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Unit 1 :- Matrices

Matrices :- A set of 'mn' no. (real or complex) arrenge in a rectangular form of 'm' rows and 'n' coloum is called Matrices of order and 'mxn' and read as 'm by n'

ex:-
$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

Types of Matrices :-

1] **Null or Zero Matrice :-** A Matrice having all element zero is called Null or Zero Matrix

ex:-
$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2] **Row Matrix :-** A Matrix having only one row is called row Matrix.

ex -
$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$$

3] **Column Matrix :-** A Matrix having only one column is called column Matrix.

ex -
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$$

4] **Square Matrix :-** A Matrix having same no. of rows and column is called Square Matrix

ex.- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$

5] **Singular and Non-Singular Matrix :-** The determinant of Matrix A is denoted by and if determinant of A is 0 then it is Singular Matrix. and if determinant of A is not zero then it is Non-Singular Matrix.

ex.- $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \; |A| = 6-4 = 2 \neq 0$

6] **Diagonal Matrix :-** A Square Matrix having all the non-diagonal element are zero is called Diagonal Matrix.

ex.-
$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

where a_{11}, a_{22}, a_{33} are called diagonal element. and the line on which they lie is called Principle diagonal line.

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7] **Scalar Matrix :-** A diagonal Matrix having all the diagonal element are same is called as Scalar Matrix.

ex-

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

8] **Unit or Identity Matrix -** A diagonal Matrix having all the diagonal element is called Identity Matrix.

ex-

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9] **Transpose Matrix -** Can be obtained by interchanging row and column and it is denoted by "A'" or "A^T"

ex-

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 3 \end{bmatrix}_{2\times3} \quad A' = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}_{3\times2}$$

10] **Symmetric Matrix -** A Square Matrix having the same corresponding element above and below the principle diagonal i.e. $a_{ij} = a_{ji}$ is called as Symmetric Matrix.

ex

$$A = \begin{bmatrix} 2 & 3 & -6 \\ 3 & 4 & 7 \\ -6 & 7 & 6 \end{bmatrix}$$

Transpose of Symmetric Matrix is same as

11] Skew - Symmetric Matrix:- A square matrix having the same corresponding element above and below the principle diagonal but with opposite sign i.e. $a_{ij} = -a_{ji}$ is called skew-symmetric matrix.

In skew - symmetric matrix all the element are zero.

ex -
$$\begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$$

★ 12] Upper triangular Matrix - A square matrix having all the element below the principle diagonal as zero is called upper triangular matrix

ex -
$$A = \begin{bmatrix} 1 & 3 & -6 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

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13] lower triangular Matrix - A Square Matrix having all the element above the principle diagonal as Zero is Called lower triangular Matrix.

ex - $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ -6 & 7 & 6 \end{bmatrix}$

* Addition or Substraction of Matrix :-

For Addition or Substraction of two Matrices Matrices should be of Same order then we add or Substract Corresponding element.

* Multiplication of two Matrices :-

Multiplication of two Matrices AB is possible if and only if Column of A is Same as row of B

ex - $(A)_{m \times n} \times (B)_{n \times p} = (AB)_{m \times p}$

* Matrix of Co-factor :- Co-factor element a_{ij} of Matrix is determinant by Surpressing row and column to which the element belong is independent of the sign of the element the arithemic sign of element is position sign of element given by $(-1)^{i+j}$

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Matrix of Co-factor can be obtain by arranging all the Co-factor in a rectangular form.

* Adjoint of Matrix :- Adjoint of Matrix A is given by Matrix of Co-factor of A

$$\text{adj } A = [\text{Matrix of Co-factor of } A]^T$$

* Inverse of Matrix :- let Matrix A and B are two Matrices of same order such that AB or BA equals to I

$$AB = BA = I$$

A is called Inverse of B
B is called Inverse of A

* Condition for Inverse :-
Inverse of Matrix A exist if and only if A is a non-singular Matrix.

* Inverse by Adjoint Method :-
Let A is Non-Singular Matrix then A inverse is given by adjoint of A determinant.

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

Q find inverse of Matrix A by adjoint Method.

if $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

$\rightarrow |A| = 1(9-16) - 3(3-4) + 3(4-3)$

$= -7 - 3(-1) + 3(1)$

$= -7 + 3 + 3$

$= -1 \neq 0$

A is non-Singular Matrix.

$a_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$ $\quad a_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$ $\quad a_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

$a_{21} \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 3$ $\quad a_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$ $\quad a_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$

$a_{31} = \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = 3$ $\quad a_{32} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$ $\quad a_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

Matrix of Co-factor of $= \begin{bmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}$

adj. A = [Matrix of Co-factor of A]^T

$\text{adj } A = \begin{bmatrix} -7 & 3 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

$A^{-1} = \dfrac{\text{adj } A}{|A|} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

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* Solution of Simultaneous equation by Gauss elimination Method:-

Consider a Simultaneous equation

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

the Matrix form of above system is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e. $AX = B$ —— ①

Where 'A' is Coefficient Matrix, 'X' is column Variable Matrix, 'B' is Column Constant Matrix

Here Augmented Matrix 'C'

$C = [A \vdots B]$

$$C = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

We Convert this Matrix in a upper triangular Matrix by using elementry row transformation

Then by using back substitution Method - we Can find the Solution.

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Q Solve by using gauss elimination Method.

① $2x + 3y + z = 13$, $x - y - 2z = -1$, $3x + y + 4z = 15$

$\Longrightarrow$ Matrix form of above system of equation is

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & -2 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \\ 15 \end{bmatrix}$$

i.e $AX = B$

augmented Matrix 'C' is

$C = [A \,\vdots\, B]$

$$\begin{bmatrix} 2 & 3 & 1 & \vdots & 13 \\ 1 & -1 & -2 & \vdots & -1 \\ 3 & 1 & 4 & \vdots & 15 \end{bmatrix}$$

Interchang R_1 with R_2 $\boxed{R_1 \longleftrightarrow R_2}$

$$\sim \begin{bmatrix} 1 & -1 & -2 & \vdots & -1 \\ 2 & 3 & 1 & \vdots & 13 \\ 3 & 1 & 4 & \vdots & 15 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & -2 & \vdots & -1 \\ 0 & 5 & 5 & \vdots & 15 \\ 0 & 4 & 10 & \vdots & 18 \end{bmatrix}$$

R_2 changes $R_2/5$

$$\sim \begin{bmatrix} 1 & -1 & -2 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & 3 \\ 0 & 4 & 10 & \vdots & 18 \end{bmatrix}$$

R_3 changes $R_3 - 4R_2$

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$$C = \left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 6 & 6 \end{array}\right]$$

$$\underset{3\times3}{\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}}$$

$x - y - 2z = -1$ — ①

$0 + y + z = 3$ — ②

$z = 1$

Substitute $z = 1$ in ②

$y + 1 = 3$

$y = 2$

Substitute $y = 2$ in ①.

$x - 2 - 2(1) = -1$

$x = -1 + 4 = 3$

Q ② $2x + 4y - 2z = 14$, $x + 3y - 4z = 16$, $-x + 2y + 3z$-

→ Matrix form of above system of equation is

$$\Rightarrow \begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 14 \\ 1 \end{bmatrix}$$

Same Sign - "-"
Opposite Sign - "+"

augmented Matrix 'C' is

$C = [A \mid B]$

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 16 \\ 2 & 4 & -2 & 14 \\ -1 & 2 & 3 & 1 \end{array}\right]$$

R_2 changes $R_2 - 2R_1$, $R_3 \to R_3 + R_1$

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$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -4 & 16 \\ 0 & -2 & 6 & -18 \\ 0 & 5 & -1 & 37 \end{array}\right]$$

$R_3 \rightarrow 2R_3 + 5R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -4 & 16 \\ 0 & -2 & 6 & -18 \\ 0 & 0 & 28 & -56 \end{array}\right]$$

$17 \times 2 + (-18)5$
$34 - 90$
-56

$$\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & 0 & 28 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -18 \\ -56 \end{bmatrix}$$

$x + 3y - 4z = 16$ — ①

$0 - 2y + 6z = -18$ — ②

$0 + 0 + 28z = -56$

$z = -56/28$

$\boxed{z = -2}$

Substitute $z = -2$ in ②

$-2y + 6(-2) = -18$

$-2y = -18 + 12 = -6$

$\boxed{y = 3}$

Substitute $y = 3$ in ①

$x + 3(3) - 4(-2) = 16$

$x = 16 - 17$

$\boxed{= -1}$

Q $x + y + z = 1,\ 3x + y - 3z = 5,\ x - 2y - 5z = 10$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

$C = A : B$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 5 \\ 1 & -2 & -5 & 10 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 2 \\ 0 & -2 & -5 & 10 \end{array}\right]$$

$R_3 = -R_1 + R_3$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 2 \\ 0 & -3 & -6 & 9 \end{array}\right]$$

$R_3 = -3/2 R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 2 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$x + y + z = 1$ —— (1)

$-2y - 6z = 2$ —— (2)

$3z = 6$

$z = 3/2$

$\Rightarrow z = 2$

Substitude $z = 2$ in (2)

$-2y - 6(2) = 2$

$\Rightarrow y = -7$

Substitude $y = -7$ in (1) +. $z = 2$ in (1)

$x + (-7) + 2 = 1$

$\Rightarrow x = 6.$

Q) $5x + 2y + z = 12,\ x + 4y + 2z = 15,\ x + 2y + 5z = 20$

$\rightarrow$
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 15 \\ 5 & 2 & 1 & 12 \\ 1 & 2 & 5 & 20 \end{array}\right]$$

$R_2 \rightarrow 5R_1 - R_2$

$$\begin{bmatrix} 1 & 4 & 2 & 15 \\ 0 & 18 & 9 & 63 \\ 1 & 2 & 5 & 20 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 15 \\ 0 & 18 & 9 & 63 \\ 0 & 2 & 3 & 5 \end{array}\right]$$

$R_3 \rightarrow 9R_3$

$$\begin{bmatrix} 1 & 4 & 2 & 15 \\ 0 & 18 & 9 & 63 \\ 0 & 18 & 27 & 45 \end{bmatrix}$$

$R_3 \rightarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 4 & 2 & 18 \\ 0 & 18 & 9 & 63 \\ 0 & 0 & -18 & 18 \end{bmatrix}$$

$x + 4y + 2z = 15$

$18y + 9z = 63$

$-18z = 18$

$z = -1$

$18y + 9 \times -1 = 63$

$18y - 9 = 63$

$y = 4$

$x + 4 \times 4 + 2 \times -1 = 15$

$x + 16 - 2 = 15$

$x + 14 = 18$

$x = 1$

H.W. Q. $2x-6y+8z=24,\ 5x+4y-3z=2,\ 3x+y+2z=16$

→

$$\begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -6 & 8 & 24 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

$R_1 \longrightarrow R_1/2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 12 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 5R_1,\ R_3 \longrightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 12 \\ 0 & 19 & -23 & -58 \\ 0 & 10 & -10 & -20 \end{array}\right]$$

$R_3 \longrightarrow 19R_3 - 10R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 12 \\ 0 & 19 & -23 & -58 \\ 0 & 0 & 40 & 200 \end{array}\right]$$

$x - 3y + 4z = 12$ —— ①

$0 + 0 + 40z = 200$

$z = 5$

$0 + 19y - 23(5) = -58$

$19y = -58 + 115$

$y = 3$

$x - 9 + 20 = 12$

$x + 11 = 12$

$x = 1$

* Inverse by Partitioning –

Consider a Matrix A after Partitioning

$$A = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right]$$

P is non-singular Matrix
i.e $|P| \neq 0$
let $A^{-1} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$

Formula –

1) let P is non-singular Matrix
i.e. $|P| \neq 0$
2) find $P^{-1} = \frac{\text{adj } P}{|P|}$ (by adjoint Method)
3) find $T = RP^{-1}$, find TQ
4) find $W = (S - TQ)^{-1}$ (by adjoint Method)
5) find $Z = -WT$
6) find $y = -P^{-1}QW$
7) find $X = P^{-1} - YT$

Q Find inverse by partitioning Method.

① $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 12 \end{bmatrix}$

$\longrightarrow$

① $A = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 3 & 5 \\ \hline 1 & 4 & 12 \end{array}\right] = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$

where, $P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $Q = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 4 \end{bmatrix}$ $S = \begin{bmatrix} 12 \end{bmatrix}$

$|P| = 3 - 2 = 1 \neq 0$

$\therefore$ P is non-Singular Matrix.

② $P^{-1} = \dfrac{\text{adj } P}{|P|}$

[diagonal change & undiagonal sign changes − to + , + to −]

$\text{adj } P = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$P^{-1} = \dfrac{\text{adj} P}{|P|} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

③ $T = RP^{-1}$

$= \begin{bmatrix} 1 & 4 \end{bmatrix}_{1\times 2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}_{2\times 2}$

$= \begin{bmatrix} 3-4 & -2+4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \end{bmatrix}_{1\times 2}$

TQ

$TQ = \begin{bmatrix} -1 & 2 \end{bmatrix}_{1\times 2} \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2\times 1}$

$= \begin{bmatrix} -3+10 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}_{1\times 1}$

④ $W = (S - TQ)^{-1}$

$W = (S - TQ)$

$= [12] - [7]$

$= [5]$

$W = (S - TQ)^{-1}$

$= \frac{1}{5} [1]$

⑤ $Z = -WT$

$= -\frac{1}{5} [1]_{1 \times 1} \begin{bmatrix} -1 & 2 \end{bmatrix}_{1 \times 2}$

$= -\frac{1}{5} \begin{bmatrix} -1 & 2 \end{bmatrix}$ or $= \frac{1}{5} \begin{bmatrix} 1 & -2 \end{bmatrix}$

⑥ $y = -P^{-1} Q W$

$= -\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \frac{1}{5} [1]$

$= -\begin{bmatrix} 9 & -10 \\ -3 & 5 \end{bmatrix}_{2 \times 1} \frac{1}{5} [1]$

$= -\frac{1}{5} \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{2 \times 1} [1]_{1 \times 1}$

$= \frac{1}{5} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

⑦ $X = P^{-1} - YT$

$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 1 \\ -2 \end{bmatrix}_{2 \times 1} \begin{bmatrix} -1 & 2 \end{bmatrix}_{1 \times 2}$

$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{-1}{5} & \frac{2}{5} \\ 2/5 & -4/5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + \frac{1}{5} & -2 \frac{-2}{5} \\ -1 \frac{-2}{5} & 1 + \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{5} & \frac{-12}{5} \\ \frac{-7}{5} & \frac{9}{5} \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 16 & -12 \\ -7 & 9 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 16 & -12 & 1 \\ -7 & 9 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

② $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$\rightarrow$

$$A = \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 1 & 4 & 3 \\ \hline 1 & 3 & 4 \end{array}\right] = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

$P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 3 \end{bmatrix}$ $\quad S = \begin{bmatrix} 4 \end{bmatrix}$

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$|P| = 4-3 = 1 \neq 0$

$\therefore$ P is non-singular matrix.

(2) $\text{adj } P = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$

$P^{-1} = \frac{\text{adj } P}{|P|} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$

(3) $T = RP^{-1}$

$= \begin{bmatrix} 1 & 3 \end{bmatrix}_{1\times 2} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}_{2\times 2}$

$= \begin{bmatrix} 4-3 & -3+3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \end{bmatrix}$

$TQ = \begin{bmatrix} 1 & 0 \end{bmatrix}_{1\times 2} \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{2\times 1}$

$= \begin{bmatrix} 3+0 \end{bmatrix}$

$= \begin{bmatrix} 3 \end{bmatrix}$

(4) $W = (S - TQ)$

$= \begin{bmatrix} 4 \end{bmatrix} - \begin{bmatrix} 3 \end{bmatrix}$

$= 1$

$= \begin{bmatrix} \frac{1}{1} \end{bmatrix}$

(5) $Z = -WT$

$= -\begin{bmatrix} \frac{1}{1} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \end{bmatrix}$

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⑥ $y = -P^{-1} QW$

$$= -\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2\times 1} [1]$$

$$= \begin{bmatrix} 12 & -9 \\ -3 & +3 \end{bmatrix} [1]$$

$$= -\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

⑦ $x = P^{-1} - YT$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} -3 \\ 0 \end{bmatrix}_{2\times 1} \begin{bmatrix} 1 & 0 \end{bmatrix}_{1\times 2}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

③ $$A = \left[\begin{array}{cc|c} 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \\ \hline 1 & 0 & 0 \end{array}\right] = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

→

$$P = \begin{bmatrix} 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}$$

$|P| = 0 - 0 = 0$

P is singular Matrix.

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now interchanging first column and

$$B = \left[\begin{array}{cc|c} -\sin\theta & \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$B = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

① where, $P = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$|P| = -\sin^2\theta - \cos^2\theta = -1 \neq 0$

$\therefore P$ is non-singular

$Q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 1 \end{bmatrix}$

$\therefore$ let $B^{-1} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$

② $\operatorname{adj} P = \begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix}$

$P^{-1} = \dfrac{\operatorname{adj} P}{|P|} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

③ $T = RP^{-1} = \begin{bmatrix} 0 & 0 \end{bmatrix}_{1\times 2} \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}_{2\times 2}$

$T = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$TQ = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$(B = A \times i13.)$

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④ $W = (S - TQ)^{-1}$.

$$(S - TQ) = [1] - [0] = [1]$$

$$W = [S - TQ]^{-1} = [1]$$

⑤ $Z = -WT$

$$= -[1]_{1\times1}[0\ 0]_{1\times2}$$

$$= [0\ \ 0]$$

⑥ $y = -P^{-1}QW$

$$= -\begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}_{2\times2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times1}[1]_{1\times1}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⑦ $X = P^{-1} - YT$

$$= \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}_{2\times2} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times1}[0\ \ 0]$$

$$X = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$\therefore B^{-1}\begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now interchanging 1^{st} row with 3^{rd} row of B^{-1} we get A^{-1}

$$A^{-1}\begin{bmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

④ $\left[\begin{array}{cc|cc} 1 & 2 & -3 & 1 \\ 1 & 3 & 3 & 2 \\ \hline 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right] = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$

Where, $P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$

$R = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$

$|P| = 3-2 = 1 \neq 0$

P is non-singular Matrix.

① $adj\, P = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

② $P^{-1} = \frac{adj\, P}{|P|} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

1 matrix row
2 matrix colom

③ $T = RP^{-1} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ $\begin{pmatrix} 6+4 & -4-4 \\ 3+1 & -2-1 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$

$T = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}$

$TQ = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6+0 & 2+0 \\ 6-3 & 2-2 \end{bmatrix}$

$= \begin{bmatrix} 6 & 2 \\ 3 & 0 \end{bmatrix}$

④ $W = (S - TQ)^{-1}$

$(S - TQ) = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} -3 & 1 \\ -2 & 1 \end{bmatrix}$

$= [1]$

④ ~~$Z = -WT$~~

$$\operatorname{adj}[S-TQ] = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$$

$$W = (S-TQ) = \begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$W = \begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$$

⑤ $Z = -WT$

$$= \begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$=$

H.W Q

$$A = \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 3 & 1 \\ \hline 1 & 2 & 4 \end{array}\right]_{3\times3} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

$\longrightarrow P = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \quad Q = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad S = \begin{bmatrix} 4 \end{bmatrix}$

$$|P| = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} = 6 - 12 = -6$$

① $P^{-1} = \dfrac{\text{Adj } P}{|P|} = \dfrac{1}{6}\begin{bmatrix} 3 & -3 \\ -4 & 2 \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} -3 & 3 \\ 4 & -2 \end{bmatrix}$

② $T = RP^{-1} = \begin{bmatrix} 1 & 2 \end{bmatrix} \dfrac{1}{6}\begin{bmatrix} -3 & 3 \\ 4 & -2 \end{bmatrix}$

$$T = \frac{1}{6}\begin{bmatrix} 5 & -1 \end{bmatrix}$$

③ $W = [S - TQ]^{-1}$

$= \left\{ [4] - \dfrac{1}{6}\begin{bmatrix} 5 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}^{-1}$

$= \left\{ [4] - \dfrac{1}{6}[19] \right\}^{-1}$

$= \left\{ [4] - [19/6] \right\}^{-1} = [5/6]^{-1} = [6/5]$

④ $Z = -WT = -[6/5] \cdot \dfrac{1}{6}\begin{bmatrix} 5 & -1 \end{bmatrix}$

$= \dfrac{1}{5}\begin{bmatrix} -5 & 1 \end{bmatrix}$

⑤ $Y = -P^{-1}QW = -\dfrac{1}{6}\begin{bmatrix} -3 & 3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{6}{5} \end{bmatrix}$

$= -1/5 \begin{bmatrix} -9 \\ 14 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 9 \\ -14 \end{bmatrix}$

⑥ $X = P^{-1} - YT = 1/6 \begin{bmatrix} -3 & 3 \\ 4 & -2 \end{bmatrix} - \dfrac{1}{5}\begin{bmatrix} 9 \\ -14 \end{bmatrix} \dfrac{1}{6}\begin{bmatrix} 5 & -1 \end{bmatrix}$

$= \dfrac{1}{6}\begin{bmatrix} -3 & 3 \\ 4 & -2 \end{bmatrix} - \dfrac{1}{6}\begin{bmatrix} 9 \\ -14 \end{bmatrix}\begin{bmatrix} 1 & -1/5 \end{bmatrix}$

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$$= \frac{1}{6}\begin{bmatrix} -3 & -3 \\ 4 & -2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 9 & -9/5 \\ -14 & 14/5 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -12 & 24/5 \\ 18 & -24/5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4/5 \\ 3 & -4/5 \end{bmatrix}$$

$$X = \frac{1}{5}\begin{bmatrix} -10 & 4 \\ 15 & -4 \end{bmatrix}$$

Now, $A^{-1} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$$

* Minor of Matrix –

Minor of order 'r' of Matrix 'A' is the determinant of it by selecting any 'r' rows are column of the given Matrix.

Their are 3 Minor of order 2

* Rank of Matrix –

The Rank of Matrix is said to be 'r' if –

i) It has atleast one non-zero minor

ii) Every minor of a of order than r is 0

It is denoted by rank of A OR $\rho(A)$

* Zero and Non-zero rule.-

If all the element in a row of Matrix are zero then it is Called Zero rule. otherwise it is non-zero rule.

* Other Method to find Rank of Matrix -

Rank of Matrix = the no. of non-zero rule in upper triangular Matrix.

* To convert given Matrix into upper triangular matrix we use Common elementry method.

Q. Find the rank of foll. Matrix?

①
$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

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$R_3 \to R_3 - R_2,\ R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_4 \to 3R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of Non-zero row are 3

$\therefore$ Rank of A = 3

② $$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$\to$ $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of Non-Zero row are 3

$\therefore$ Rank of $A = 3$

③ $$\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$$

$R_2 \to 5R_2 - 6R_1$

$$\sim \begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & -1 & -2 & -9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$$

$R_3 \rightarrow 5R_3 - 11R_1$
$R_4 \rightarrow 5R_4 - 16R_1$

$$\sim \begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & -1 & -2 & 9 \\ 0 & -6 & -12 & -18 \\ 0 & -11 & -22 & -33 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_1$

$$\sim \begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & -1 & -2 & -9 \\ 0 & 0 & -5 & -10 \\ 0 & -11 & -22 & -33 \end{bmatrix}$$

$R_4 \rightarrow R_4 + 11R_2$

$$\sim \begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & -1 & -2 & -9 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & & \end{bmatrix}$$

No. of Non-Zero Row are 3
$\therefore$ Rank of $A = 3$

(4)
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

$\rightarrow$

$R_2 \rightarrow R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$.

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & -10 & -15 \end{bmatrix}$$

Interchange $R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & -15 & -10 & -15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 - 15R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of non-zero rows are 2

$\therefore$ Rank is = 2.

* Consistency of system of non homogeneous equation -

Consider a system of equation after reduction to matrix form

$$AX = B$$

where, A is Coefficient Matrix
X is Variable Matrix
B is Constant Matrix

- If $B = 0$, then system of equation is said to be homogenous equation.

If $B \neq 0$, then system of equation is said to be non-homogenous equation.

* Let system of equation $AX = B$ where,
A is coefficient matrix
and augmented matrix 'C' is given by,
then system of equation is said to be consistance if and only if rank of coefficient matrix is equals to rank of augmented matrix.

i.e. $\rho(A) = \rho(C)$

* Let 'n' be the no. of variable of system of equation and system is consistance rank of 'A' is same as rank of 'C'

i.e. $\rho(A) = \rho(C) = r$

1) if $r = n$, then system will have unique solution.
2) if $r < n$, then system will have infinite solution.
3) if $\rho(A) \neq \rho(C)$, where system is inconsistance and solution does not exist.

* If $r < n$ Nature of solution.
let system of equation

$AX = B$

and rank of 'A' = rank of 'C'
n be the no. of variable.
and $r < n$. In this case we will have 'n' unknown and 'r' equation.
now $n - r$ variable assign with arbitrary constant.

remaing 'r' variable will determine in terms of arbitrary constant.

Q. Test the consistency and solve it.

① $x + y + z = 6$, $2x + y + 3z = 13$, $5x + 2y + z = 12$

→ Matrix form of above system of equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 12 \end{bmatrix}$$

i.e. $AX = B$ — ①

Augmented matrix C is given by

$$C = [A \,|\, B]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \\ 5 & 2 & 1 & 12 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & -3 & -4 & -18 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -7 & -21 \end{array}\right] \quad \text{—②}$$

$\Rightarrow \rho(A) = 3$, $\rho(C) = 3$

$\therefore \rho(A) = \rho(C) = r = 3$

$\therefore$ System is consistance.

Here, 'n' be no. of variable

$n = 3$

and clearly $r = n$

∴ System will have unique solution.
from equation ②

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -21 \end{bmatrix}$$

① $x + y + z = 6$ — ⓐ

② $0 + y + z = 1$ — ⓑ

③ $0 + 0 + 7 = -21$

$-7 = -21$ $\boxed{z = 3}$

b) $-y + 3 = 1$

$-y = -2 = 1$ $\boxed{y = 2}$

a) $x + 2 + 3 = 6$ $\boxed{x = 6 - 5 = 1}$

② $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$

→ Matrix form of above system of equation is

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$AX = B$

Augmented Matrix C is given by

$C = [A \mid B]$

$$= \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$R_2 \to 5R_2 - 3R_1$, $R_3 \to 5R_3 - 7R_1$

$$\sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{array}\right]$$

$R_3 \to 11R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{--- (2)}$$

$\Rightarrow \rho(A) = 2, \rho(C) = 2$

$\therefore \rho(A) = \rho(C) = r = 2$

$\therefore$ System is Consistance.

$\therefore$ No. of Variable $n = 3$

Clearly $r < n$

System will have infinite Solution.

$n - r = 3 - 2 = 1$

let $z = k$

from equation (2)

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \\ 0 \end{bmatrix}$$

$5x + 3y + 7z = 4$ --- (3)

$0 + 121y - 11z = 33$

$\boxed{z = k}$

$121y - 11k = 33$

$121y = 33 + 11k$

$y = \frac{11(3+k)}{121}$

$\boxed{y = \frac{3+k}{11}}$

$5x + 3\left(\frac{3+k}{11}\right) + 7k = 4$

$55x + 9 + 3k + 77k = 44$

$55x = 44 - 9 - 80k$

~~$55x = 44 - 9 - 80k$~~

$55x = 35 - 80k = 5(7 - 16k)$

$\boxed{x = \frac{\cancel{5}(7-16k)}{\cancel{55}\ 11}}$

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Q Determine Value of λ and μ such that the equation $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ Have 1] no solution 2] unique Solution 3] Infinite Solution.

→ Matrix form of above system of equation is -

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented Matrix C is given by - $C = [A | B]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-2 & \mu-10 \end{array}\right]$$

1] If $\lambda \neq 3$, μ may have any value.

$\therefore \rho(A) = 3$, $\rho(C) = 3$

$\rho(A) = \rho(C) = r = 3$

System is Consistance.

n be no. of Variable

$n = 3$, $r = n$

System is unique Solution.

2) $\lambda = 3$ $\mu \neq 10$

Rank of A is 2

Rank of C is 3

$\rho(A) = 2$, $\rho(C) = 3$

$\rho(A) \neq \rho(C)$

System is inconsistance

Solution does not exist.

System will have no solution.

3) $\lambda = 3$ $\mu = 10$

$\rho(A) = 2$ $\rho(C) = 2$

$\rho(A) = \rho(C) = r = 2$

System is consistance.

$n = 3$, $r < n$

System will have infinite solution.

* **System of Homogenous equation**

A system of equation $AX = B$ is said to be homogenous if $B = 0$

If determinant of $|A| \neq 0$

$$A^{-1}AX = A^{-1}0$$

$$X = 0$$

$$\Rightarrow x = 0,\ y = 0,\ z = 0$$

is called trivial solution (System is always consistance.)

* The system of equation $AX = 0$ will have non-trivial solution if an only if coefficient matrix is singular determinant $|A| = 0$

Q For what Value of λ is system $3x+\lambda y+5z=0$, $4\lambda x-2y-\lambda z=0$, $18x-4y+7z=0$ will have non-trivial solution also find solution.

→ Matrix form is

~~4λ 2 λ 0~~

$$\begin{bmatrix} 3 & \lambda & 5 \\ 4\lambda & -2 & -\lambda \\ 18 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if $|A|=0$

$$\begin{bmatrix} 3 & \lambda & 5 \\ 4\lambda & -2 & -\lambda \\ 18 & -4 & 7 \end{bmatrix}=0$$

$\Rightarrow 3(14-4\lambda)-\lambda(28\lambda^2+18\lambda)+5(-16\lambda+36)=0$

$\Rightarrow -42-12\lambda-46\lambda^2-80\lambda+180=0$

$\Rightarrow -46\lambda^2-92\lambda+138=0$

$\boxed{\lambda=1}$, $\boxed{\lambda=-3}$

For $\lambda=1$, $\lambda=-3$

System will have non-trivial solution.

1) for $\lambda=1$

augmented matrix C is

$$C=\left[\begin{array}{ccc|c} 3 & 1 & 5 & 0 \\ 4 & -2 & -1 & 0 \\ 18 & -4 & 7 & 0 \end{array}\right]$$

$R_1 \to R$

* Linear Depedance -

Vector - An order set of 'n' number $(x_1, x_2 \ldots x_n)$ is called vector.

eg - $X = (x_1, x_2 \ldots x_n)$ is a row vector.

Linear Depedance - Set of vector $X_1, X_2 \ldots X_n$ is said to be linear Depedance if their exist scalar $k_1, k_2 \ldots k_n$ not all zero. Such that $k_1 X_1 + k_2 X_2 \ldots + k_n X_n = 0$ where, '0' is called null vector. and if $k_1 = k_2 = \ldots = k_n = 0$ then system set of vector is called linearly independent.

Q/ Invistigate the linear depedance of vector and find relation. $X_1 = (2, -1, 3, 2)$ $X_2 = (1, 3, 4, 2)$ $X_3 = (3, -5, 2, 2)$

→

$X_1 = (2, -1, 3, 2)$ $X_2 = (1, 3, 4, 2)$ $X_3 = (3, -5, 2, 2)$

$X_1 - 2X_2 = (0, -7, -5, -2)$ —— ①

$X_3 - 3X_2 = (0, -14, -10, -4)$ —— ②

From eq. ② $-2 \times$ eq ①

$(X_3 - 3X_2) - 2 \times (X_1 - 2X_2) = (0, 0, 0, 0)$

$X_3 - 3X_2 - 2X_1 + 4X_2 = 0$

$-2X_1 + X_2 + X_3 = 0$ —— ③

$k_1 X_1 + k_1 X_2 + k_3 X_3 = 0$

$k_1 = -2, \ k_2 = 1, \ k_3 = 1$

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∴ given set of Vector are linearly dependance.

∴ $-2X_1 + X_2 + X_3 = 0$ required equation.

Q $X_1 = (1,1,1,3)$, $X_2 = (1,2,3,4)$, $X_3 = (2,3,4,7)$

→ $(X_2 - X_1) = (0,1,2,1)$ — ①

$(X_3 - 2X_1) = (0,1,2,1)$ — ②

eq. (1) − (2), we get.

$(X_2 - X_1) - (X_3 - 2X_1) = (0,0,0,0)$

$3X_2 - X_1 - X_3 + 2X_1 = 0$

$3X_1 + 3X_2 + 3X_3 = 0$

$K_1X_1 + K_2X_2 + K_3X_3 = 0$

$K_1 = 1$, $K_2 = 1$, $K_3 = (-1)$

∴ given set of Vector are linearally dependance.

$3X_1 + 3X_2 + 3X_3 = 0$ required equation.

Q $X_1 = (1,2,4)$, $X_2 = (2,-1,3)$, $X_3 = (0,1,2)$, $X_4 = (-3,7,2)$

→ $(X_2 - 2K_1) = (0,-5,-5)$ — ①

$(X_4 + 3X_1) = (0,13,14.)$ — ②

eq. ① $+ 5X_3$

$X_2 - 2X_1 + 5X_3 = (0,0,5)$ — ③

~~eq. ② $- 13X3 = (0,0,-12)$ — ④~~

~~eq. ②~~ ~~$X12 =$~~ $X_4 + 3X_1 - 13X_3 = (0,0,-13)$ — ④

eq. ③ $\times 12$ + eq. ④ $\times 5$

$12 \times [X_2 - 2X_1 + 5X_3] + 5 \times [X_4 + 3X_1 - 13X_3] = [0,0,0]$

$12X_2 - 24X_1 + 60X_3 + 5X_4 + 15X_1 - 65X_3 = 0$

$-9X_1 + 12X_2 - 5X_3 + 5X_4 = 0$ — ⑤

$K_1X_1 + K_2X_2 + K_3X_3 + K_4X_4 = 0$

$K_1 = -9$, $K_2 = 12$, $K_3 = -5$, $K_4 = 5$.

∴ given set of Vector are Linearly dependance.

$-9X_1 + 12X_2 - 5X_3 + 5X_4 = 0.$

Q) $X_1 = (1, 2, -1, 3)$ $X_2 = (2, -1, 3, 2)$ $X_3 = (-1, 8, -9, 5)$

→ $X_2 - 2X_1 = (0, -5, 5, 4)$ — (1)

$X_3 + X_1 = (0, 10, -10, 8)$ — (2)

eq^n (1) $\times -2$

$-2X_2 + 4X_1 = (0, 10, -10, 8)$ — (3)

Eq^n (2) − Eq^n (3)

$X_3 + X_1 + 2X_2 - 4X_1 = (0, 0, 0, 0)$

$-3X_1 + 2X_2 + X_3 = (0, 0, 0, 0)$

Set of eq^n is Linear dependance.

$K_1 = -3$, $K_2 = 2$, $K_3 = 1$

Q) $X_1 = (1, 2, 2)$, $X_2 (2, 1, -2)$, $X_3 = (2, -2, 1)$

→ $X_1 = (1, 2, 2)$ $X_2 = (2, 1, -2)$ $X_3 = (2, -2, 1)$

$X_2 \to X_2 - 2X_1$

$X_3 \to X_3 - 2X_1$

$X_2 = (0, -3, -6)$ — ① ①

$X_3 = (0, -6, -3)$ — ② ②

∴ $X_2 - 6X_3$

$(0, -18, -36)$ $(0, -36, -18)$ $-18)$

$(0, -18, -18)$ $(0, -18, 18)$ $18)$

$\therefore (0, 0, 0)$

$6x_2 - 12x_1 - 6x_3 + 12x_1 = (0, 0, 0)$

$0x_1 + 6x_2 - 6x_3 = (0, 0, 0)$

$k_1 = 0 \quad k_2 = 6 \quad k_3 = -6.$

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Unit 2 : Matrices

* **Characteristic equation** - If A is a Square matrix is order here then the Matrix $[A-\lambda I]$ where, λ is a Scalar, is called Characteristic Matrix.

The determinant of this Matrix is equals to zero is called determinant of
$|\lambda I - A| = 0$ or $|A-\lambda I| = 0$

* **Eigen Values** - The roots of the characteristic equation i.e $|A-\lambda I| = 0$ say $\lambda_1, \lambda_2 \dots \lambda_n$ are said to be eigen values or characteristic roots or latent root.

* **Eigen Vector** - If $X = \begin{bmatrix} x_1 \\ \vdots \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2 \dots x_n]$

is non-zero column vector such that
$AX = \lambda X \Rightarrow [A-\lambda I] X = 0$ — ①
equation ① form system of homogenous equation and we have non-trivian solution if coefficient matrix is singular ie $[A-\lambda I] X = 0$

* Reduction to diagonal form (Canonical)

If A is Square Matrix of order n having n linearly independent I vector then a non-Singular Square Matrix B can be found such that $B^{-1}AB$ is a diagonal form.

If A is Square Matrix of order 3 having 3 linearly independent eigen vector corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$ then

$$B^{-1}AB = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

is called diagonal form. $B^{-1}AB$ where, B is non-Singular. Matrix B is a eigen vector Matrix. is called Model Matrix.

Note- 1] B must be non-Singular matrix i.e determinant of B not equals to zero.

2] For same eigen values eigen ~~vector~~ taken different eign vector.

If $\lambda_1 = \lambda_2$ then corresponding eigen vector X_1, X_2, should not be scalar multiple of each other.

Find eigen values, vectors, and modal Matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & 4 \\ 2 & -4 & 3 \end{bmatrix}$$

First step - characteristic equation is
$|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$ — ①

S_1 = Sum of diagonal element of A
$= 8+7+3 = 18$

S_2 = Sum of minor of diagonal element of A

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= 12 - 16 + 24 - 4 + 56 - 36$
$= 5 + 20 + 20$
$= 45$

$|A| = 0$

(1) $\Rightarrow \lambda^3 - 18\lambda^2 + 45 - 0 = 0$
$ax^3 + bx^2 + cx + d = 0$

$\lambda = 15, 3, 0$

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$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (8-\lambda)x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + (7-\lambda)x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right] \text{— ②}$$

for $\lambda = 15$, we get.

$-7x_1 - 6x_2 + 2x_3 = 0$ — ①

$-6x - 8x_2 - 4x_3 = 0$ — ②

$2x_1 - 4x_2 - 12x_3 = 0$ — ③

on solving ② & ③, we get,

$$\frac{x}{\begin{vmatrix} -8 & -4 \\ -4 & -12 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & -12 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & -8 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{96-16} = \frac{-x_2}{72+8} = \frac{x_3}{24+16}$$

$$\frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$x_1 = 2, \quad x_2 = -2, \quad x_3 = 1$

which satisfying equation ①

for $\lambda = 15$, $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

from eq. ②, $\lambda = 3$.

$5x_1 - 6x_2 + 2x_3 = 0$ — ①

$-6x_1 + 4x_2 - 4x_3 = 0$ — ②

$2x_1 - 4x_2 + 0x_3 = 0$ — ③

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on solving ② & ③, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -4 \\ 4 & 0 \end{vmatrix}} = \frac{-x}{\begin{vmatrix} -6 & -4 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8}$$

$$\frac{x_1}{16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}.$$

$x_1 = 2$, $x_2 = 1$, $x_3 = -2$

Which satisfaying equation ①.

for $\lambda = 3$, $X = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

from eq ② $\lambda = 0$.

$8x_1 - 6x_2 + 2x_3 = 0$ — ①

$-6x_1 + 7x_2 - 4x_3 = 0$ — ②

$2x_1 - 4x_2 + 3x_3 = 0$ — ③.

On solving ② & ③, we get.

$$\frac{x_1}{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{21-16} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14}.$$

$$\frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$x_1 = 1$, $x_2 = 2$, $x_3 = 2$.

Which satisfy ①.

for $\lambda = 0$ $X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Model Matrix (Q)

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$$B = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

Q)

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Characteristic eq. is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad — (1).$$

$$S_1 = -2 + 1 + 0 = -1$$

$$S_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 0 - 12 + 0 - 3 (-2 - 4)$$

$$= -12 - 3 - 6$$

$$S_2 = -21$$

$$|A| = 45$$

$$(1) \Rightarrow \lambda^3 - \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda_1 = 5, -3, -3.$$

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be given.

Vector.

$$\therefore [A - \lambda I] \cdot X = 0$$

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$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (-2-\lambda)x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + (1-\lambda)x_2 - 6x_3 &= 0 \\ -1x_1 - 2x_2 + (0-\lambda)x_3 &= 0 \end{aligned}\right] \text{ — } (2).$$

for $\lambda = 5$, we get.

$-7x_1 + 2x_2 - 3x_1 = 0$ — (1)

$2x_1 - 4x_2 - 6x_3 = 0$ — (2).

$-x_1 - 2x_2 - 5x_3 = 0$ — (3).

Solving eq. (2) & (3).

$$\frac{x_1}{\begin{vmatrix} -4 & -6 \\ -2 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & -6 \\ -1 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -4 \\ -1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{20-12} = \frac{-x_2}{-10-6} = \frac{x_3}{-4-2}.$$

$$\frac{x_1}{8} = \frac{+x_2}{+16} = \frac{x_3}{-8}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$x_1 = 1$, $x_2 = 2$, $x_3 = -1$

Which satisfies (i)

for $\lambda = 5$, $X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\lambda = 3$ from (2).

$x_1 + 2x_2 - 3x_3 = 0$ — (1)

$2x_1 + 4x_2 - 6x_3 = 0 \Rightarrow x_1 + 2x_2 - 3x_2 = 0$ — (2)

$-x_1 - 2x_2 + 3x_3 = 0 \Rightarrow x_1 + 2x_2 - 3x_3$ — (3)

Here we have 3 unknown i.e. x_1 x_2 x_3 and 1 eq. therefore we have to assign two variable assign with arbitrary constant.

$\therefore [x_1 + 2x_2 - 3x_3 = 0]$

let, $x_2 = 0$

$x_1 = 3x_3$

$\frac{x_1}{3} = \frac{x_3}{1}$

$x_1 = 3,\ x_2 = 0,\ x_3 = 1$

for $\lambda = -3$, $X = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

for $\lambda = -3$ from (2)

$x_1 + x_2 - 3x_3 = 0$

let $x_3 = 0$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$\frac{x_1}{2} = \frac{x_2}{-1}$

$x_1 = 2,\ x_2 = -1,\ x_3 = 0$

for $\lambda = -3$, $X = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Modal Matrix is,

$B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

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Q. find eigen Value, Vectors, Model Matrix reduce in diagonal form. and $B^{-1}AB$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

characteristic eq. is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\lambda = 4, -1$$

let, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\therefore [A - \lambda I]\, x = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x_1 + 2x_2 &= 0 \\ 3x_1 + (2-\lambda)x_2 &= 0 \end{aligned}\right] \text{ --- } (1)$$

for $\lambda = 4$, (1) be common

$$-3x_1 + 2x_2 = 0 \Rightarrow 3x_1 - 2x_2 = 0$$

$$3x_1 - 2x_2 = 0$$

$$\Rightarrow 3x_1 - 2x_2 = 0$$

$$3x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{3}$$

$$x_1 = 2, \quad x_2 = 3$$

for $\lambda = 4$, $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

for $\lambda = -1$ from ①

$2x_1 + 2x_2 = 0 \Rightarrow x_1 + x_2 = 0$

$3x_1 + 3x_2 = 0 \Rightarrow x_1 + x_2 = 0$

$x_1 = -x_2$

$\frac{x_1}{1} = \frac{x_2}{-1}$

for $\lambda = -1$, $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Modal Matrix B is,

$$B = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$|B| = -2 - 3 = -5 \neq 0$$

$$B^{-1} \frac{\text{adj } B}{|P|} = \frac{-1}{5} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$$

$$B^{-1}AB = \frac{-1}{5} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{-1}{5} \begin{bmatrix} -1-3 & -2-2 \\ -3+6 & -6+4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$B^{-1}AB = \frac{-1}{5} \begin{bmatrix} -4 & -4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$B^{-1}AB = \frac{-1}{5} \begin{bmatrix} -20 & 0 \\ 0 & 5 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

* Q $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

characteristic eq. is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(-3-\lambda) - 16 = 0$$

$$-9 - 3\lambda + 3\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 25 = 0 \Rightarrow \lambda^2 = 25$$

$$\lambda = 5, -5$$

$\lambda = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigen Vector.

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (3-\lambda) x_1 + 4x_2 &= 0 \\ 4x_1 + (-3-\lambda) x_2 &= 0 \end{aligned}\right] \text{— (I)}$$

for, $\lambda = 5$

$$-2x_1 + 4x_2 = 0 \Rightarrow x_1 - 2x_2 = 0$$

$$4x_1 - 8x_2 = 0 \Rightarrow x_1 - 2x_2 = 0$$

$$x_1 - 2x_2 = 0$$

$$x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{1}$$

$$x_1 = 2, \; x_2 = 1$$

for $\lambda = 5$, $\lambda = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

for $\lambda = -5$ from (I)

$$8x_1 + 4x_2 = 0$$

$$\Rightarrow 2x_1 + x_2 = 0$$

$$4x_1 + 2x_2 = 0$$
$$\Rightarrow 2x_1 + x_2 = 0$$
$$2x_1 + x_2 = 0$$
$$\Rightarrow 2x_1 = -x_2$$
$$\frac{x_1}{-1} = \frac{x_2}{2}$$

$\therefore$ Model Matrix B is

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1} AB = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

Properties of eigen Values -

1) Any square Matrix A and its transpose A^{-T} have same eigen values

2) The eigen values of triangular Matrix are just the diagonal element of the Matrix.

3) The eigen value of an idempotent Matrix are either zero or unity ($A^2 = A$)

4) The sum of eigen values of a Matrix is the sum of the elements of the principle diagonal.

5] The product of eign values of Matrix A is equal to its determinant

6] If λ is eign value of Matrix A then $\frac{1}{\lambda}$ is an eign value of A^{-1} inverse.

7] If λ is an eign value of an orthogonal Matrix then, $\frac{1}{\lambda}$ is also its eign values $(A \cdot A^{-} = B)$

8] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eign value of Matrix A, then A^m has eign values $\lambda_1^m, \lambda_2^m, \ldots \lambda^n$ where m is +ve integer.

Q Find the eign values of Matrix represented by $A^3 - 6A^2 + 3A - 2I$ if $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

⇒ characteristic eq. is $A - \lambda I = 0$

$$A = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix}$$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{—— (1)}$$

$s_1 = 1 + 2 + 3 = 6$

$s_2 = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -9 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$

$= 6 - 4 + 5 + 4 + 2 + 0$

$= 2 + 7 + 2$

$= 11$

$|A| = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, 2, 3$

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By using property (1) eign values of A^2 are

$$A^2 \rightarrow 1, 4, 9$$

eign value of A^3 are

$$A^3 \rightarrow 1, 8, 27$$

eign value of I are

$$I \rightarrow 1, 1, 1$$

$\therefore$ eign value of $A^3 - 6A^2 + 3A - 2I$ are first eign value are.

$$= 1 - 6(1) + 3(1) - 2(1)$$
$$= -4$$

Second eign values are

$$= 8 - 6(4) + 3(2) - 2(1) = -12$$

Third eign values are.

$$= 27 - 6(3) + 3(3) - 2(1)$$
$$= -20$$

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* Cayley's - Hamilton's theorem -

Every square matrix satisfies its own characteristic equation.

Q Verify Cayley's Hamilton's theorem for given matrix A.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$\rightarrow$ Characteristic equation.

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$ —— ①

$S_1 = 3 + 5 + 3 = 11$

$S_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix}$

$= 15 - 1 + 9 - 1 + 15 + 1$

$= 14 + 8 + 14$

$= 38$

$|A| = 3(14) - 1(-2) + 1(-4)$

$= 42 + 2 - 4$

$= 40$

Replacing λ by A in LHS — ②
We get,

$A^2 - 11A^2 + 38A - 40I$ —— ③

$A^2 = \begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix}$ $A^3 = \begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix}$

$\begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix} - 11 \begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix}$

$+ 38 \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore A^3 - 11A^2 + 38A - 40I = 0$ —— ④.

$\therefore$ Calyley's - Hamiltons Theorem Verified.

Pre - Multiplying A^{-1} on B.S

$A^{-1}.A^3 - 11A^{-1}A^2 + 38A^{-1}A - 40A^{-1}I = 0.$

$A^2 - 11A + 38I = 40A^{-1}$

$= \frac{1}{40} \begin{bmatrix} 14 & -4 & -6 \\ 2 & 8 & 2 \\ -4 & 4 & 16 \end{bmatrix}$

Q Find the characteristic equation of Matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the Matrix represented by

1] $A^4 - 5A^3 + 8A^2 - 2A + I$

2] $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$

→ characteristic eq. is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$S_1 = 2+1+2 = 5$

$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= 2-0+4-1+2-0$

$= 2+3+2 = 7$

$S_2 = 7$

$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$

$= 2(2) - 1(0) + 1(-1)$

$= 4 - 1$

$|A| = 3$

By using Cayley's Hamilton theorem -

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{—— (1)}$$

i] $A^4 - 5A^3 + 8A^2 - 2A + I$

2] $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$

1] $A^4 - 5A^3 + 7A^2 - 3A + I + A^2 + A$

$A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$A(0) + A^2 + A + I$

$0 + A^2 + A + I \quad \text{—— (2)}$

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$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

2) $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^4 - 5A^3 + 8A^2 - 2A + I$

$A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 17A - 3I) + A^2 + A$

$A(0) + A(0) + A^2 + A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$ — from ①

Q Verify Cayley's Hamiltons theorem and express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as linear polynomial as $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

⇒ Char. eq. $|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix} = 0$$

$(1-\lambda)(3-\lambda) + 2 = 0$

$3 - \lambda - 3\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$ — ①

Replace λ by A in LHS of ①.

$A^2 - 4A + 5I = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^2 - 4A + 5I = 0$ — ②

$\therefore$ Cayley's Hamiltons Theorem verified.

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$A^6 - 4A^5 + 5A^4 + 3A^4 - 12A^3 + 14A^2$

$A^4(A^2 - 4A + 5I) + 3A^4 - 12A^3 + 14A^2$

$A^4(0) + 3A^4 - 12A^3 + 15A^2 - A^2$

$0 + 3A^2(A^2 - 4A + 5I) - A^2$ by ②

$$3A^2(0) - A^2 = -4A - 5I$$

Q Verify cayley's Hamilton's theorem and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

⇒ characteristic eq. is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

Replace λ by A LHS — ①.

$$A^2 - 4A - 5I = 0 \quad \text{— ①}$$

$$\begin{bmatrix} 9 & 16 \\ 8 & 7 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 - 4A - 5I = 0 \quad \text{— ②}$$

Cayley's Hamilton's theorem verified

$$A^2 - 4A - 5I = 0$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$$

$$A^5 - 4A^4 - 5A^3 - 2A^3 + 11A^2 - A - 10I$$

$$A^3(A^2 - 4A - 5I) - 2A^3 + 11A^2 - A - 10I$$

$$A^3(0) - 2A^3 + 8A^2 + 10A + 3A^2 - 11A - 10I$$

$$-2A(A^2 - 4A - 5I) + 3A^2 - 11A - 10I$$

$$-2A(0) + 3(4A + 5I) - 11A - 10I$$

$$12A + 15I - 11A - 10I = A + 5I$$

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HW

Q Verify Cayley's Hamilton's theorem and Hence find A^{-1} $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

$\Rightarrow$ Char. eq. $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 4 \\ 2 & 1-\lambda & 2 \\ 4 & 2 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - P\lambda^2 + Q\lambda - |A| = 0$

$S_1 = 1+1+1 = 3$

$S_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$

$= -3 - 15 - 3$

$S_2 = -21$

$|A| = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

$= 1(-3) - 2(-6) + 4(0)$

$= -3 + 12 + 0$

$|A| = 9$

$\lambda^3 - 3\lambda^2 - 21\lambda - 9 = 0$

By cayleys Hamiltons theorem.

$A^3 - 3A^2 - 21A - 9I = 0$

$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+4+16 & 2+2+8 & 4+4+4 \\ 2+2+8 & 4+1+4 & 8+2+2 \\ 4+4+4 & 8+2+2 & 16+4+1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 21 & 12 & 12 \\ 12 & 9 & 12 \\ 12 & 12 & 21 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 21 & 12 & 12 \\ 12 & 9 & 12 \\ 12 & 12 & 21 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 93 & 78 & 120 \\ 78 & 57 & 78 \\ 120 & 78 & 93 \end{bmatrix}$$

$$LHS = A^3 - 3A^2 - 21A - 9I$$

$$= \begin{bmatrix} 93 & 78 & 120 \\ 78 & 57 & 78 \\ 120 & 78 & 93 \end{bmatrix} - 3 \begin{bmatrix} 21 & 12 & 12 \\ 12 & 9 & 12 \\ 12 & 12 & 21 \end{bmatrix} -$$

$$-21 \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix} - 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 93 & 78 & 120 \\ 78 & 57 & 78 \\ 120 & 78 & 93 \end{bmatrix} - \begin{bmatrix} 63 & 36 & 36 \\ 36 & 27 & 36 \\ 36 & 36 & 63 \end{bmatrix}$$

$$- \begin{bmatrix} 21 & 42 & 84 \\ 42 & 21 & 42 \\ 84 & 42 & 21 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$$= RHS$$

Cayleys Hamiltons theorem Verified.

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$$A^3 - 3A^2 - 21A - 9 = 0$$

Multiply by A^{-1}

$$A^{-1}A^3 - 3A^{-1}A^2 - 21A^{-1}A - 9A^{-1} = 0$$

$$A^2 - 3A - 21 - 9A^{-1} = 0$$

$$A^{-1} = \frac{1}{9}(A^2 - 3A - 21I)$$

$$= \frac{1}{9}\begin{bmatrix} 21 & 12 & 12 \\ 12 & 9 & 12 \\ 12 & 12 & 21 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix} - 21\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 21 & 12 & 12 \\ 12 & 9 & 12 \\ 12 & 12 & 21 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 12 \\ 6 & 3 & 6 \\ 12 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

$$\Rightarrow \frac{1}{9}\begin{bmatrix} -3 & 6 & 0 \\ 6 & -15 & 6 \\ 0 & 6 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 0 \\ 2 & -5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

★ **Sylvester's theory** - If A is square matrix as eigen value $\lambda_1, \lambda_2 \dots \lambda_n$ or are distinct and $P(A)$ is a polynomial in matrix a $A[A]$ of form

$$P(A) = C_0 I + C_1 A + C_2 A^2 + \dots C_K A^K.$$

where $C_0, C_1 \dots C_K$ all are constant and I is identity matrix of order n.

The $P(A)$ can be express as

$$P(A) = \sum_{r=1}^{n} P(\lambda_r) Z(\lambda_r)$$

where, $$Z(\lambda) = \left[\frac{\text{adj}[\lambda I - A]}{\phi(\lambda)}\right]$$

and
$$\phi(\lambda) = |\lambda I - A|$$

Q Using Sylvester theorem find the value of A^{50} where $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow [\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$[\lambda I - A] = \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{bmatrix}$

$\text{adj}[\lambda I - A] = \begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-1 \end{bmatrix}$

characteristic eq. $(\lambda) = |\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$\lambda_1 = 1 \quad \lambda_2 = 3$

Here, $\phi(\lambda) = \lambda^2 - 4\lambda + 3$

$\phi'(\lambda) = 2\lambda - 4$

Here, $P(A) = A^{50}$, by Sylvester theorem

$P(A) = P(\lambda_1) z(\lambda_1) + P(\lambda_2) z(\lambda_2)$ — ①

$z(\lambda) = \left[\dfrac{\text{adj}[\lambda I - A]}{\phi'[\lambda]}\right]$

$z(\lambda) = \dfrac{\begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-1 \end{bmatrix}}{2\lambda - 4}$

$\therefore$ eq. ① becomes

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$$A^{50} = 1^{50}\frac{\begin{bmatrix}1-3 & 0\\ 0 & 1-1\end{bmatrix}}{2(1)-4} + 3^{50}\frac{\begin{bmatrix}3-3 & 0\\ 0 & 3-1\end{bmatrix}}{2(3)-4}$$

$$= \frac{\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}}{(-2)} + 3^{50}\frac{\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}}{2}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 3^{50}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 3^{50}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 3^{50}\end{bmatrix}$$

Q By using Sylvester's theorem show that $3\tan A = (\tan 3)A$

where, $A = \begin{bmatrix}-1 & 4\\ 2 & 1\end{bmatrix}$

$$z(\lambda) = \left[\frac{\text{adj}\,[\lambda I - A]}{\phi(\lambda)}\right]$$

$\rightarrow$ $[\lambda I - A] = \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}-1 & 4\\ 2 & 1\end{bmatrix}$

$$[\lambda I - A] = \begin{bmatrix}\lambda+1 & -4\\ -2 & \lambda-1\end{bmatrix}$$

$$\text{adj}\,[\lambda I - A] = \begin{bmatrix}\lambda-1 & 4\\ 2 & \lambda+1\end{bmatrix}$$

characteristic eq. $\phi(\lambda) = |\lambda I - A| = 0$

$$\begin{bmatrix}\lambda+1 & -4\\ -2 & \lambda-1\end{bmatrix} = 0$$

$$(\lambda+1)(\lambda-1) - 8 = 0$$

$$\lambda^2 - 1 - 8 = 0$$

$$\lambda^2 - 9 = 0$$

$$\Rightarrow \lambda = 3, -3$$

$$\phi(\lambda) = \lambda^2 - 9$$

$$\phi'(\lambda) = 2\lambda$$

Here,

$P(A) = 3\tan A$

By Sylvester's theorem these $P(A)$ can be expressed as $P(A) = P(\lambda_1) + P(\lambda_2)$

$z(\lambda_i)$ —— ①

where,

$$z(\lambda) = \frac{\operatorname{adj}[\lambda I - A]}{\phi(\lambda)}$$

$$z(\lambda) = \frac{\begin{bmatrix} \lambda-1 & 4 \\ 2 & \lambda+1 \end{bmatrix}}{2\lambda}$$

① $\Rightarrow$ $3\tan A = \cancel{3}\tan 3 \dfrac{\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}}{\cancel{6}_2} + \cancel{3}\tan(-3) \dfrac{\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}}{(\cancel{-6}_2)}$

$= \tan 3 \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \tan(-3) \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$

$= \tan 3 \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} - \tan 3 \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$ $\quad \tan(-\theta) = -\tan\theta$

$= \tan 3 \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \right\}$

$= \tan 3 \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = (\tan 3)_A = \text{RHS}$

* Q $e^A = e^x \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$

where, $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$

$\Rightarrow [\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} x & x \\ x & x \end{bmatrix}$

$[\lambda I - A] = \begin{bmatrix} \lambda - x & -x \\ -x & \lambda - x \end{bmatrix}$

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$$\operatorname{adj}[\lambda I - A] = \begin{bmatrix} \lambda - x & x \\ x & \lambda - x \end{bmatrix}$$

characteristic equation is $\phi(\lambda) = |\lambda I - A| = 0$

$$\begin{vmatrix} \lambda - x & -x \\ -x & \lambda - x \end{vmatrix} = 0$$

$(\lambda - x)^2 - x^2 = 0$

$\lambda^2 - 2\lambda x + x^2 - x^2 = 0$

$\lambda(\lambda - 2x) = 0$

$\Rightarrow \lambda = 0, 2x.$

$\phi(\lambda) = \lambda^2 - 2\lambda x.$

$\phi'(\lambda) = 2\lambda - 2x.$

$P(A) = e^A$

By sylvester's theorem $P(A)$ can be expressed as $P(A) = P(\lambda_1) Z(\lambda_1) + P(\lambda_2) Z(\lambda_2)$ — ①

where,

$$Z(\lambda) = \frac{\operatorname{adj}[\lambda I - A]}{\phi'(\lambda)}$$

$$Z(\lambda) = \frac{\begin{bmatrix} \lambda - x & x \\ x & \lambda - x \end{bmatrix}}{2\lambda - 2x}$$

$$① \Rightarrow e^A = e^0 \frac{\begin{bmatrix} -x & x \\ x & -x \end{bmatrix}}{0 - 2x} + e^{2x} \frac{\begin{bmatrix} x & x \\ x & x \end{bmatrix}}{2x}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{e^{2x}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (e^0 = 1)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 + e^{2x} & -1 + e^{2x} \\ -1 + e^{2x} & 1 + e^{2x} \end{bmatrix}\right\}$$

$$= \frac{e^{x} \cdot e^{-x}}{2}\begin{bmatrix} 1 + e^{2x} & -1 + e^{2x} \\ -1 + e^{2x} & 1 + e^{2x} \end{bmatrix} \quad e^0 = 1$$

$$= e^{x}\begin{bmatrix} \frac{e^{-x}+e^{x}}{2} & \frac{-e^{-x}+e^{x}}{2} \\ & \end{bmatrix}$$

$$e^{A} = e^{x}\begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix} = RHS.$$

Q $\log_e e^{A} = A.$ if $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

→ $[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$[\lambda I - A] = \begin{bmatrix} \lambda-3 & -2 \\ -2 & \lambda-3 \end{bmatrix}$

adj $[\lambda I - A] = \begin{bmatrix} \lambda-3 & -2 \\ -2 & \lambda-3 \end{bmatrix}$

characteristic eq. is $\phi(\lambda) = |\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-3 & -2 \\ -2 & \lambda-3 \end{vmatrix} = 0$$

$(\lambda-3)^2 - 4 = 0$

$\lambda^2 - 6\lambda + 9 - 4 = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$\lambda_1 = 1, 5$

$\phi(\lambda) = \lambda^2 - 6\lambda + 5$

$\phi'(\lambda) = 2\lambda - 6$

$P(A) = \log_e e^{A}$

By Sylvesters theorem these $P(A)$ can be expressed as $P(A) = P(\lambda_1) Z(\lambda_1) + P(\lambda_2) Z$

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where,

$$z(\lambda) = \frac{\text{adj}\,[\lambda I - A]}{\phi'(\lambda)}$$

$$z(\lambda) = \frac{\begin{bmatrix} \lambda-3 & 2 \\ 2 & \lambda-3 \end{bmatrix}}{2\lambda-6}$$

(1)

$$\log_e e^A = \log_e e^1 \frac{\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}}{(-4)} + \log_e e^5 \frac{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}{4}$$

$$= \frac{1}{4}\left\{ 1 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} + 5 \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right\}$$

$$\log_e e^A = \frac{1}{4}\begin{bmatrix} 12 & 8 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = A.$$

* Solution of linear differential equation with Constant co-efficient by Matrix Method.-

1] Model Matrix Method- Reduce the given eq. of second order of the form.

II^{nd} order of form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d}{dt} X = AX \quad \text{—— (1)}$$

where, $\frac{dy}{dy} = x_1$, $y = x_2$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

where, 'A' is coefficient Matrix

2] the characteristic eq. is given by $|A - \lambda I| = 0$ and solve it.

Let distinct eign value be λ_1 and λ_2 and Model Matrix 'B' can be find.

Let $B = \begin{bmatrix} l_1 & M_1 \\ l_2 & M_2 \end{bmatrix}$, $|B| \neq 0$

the solution of equation are is given by $X = B\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} l_1 & M_1 \\ l_2 & M_2 \end{bmatrix}\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

$$\begin{bmatrix} \frac{dy}{dt} \\ y \end{bmatrix}_{2\times1} = \begin{bmatrix} l_1 c_1 e^{\lambda_1 t} + m_1 c_2 e^{\lambda_2 t} \\ l_2 c_1 e^{\lambda_1 t} + M_2 c_2 e^{\lambda_2 t} \end{bmatrix}_{2\times1}$$

the value of c_1 & c_2 will be determine from the initial condition.

$\boxed{\text{Note}}$ These Method can be only use when the eigen values are different and A is constant Matrix.

Q. Solve by Matrix Method. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0$ given $y(0) = 2$, $y'(0) = 2$.

$\Rightarrow$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0 \quad \text{--- } ①$$

put $\frac{dy}{dt} = x_1$, $y = x_2$

$$\frac{d^2y}{dt^2} = \frac{dx_1}{dt}, \quad \frac{dy}{dt} = \frac{dx_2}{dt}$$

$$① \Rightarrow \frac{dx}{dt} - 4x_1 + 3x_2 = 0$$

$$\frac{dy_1}{dt} = 4x_1 - 3x_2$$

$$\frac{dx_2}{dt} = x_1 + 0x_2$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{dx}{dt} = AX \quad \text{--- } ②$$

where $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$

characteristics eq. is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-\lambda) + 3 = 0$$

$$(\lambda^2 - 4) + 3 = 0$$

$d = 1, 3.$

$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ be eigen vector.

$\therefore [A - dI] Z = 0$

$$\begin{bmatrix} 4-d & -3 \\ 1 & -d \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = 0$$

$$\left.\begin{aligned} (4-d) Z_1 - 3Z_2 &= 0 \\ Z_1 - dZ_2 &= 0 \end{aligned}\right] \text{ — } (3)$$

for $d = 1$, in (3)

$3Z_1 - 3Z_2 = 0 \Rightarrow Z_1 - Z_2 = 0 \Rightarrow Z_1 - Z_2 = 0$

$\Rightarrow Z_1 - Z_2 = 0 \Rightarrow Z_1 = Z_2$

$\frac{Z_1}{1} = \frac{Z_2}{1}$

for $d = 1$, $Z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $d = 3$ in (3)

$Z_1 = 3Z_2 = 0$

$Z_1 - 3Z_2 = 0$

$Z_1 - 3Z_2 = 0$

$Z_1 = 3Z_1$

$\frac{Z_1}{3} = \frac{Z_2}{1}$

for $d = 3$, $Z = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$

$|B| = 1 - 3 = -2 \neq 0$

Solution of (2) is

$$X = B \begin{bmatrix} c_1 e^{d_1 t} \\ c_2 e^{d_2 t} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{t} \\ c_2 e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} \frac{dy}{dt} \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{t} + 3c_2 e^{3t} \\ c_1 e^{t} + c_2 e^{3t} \end{bmatrix}$$

$$\frac{dy}{dt} = c_1 e^{t} + 3c_2 e^{3t} \quad \text{— (4)}$$

$$y = c_1 e^{t} + c_2 e^{3t} \quad \text{— (5)}.$$

$t = 0$ & $\frac{dy}{dt} = 2$ in (4).

$2 = c_1 e^{0} + 3c_2 e^{0} - (e^{0} = 1)$

$2 = c_1 + 3c_2 \Rightarrow c_1 = 2 - 3c_2$.

& $t = 0$ & $y = 2$ in (5).

$2 = c_1 e^{0} + c_2 e^{0}$

$2 = c_1 + c_2$

$2 = 2 - 3c_2 + c_2$

$2 = 2 - 2c_2 \Rightarrow 2c_2 = 2 - 2 \quad \boxed{c_2 = 0}$

$\therefore c_1 = 2 - 3(0) \quad \therefore \boxed{c_1 \Rightarrow 2}$

$\therefore c_1 = 2$.

$\therefore y = 2e^{t}$.

Q $\frac{d^2y}{dt^2} - \frac{3dy}{dt} - 10y = 0$. given, $y(0) = 3$, $y'(0) = 15$

$\Rightarrow \frac{d^2y}{dt^2} - \frac{3dy}{dt} - 10y = 0 \quad \text{— (1)}$

~~given~~ put $\frac{dy}{dt} = x_1$, $y = x_2$

$\frac{d^2y}{dt^2} = \frac{dx_1}{dt}$, $\frac{dy}{dt} = \frac{dx_2}{dt}$

$$\therefore \text{①} \Rightarrow \frac{dx_1}{dt} - 3x_1 - 10x_2 = 0$$

$$\frac{dx_1}{dt} = 3x_1 + 10x_2$$

$$\frac{dx_2}{dt} = x_1 + 0x_2$$

$$\therefore \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where, $A = \begin{bmatrix} 3 & 10 \\ 1 & 0 \end{bmatrix}$

characteristic eq. is $[A - d I] = 0$.

$$\begin{bmatrix} 3-d & 10 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$(3-d)(-d) - 10 = 0$$

$$d^2 - 3d - 10 = 0$$

$$d = -2, 5$$

let $z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ be eigen vector.

$$\therefore [A - dI]z = 0$$

$$\begin{bmatrix} 3-d & 10 \\ 1 & -d \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (3-d)z_1 + 10z_2 &= 0. \\ Z_1 - d z_2 &= 0 \end{aligned}\right] — ③$$

$\therefore d = 2$ in ③

$$5Z_1 + 10Z_2 = 0$$

$$Z_1 + 2Z_2 = 0$$

$$Z_1 + 2Z_2 = 0.$$

$$Z_1 = 2Z_2$$

$$\frac{Z_1}{-2} = \frac{Z_2}{1}$$

for $d = -2$, $Z = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

for $d=5$ in (3)

$$-2Z_1+10Z_2=0$$

$$Z_1-5Z_2=0$$

$$Z_1-5Z_2=0$$

$$Z_1=5Z_2.$$

$$\frac{Z_1}{5}=\frac{Z_2}{1}$$

$$\therefore B=\begin{bmatrix}-2 & 5\\ 1 & 1\end{bmatrix}$$

$$|B|=-2-5$$

$$=-7\neq 0$$

$\therefore$ Sol^n. of (2) is

$$X=B\begin{bmatrix}c_1e^{d_1t}\\ c_2e^{d_2t}\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-2 & 5\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1e^{d_1t}\\ c_2e^{d_2t}\end{bmatrix}$$

$$\begin{bmatrix}\frac{dy}{dt}\\ y\end{bmatrix}\begin{bmatrix}-2c_1e^{-2t}+5c_2e^{5t}\\ c_1e^{-2t}+c_2e^{5t}\end{bmatrix}$$

$$\therefore \frac{dy}{dt}=-2c_1e^{-2t}+5c_2e^{5t} \quad (4).$$

$$y=c_1e^{-2t}+c_2e^{5t} \quad (5)$$

where, $t=0$, $y=3$, $\frac{dy}{dt}=15.$

$c_1=0$, $c_2=3$

$$y=3e^{5t}$$

* Method of expansion-

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$$

After reduction given second order differentia eq. we have $\frac{dx}{dt} = Ax$.

$$\frac{dx}{x} = A\,dt.$$

On Integrating,

$$\log x = At + \log c$$
$$\log x - \log c = At$$
$$\log \frac{x}{c} = At$$
$$\frac{x}{c} = e^{At}$$
$$x = e^{At} c \quad \text{——— (1)}$$

'c' will be determine by initial condition ie $y'(0)$

$$c = x(0) = \begin{bmatrix} y'(0) \\ y(0) \end{bmatrix}$$

and, $e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots\dots$

from eq. (1).

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} \right] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

[Note] this Method is used when characteristic equation are same i.e. λ equals.

Q. Solve by Matrix Method.

$\frac{dx_1}{dt} = x_1 + x_2, \quad \frac{dx_2}{dt} = x_2$

given, $x_1(0) = 1, \quad x_2(0) = 1.$

⇒ Matrix form is

We have,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{dx}{dt} = AX \quad \text{—— (1)}$$

where, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

characteristic eq. is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(1-\lambda) - 0 = 0$

$\lambda = 1, \ \lambda = 1$

Here, eigen value are same.

∴ We used expansion Method.

Solution of (1) is,

$$X = e^{At}.C \quad \text{—— (2)}$$

where,

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\dots$$

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and.

$$C = x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

from ② eq.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{—③}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

from ③ eq.

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & t \\ 0 & t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} t^2 & 2t^2 \\ 0 & t^2 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} t^3 & 3t^3 \\ 0 & t^3 \end{bmatrix} + \dots \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots & 0 + t + t^2 + \frac{t^3}{2} + \dots \\ 0 & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dots$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots & 0 + t + t^2 + \frac{t^3}{2} + \dots \\ 0 & 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots + 0 + t + t^2 + \frac{t^3}{3} + \dots \\ 0 + 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \end{bmatrix}$$

$$x_1 = 1 + 2t + \frac{3t^2}{2} + \frac{2}{3}t^3 + \dots$$

$$x_2 = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}$$

Q Solve by Matrix Method.

$$\frac{d^2y}{dt^2} + 4y = 0, \text{ given, } y = 8, \frac{dy}{dt} = 0 \text{ where}$$

$t = 0$.

⇒

$$\frac{d^2y}{dt} + 4y = 0 \quad \text{—— (1)}$$

put, $\frac{dy}{dt} = x_1$, $y = x_2$

$$\frac{d^2y}{dt^2} = \frac{dx_1}{dt}, \quad \frac{dy}{dt} = \frac{dx_2}{dt}$$

From eq. (1).

$$\frac{dx_1}{dt} + 4x_2 = 0$$

$$\frac{dx_1}{dt} = 0x_1 - 4x_2.$$

$$\frac{dx_2}{dt} = x_1 + 0x_2.$$

$$\therefore \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{dx}{dt} = Ax \quad \text{—— (2)}$$

where,

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

Characteristics eq. is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -4 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 4 = 0$$

$\lambda^2 = -4 = i^2 4$ $(i^2 = -1)$

$\lambda = 2i, -2i$ $(i = \sqrt{-1})$

let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ be eigen Vector.

$\therefore [A - \lambda I] z = 0$

$$\left.\begin{array}{l} -\lambda z_1 - 4z_2 = 0 \\ z_1 - \lambda z_2 = 0 \end{array}\right] \text{--- (3)}$$

for $\lambda = 2i$ in (3)

$-2iz_1 - 4z_2 = 0.$

$-2iz_1 + i4z_2 = 0$ $(-i - i^2)$

$-2i(z_1 - 2iz_2) = 0$

$z_1 - 2iz_2 = 0$

& $z_1 - 2iz_2 = 0$

$z_1 = 2iz_2$

$\frac{z_1}{2i} = \frac{z_2}{1}$

for $\lambda = 2i$, $z = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

for $\lambda = -2i$

$2iz_1 - 4z_2 = 0.$

$\Rightarrow 2iz_1 + i^2 4z_2 = 0$

$2i(z_1 + 2iz_2) = 0$

$z_1 + 2iz_2 = 0$

& $z_1 + 2iz_2 = 0$

$z_1 = -2iz_2$

$\frac{z_1}{-2i} = \frac{z_2}{1}$

for $\lambda = -2i$, $z = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

Model Matrix $B = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}$

$|B| = 2i + 2i = 4i \neq 0$

Solution of (2) is,

$$X = B \begin{bmatrix} c_1 e^{d_1 t} \\ c_2 e^{d_2 t} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4e^{2it} \\ c_2 e^{-2it} \end{bmatrix}$$

$$\begin{bmatrix} \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2ic_1 e^{2it} - 2ic_2 e^{-2it} \\ c_1 e^{2it} + c_2 e^{-2it} \end{bmatrix}$$

$$\frac{dy}{dt} = 2ic_1 e^{2it} - 2ic_2 e^{-2it} \quad \text{—— (4)}$$

$$y = c_1 e^{2it} + c_2 e^{-2it} \quad \text{—— (5)}$$

when $t = 0$, $\frac{dy}{dt} = 0$

$0 = 2ic_1 e^0 - 2ic_2 e^0 \quad (e^0 = 1)$

$0 = 2i(c_1 - c_2)$

$\Rightarrow c_1 - c_2 = 0$

$c_1 = c_2$

where $t = 0$, $y = 8$.

from (5),

$\Rightarrow 8 = c_1 e^0 + c_2 e^0$

$8 = c_1 + c_2$

$8 = c_2 + c_2$

$8 = 2c_2$

$\Rightarrow c_2 = 4$

$c_1 = 4$.

from (5),

$y = 4e^{2it} + e^{-2it}$

$$y = 4 \times 2 \left(\frac{e^{2it} + e^{-2it}}{2} \right)$$

$y = 8 \cos 2t.$

Q $\frac{d^2x}{dt^2} + x = 0$ given that $x = 0$ $\frac{dx}{dt} = 1$ when $t = 0$ by Matrix Method.

Ans - $x = \sin t.$

Q Solve by Matrix Method $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ given that $y(0) = 2$, $y'(0) = 5$

Ans - $y = e^{2x} + e^{3x}.$

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Unit 3 Successive Differentiation.

let $y = f(x)$
Differentiate with respect to x we get, $\frac{dy}{dx}$ and is called first differential Coefficient of y. with respect to x and also function of x. Again differentiate w.r.t x we get $\frac{d^2y}{dx^2}$ is called second differential Coefficient function of y. w.r.t x and is also function of x.

Similarly $\frac{d^2y}{dx^n}$ with respect to x. and so on.
this process of differentiation is called Successive differentiation

* Notation -

$\frac{dy}{dx}$ $\frac{d^2y}{dx^2}$ .... $\frac{d^ny}{dx^n}$ ....
y' y'' .... y'^n ....
y_1 y_2 .... y_n ....
Dy D^2y .... D^ny ....

* n^{th} derivative by leibnitz Rule.

let 'u' and 'v' are function of 'x' then n^{th} derivative of product uv is ~~Gatts~~ Calculate by leibnitz Rule.

$D^n(u \cdot v)$

$$D^n(\underset{I}{u}\,\underset{II}{v}) = (D^n u)v + {}^nC_1(D^{n-1}u)Dv + {}^nC_2(D^{n-2}u) + \ldots + u(D^n v)$$

$\boxed{\text{Note}}$

① ${}^nC_1 = n$, ${}^nC_2 = \frac{n(n-1)}{2!}$, ${}^nC_3 = \frac{n(n-1)(n-2)}{3!}$

② Select 2nd function i.e. 'v' which has finite derivative i.e. whose derivative eliminate.

③ While calculating n^{th} derivative avoid denominator term / function.

$y = \frac{\sin x}{x^2} \longrightarrow x^2 \cdot y = \sin x.$

④ Dependend Variable should be on left hand side and should be single.

Q If $y = a\cos(\log x) + b\sin(\log x)$ prove that $x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n$.

$\Rightarrow$ $y = a\cos(\log x) + b\sin(\log x)$ — ①

Diff. w.r. to x, on both side, we get,

$y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}$

$$xy_1 = -a\sin(\log x) + b\cos(\log x) \quad \text{— (2)}$$

again diff. w.r. to x, we get,

$$xy_2 + y_1 \cdot 1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}.$$

$$x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$$

$$x^2y_2 + xy_1 = -y \quad (\text{from (1)})$$

$$x^2y_2 + xy_1 + y = 0 \quad \text{— (3)}$$

Diff. n^{th} times by leibnitz Rule.

$$D^n[\underset{\text{II}}{x^2}\underset{\text{I}}{y_2}] + D^n[\underset{\text{II}}{x}\underset{\text{I}}{y_1}] + D^n(y) = 0$$

$$[(D^ny_2)x^2 + {}^nC_1(D^{n-1}y_2)(2x) + {}^nC_2(D^{n-2}y_2)2]$$

$$+ [(D^ny_1)x + {}^nC_1(D^{n-1}y_1)\cdot 1] + y_n = 0.$$

$$\underbrace{y_{n+2}}_{\text{common}} x^2 + n\, y_{n+1}\, 2x + \frac{n(n-1)}{\cancel{2}!}\, y_n \cancel{2}$$

$$+ [y_{n+1}\, x + n\, y_n] + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)\, x\, y_{n+1} + n^2y_n - n\cancel{y_n} + n\cancel{y_n}$$

$$+ y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2+1)\, y_n = 0$$

Q If $y = (\sin^{-1}x)^2$ $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

→ Diff. ① w.r.to x,

$$y_1 = 2\sin^{-1}x \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = 2\sin^{-1}x \quad — ②$$

again diff. w.r. to x,

$$\sqrt{1-x^2}\, y_2 + y_1 \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = 2\frac{1}{\sqrt{1-x^2}}$$

Multiplying by $\sqrt{1-x^2}$

$$(1-x^2)y_2 - xy_1 = 2 \quad — ③.$$

Diff. n^{th} times by using leibnitz rule,

$$D^n[\underset{II}{(1-x^2)}\, y_2] - D^n[\underset{II}{x}y_1] = 0.$$

$$[(D^ny_2)(1-x^2) + {}^nC_1(D^{n-1}y_2)(-2x) + {}^nC_2(D^{n-2}y_2)(-2)$$

$$-[(D^ny_1)x + {}^nC_1(D^{n-1}y_1)\cdot 1] = 0$$

$$y_{n+2}(1-x^2) + n\, y_{n+1}(-2x) + \frac{n(n-1)}{\cancel{2!}} y_n(-\cancel{2})$$

$$-y_{n+1}x - ny_n = 0.$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\, y_{n+1}$$
$$-n^2y_n + \cancel{ny_n} - \cancel{ny_n} = 0$$

Q If $y = e^{\tan^{-1}x}$ prove that $(1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1}$

$+ n(n+1)y_n = 0$

⇒

$y = e^{\tan^{-1}x}$ — (1)

Diff. w.r. to x,

$$y_1 = e^{\tan^{-1}x}\cdot\frac{1}{1+x^2}$$

$(1+x^2)y_1 = e^{\tan^{-1}x} = y$ (from (1)) — (2)

Diff. w.r. to x,

$(1+x^2)y_2 + y_1(2x) = y_1$ — (3)

Diff. n^{th} times by Leibnitz Rule.

$$D^n[(1+x^2)\underset{\text{II}}{y_2}] + 2D^n[x\underset{\text{II}}{y_1}] - D^n(y_1) = 0$$

$$[(D^n y_2)(1+x^2) + {}^nC_1(D^{n-1}y_2)(2x) + {}^nC_2(D^{n-2}y_2)(2)]$$

$$+ 2((D^n y_1)x + {}^nC_1(D^{n-1}y_1)\cdot 1] - y_{n+1} = 0.$$

$$\underbrace{y_{n+2}(1+x^2)}_{\text{common}} + n\,\underbrace{y_{n+1}(2n)}_{\text{common}} + \frac{n(n-1)}{\cancel{2!}}y_n\cancel{(2)}$$

$$\underbrace{+2y_{n+1}x}_{\text{common}} + 2n\,y_n \underbrace{- y_{n+1}}_{\text{common}} = 0$$

$$(1+x^2)\,y_{n+2} + 2n(n+1)\,y_{n+1} + n^2y_n - ny_n + 2ny_n$$
$$- y_{n+1} = 0$$

$$(1+x^2)y_{n+2} + [2x(n+1)-1]\,y_{n+1} + n^2y_n + ny_n = 0$$

$$(1+x^2)\,y_{n+2} + [2x(n+1)-1]\,y_{n+1} + n(n+1)\,y_n = 0$$

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Q. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ prove that

$x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0$

$\Rightarrow \quad \frac{y}{b} = \cos\left[n \log\left(\frac{x}{n}\right)\right]$

$y = b\cos\left[n(\log x - \log n)\right]$ —— ①

① Diff. w.r. to x,

$y_1 = -b\sin\left[n(\log x - \log n)\right] n\left(\frac{1}{x} - 0\right)$

$y_1 = -b\sin\left[n(\log x - \log n)\right]\left(\frac{n}{x} - 0\right)$

$xy_1 = -nb\sin\left[n(\log x - \log n)\right]$ —— ②

Again diff. w.r. to x,

$xy_2 + y_1 \cdot 1 = -nb\cos\left[n(\log x - \log n)\right]\frac{n}{x}$

$x^2y_2 + xy_1 = -n^2 b\cos\left[n(\log x - \log n)\right]$

Solve

$x^2y_2 + xy_1 = -n^2 y$ (from ①)

$x^2y_2 + xy_1 + n^2y = 0$

Q If $x = \sin\left(\frac{1}{a}\log y\right)$ prove that $(1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} - (n^2+a^2)\, y_n = 0$.

→. $\frac{1}{a}\log y = \sin^{-1}x$.

$\log y = a\sin^{-1}x$

$y = e^{a\sin^{-1}x}$.

Diff. w.r. to x.

$y_1 = e^{a\sin^{-1}x} \frac{a}{\sqrt{1-x^2}}$

solve ↓

$\sqrt{1-x^2}\, y_1 = ae^{a\sin^{-1}x}$.

$\sqrt{1-x^2}\, y_2 + \frac{y_1}{2\sqrt{1-x^2}}(-2x) = ae^{a\sin^{-1}x} \frac{a}{\sqrt{1-x^2}}$.

$(1-x^2)y_2 - xy_1 = a^2e^{a\sin^{-1}x}. = a^2y$

$(1-x^2)y_2 - xy_1 - a^2y = 0$.

Q. $y = \cos(a \sin^{-1} x)$ prove that

$(1-x^2) y_{n+2} - (2n+1) x\, y_{n+1} - (n^2 - a^2) y_n = 0$

* Expansion of function of one Variable - Taylor's

1) ~~Scalar~~ Series - expansion of function $f(x+h)$ in ascending power of 'h' where, h is small increment

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots\dots\dots$$

2) Taylor's Series - expansion of function $f(x)$ in ascending power of $(x-a)$ or about $x=a$ is given by

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots\dots\dots$$

3) Maclaurin's Series - expansion of function $f(x)$ in ascending power of x or about $x=0$ or about origin.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots$$

for e.

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots\dots$$

Q Apply Taylor's series.
$f\left(\frac{11}{10}\right)$, $f(x) = x^3 + 3x^2 + 15x - 10$.

$\rightarrow \frac{11}{10} = \frac{10+1}{10} = 1 + 0.1 = x + h$

We have, Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) \ldots \quad (1)$$

$$f\left(\frac{11}{10}\right) = f(1) + (0.1) f'(1) + \frac{(0.1)^2}{2} f''(1) + \frac{(0.1)^3}{6} f'''(1) + \ldots \quad (2)$$

Here, $f(x) = x^3 + 3x^2 + 15x - 10 \Rightarrow f(1) = 9$
$f'(x) = 3x^2 + 6x + 15 \Rightarrow f'(1) = 24$
$f''(x) = 6x + 6 \Rightarrow f''(1) = 12$
$f'''(x) = 6 \Rightarrow f'''(1) = 6$

(2) $\Rightarrow f\left(\frac{11}{10}\right) = 9 + (0.1) \times 24 + \frac{(0.1)^2}{2} \times 12 + \frac{(0.1)^3}{6} \times 6 + \ldots$
$f\left(\frac{11}{10}\right) \simeq 11.461$

Q Apply Taylor's series
$f\left(\frac{11}{10}\right)$, $f(x) = x^3 + 8x^2 + 15x - 24$

$\rightarrow \frac{11}{10} = \frac{10+1}{10} = 1 + 0.1 = x + h$

We have Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) \ldots \quad (1)$$

$$f\left(\frac{11}{10}\right) = f(1) + (0.1) f'(1) + \frac{(0.1)^2}{2} f''(1) + \frac{(0.1)^3}{6} f'''(1) + \ldots \quad (2)$$

Here, $f(x) = x^3 + 8x^2 + 15x - 24 \Rightarrow f(1) = 0$
$f'(x) = 3x^2 + 16x + 15 \Rightarrow f'(1) = 34$
$f''(x) = 6x + 16 \Rightarrow f''(1) = 22$
$f'''(x) = 6 \Rightarrow f'''(1) = 6$

(2) $\Rightarrow \simeq 3.511$

$F\left(\frac{11}{10}\right) = 0 + (0.1) \times 34 + \frac{(0.1)^2}{2} \times 22 + \frac{(0.1)^3}{6} \times 6$

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Q) Expand $2x^3+7x^2+x-7$ in power of $(x-2)$ by using taylor's theorem.
$(x+3)$
$(x-3)$
(any)

→ By taylor's series we have,

$$\cancel{f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) \ldots}$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots \quad (1)$$

Here, $f(x) = 2x^3+7x^2+x-7$, $a = -2$

$f(2) = 39$

$f'(x) = 6x^2+14x+1$, $f'(2) = 53$

$f''(x) = 12x+14 \Rightarrow f''(2) = 38.$

(1) ⇒

$$2x^3+7x^2+x-7 = 39 + (x-2)\times 53 + \frac{(x-2)^2}{2}\times 38 + \frac{(x-2)^3}{\cancel{6}} \times \cancel{12}^{2} + \ldots$$

$$= 2(x-2)^3 + 19(x-2)^2 + 53(x-2) + 39.$$

Q By using taylor's theorem tan 46° correct upto 4 decimal places.

→ $46° = 45° + 1° = 45 + \left(1 \times \frac{\pi}{180}\right)($

$46° = 45° + 0.0175$
↓ x ↓ h

$f(x+h) = \tan(x+h)$

$f(x) = \tan x.$

By Taylors theorem we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots \quad (1)$$

$$\tan 46^\circ = f(45^\circ) + (0.0175) f'(45) + \frac{(0.0175)^2}{2} f''(45) + \frac{(0.0175)^3}{6} f'''(45^\circ) + \ldots \quad (2)$$

$f(x) = \tan x \Rightarrow f(45^\circ) = \tan 45^\circ = \boxed{1}$

$f'(x) = \sec^2 x \Rightarrow f(45^\circ) = (\sec 45^\circ)^2 = (\sqrt{2})^2 = \boxed{2}$

$f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$

$f''(45) = 2.2.1 = \boxed{4}$

$f'''(x) = 2[\sec^2 x \sec^2 x + \tan x \, 2 \sec^2 x \tan x]$

$f'''(45^\circ) = 2[2.2 + 1.2.2.1] = \boxed{16}$

(2) $\Rightarrow$

$$\tan 46^\circ = 1 + (0.0175) \times 2 + \frac{(0.0175)^2}{2} \times 4 + \frac{(0.0175)^3}{6} \times 16$$

$\Rightarrow \tan 46^\circ \cong 1.0356.$

Q By using Taylors theorem find Cos 64° Correct upto 4 decimal places.

$\rightarrow$

Q Expand $\log_e(\cos x)$ in ascending power of x. upto and including the term of x^4. and Calculate $\log_{10}\left(\cos\frac{\pi}{12}\right)$ upto 3 decimal places.

→ By Maclaurins we have,

$$y = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) \ldots\ldots \quad (1)$$

Here, $y = \log_e(\cos x) \Rightarrow y(0) = 0$

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$$y_1 = \frac{1}{\cos x} \times (-\sin x) = -\tan x \Rightarrow y_1(0) = 0$$

$$y_2 = -\sec^2 x \Rightarrow y_2(0) = -1$$

$$y_3 = -2\sec^2 x \tan x \Rightarrow y_3(0) = 0$$

$$y_4 = -2\left[\sec^2 x \sec^2 x + \tan x \, 2\sec^2 x \tan x\right]$$
$$= -2[1 \cdot 1 + 0] = -2$$

eq. ① $\Rightarrow$

$$\log_e(\cos x) = 0 + 0 + \frac{x^2}{2} \times (-1) + 0 + \frac{x^4}{24} \times (-2)$$

$$\log_e \cos x = \frac{-x^2}{2} - \frac{x^4}{12} \ldots\ldots$$

$$\log_e \cos\left(\frac{\pi}{12}\right) = -\frac{(\pi/12)^2}{2} - \frac{(\pi/12)^4}{12}.$$

$$\log_e \cos\left(\frac{\pi}{12}\right) = -0.0349.$$

$$\boxed{\log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\quad}{2.3025}}$$

$$\log_{10} \cos(\pi/12) = \frac{-0.0347}{2.3027}$$

$$= -0.0151$$

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Q If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then show that $y_{n+2}(0) = (n+1)^2 y_n(0)$ hence prove that $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2x^3}{3} + \frac{8}{15}x^5 + \ldots$

→ $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ —— ①

$\sqrt{1-x^2}\, y = \sin^{-1}x$

differentiate with respect to x,

$\sqrt{1-x^2}\; y_1 + y \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1}{\sqrt{1-x^2}}$

Multiplying by $\sqrt{1-x^2}$

$(1-x^2)\, y_1 - xy = 1$ —— ②.

$(1-x^2)\, y_2 + y_1(-2x) - [x\, y_1 + y \cdot 1] = 0.$

$(1-x^2)\, y_2 - 2xy_1 - xy_1 - y = 0$

$(1-x^2)\, y_2 - 3xy_1 - y = 0$ —— ③.

Differentiate n^{th} times by leibnitz.

$D^n[(1-x^2)\underset{\mathrm{I}}{y_2}] - 3D^n[x\underset{\mathrm{I}}{y_1}] - D^n[y] = 0.$

$(D^n y_2)(1-x^2) + {}^nC_1 (D^{n-1}y_2)(-2x) + {}^nC_2 (D^{n-2}y_2)(-2)$

$-3[(D^n y_1)\cdot x + {}^nC_1 (D^{n-1}y_1) \cdot 1] - y_n = 0.$

$y_{n+2}(1-x^2) + n\, y_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2)$

$-3y_{n+1}x - 3n\, y_n - y_n = 0.$

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$(1-x^2) y_{n+2} - (2n+3) x\, y_{n+1} - n^2 y_n$.
$+ n y_n - 3 n y_n - y_n = 0$

$(1-x^2) y_{n+2} - (2n+3) x y_{n+1} - n^2 y_n - 2n y_n - y_n = 0$.

$(1-x^2) y_{n+2} - (2n+3) x y_{n+1} - (n^2+2n+1) y_n = 0$

$(1-x^2) y_{n+2} - (2n+3) x y_{n+1} - (n+1)^2 y_n = 0$ —— (4).

put $x = 0$ in all (4) equation

(1) $\Rightarrow$ $y(0) = 0$

(2) $\Rightarrow$ $y_1(0) = 1$

(3) $\Rightarrow$ $y_2(0) - y(0) = 0$ $\therefore y_2(0) = y(0) = 0$

(4) $\Rightarrow$ $y_{n+2}(0) - (n+1)^2 y_n(0) = 0$

$y_{n+2}(0) = (n+1)^2 y_n(0)$ —— (5)

put $n=1$ in (5)

$y_3(0) = 4\, y_1(0) = 4\times 1 = 4$

put $n=2$ in (5)

$y_4(0) = 9\, y_2(0) = 0$

put $n = 3$ in (5)

$y_5(0) = 16 y_3(0) = 16 \times 4 = 64$

By Maclaurin Series,

$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0)$

$+ \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \ldots$

$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = 0 + x \times 1 + 0 + \frac{x^3}{6} \times 4 + 0 + \frac{x^5}{120} \times 64 + \ldots$

$= x + \frac{2}{3} x^3 + \frac{8}{15} x^5 + \ldots$

Unit 4 Partial Differentiation.

let z is function of x & y.

$$z = f(x, y)$$

differentiate z partially with respect to x and keeping y constant. we get,

$$\frac{\partial z}{\partial x}.$$

differentiate z partially with respect to y and keeping x constant. we get,

$$\frac{\partial z}{\partial y}$$

Q (1) Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

$$u = (x^2+y^2+z^2)^{-1/2}$$

→ given that $u = (x^2+y^2+z^2)^{-1/2}$ —— (1).

Diff. p. w. r. to. x we get,

$$\frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x+0+0)$$

$$\frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$$

again diff. p. w. r. to x we get,

$$\frac{\partial^2 u}{\partial x} = -\left[x\left(\frac{-3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-5/2}(\cancel{2}x) + (x^2+y^2+z^2)^{-3/2} \cdot 1\right]$$

$$\frac{\partial^2 y}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{—— (2)}$$

② Similarly,
$$\frac{\partial^2 u}{\partial y^2} = 3y^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad —③$$
$$\frac{\partial^2 u}{\partial z^2} = 3z^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad —④.$$
Adding ② ③ ④ we get,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2}$$
$$-3(x^2+y^2+z^2)^{-3/2}$$
$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$$
$$= RHS.$$

Q $x^x y^y z^z = c$
S.T. at $x = y = z$ $\frac{\partial^2 z}{\partial x \partial y} = -(x \log_e x)^{-1}$

→ Taking log on b.s.
$x \log x + y \log y + z \log z = \log c$
diff. p. w.r. to x. on b.s.
$$x \cdot \frac{1}{x} + \log x \times 1 + 0 + z \frac{1}{z} \frac{\partial z}{\partial x} + \log z \frac{\partial z}{\partial x} = 0.$$
$$1 + \log x + \frac{\partial z}{\partial x} + \log z \frac{\partial z}{\partial x} = 0$$
$$(1+\log x) + \frac{\partial z}{\partial x}(1+\log z) = 0$$
$$\frac{\partial z}{\partial x}(1+\log z) = -(1+\log x)$$
$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \quad —①$$

- $\log e = 1$
- $\log 1 = 0$
- $\log 0 = -\infty$
- $\log \infty = \infty$

- $\log x^n = n \log x$
- $\log AB = \log A + \log B$

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③ Similarly $\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}$ —— ②.

Diff ① p. w. r to y, we get.

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log x)\left[\frac{-1}{(1+\log z)^2}\frac{1}{z}\frac{\partial z}{\partial y}\right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = (1+\log x)\left\{\frac{1}{(1+\log z)^2}\frac{1}{z}\left[\frac{-(1+\log y)}{(1+\log z)}\right]\right\} \quad \text{by ②}$$

at $x = y = z$, we get

$$\frac{\partial^2 z}{\partial x \partial y} = -\cancel{(1+\log x)}\frac{1}{(1+\log x)^2}\cdot\frac{1}{x}\frac{\cancel{(1+\log x)}}{\cancel{(1+\log x)}}$$

$$= -\left[\frac{1}{x(1+\log x)}\right] = -\left[x(1+\log x)\right]^{-1}$$

$$= -\left[x(\log e + \log x)\right]^{-1} = -\left[x \log ex\right]^{-1}$$

5M

④ If $u = \log(\tan x + \tan y + \tan z)$ P.T. $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

$u = f(xyz)$

→ ~~Differentiate~~ $u = \log(\tan x + \tan y + \tan z)$ —— ①

Differentiate ① w. r. to x.

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x + 0 + 0)$$

Multiply by $\sin 2x$ on b.s.

$$\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2 \sin x \cos x \frac{1}{\cos^2 x}}{\tan x + \tan y + \tan z} = \frac{2\tan x}{\tan x + \tan y + \tan z} \quad \text{——} (2)$$

|ly,

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z} \quad \text{——} (3)$$

$$\sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z} \quad \text{——} (4)$$

Adding (2) (3) (4), we get.

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{2(\cancel{\tan x + \tan y + \tan z})}{\cancel{\tan x + \tan y + \tan z}}$$

$= 2 = RHS.$

③ Q. If, $z(x+y) = x^2+y^2$ S.T. $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$

$$= 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

→ $z(x+y) = x^2+y^2$ $\cancel{\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2}$ — ①

Here, $z = f(x, y)$

$$z = \frac{x^2+y^2}{x+y}$$

Differentiate p. w. to x.

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x+0) - (x^2+y^2)(1+0)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$ — ②

Diff. p. w. r. to y

$$\frac{\partial z}{\partial y} = \frac{(x+y)(0+2y) - (x^2+y^2)(0+1)}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}$$ — ③

LHS =

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{(y^2+2xy-x^2)}{(x+y)^2}\right]$$

$$= \left[\frac{x^2+\cancel{2xy}-y^2-y^2-\cancel{2xy}+x^2}{(x+y)^2}\right]^2$$

(6)

$$= \left[\frac{2x^2 - 2y^2}{(x+y)^2}\right]^2 = (2)^2\left[\frac{x^2-y^2}{(x+y)^2}\right]^2$$

$$= 4\left[\frac{(x-y)\cancel{(x+y)}}{(x+y)^{\cancel{2}}}\right]^2 = 4\,\frac{(x-y)^2}{(x+y)^2} \quad \text{—— (4)}$$

RHS =

$$4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

$$= 4\left[1 - \frac{(x^2+2xy-y^2)}{(x+y)^2} - \frac{(y^2+2xy-x^2)}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{\cancel{x^2} + \cancel{2xy} + y^2 - \cancel{x^2} - 2xy + \cancel{y^2} - \cancel{y^2} - 2xy + x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2 - 2xy + y^2}{(x+y)^2}\right] = 4\left[\frac{(x-y)^2}{(x+y)^2}\right] \quad \text{—— (5)}$$

From (4) (5) LHS = RHS.

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Q. Find Value of n that the equation

$v = r^n [3\cos^2\theta - 1]$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = 0$$

$\Rightarrow$ Ans $\Rightarrow n = 2,\ n = -3$

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Homogeneous function's and Euler's theorem on homogenous functions -

Let 'z' is function of x and y is said to be homogenous function of degree 'n' if sum of indices (power's) of x and y in every term is same and is equal to 'n'

let $z = f(x,y)$ degree 'n'

eg - $Z = (x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_n y^n$

Where, $a_0, a_1, \ldots a_n$ are Constant, is a homogenous function of degree 'n'

$$Z = x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \frac{y^2}{x^2} + \ldots + a_n \frac{y^n}{x^n} \right]$$

$$Z = x^n f\left(\frac{y}{x}\right) \quad \underline{or}$$

$$Z = y^n \left[a_0 \frac{x^n}{y^n} + a_1 \frac{x^{n-1}}{y^{n-1}} + \ldots + a_n \right]$$

$$Z = y^n f\left(\frac{x}{y}\right).$$

Euler's Theorem on Homogenous function -

If 'f' is homogenous function in x and y of degree 'n' then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$

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Q If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 x}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$$

→ Where, 'u' is not Homogenous function

$$\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left[1+\frac{y^3}{x^3}\right]}{x\left[1-y/x\right]}$$

$$z = \tan u = x^2 f\left(\frac{y}{x}\right)$$

Here, $z = \tan$ 'u' is Homogenous function in x & y of degree $n = 2$ then by Eulers theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x\sec^2 u \frac{\partial u}{\partial x} + y\sec^2 u \frac{\partial u}{\partial y} = 2\tan u.$$

Dividing by $\sec^2 u$ on b.s.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u}\cos^2 u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \quad \text{—— (1)}$$

Diff. (1) p w.r. to x on b.s.

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}1 + y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y}\times 0 = 2\cos 2u \frac{\partial u}{\partial x}$$

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Multiplying by x on b.s. (8)

$$x^2 \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u\left[x\frac{\partial u}{\partial x}\right] \quad \text{--- (2)}$$

Diff (1) p.w.r. to y on b.s.

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\times 0 + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\cdot 1 = 2\cos 2u\left(\frac{\partial u}{\partial y}\right)$$

Multiply by y on b.s.

$$xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2\cos 2u\left(y\frac{\partial u}{\partial y}\right) \quad \text{--- (3)}$$

Adding (2) and (3), we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$
$$= 2\cos 2u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$\frac{x^2\partial^2 u}{\partial x} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 2u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] - \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 2u\,[\sin 2u] - \sin 2u$$

--- (from (1))

$$= \sin 4u - \sin 2u.$$
$$= 2\cos (4+2)/_2\, u \sin (4-2)/2u.$$
$$= 2\cos 3u \cdot \sin u.$$

Q If $u = \log\left(\dfrac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$ find,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

→ 'u' is not homogenous function.

$$e^u = \frac{x^2+y^2}{x^{1/2}+y^{1/2}} = \frac{x^2\left[1+y^2/x^2\right]}{x^{1/2}\left[1+y^{1/2}/x^{1/2}\right]}$$

$$z = e^u = x^2 x^{-1/2} f\left(\frac{y}{x}\right) = x^{3/2} f\left(\frac{y}{x}\right)$$

Here, $z = e^u$ is homogenous function in x and y of degree $n = 3/2$ then by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = \frac{3}{2}e^u$$

$\cancel{x e^u \frac{\partial y}{\partial x}}$

$$x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = \frac{3}{2}e^u$$

Dividing e^u, ~~on b.s.~~

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2} \quad \text{---} \quad (1)$$

Diff (1) p. w. r. to x on b.s.

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.1 + y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y}\cdot 0 = 0$$

Multiply by x on b. S.

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = 0 \quad \text{--- (2)}.$$

Diff. (1) p. w. r. to y on b. S.

$$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} \times 0 + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \times 1 = 0.$$

Multiply by y on b. S.

$$xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = 0 \quad \text{--- (3)}.$$

Adding (2) and (3), we get,

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right]$$

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right] = \frac{-3}{2}$$

$\longleftarrow$ (from (1))

Q If $u = \operatorname{Cosec}^{-1}\left[\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right]^{1/2}$ then find,

$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x^2 y} + y^2 \frac{\partial^2 u}{\partial y^2}$

→ u is not Homogenous function

$\therefore \operatorname{Cosec} u = \left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$

$\operatorname{Cosec} u = \frac{(x^{1/2})^{1/2}}{(x^{1/3})^{1/2}}\left[\frac{1+y^{1/2}/x^{1/2}}{1+y^{1/3}/x^{1/3}}\right]^{1/2}$

$\operatorname{Cosec} u = x^{\frac{1}{4}} x^{-\frac{1}{6}} f\left(\frac{y}{x}\right) = x^{\frac{1}{12}} f\left(\frac{y}{x}\right)$

Here, Cosec u is Homogenous function in x and y of degree $n = \frac{1}{12}$

By Euler's theorem

$x \frac{\partial}{\partial x}(\operatorname{Cosec} u) + y \frac{\partial}{\partial y}(\operatorname{Cosec} u) = \frac{1}{12} \operatorname{Cosec} u$

$x(-\operatorname{Cosec} u \operatorname{Cot} u)\frac{\partial u}{\partial x} + y(-\operatorname{Cosec} u \operatorname{Cot} u)\frac{\partial u}{\partial y} = \frac{\operatorname{Cosec} u}{12}$

Dividing $(-\operatorname{Cosec} u \cot u)$ on B.S.

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{-1}{12 \operatorname{Cot} u} = \frac{-1}{12} \tan$ — ①

Diff. Op. w. r. to x

$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \times 1 + y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} \times 0 = \frac{-1}{12} \sec$

$\left(\frac{\partial u}{\partial x}\right)$

Multiply by x on B.S.

$$\frac{x^2\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = -\frac{\sec^2 u}{12}\left(x\frac{\partial u}{\partial x}\right) \quad \text{—(2)}$$

Diff. (1) p. w. r. to y.

$$x\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial x}\times 0 + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial y}{\partial y}\times 1$$
$$= -\frac{\sec^2 y}{12}\left(\frac{\partial u}{\partial y}\right)$$

Multiply by y, on B.S.

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 y}{\partial y^2} + y\frac{\partial u}{\partial y}$$
$$= \frac{-\sec^2 y}{12}\left(y\frac{\partial u}{\partial y}\right) \quad \text{—(3)}$$

Adding (2) & (3)

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$
$$= -\frac{\sec^2 u}{12}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sec^2 u}{12}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$
$$- \left(x\frac{\partial y}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= -\frac{\sec^2 u}{12}\left[-\frac{\tan u}{12}\right] - \left[-\frac{\tan u}{12}\right]$$

[by (1)]

$$= \frac{\tan u}{12}\left[\sec^2 u + 12\right]$$

$$= \frac{\tan u}{144}\left[1 + \tan^2 u + 12\right]$$

$$= \tan u \left[\tan^2 \cancel{u} + 13\right]$$

Q If $u = f(v)$ being homogenous function of n^{th} degree of x & y then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nv\,f'(v)$

hence, $u = \log v$ P.T. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$

→ given that v is Homogenouous function x & y of degree n then By ~~Tylor~~ Euler's theorem.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv \quad \text{— ①}$$

$$u = f(v)$$

$$\frac{\partial u}{\partial x} = f'(v)\frac{\partial v}{\partial x}$$

$$x\frac{\partial u}{\partial x} = f'(v)\left(x\frac{\partial v}{\partial x}\right) \quad \text{— ②}$$

& $\frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y}$

$$y\frac{\partial u}{\partial y} = f'(v)\left(y\frac{\partial v}{\partial y}\right) \quad \text{— ③}$$

Add ② & ③

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = f'(v)\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right]$$

$= f'(v)\,nv$ = RHS (by ①) — ④

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$u = \log v$

$f(v) = \log v \quad (\because u = f(v))$

$f'(v) = \frac{1}{v}$

(ii) $\Rightarrow$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{\cancel{v}} n\cancel{v} = n.$$

Hence proved.

Q. If $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$ prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

→ Let $z = u + v$

Where, $u = x^n f_1\left(\frac{y}{x}\right)$

& $v = y^{-n} f_2\left(\frac{x}{y}\right)$

where, u & v are the homogenous function in x & y. of degree n and $-n$ respectively then by euler's theorem

$$\left.\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= nu \\ x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} &= -nv \end{aligned}\right] - (1)$$

Solve

Now, $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(u+v)$

$x\frac{\partial z}{\partial x}$

Unit 5. Vectors.

* **Scalar** - A physical quantity is not related specific direction in a space is called scalar.
  ex- Mass, length, Volume.

* **Vector** - A physical quantity which have magnetude and related to specific direction in a space is called vector.
  ex- velocity, force.

* **Modulas** - Modulas of vector
$$|\bar{a}| = a.$$

* **Unit** - let $\bar{a}$ be vector and a is its magnitude then unit vector of $\bar{a}$ in the direction of $\bar{a}$ is denoted by
$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

* **positive vector** - let $p(x, y, z)$ position vector with reference to origine 'o' and given by '$\overline{OP}$'

$$P(x, y, z) \quad O = (0,0,0)$$
$$\overline{OP} = (x-0)\hat{i} + (y-0)\hat{j} + (z-0)\hat{k}$$
$$\overline{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

let $A = (x_1\ y_1\ z_1) \quad B = (x_2\ y_2\ z_2)$
$$\overline{AB} = B-A = (x_2 - x_1) + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

Topic (I) :- Vector differential Opearator

$$\nabla(\text{del}) = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\Delta = \frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}$$

* Gradient - It is used to measured the length of defined the measured of slope of straight line is scalar. If $\phi(x,y,z)$ ↑ and it continously differentiable then

$$\nabla\phi = \left(\frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}\right)\phi$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Note: $\nabla\phi$ is vector Normal.

* Directional Derivative - If $\phi(x,y,z)=c$ is scalar point function the directional derivative of ϕ in direction of $\bar{a}$ is given by

$$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|} = \nabla\phi \cdot \hat{a}$$

- Magnitude of Max direction derivative is given by $|\nabla\phi|$

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- Surfaces are orthogonal then their vector.
- ϕ_1 and ϕ_2 are two surface of orthogonal then $\Delta\phi_1$ & $\Delta\phi_2$ is equal to zero
- The angle between two surfaces is the angle between their vector normal at point of interaction.

★ $\Delta\phi_1 \cdot \Delta\phi_2 = |\Delta\phi_1||\Delta\phi_2|\cos\theta$

★ $\theta = \cos^{-1}\left[\frac{\Delta\phi_1 \cdot \Delta\phi_2}{|\Delta\phi_1| \cdot |\Delta\phi_2|}\right]$

- where ϕ is angle between ϕ_1 & ϕ_2

Q Find the directional derivative of $\phi(x,y,z) = x^2 - 2y^2 + 4z^2$ at point $(1,1,-1)$ in direction of $(2\hat{i}+\hat{j}-\hat{k})$ in what direction will the directional derivative is maximum & magnitude?

→ given that

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 - 2y^2 - 4z^2)$$

diff w.r.to x, y, z

$(2x - 0 + 0)\hat{i} + (0 - 4y + 0)\hat{j} + (0 - 0 + 8z)\hat{k}$

At point $(1,1,-1)$

put $x = 1$, $y = 1$, $z = (-1)$

$2\hat{i} - 4\hat{j} - 8\hat{k}$

$\therefore \nabla\phi = 2\hat{i} - 4\hat{j} - 8\hat{k}$ —— ①

$\therefore \bar{a} = 2\hat{i} + \hat{j} - \hat{k}$ —— (given)

unit vector $= \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{4+1+1}} = \frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$

$$\hat{a} = \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}} \quad \text{—— ②}$$

2 Direction derivitive of given by the formula ϕ in direction of $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$

$$\nabla\phi \cdot \hat{a} = (2\hat{i} - 4\hat{j} - 8\hat{k}) \frac{(2\hat{i} + \hat{j} + \hat{k})}{\sqrt{6}} = \frac{-4-4+8}{\sqrt{6}}$$

$$\nabla\phi \cdot \hat{a} = \frac{8}{\sqrt{6}}$$

We know that Direction Derivative Max in direction $\nabla\phi = 2\hat{i} - 4\hat{j} - 8\hat{k}$

and mangnitude $|\nabla\phi| = \sqrt{4+16+64}$

$$= \sqrt{84}$$

Q Find the direction derivative of $\phi = 4e^{2x-y+z}$ at point $(1,1,-1)$ in direction towards at point $(-3, 5, 6)$:

→ $$\nabla\phi = \left(\frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}\right)\phi$$

$$\nabla\phi = \left(\frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}\right)(4e^{2x-y+z})$$

$$\nabla\phi = 4e^{2x-y+z}(2-0+0) + 4e^{2x-y+z}(0-1+0) + 4e^{2x-y+z}(0-0+1)\hat{k}$$

At point $(1,1,-1)$

$$\nabla\phi = 8e^{0}\hat{i} - 4e^{0}\hat{j} + 4e^{0}\hat{k}$$

$$\nabla\phi = 8\hat{i} - 4\hat{j} + 4\hat{k} \quad \text{—— (1)}$$

Vector joining point $P(1,1,-1)$ $(-3,5,6)$ towards Q is

$$\overline{PQ} = \overline{Q} - \overline{P}$$

$$= (-3-1)\hat{i} + (5-1)\hat{j} + (6+1)\hat{k}$$

$$= -4\hat{i} + 4\hat{j} + 7\hat{k}$$

$$\overline{PQ} = -4\hat{i} + 4\hat{j} + 7\hat{k}$$

$\therefore$ unit Vector $\hat{n} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{\sqrt{16+16+49}}$

$= \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{\sqrt{81}}$

$= \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{9}$ —— ②

Required directional derivative of given ϕ in direction at point Q.

$\therefore \Delta Q \cdot \hat{n} = (8\hat{i} - 4\hat{j} + 4\hat{k}) \left(\frac{-4\hat{i}+4\hat{j}+7\hat{k}}{9}\right)$

$= \frac{-32+28-16}{9}$

$= \frac{-20}{9}.$

Q If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ Show that –
1) grad $r = \frac{\bar{r}}{r}$ 2) grad $\frac{1}{(r)} = \frac{-\bar{r}}{r^3}$ 3) $\nabla r^n = n r^{n-2} \bar{r}$

→

$\bar{r} = x\hat{i} + y\hat{i} + z\hat{i}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2 + y^2 + z^2$

Diff p. w. r. to x.

$\cancel{2}r \frac{\partial r}{\partial x} = \cancel{2}x + 0 + 0$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$ —— ①

1) grad $(r) = \nabla(r) = \frac{\partial r}{\partial r}\hat{i} + \frac{\partial r}{\partial y}\hat{i} + \frac{\partial r}{\partial z}\hat{k}$

$\nabla(r) = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}.$

$$= \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla(r) = \frac{\bar{r}}{r}$$

2) $\text{grad}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}$

$$= -\frac{1}{r^2}\frac{\partial r}{\partial x}\hat{i} - \frac{1}{r^2}\frac{\partial r}{\partial y}\hat{j} - \frac{1}{r^2}\frac{\partial r}{\partial z}\hat{k}$$

$$= -\frac{1}{r^2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) \text{ — by ①}$$

$$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{-\bar{r}}{r^3}$$

3) $\nabla(r^n) = \frac{\partial r^n}{\partial x}\hat{i} + \frac{\partial}{\partial y}r^n\hat{j} + \frac{\partial}{\partial z}r^n\hat{k}$

$$= \frac{-1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\bar{r}}{r^3}$$

$$= n r^{n-1}\frac{\partial r}{\partial x}\hat{i} + n r^{n-1}\frac{\partial r}{\partial y}\hat{j} + n r^{n-1}\frac{\partial r}{\partial z}\hat{k}$$

$$= n r^{n-1}\left[\frac{x\hat{i}}{r} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \frac{n r^{n-1}}{r}[x\hat{i} + y\hat{j} + z\hat{k}] = n r^{n-2}\bar{r}$$

Q Find value of a, b, c so that d.d of $\phi = axy^2 + byz + cz^2x^3$ at a point $(1, 2, -1)$ has max. magnitude 64 in direction parallel to z-axis.

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5 → $\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(axy^2 + byz + cz^2x^3)$

$= (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$

at $(1, 2, -1)$

$\nabla\phi = (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$ — ①

We know that d.d is Max. in direction of $\nabla\phi$ then

But given that d.d. is Max. in direction parallel to z-axis i.e. along $\hat{k}$

∴ Coefficient of $\hat{i}$ & $\hat{j}$ in $\nabla\phi$ becomes '0' & coefficient of $\hat{k}$ is +ve.

∴ from ①

$4a + 3c = 0$ — ②

$4a - b = 0$ — ③

$2b - 2c > 0$ — ④

① → $\nabla\phi = (2b - 2c)\hat{k}$

$|\nabla\phi| = \sqrt{(2b - 2c)^2}$

$|\nabla\phi| = 2b - 2c$

But given that d.d has Max. Magnitude 64

∴ $|\nabla\phi| = 64$

$2b - 2c = 64$

$b - c = 32$ — ⑤

Add eq. ③ & ⑤

$4a - \cancel{b} = 0$

$-c + \cancel{b} = 0$

$4a - c = 32$

$a = 6$

$b = 24$

$c = -8$

Q Find the directional derivative of $\frac{1}{r}$ in direction of $\bar{r}$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\rightarrow$ $\frac{1}{r} = \nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}$

$= \frac{-1}{r^2}\frac{\partial u}{\partial x}\hat{i} - \frac{1}{r^2}\frac{\partial u}{\partial y}\hat{j} - \frac{1}{r^2}\frac{\partial u}{\partial z}\hat{k}$

$= \frac{-1}{r^2}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$

$= \frac{-1}{r}(x\hat{i} + y\hat{j} + z\hat{k})$

$= \frac{-\bar{r}}{r^3}$

$\hat{r} = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{r}}{r}$

$\nabla\left|\frac{1}{r}\right|\hat{r} = -\frac{\bar{r}}{r^3}\cdot\frac{\bar{r}}{r}$

$= -\frac{r^2}{r^4}$

$= \frac{-1}{r^2}$

Q Find constant m & n such that surfaces $mx^2 - 2nyz = (m+4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, -1, 2)

$\rightarrow$ $\phi_1 = mx^2 - 2nyz - (m+4)x = 0$

$\phi_2 = 4x^2y + z^3 - 4 = 0.$

ϕ_1 & ϕ_2 are orthogonal at point (1, -1, 2)

$\therefore$ point (1, -1, 2) lies on the both the surfaces

$m.1 - 2n(-1)2 = (m+4)1$

$\cancel{m} + 4n = \cancel{m} + 4$

$n = 1$

$\hat{n} = \frac{\nabla}{|\nabla|}$ ⇒ $\frac{\partial \phi}{\partial n} = \nabla\phi \cdot \hat{n}$

2 Surfaces are orthogonal if their vector normal are Perpendicular.

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$ — ①.

$\nabla\phi_1 = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \cdot [mx^2 - 2nyz - (m+4)x]$

$\nabla\phi_1 = [2mx - (m+4)]\hat{i} + [-2nz]\hat{j} + [-2ny]\hat{k}$

$\nabla\phi_2 = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})\ (4x^2y + z^3 - 4)$

$\nabla\phi_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

At pt $(1, -1, 2)$

$\nabla\phi_1 = (2m - (m+4)]\hat{i} - 4n\hat{j} + 2n\hat{k}$.

$\nabla\phi_2 = -8\hat{i} + 4\hat{j} + 12\hat{k}$

① ⇒ $[(m-4)\hat{i} - 4\hat{j} + 2\hat{k}] \cdot [-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$.

$-8(m-4) - 16 + 24 = 0$

$-8m + 32 + 8 = 0$

$+8m = +40$

$m = 5$.

New

Q Find directional derivative of function $\phi\ xy^2 + yz^3$ at point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at point $(-1, 2, 1)$

Ans – $\frac{15}{\sqrt{17}}$

Q Find angle between surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$

→ $\phi_1 = x^2 + y^2 + z^2 - 9 = 0$

$\phi_2 = x^2 + y^2 - z - 3 = 0$

$$\phi = \cos^{-1}\left[\frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}\right] \quad \text{—} ①$$

$$\nabla\phi_1 = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2-9)$$

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

at pt $(2,-1,2)$

$$\cancel{\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}}$$

$$\nabla\phi_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\nabla\phi_2 = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$|\nabla\phi_1| = \sqrt{(4)^2+(2)^2+(4)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$|\nabla\phi_2| = \sqrt{(4)^2+(2)^2+(1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

$$\theta = \cos^{-1}\left[\frac{(4\hat{i}+2\hat{j}+4\hat{k})(4\hat{i}-2\hat{j}-\hat{k})}{6\sqrt{21}}\right]$$

$$\theta = \cos^{-1}\left(\frac{\overset{8}{\cancel{16}}+\cancel{4}-\cancel{4}}{3\,\cancel{6}\sqrt{21}}\right)$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

* Divergance of Vector pt. function-

Divergance of vector point function is denoted and defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \vec{V}$$

* Solenoidal Vector -

A vector $\vec{V}$ is said to be solenoidal vector if diversion (div $\vec{V} = 0$)

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* Curl of Vector point function-

Curl of Vector point function $\bar{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is a denoted and define as Curl $\bar{V} = \nabla \times \bar{V} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

} fix
} only $\bar{V}$ change.

Curl off Vector point function $\bar{V} = 0$ than $\bar{V}$ is said to be Irrotational Vector and vice versa

If $\bar{V}$ is a Irrotational Vector than $\bar{V}$ can always be empress as a
$\bar{V} = \nabla \phi$ (where ϕ is scalar Potential).

* **Conservative field-** A force $\bar{F}$ is said to Conservative if work done by it by moving its point of application from point a to point b is only depend on and is a independent upon path joining A and B.

* Condition for $\bar{F}$ Conservative -

- $\bar{F}$ is Conservative than Curl of $\bar{F}$ is $= 0$ (Curl $\bar{F} = 0$) that is $\bar{F}$ is Irrotational
- If $\bar{F}$ is conservative than their exist scalar potential ϕ such that $\bar{F} = \nabla\phi$

Q Show that $\overline{F} = (y^2 \cos x + z^3)\hat{i} + (2y \sin x - y)\hat{j} + (3xz^2 + 2)\hat{k}$ be conservative. Vector field

ii) And find the function ϕ such that $\overline{F} = \nabla \phi$

iii) Also find work done moving the particle from $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$

→

We know that to show $\overline{F}$ is a conservative we have to show that curl $\overline{F} = 0$

$$\text{Curl } \overline{F} = \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y\cos x + z) & (2y \sin x - y) & (3xz^2 + z) \end{vmatrix}$$

$$= \hat{i}[0-0] - \hat{j}[3z^2 - 3z^2] + \hat{k}[2y\cos x - 2y \cos x]$$
$$= 0$$

$\therefore$ F is conservative

$\therefore$ We can express $\overline{F} = \nabla \phi$

$$(y^2 \cos x + z^3)\hat{i} + (2y \sin x - y)\hat{j} + (3xz^2 + 2)\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

Comparing coefficient of i, j, k of both side

$$\frac{\partial \phi}{\partial x} = y^2 \cos x + z^3$$

$$\frac{\partial \phi}{\partial y} = 2y \sin x - y$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 + 2$$

On a integrating partially with respect to x, y, z respectively.

$\int \frac{\partial \phi}{\partial x} = y^2 \cos x + z^3$ | $\phi = y^2 \sin x + z^2 x$

$\int \frac{\partial \phi}{\partial y} = 2y \sin x - y$ | $\phi = \cancel{2}\frac{y^2}{\cancel{2}} \sin x - yy$

$\int \frac{\partial \phi}{\partial z} = 3xz^2 + 2.$ | $\phi = \cancel{3}\frac{xz^3}{\cancel{3}} + 2x.$

Adding (3) functions.

(y, z)

(x, z)

(x, y).

$\phi_1 (y, z)$ = term in ϕ independent of x

$= -4y + 2z$

$\phi_2 (x, z)$ = term in ϕ independent of y

$= xz^3 + 2z$

$\phi_3 (x, y)$ = term in ϕ independent of z

$= y^2 \sin x - 4y$

Req. Scalar ϕ potentials to

$\phi = y^2 \sin x + z^3 x - 4y + 2z$

Workdone $= \phi_B (\pi/2, 1, 2) - \phi_A (0, 1, -1)$

$= \left[1 \cdot \sin \frac{\pi}{2} + \frac{\pi}{2} \times 8 - 4(-1) + 4\right] - \left[0 + 0 - 4 \cdot 1 + 2(-1)\right]$

$= [1 + 4\pi + 4 + 4] + 6$

$= 4\pi + 15$

Q Find value of 'a' so that vector field $\bar{A} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is an irrotational & hence find scalar potential?

$\Rightarrow$ $\bar{A}$ is irrotational.

Curl of $\bar{A} = \nabla \times \bar{A} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$= \hat{i}[0-0] - \hat{j}[(1-a)z^2 + 3z^2]\hat{j} - \hat{k}$

$[2(a-2)x - ax] = 0\hat{i} + 0\hat{j} + 0\hat{k}$

Comparing coefficient of i, j, k on both side

$-[(1-a)z^2 + 3z^2] = 0$

$-(1-a)z^2 - 3z^2 = 0$

$(-1 + a - 3)z^3 = 0$

$a - 4 = 0$

$a = 4$

Since $\bar{A}$ is irrotational that $\bar{A}$ can be express a $\bar{A} = \nabla\phi$

$(4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Compare coefficient of i, j, k of both side we get $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 4xy - z^3$ —— ①

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$$\frac{\partial \phi}{\partial y} = 2x^2 \quad \text{— (2)}.$$

$$\frac{\partial \phi}{\partial z} = -3xz^2 \quad \text{— (3)}.$$

on integrating paratially eq. (1). (2) (3).

$$\phi = 4\frac{x^2}{2}y - z^3 x + \phi_1(y, z)$$

$$\phi = 2x^2 y + \phi_2(x, z).$$

$$\phi = -\cancel{3} x \frac{z^3}{\cancel{3}} + \phi_3(x, y)$$

$\phi_1 = (y, z) =$ term in ϕ independent of $x = 0$
$\phi_2 = (x, z) =$ term in ϕ independent of $y = -xz^2$
$\phi_3 = (x, y) =$ term in ϕ independent of $z = 2x^2y$

$$\phi = 2x^2y - xz^3.$$

* Vector differentiation -

$$\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$\bar{r}$ be the position vector.
t be scalar parameter.

velocity, $\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

acc.^{n}, $\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt}$

$$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}.$$

* Formula -

The tangentatial component of acceleration $a_t = \bar{a} \cdot t^{\wedge}$

where, $t^{\wedge}$ is unit tangent vector to the curve

The normal component of acceleration is given by $a_n = |t^{\wedge} \times \bar{a}|$

* $\bar{v} = \bar{v} = \frac{d\bar{r}}{dt}$

$\bar{v}$ is vector target to curve
$\bar{r}$ is vector

* $t^{\wedge} = \frac{\bar{v}}{|\bar{v}|}$

$t^{\wedge}$ is unit target vector to curve.

5M Imp ① A particle move along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ find compo of its velocity & acceleration at $t = 1$, in direction of $i^{\wedge} - 3j^{\wedge} + 2k^{\wedge}$

⇒ let $\bar{r}$ be the position vector at point P to curve

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$P(x, y, z)$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{r} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$

Velocity $\bar{v} = \frac{d\bar{r}}{dt} = 4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$.

acc. $\bar{a} = \frac{d\bar{v}}{dt} = 4\hat{i} + 2\hat{j} + 0\hat{k}$

at $t = 1$

$\bar{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{a} = 4\hat{i} + 2\hat{j} + 0\hat{k}$.

unit vector of direction $i - 3j + 2k$ is $\hat{n}$

$$\hat{n} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1+9+4}} = \frac{\hat{i} - 3\hat{j} - 2\hat{k}}{\sqrt{14}}$$

Component of velocity in direction of $i - 3j + 2k$ is given by $\bar{v} \cdot \hat{n} = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{(\hat{i} - 3\hat{j} - 2\hat{k})}{\sqrt{14}}$

$$= \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

Component of acceleration in direction of $i - 3j + 2k$ is given by

$$\bar{a} \cdot \hat{n} = (4\hat{i} + 2\hat{j} + 0\hat{k}) \cdot \frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}}$$

$$\bar{a} \cdot \hat{n} = \frac{4 - 6 + 0}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$$

Q Particle move along curve $\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$ where t is the time. find the Magnititude of tangentalial and normal Component its acceleration. at $t = 2$

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→ velocity $\bar{V} = (3t^2-4)\hat{i} + (2t+4)\hat{j} + (16t-9t^2)\hat{k}$

acceleration, $\bar{a} = \frac{d\bar{V}}{dt} = 6t\,\hat{i} + 2\hat{j} + (16-18t)\hat{k}$

at $t = 2$.

$$\bar{V} = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$$\bar{a} = 12\hat{i} + 2\hat{j} - 20\hat{k}$$

$$\hat{t} = \frac{\bar{V}}{|\bar{V}|} = \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{64+64+16}} = \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{144}}$$

$$= \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{12}$$

tangential component of acceleration is given by $a_t = \bar{a}\cdot\hat{t} = (12\hat{i} + 2\hat{j} - 20\hat{k}).\frac{(8\hat{i} + 8\hat{j} - 4\hat{k})}{12}$

$$= \frac{96+16+80}{12}$$

$$= \frac{192}{12} = 16$$

Normal component of acceleration is given by $a_n = |\hat{t} \times \bar{a}|$

$$\hat{t} \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cancel{8/12}\ 2/3 & \cancel{8/12}\ 2/3 & \cancel{-4/12}\ -1/3 \\ 12 & 2 & -20 \end{vmatrix}$$

$$= \frac{1}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ 12 & 2 & -20 \end{vmatrix}$$

$= \frac{1}{3}\{\hat{i}[-40+2] - \hat{j}[-40+12]$

$+\hat{k}[4-24]\} = \frac{1}{3}\{\hat{i}[(-38)\hat{i} + 28\hat{j} + 20\hat{k}]$

$$= \frac{1}{3}\sqrt{(-38)^2 + (28)^2 + (28)^2} = 2\sqrt{73}$$

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Q. A particle moves along the curve $x = t^2+1$, $y = t^2$, $z = 2t+5$ where t is time. Find component of velocity & acceleration at $t = 1$ in direction of $\hat{i}+\hat{j}+3k$

⇒ Component of velocity $\frac{10}{\sqrt{11}}$ Component of acc. $\frac{4}{\sqrt{11}}$

Q. Find the tangential & normal component of acceleration at any time t of a particle whose position at time t is given by $x = e^t \times \cos t$, $y = e^t \times \sin t$.

⇒ tangential component of acc. $\sqrt{2}\, e^t$
Normal component of acceleration $\sqrt{2}\, e^t$.

* Total Differentiation and chain Rule -

Let 'z' is function of x & y where x is function of t & y is also function of t.

$$z = f(x, y)$$
$$x = \phi(t), \quad y = \phi(t).$$

then, total differentiation of z w.r. to t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

* Chain Rule -

1) $z \to x, y \to t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

2)
$$\begin{tikzcd} z \arrow[r] \arrow[r, bend left, "RHS"] \arrow[rr, bend right, "LHS"'] & t \arrow[r] \arrow[r, bend left, "RHS"] & x, y \end{tikzcd}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t} \frac{\partial t}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial y}$$

3) $z \to x, y \to r, t$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

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Q If $u = f(x-y, y-z, z-x)$ p.t. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

→ Here, $t_1 = x-y$, $t_2 = y-z$, $t_3 = z-x$

$u \rightarrow t_1, t_2, t_3 \rightarrow x, y, z$

By using chain rule.

$u \rightarrow t_1, t_2, t_3 \rightarrow x, y, z.$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial x}\\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial y}\\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial z} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial z} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial z}\end{aligned}\right\} \quad \text{--- (1)}$$

$t_1 = x-y$, $t_2 = y-z$, $t_3 = z-x$.

$\frac{\partial t_1}{\partial x} = 1-0$, $\frac{\partial t_2}{\partial x} = 0$, $\frac{\partial t_3}{\partial x} = 0-1$

$\frac{\partial t_1}{\partial y} = -1$, $\frac{\partial t_2}{\partial y} = 1$, $\frac{\partial t_3}{\partial y} = 0$.

$\frac{\partial t_1}{\partial z} = 0$, $\frac{\partial t_2}{\partial z} = -1$, $\frac{\partial t_3}{\partial z} = 1$

(1) ⇒

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\times 1 + \frac{\partial u}{\partial t_2}\times 0 + \frac{\partial u}{\partial t_3}\times(-1)$ $\quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3}$ — (2)

&

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}(-1) + \frac{\partial u}{\partial t_2}\times 1 + \frac{\partial u}{\partial t_3}\times 0$ $\quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2}$ — (3).

&

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_1}\times 0 + \frac{\partial u}{\partial t_2}(-1) + \frac{\partial u}{\partial t_3}$ $+ \quad \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial t_3}$ — (4)

Adding eq. (2) & (3) & (4).

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3} - \frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} - \frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial t_3} = 0$$

Q. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

$\Rightarrow$ Here, $t_1 = \frac{x}{y}$, $t_2 = \frac{y}{z}$, $t_3 = \frac{z}{x}$.

$$u = f(t_1, t_2, t_3).$$

$$u \longrightarrow t_1, t_2, t_3 \rightarrow x, y, z.$$

By chain Rule,

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial x} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial y} \\
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial z} + \frac{\partial u}{\partial t_2} + \frac{\partial t_2}{\partial z} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial z}
\end{aligned}\right\} \text{--- (1)}$$

$t_1 = \frac{x}{y}$, $t_2 = \frac{y}{z}$, $t_3 = \frac{z}{x}$.

$\frac{\partial t_1}{\partial x} = \frac{1}{y}$, $\frac{\partial t_2}{\partial x} = 0$, $\frac{\partial t_3}{\partial x} = \frac{-z}{x^2}$

$\frac{\partial t_2}{\partial y} = \frac{-x}{y^2}$, $\frac{\partial t_2}{\partial y} = \frac{1}{z}$, $\frac{\partial t_3}{\partial y} = 0$

$\frac{\partial t_3}{\partial z} = 0$, $\frac{\partial t_3}{\partial z} = \frac{-y}{z^2}$, $\frac{\partial t_3}{\partial z} = \frac{1}{x}$

① ⇒

$$\frac{du}{dx} = \frac{\partial u}{\partial t_1} \times \frac{1}{y} + 0 + \frac{\partial u}{\partial t_3}\left(\frac{-z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial t_1} - \frac{z}{x}\frac{\partial u}{\partial t_3} \quad \text{— ②}$$

&

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial t_2}\left(\frac{1}{z}\right) + 0$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial t_1} + \frac{y}{z}\frac{\partial u}{\partial t_2} \quad \text{— ③}$$

&

$$\frac{du}{\partial z} = 0 + \frac{\partial u}{\partial t_2}\cdot\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial t_3}\left(\frac{1}{x}\right)$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial t_2} + \frac{z}{x}\frac{\partial u}{\partial t_3} \quad \text{— ③}$$

Adding eq. ② ③ ④.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial t_1} - \frac{z}{x}\frac{\partial u}{\partial t_3} - \frac{x}{y}\frac{\partial u}{\partial t_1} + \frac{y}{z}\frac{\partial u}{\partial t_2} - \frac{y}{z}\frac{\partial u}{\partial t_2} + \frac{z}{x}\frac{\partial u}{\partial t_3} = 0.$$

Imp

Q If $\phi = f(x, y, z)$ & $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$
s.t. $u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w} = x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}$

→ $\phi \longrightarrow x, y, z \longrightarrow u, v, w$

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial\phi}{\partial v} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$\frac{\partial\phi}{\partial w} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial w} \quad \text{— (1)}$$

$x = \sqrt{v}\sqrt{w}$, $y = \sqrt{w}\sqrt{u}$, $z = \sqrt{u}\sqrt{v}$

$\frac{\partial x}{\partial u} = 0$, $\frac{\partial y}{\partial u} = \frac{\sqrt{w}}{2\sqrt{u}}$, $\frac{\partial z}{\partial u} = \frac{\sqrt{v}}{2\sqrt{u}}$

$\frac{\partial x}{\partial v} = \frac{\sqrt{w}}{2\sqrt{v}}$, $\frac{\partial y}{\partial v} = 0$, $\frac{\partial y}{\partial v} = \frac{\sqrt{u}}{2\sqrt{v}}$

$\frac{\partial x}{\partial w} = \frac{\sqrt{v}}{2\sqrt{w}}$, $\frac{\partial y}{\partial w} = \frac{\sqrt{v}}{2\sqrt{w}}$, $\frac{\partial z}{\partial w} = 0$

(1) $\Rightarrow$

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\cdot 0 + \frac{\partial\phi}{\partial y}\cdot\frac{\sqrt{w}}{2\sqrt{u}} + \frac{\partial\phi}{\partial z}\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$$\frac{u\partial\phi}{\partial x} = \frac{\sqrt{uw}}{2}\frac{\partial\phi}{\partial y} + \frac{\sqrt{uv}}{2}\frac{\partial\phi}{\partial z} \quad \text{— (2)}$$

$$\& \quad \frac{u\partial\phi}{\partial x} = \frac{y\partial\phi}{2\partial y} + \frac{z}{2}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial v} = \frac{\partial\phi}{\partial x}\times\frac{\sqrt{w}}{2\sqrt{v}} + 0 + \frac{\partial\phi}{\partial z}\cdot\frac{\sqrt{u}}{2\sqrt{v}}$$

$$\frac{v\partial\phi}{\partial v} = \frac{\sqrt{vw}}{2}\frac{\partial\phi}{\partial x} + \frac{\sqrt{uv}}{2}\frac{\partial\phi}{\partial z} \quad \text{— (3)}$$

$$\frac{v\partial\phi}{\partial v} = \frac{x}{2}\frac{\partial\phi}{\partial v} + \frac{z}{2}\frac{\partial\phi}{\partial z}$$

$$\& \quad \frac{\partial\phi}{\partial w} = \frac{\partial\phi}{\partial x}\cdot\frac{\sqrt{v}}{2\sqrt{w}} + \frac{\partial\phi}{\partial y}\cdot\frac{\sqrt{u}}{2\sqrt{w}} + 0$$

$$\frac{w\partial\phi}{\partial w} = \frac{\sqrt{vw}}{2}\frac{\partial\phi}{\partial x} + \frac{\sqrt{uw}}{2}\frac{\partial\phi}{\partial y} \quad \text{— (4)}$$

$$\frac{w\partial\phi}{\partial w} = \frac{x}{2}\frac{\partial\phi}{\partial x} + \frac{y}{2}\frac{\partial\phi}{\partial x} + \frac{y}{2}\frac{\partial\phi}{\partial y}$$

Adding ② ③ ④ on both side.

H/p

Q $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$ then p.t.

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$

Q $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ find value of $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$

Ans = 0

Soln

$f_1 = \frac{y-x}{xy}$, $f_2 = \frac{z-x}{xz}$

$f_1 = \frac{1}{x} - \frac{1}{y}$

$f_2 = \frac{1}{x} - \frac{1}{z}$

* Jacobian- If 'u' and 'v' are function of 'x' & 'y' then Jacobian of u and v with respect to xy is denoted and defined as

$$J\left[\frac{u,v}{x,y}\right] = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

* Properties -

1] $u,v \rightarrow x,y \rightarrow r,s$

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,s)}$$

2] If $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J' = \frac{\partial(x,y)}{\partial(u,v)}$

then, $JJ' = 1$

3] If u, v, w, r function of independent variable xyz are not independent

then, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Q If $x = u(1-v)$, $y = uv$ pt. $JJ' = 1$, where $J = \frac{\partial(x,y)}{\partial(u,v)}$

→ By defination of Jacobian we have,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{--- } ①$$

$x = u - uv$, $y = uv$

$\frac{\partial x}{\partial u} = 1 - v$ $\quad \frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = -u.$ $\quad \frac{\partial y}{\partial v} = u$

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(1) ⇒

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u(1-v) + uv$

$= u - uv + uv = u$ —— (2)

$x+y = u - \cancel{uv} + \cancel{uv}$

$\Rightarrow u = x+y$

$v = \frac{y}{u} = \frac{y}{x+y}$

By defination $J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ —— (3)

$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1$

$\frac{\partial v}{\partial x} = y\left[\frac{-1}{(x+y)^2} \cdot 1\right] = \frac{-y}{(x+y)^2}$

$\frac{\partial v}{\partial x} = \frac{(x+y) \cdot 1 - y(1)}{(x+y)^2} = \frac{x}{(x+y)^2}$

(3) ⇒

$$J' = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$

$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}.$

Consider LHS $JJ' = u \cdot \frac{1}{u} = 1.$

Q If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ find,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

→ Here, $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

$$J' = \frac{\partial(u,v,w)}{\partial(v,y,z)}$$

By defination Jacobian

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad \text{——} (1)$$

$$u = \frac{yz}{x}$$

$$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial u}{\partial z} = \frac{y}{x}$$

$$v = \frac{xz}{y}$$

$$\frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial v}{\partial y} = \frac{-xz}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{x}{y}$$

$$w = \frac{xy}{z}$$

$$\frac{\partial w}{\partial x} = \frac{y}{z}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}, \quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$$

$$(1) \rightarrow \quad J' = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-x^2}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

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$$= \frac{1}{x^2} \times \frac{1}{y^2} \times \frac{1}{z^2} \begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$R_1 \rightarrow R_1 + R_2$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} 0 & 0 & 2xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\left[0 - 0 + 2xy\,(xyz^2 + xyz^2)\right]$$

$$J' = \frac{1}{x^2y^2z^2}\left[2xy\,(2xyz^2)\right] = \frac{4x^2y^2z^2}{x^2y^2z^2} = 4$$

By property we have $JJ' = 1$

$$\therefore\ J = \frac{1}{J'}$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{4}$$

Q If $u = x+y+z$ $u^2v = y+z$ $u^3w = z$ find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ find

~~J of u^3w uvw wx~~

J' i.e $\frac{\partial(x, yz)}{\partial(u, v, w)}$

$\Rightarrow J' = 5$ $J = u^{-5}$

* If it is given that $\frac{\partial(u\,v\,w)}{\partial(x\,y\,z)}$ & $u\,v\,w$ are not independent ~~of~~ one another are connected by relation $u, v, w = 0$ then they are

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5m Q If $u = 3x+2y-z$, $v = x-2y+z$, $w = x(x+2y-z)$ show that they are functionality related & find the relation.

→ By Definition

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad \text{—— (1)}$$

$u = 3x + 2y - z$

$$\frac{\partial u}{\partial x} = 3, \quad \frac{\partial u}{\partial y} = 2, \quad \frac{\partial u}{\partial z} = -1$$

$v = x - 2y + z$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -2, \quad \frac{\partial v}{\partial z} = 1$$

$w = x^2 - 2xy - xz$

$$\frac{\partial w}{\partial x} = 2x + 2y - z, \quad \frac{\partial w}{\partial y} = 2x, \quad \frac{\partial w}{\partial z} = -x$$

(1) ⇒

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 4 & 0 & 0 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix}$$

$$= 4(2x - 2x) - 0 + 0$$

$$= 0$$

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$\therefore$ u v w are functionally related.

$$u+v = 3x + \cancel{2y} - \cancel{z} + x - \cancel{2y} + \cancel{z} = 4x$$

$$u - v = 3x + 2y - z = x + 2y - z$$

$$= 2x + 4y - 2z = 2(x + 2y - z)$$

$$(u+v)(u-v) = 4x \times 2\,(x + 2y - z)$$

$$u^2 - v^2 = 8x\,(x + 2y - z) = 8w$$

Q Test whether $u = \dfrac{x+y}{x-y}$, $v = \dfrac{xy}{(x-y)^2}$ are functionally dependent if so state the dependent.

→ By defination. We have,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

$$u^2 = \frac{(x+y)^2}{(x-y)^2} = \frac{x^2+2xy+y^2}{(x-y)^2}$$

$$u^2-1 = \frac{x^2+2xy+y^2}{(x-y)^2} - 1$$

$$= \frac{(x^2+2xy+y^2)-(x-y)^2}{(x-y)^2}$$

$$= \frac{\cancel{x^2}+2xy+\cancel{y^2}-(\cancel{x^2}-2xy+\cancel{y^2})}{(x-y)^2} = \frac{4xy}{(x-y)^2}$$

$$\Rightarrow u^2-1 = 4v$$

Q If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ check whether u and v are if so find relation.

→ By Defination.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0.$$

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$\tan v = \tan\left[\tan^{-1}x + \tan^{-1}y\right]$

$= \dfrac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)}$

$\tan v = \dfrac{x+y}{1-xy} = u.$

Maxima & Minima (Two Variable)

Procedure :-

1) $f(x,y)$

find $\dfrac{\partial f}{\partial x} = 0$ $\dfrac{\partial f}{\partial y} = 0$

equate the value with 0 and then solve it. let roots of equation $x = a$ & $y = b$ i.e (a,b) the (a,b) is called as stationary value.

2) Find $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$ and check

(a,b)

3) 1) $rt - s^2 > 0$, $r < 0$ then $f(x,y)$ is ~~Min~~ Max. at (a,b)

2) $rt^2 - s^2 > 0$, $r > 0$ then $f(x,y)$ is min at (a,b)

3) $rt - s^2 < 0$ then f is neither max nor min then point (a,b) is called saddle point.

4) $rt - s^2 = 0$ then we cannot say anything about Max or Min.

Q Find the points on the surface $z^2 = xy + 1$ nearest to the origin.

→ given surface is

$$z^2 = xy + 1 \quad \text{— (1)}$$

let point $p(x\ y\ z)$ be any point on the give surface

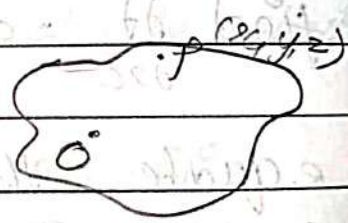
∴ its distance from origin given by

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$f(x, y, z) = d^2 = x^2 + y^2 + z^2$$

$$f(x, y) = x^2 + y^2 + xy + 1 \quad (\text{by } (1))$$

$$\frac{\partial f}{\partial x} = 2x + y, \quad \frac{\partial f}{\partial y} = 2y + x$$



for Max. OR Min.

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$2x + y = 0, \quad x + 2y = 0$$

$$x = 0, \quad y = 0$$

∴ $(0,0)$ is stationary value.

$$r = \frac{\partial^2 f}{\partial x^2} = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 1, \quad t = \frac{\partial^2 f}{\partial y^2} = 2$$

At $(0,0)$ $r = 2$, $s = 1$, $t = 2$

$$rt - s^2 = (2)(2) - 1 = 3 > 0$$

$$r = 2 > 0$$

∴ $f(x,y)$ is Min at $(0,0)$

$$z^2 = xy + 1 = 0 + 1 = 1$$

$$z = \pm 1$$

∴ $(0, 0, \pm 1)$ is nearest point on the origin on given surface.

Q. Divide 24 into 3 parts. Such that continue product of first square of second cube of third is Maximum

→ let x, y, z be three parts of 24

$\therefore \; x+y+z = 24$

$z = 24 - x - y$

$f(x,y,z) = z x^2 y^3$

$f(x,y) = (24-x-y)\, x^2 y^3$

$f(x,y) = 24x^2y^3 - x^3y^3 - x^2y^4$

$\frac{\partial f}{\partial x} = 48xy^3 - 3x^2y^3 - 2xy^4$

$\frac{\partial f}{\partial y} = 72x^2y^2 - 3x^2y^2 - 4x^2y^3$

for Max. or Min.

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0.$

$48xy^3 - 3x^2y^3 - 2xy^4 = 0$

$xy^3(48-3x-2y) = 0$

$48-3x-2y = 0 \Rightarrow 3x+2y = 48$ — (2)

& $72x^2y^2 - 3x^3y^2 - 4x^2y^3 = 0$

$x^2y^2(72-3x-4y) = 0$

$72-3x-4y = 0 \Rightarrow 3x+4y = 72$ — (3)

$x = 8, \; y = 12$

$(8, 12)$

$r = \frac{\partial^2 f}{\partial x^2} = 48y^3 - 6xy^3 - 2y^4$

$s = \frac{\partial^2 f}{\partial x \partial y} = 144xy^2 - 9x^2y^2 - 8xy^3$

$t = \frac{\partial^2 f}{\partial y^2} = 144x^2y - 6x^3y - 12x^2y^2$

At $(8,12)$

$r = -41472$

$s = -27648$

$t = -36864$

$rt - s^2 = 764411904 > 0$

$r < 0$

$\therefore f(x,y)$ is Max at $(8,12)$ from ①

$z = 24 - 8 - 12 = 4$

$\therefore$ Req. parts of 24 are $4, 8, 12$