7.** $\frac{1}{(n+1)(n+2)}$

8. $\frac{\pi}{8}\log 2$ **9.** $\frac{16\sqrt{2}}{15}$ **10.** $\frac{\pi}{2}\log\frac{1}{2}$ **11.** $\frac{\pi}{2}$

12. π	**13.** 0	**14.** 0	**15.** 0
16. $-\pi \log 2$	**17.** $\frac{a}{2}$	**18.** 5	**20.** C
21. C			

Miscellaneous Exercise on Chapter 7

1. $\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$ **2.** $\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$

3. $-\frac{2}{a}\sqrt{\frac{(a-x)}{x}}+C$ **4.** $-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$

5. $2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log(1+x^{\frac{1}{6}})+C$

6. $-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+9)+\frac{3}{2}\tan^{-1}\frac{x}{3}+C$

7. $\sin a\log|\sin(x-a)|+x\cos a+C$ **8.** $\frac{x^3}{3}+C$

9. $\sin^{-1}\left(\frac{\sin x}{2}\right)+C$ **10.** $-\frac{1}{2}\sin 2x+C$

11. $\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$ **12.** $\frac{1}{4}\sin^{-1}(x^4)+C$

13. $\log\left(\frac{1+e^x}{2+e^x}\right)+C$ **14.** $\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$

15. $-\frac{1}{4}\cos^4 x+C$ **16.** $\frac{1}{4}\log(x^4+1)+C$

17. $\frac{[f(ax+b)]^{n+1}}{a(n+1)}+C$ **18.** $\frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}}+C$

19. $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2\sqrt{x-x^2}}{\pi}-x+C$

20. $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$

21. $e^x \tan x + C$

22. $-2\log|x+1|-\frac{1}{x+1}+3\log|x+2|+C$

23. $\frac{1}{2}\left[x\cos^{-1}x-\sqrt{1-x^2}\right]+C$

24. $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$

25. $e^{\frac{\pi}{2}}$

26. $\frac{\pi}{8}$

27. $\frac{\pi}{6}$

28. $2\sin^{-1}\frac{(\sqrt{3}-1)}{2}$

29. $\frac{4\sqrt{2}}{3}$

30. $\frac{1}{40}\log 9$

31. $\frac{\pi}{2}-1$

32. $\frac{\pi}{2}(\pi-2)$

33. $\frac{19}{2}$

40. $\frac{1}{3}\left(e^2-\frac{1}{e}\right)$

41. A

42. B

43. D

44. B

EXERCISE 8.1

1. $\frac{14}{3}$

2. $16-4\sqrt{2}$

3. $\frac{32-8\sqrt{2}}{3}$

4. 12π

5. 6π

6. $\frac{\pi}{3}$

7. $\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$

8. $(4)^{\frac{2}{3}}$

9. $\frac{1}{3}$

10. $\frac{9}{8}$

11. $8\sqrt{3}$

12. A

13. B

EXERCISE 8.2

1. $\dfrac{\sqrt{2}}{6}+\dfrac{9}{4}\sin^{-1}\dfrac{2\sqrt{2}}{3}$ 2. $\left(\dfrac{2\pi}{3}-\dfrac{\sqrt{3}}{2}\right)$

3. $\dfrac{21}{2}$ 4. 4 5. 8

6. B 7. B

Miscellaneous Exercise on Chapter 8

1. (i) $\dfrac{7}{3}$ (ii) 624.8

2. $\dfrac{1}{6}$ 3. $\dfrac{7}{3}$ 4. 9 5. 4

6. $\dfrac{8}{3}\dfrac{a^2}{m^3}$ 7. 27 8. $\dfrac{3}{2}(\pi-2)$

9. $\dfrac{ab}{4}(\pi-2)$ 10. $\dfrac{9}{2}$ 11. 2 12. $\dfrac{1}{3}$

13. 7 14. $\dfrac{7}{2}$ 15. $\dfrac{9\pi}{8}-\dfrac{9}{4}\sin^{-1}\left(\dfrac{1}{3}\right)+\dfrac{1}{3\sqrt{2}}$

16. D 17. C 18. C 19. B

EXERCISE 9.1

1. Order 4; Degree not defined 2. Order 1; Degree 1
3. Order 2; Degree 1 4. Order 2; Degree not defined
5. Order 2; Degree 1 6. Order 3; Degree 2
7. Order 3; Degree 1 8. Order 1; Degree 1
9. Order 2; Degree 1 10. Order 2; Degree 1
11. D 12. A

EXERCISE 9.2

11. D 12. D

EXERCISE 9.3

1. $y'' = 0$
2. $xy\, y'' + x\,(y')^2 - y\, y' = 0$
3. $y'' - y' - 6y = 0$
4. $y'' - 4y' + 4y = 0$
5. $y'' - 2y' + 2y = 0$
6. $2xyy' + x^2 = y^2$
7. $xy' - 2y = 0$
8. $xyy'' + x(y')^2 - yy' = 0$
9. $xyy'' + x(y')^2 - yy' = 0$
10. $(x^2 - 9)\,(y')^2 + x^2 = 0$
11. B
12. C

EXERCISE 9.4

1. $y = 2\tan\dfrac{x}{2} - x + C$
2. $y = 2\sin(x + C)$
3. $y = 1 + Ae^{-x}$
4. $\tan x \tan y = C$
5. $y = \log(e^x + e^{-x}) + C$
6. $\tan^{-1} y = x + \dfrac{x^3}{3} + C$
7. $y = e^{cx}$
8. $x^{-4} + y^{-4} = C$
9. $y = x\sin^{-1}x + \sqrt{1-x^2} + C$
10. $\tan y = C\,(1 - e^x)$
11. $y = \dfrac{1}{4}\log\left[(x+1)^2\,(x^2+1)^3\right] - \dfrac{1}{2}\tan^{-1}x + 1$
12. $y = \dfrac{1}{2}\log\left(\dfrac{x^2-1}{x^2}\right)$
13. $\cos\left(\dfrac{y-2}{x}\right) = a$
14. $y = \sec x$
15. $2y - 1 = e^x(\sin x - \cos x)$
16. $y - x + 2 = \log(x^2\,(y+2)^2)$
17. $y^2 - x^2 = 4$
18. $(x + 4)^2 = y + 3$
19. $(63t + 27)^{\frac{1}{3}}$
20. 6.93%
21. Rs 1648
22. $\dfrac{2\log 2}{\log\left(\dfrac{11}{10}\right)}$
23. A

EXERCISE 9.5

1. $(x-y)^2 = Cx\, e^{\frac{-y}{x}}$
2. $y = x\log|x| + Cx$

3. $\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log(x^2+y^2)+C$

4. $x^2 + y^2 = Cx$

5. $\frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right|=\log|x|+C$

6. $y+\sqrt{x^2+y^2}=Cx^2$

7. $xy\cos\left(\frac{y}{x}\right)=C$

8. $x\left[1-\cos\left(\frac{y}{x}\right)\right]=C\sin\left(\frac{y}{x}\right)$

9. $cy=\log\frac{y}{x}-1$

10. $ye^{\frac{x}{y}}+x=C$

11. $\log(x^2+y^2)+2\tan^{-1}\frac{y}{x}=\frac{\pi}{2}+\log 2$

12. $y + 2x = 3x^2 y$

13. $\cot\left(\frac{y}{x}\right)=\log|ex|$

14. $\cos\left(\frac{y}{x}\right)=\log|ex|$

15. $y=\frac{2x}{1-\log|x|}(x\neq 0,\ x\neq e)$

16. C

17. D

EXERCISE 9.6

1. $y=\frac{1}{5}(2\sin x-\cos x)+C\,e^{-2x}$

2. $y = e^{-2x} + Ce^{-3x}$

3. $xy=\frac{x^4}{4}+C$

4. $y(\sec x + \tan x) = \sec x + \tan x - x + C$

5. $y = (\tan x - 1) + C e^{-\tan x}$

6. $y=\frac{x^2}{16}(4\log x-1)+Cx^{-2}$

7. $y\log x=\frac{-2}{x}(1+\log x)+C$

8. $y=(1+x)^{-2}\ \log|\sin x|+C(1+x^2)^{-1}$

9. $y=\frac{1}{x}-\cot x+\frac{C}{x\sin x}$

10. $(x + y + 1) = C\,e^{y}$

11. $x=\frac{y^2}{3}+\frac{C}{y}$

12. $x = 3y^2 + Cy$

13. $y = \cos x - 2\cos^2 x$ **14.** $y(1 + x^2) = \tan^{-1} x - \dfrac{\pi}{4}$

15. $y = 4\sin^3 x - 2\sin^2 x$ **16.** $x + y + 1 = e^x$

17. $y = 4 - x - 2e^x$ **18.** C **19.** D

Miscellaneous Exercise on Chapter 9

1. (i) Order 2; Degree 1 (ii) Order 1; Degree 3
(iii) Order 4; Degree not defined

3. $y' = \dfrac{2y^2 - x^2}{4xy}$ **5.** $(x + yy')^2 = (x - y)^2 (1 + (y')^2)$

6. $\sin^{-1}y + \sin^{-1}x = C$ **8.** $\cos y = \dfrac{\sec x}{\sqrt{2}}$

9. $\tan^{-1} y + \tan^{-1}(e^x) = \dfrac{\pi}{2}$ **10.** $e^{\frac{x}{y}} = y + C$

11. $\log |x - y| = x + y + 1$ **12.** $y e^{2\sqrt{x}} = (2\sqrt{x} + C)$

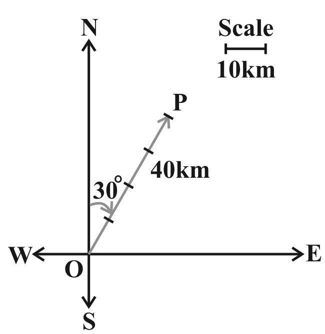
13. $y \sin x = 2x^2 - \dfrac{\pi^2}{2}$ $(\sin x \neq 0)$ **14.** $y = \log\left|\dfrac{2x+1}{x+1}\right|, x \neq -1$

15. 31250 **16.** C

17. C **18.** C

EXERCISE 10.1

1. In the adjoining figure, the vector $\overrightarrow{OP}$ represents the required displacement.



2. (i) scalar (ii) vector (iii) scalar (iv) scalar (v) scalar
(vi) vector
3. (i) scalar (ii) scalar (iii) vector (iv) vector (v) scalar
4. (i) Vectors $\vec{a}$ and $\vec{b}$ are coinitial
(ii) Vectors $\vec{b}$ and $\vec{d}$ are equal
(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal
5. (i) True (ii) False (iii) False (iv) False

EXERCISE 10.2

1. $|\vec{a}|=\sqrt{3}, |\vec{b}|=\sqrt{62}, |\vec{c}|=1$
2. An infinite number of possible answers.
3. An infinite number of possible answers.
4. $x = 2, y = 3$
5. -7 and 6; $-7\hat{i}$ and $6\hat{j}$
6. $-4\hat{j}-\hat{k}$
7. $\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$
8. $\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$
9. $\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$
10. $\frac{40}{\sqrt{30}}\hat{i}-\frac{8}{\sqrt{30}}\hat{j}+\frac{16}{\sqrt{30}}\hat{k}$
12. $\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}$
13. $-\frac{1}{3},-\frac{2}{3},\frac{2}{3}$
15. (i) $-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$ (ii) $-3\hat{i}+3\hat{k}$
16. $3\hat{i}+2\hat{j}+\hat{k}$
18. (C)
19. (D)

EXERCISE 10.3

1. $\frac{\pi}{4}$
2. $\cos^{-1}\left(\frac{5}{7}\right)$
3. 0
4. $\frac{60}{\sqrt{114}}$
6. $\frac{16\sqrt{2}}{3\sqrt{7}},\frac{2\sqrt{2}}{3\sqrt{7}}$
7. $6|\vec{a}|^2+11\vec{a}.\vec{b}-35|\vec{b}|^2$
8. $|\vec{a}|=1,|\vec{b}|=1$
9. $\sqrt{13}$
10. 8

12. Vector $\vec{b}$ can be any vector **13.** $\frac{-3}{2}$

14. Take any two non-zero perpendicular vectors $\vec{a}$ and $\vec{b}$

15. $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$ **18.** (D)

EXERCISE 10.4

1. $19\sqrt{2}$ **2.** $\pm\frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$ **3.** $\frac{\pi}{3}; \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

5. $3, \frac{27}{2}$ **6.** Either $|\vec{a}|=0$ or $|\vec{b}|=0$

8. No; take any two nonzero collinear vectors

9. $\frac{\sqrt{61}}{2}$ **10.** $15\sqrt{2}$ **11.** (B) **12.** (C)

Miscellaneous Exercise on Chapter 10

1. $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

2. $x_2 - x_1, y_2 - y_1, z_2 - z_1; \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

3. $\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

4. No; take $\vec{a}$, $\vec{b}$ and $\vec{c}$ to represent the sides of a triangle.

5. $\pm\frac{1}{\sqrt{3}}$ **6.** $\frac{3}{2}\sqrt{10}\,\hat{i} + \frac{\sqrt{10}}{2}\hat{j}$ **7.** $\frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}$

8. $2:3$ **9.** $3\vec{a} + 5\vec{b}$ **10.** $\frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k});\ 11\sqrt{5}$

12. $\frac{1}{3}(160\hat{i} - 5\hat{j} + 70\hat{k})$ **13.** $\lambda = 1$ **16.** (B)

17. (D) **18.** (C) **19.** (B)

EXERCISE 11.1

1. $0, \dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$ **2.** $\pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}$ **3.** $\dfrac{-9}{11}, \dfrac{6}{11}, \dfrac{-2}{11}$

5. $\dfrac{-2}{\sqrt{17}}, \dfrac{-2}{\sqrt{17}}, \dfrac{3}{17}; \dfrac{-2}{\sqrt{17}}, \dfrac{-3}{\sqrt{17}}, \dfrac{-2}{\sqrt{17}}; \dfrac{4}{\sqrt{42}}, \dfrac{5}{\sqrt{42}}, \dfrac{-1}{\sqrt{42}}$

EXERCISE 11.2

4. $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$, where λ is a real number

5. $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and cartesian form is

$\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z-4}{-1}$

6. $\dfrac{x+2}{3} = \dfrac{y-4}{5} = \dfrac{z+5}{6}$

7. $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

8. Vector equation of the line: $\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$;

Cartesian equation of the line: $\dfrac{x}{5} = \dfrac{y}{-2} = \dfrac{z}{3}$

9. Vector equation of the line: $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda(11\hat{k})$

Cartesian equation of the line: $\dfrac{x-3}{0} = \dfrac{y+2}{0} = \dfrac{z+5}{11}$

10. (i) $\theta = \cos^{-1}\left(\dfrac{19}{21}\right)$ (ii) $\theta = \cos^{-1}\left(\dfrac{8}{5\sqrt{3}}\right)$

11. (i) $\theta = \cos^{-1}\left(\dfrac{26}{9\sqrt{38}}\right)$ (ii) $\theta = \cos^{-1}\left(\dfrac{2}{3}\right)$

12. $p = \dfrac{70}{11}$ **14.** $\dfrac{3\sqrt{2}}{2}$ **15.** $2\sqrt{29}$

16. $\dfrac{3}{\sqrt{19}}$ **17.** $\dfrac{8}{\sqrt{29}}$

EXERCISE 11.3

1. (a) 0, 0, 1; 2 (b) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}$

(c) $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}; \frac{5}{\sqrt{14}}$ (d) 0, 1, 0; $\frac{8}{5}$

2. $\vec{r} \cdot \left(\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}} \right) = 7$

3. (a) $x + y - z = 2$ (b) $2x + 3y - 4z = 1$

(c) $(s - 2t)x + (3 - t)y + (2s + t)z = 15$

4. (a) $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29} \right)$ (b) $\left(0, \frac{18}{5}, \frac{24}{5} \right)$

(c) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$ (d) $\left(0, \frac{-8}{5}, 0 \right)$

5. (a) $[\vec{r} - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0$; $x + y - z = 3$

(b) $[\vec{r} - (\hat{i} + 4\hat{j} + 6\hat{k})] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$; $x - 2y + z + 1 = 0$

6. (a) The points are collinear. There will be infinite number of planes passing through the given points.

(b) $2x + 3y - 3z = 5$

7. $\frac{5}{2}$, 5, –5 **8.** $y = 3$ **9.** $7x - 5y + 4z - 8 = 0$

10. $\vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$ **11.** $x - z + 2 = 0$

12. $\cos^{-1}\left(\frac{15}{\sqrt{731}} \right)$

13. (a) $\cos^{-1}\left(\frac{2}{5} \right)$ (b) The planes are perpendicular

(c) The planes are parallel (d) The planes are parallel

(e) 45°

14. (a) $\frac{3}{13}$ (b) $\frac{13}{3}$

(c) 3 (d) 2

Miscellaneous Exercise on Chapter 11

3. 90° **4.** $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ **5.** $\cos^{-1}\left(\frac{5}{\sqrt{187}}\right)$

6. $k=\frac{-10}{7}$ **7.** $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}+2\hat{j}-5\hat{k})$

8. $x+y+z=a+b+c$ **9.** 9

10. $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$ **11.** $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$ **12.** $(1, -2, 7)$

13. $7x-8y+3z+25=0$ **14.** $p=1$ or $\frac{7}{3}$

15. $y-3z+6=0$ **16.** $x+2y-3z-14=0$

17. $33x+45y+50z-41=0$ **18.** 13

19. $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(-3\hat{i}+5\hat{j}+4\hat{k})$

20. $\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$ **22.** D

23. B

EXERCISE 12.1

1. Maximum Z = 16 at (0, 4)

2. Minimum Z = – 12 at (4, 0)

3. Maximum Z = $\frac{235}{19}$ at $\left(\frac{20}{19}, \frac{45}{19}\right)$

4. Minimum Z = 7 at $\left(\frac{3}{2}, \frac{1}{2}\right)$

5. Maximum Z = 18 at (4, 3)

6. Minimum Z = 6 at all the points on the line segment joining the points (6, 0) and (0, 3).

7. Minimum Z = 300 at (60, 0);

Maximum Z = 600 at all the points on the line segment joining the points (120, 0) and (60, 30).

8. Minimum Z = 100 at all the points on the line segment joining the points (0, 50) and (20, 40);

Maximum Z = 400 at (0, 200)

9. Z has no maximum value

10. No feasible region, hence no maximum value of Z.

EXERCISE 12.2

1. Minimum cost = Rs 160 at all points lying on segment joining $\left(\frac{8}{3},0\right)$ and $\left(2,\frac{1}{2}\right)$.

2. Maximum number of cakes = 30 of kind one and 10 cakes of another kind.

3. (i) 4 tennis rackets and 12 cricket bats

(ii) Maximum profit = Rs 200

4. 3 packages of nuts and 3 packages of bolts; Maximum profit = Rs 73.50.

5. 30 packages of screws A and 20 packages of screws B; Maximum profit = Rs 410

6. 4 Pedestal lamps and 4 wooden shades; Maximum profit = Rs 32

7. 8 Souvenir of types A and 20 of Souvenir of type B; Maximum profit = Rs 1600.

8. 200 units of desktop model and 50 units of portable model; Maximum profit = Rs 1150000.

9. Minimise Z = $4x + 6y$

subject to $3x + 6y \geq 80$, $4x + 3y \geq 100$, $x \geq 0$ and $y \geq 0$, where x and y denote the number of units of food F_1 and food F_2 respectively; Minimum cost = Rs 104

10. 100 kg of fertiliser F_1 and 80 kg of fertiliser F_2; Minimum cost = Rs 1000

11. (D)

Miscellaneous Exercise on Chapter 12

1. 40 packets of food P and 15 packets of food Q; Maximum amount of vitamin A = 285 units.

2. 3 bags of brand P and 6 bags of brand Q; Minimum cost of the mixture = Rs 1950

3. Least cost of the mixture is Rs 112 (2 kg of Food X and 4 kg of food Y).

5. 40 tickets of executive class and 160 tickets of economy class; Maximum profit = Rs 136000.
6. From A : 10,50, 40 units; From B: 50,0,0 units to D, E and F respectively and minimum cost = Rs 510
7. From A: 500, 3000 and 3500 litres; From B: 4000, 0, 0 litres to D, E and F respectively; Minimum cost = Rs 4400
8. 40 bags of brand P and 100 bags of brand Q; Minimum amount of nitrogen = 470 kg.
9. 140 bags of brand P and 50 bags of brand Q; Maximum amount of nitrogen = 595 kg.
10. 800 dolls of type A and 400 dolls of type B; Maximum profit = Rs 16000

EXERCISE 13.1

1. $P(E|F) = \frac{2}{3}, P(F|E) = \frac{1}{3}$ 2. $P(A|B) = \frac{16}{25}$

3. (i) 0.32 (ii) 0.64 (iii) 0.98

4. $\frac{11}{26}$

5. (i) $\frac{4}{11}$ (ii) $\frac{4}{5}$ (iii) $\frac{2}{3}$

6. (i) $\frac{1}{2}$ (ii) $\frac{3}{7}$ (iii) $\frac{6}{7}$

7. (i) 1 (ii) 0

8. $\frac{1}{6}$ 9. 1 10. (a) $\frac{1}{3}$, (b) $\frac{1}{9}$

11. (i) $\frac{1}{2}, \frac{1}{3}$ (ii) $\frac{1}{2}, \frac{2}{3}$ (iii) $\frac{3}{4}, \frac{1}{4}$

12. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ 13. $\frac{5}{9}$

14. $\frac{1}{15}$ 15. 0 16. C 17. D

EXERCISE 13.2

1. $\frac{3}{25}$ 2. $\frac{25}{102}$ 3. $\frac{44}{91}$
4. A and B are independent 5. A and B are not independent
6. E and F are not independent
7. (i) $p = \frac{1}{10}$ (ii) $p = \frac{1}{5}$
8. (i) 0.12 (ii) 0.58 (iii) 0.3 (iv) 0.4
9. $\frac{3}{8}$ 10. A and B are not independent
11. (i) 0.18 (ii) 0.12 (iii) 0.72 (iv) 0.28
12. $\frac{7}{8}$ 13. (i) $\frac{16}{81}$, (ii) $\frac{20}{81}$, (iii) $\frac{40}{81}$
14. (i) $\frac{2}{3}$, (ii) $\frac{1}{2}$ 15. (i), (ii) 16. (a) $\frac{1}{5}$, (b) $\frac{1}{3}$, (c) $\frac{1}{2}$
17. D 18. B

EXERCISE 13.3

1. $\frac{1}{2}$ 2. $\frac{2}{3}$ 3. $\frac{9}{13}$ 4. $\frac{12}{13}$
5. $\frac{198}{1197}$ 6. $\frac{4}{9}$ 7. $\frac{1}{52}$ 8. $\frac{1}{4}$
9. $\frac{2}{9}$ 10. $\frac{8}{11}$ 11. $\frac{5}{34}$ 12. $\frac{11}{50}$
13. A 14. C

EXERCISE 13.4

1. (ii), (iii) and (iv) 2. X = 0, 1, 2; yes 3. X = 6, 4, 2, 0

4. (i)

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(ii)

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(iii)

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

5. (i)

X	0	1	2
P(X)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

(ii)

X	0	1
P(X)	$\frac{25}{36}$	$\frac{11}{36}$

6.

X	0	1	2	3	4
P(X)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

7.

X	0	1	2
P(X)	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

8. (i) $k = \frac{1}{10}$ (ii) $P(X < 3) = \frac{3}{10}$ (iii) $P(X > 6) = \frac{17}{100}$

(iv) $P(0 < X < 3) = \frac{3}{10}$

9. (a) $k = \frac{1}{6}$ (b) $P(X < 2) = \frac{1}{2}, P(X \le 2) = 1, P(X \ge 2) = \frac{1}{2}$

10. 1.5 **11.** $\frac{1}{3}$ **12.** $\frac{14}{3}$

13. Var(X) = 5.833, S.D = 2.415

14.

X	14	15	16	17	18	19	20	21
P(X)	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

Mean = 17.53, Var(X) = 4.78 and S.D(X) = 2.19

15. E(X) = 0.7 and Var (X) = 0.21 **16.** B **17.** D

EXERCISE 13.5

1. (i) $\frac{3}{32}$ (ii) $\frac{7}{64}$ (iii) $\frac{63}{64}$

2. $\frac{25}{216}$ **3.** $\left(\frac{29}{20}\right)\left(\frac{19}{20}\right)^9$

4. (i) $\frac{1}{1024}$ (ii) $\frac{45}{512}$ (iii) $\frac{243}{1024}$

5. (i) $(0.95)^5$ (ii) $(0.95)^4 \times 1.2$ (iii) $1 - (0.95)^4 \times 1.2$
(iv) $1 - (0.95)^5$

6. $\left(\frac{9}{10}\right)^4$ **7.** $\left(\frac{1}{2}\right)^{20}\left[20C_{12} + {}^{20}C_{13} + ... + {}^{20}C_{20}\right]$

9. $\frac{11}{243}$

10. (a) $1-\left(\frac{99}{100}\right)^{50}$ (b) $\frac{1}{2}\left(\frac{99}{100}\right)^{49}$ (c) $1-\frac{149}{100}\left(\frac{99}{100}\right)^{49}$

11. $\frac{7}{12}\left(\frac{5}{6}\right)^5$ **12.** $\frac{35}{18}\left(\frac{5}{6}\right)^4$ **13.** $\frac{22\times 9^3}{10^{11}}$

14. C **15.** A

Miscellaneous Exercise on Chapter 13

1. (i) 1 (ii) 0

2. (i) $\frac{1}{3}$ (ii) $\frac{1}{2}$

3. $\frac{20}{21}$

4. $1-\sum_{r=7}^{10} {}^{10}C_r (0.9)^r (0.1)^{10-r}$

5. (i) $\left(\frac{2}{5}\right)^6$ (ii) $7\left(\frac{2}{5}\right)^4$ (iii) $1-\left(\frac{2}{5}\right)^6$ (iv) $\frac{864}{3125}$

6. $\dfrac{5^{10}}{2\times 6^9}$	**7.** $\dfrac{625}{23328}$	**8.** $\dfrac{2}{7}$
9. $\dfrac{31}{9}\left(\dfrac{2}{3}\right)^4$	**10.** $n \geq 4$	**11.** $\dfrac{11}{216}$
12. $\dfrac{1}{15}, \dfrac{2}{5}, \dfrac{8}{15}$	**13.** $\dfrac{14}{29}$	**14.** $\dfrac{3}{16}$
15. (i) 0.5 (ii) 0.05		**16.** $\dfrac{16}{31}$
17. A	**18.** C	**19.** B