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Unit-1 Integral Calculus

Gamma function: Gamma function is defined and denoted as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Properties of Gamma function

1. $\Gamma 1 = 1$

Proof: $\Gamma 1 = \int_0^\infty e^{-x} x^{1-1} dx$

$$= \left. \frac{e^{-x}}{(-1)} \right]_0^\infty$$

$$= \frac{1}{(-1)} [e^{-\infty} - e^0] = -[0-1] = 1$$

2. $\Gamma n+1 = n \Gamma n = n!$

Ex: $\Gamma 5 = \Gamma 4+1 = 4 \Gamma 4 = 4 \Gamma 3+1$

$$= 4.3 \Gamma 3$$

$$= 4 \cdot 3 \Gamma 2+1$$

$$= 4 \cdot 3 \cdot 2 \Gamma 2$$

$$= 4 \cdot 3 \cdot 2 \Gamma 1+1$$

$$= 4 \cdot 3 \cdot 2 \cdot 1 \Gamma 1$$

$$= 4!$$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
4. $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Q. Evaluate $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$

$$I = \int_{x=0}^{\infty} x^{1/4} e^{-\sqrt{x}}\, dx$$

put $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$I = \int_{t=0}^{\infty} (t^2)^{1/4} e^{-t} (2t\,dt)$$

$$= 2\int_{t=0}^{\infty} t^{1/2} e^{-t}\, t\, dt$$

$$= \underbrace{2\int_{t=0}^{\infty} t^{3/2} e^{-t}\, dt}_{\Gamma(n)}$$

$n - 1 = \frac{3}{2} \Rightarrow n = \frac{5}{2}$

$$= 2\,\Gamma\left(\frac{5}{2}\right) \longrightarrow \text{(By defination)}$$

$$= \cancel{2}\cdot\frac{3}{\cancel{2}}\,\Gamma\left(\frac{3}{2}\right) = 3\,\Gamma\left(\frac{1}{2}+1\right) = 3\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

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Q2 $I = \int_0^{\infty} e^{-x^2} x^2 \, dx$

put $x^2 = t \Rightarrow x = \sqrt{t}$

$2x\,dx = dt$

$dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$I = \int_{t=0}^{\infty} e^{-t}\, t \, \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2} \int_{t=0}^{\infty} e^{-t}\, t^{1-1/2} \, dt$$

$$= \frac{1}{2} \int_{t=0}^{\infty} e^{-t}\, t^{+1/2} \, dt$$

$n - 1 = +\frac{1}{2} \Rightarrow n = \frac{3}{2}$

$$= \frac{1}{2} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{2} \sqrt{\pi}\, \Gamma\left(\frac{1}{2} + 1\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} \sqrt{\pi}$$

Q3 For $c > 1$, show that $\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$

$$L.H.S = I = \int_0^\infty \frac{x^c}{c^x} dx$$

$$c^x = e^{\log c^x}$$

$$I = \int_0^\infty \frac{x^c}{e^{\log c^x}} dx$$

$$I = \int_0^\infty x^c e^{-x \log c} dx$$

put $x \log c = t$

$$dx = \frac{dt}{\log c}$$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$I = \int_{t=0}^\infty e^{-t} \left(\frac{t}{\log c}\right)^c \frac{dt}{\log c}$$

$$I = \frac{1}{(\log c)^{c+1}} \int_{t=0}^\infty e^{-t} t^c \, dt$$

$n = c+1$

$$I = \frac{\Gamma(c+1)}{(\log c)^{c+1}} = R.H.S$$

$\log e = 1$ $\quad$ $\log 0 = -\infty$

$\log 1 = 0$ $\quad$ $\log \infty = \infty$

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Q4 Evaluate $\int_0^1 (x \log x)^3 \, dx$

put $\log x = -t$

$$x = e^{-t} \Rightarrow dx = -e^{-t} dt$$

when $x = 0 \Rightarrow t = \infty$

$x = 1 \Rightarrow t = 0$

$$I = \int_{t=\infty}^{0} (e^{-t}(-t))^3 (-e^{-t} dt)$$

$$I = \int_{t=\infty}^{0} e^{\cancel{t}-3t} t^3 e^{-t} \, dt$$

$$I = \int_{\infty}^{0} e^{-4t} t^3 \, dt$$

put $4t = u$

$t = \frac{u}{4}$

$dt = \frac{du}{4}$

when $t = 0 \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

$$I = \int_{u=\infty}^{0} e^{-u} \left(\frac{u}{4}\right)^3 \frac{du}{4}$$

$n - 1 = 3$

$$I = -\frac{1}{4^4} \int_0^\infty e^{-u} u^3 \, du$$

$$= -\frac{1}{4^4} \Gamma 4$$

$$= -\frac{1}{4^4} 3 \Gamma 3$$

$$= -\frac{1}{4^4} 3 \Gamma (2+1)$$

$$= -\frac{1}{4^4} . 3 . 2 . 1$$

$$= \frac{+6}{64} = -\frac{3}{128}$$

Q5 Prove that $\int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1} dx = \frac{\Gamma m}{n^m}$

put $\log\frac{1}{x} = t \Rightarrow \cancel{\log 1}^{0} - \log x = t$

$\cancel{\frac{1}{x} = e^{t}}$

$-\log x = t$

$\log x = -t$

$x = e^{-t}$

$dx = -e^{-t}\,dt$

where, $x=0 \Rightarrow t=\infty$

$x = 1 \Rightarrow t = 0$

$$I = \int_{t=\infty}^{0} (e^{-t})^{n-1}\, t^{m-1}\,(-e^{-t}\,dt)$$

$\cancel{n-1=m-1}$

$$I = \int_{t=0}^{\infty} e^{-nt}\, t^{m-1}\,dt \quad \text{---} \quad \left[\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

put $nt = y$

$t = \frac{y}{n}$

$dt = \frac{dy}{n}$

$I = \int_0$ when $t=0 \Rightarrow y=0$

$t=\infty \Rightarrow y=\infty$

$$I = \int_0^{\infty} e^{-y} \left(\frac{y}{n}\right)^{m-1} \frac{dy}{n}$$

$$I = \frac{1}{n^m} \int_0^{\infty} e^{-y} y^{m-1}\, dy$$

$$n-1 = m-1$$
$$n = m$$

$$I = \frac{1}{n^m} \Gamma m$$

$$I = \frac{\Gamma m}{n^m} = R.H.S$$

Q6 $\int_0^{\infty} x^2 e^{-x^4} dx \times \int_0^{\infty} e^{-x^4} dx$

put $x^4 = t \Rightarrow x = t^{1/4}$

$$dx = \frac{1}{4} t^{-3/4}\, dt$$

when, $x = 0 \Rightarrow t = 0$
$x = \infty \Rightarrow t = \infty$

$$I = \int_{t=0}^{\infty} (t^{1/4})^2 e^{-t} \frac{t^{-3/4}}{4} dt \times \int_{t=0}^{\infty} e^{-t} \frac{t^{-3/4}}{4} dt$$

$$= \frac{1}{16} \left(\int_0^{\infty} t^{-1/4} e^{-t} dt \times \int_{t=0}^{\infty} e^{-t} t^{-3/4} dt \right)$$

$n-1 = -\frac{1}{4}$ $\qquad$ $n-1 = -\frac{3}{4}$

$n = \frac{3}{4}$ $\qquad$ $n = \frac{1}{4}$

$$= \frac{1}{16}\left(\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}\right)$$

$$= \frac{1}{16}\,\Gamma\frac{3}{4}\,\Gamma 1-\frac{3}{4}$$

$$= \frac{1}{16}\,\frac{\pi}{\sin\frac{3}{4}\pi} = \frac{16}{16}\,\frac{\pi}{1/\sqrt{2}} = \frac{\pi\sqrt{2}}{16}$$

Q7 S.T. if $n > -1$

$$\int_0^\infty x^n e^{-k^2x^2}\,dx = \frac{1}{2k^{n+1}}\,\Gamma\frac{n+1}{2}$$

hence, $\int_\infty^\infty e^{-x^2}\,dx$

$$L.H.S = I = \int_0^\infty x^n e^{-k^2x^2}\,dx$$

put $k^2x^2 = t \Rightarrow \boxed{x = \sqrt{\frac{t}{k^2}}}$

$$x^2 = \frac{t}{k^2}$$

$$2x\,dx = \frac{t}{k^2}$$

$$dx = \frac{dt}{2xk^2}$$

$$dx = \frac{dt}{2\sqrt{\frac{t}{k^2}}\,k^2}$$

$$\boxed{dx = \frac{dt}{2\sqrt{t}\,k}}$$

Q. $\int_0^\infty \frac{x^7}{7^x}dx \Rightarrow$ We know that $\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$ | $= \int_0^\infty \frac{x^7}{7^x}dx = \frac{\Gamma 8}{(\log 7)^8} = \frac{7!}{(\log c)^8}$

when $x = \infty \Rightarrow t = \infty$
$x = 0 \Rightarrow t = 0$

$$I = \int_{t=0}^{\infty}\left(\sqrt{\frac{t}{K^2}}\right)^n e^{-t}\,\frac{dt}{2\sqrt{t}\,K}$$

$$I = \frac{1}{2K^{n+1}}\int_{t=0}^{\infty} t^{\frac{n-1}{2}}e^{-t}\,dt$$

$n-1 = \frac{n\cancel{x}-1}{2}$

$$I = \frac{1}{2K^{n+1}}\Gamma\frac{n+1}{2}$$

$n = \frac{n-1}{2}+1$

$n = \frac{n+1}{2}$

Hence proved

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{\infty} e^{-x^2}dx$$
$$= \int_{-\infty}^{\infty} e^{-x^2}dx$$
$$= \int_{-\infty}^{\infty} 1\cdot e^{-1^2x^2}dx$$
$$= \int_{-\infty}^{\infty} x^0 e^{-1^2x^2}dx$$

We know that
$\int_{-\infty}^{\infty} dx = 2\int_0^\infty dx \rightarrow$ property

$\therefore \int_{-\infty}^{\infty} x^0 e^{-1^2x^2}dx = 2\int_0^\infty x^0 e^{-1^2x^2}dx$

$K=1 \quad n=0$

$$= 2\,\frac{1}{2(1)^{0+1}}\Gamma\frac{0+1}{2}$$
$$= 2\cdot\frac{1}{2}\Gamma\frac{1}{2}$$
$$= \sqrt{\pi}$$

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→ **Beta function** : Beta function is defined and denoted as

$$\beta(m,n) = \int_{x=0}^{1} x^{m-1}(1-x)^{n-1}\,dx$$

$m, n > 0$

→ **Property of beta function** :

1] $\beta(m,n) = \beta(n,m)$

2] $\displaystyle\int_{x=0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$

3] $\displaystyle\int_{0}^{1} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$

4] $\displaystyle\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta = \frac{1}{2}\beta(m,n)$ —— ①

$2m = p$, $2n = q$ (in ①)

$m = \frac{p}{2}$, $2n = \frac{q}{2}$

① $\Rightarrow \displaystyle\int_{\theta=0}^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p}{2},\frac{q}{2}\right)$

$2m-1 = p$ $\quad$ $2n-1 = q$

$2m = p+1$ $\quad$ $2n = q+1$

$m = \frac{p+1}{2}$ $\quad$ $n = \frac{q+1}{2}$

$$(1) \Rightarrow \int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad \text{--- } (2)$$

if, $p = 0$

$$\int_{\theta=0}^{\pi/2} \cos^q\theta = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{q+1}{2}\right)$$

if, $q = 0$

$$\int_{\theta=0}^{\pi/2} \sin^p\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1}{2}\right)$$

→ <u>Relation between Gamma function and Beta function</u>

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Q8 Evaluate $\displaystyle\int_{x=0}^{3} \frac{x^3}{\sqrt{3-x}}\,dx$

$$I = \int_{x=0}^{3} \frac{x^3}{\sqrt{3-x}}\,dx$$

$$I = \int_{x=0}^{3} \frac{x^3}{\sqrt{3}\left(1-\frac{x}{3}\right)^{1/2}}\,dx$$

$$I = \frac{1}{\sqrt{3}}\int_{x=0}^{3} \frac{x^3}{\sqrt{3}} \left(1-\frac{x}{3}\right)^{-1/2} dx$$

put $\frac{x}{3} = t \Rightarrow x = 3t$

$$dx = 3\,dt$$

when $x = 3 \Rightarrow t = 1$

$x = 0 \Rightarrow t = 0$

$$I = \frac{1}{\sqrt{3}} \int_{t=0}^{1} (3t)^3 \left(1 - t\right)^{-1/2} 3\,d\cancel{t}$$

$$I = \frac{3^4}{\sqrt{3}} \int_{t=0}^{1} t^3 \left(1 - t\right)^{-1/2} d\cancel{t}$$

$m - 1 = 3 \qquad n - 1 = -\frac{1}{2}$

$m = 4 \qquad n = \frac{1}{2}$

$$= \frac{3^4}{\sqrt{3}} \beta\left(4, \frac{1}{2}\right)$$

$$= \frac{3^4}{\sqrt{3}} \frac{\Gamma 4 \; \Gamma \frac{1}{2}}{\Gamma \left(4 + \frac{1}{2}\right)}$$

$$= \frac{3^4}{\sqrt{3}} \frac{3! \sqrt{\pi}}{\Gamma \frac{9}{2}}$$

$$= \frac{3^4}{\sqrt{3}} \frac{3! \sqrt{\pi}}{\Gamma \left(\frac{7}{2} + 1\right)}$$

$$= \frac{3^4}{\sqrt{3}} \frac{3! \sqrt{\pi}}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2}} = \frac{3^4}{\sqrt{3}} \frac{3! \cancel{\sqrt{\pi}}}{\frac{3 \times \cancel{105}\, 35 \cancel{\sqrt{\pi}}}{816}} = \frac{3^4}{\sqrt{3}} \times \frac{16 \times 2}{35}$$

$$= 42.757$$

Q9 Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$I = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta$$

$$I = \int_0^{\pi/2} \sin\theta^{1/2} \cos\theta^{-1/2}\, d\theta$$

$$2m-1 = \frac{1}{2} \qquad 2n-1 = \frac{-1}{2}$$

$$m = \frac{3}{4} \qquad n = \frac{1}{4}$$

$$I = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$I = \frac{1}{2} \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2} \frac{\Gamma\left(1-\frac{1}{4}\right)\Gamma\frac{1}{4}}{\Gamma 1}$$

$$I = \frac{1}{2} \frac{\pi}{\sin\pi/4}$$

$$I = \frac{1}{2} \frac{\pi}{1/\sqrt{2}}$$

$$I = \frac{\pi\sqrt{2}}{2}$$

$$I = \frac{\pi}{\sqrt{2}}$$

Q10 Evaluate $\int_{\theta=0}^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$$I = \int_{0}^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta$$

$$I = \int_{0}^{\pi/2} \sin^{-1/2} \cos^{1/2} d\theta$$

$2m-1 = -\frac{1}{2}$ $\qquad$ $2n-1 = \frac{1}{2}$

$m = \frac{1}{4}$ $\qquad$ $n = \frac{3}{4}$

$$I = \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$I = \frac{1}{2} \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}$$

$$I = \frac{1}{2} \pi\sqrt{2}$$

$$I = \frac{\pi}{\sqrt{2}}$$

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Q11 Prove that $\int_{\theta=0}^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_{\theta=0}^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

$$I = \int_{\theta=0}^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_{\theta=0}^{\pi/2} \sqrt{\sin\theta}\, d\theta = L.H.S$$

$$I = \underbrace{\int_{\theta=0}^{\pi/2} \sin^{-1/2}\theta\, d\theta}_{m=\frac{1}{4},\ n=\frac{1}{2}} \times \underbrace{\int_{\theta=0}^{\pi/2} \sin^{1/2}\theta\, d\theta}_{m=\frac{3}{4},\ n=\frac{1}{2}}$$

$$I = \frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right) \times \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{2}\right)$$

$$I = \frac{1}{4} \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \times \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}$$

$$I = \frac{1}{4} \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}\,(\sqrt{\pi})^2}{\Gamma\frac{3}{4} \times \Gamma\frac{5}{4}}$$

$$I = \frac{1}{4} \frac{\pi\sqrt{2}\,(\sqrt{\pi})^2}{\Gamma\frac{3}{4}\,\Gamma\left(\frac{1}{4}+1\right)}$$

$$I = \frac{\cancel{1}}{\cancel{4}} \frac{\pi^2\sqrt{2}}{\Gamma\frac{3}{4}\cdot\frac{\cancel{1}}{\cancel{4}}\cdot\Gamma\frac{1}{4}}$$

$$I = \frac{\pi^2\sqrt{2}}{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}} = \frac{\pi^{\cancel{2}}\cancel{\sqrt{2}}}{\cancel{\pi}\,\cancel{\sqrt{2}}} = \pi = R.H.S$$

Q12 Show that $\beta(n, n+1) = \frac{1}{2}\frac{(\Gamma n)^2}{\Gamma 2n}$ & hence deduce that $\int_0^{\pi/2}\left(\frac{1}{\sin^3\theta} - \frac{1}{\sin^2\theta}\right)^{1/4}\cos\theta \, d\theta = \frac{\Gamma \frac{1}{4}^2}{2\sqrt{\pi}}$

Soln: We know that

$$\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

replace m by n and n by (n+1)

$$\beta(n, n+1) = \frac{\Gamma n \, \Gamma n+1}{\Gamma \cancel{m} + n+1}$$

$$\beta(n, n+1) = \frac{\Gamma n \; n\Gamma n}{\Gamma 2n+1}$$

$$\beta(n, n+1) = \frac{\cancel{n}(\Gamma n)^2}{2\cancel{n}\Gamma 2n}$$

$$\beta(n, n+1) = \frac{1}{2}\frac{(\Gamma n)^2}{\Gamma 2n} \quad \text{—— (1)}$$

Consider L.H.S

$$\int_{\theta=0}^{\pi/2}\left(\frac{1}{\sin^3\theta} - \frac{1}{\sin^2\theta}\right)^{1/4}\cos\theta \, d\theta$$

$$\int_0^{\pi/2} \frac{(1-\sin\theta)^{1/4}}{(\sin^3\theta)^{1/4}} \cos\theta \, d\theta$$

put $\sin\theta = t$

$\cos\theta \, d\theta = dt$

$\int_0^{\pi/2}$ when $\theta = \pi/2 \Rightarrow t = 1$

$\theta = 0 \Rightarrow t = 0$

$$\int_{t=0}^{1} \frac{(1-t)^{1/4}}{t^{3/4}} dt$$

$$\int_0^{1} t^{-3/4}(1-t)^{1/4} dt$$

$m-1 = -\frac{3}{4}$ $\qquad$ $n-1 = \frac{1}{4}$

$m = \frac{1}{4}$ $\qquad$ $n = \frac{5}{4}$

$$= \beta\left(\frac{1}{4}, \frac{5}{4}\right)$$

$$= \beta\left(\frac{1}{4}, \frac{1}{4}+1\right) \quad \text{(by (1))}$$

$$= \frac{1}{2} \frac{\left(\Gamma\frac{1}{4}\right)^2}{\Gamma 2\cdot\frac{1}{4}} \quad \text{(by (1))}$$

$$= \frac{\left(\Gamma\frac{1}{4}\right)^2}{2\sqrt{\pi}}$$

Q13 Evaluate $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx$

Solⁿ: $I = \int_{x=0}^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx$

$$I = \int_0^\infty \frac{1}{a^{m+n}} \frac{x^{m-1}}{\left(1 + \frac{b}{a}x\right)^{m+n}} dx$$

put $\frac{b}{a}x = t \Rightarrow x = \frac{a}{b}t$

$dx = \frac{a}{b}dt$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$I = \frac{1}{a^{m+n}} \int_0^\infty \frac{\left(\frac{a}{b}t\right)^{m-1}}{(1+t)^{m+n}} \frac{a}{b} dt$$

$$I = \frac{1}{a^{m+n}} \left(\frac{a}{b}\right)^{m-1} \left(\frac{a}{b}\right) \int_0^\infty \frac{(t)^{m-1}}{(1+t)^{m+n}} dt$$

$$I = \frac{1}{a^n b^m} \beta(m, n)$$

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Q14 $\int_0^2 x(8-x^3)^{1/3}\,dx$

$$I = \int_0^2 x\,8^{1/3}\left(1-\frac{x^3}{8}\right)^{1/3} dx$$

$$I = 8^{1/3}\int_0^2 x\left(1-\frac{x^3}{8}\right)^{1/3} dx$$

put $\frac{x^3}{8} = t \Rightarrow x = \sqrt[3]{t}\cdot 2$

$$dx = \frac{2}{3}\,t^{-2/3}\,dt$$

when $x=2 \Rightarrow t=1$

$x=0 \Rightarrow t=0$

$$I = 2\int_0^1 2\sqrt[3]{t}\,(1-t)^{1/3}\,\frac{2}{3}\,t^{-2/3}\,dt$$

$$I = \frac{8}{3}\int_0^1 t^{1/3-2/3}(1-t)^{1/3}\,dt$$

$$I = \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3}\,dt$$

$$I = \frac{8}{3}\,\beta\left(\frac{2}{3},\frac{4}{3}\right)$$

$$I = \frac{8}{3}\,\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma 2}$$

$$I = \frac{8}{3}\,\Gamma\frac{2}{3}\,\Gamma\frac{4}{3} = \frac{8}{3}\,\frac{1}{3}\,\Gamma\frac{1}{3}\cdot\frac{2}{3} = \frac{8\times 2\pi}{9\;\sqrt{3}} = \frac{16\pi}{9\sqrt{3}}$$

Differentiation of definite integral:

The value of definite integral $\int_a^b f(x,\alpha)\,dx$ is the function of alpha (parameter) say $F(\alpha)$.

Now to find $F'(\alpha)$ we have to 1^{st} evaluate integral $\int_a^b f(x,\alpha)\,dx$ and then we have to differentiate w.r.t α.

However it is not always possible to evaluate the integral and then differentiate. In that case we 1^{st} partially differentiate $f(x,\alpha)$ and then integrate.

Leibnitz Rule

If $f(x,\alpha)$ & $\frac{\partial}{\partial\alpha} f(x,\alpha)$ be the continuou function of x and α then

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$$

Q15 Evaluate by, using differentiation under the integral ~~stide~~ sign $I = \int_{x=0}^{1} \frac{x^{\alpha}-1}{\log x}\, dx,\ \alpha \geq 0$ —— ①

Soln: Differentiate w.r.t α on b.s.

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\left[\int_0^1 \frac{x^{\alpha}-1}{\log x}\, dx\right]$$

$$= \int_0^1 \frac{\partial}{\partial\alpha} \frac{x^{\alpha}-1}{\log x}\, dx$$

$$= \int_{x=0}^{1} \frac{1}{\cancel{\log x}} (x^{\alpha}\cancel{\log x} - 0)\, dx$$

$$= \int_{x=0}^{1} x^{\alpha}\, dx$$

$$= \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1$$

$$= \frac{1^{\alpha+1}}{\alpha+1} - 0$$

$$= \frac{1}{\alpha+1} \qquad \left(\int \frac{f'(x)}{f(x)} = \log f(x)\right)$$

Integrate w.r.t α

$$I = \int \frac{1}{\alpha+1}\, d\alpha = \log(\alpha+1) + \cancel{c}\ \log c \quad \text{—— ②}$$

for $\alpha = 0$ from ①

$$I = 0$$

put, $\alpha = 0$ in ②

$$0 = \log(0+1) + \log C$$

$$0 = 0 + \log C$$

$$\log C = 0$$

from ② $\Rightarrow$ $I = \log(\alpha+1)$

Q16 Evaluate by differentiation under the integral sign $F(a) = \int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$

hence, S.T. $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, Also find $F(1)$

Soln: $F(a) = \int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$ ——— ①

Differentiate w.r.t 'a' on B.S

$$\frac{dF}{da} = \frac{d}{da}\left[\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx\right]$$

$$= \int_{x=0}^\infty \frac{\partial}{\partial a} \frac{e^{-ax}}{x}\sin x\,dx$$

(Leibnitz Rule)

$\rightarrow \int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}\left[a \sin bx + b \cos bx\right]$

$$= \int_{x=0}^{\infty} \frac{\sin x}{x} e^{-ax} (-x)\, dx$$

$$= -\int_{x=0}^{\infty} \underset{\mathrm{II}}{e^{-ax}} \underset{\mathrm{I}}{\sin x}\, dx \qquad (a = -a \;,\; b = 1)$$

$$\frac{dF}{da} = -\left\{ \frac{e^{-ax}}{a^2+1} \left[(-a) \sin x - \cos x\right] \right\}_0^{\infty}$$

$$\frac{dF}{da} = \frac{-1}{a^2+1} \left\{ e^{-\infty} - e^{0}\left[0 - 1\right] \right\}$$

$$e^{-\infty} = 0$$

$$\frac{dF}{da} = \frac{-1}{a^2+1}$$

Integrate w.r.t a,

$$F(a) = -\int \frac{1}{1+a^2}\, dx$$

$$F(a) = -\tan^{-1}(a) + c \quad \text{—— } (2)$$

put $a = \infty$ from (1) $\quad F(a) = 0$

put $a = \infty$ in (2)

$$0 = -\tan^{-1}(\infty) + c$$

$$0 = -\frac{\pi}{2} + c$$

$$\frac{\pi}{2} = c$$

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$$② \Rightarrow \quad F(a) = -\tan^{-1}(a) + \frac{\pi}{2}$$

$$F(a) = \frac{\pi}{2} - \tan^{-1}(a)$$

from ② $\Rightarrow$ $F(1) = \frac{\pi}{2} - \tan^{-1}(1)$

$$F(1) = \frac{\pi}{2} - \frac{\pi}{4}$$

$$\therefore \quad F(1) = \frac{\pi}{4}$$

For $a = 0$ from ①, we get.

$$F(0) = \int_0^{a} \frac{\sin x}{x}\, dx \quad \text{——— ③}$$

put $a = 0$ in ②

$$F(0) = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2} \quad \text{——— ④}$$

from ③ and ④, $\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

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Q17 Evaluate by differentiation & $\int_0^{\infty}$ eqn's under integral $\int_0^{\infty} \frac{e^{-x}\sin bx}{x}\,dx$

Soln:
$$I = \int_{x=0}^{\infty} \frac{e^{-x}\sin bx}{x}\,dx \quad \text{——— (1)}$$

Differentiate w.r.t 'b' b.s

$$\frac{dI}{db} = \frac{d}{db}\left[\int_0^{\infty} \frac{e^{-x}\sin bx}{x}\,dx\right]$$

$$\frac{dI}{db} = \int_0^{\infty} \frac{\partial}{\partial b}\,\frac{e^{-x}\sin bx}{x}\,dx$$

$$\frac{dI}{db} = \int_{x=0}^{\infty} \frac{e^{-x}\cancel{x}\cos bx}{\cancel{x}}\,dx$$

$$\left\{ \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] \right\}$$

$a=-1$, $b=b$

$$\frac{dI}{db} = \left(\frac{e^{-x}}{1+b^2}\left[-\cos bx + b\sin bx\right]\right)_0^{\infty}$$

$$\frac{dI}{db} = e^{-\infty} - \frac{e^0}{1+b^2}\left[(-1)\cdot 1 + 0\right]$$

$$\frac{dI}{db} = \frac{1}{1+b^2}$$

Integrate w.r.t b

$$I = \int \frac{1}{1+b^2}\, db$$

$$I = \tan^{-1}(b) + C \quad \text{—— (2)}$$

For $b=0$ from (1) $I=0$

put $b=0$ from (2) $\Rightarrow C=0$

$\therefore$ (2) $\Rightarrow$ $I = \tan^{-1}(b)$

Q18 Evaluate $\int_0^\infty e^{-(x^2+a^2/x^2)}\, dx$ by differentiation under the integral sign given that $\int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$

$$I = \int_0^\infty e^{-(x^2+a^2/x^2)}\, dx \quad \text{—— (1)}$$

Differentiate w.r.t a on b.s

$$\frac{dI}{da} = \frac{d}{da}\int_0^\infty e^{-(x^2+a^2/x^2)}\, dx$$

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a} e^{-(x^2+a^2/x^2)}\, dx$$

$$\frac{dI}{da} = \int_0^\infty e^{-(x^2+a^2/x^2)} \left[\cancel{2x} + \frac{2a}{x^2}\right] dx$$

$\frac{dI}{da}$ #

put, $\frac{a}{x} = t \Rightarrow \frac{x}{a} = \frac{1}{t} \Rightarrow x = \frac{a}{t}$

$$dx = -\frac{a}{t^2}\,dt$$

when $x = \infty \Rightarrow t = 0$

$x = 0 \Rightarrow t = \infty$

$$\frac{dI}{da} = -\int_{t=\infty}^{0} e^{-(a^2/t^2 + t^2)} \left[\frac{2a}{\frac{a^2}{t^4}}\right]\left(-\frac{a}{t^2}\right)dt$$

$$\frac{dI}{da} = 2\int_{t=\infty}^{0} e^{-(a^2/t^2+t^2)}\, dt$$

$$\frac{dI}{da} = -2\int_{0}^{\infty} e^{-(a^2/t^2+t^2)}\, dt$$

$$\frac{dI}{da} = -2I$$

Separating the variable

$$\frac{dI}{I} = -2da$$

on ~~put~~ Integration

$$\log I = -2a + \log c$$

$$\log I - \log c = -2a$$

$$\log \frac{I}{c} = -2a$$

$$\frac{I}{c} = e^{-2a}$$

$$I = ce^{-2a} \quad \text{--- (2)}$$

put $a=0$ in (1)

$$I = \int_0^{\infty} e^{-x^2}\, dx = \sqrt{\frac{\pi}{2}} \quad \text{given}$$

put $a=0$ (2) $\Rightarrow \sqrt{\frac{\pi}{2}} = ce^0 \Rightarrow c = \sqrt{\frac{\pi}{2}}$

(2) $\Rightarrow \quad I = \sqrt{\frac{\pi}{2}}\, e^{-2a}$

$\left[1-\cos x = 2\sin^2\frac{x}{2}\right]$ $\left[1+\cos x = 2\cos^2\frac{x}{2}\right]$

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$\sin 2x = 2\sin x \cos x$

$\sin x = 2\sin x/2 \cos x/2$

Q19 Express $\int_0^1 x^m (1-x^n)^p \, dx$ in terms of Beta function $\frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$ and hence evaluate $\int_0^1 x^5 (1-x^3)^{10} \, dx$ $1/396$

Solⁿ

$$I = \int_0^1 x^m (1-x^n)^p \, dx$$

put $x^n = t$

$x = t^{1/n}$

$dx = \frac{t^{\frac{1-n}{n}}}{n} dt$

when $x = 1 \Rightarrow t = 1$

$x = 0 \Rightarrow t = 0$

$$= \int_{t=0}^{1} t^{\frac{m}{n}} (1-t)^p \frac{t^{\frac{1-n}{n}}}{n} dt$$

$$= \frac{1}{n}\int_{t=0}^{1} t^{\frac{m+1-n}{n}} (1-t)^p \, dt$$

$m-1 = \frac{m+1-n}{n}$ $\qquad$ $n-1 = p$

$m = \frac{m+1}{n}$ $\qquad$ $n = p+1$

$$I = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

$$\int_{x=0}^{1} x^5 (1-x^3)^{10} dx$$

$$\int_0^1 x^m (1-x^n)^p = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

$$\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\beta\left(\frac{5+1}{3}, 10+1\right)$$

$$= \frac{1}{3}\beta(2, 11)$$

$$= \frac{1}{3}\frac{\Gamma 2\,\Gamma 11}{\Gamma(2+11)}$$

$$= \frac{1}{3}\frac{1 \cdot 10!}{12!}$$

$$= \frac{1}{3}\frac{1}{12 \times 11}$$

$$= \frac{1}{396}$$

Q20 Evaluate $\int_0^{\pi} \sin^5 x(1-\cos x)^3 dx$

$$I = \int_0^{\pi} \sin^5 x\,(1-\cos x)^3\,dx$$

$$I = \int_0^{\pi} \sin^5 x \cdot \left(2\sin^2\frac{x}{2}\right)^3 dx$$

put $\frac{x}{2} = \theta$

$x = 2\theta$

$dx = 2\,d\theta$

when, $x = \pi \Rightarrow \theta = \frac{\pi}{2}$

$x = 0 \Rightarrow \theta = 0$

$$I = \int_{\theta=0}^{\pi/2} \sin^5 2\theta \cdot (2\sin^2\theta)^3 2d\theta$$

$$I = \int_{\theta=0}^{\pi/2} (2\sin\theta\cos\theta)^5 (2\sin^2\theta)^3 2d\theta$$

$$I = \int_{\theta=0}^{\pi/2} 2^5 \sin^5\theta \cos^5\theta \cdot 2^3 \sin^6\theta \cdot 2d\theta$$

$$I = 2^5 \cdot 2^3 \cdot 2 \int_0^{\pi/2} \sin^{11}\theta \cos^5\theta \, d\theta$$

$$I = \cancel{2}\; 2^9 \int_0^{\pi/2} \sin^{11}\theta \; \cos^5\theta \, d\theta$$

$$2m-1 = 11 \qquad\qquad 2n-1 = 5$$
$$m = 6 \qquad\qquad n = 3$$

$$I = 2^9 \cdot \frac{1}{2} \beta(6, 3)$$

$$I = 2^9 \cdot \frac{1}{2} \cdot \frac{\Gamma 6 \; \Gamma 3}{\Gamma(6+3)}$$

$$I = 2^8 \; \frac{5! \; 2!}{\Gamma 9}$$

$$I = 2^8 \; \frac{5! \cdot 2!}{8!}$$

$$I = 2^8 \; \frac{\cancel{5!} \; \cancel{2!}}{8 \times 7 \times \underset{3}{\cancel{6}} \times \cancel{5!}} = \frac{2^5 \cdot \cancel{2^3}}{\cancel{8} \times 7 \times 3} = \frac{32}{21}$$

Q21 $\int_0^{2a} x\sqrt{2ax-x^2}\,dx = \frac{a^3\pi}{2}$

$$L.H.S = \int_0^{2a} x\,(2ax-x^2)^{1/2}\,dx$$

$$= \int_0^{2a} (2a)^{1/2} x^{3/2}\left(1-\frac{x}{2a}\right)^{1/2} dx$$

$$= \int_0^{2a} (2a)^{1/2}\, x^{3/2}\left(1-\frac{x}{2a}\right)^{1/2} dx$$

put $\frac{x}{2a} = t$

$x = 2at$

$dx = 2a\,dt$

when, $x = 2a \Rightarrow t = 1$

$x = 0 \Rightarrow t = 0$

$$= (2a)^{1/2}\int_{t=0}^{1} (2at)^{3/2}\,(1-t)^{1/2}\, 2a\,dt$$

$$= (2a)^3 \int_{t=0}^{1} t^{3/2}(1-t)^{1/2}\,dt$$

$m = \frac{5}{2}$ $n = \frac{3}{2}$

$$= (2a)^3 \frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{\Gamma 4}$$

$$= (2a)^3 \frac{\frac{3}{2}\Gamma\frac{3}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{3!}$$

$$= (2a)^3 \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{3!}$$

$$= (2a)^3 \frac{3 \times 1 \times 1 \cdot \sqrt{\pi}\ \sqrt{\pi}}{2 \times 2 \times 2 \times 3!}$$

$$= (2a)^3 \frac{\cancel{8}\,\pi}{16 \times \cancel{3}}$$

$$= \frac{a^3 \pi}{2} = R.H.S$$

Q22 Evaluate differentiation under the integral side

$$I = \int_0^1 \frac{x^a - x^b}{\log x}\, dx$$

$$I = \int_0^1 \frac{x^a - x^b}{\log x}\, dx \quad \text{———} \ (1)$$

$$I = \int_0^1 \frac{x^a}{\log x}\, dx - \int_0^1 \frac{x^b}{\log x}\, dx$$

$$I = I_a - I_b$$

Differentiate I_a w.r.t a and I_b w.r.t b

$$I = \frac{d}{da}\int_0^1 \frac{x^a}{\log x}\, dx - \frac{d}{db}\int_0^1 \frac{x^b}{\log x}\, dx$$

$$I = \int_0^1 \frac{\partial}{\partial a}\frac{x^a}{\log x}\, dx - \int_0^1 \frac{\partial}{\partial b}\frac{x^b}{\log x}\, dx$$

$$I = \int_0^1 x^a \, dx - \int_0^1 x^b \, dx$$

$$I = \left[\frac{x^{a+1}}{a+1}\right]_0^1 - \left[\frac{x^{b+1}}{b+1}\right]_0^1$$

$$I = \frac{1}{a+1} - \frac{1}{b+1}$$

Integrate I_a w.r.t 'a' and I_b w.r.t 'b'

$$I = \log(a+1) - \log(b+1) + \log c \quad \text{— (2)}$$

In (1) if $a=0$ and $b=0$ then,

$$I = 0$$

In (2) if $a=0$ and $b=0$

$$0 = 0 - 0 + \log c$$

$$\log c = 0$$

(2) $\Rightarrow$ $I = \log(a+1) - \log(b+1)$

$$I = \log\frac{(a+1)}{(b+1)}$$