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Unit 2 : Multiple Integral & Their Application

Double integral

Double integral over region and R may be evaluated by 2 successive integration.

Method 1 :

If R describes

$$y_1 \le y \le y_2 \Rightarrow f(x_1) \le y \le f(x_2)$$
$$a \le x \le b$$

$$\iint f(x,y)\,dx\,dy = \int_a^b \left[\int_{y=y_1}^{y_2} f(x,y)\,dy \right] dx$$

Here, we 1st integrate w.r.t y from $y=y_1$ to $y=y_2$ Keeping x constant than we integrate w.r.t from $x=a$ to $x=b$

Method 2 :

If R describes

$$x_1 \le x \le x_2 \Rightarrow f(y_1) \le x \le f(y_2)$$
$$c \le y \le d$$

$$\iint f(x,y)\,dx\,dy = \int_{y=c}^{d} \left[\int_{x=x_1}^{x_2} f(x,y)\,dx \right] dy$$

Here we 1st integrate w.r.t x from $x=x_1$ then to $x=x_2$ Keeping y constant integrate from $y=c$ to $y=d$

Note: If limit are the function of x then 1st integrate w.r.t y

- If limit are the function of y then 1st integrate w.r.t x
- If all 4 limits are constant than f^n $f(x,y)$ can be integrated w.r.t any f^n

Q1 $\int_0^1 \int_0^x e^{y/x}\, dx\, dy$

$$\int_{x=0}^{1} \left[\int_{y=0}^{x} e^{y/x}\, dy \right] dx$$

$$\int_{x=0}^{1} \left[\frac{e^{y/x}}{1/x} \right]_0^x dx$$

$$\int_{x=0}^{1} xe - x\, dx$$

$$(e-1) \int_{x=0}^{1} x\, dx$$

$$(e-1) \left[\frac{x^2}{2} \right]_0^1$$

$$\frac{(e-1)}{2} [1-0] = \frac{(e-1)}{2}$$

Q2 Evaluate $\int_0^1 \int_0^y xye^{-x^2}\, dy\, dx$

$$\int_0^1 \left[\int_{x=0}^{y} xye^{-x^2}\, dx\right] dy$$

$$\int_0^1 y\left[\int_{x=0}^{y} xe^{-x^2}\, dx\right] dy$$

$$\int e^{f(x)} f'(x)\, dx = e^{f(x)}$$

$$\int_0^1 \frac{y}{(-2)}\left[\int_{x=0}^{y} -2x\, e^{-x^2}\, dx\right] dy$$

$$\int_0^1 \frac{y}{(-2)}\left[e^{-x^2}\right]_0^y dy$$

$$\int_0^1 \frac{y}{(-2)}\left[e^{-y^2} - 1\right] dy$$

$$-\frac{1}{2}\int_{y=0}^{1} y\left(e^{-y^2}-1\right) dy$$

$$-\frac{1}{2}\int_0^1 ye^{-y^2} + \int_0^1 y\, dy$$

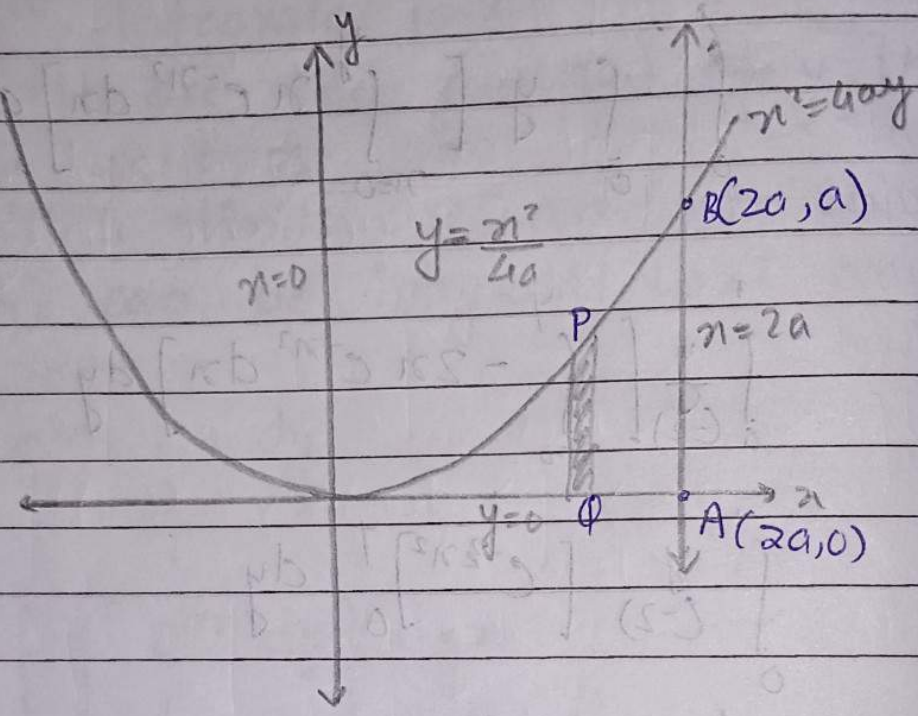
$$-\frac{1}{2}\,\frac{1}{(-2)}\int_{y=0}^{1} -2ye^{-y^2} dy + \frac{1}{2}\left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{4}\left[e^{-y^2}\right]_0^1 + \frac{1}{4}\left[1-0\right] = \frac{1}{4}\left[e^{-1}-e^{0}\right] + \frac{1}{4} = \frac{1}{4e}$$

Q3 Evalute $\iint_A xy\,dx\,dy$

where A is a domain bounded by x-axis, ordinate $x=2a$, and parabola $x^2=4ay$

Soln:



$x=2a \qquad x^2=4ay$

$(2a)^2=4ay$

$y=a$

Here line $x=2a$ and parabola intersect at point B (2a, a).

Here region of intergration is AOB as shown in fig.

Here we 1st integrate w.r.t y or strip PQ from $y=0$ to $y=x^2/4a$ then we integrate w.r.t x from $x=0$ to $x=2a$.

$$\int_{x=0}^{2a} x \left[\int_{y=0}^{x^2/4a} y\,dy \right] dx$$

$$\int_{x=0}^{2a} x \left[\frac{y^2}{2} \right]_0^{x^2/4a} dx$$

$$\frac{1}{2} \int_{x=0}^{2a} x \left[\frac{x^4}{16a^2} \right] dx$$

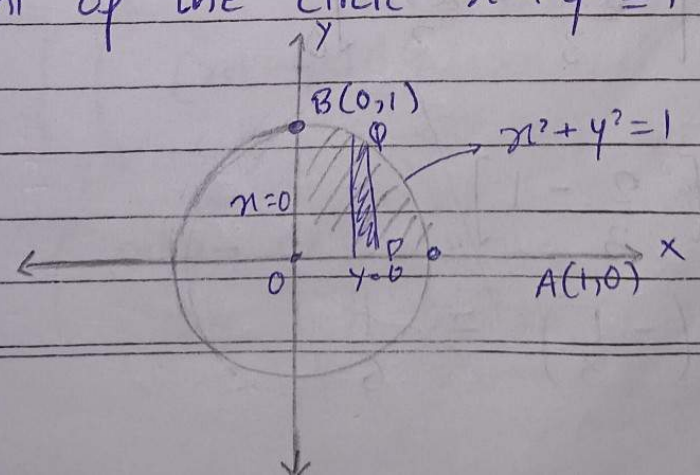
$$\frac{1}{32a^2} \int_{x=0}^{2a} x^5\,dx$$

$$\frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a}$$

$$\frac{1}{\cancel{32}\,a^{\cancel{2}}} \; \frac{\cancel{64}^{\,2}\,a^{\cancel{6}\,4} - 0}{\cancel{6}\;3}$$

$$= \frac{a^4}{3}$$

Q4 Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$



$y=\sqrt{1-x^2}$

Here region of integration is OAB. Here we 1st integrate w.r.t y from $y=0$ to $y=\sqrt{1-x^2}$ then we integrate w.r.t $x=0$ to $x=1$.

$$\int_0^1 x\left[\int_{y=0}^{\sqrt{1-x^2}} \frac{y}{\sqrt{1-y^2}}\,dy\right]dx \qquad \left[\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)}\right]$$

$$\int_0^1 x\left[\int_{y=0}^{\sqrt{1-x^2}} \frac{-2}{-2}\frac{y}{\sqrt{1-y^2}}\,dy\right]dx$$

$$\int_0^1 x\,\frac{1}{-2}\left[2\sqrt{1-y^2}\right]_0^{\sqrt{1-x^2}}$$

$$-\frac{1}{2}\int_0^1 x\left[2\sqrt{1-1-x^2} - 2\sqrt{1-0}\right]$$

$$-\frac{1}{2}\int_0^1 x\left[2x-2\right]$$

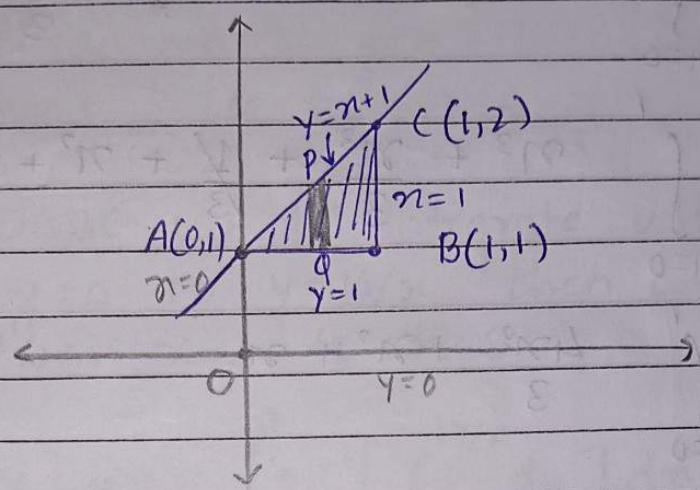
$$-\frac{1}{2}\int_0^1 2x^2-2x$$

$$-\frac{1}{2}\left[\frac{2x^3}{3}-\frac{2x^2}{2}\right]_0^1$$

$$-\frac{1}{2}\left[\frac{2}{3}-1\right]$$

$$-\frac{1}{2}\left(-\frac{1}{3}\right) = \frac{1}{6}$$

Q5 Evaluate $\iint x^2+y^2 \, dx\,dy$ over the triangle the quad coordinate of whose vertices are $(0,1), (1,1), (1,2)$



Eqn of line : $y - y_1 = m(x - x_1)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 1 = \frac{2-1}{1-0}(x - 0)$$

$$y = \cancel{1/1}(x) + 1$$

$$y = x + 1$$

Here we 1st integrate w.r.t y from $y = 1$ to $y = x+1$ the we integrate w.r.t. x from $x = 0$ to $x = 1$

$$\int_{x=0}^{1} \left[\int_{y=1}^{x+1} (x^2+y^2)\,dy \right] dx$$

$$\int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_1^{x+1}$$

$$\int_{x=0}^{1} \left[x^2(x+1) + \frac{(x+1)^3}{3} - x^2 \mp \frac{1}{3} \right]$$

$$\int_{x=0}^{1} x^3 + \cancel{x^2} + x^3 + \cancel{1} + \frac{3x^2 + 3x}{3} - \cancel{x^2} \mp \frac{1}{3}$$

$$\int_{x=0}^{1} x^3 + \frac{x^3}{3} + \cancel{\frac{1}{3}} + x^2 + x \mp \cancel{\frac{1}{3}}$$

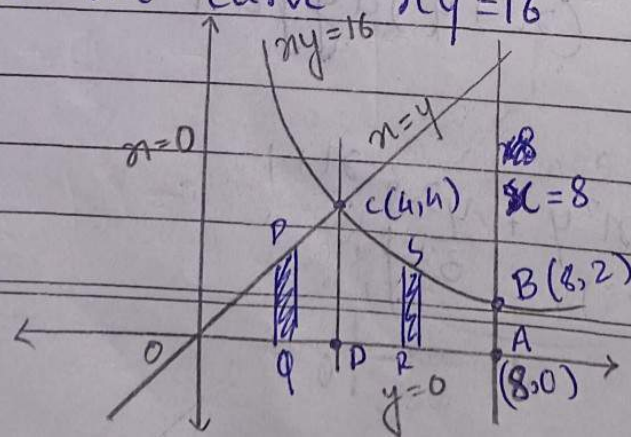
$$\int_{x=0}^{1} \frac{4x^3}{3} + x^2 + x$$

$$= \left[\frac{\cancel{4}x^4}{3 \cdot \cancel{4}} + \frac{x^3}{3} + \frac{x^2}{2} \right]_0^1$$

$$= \left[\frac{1}{3} + \frac{1}{3} + \frac{1}{2} - 0 \right]$$

$$= \frac{7}{6}$$

Q6 Evaluate $\iint_R x^2 \, dx \, dy$ where R is the region in the 1st quadrant bounded by the lines $x=y$, $y=0$, $x=8$ and the curve $xy=16$



Here region of integration is OABC.

To evaluate integral we divide region into 2 portion ODC & DABC in region ODC we 1st integrate w.r.t 'y' from $y=0$ to $y=x$ then we integrate w.r.t. x from $x=0$ to $x=4$.

In region DABC we 1st integrate with y or strip RS $y=0$ to $y=16/x$ then we integrate w.r.t x from $x=4$ to $x=8$

$$\int_{x=0}^{4} x^2 \left[\int_{y=0}^{x} dy\right] dx + \int_{x=4}^{8} x^2 \left[\int_{y=0}^{16/x} dy\right] dx$$

$$\int_{x=0}^{4} x^2 \left[y\right]_0^x dx + \int_{x=4}^{8} x^2 \left[y\right]_0^{16/x} dx$$

$$\int_{x=0}^{4} x^2 . x \, dx + \int_{x=4}^{8} x^{\cancel{2}} . \frac{16}{x} \, dx$$

$$\left[\frac{x^4}{4}\right]_0^4 + \cancel{16}^{8} \left[\frac{x^2}{2}\right]_4^8$$

$$\frac{256}{4} + 8\left[64-16\right]$$

$$64 + 384$$

$$448$$

Q7 Evaluate: $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

$$\int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx \right] dy$$

$$\left\{ \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right\}$$

$$\int_{y=0}^{a} \left[\frac{x\sqrt{(a^2-y^2)-x^2}}{2} + \frac{(a^2-y^2)}{2} \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right) \right]_0^{\sqrt{a^2-y^2}} dy$$

$$\int_{y=0}^{a} \left[0 + \left(\frac{a^2-y^2}{2}\right) \sin^{-1}(1) - 0 + \left(\frac{a^2-y^2}{2}\right) \sin^{-1}(0) \right] dy$$

$$\int_{y=0}^{a} \frac{a^2-y^2}{2} \left[\frac{\pi}{2} - 0\right] dy$$

$$\int_{y=0}^{a} (a^2-y^2) \frac{\pi}{4}\, dy$$

$$\frac{\pi}{4} \left[\frac{a^2y}{1} - \frac{y^3}{3} \right]_0^a$$

$$\frac{\pi}{4} \left[a^3 - \frac{a^3}{3} - 0 + 0 \right]$$

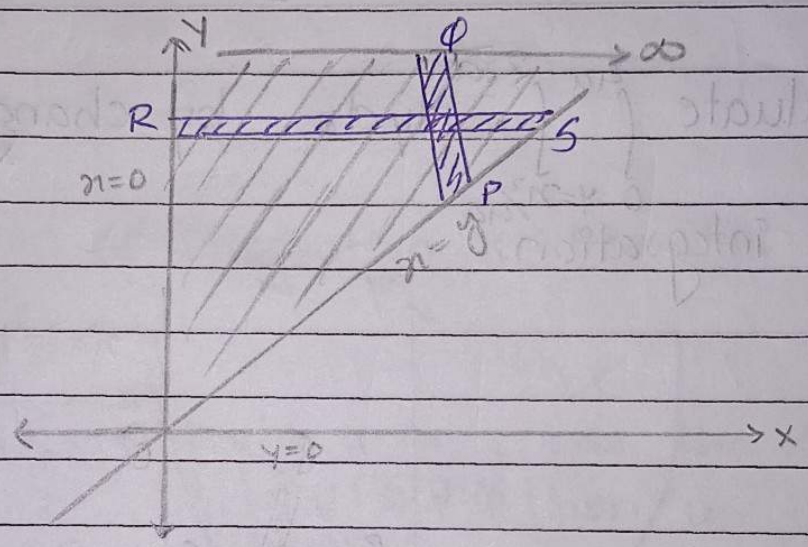
$$\frac{\pi}{2\cancel{4}} \left(\frac{\cancel{2}a^3}{3}\right) = \frac{\pi a^3}{6}$$

Change the order of integration

In double integration with variable limits change the order of integration changes the limit of integration.

Q8 On changing the order of integration evaluate

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dx\,dy$$



Here region of integration is shown by shaded line in fig.

Given that here we 1st integrate w.r.t y over $y=x$ to $y=\infty$ then we integrate from $x=0$ to $x=\infty$.

On changing the order of integration we 1st integrate w.r.t x from $x=0$ to $x=y$ over a strip RS then we integrate w.r.t from $y=0$ to $y=\infty$

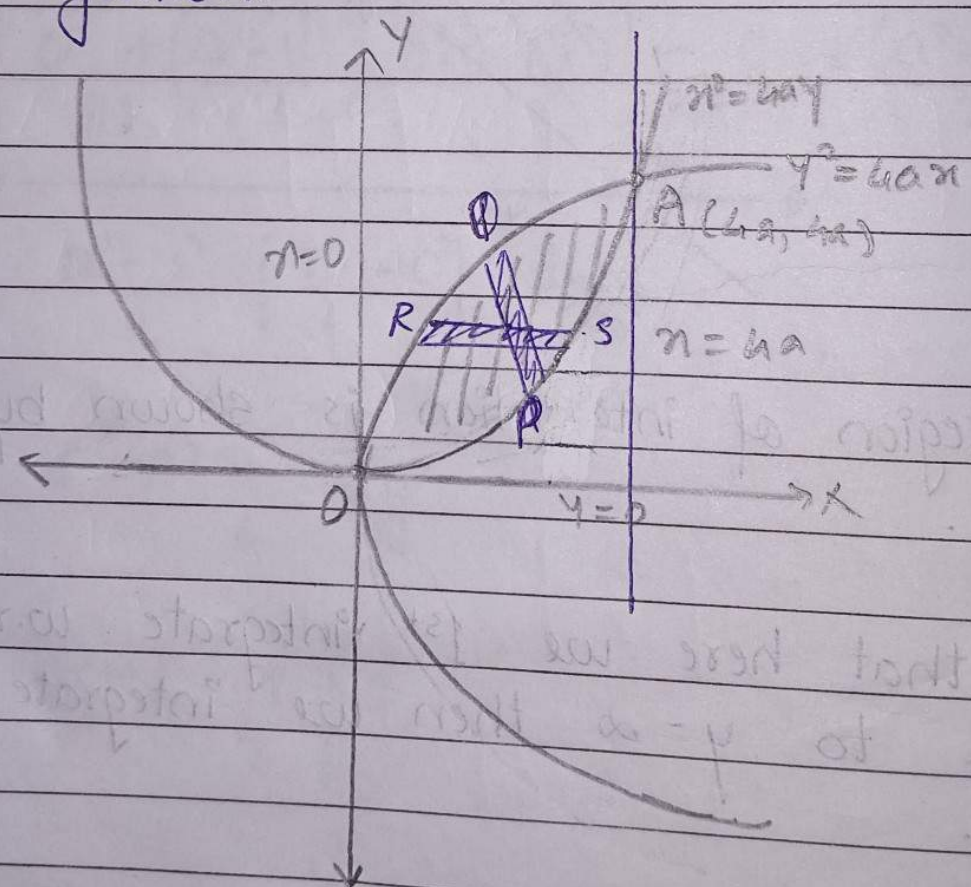
$$\int_{y=0}^{\infty} \frac{e^{-y}}{y}\left[\int_{0}^{y} dx\right] dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}\left[x\right]_{0}^{y} dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}\left[y-0\right] dy$$

$$= \left[\frac{e^{-y}}{(-1)}\right]_{0}^{\infty}$$

$$= 1$$

Q9 Evaluate $\int_{0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration.



$x^2 = 4ay$, $y^2 = 4ax$

$y = \frac{x^2}{4a}$

$\left(\frac{x^2}{4a}\right)^2 = 4ax$

$\frac{x^4}{16a^2} = 4ax \Rightarrow x^4 = 64a^3x$

$x^3 = (4a)^3 \Rightarrow x = 4a$

$y = \frac{16a^2}{4a}$

$y = 4a$

Here region of integration is bounded by 2 parabolas

Here we 1st integrate w.r.t y over the strip PQ from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ then we ingreate w.r.t x from $x = 0$ to $x = 4a$

On changing the order of integration we 1st integrate w.r.t x from $x = y^2/4a$ to $x = 2\sqrt{ay}$ over the strip RS then we integrate w.r.t from $y = 0$ to $y = 4a$.

$$\int_{y=0}^{4a} \left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx \right] dy$$

$$\int_{y=0}^{4a} \left[x \right]_{y^2/4a}^{2\sqrt{a}\,y} dy$$

$$\int_{y=0}^{4a} \left[2\sqrt{a}\,y - \frac{y^2}{4a} \right] dy$$

$$2\sqrt{a} \int_{y=0}^{4a} y^{1/2} dy - \frac{1}{4a} \int_{y=0}^{4a} y^2 dy$$

$$2\sqrt{a} \left[\frac{2y^{3/2}}{3} \right]_0^{4a} - \frac{1}{4a} \left[\frac{y^3}{3} \right]_0^{4a}$$

$$2\sqrt{a} \times 2 \times \frac{(4a)^{3/2}}{3} - \frac{1}{\cancel{4a}} \frac{(4a)^{\cancel{3}2}}{3}$$

$$\frac{4^{1+3/2} \times a^{1/2+3/2}}{3} - \frac{(4a)^2}{3}$$

$$\frac{4^{5/2} \times a^2 - 4^2 a^2}{3}$$

$$= \frac{a^2}{3} \left(4^{5/2} - 4^2 \right)$$

$$= \frac{16a^2}{3}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

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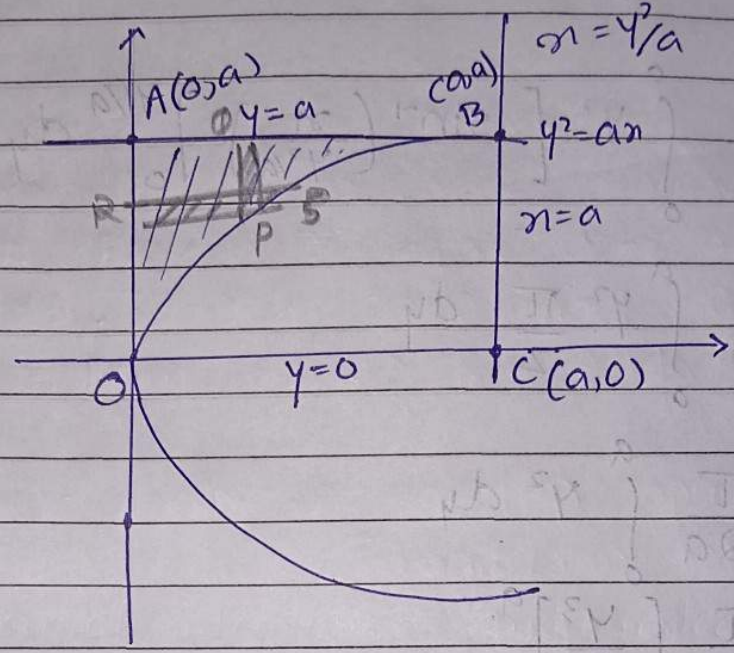
Q10 By changing the order of integration ~~limit~~ $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}}$



Region of integration is OAB

Given that 1st integration over strip PQ w.r.t y PQ, $y=\sqrt{ax}$ to $y=a$ ~~over strip PQ~~ then we integrate w.r.t x from $x=0$ to $x=a$.
On changing the order of integration we 1st integrate over the strip RS w.r.t x from $x=0$ to $x=\frac{y^2}{4a}$ and then we integrate w.r.t y from $y=0$ to $y=a$.

$$\int_{y=0}^{a} y^2 \left[\int_{x=0}^{y^2/4a} \frac{dx}{\sqrt{y^4-a^2x^2}} \right] dy$$

$$\int_{y=0}^{a} \frac{y^2}{a} \left[\int_{x=0}^{y^2/4a} \frac{dx}{\sqrt{\frac{y^4}{a^2}-x^2}} \right] dy$$

$$\int_0^a \frac{y^2}{a}\left[\int_0^{y^2/a} \frac{dx}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\right] dy$$

$$= \frac{1}{a}\int_0^a y^2 \left[\sin^{-1}\left(\frac{x}{y^2/a}\right)\right]_0^{y^2/a} dy$$

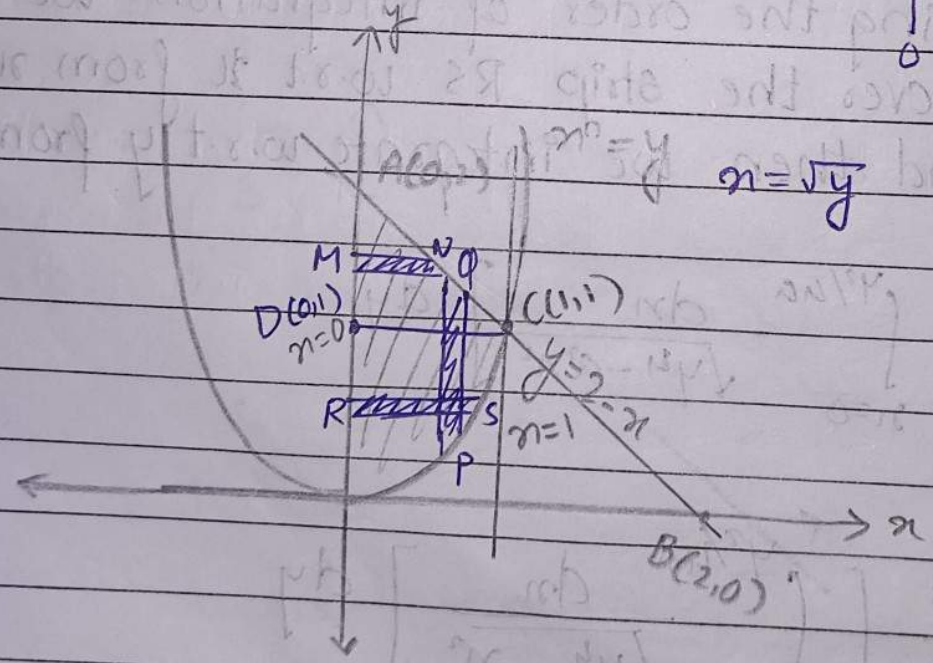
$$= \frac{1}{a}\int_0^a y^2 \frac{\pi}{2}\, dy$$

$$= \frac{\pi}{2a}\int_0^a y^2\, dy$$

$$= \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi a^2}{6}$$

Q11 Change the order of integration $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$



$x^2 = y,\ y = 2-x$

$x^2 = 2-x$

$x^2 + x = 2$

$x = 1, -2$

Here region of integration is OCA on changing the order of integration to evaluate region into 2 parts OCD and CDA.

In OCD we 1st integrate w.r.t x over a strip RS from $x=0$ to $x=\sqrt{y}$, then we integrate from $y=0$ to $y=1$.

In region CDA we 1st integrate ~~we~~ w.r.t y over a strip MN from $x=0$ to $x=2-y$, then we integrate $y=1$ ~~to~~ $y=2$.

$$\int_{y=0}^{1} y\left[\int_{x=0}^{\sqrt{y}} x\,dx\right]dy + \int_{y=1}^{2} y\left[\int_{x=0}^{2-y} x\,dx\right]dy$$

$$\int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$\frac{1}{2}\int_0^1 y.y\,dy + \int_0^2 y\left[\frac{(2-y)^2}{2}\right] dy$$

$$\frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_0^2 4y + y^3 - 4y^2\,dy$$

$$\frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\left[\frac{4y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_0^2$$

$$\frac{1}{6} + \frac{1}{2}\left[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1)\right]$$

$$\frac{1}{6} + \frac{1}{2}\left[\frac{72+45-112}{12}\right] = \frac{1}{6} + \frac{5}{24} = \frac{3}{28}$$

Double Integration in polar co-ordinates.

$$\int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1}^{r_2} f(r,\theta)\,dr\,d\theta$$

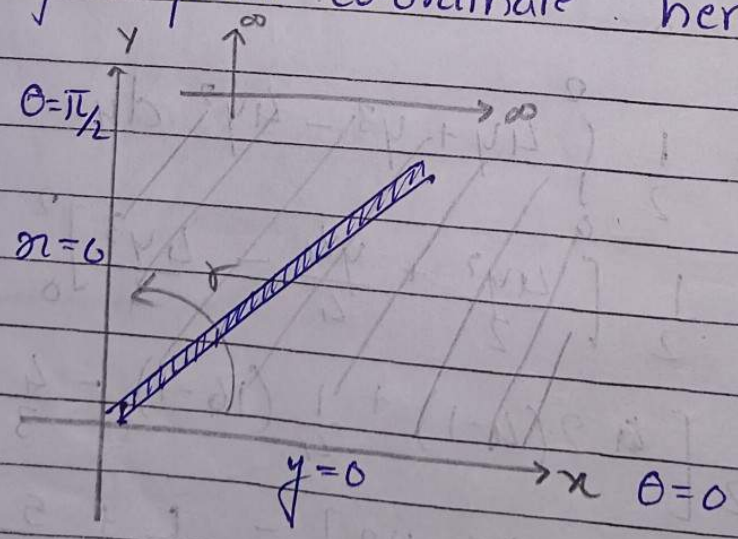
To evaluate here we 1st integrate w.r.t r from $r=r_1$ to $r=r_2$ keeping θ constant then we integrate w.r.t θ from $\theta=\theta_1$ to $\theta=\theta_2$. Here r is the function of θ and θ is a constant.

To change cartesian co-ordinate $(x,y) \rightarrow (r,\theta)$ to polar co-ordinate

we put,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2+y^2=r^2$$
$$dx\,dy = r\,dr\,d\theta$$

Q12 Evaluate integration $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar co-ordinate hence, S.T. $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$



Here region of integration is +ve quadrant of x-y plane.

Here we 1st integrate w.r.t r from $r=0$ to $r=\theta$ then we integrate w.r.t θ from $\theta=\frac{\pi}{2}$ to $\theta=0$

put $x=r\cos\theta$, $y=r\sin\theta$,
$x^2+y^2=r^2$, $dxdy=rdrd\theta$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$-\frac{1}{2}\int_{0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}(-2)\, r\,dr\,d\theta$$

$$-\frac{1}{2}\int_{0}^{\pi/2}\left[e^{-r^2}\right]_0^{\infty} d\theta$$

$$-\frac{1}{2}\int_{\theta=0}^{\pi/2}\left(e^{-\infty}-e^{0}\right)d\theta$$

$$-\frac{1}{2}\int_{0}^{\pi/2}(0-1)\,d\theta$$

$$\frac{1}{2}\left(\theta\right)_0^{\pi/2}=\frac{1}{2}\left[\frac{\pi}{2}-0\right]=\frac{\pi}{4}$$

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

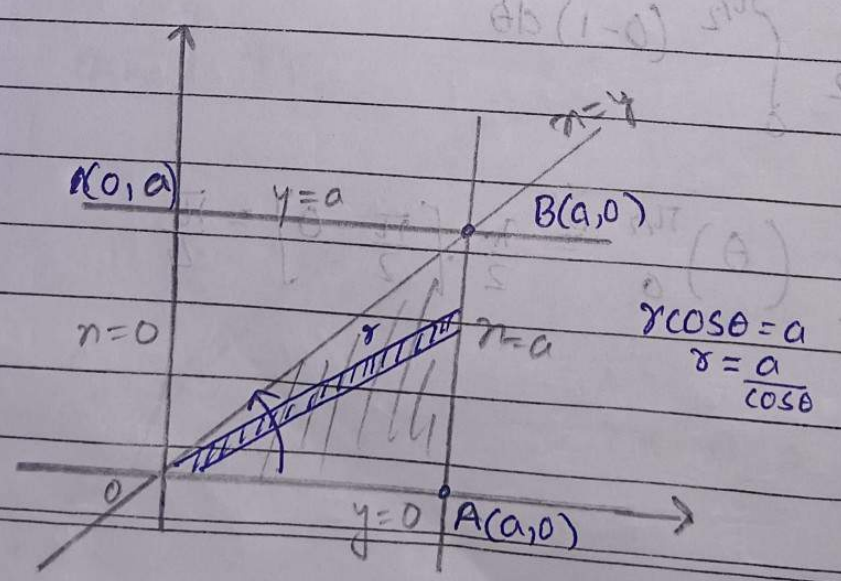
$$\int_{x=0}^{\infty} e^{-x^2}\,dx \times \int_{y=0}^{\infty} e^{-y^2}\,dy = \frac{\pi}{4}$$

$$\int_{x=0}^{\infty} e^{-x^2}\,dx \times \int_{0}^{\infty} e^{-x^2}\,dx = \frac{\pi}{4}$$

$$\left(\int_{0}^{\infty} e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Q13 Evaluate $\int_{y=0}^{a}\int_{x=y}^{a} \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy$ by changing to the polar co-ordinate.



$r\cos\theta = a$

$r = \frac{a}{\cos\theta}$

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Here we integrate w.r.t r from $r=0$ to $r=a/\cos\theta$
then we integrate w.r.t θ from $\theta=0$ to $\theta=\frac{\pi}{4}$

put,

$$x=r\cos\theta \quad , \quad y=r\sin\theta$$

$$x^2+y^2=r^2 \quad , \quad dx\,dy=r\,dr\,d\theta$$

$$\int_{\theta=0}^{\pi/4}\int_{r=0}^{a/\cos\theta} \frac{r^2\cos^2\theta}{(r^2)^{3/2}}\, r\,dr\,d\theta$$

$$\int_{\theta=0}^{\pi/4}\cos^2\theta\int_{r=0}^{a/\cos\theta} dr\,d\theta$$

$$\int_{\theta=0}^{\pi/4}\cos^2\theta\,\left[r\right]_0^{a/\cos\theta}\,d\theta$$

$$\int_{\theta=0}^{\pi/4}\cos^2\theta\left[\frac{a}{\cos\theta}-0\right]d\theta$$

$$\int_{\theta=0}^{\pi/4}\cos\theta\, a\,d\theta$$

$$a\left[\sin\theta\right]_0^{\pi/4}$$

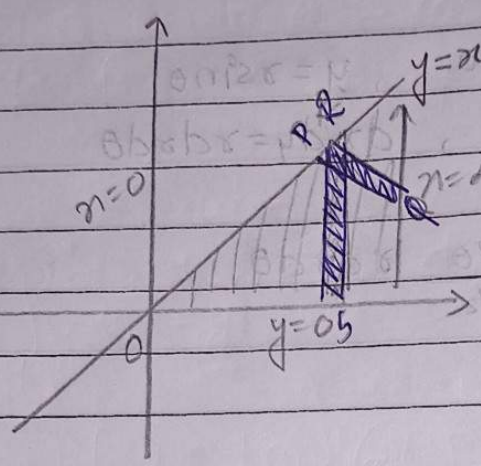
$$a\left[\sin\frac{\pi}{4}-\sin 0\right]$$

$$a\left[\frac{1}{\sqrt{2}}-0\right]$$

$$\frac{a}{\sqrt{2}}$$

H.W

Q14 Evalute $\int_{x=0}^{\infty}\int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$ by changing the order of integration.



Here region of integration is shaded region.

Here 1st we integrate w.r.t y over the strip PQ from $y=0$ to $y=x$ then we integrate w.r.t x from $x=0$ to $x=\infty$. On changing the order we 1st integrate w.r.t x over the strip RS from $x=y$ to $x=\infty$ then we integrate w.r.t y from $y=0$ to $y=\infty$

$$\int_{y=0}^{\infty}\int_{x=y}^{x=\infty} x e^{-x^2/y}\, dy\, dx = \int_{y=0}^{y=\infty} e^{1/y} \int_{x=y}^{x=\infty} x e^{-x^2}\, dx\, dy$$

$$= \int_{y=0}^{y=\infty} \frac{e^{1/y}}{-2} \int_{x=y}^{\infty} -2x e^{-x^2}\, dx\, dy$$

$$= \int_{y=0}^{y=\infty} \frac{e^{1/y}}{-2} \left[e^{-x^2}\right]_{y}^{\infty} dy$$

$$= \int_{y=0}^{y=\infty} \frac{e^{1/y}}{-2} \left(-e^{-y^2}\right) dy$$

$$= \frac{1}{2}\int_{y=0}^{y=\infty} e^{-y}\, dy$$

$$= \frac{1}{2}\left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-\infty} - e^{0}}{-1}\right]$$

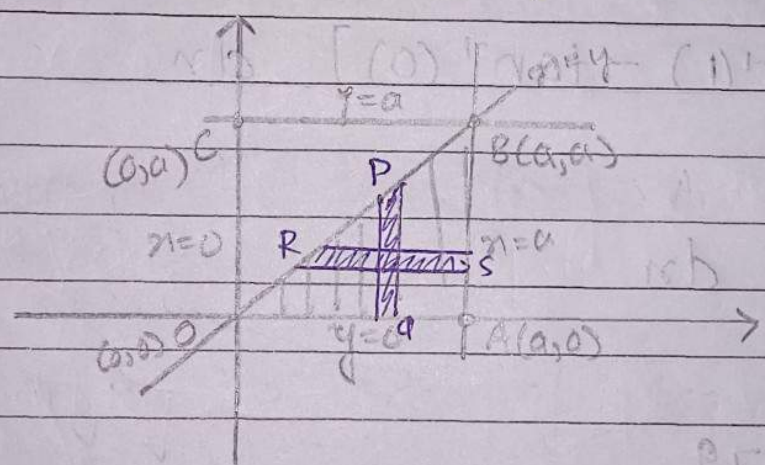
$$= \frac{1}{2}$$

Q15 Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dy\, dx$

by changing the order of integration $\int \frac{1}{a^2+x^2}\, dx$

$\Downarrow$

$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$



Here region of integration is OAB

Here we 1st integrate w.r.t x over the strip PQ from $x=y$ to $x=a$ then we integrate w.r.t. $y=0$ to $y=a$.
On changing the order we 1st integrate w.r.t y over the strip RS ~~w.r.t~~ from $y=0$ to $y=x$ then we integrate w.r.t x from $x=0$ to $x=a$.

$$\int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2} \, dy \, dx$$

$$\int_{x=0}^{a} x \int_{y=0}^{x} \frac{1}{x^2+y^2} \, dy \, dx$$

$$\int_{x=0}^{a} x \int_{y=0}^{x} \frac{1}{x^2+y^2} \, dy \, dx$$

$$\int_{x=0}^{a} x \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x dx$$

$$\int_{x=0}^{a} \left[\tan^{-1}(1) - \tan^{-1}(0) \right] dx$$

$$\int_{x=0}^{a} \frac{\pi}{4} \, dx$$

$$\frac{\pi}{4} [x]_0^a$$

$$\frac{\pi}{4} (a-0)$$

$$\frac{a\pi}{4}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

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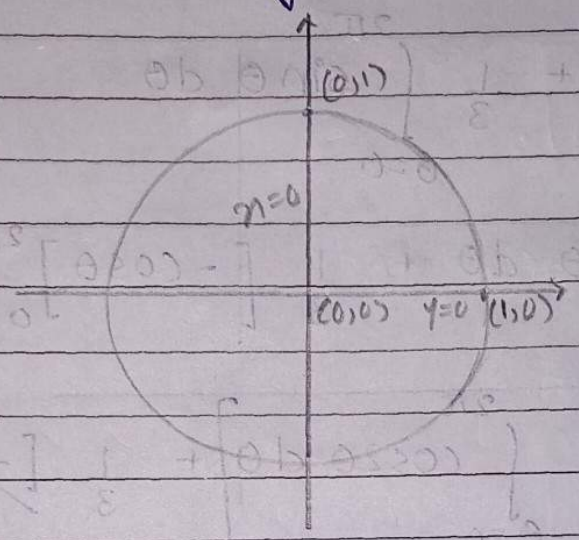
Q16 Evaluate the following integral double integration

$\iint_R (x^2+y)\,dx\,dy$ by changing into polar form.

Where R is the region $x^2+y^2=1$



put

$x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2=r^2$, $dx\,dy = r\,dr\,d\theta$

r varies from $r=0$ to $r=1$

θ varies from $\theta=0$ to $\theta=2\pi$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r^2\cos^2\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$\int_{\theta=0}^{2\pi}\left[\cos^2\theta\int_{r=0}^{1} r^3\,dr + \sin\theta\int_{r=0}^{1} r^2\,dr\right]d\theta$$

$$\int_{\theta=0}^{2\pi} \left[\cos^2\theta \left[\frac{r^4}{4} \right]_0^1 + \sin\theta \left[\frac{r^3}{3} \right]_0^1 \right] d\theta$$

$$\int_{\theta=0}^{2\pi} \left[\frac{\cos^2\theta}{4} + \frac{\sin\theta}{3} \right] d\theta$$

$$\frac{1}{4}\int_{\theta=0}^{2\pi} \cos^2\theta \, d\theta + \frac{1}{3}\int_{\theta=0}^{2\pi} \sin\theta \, d\theta$$

$$\frac{1}{4}\int_{\theta=0}^{2\pi} \frac{1+\cos 2\theta}{2} \, d\theta + \frac{1}{3}\left[-\cos\theta \right]_0^{2\pi} d\theta$$

$$\frac{1}{8}\left[\int_{\theta=0}^{2\pi} 1 \, d\theta + \int_{\theta=0}^{2\pi} \cos 2\theta \, d\theta \right] + \frac{1}{3}\left[\cancel{-1+1} \right] \to 0$$

$$\frac{1}{8}\left[\left[\theta \right]_0^{2\pi} + \left[\frac{\sin 2\theta}{2} \right]_0^{2\pi} \right]$$

$$\frac{1}{8}\left[2\pi + 0 \right]$$

$$\frac{\pi}{4}$$

$(x-h)^2+(y-k)^2=r^2$ $\quad r \rightarrow$ radius

$(h,k) \rightarrow$ centre

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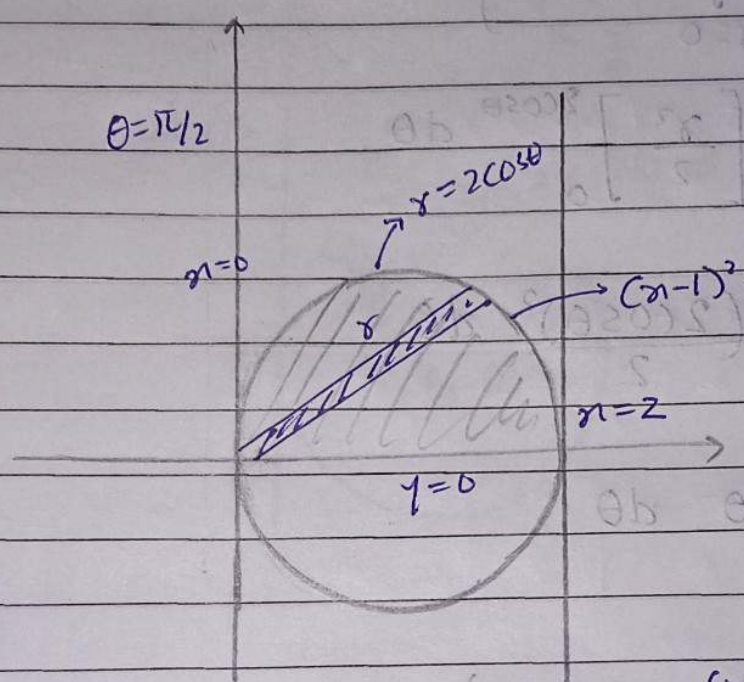
Q17 Evaluate changing by polar coordinate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

$\theta = \frac{\pi}{2} \rightarrow$ upper half



$y=\sqrt{2x-x^2}$

$y^2 = 2x-x^2$

$x^2-2x+y^2=0$

$x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1$

$(h,k) \rightarrow (1,0) \rightarrow$ centre

$r \rightarrow 1 \rightarrow$ radius

Here $y=\sqrt{2x-x^2}$ is a circle centre a $(1,0)$ and radius is 1

put

$x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2=r^2$, $dr\,dy = r\,dr\,d\theta$

$r \longrightarrow r=0$ to $r=2\cos\theta$

$\theta \longrightarrow \theta=0$ to $\theta=\pi/2$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} \frac{r\cos\theta\, \cancel{r}\, dr\, d\theta}{\cancel{\sqrt{r^2}}}$$

$$\int_{\theta=0}^{\pi/2} \cos\theta \left[\int_{r=0}^{2\cos\theta} r \, dr\right] d\theta$$

$$\int_{\theta=0}^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$\int_{\theta=0}^{\pi/2} \cos\theta \, \frac{(2\cos\theta)^2}{2} \, d\theta$$

$$\int_{\theta=0}^{\pi/2} \frac{4\cos^3\theta}{2} \, d\theta$$

$$2\int_{\theta=0}^{\pi/2} \left(\frac{\cos 3\theta + 3\cos\theta}{4}\right) d\theta$$

$$\frac{1}{2}\left(\int_{\theta=0}^{\pi/2} \cos 3\theta \, d\theta + \int_{\theta=0}^{\pi/2} 3\cos\theta \, d\theta\right)$$

$$\frac{1}{2}\left\{\left[\frac{\cancel{\cos 3\theta}}{}\right] \left[\frac{\sin 3\theta}{3}\right]_0^{\pi/2} + 3\left[\sin\theta\right]_0^{\pi/2}\right\}$$

$$\frac{1}{2}\left\{\frac{1}{3}\left[-1-0\right] + 3\left[1-0\right]\right\}$$

$$\frac{1}{2}\left\{-\frac{1}{3} + 3\right\} = \frac{4}{3}$$

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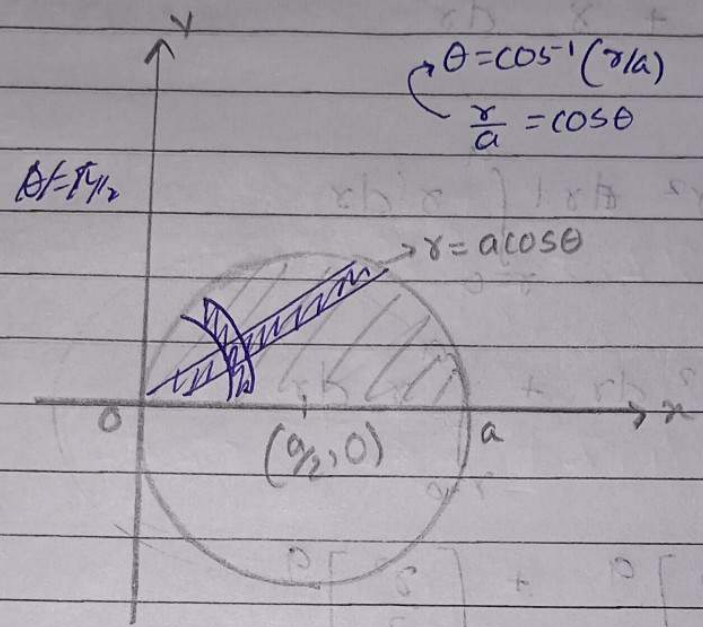
Q18 Evalute following integral by changing order of integration. $\int_{\theta=0}^{\pi/2}\int_{0}^{a\cos\theta} r\sin\theta \, dr\, d\theta$



$r^2 = ar$

$x^2+y^2 = ax$

$x^2 - ax + y^2 = 0$

$x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

$(x-\frac{a}{2})^2 + y^2 = (\frac{a}{2})^2$

$(h,k) \to (\frac{a}{2}, 0) \to$ centr

$r \to \frac{a}{2} \longrightarrow$ radius

On changing the order of integration we 1st integrate w.r.t θ from $\theta = 0$ to $\theta = \cos^{-1}(r/a)$ the we integrate w.r.t r from $r=0$ to $r=a$

$$\int_{r=0}^{a} r\left[\int_{\theta=0}^{\cos^{-1}(r/a)} \sin\theta \, d\theta\right] dr$$

$$\int_{r=0}^{a} r\left[-\cos\theta\right]_{0}^{\cos^{-1}(r/a)} dr$$

$$\int_{r=0}^{a} r\left[-\cos\left(\cos^{-1}\frac{r}{a}\right) + 1\right] dr$$

$$\int_{r=0}^{a} r\left[-\frac{r}{a}+1\right] dr$$

$$\int_{r=0}^{a} -\frac{r^2}{a} + r \quad dr$$

$$-\frac{1}{a}\int_{r=0}^{a} r^2 \, dr + \int_{r=0}^{a} r \, dr$$

$$-\frac{1}{a}\int_{r=0}^{a} r^2 \, dr + \int_{r=0}^{a} r \, dr$$

$$-\frac{1}{a}\left[\frac{r^3}{3}\right]_0^a + \left[\frac{r^2}{2}\right]_0^a$$

$$-\frac{1}{a}\cdot\frac{a^3}{3} + \frac{a^2}{2}$$

$$\frac{-2a^2+3a^2}{6}$$

$$\frac{a^2}{6}$$

Triple integration

Let, $x_1 = f_1(y, z)$,
$x_2 = f_2(y, z)$,
$y_1 = g_1(z)$,
$y_2 = g_2(z)$
$z_1 = a$, $z_2 = b$

$$\int_{z=a}^{b} \left\{ \int_{y=y_1}^{y_2} \left[\int_{x=x_1}^{x_2} f(x, y, z)\, dx \right] dy \right\} dz$$

Here we 1st integrate w.r.t x from $x = x_1$ to $x = x_2$ keeping y and z constant.
Then we integrate w.r.t y from $y = y_1$ to $y = y_2$ keeping z constant.
Then we integrate w.r.t z from $z = a$ to $z = b$.

Q19 $$\int_{z=-1}^{1} \int_{x=0}^{z} \int_{y=x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$$

$$\int_{z=-1}^{1} \int_{x=0}^{z} \left[\int_{y=x-z}^{x+z} (x+y+z)\, dy \right] dx\, dz$$

$$\int_{-1}^{1} \int_{x=0}^{z} \left[xy + \frac{y^2}{2} + zy \right]_{x-z}^{x+z} dx\, dz$$

$$\int_{-1}^{1}\int_{x=0}^{z}\left\{ x\left[(x+z)-(x-z)\right]+\frac{1}{2}\left[(x+z)^2-(x-z)^2\right]+z\left[(x+z)-(x-z)\right]\right\}dx\,dz$$

$$\int_{-1}^{1}\int_{x=0}^{z}\left\{2xz+\frac{1}{2}\left[x^2+z^2+2xz-x^2-z^2+2xz\right]+2z^2\right\}dx\,dz$$

$$\int_{-1}^{1}\int_{x=0}^{z}\left\{4xz+2z^2\right\}dx\,dz$$

$$\int_{-1}^{1}\left[\frac{4zx^2}{2}+2z^2x\right]_0^z dz$$

$$\int_{-1}^{1}\left[2z\left[z^2-0\right]+2z^2\left[z-0\right]\right]dz$$

$$\int_{-1}^{1}\left[2z^3+2z^3\right]dz$$

$$\int_{-1}^{1}4z^3\,dz$$

$$\left[\frac{4z^4}{4}\right]_{-1}^{1}$$

$$1^4-(-1)^4$$

$$0$$

Q20 Evaluate $\int_{y=0}^{1}\int_{x=y^2}^{1}\int_{z=0}^{1-x} x\,dz\,dx\,dy$

$$\int_{y=0}^{1}\int_{x=y^2}^{1}\left[\int_{z=0}^{1-x} x\,dz\right]dx\,dy$$

$$\int_{y=0}^{1}\int_{x=y^2}^{1}\left[xz\right]_0^{1-x}dx\,dy$$

$$\int_{y=0}^{1}\int_{x=y^2}^{1}\left[x(1-x)-x(0)\right]dx\,dy$$

$$\int_{y=0}^{1}\int_{x=y^2}^{1}\left[x-x^2\right]dx\,dy$$

$$\int_{y=0}^{1}\left[\int_{x=y^2}^{1} x-x^2\,dx\right]dy$$

$$\int_{y=0}^{1}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^{1}dy$$

$$\int_{y=0}^{1}\left\{\frac{1}{2}\left(1-(y^2)^2\right)-\frac{1}{3}\left(1-(y^2)^3\right)\right\}dy$$

$$\int_{y=0}^{1}\left\{\frac{1-y^4}{2}-\frac{1+y^6}{3}\right\}dy=\left[\frac{1}{2}\left[y-\frac{y^5}{5}\right]-\frac{1}{3}\left[y-\frac{y^7}{7}\right]\right]_0^1$$

$$\left[\frac{1}{2}\left(1-\frac{1}{5}\right)-\frac{1}{3}\left(1-\frac{1}{7}\right)\right]=\frac{4}{35}$$

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Q21 $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\int_{z=0}^{\sqrt{1-x^2-y^2}} z\,dz\right]dx\,dy$$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left(\frac{1-x^2-y^2}{2}\right)dx\,dy$$

$$\int_{x=0}^{1}\frac{x}{2}\left(\int_{y=0}^{\sqrt{1-x^2}} y-x^2y-y^3\,dy\right)dx$$

$$\int_{x=0}^{1}\frac{x}{2}\left[\frac{y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$\int_{x=0}^{1}\frac{x}{2}\times\frac{1}{4}\left[2y^2-2x^2y^2-y^4\right]_0^{\sqrt{1-x^2}} dx$$

$$\int_{x=0}^{1}\frac{x}{8}\left[2(1-x^2)-2x^2(1-x^2)-(1-x^2)^2\right]dx$$

$$\int_{x=0}^{1}\frac{x}{8}\left[2-2x^2-2x^2+2x^4-(1+x^4-2x^2)\right]dx$$

$$\int_{x=0}^{1} \frac{x}{8}\left[2-4x^2+2x^4-1-x^4+2x^2\right]dx$$

$$\int_{x=0}^{1} \frac{x}{8}\left[1-2x^2+x^4\right]dx$$

$$\frac{1}{8}\int_{x=0}^{1} x-2x^3+x^5\,dx$$

$$\frac{1}{8}\left[\frac{x^2}{2}-2\frac{x^4}{4}+\frac{x^6}{6}\right]_0^1$$

$$\frac{1}{8}\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right]$$

$$\frac{1}{48}$$

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Q22 $\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\int_{z=0}^{\frac{a^2-r^2}{a}} r\,dr\,d\theta\,dz$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r\left[\int_{z=0}^{\frac{a^2-r^2}{a}} dz\right] dr\,d\theta$$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r\left[z\right]_0^{\frac{a^2-r^2}{a}} dr\,d\theta$$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r\,\frac{a^2-r^2}{a}\,dr\,d\theta$$

$$\int_{\theta=0}^{\pi/2}\frac{1}{a}\int_{r=0}^{a\sin\theta} ra^2-r^3\,dr\,d\theta$$

$$\int_{\theta=0}^{\pi/2}\frac{1}{a}\left[\frac{a^2r^2}{2}-\frac{r^4}{4}\right]_0^{a\sin\theta} d\theta$$

$$\int_{\theta=0}^{\pi/2}\frac{1}{\cancel{a}}\left[\frac{a^{\cancel{2}}}{2}(a^2\sin^2\theta-0)-\frac{(a\sin\theta)^4}{4}\right] d\theta$$

$$\int_{\theta=0}^{\pi/2}\frac{a^3\sin^2\theta}{2}-\frac{a^3\sin^4\theta}{4}\,d\theta$$

$$\frac{1}{4}\int_{\theta=0}^{\pi/2} 2a^3\sin^2\theta-a^3\sin^4\theta\,d\theta$$

$$\frac{a^3}{4}\int_0^{\pi/2} 2\sin^2\theta-\sin^4\theta\,d\theta$$

$$\frac{1}{3}\int_0^{\pi/2}\left(3a^2\sin\theta - a^2\left[\frac{3\sin\theta - \sin 3\theta}{4}\right]\right)d\theta$$

$$\frac{a^3}{4}\int_0^{\pi/2} \frac{2(1+\cos 2\theta)}{2} - \frac{(1+\cos 2\theta)^2}{4}\, d\theta$$

$$\frac{a^3}{4}\int_0^{\pi/2} 1+\cos 2\theta - \frac{1}{4}(1+\cos^2 2\theta + 2\cos 2\theta)\, d\theta$$

$$\frac{a^3}{4}\int_0^{\pi/2} 1+\cos 2\theta - \frac{1}{4}\left(1+\frac{1+\cos 4\theta}{2} + 2\cos 2\theta\right) d\theta$$

$$\frac{a^3}{4}\int_0^{\pi/2} 1+\cos 2\theta - \frac{1}{4} - \frac{1}{8} + \frac{\cos 4\theta}{8} - \frac{\cos 2\theta}{2}\, d\theta$$

$$\frac{a^3}{4}\int_0^{\pi/2} \frac{8-2-1}{8} + \frac{\cos 2\theta}{2} - \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}\, d\theta$$

$$\frac{a^3}{4}\int_0^{\pi/2} \frac{5}{8} + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}\, d\theta$$

$$\frac{a^3}{4}\left[\frac{5}{4}\theta + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{16}\right]_0^{\pi/2} d\theta$$

$$\frac{a^3}{4}\left[\frac{5}{8}\left[\frac{\pi}{2}-0\right] + \frac{1}{4}\left[\cancel{\sin 2\frac{\pi}{2}}^{0} - \cancel{\sin 2(0)}^{0}\right] + \frac{1}{16}\left[\cancel{\sin 4\frac{\pi}{2}}^{0} - \cancel{\sin 4(0)}^{0}\right]\right]$$

$$\frac{a^3}{4}\left(\frac{5}{8}\times\frac{\pi}{2}\right)$$

$$\frac{5a^3\pi}{64}$$

Application of Integration

- Area by double integration

Q2) Evaluate $\iint r^3\,dr\,d\theta$, over the area included between the circle $r=2\sin\theta$, $r=4\sin\theta$

$x^2+y^2=r^2$

Sol^n: $y=r\sin\theta$

$r=2\frac{y}{r} \Rightarrow r^2=2y \Rightarrow x^2+y^2=2y$

$x^2+y^2-2y=0$

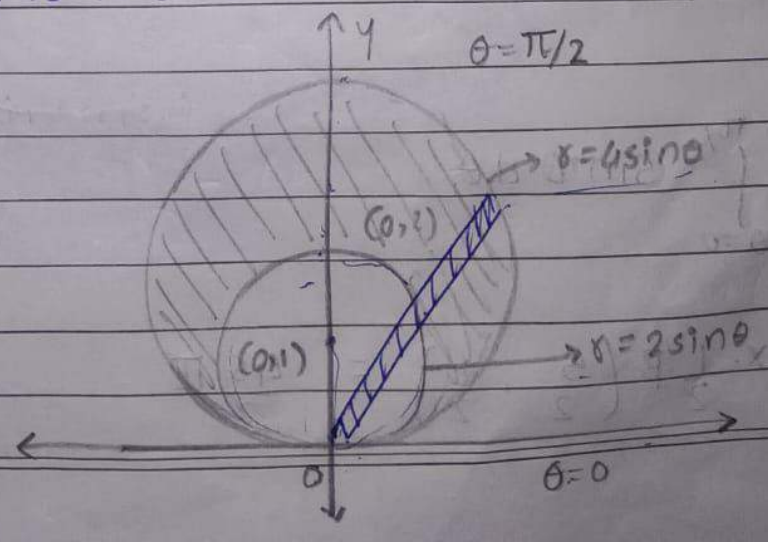
$x^2+y^2-2y+1=1$

$x^2+(y-1)^2=1$

$\therefore$ $r=2\sin\theta$ is a circle having centre at $(0,1)$ & radius at 1

Similarly,

$r=4\sin\theta$ is a circle center at $(0,2)$ & radius at 2.



$r \rightarrow \quad r = 2\sin\theta \quad$ to $\quad r = 4\sin\theta$

$\theta \rightarrow \quad \theta = 0 \quad$ to $\quad \theta = \pi$

$$\int_{2\sin\theta}^{4\sin\theta} \int_{0}^{\pi} r^3 \, dr \, d\theta$$

$$\int_{0}^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \int_{0}^{\pi/2} \left[\frac{r^4}{4}\right]_{2\sin}^{4\sin\theta} + \int_{0}^{\pi/2}$$

$$2 \cdot \int_{0}^{\pi/2} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$\frac{1}{2} \int_{\theta=0}^{\theta=\pi/2} \left[r^4\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$\frac{1}{2} \int_{0}^{\pi/2} \left[(4\sin\theta)^4 - (2\sin\theta)^2\right] d\theta$$

$$\frac{1}{2} \int^{\pi/2} 256\sin^4\theta - 16\sin^4\theta \, d\theta$$

$$\frac{1}{2} \int_{0}^{\pi/2} 240 \sin^4\theta \, d\theta$$

$$\frac{1}{2} \times 240 \int_{\theta=0}^{\pi/2} \sin^4\theta \, d\theta$$

$$\frac{1}{2} \cancel{B}\, 120 \times \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{45}{2}\pi$$

Q25 Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the st. line $2x + 3y = 6$

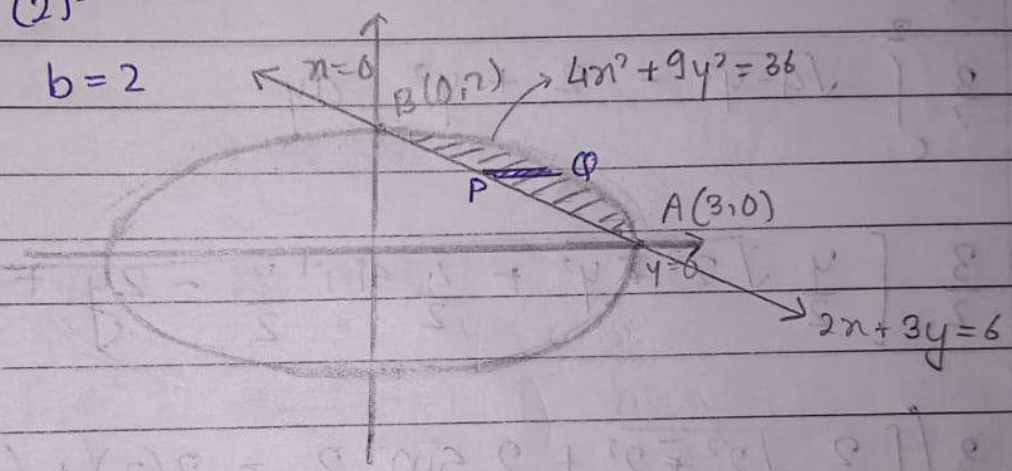
$4x^2 + 9y^2 = 36$

dividing by 36

$\frac{4x^2}{36} + \frac{9y^2}{36} = \frac{36}{36}$

$\frac{x^2}{(3)^2} + \frac{y^2}{(2)^2} = 1$

$a = 3 \quad b = 2$



take $x = 0$,

$2(0) + 3y = 6$

$y = 2$

$P = 2x + 3y = 6$

$x = \frac{6-3y}{2}$

take $y = 0$,

$2x + 3(0) = 6$

$x = 2$

$Q = 4x^2 + 9y^2 = 36$

$x = \frac{3}{2}\sqrt{4-y^2}$

$x = \frac{6-3y^2}{2}$ to $x = \frac{3}{2}\sqrt{4-y^2}$

$y = 0$ to $y = 2$

$$\int_{y=0}^{2} \int_{x=\frac{6-3y^2}{2}}^{\frac{3}{2}\sqrt{4-y^2}} dx\, dy$$

$$\int_{y=0}^{2} \left[x\right]_{\frac{6-3y^2}{2}}^{\frac{3}{2}\sqrt{4-y^2}} dy$$

$$\int_{y=0}^{y=2} \left[\frac{3}{2}\sqrt{4-y^2} - \left(\frac{6-3y^2}{2}\right)\right] dy$$

$$\frac{3}{2}\int_{0}^{2} \sqrt{4-y^2} - 2 + y^2 \, dy$$

$$\frac{3}{2}\left[\frac{y}{2}\sqrt{2^2 - y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2} - 2y + \frac{y^3}{3}\right]_0^2$$

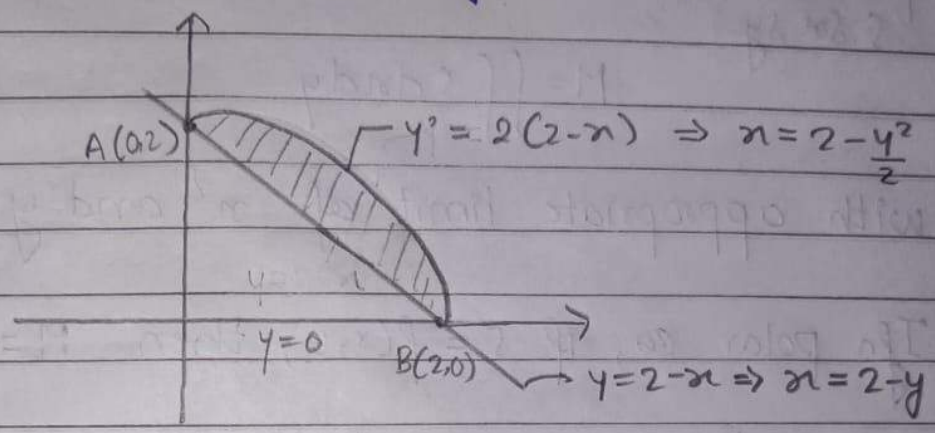
$$\frac{3}{2}\left(\left[\frac{2}{2}\sqrt{2^2 - 2^2} + 2\sin^{-1}\frac{2}{2} - 2(2) + \frac{(8)}{3}\right] - \left[0 + \sin^{-1}(0) - 0 + 0\right]\right)$$

$$\frac{3}{2}\left(\left[0 + \frac{2\pi}{2} - 4 + 2\right] - [0]\right)$$

$$\frac{3}{2}\left(\pi - 2\right)$$

$$\frac{3\pi}{2} - 3$$

Q26 Find by double integration the area enclosed by the pair of curves $y = 2 - x$, $y^2 = 2(2-x)$



$x \rightarrow x - 2 - y$ to $x = 2 - \frac{y^2}{2}$

$y \rightarrow y = 0$ to $y = 2$

$$\int_{y=0}^{2} \left[\int_{x=2-y}^{2-\frac{y^2}{2}} dx \right] dy$$

$$\int_{0}^{2} \left[x \right]_{2-y}^{2-\frac{y^2}{2}} dy$$

$$\int_{0}^{2} 2 - \frac{y^2}{2} - (2 - y) \, dy$$

$$\int_{0}^{2} \cancel{2} - \frac{y^2}{2} - \cancel{2} + y \, dy$$

$$\left[\frac{y^2}{2} - \frac{y^3}{6} \right]_0^2$$

$$\left[\frac{4}{2} - \frac{8}{6} \right] = \frac{2}{3}$$

For the plane if the surface density $\rho = f(x,y)$ at pt $P(x,y)$ then elementary mass is given by $\rho\, \delta x\, \delta y$

$$M = \iint \rho\, dx\, dy$$

with appropriate limit of x and y.

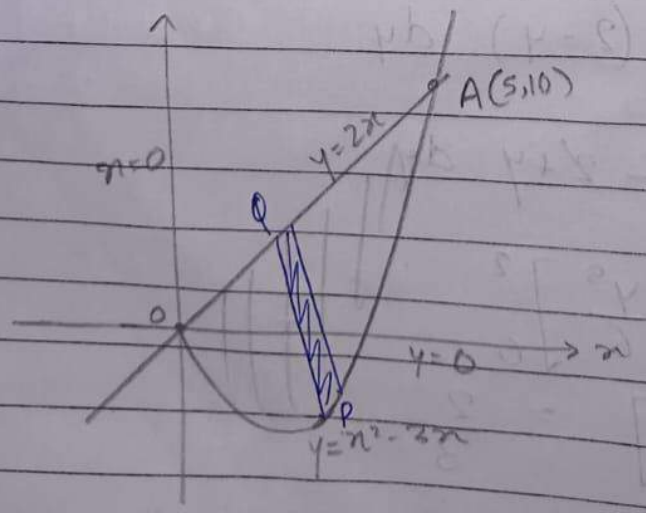
→ In polar co; if $\rho = f(r,\theta)$ then $M = \iint \rho\, r\, dr\, d\theta$

→ If the density at pt p is $\rho = f(x,y,z)$ (for solid) then total mass is given by

$$M = \iiint \rho\, dx\, dy\, dz.$$

with appropriate limit of x, y and z.

Q2) A lamina is bounded by the curve $y = x^2 - 3x$ and $y = 2x$. If density at any pt is given by λxy. Find the mass of lamina.



$y \longrightarrow \quad y = x^2 - 3x$ to $y = 2x$

$x \longrightarrow \quad x = 0$ to $x = 5$

given, $\rho = \lambda x y$

Required mass is

$$M = \iint \rho \, dx\, dy = \int_{x=0}^{5} \int_{y=x^2-3x}^{2x} \lambda x y \, dy\, dx$$

$$= \lambda \int_{x=0}^{5} x \left[\frac{y^2}{2} \right]_{x^2-3x}^{2x} dx$$

$$= \frac{\lambda}{2} \int_{x=0}^{5} x \left[4x^2 - (x^2 - 3x)^2 \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{5} x \left[4x^2 - x^4 - 9x^2 + 6x^3 \right] dx$$

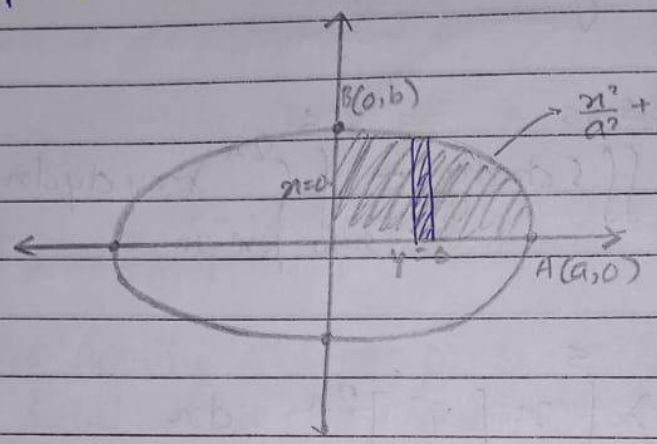
$$= \frac{1}{2} \int_{x=0}^{5} -5x^3 - x^5 + 6x^4 \, dx$$

$$= \frac{\lambda}{2} \left[-\frac{5x^4}{4} - \frac{x^6}{6} + \frac{6x^5}{5} \right]_0^5$$

$$= \frac{\lambda}{2} \left[-\frac{5(5)^4}{4} - \frac{5^6}{6} + \frac{6(5)^5}{5} \right] = 182.29\lambda$$

$$= \lambda 5^4 \left[-5 - 25 + 30 \right] = \frac{5^4}{\ldots} \left[\ldots \right]$$

Q28 If the mass of ellipse plane $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if the density of any point on it is $\rho = \mu xy$



$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

Let a required mass m is the ~~mas~~ 4 times mass of the 1st quadrant

$$M = 4\int\int \rho\, dx\, dy$$

$$= 4\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \mu xy\, dx\, dy$$

$$= 4\mu\int_{x=0}^{a} x\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y\, dx\, dy$$

$$= 4\mu\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}$$

$$= 4\mu\int_{x=0}^{a} x\left[\frac{\frac{b^2}{a^2}(a^2-x^2)}{2}\right]$$

$$= \frac{4}{2}\frac{\mu b^2}{a^2}\int_{x=0}^{a} a^2x - x^3$$

$$= \frac{4}{2}\mu\frac{b^2}{a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{4}{2}\mu\frac{b^2}{a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

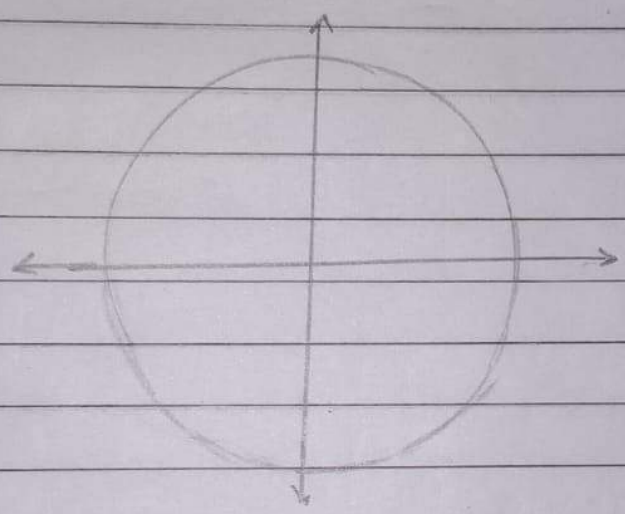
$$= \frac{\cancel{4}\mu b^2}{2\,\cancel{a^2}}\left(\frac{a^{\cancel{4}\,2}}{\cancel{4}}\right)$$

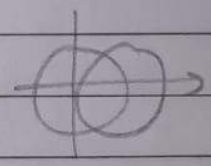
$$= \frac{b^2a^2\mu}{2}$$

Q29 Find the mass of 1st quadrant of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Q30 Find the mass of the plate which is inside of the circle $r = 2a\cos\theta$ and outside the circle $r = a$. If the density varies as the distance from the pole.

Soln: Here, given that density varies as the distance from the pole i.e. $\rho \propto r \Rightarrow \rho = Kr$ (K = constant)





$x = r\cos\theta$, $x^2 + y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

$r = 2a\cos\theta$

$r = 2a\dfrac{x}{r}$

$r^2 = 2ax$

$x^2 + y^2 = 2ax$

$x^2 - 2ax + y^2 = 0$

$x^2 - 2ax + a^2 - a^2 + y^2 = 0$

$(x-a)^2 + y^2 = a^2$

$\therefore r = 2a\cos\theta$

centre $(a, 0)$ & radius a.

& $r = a \Rightarrow \sqrt{x^2+y^2} = a \Rightarrow x^2+y^2=a^2$ is circle with centre $(0,0)$ & radius a