Unit - 2 Probability

Random Variable (R.V.)

If we assign a real number x to each sample point of sample space S then we have defined a function on that sample space. This function is called Random variable. It is denoted by 'X' or 'Y'

Ex1: Suppose that coin is tossed twice. Then sample space S is given by.

$$S = \{HH, HT, TH, TT\}$$

Let the random variable be x which denotes no. of heads

$X \rightarrow$ no. of heads

$X = 2$ [for HH]

$X = 1$ [for HT, TH]

$X = 0$ [for TT]

$\therefore$ X takes value $X = 0, 1, 2$

Ex 2: Suppose that we tossed a fair coin 2 times then sample space is

$$S = \{HH, HT, TH, TT\}$$

X is the random variable which is given by

$$X = (\text{no. of heads})^2 - \text{no. of tails}$$

For HH, $X = 2^2 - 0 = 4$
For HT & TH, $X = 1^2 - 1 = 0$
For TT, $X = 0 - 2 = -2$

$\therefore$ X takes values
$X = -2, 0, 4$.

Discrete Random variable (D.R.V)
If the variable takes finite number or countable infinite numbers of values then it is called as discrete random variable.

Ex: $X = \{x_1, x_2, ---- x_n\}$
$X = \{-\infty --- -2, -1, 0, 1, 2 --- \infty\}$

Probability function or Probability distribution for discrete R.V.

Let the discrete R.V. X takes the values
$x_1, x_2, -----, x_n$
with probabilities $f(x_1), f(x_2) ----- f(x_n)$ resp.
Then $f(x)$ is called probability f^n or probability distribution if
i) $f(x) \geq 0$
ii) $\sum_x f(x) = 1$ for all x

Ex 3: Find the probability f^n for the event given in Ex 2.
Sol^n: $X = -2, 0, 4$
$P(X = -2) = \frac{1}{4}$

$$P(X=0) = \frac{2}{4} = \frac{1}{2}$$

$$P(X=4) = \frac{1}{4}$$

Probability f^n is given by

x	-2	0	4
$f(x)$	$1/4$	$1/2$	$1/4$

Distribution for discrete R.V.

Let the D.R.V. x takes value $x_1, x_2 - - - - - x_n$ with probabilities $f(x_1), f(x_2), - - - - f(x_n)$ resp. then distribution f^n $F(x)$ for discrete random variable x is defined as.

$$\text{Ex:} \begin{cases} 0 & -\infty < x \le x_1 \\ f(x_1) & x_1 \le x \le x_2 \\ f(x_1) + f(x_2) & x_2 \le x \le x_3 \\ \vdots & \vdots \\ f(x_1) + f(x_2) + \cdots f(x_n) = 1 & x_n < x < \infty \end{cases}$$

Ex 4: Find distribution f^n for Ex no. 3

Sol^n: distribution is given by

$$F(x) = \begin{cases} 0 & -\infty < x \le -2 \\ 1/4 & -2 \le x \le 0 \\ 1/4 + 1/2 = 3/4 & 0 \le x \le 4 \\ 1/4 + 1/2 + 1/4 = 1 & 4 \le x < \infty \end{cases}$$

Ex5: Find Probability function and distribution for Ex:1.

Sol^n: Probability function is given by

$X = 0, 1, 2$

$P(X=0) = \frac{1}{4}$

$P(X=1) = \frac{2}{4} = \frac{1}{2}$

$P(X=2) = \frac{1}{4}$

x	0	1	2
$f(x)$	1/4	1/2	1/4

Distribution function is given by

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1/4 & 0 \le x < 1 \\ 1/4 + 1/2 = 3/4 & 1 \le x < 2 \\ 1/4 + 1/2 + 1/4 = 1 & 2 \le x < \infty \end{cases}$$

Ex6: A random variable X has the following probability function

X	0	1	2	3	4	5	6	7	8
f(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

i] Determine the value of a

ii] $P(x < 4)$

iii] $P(x > 5)$

iv] Distribution function

Soln: i] We know that

$$\sum_x f(x) = 1$$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$a = \frac{1}{81}$$

ii] $P(x \le 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= a + 3a + 5a + 7a + 9a$

$= 25a$

$P(x \le 4) = 25/81$

iii] $P(x > 5) = P(x=6) + P(x=7) + P(x=8)$

$= 13a + 15a + 17a$

$= 45a$

$= \frac{45}{81} = \frac{5}{9}$

iv] Distribution function is given by $F(x)$

$$F(x) = \begin{cases} 0 & -\infty < x \le 0 \\ a = 1/81 & 0 \le x \le 1 \\ 4a = 4/81 & 1 \le x \le 2 \\ 9a = 1/9 & 2 \le x \le 3 \\ 16a = 16/81 & 3 \le x \le 4 \\ 25a = 25/81 & 4 \le x \le 5 \\ 36a = 4/9 & 5 \le x \le 6 \\ 49a = 49/81 & 6 \le x \le 7 \\ 64a = 64/81 & 7 \le x \le 8 \\ 81a = 1 & 8 \le x \le \infty \end{cases}$$

Ex 7: A discrete random variable x has following probability function.

x	0	1	2	3	4	5	6	7
$f(x)$	0	K	$2K$	$2K$	$3k$	K^2	$2K^2$	$7K^2+K$

i] Find the value of k, $P(x>6)$ and distribution function.

Sol^n: i]. $\sum_x f(x) = 1$

$$K+2K+2K+3K+K^2+2K^2+7K^2+K=1$$

$$8K+K+10K^2=1$$

$$9K+10K^2=1$$

$$\cancel{K(9+10/k)=1} \qquad 10K^2+9K-1=0$$

$$K=-1 \qquad 10K=\cancel{-8}\,1$$

$$K=-1 \qquad k=1/10$$

$\therefore$ We consider $k=1/10$ since $k \neq -1$ as $f(x)\geq 0$)

ii] $P(x>6) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$
$P(x=5) + P(x=6)$

$$= 0 + K + 2K + 2K + 3K + K^2$$

$$= 8K + K^2$$

$$= \frac{8}{10} + \frac{1}{100}$$

$$= \frac{81}{100}$$

$$= 7K^2 + K = \frac{17}{100}$$

iii] Distribution function is given by $F(x)$

$$F(x) = \begin{cases} 0 & -\infty < x \le 0 \\ 0 & 0 \le x \le 1 \\ k = 1/10 & 1 \le x \le 2 \\ 3k = 3/10 & 2 \le x \le 3 \\ 5k = 5/10 = 1/2 & 3 \le x \le 4 \\ 8k = 4/5 & 4 \le x \le 5 \\ 8k + k^2 = 4/5 + 1/100 = 81/100 & 5 \le x \le 6 \\ 8k + 3k^2 = 83/100 & 6 \le x \le 7 \\ 9k + 10k^2 = 1 & 7 \le x \le \infty \end{cases}$$

Ex8. The probability function of random variable x is given by

$$f(x) = \begin{cases} 2P & x = 1 \\ P & x = 2 \\ 4P & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

i] $P(0 \le x \le 3)$
ii] $P(x > 1)$
iii] $P(x \le 2)$

Soln:
$$\sum_x f(x) = 1$$
$$2P + P + 4P + 0 = 1$$
$$7P = 1$$
$$P = \frac{1}{7}$$

i] $P(0 \le x \le 3) = P(x=1) + P(x=2) + P(x=3)$
$= 2P + P + 4P$
$= 7P$
$= 1$

ii] $P(x > 1) = P(x=2) + P(x=3)$
$= P + 4P$
$= 5P$
$= \frac{5}{7}$

iii] $P(x \le 2) = P(x=1) + P(x=2)$
$= 2P + P$
$= 3P$
$= \frac{3}{7}$

Ex:9 Let X ~~by~~ be the random variable giving aces in random draw of 4 cards from a pack of 52 cards. Find the probability function and distribution function for X.

Given that X is a random varible which gives no. of aces

4 cards can be drawn from 52 cards in

$${}^{52}C_4 = \frac{52 \times 51 \times 50 \times 49}{4}$$

$${}^{52}C_4 = 270725 \text{ ways}$$

$\therefore$ No of sample points in sample space is = 270725 ways

1] Zero aces cards can be drawn in

$$^{48}C_4\,^{4}C_0 = \frac{48 \times 47 \times 46 \times 45}{4 \times 3 \times 2 \times 1}$$

$$= 194580 \text{ ways}$$

$$P(x=0) = \frac{194580}{270725}$$

2] 1 ace card can be drawn in

$$^{48}C_3\,^{4}C_1 = \frac{48 \times 47 \times 46}{3 \times 2 \times 1} \times 4$$

$$= 69184 \text{ ways}$$

$$P(x=1) = \frac{69184}{270725}$$

3] 2 aces cards can be drawn in

$$^{48}C_2\,^{4}C_2 = \frac{48 \times 47}{2 \times 1} \times \frac{4 \times 3}{2 \times 1}$$

$$= 1128 \times 6$$

$$P(x=2) = \frac{1128 \times 6}{270725} = \frac{6768}{270725}$$

4] 3 aces cards ~~be dro~~ can be drawn in

$$^{48}C_1 \; ^4C_3 = \frac{48}{1} \times \frac{4 \times \cancel{3} \times \cancel{2}}{\cancel{3} \times \cancel{2} \times 1}$$

$$= 192 \text{ ways}$$

$$P(X=3) = \frac{192}{270725}$$

5] 4 aces cards can be drawn in

$$^{48}C_0 \; ^4C_4 = 1$$

$$P(X=4) = \frac{1}{270725}$$

x	0	1	2	3	4
$f(x)$	$\frac{194580}{270725}$	$\frac{69184}{270725}$	$\frac{6768}{270725}$	$\frac{192}{270725}$	$\frac{1}{270725}$

$$F(x) = \begin{cases} 0 & -\infty < x \le 0 \\ \frac{194580}{270725} & 0 \le x \le 1 \\ \frac{263764}{270725} & 1 \le x \le 2 \\ \frac{270532}{270725} & 2 \le x \le 3 \\ \frac{270724}{270725} & 3 \le x \le 4 \\ 270725/270725 = 1 & 4 \le x < \infty \end{cases}$$

Ex10: An urn holds 5 white 3 black marbles. If 2 marbles are drawn at rondom at replacement

i Find probability function and distbution function

given that X dinotes no. of whites marbles

$\therefore X = \{0, 1, 2\}$

2 marbles can be drawn with replacement in

$^8C_1 \, ^8C_1 = 8 \times 8 = 64$ ways

$\therefore$ Sample space S constains 64 sample points

1] Zero White marbles can be drawn in

$^3C_1 \, ^3C_1 = 3 \times 3 = 9$ ways

$P(X=0) = \frac{9}{64}$

2] 1 white

$^5C_1 \, ^3C_1 = 5 \times 3 = 15$ ways

$^3C_1 \, ^5C_1 = 5 \times 3 = 15$ ways

$P(X=1) = \frac{15+15}{64} = \frac{30}{64}$

3] Two white

$^5C_1 \, ^5C_1 = 5 \times 5 \times 25$ ways

$P(X=2) = \frac{25}{64}$

x	0	1	2
$f(x)$	9/64	30/64	25/64

$$F(x) = \begin{cases} 9/64 & -\infty < x \le 0 \\ 39/64 & 0 \le x \le 1 \\ 1 & 1 \le x \le 2 \\ & 2 \le x \le \infty \end{cases}$$

Ex11: Suppose that a coin is tossed 3 times X denotes no. of ~~phase~~ H. Find the probability function and distribution function. Also draw the graph of PF and PDF.

Ex= $S = \{TTT, TTH, THT, HTT, HHT, HTH, THH, HHH\}$

$X = 0, 1, 2, 3$ = No. of heads

$P(X=0) = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

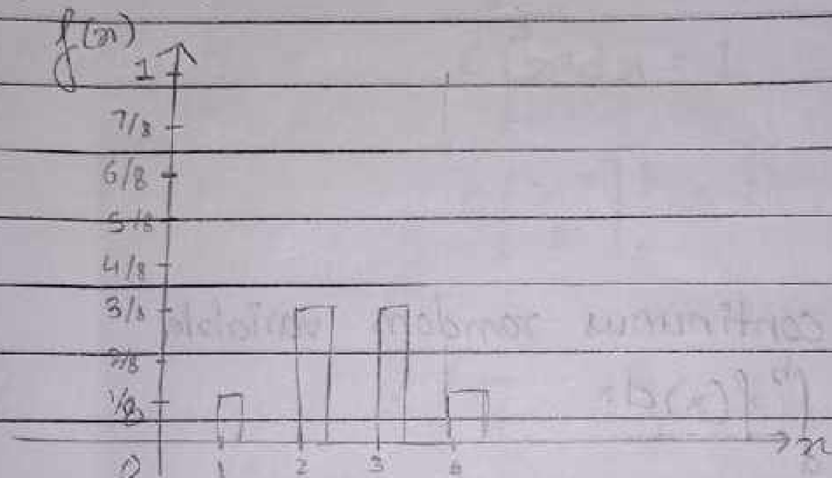
$P(X=2) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$

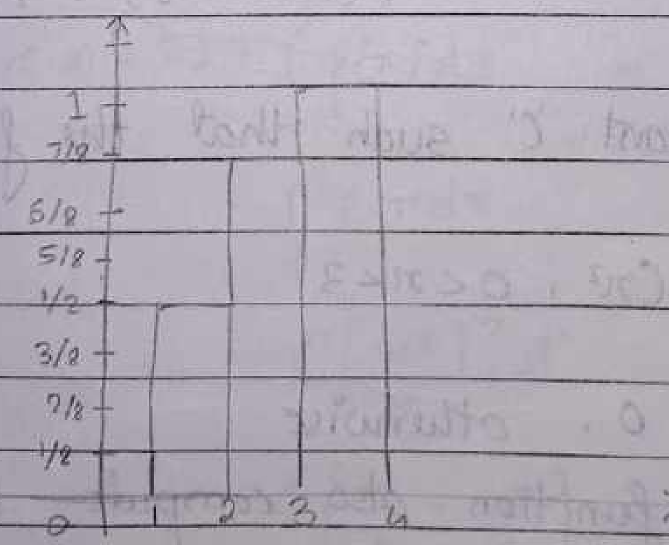
x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

$$F(x) = \begin{cases} 0 & -\infty < x \le 0 \\ 1/8 & 0 \le x \le 1 \\ 4/8 = 1/2 & 1 \le x \le 2 \\ 7/8 & 2 \le x \le 3 \\ 1 & 3 \le x < \infty \end{cases}$$

Graph for prob function:



Graph for pdf:



Continuous random variable

→ Probability function for continuous random variable

let x be the continuous random variable. $f(x)$ is called probability function or density function or continuous random variable.
If it satisfies

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Note:
In case of continuous random variable
$P(a \leq x \leq b) = \int_a^b f(x)\, dx$

Here we can replace '$\leq$' by '$<$'

$$\therefore P(a \leq x \leq b) = P(a < x \leq b)$$
$$= P(a \leq x < b) = P(a < x < b)$$

E12: Find the constant 'C' such that the f^n $f(x)=$

$$f(x) = \begin{cases} Cx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

is a density function also compute $P(1 < x < 2)$

We know that

$$f(x) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{3} cx^2\,dx + 0 = 1$$

$$c\int_{0}^{3} x^2\,dx = 1$$

$$c\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$c\left(\frac{27}{3} - 0\right) = 1$$

$$c9 = 1$$

$$c = \frac{1}{9}$$

$$P(1 < x < 2) = \int_{1}^{2} f(x)\,dx$$

$$= \int_{1}^{2} cx^2\,dx$$

$$= c\left[\frac{x^3}{3}\right]_1^2$$

$$= c\left[\frac{8}{3} - \frac{1}{3}\right]$$

$$= c\frac{7}{3} = \frac{1}{9} \times \frac{7}{3} = \frac{7}{27}$$

Ex13: A continuous random variable x that can assume values between

$X = 2$ and $X = 5$

has a density f^n given by

$f(x) = c(1+x)$

Find i) c ii) $P(x<4)$ iii) $P(3<x<4)$

We know that

$$f(x) = \int_{-\infty} f(x)\,dx = 1$$

$$\int_2^5 c(1+x)\,dx = 1$$

$$c\int_2^5 (1+x)\,dx = 1$$

$$c\left[x+\frac{x^2}{2}\right]_2^5 = 1$$

$$c\left[3+10.5\right] = 1$$

$$c\,13.5 = 1$$

$$c = \frac{2}{27}$$

ii) $P(x<4) = P(-\infty < x < 4)$

$$P(-\infty < x < 4) = \int_{-\infty}^{4} f(x)\,dx$$

$$= \int_{2}^{4} c(1+x)\,dx$$

$$= \frac{2}{27} \int_{2}^{4} (1+x)\,dx$$

$$= \frac{2}{27} \times \left[x + \frac{x^2}{2} \right]_{2}^{4}$$

$$= \frac{2}{27} \times \left[2 + \frac{12}{2} \right]$$

$$= \frac{2 \times 8}{27}$$

$$= \frac{16}{27}$$

$$P(3<x<4) = \int_{3}^{4} f(x)\,dx$$

$$= \int_{3}^{4} c(1+x)\,dx$$

$$= \frac{2}{27} \left[x + \frac{x^2}{2} \right]_{3}^{4}$$

$$= \frac{2}{27} \times \left[1 + \frac{7}{2} \right]$$

$$= \frac{1}{3}$$

* Distribution function of continuous R.V.

$$F(x) = P(X \le x) = P(-\infty < x \le x) = \int_{-\infty}^{x} f(x)\,dx$$

Note:

i) $P(a \le X \le b) = P(X \le b) - P(X \le a)$

$\qquad\qquad\qquad = F(b) - F(a)$

ii) $\frac{d}{dx} F(x) = f(x)$ density f^n

Ex14: A random varible x has density function

$$f(x) = \begin{cases} kx^2 & 1 \le x \le 2 \\ kx & 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

then find distribution function

Sol^n: We know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{1}^{2} kx^2\,dx + \int_{2}^{3} kx\,dx + 0 = 1$$

$$k\int_{1}^{2} x^2\,dx + k\int_{2}^{3} x\,dx = 1$$

$$k\left[\frac{x^3}{3}\right]_1^2 + k\left[\frac{x^2}{2}\right]_2^3 = 1$$

$$k\,\frac{7}{3} + k\,\frac{5}{2} = 1$$

$$k\left[\frac{14+15}{6}\right] = 1$$

$$k\,\frac{29}{6} = 1$$

$$k = \frac{6}{29}$$

$\therefore$ Density f^n is

$$f(x) = \begin{cases} 6/29\, x^2 , & 1 \le x \le 2 \\ 6/29\, x , & 2 < x < 3 \\ 0 , & \text{otherwise} \end{cases}$$

To find distribution function :

For $1 \le x \le 2$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{1}^{x} \frac{6}{29} x^2\,dx$$

$$= \frac{6}{29}\int_{1}^{x} x^2\,dx$$

$$F(x) = \frac{6}{29}\left[\frac{x^3}{3}\right]_1^x$$

$$= \frac{6}{29}\left[\frac{x^3-1}{3}\right] = \frac{2(x^3-1)}{29}$$

$$F(x) = \frac{2(x^3-1)}{29}$$

For $2 < x < 3$

$$F(x) = \int_{\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx = \frac{6}{29}\left(\frac{x^2}{2} + \frac{1}{3}\right)$$

$$= \frac{3x^2+2}{29}$$

$x \geq 3$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$= 0 + \int_{1}^{2} kx^2\,dx + \int_{2}^{3} kx\,dx + \int_{3}^{x} kx^2\,dx = \frac{3x^2+2}{29}$$

$$= \frac{\cancel{6}}{\cancel{29}} \times \frac{7}{\cancel{3}} + \frac{\cancel{5}}{2} \times \frac{\cancel{6}}{\cancel{29}} \cancel{x} \left(\frac{6}{\cancel{29}}\left[\frac{x^{\cancel{2}}}{3} - \cancel{1}\right]\right) \neq 0 \quad \phi$$

$$= \frac{14}{29} + \frac{15}{29} + \frac{6}{29}\left[\frac{x^3 - 1}{3}\right]$$

$$= 1 + \frac{6}{29}\left[\frac{x^3 - 1}{3}\right] = \frac{6}{2}$$

$\therefore$ Distribution f^n

$$F(x) = \begin{cases} 0 & x < 1 \\ \dfrac{2(x^3-1)}{29}, & 1 \le x \le 2 \\ \dfrac{3x^2+2}{29}, & 2 < x < 3 \\ 1, & x \ge 3 \end{cases}$$

Ex16: The distribution f^n for R.V. x is

$$F(x) = \begin{cases} 1-e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

i) Find density f^n

ii) $P(x > 2)$

iii) $P(-3 < X \leq 4)$

Sol^n: Given that

$$F(x) = \begin{cases} 1-e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \text{---} \ (1)$$

we know that

$$f(x) = \frac{d}{dx} F(x)$$

Diff. eq^n (1) w.r. to x, we get

$$f(x) = \begin{cases} 0-(-2)e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\therefore$ Density fu^n is

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

ii) $P(X>2) = P(2<X<\infty)$
$= F(\infty) - F(2)$
$= (1-e^{-\infty}) - (1-e^{-4})$
$P(X>2) = e^{-4}$

iii) $P(-3<X\leq 4) = F(4) - F(3)$
$= 1 - e^{-8}$

Q5] Can the function

$$F(x) = \begin{cases} C(1-x^2) & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

be a distribution Function. Explain

Suppose that $F(x)$ be a distribution f^n and $f(x)$ be a corresponding density f^n

Given that:

$$F(x) = \begin{cases} C(1-x^2) & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases} \quad \text{—— (1)}$$

As, $f(x) = \dfrac{d}{dx} F(x)$

$$\text{eq}^n (1) \Rightarrow f(x) = \begin{cases} C(0-2x), & 0 \leq x \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} -2Cx & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases} \quad \text{—— (2)}$$

We know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 c-2cx)\,dx = 1$$

$$-2c\int_0^1 x\,dx = 1$$

$$-2c\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$-c[1-0] = 1$$

$$c = -1$$

$$eq^n (2) \Rightarrow f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

For $x > 0.5$ we get $f(x) > 1$ which is impposible as probability never exceeds 1.

$\therefore$ $f(x)$ is not a density f^n and hence $F(x)$ is not a distribution f^n

Mathematical Expectation:

1] For discrete random variable

$$E(X) = \Sigma x f(x)$$

2] For continuous random variable

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Theorem 1:
If c is any constant in $E(cX) = cE(X)$

Theorem 2:
If X and Y are any R.V then
$$E(X+Y) = E(X) + E(Y)$$

Theorem 3:
If X and Y are independent R.V :
$$E(XY) = E(X)E(Y)$$

Variance:

$$Var(X) = E\left[(X-\mu)^2\right] \qquad \boxed{\cancel{Var. X = E(X^2)}}$$
$$Var. X = E(X^2) - [E(x)]^2$$

where μ is the expectation of a R.V X i.e.
$$\mu = E(X)$$

The variance is a non-negative number.

Standard deviation:
The +ve square root of the variance is called standard deviation and it is given by
$$\sigma_x = \sqrt{Var(x)}$$

Standard deviation is often denoted by 'σ' and in that case variance is denoted by 'σ^2'

Q18 A random variable x is defined by

$$X = \begin{cases} -2 & , \text{prob } \frac{1}{3} \\ 3 & , \text{prob } 1/2 \\ 1 & , \text{prob } 1/6 \end{cases}$$

Find i) $E(x)$ ii) $E(2x+3)$ iii) $E(x^2)$
iii) $E(x^2+5x)$ v) Var. X vi) σ_x

Soln:

i) $E(X) = \sum x f(x)$

$= (-2)\frac{1}{3} + 3\left(\frac{1}{2}\right) + 1\left(\frac{1}{6}\right)$

$= \frac{-4 + 9 + 1}{6}$

$= 1$

ii) $E(2x+3) = 2E(X) + 3$

$= 2 \times 1 + 3$

$= 5$

iii) $E(x^2) = \sum x^2 f(x)$

$= (-2)\left(\frac{1}{3}\right) + 3^2\left(\frac{1}{2}\right) + 1^2\left(\frac{1}{6}\right)$

$= 4$

iv) $E(x^2+5x) = E(x^2) + 5E(x)$

$= 4 + 5(1)$

$= 9$

v) $Var(X) = E(X^2) - [E(X)]^2$
$= 4 - [1]^2$
$= 3$

vi) $\sigma_x = \sqrt{Var(X)}$
$= \sqrt{3}$

Q.19 The density f^n of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

find i) $E(X)$ ii) $E(3X^2 - 2X)$
iii) $Var(X)$ iv) Standard deviation

i) $E(X) = \int_{-\infty}^{\infty} \cancel{\Sigma} x f(x)\, dx$

$= \int_0^2 x f(x) dx$

$= \cancel{\frac{1}{2}\left[\frac{x^2}{2}\right]_0^2}$ $\int_0^2 x \frac{1}{2} \cdot x\, dx$

$= \cancel{\frac{1}{2}}$ $\frac{1}{2}\int_0^2 x^2 dx$

$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$

$= \frac{4}{3}$

ii) $E(3x^2-2x) = 3E(x^2) - 2E(x)$

or

$$= \int_{-\infty}^{\infty} (3x^2-2x)\, f(x)\, dx$$

$$= \int_{0}^{2} (3x^2-2x)\, f(x)\, dx$$

$$= \frac{1}{2}\int_{0}^{2} (3x^3-2x^2)\, dx$$

$$= \frac{1}{2}\left[3\frac{x^4}{4} - 2\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{9x^4-8x^3}{12}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{9\times16-8\times8}{12}\right]$$

$$= \frac{1}{2}\left[\frac{20}{3}\right]$$

$$= \frac{10}{3}$$

iii) $Var(x) = E(x^2) - [E(x)]^2$

$$= \int_0^2 x^2\, dx - 8\left(\frac{4}{3}\right)^2$$

$$= \left[\frac{x^3}{3}\right]_0^2 dx - \frac{16}{9}$$

$$= \frac{8}{3} - \frac{16}{9}$$

$$= \frac{36-16}{9}$$

$$= \frac{20}{9} = \frac{10}{3}$$

iv) $\sigma_x = \sqrt{Var(X)} = \sqrt{\frac{20}{9}} = \frac{\sqrt{20}}{3}$

Q.20 let X be the random variable

$$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $E(X)$ & Var(X)

i) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

$= \int_0^1 x \cdot 3x^2\, dx$

$= 3 \cdot \left[\frac{x^4}{4}\right]_0^1$

$E(X) = \frac{3}{4}$

ii) $Var(X) = E(X^2) - [E(X)]^2$

$= \int_{-\infty}^{\infty} x^2 f(x) \cdot dx - \left(\frac{3}{4}\right)^2$

$= \int_0^1 x^2 \cdot 3x^2 \cdot dx - \left(\frac{3}{4}\right)^2$

$= 3 \cdot \left[\frac{x^5}{5}\right]_0^1 - \frac{9}{16}$

$= 3\left[\frac{1}{5} - 0\right] - \frac{9}{16}$

$= \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$

Ex 21: A continuous R.V. X has probability density given by ~~f(x) is~~

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

find i) $E(X)$ ii) $E(X^2)$ iii) $Var(X)$
iv) Standard deviation

Soln: $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^{\infty} x \cdot 2e^{-2x}\, dx$

$= 2\int_0^{\infty} \underset{1}{x}\ \underset{2}{e^{-2x}}\, dx$

$\int u.v = u\int v - \int (u'.\int v)$

$E(X) = 2\left[x\left\{\frac{e^{-2x}}{-2}\right]_0^{\infty} - \int\left(1\cdot\left[\frac{e^{-2x}}{2}\right]_0^{\infty}\right) \to 0$

$= 2\left[x\,\frac{e^{-2x}}{-2}\right]_0^{\infty} - 2\int[0]$

$= \cancel{2\times 0} = \frac{1}{2}$

$= 0$

ii) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \int_0^{\infty} x^2 \cdot 2e^{-2x}\, dx$

$x^2 \quad e^{-2x}$
$1 \quad e^{-2x}/2$
$0 \quad e$

$$= 2\int_0^{\infty} x^2 e^{-2x} dx$$

$$= 2\left\{\left[x^2 \frac{e^{-2x}}{-2}\right]_0^{\infty} - \int_0^{\infty} 2x \frac{e^{-2x}}{-2} dx\right\}$$

$$= 2\left\{0 + \int_0^{\infty} x e^{-2x} dx\right\}$$

$$= 2\left\{0 + \frac{1}{4}\right\}$$

$$= 2 * \frac{1}{4}$$

$$= \cancel{2.5}\ 1/2$$

iii) $Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{1}{2} \cancel{2.5} - \left(\frac{1}{2}\right)^2$$

$$= 0.5 - 0.25$$

$$= \cancel{2.125}\ 0.25$$

iv) $\sigma_x = \sqrt{Var X} = \sqrt{0.25} = 0.5$

~~Ex22~~ HOW

Ex22: A R.V. X has density function

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find i) $E(X)$

ii) $E(X^2)$

iii) $E(e^{2X/3})$

Q23: Let X be the R.V. with density function $f(x) =$

$$f(x) = \begin{cases} x^2/9 , & 0 \le x \le 3 \\ 0 , & \text{otherwise} \end{cases}$$

Find : i) $E(X)$ ii) $Var(X)$

Moment generating function :
It is defined by $M_x(t) = E(e^{tx})$

For discrete R.V. X
$$M_x(t) = \sum e^{tx} f(x)$$

For continuous R.V. X.
$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Ex24: Find the moment generating function of the R.V.
$$X = \begin{cases} 1/2, & \text{prob. } 1/2 \\ -1/2, & \text{prob. } 1/2 \end{cases}$$
Hence find 1st 4 moments about ~~origin~~ origin

Sol^n: Moment generating f^n for discrete R.V. is given by
$$M_x(t) = \sum e^{tx} f(x)$$
$$= e^{1/2 t} \frac{1}{2} + e^{t(-1/2)} \cdot \frac{1}{2}$$
$$M_x(t) = \frac{1}{2}\left(e^{t/2} + e^{-t/2}\right) \quad \text{—— (1)}$$

$$e^{t/2} = 1 + \frac{t}{2} + \frac{(t/2)^2}{2!} + \frac{(t/2)^3}{3!} + \frac{(t/2)^4}{4!} + \dots \quad \text{—— (2)}$$

$$e^{-t/2} = 1 - \frac{t}{2} + \frac{(-t/2)^2}{2!} + \frac{(t/2)^3}{3!} + \frac{(-t/2)^4}{4!} + \dots \quad \text{—— (3)}$$

eq^n (1) $\Rightarrow$

$$M_x(t) = \frac{1}{2}\left[2 + 2\frac{(t/2)^2}{2!} + 2\frac{(t/2)^2}{4!} + \dots\right] \quad \text{—— (4)}$$

$$M_x(t) = 1 + \frac{t^2}{8} + \frac{t^4}{384} + \cdots$$

We know that

$$M_x(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \cdots \quad \text{--- (5)}$$

$\mu_1' = 0$ $\quad\quad$ $\mu_3' = 0$

$\mu_2' = 1/4$ $\quad\quad$ $\mu_4' = 1/16$

Q25 The R.V. X can assume the value 1 and -1 with probability 1/2 each find moment generating f^n and 1^{st} 4 moments about origin.

Find moment generating f^n for R.V. X having density f^n

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0; & x < 0 \end{cases}$$

and determine 1^{st} 4 moments about origin

Sol^n: Moment generating f^n is given by : $M_x(t) = E(e^{tx})$

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{0}^{\infty} e^{tx} e^{-x}\,dx$$

$$= \int_0^\infty e^{x(t-1)}\,dx$$

$$= \left[\frac{e^{(t-1)x}}{(t-1)}\right]_0^\infty$$

$$= 0 - \frac{1}{t-1}$$

$$= \cancel{\frac{1}{1-t}} - \frac{1}{t-1}$$

$\therefore$ This is the Moment generating f^n $M_x(t) = -\frac{1}{t-1}$

$$M_x(t) = \frac{1}{1-t}$$

$$M_x(t) = (1-t)^{-1}$$

$$M_x(t) = 1 + t + t^2 + t^3 + t^4 + \cdots \infty \quad \text{—— (1)}$$

$$\because (1-a)^{-1} = 1 + a + a^2 + a^3 + \cdots \infty$$

Expansion of no. of f is

$$M_x(t) = 1 + \mu_1' t + \mu_2' \frac{t}{2!} + \mu_3' \frac{t^3}{3!} \cdots \infty \quad \text{—— (2)}$$

By comparing eq^n (1) & (2), we get

$$\mu_1' = 1$$

$$\frac{\mu_2'}{2!} = 1$$

$$\mu_2' = 2$$

$$\frac{\mu_3'}{3!} = 1 \Rightarrow \mu_3' = 6$$

$$\frac{\mu_4'}{4!} = 1$$

$$\mu_4' = 24$$

$\therefore$ First 4 moments along origin are

$\mu_1' = 1, \; \mu_2' = 2, \; \mu_3' = 6, \; \mu_4' = 24$

HW

~~Ex27~~. Find MGF of R.V. X having desity f^n

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Also find 1^{st} 4 moments of MGF

Ex28: Find the MGF for f

$$f(x) = \begin{cases} \frac{1}{b-a} , & a < x < b \\ 0 , & \text{otherwise} \end{cases}$$

Also find 1st f 4 moments about origin

Soln: 1st 4 moments are given by

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) \, dx$$

$$= \int_{a}^{b} e^{tx} \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \int_{a}^{b} e^{tx} \, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$= \frac{1}{b-a} \left(\frac{e^{tb}}{t} - \frac{e^{ta}}{t} \right)$$

$$= \frac{1}{b-a} \left(\frac{e^t}{t} (e^b - e^a) \right)$$

This required m.g.f

$$Mx(t) = \frac{1}{t(b-a)}\left[\left(1 + bt + \frac{b^2t^2}{2!} + \frac{b^3t^3}{3!} + \frac{b^4t^4}{4!} \cdots \infty\right)\right.$$

$$\left. - \left(1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!}\right)\right]$$

$$Mx(t) = \frac{1}{t(b-a)}\left[(b-a)t + (b^2-a^2)\frac{t^2}{2!} + (b^3-a^3)\frac{t^3}{3!}\right.$$

$$\left. + (b^4-a^4)\frac{t^4}{4!}\right]$$

$$= 1 + \frac{(b^2-a^2)}{b-a}\frac{t}{2} + \frac{b^3-a^3}{(b-a)}\frac{t^2}{6} +$$

$$\frac{(b^4-a^4)}{b-a}\frac{t^3}{24} + \cdots \infty \quad (1)$$

We know that,

$$Mx(t) = 1 + \mu_1' t + \mu_2' \frac{t}{2!} + \mu_3' \frac{t}{3!} \cdots \infty \quad (2)$$

Comparing eqn (1) and (2)

$$\frac{(b^2-a^2)}{b-a} = \mu_2' \qquad \mu_1' = 0$$

$$\mu_3' = \frac{b^3-a^3}{b-a}$$

$$\mu_4' = \frac{b^4-a^4}{b-a}$$

Unit 3 : Probability distribution

Consider a random experiment with only 2 possible outcomes say success

The P denotes probability of success and $q = 1-p$ is probability of failure. Suppose that the experiment is independently repeated and X represents No. of success in an trials. Hence X assumes the values 0, 1, 2, - - - , n. then probability of exactly x successes in n trials is given by the probability function

$$f(x) = P(X=x) = {}^nC_x \, p^x q^{n-x},$$

$$x = 0, 1, 2, - - - n$$

* Properties of binomial distribution :

1. If the R.V. X is binomially distributed then

The mean of X is : $\mu = np$
The variance of X is : $\sigma^2 = npq$
The standard deviation is : $\sigma = \sqrt{npq}$

Q. Out of 800 families with 5 children each, how many would you expect to have :
(i) 3 boys (ii) 5 girls (iii) either 2 or 3 boys

assume different probabilities for boys and girls

Sol^n: Let P and $\cdot$ be the probabilities of success and respectively. Let n be the no. of trials

We know that

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

i) $P(3 \text{ boys}) = P(X=3)$

$$= {}^5C_3 (1/2)^3 (1/2)^2$$

$$= \frac{5\times4\times3}{3\times2\times1} \times \frac{1}{8} \times \frac{1}{4} = \frac{5}{16}$$

$\therefore$ The number of families having 3 boys is $\frac{5}{16} \times 800 = 250$

ii) $P(5 \text{ girls}) = {}^5C_5 (1/2)^5 (1/2)^0$

$$= 1 \times 1/32 \times 1$$

$$= 1/32$$

iii) $P(\text{either } 2 \text{ or } 3 \text{ boys}) = P(X=2) + P(X=3)$

$$= 10/16 = \cancel{8}\ 5/8$$

Q. An insurance sales man sales policy to 5 males. All of identical age and in good health. The probability that the man of this particular will be alive 30 years is 2/3. FTPT in 30 years

(i) All 5 men
(ii) Atleast 3 men
(iii) Only 2 men
(iv) Atmost 1 men
(v) Atleast 1 men will be alive

We have no. of trials

$p = \frac{2}{3} \qquad q = \frac{1}{3}$

We know that

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

i) $P(\text{all 5 men}) = P(X=5)$

$= {}^5C_5 \left(2/3\right)^5 \left(1/3\right)^0$

$= \frac{32}{243}$

ii) $P(\text{At least 3 men}) = P(X=3) + P(X=4) + P(X=5)$

$= {}^5C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 + {}^5C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^1 + {}^5C_5 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^0$

$= \cancel{\frac{194}{2}}\ \frac{21632}{729}$

iii) $P(\text{Only 2 men}) = {}^5C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^3$

$= \frac{10 \times 4}{9 \times 7}$

$= \frac{40}{243}$

iv) $P(\text{Atmost 1 man}) = P(x=0) + P(x=1)$

$= {}^5C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^5 + {}^5C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^4$

$= \frac{11}{243}$

v] At least one man will be alive.

$= P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= {}^5C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^4 + {}^5C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^3 + {}^5C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 + {}^5C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^1$

$+ {}^5C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^0$

$= \frac{5 \times 2}{243} + \frac{40}{243} + \frac{80}{243} + \frac{33}{243} = \frac{243}{243}$

$= 1$

Q. The probability that a bomb dropped from a plane will strike a target is 1/5. If 6 bombs are dropped, FTPT
i) exactly 2 will strike the target
ii) atleast 2 will strike

Soln: We have number of trials $n = 6$
The probability of succes $p = \frac{1}{5}$, $q = 1-p = 1-\frac{1}{5} = \frac{4}{3}$

i] exactly 2 strike the target

$$P(X=2) = {}^6C_2\, p^2 q^{6-2}$$

$$= \frac{6\times5}{2\times1}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4$$

$$= 15 \times \frac{1}{25} \times \frac{256}{625} = 0.2457$$

ii] At least 2 will strike the target

$$P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= {}^6C_2\, p^2 q^{6-2} + {}^6C_3\, p^3 q^{6-3} + {}^6C_4\, p^4 q^{6-4} + {}^6C_5\, p^5 q^{6-5} + {}^6C_6\, p^6 q^{6-6}$$

$$= 15\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4 + 20\times\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3 + 15\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^2 + 6\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)^6\left(\frac{1}{5}\right)^0$$

$$= \frac{15}{25}\times\frac{256}{625} + \frac{20\times}{125}\frac{64}{125} + \frac{15}{625}\times\frac{16}{25} + \frac{6}{3125}\times\frac{4}{5} + \frac{1}{15625}$$

$$= \frac{3840}{15625} + \frac{1280}{15625} + \frac{240}{15625} + \frac{96}{15625} + \frac{1}{15625}$$

$$= \frac{5457}{15625}$$

$$= 0.3492$$

Q. The probability that an ~~anthor~~ entering student will gradute is 0.4. Determine TPT out of 5 students

i) none

ii) one

iii) atleast one

Solution: We have number of trials $n=5$

$p = 0.4$ $\quad q = 1 - 0.4$ $\quad q = 0.6$

We know that

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

i) none

$= P(X=0)$

$= {}^5C_0\, p^0 q^{5-0}$

$= 1 \times (0.4^0) \times (0.6)^5$

$= 0.0777$

ii) $1 = P(X=1)$

$= {}^5C_1\, p^1 q^{5-1}$

$= 5 \times (0.4)^1 \times (0.6)^4$

$= \cancel{0.05184}\ 0.2592$

iii) atleast one

$1 - (0.6)^5 = 0.9221$

Q The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that
i] exactly 2 will be defective.
ii] at least 2 will be defective.
iii] none will be defective.

→ The probability of defective pen,

$$p = \frac{1}{10} \qquad q = 1 - \frac{1}{10} = \frac{9}{10}$$

$$p = 0.1 \qquad q = 0.9$$

Here number of trials $n = 12$

We know that

$$P(X = x) = {}^nC_x p^x q^{n-x}$$

i] exactly 2 will be defective.

$$P(X=2) = {}^{12}C_2 p^2 q^{12-2}$$
$$= \frac{12 \times 11}{2 \times 1} (0.1)^2 (0.9)^{10}$$
$$= 66 \times 0.01 \times 0.3486$$
$$= 0.2300$$

ii] ~~that~~ at two will be defective.

$$= 1 - [P(X=0) + P(X=1)]$$
$$= 1 - [{}^{12}C_0 p^0 q^{12-0} + {}^{12}C_1 p^1 q^{12-1}]$$
$$= 1 - [1 \times (0.1)^0 (0.9)^{12} + 12 (0.1)(0.9)^{11}]$$
$$= 1 - [0.2824 + 0.3765] = 1 - 0.6589 = 0.3411$$

iii] None will be defective.

$P(X=0) = {}^{12}C_0 p^0 q^{12-0}$

$= 1 (0.1)^0 (0.9)^{12}$

$= 1 \times 1 \times 0.2824$

$= 0.2824$

Q6. In sampling a large number at parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 deffective parts

→

Number of trials, $n = 20$

Mean number of defective $= 2$

$\mu = 2$

But $\mu = np$

$\therefore np = 2$

$20p = 2$

$p = \frac{1}{10}$

$p = 0.1$

$q = 1 - p$

$= 1 - \frac{1}{10}$

$q = \frac{9}{10}$

$q = 0.9$

We know that

$P(X=x) = {}^nC_x p^x q^{n-x}$

P(at least 3 defective parts).

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - [{}^{20}C_0 p^0 q^{20-0} + {}^{20}C_1 p^1 q^{20-1} + {}^{20}C_2 p^2 q^{20-2}]$

$= 1 - [1 \times 1 \times (0.9)^{20} + 20 \times 0.1 \times (0.9)^{19} + \frac{20 \times 19}{2 \times 1} (0.1)^2 (0.9)^{18}]$

$= 1-[0.1215 + 20 \times 0.1 \times 0.7350 + 190 \times (0.1)^2 (0.1500)]$
$= 1-[0.1215 + 0.27 + 0.285]$
$= 1-0.6765$
$= 0.3235$

Q ~~Ques~~ If the chance that 1 of the 10 telephone lines is busy at an instant is 0.2, what is the ~~prop~~ chance that -

a] 5 at the lines are busy

b] What is the most probable number of busy line and what is probability of this number.

c] What is the probability that all the lines are busy.

→

Number of trials, $n=10$

$p=0.2 \qquad q=1-0.2 \qquad q=0.8$

We know that

$P(X=x) = {}^nC_x\, p^x q^{n-x}$

a] $P(X=5) = {}^{10}C_5\, p^5\, q^{10-5}$
$= (0.2)^5\ (0.8)^5$
$= 252 \times 0.00032 \times 0.3276$
$= 0.0264$

b] Expected number of busy lines
$E(X) = np$
$= 0.2 \times 10 = 2$

$P(X=2) = {}^nC_x\, p^x q^{n-x} = {}^{10}C_2\, (0.2)^2 (0.8)^8 = 45$

Page

c] P(all lines are busy) = P(no line is free)

$\therefore P(X=0) = {}^{10}C_0 p^{10} q^0$

$= {}^{10}C_0 (0.2)^{10} (0.8)^0 = 0.1024 \times 10^{-6}$

$\therefore$ The probability of all lines are busy is $0.1024 \times$

Poisson Distribution :

If the D.R.V. has the probability function

$$f(x) = P(X=x) = \frac{\lambda^{x} e^{-\lambda}}{x}$$

$$x = 0, 1, 2 \ldots\ldots$$

where λ is positive constant then X is said to be poisson distributed and the probability distribution $f(x)$ is called poisson distribution.

* Properties of poisson distribution

- If R.V. x is poisson distributed then

1. the mean of X is $\mu = \lambda$
2. the variance of X is $\sigma^2 = \lambda$
3. S.D is $\sigma = \sqrt{\lambda}$

Relation between the binomial and poisson distribution

If the no. of trial 'n' in the B.D. is large and the probability of success p ~~is~~ is closer to zero then the B.D. can be closely approximated by the poisson distribution with mean $\mu = \lambda = np$.

Q If 3% of electric bulbs manufactured by the company are defective find the probability that in the sample of 100 bulbs.

1. exactly 2
2. more than 5
3. between 1 and 3

Date
Page

④ at most 2
⑤ atleast 2 bulbs will be defective.

let X denotes the no. of defective bulbs

we have $p = \frac{3}{100} = 0.03 \quad q = \frac{97}{100} = 0.97$

here no. of trials, $n = 100$

The probability of success is very small and no of trials is very large ∴ we will use P.D.

$$\lambda = np = 100 \times \frac{3}{100} = 3$$

i] $P(X=2)$

$$\frac{3^2 e^{-3}}{2!} = \frac{9e^{-3}}{2} = 0.22404$$

ii] $P(X>5) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)]$

$$= 1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} + \frac{3^3 e^{-3}}{3!} + \frac{3^4 e^{-3}}{4!}\right]$$

$$= 1 - \left[e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}\right)\right]$$

$$= 1 - \left[e^{-3}(14.125)\right]$$

$$= 0.19697$$

ii] between 1 and 3

$P(1 \le X \le 3) = P(X=1) + P(X=2) + P(X=3)$

$= \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} + \frac{3^3 e^{-3}}{3!}$

$= e^{-3}\left[3 + \frac{9}{2} + \frac{27}{6}\right]$

$= 0.0497 \times [3 + 4.5 + 4.5]$

$= 0.0497 \times 12$

$= 0.5964$

iii] $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!}$

$= e^{-3}\left[1 + 3 + \frac{9}{2}\right]$

$= 0.4224$

iv] $P(X \ge 2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \cancel{\frac{3^2 e^{-3}}{2!}}\right]$

$= \cancel{e^{-3}}[\cdot 1 - [1+3] e^{-3}$

$= 1 - 0.1988$

$= 0.8012.$

Q. Between the hours 2pm and 4pm the average number of phone calls per minute coming into the switch board of a company is 2.35. find the probability that during 1 ~~practicah~~ particular min there will be atmost 2 phone

If X denotes the no. of phone calls per min then it follows poisson distribution with $\lambda = 2.35$

We know that

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{(2.35)^0 e^{-2.35}}{0!} + \frac{(2.35)^1 e^{-2.35}}{1!} + \frac{(2.35)^2 \cdot e^{-2.35}}{2!}$$

$$= e^{-2.35}\left(1 + 2.35 + \frac{5.5225}{2}\right)$$

$$= e^{-2.35}(1 + 2.35 + 2.7612$$

$$= e^{-2.35}(6.112)$$

$$= 0.5828$$

Q. A certain screw making machine produces on average 2 detective screws out of 100 and packs them in boxes of 500, find the probability that a box contain 15 detective screws.

Given that : machine produces 2 detective screws out of 100.

$n = 500 \quad p = 0.02 \quad \lambda = np = \frac{500 \times 2}{100}$

$$\lambda = 10$$

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$P(X=15) = \frac{10^{15} e^{-10}}{15!} = \frac{10^{15} \times 4.5399 \times 10^{-5}}{15!} = 0.0347$$

Q If the probability at a bad reaction from a certain injection of 2000 individuals more than 2 will get a bad reaction.

$p = 0.001 \qquad n = 2000$

$\lambda = np$

$\lambda = 2000 \times 0.001$

$\lambda = 2$

We know that

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$P(X > 2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!}\right]$$

$$= 1 - e^{-2}[1 + 2 + 2]$$

$$= 1 - e^{-2} \times 5$$

$$= 1 - 5e^{-2}$$

$$= 1 - 0.6776$$

$$= 0.3234$$

Q In a certain factory turning out razor blades, their is a small change at 0.002 for any blade to be defective. The blades are supplied in packets of 10, use poisson distribution to calculate the approximate number of packets containing no detective, 1 defective and 2 defective blades respectively in a consignment of 10,000 packets.

$p = 0.002$

$n = 10$

$\lambda = np$

$\lambda = 10 \times 0.002 = 0.02$

We know that $P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$$P(X=0) = \frac{(0.02)^1 e^{-0.02}}{0!}$$
$$= 1 \times 0.9801$$
$$= 0.9801$$

$$P(X=1) = \frac{(0.02)^1 e^{-0.02}}{1!}$$
$$= 0.9801 \times 0.02$$
$$= 0.0196$$
$$= 0.0196 \times 10{,}000$$
$$= 196$$

$$P(X=2) = \frac{(0.02)^2 \times e^{-0.02}}{2!}$$
$$= \frac{4 \times 10^{-4} \times 0.9801}{2}$$
$$= 1.9602 \times 10^{-4}$$

10,000 consignment of packets

$$= 1.9602 \times 10^{-4} \times 10000$$
$$= 1.9602$$

Q A manufacturer knows that the condenser's he make contain on the average 1% defective he packs them in boxes of 100. What is the probability that a box picked at random contain 3 or more faulty condenser.

$\lambda = \frac{1}{100} = 0.01$

$P(x \geq 3) = 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - \left[\frac{(0.01)^0 e^{-0.01}}{0!} + \frac{(0.01)^1 e^{-0.01}}{1!} + \frac{(0.01)^2 e^{-0.01}}{2!}\right]$

$= 1 - e^{-0.01}[1 + 0.01 + 5 \times 10^{-5}]$

$= 1 - e^{-0.01}(1.01005)$

$= 1 - 0.999$

$= 0.001$

Q A car hire firm has 2 cars which it hires out day by day. The number of depends for a car on each day is distributed as a poisson distribution with mean, $\mu = 1.5$. Calculate the proportion of base

i] On which their is no demand. $P(X=0)$

ii] On which demand is refused P.

$\lambda = 1.5$

$P(x=0) = \frac{(1.5)^0 e^{-1.5}}{0!}$

$= 0.2231$

Proportion of days on which neither car is used is 22.31%.

$$P(x>2) = 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - \left[\frac{(1.5)^0 e^{-1.5}}{0!} + \frac{(1.5)^1 e^{-1.5}}{1!} + \frac{(1.5)^2 e^{-1.5}}{2!}\right]$$

$$= 1 - e^{-1.5}\left[1 + 1.5 + \frac{2.25}{2}\right]$$

$$= 1 - e^{-1.5}(3.625)$$

$$= 1 - 0.8087$$

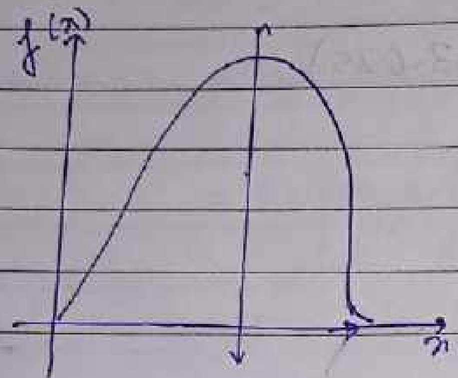
$$= 0.1913$$

Proportion of days on which demand is refused = 19.31%

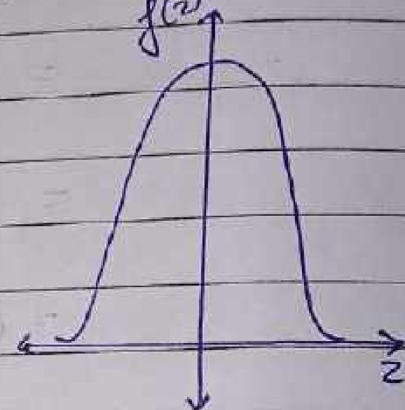
Normal Distribution:

If Z is standardized variable corresponding to X then $Z = \frac{X-\mu}{\sigma}$.

where μ and σ are mean and S.D. resp, X is said to be normally distributed and probability distribution if it is given by:



Normal Curve



Standard curve

Q If the diameter of ball bearings are normally distributed with mean 15.60mm and standard deviation 0.06mm. Determine the percentage of ball bearings with diameter

a] between 15.50 and 15.70mm
b] greater than 15.70mm
c] less than 15.40mm
d] Equal to 15.60 mm.

Assume the measurements to be recorded to the nearest 0.01mm

Sol^n: X denotes diameters of ball bearings which is normally distributed with mean $\mu = 15.60$ and S.D. $\sigma = 0.06$

Let X be the standarize R.V given by

$$Z = \frac{X-\mu}{\sigma} = \frac{X - 15.60}{0.06}$$

a] between 15.50 and 15.70 mm

$(15.49 \text{ in std units}) = \dfrac{15.49 - 15.60}{0.06}$

$= -1.833$

$(15.71) \text{ in std units} = \dfrac{15.71 - 15.60}{0.06}$

$= 1.833$

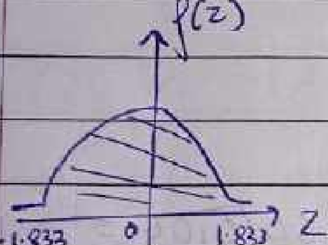
$P(15.50 \le X \le 15.70) = P(-1.833 \le Z \le 1.833)$

= area between $Z = -1.833$ and $Z = 1.833$

$= 2(\text{area between } Z = 0 \text{ and } Z = 1.833)$

$= 2(0.4664)$

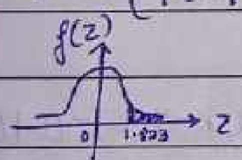
$= 0.9328$



$\therefore$ % of ball bearings with diameter between 15.50 and 15.70mm = 93.28%

b] greater than 15.70

$(15.71 \text{ in standard units}) = \dfrac{15.71 - 15.60}{0.06}$



$= 1.833$

$P(X > 15.70) = P(Z > 1.883)$

$= 0.5 - (\text{area between } Z = 0 \text{ and } Z = 1.833)$

$= 0.5 - 0.4664$

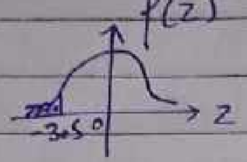
$= 0.0336$

$\therefore$ Percentage of ball bearings with diameter greater than 15.70 mm = 3.36%

c] less than 15.40mm

$(15.39 \text{ in std units}) = \dfrac{15.39 - 15.60}{0.06}$

$= -3.5$



$P(X < 15.40) = P(Z < -3.5)$

$= 0.5 - (\text{area between } z=0 \text{ and } z=3.5)$

$= 0.5 - 0.4998$

$= 0.0002$

∴ percentage of ball bearings with diameter less than 15.40 mm is 0.02%.

d] equal to 15.60mm

$(15.61 \text{ in std units}) = 15.61 - 15.60/0.06 = 0.167$

$(15.59 \text{ in std units}) = \dfrac{15.59 - 15.60}{0.06} = -0.167$

$P(15.59 \leq X \leq 15.61) = P(X = 15.60)$

$P(X = 15.60) = \text{area between } z = -0.167 \text{ and } z = 0.167$

$= 2(\text{area between } z=0 \text{ and } z=0.167)$

$= 2 \times 0.0675$

$= 0.135$

∴ % of ball bearings with diameter equal to 15.60mm is 13.5%.

Q. X is a normal variate with mean 30 and $\sigma = 5$ find the probability that

i) $26 \le X \le 40$

ii) $X \ge 45$

iii) $|X - 30| > 5$

0.2881 + 4772

Soln: Given that $\mu = 30$ and $\sigma = 5$

$$Z = \frac{X-\mu}{\sigma} = \frac{X-30}{5} = \frac{X-30}{5}$$

i) $26 \le X \le 40$

$$(26 \text{ in std terms}) = \frac{26-30}{5} = -0.8$$

$$(40 \text{ in std terms}) = \frac{40-30}{5} = 2$$

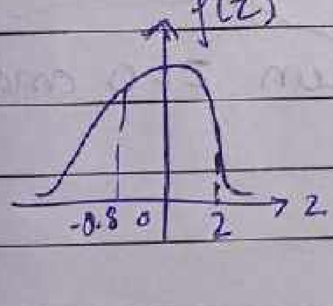
$$P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$$

= area between $z = -0.8$ and $z = 2$

= (area of $z = -0.8$) + (area of $z = 2$)

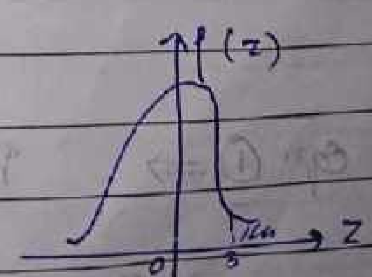
= .2881 + .4772

= 0.7653



ii) $X \ge 45$

$$X = 45, \quad Z = \frac{45-30}{5} = 3$$



$$P(X \ge 45) = P(Z \ge 3)$$

$$P(X \geq 45) = P(X \geq 3)$$
$$= 0.5 - (\text{area between } Z = 0 \text{ and } z = 3)$$
$$= 0.5 - 0.4987 = 0.0013$$

iii] $P(|X-30| > 5) = 1 - P(|X-30| \leq 5)$

$|X-30| \leq 5$

$X - 30 = 5 \Rightarrow X = 35$

$-(X-30) = 5 \Rightarrow X = 25$

$25 \leq X \leq 35$

$P(|X-30| \leq 5) = P(25 \leq X \leq 35)$

when, $X = 25$, $Z = \frac{25-30}{5} = -1$

when, $X = 35$ $Z = \frac{35-30}{5} = 1$

$$P(|X-30| \leq 5) = P(25 \leq X \leq 35)$$
$$= P(-1 \leq Z \leq 1)$$
$$= 2(\text{area between } Z = 0 \text{ and } z = 1)$$
$$= 2 \times 0.3413$$
$$= 0.6826$$

eqⁿ ① $\Rightarrow$ $P(|X-30| > 5) = 1 - P(|X-30| \leq 5)$
$$= 1 - 0.6826$$
$$= 0.3174$$

Q. In a test on 2000 electric bulb it was found that the life of a proticular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for 2

a] more than 2150 hours

b] less than 1950 hours

c] more than 1920 hours but less than 2160 hours

$\sigma = 60$ hours $\qquad \mu = 2040$ hours

X denotes the, number of bulbs like to burn for

$$Z = \frac{X-\mu}{\sigma} = \frac{X-2040}{60}$$

a] $P(X > 2150)$

We change to the standard normal variate.

The total area to the right of $Z=0$ is 0.5.

The area between $Z=0$ and 1.833 is 0.4664

So $P(Z>1.833) = 0.5 - 0.4664 = 0.0336$

The number of bulbs likely to burn for more than 2150 hours is $2000 \times 0.0336 = 67.2 \sim 67$ bulbs

b] $P(X<1950) = P(Z<-1.5)$

$= 0.5 - $ (The area between $Z = -1.5$ and $Z=0$)

$= 0.5 - 0.4332$

$= 0.0668$

Hence the number of bulbs likely to burn for less than 1950 hours is $2000 \times 0.0668 = 133.6 \sim 134$ bulbs c] $P(1920 < X < 2160)$

$$(1920 \text{ in std units}) = \frac{1920-2140}{60} = -2$$

$$(2160 \text{ in std units}) = \frac{2160-2140}{60} = 2$$

$$P(1920 < X < 2160) = P(-2 \leq Z \leq 2)$$
$$= 2(\text{area between } z=0 \text{ and } z=2)$$
$$= 2(0.4772)$$
$$= 0.9544$$

The number of bulb's likely to burn for more that 1920 hours but less than 2160 hours is 0.9544×2000
$= 1908.8 \approx 1909$ bulbs

Unit I

Satistics

Introduction to measures of central tendancy :

Measures of central tendancy

- Mean
- Median
- Mode

Mean : If the ~~numbers~~ $x_1, x_2, \text{-----} x_n$ ~~occurs~~ are the numbers given then mean is given by

$$\bar{x} = \frac{\Sigma x}{n}$$

Median : The median of the set of numbers arranged in order of magnitude is either a middle value or the arithematic mean of 2 middle values

Mode : The mode of a set of numbers is that value which occurs with the greatest frequency.

Note → The mode may not exist and even it does exist it may not unique.

Q1] Find the mean, median, mode for the sets

a] 3, 5, 2, 6, 5, 9, 5, 2, 8, 6

b] 51.6, 48.7, 50.3, 49.5, 48.9

a] $\text{Mean} = \frac{3+5+2+6+5+9+5+2+8+6}{10} = \frac{51}{10} = 5.1$

Median = 5

2, 2, 3, 5, 5, 5, 6, 6, 8, 9

Mode = 5

b] Mean

$$\overline{X} = \frac{51.6 + 48.7 + 50.3 + 49.5 + 48.9}{5} = \frac{249.0}{5} = 49.8$$

Median = 49.5

Mode =

* Curve fitting

→ Least square method:

Fitting of a straight line:

The equation of straight line is : $y = ax + bx$

Normal equations are

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

Q2 Find Mean, Median, Mode of set 4, 2, 3, 5, 4, 4, 7, 9, 7, 7

$$\text{Mean} = \frac{4+2+3+5+4+4+7+9+7+7}{10} = 5.2$$

Median = Mode = 7, 4

M

Q3 Fit a straight line to the following data

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

Soln: Equation of the straight line is

$$y = a + bx \quad \text{--- ①}$$

Normal equations are

$$\Sigma y = na + b\Sigma x \quad \text{--- ②}$$

$$\cancel{\Sigma xy = na\Sigma x + b\Sigma x^2} \quad \text{--- ③}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

x	y	xy	x^2
1	2.4	2.4	1
2	3	6	4
3	3.6	10.8	9
4	4	16	16
6	5	30	36
8	6	~~30~~ 48	64
8	6	48	
$\Sigma x = 24$	$\Sigma y = 24$	$\Sigma xy = 113.2$	$\Sigma x^2 = 130$

Here, $n = 6$

eqn ② $\Rightarrow$

$$24 = 6a + 24b$$

eq^n ③ $\Rightarrow 113.2 = 24a + 130b$

$a = 1.97 \quad b = 0.506$

Q4. Fit a curve $y = ax + bx^2$ for the following data

x	1	2	3	4	5	6
y	2.51	5.82	9.93	14.84	20.55	27.06

Soln: Given eqn of curve is

$$y = ax + bx^2$$

Normal eqn are

$$\Sigma xy = a\Sigma x^2 + b\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^3 + b\Sigma x^4$$

x	y	x^2	x^3	xy	x^4	x^2y
1	2.51	1	1	2.51	1	2.51
2	5.82	4	8	118.04	16	23.28
3	9.93	9	27	29.79	81	83.87
4	14.84	16	64	59.36	256	237.44
5	20.55	25	125	102.75	625	513.75
6	27.06	36	216	162.36	1296	974.16
$\Sigma x = 21$	$\Sigma y = 80.71$	$\Sigma x^2 = 91$	$\Sigma x^3 = 441$	$\Sigma xy = 368.41$	$\Sigma x^4 = 2275$	$\Sigma x^2y = 1840.5$

$368.41 = 91a + 441b$ $\quad a = 2.11$

$1840.5 = 441b + 2275b$ $\quad b = 0.399$

eq^n ① $\Rightarrow y = 2.11x + 0.4x^2$

Q5- Fit a parabola $y = a + bx^2$ for the following data by least square method.

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	14.8

Given eqn is

$$y = a + bx^2 \quad — ①$$

Normal eqn

$$\Sigma y = na + b\Sigma x^2 \quad — ②$$

$$\cancel{\Sigma x^2 y = na\Sigma x^2 + b\Sigma x^3} \quad — ③$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^4$$

x	y	x^2	x^2y	x^3	x^4
1	1.8	1	1.8	1	1
2	5.1	4	10.2	8	16
3	8.9	9	26.7	27	81
4	14.1	16	56.4	64	256
5	14.8	25	74	125	625
$\Sigma x = 15$	$\Sigma y = 44.7$	$\Sigma x^2 = 55$	$\Sigma x^2 y = 691.9$	$\Sigma x^3 = 225$	Σx^4 [illegible]

$$44.7 = 5a + 55b$$

$$697.9 = 55a + 979b$$

$$a = 2.87$$

$$b = 0.5513$$

$$\text{eqn ①} \Rightarrow y = 2.87 + 0.551x^2$$

Q6

x	12	16	20	22	24	26	30
y	6.44	7.5	6.9	10.76	10.76	11.76	14.0

$$y = ax^2 + b \quad \text{—— (1)}$$

$$\sum x^2 y = a\sum x^4 + b\sum x^2 \quad \text{—— (2)}$$

$$\sum y = a\sum x^2 + nb \quad \text{—— (3)}$$

x	y	x^2	x^2y	x^4
12	6.44	144	927.36	20736
16	7.5	256	120	65536
20	6.9	400	138	160000
22	10.76	484	236.72	234256
24	10.76	576	258.24	331776
26	11.76	676	305.76	456976
30	14.0	900	420	810000
	$\sum y =$ 68.12	$\sum x^2 =$ 3436	$\sum x^2y =$ 2100.32	$\sum x^4 = 2079280$

eq^n (2) $\Rightarrow$ $2100.32 = 2079280a + 3436b$

eq^n (3) $\Rightarrow$ $68.12 = 3436a + 7b$

$a = 0.02044 \quad b = 0.001631$

Q. Fit the curve $y = ax + \frac{b}{x}$ be the following

x	1	2	3	4	5	6	7	8
y	5.43	6.28	8.23	10.23	12.63	14.86	17.27	19.51

Given curve is

$$y = ax + \frac{b}{x}$$

Normal equations are

$$\sum xy = a\sum x^2 + nb$$

$$\sum \frac{y}{x} = na + b\sum \frac{1}{x^2}$$

Here, $n = 8$

x	y	xy	x^2	y/x	$1/x^2$
1	5.43	5.43	1	5.43	1
2	6.28	12.56	4	3.14	0.25
3	8.23	24.69	9	2.74	0.11
4	10.23	40.92	16	2.56	0.0625
5	12.63	63.15	25	2.53	0.04
6	14.86	89.16	36	2.48	0.028
7	17.27	120.89	49	2.47	0.0204
8	19.51	156.08	64	2.44	0.015625
$\sum x = 36$	$\sum y = 94.44$	$\sum xy = 512.88$	$\sum x^2 = 204$	$\sum y/x = \cancel{2.623}$ 23.79	$\sum 1/x^2 = \cancel{0.0049}$ 1.525

$$512.88 = a\,204 + 8b$$

$$23.79\cancel{2.623} = 8a + b\,\cancel{0.0049}\,1.525$$

$\cancel{a=}$ 2.98

$b = \cancel{-11.906}$ 3.03

$a = 2.395$

Q8 Fitting of the curve of the type $Y = ax^b$, $y = ab^x$ and $y = ae^{bx}$

Q9 Fit the curve $y = ax^b$ to the following data

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.10	6.80	7.50

Given curve is

$$y = ax^b \quad \text{——} \quad (1)$$

Taking log on both side, we get

$$\log y = \log(ax^b)$$

$$\log y = \log a + b \log x$$

$$Y = A + bX \quad \text{——} \quad (2)$$

Normal equations are

$$\Sigma Y = nA + b\Sigma X$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2$$

x	y	$X = \log x$	$Y = \log y$	XY	X^2
1	2.98	0	0.4742	0	0
2	4.26	0.303	0.6294	0.1907	0.0918
3	5.21	0.477	0.7168	0.3419	0.2275
4	6.10	0.602	0.7853	0.4727	0.3624
5	6.80	0.698	0.8325	0.5810	0.4872
6	~~6.10~~ 7.50	0.778	0.8750	0.6807	0.6052
$\Sigma x = 21$	$\Sigma y = 32.85$	$\Sigma X = 2.858$	$\Sigma Y = 4.3132$	$\Sigma XY = 2.267$	$\Sigma X^2 = 1.7741$

$$\Sigma Y = nA + b\Sigma X$$
$$4.3132 = 6A + b\,2.858$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2$$
$$2.267 = A\,2.858 + b\,1.7741$$

$A = 0.473$ $\quad$ $b = 0.514$

$$A = \log a$$
$$a = \text{antilog}\, A$$
$$a = \text{antilog}\,(0.473)$$
$$a = 2.971$$

Q. Fit a curve $y = ae^{bx}$

x	1	2	3	4	5	6
y	2.98	4.61	7.93	18.54	51.83	128.92

Given curve is

$$y = ae^{bx} \quad \text{—— (1)}$$

Taking log on b.s., we get

$$\log y = \log(ae^{bx})$$
$$= \log a + \log e^{bx}$$
$$\log y = \log a + bx \log e$$
$$Y = A + Bx \quad \text{—— (2)}$$

Normal eq^n s are

$\Sigma Y = nA + B\Sigma x$ ——— (3)

$\Sigma xY = A\Sigma x + B\Sigma x^2$ ——— (4)

x	y	$Y = \log y$	xY	x^2
1	2.98	0.4742		1
2	4.61	0.6191		4
3	7.93	0.8992		9
4	18.54	1.2681		16
5	51.83	1.7145		25
6	128.92	2.1103		36
$\Sigma x = 21$	$\Sigma y =$	$\Sigma Y = 7.1297$	$\Sigma xY = 30.8051$	$\Sigma x^2 = 91$

eq^n (3)

$$7.1297 = 6A + 21B$$

$$30.8059 = 21A + 91B$$

$$A = 0.0178 \qquad B = 0.3343$$

$A = \log a$

$a = \text{antilog } A$

$a = \text{antilog}(0.0183)$

$a = 1.043$

$B = \log b \log e$

~~$b = \text{antilog } B$~~

$b = 1.2991$

Q10 Fit a curve $y = ab^x$ for the following data

x	2	3	4	5	6
y	144	172.3	207.4	248.8	298.5

Given curve is

$$y = ab^x \quad \text{—— (1)}$$

Taking log on both side

$$\log y = \log(ab^x)$$

$$\log y = \log a + x\log b$$

$$Y = A + xB \quad \text{—— (2)}$$

$Y = \log y \qquad A = \log a \qquad B = \log b$

Normal eqn is

$$\Sigma Y = nA + \Sigma x B \quad \text{—— (3)}$$

$$\Sigma xY = \Sigma x A + \Sigma x^2 B \quad \text{—— (4)}$$

x	y	$Y = \log y$	x^2	ΣxY
2	144	2.1583	4	4.3166
3	172.3	2.2362	9	6.7086
4	207.4	2.3168	16	~~8.29.6~~ 9.2672
5	248.8	2.3958	25	11.979
6	298.5	2.4749	36	14.8494
$\Sigma x = 20$	$\Sigma y =$	$\Sigma Y = 11.582$	$\Sigma x^2 = 90$	$\Sigma xY = 47.1208$

$$11.582 = 5n + 20B$$

$$47.1208 = 20A + 90B$$

$A = 1.999$ $B = 0.0792$

$A = \log a$

$a = \text{antilog} A$

$a = 99.77$

$B = \log b$

$b = \text{antilog} B$

$b = 1.20005$

Correlations : To variable 'x' & 'y' said to co-related if increase or decrease in one variable is compared by increases and decreases in other variable.

Coefficient of Correlation :

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \, \Sigma Y^2}}$$ here $X = x - \bar{x}$, $Y = y - \bar{y}$

$$\sigma_x^2 = \frac{\Sigma X^2}{x} \quad , \quad \sigma_y^2 = \frac{\Sigma Y^2}{n}$$

X and Y are the deviations of variable x & y from their respective mean.

Here σx and σy are the standard deviation of x and y resp.

Q1 Find coeffient of correlation of following data

x	60	60	65	70	75	75	75	70	70	6[illegible]
y	21	23	24	26	26	25	25	24	23	2[illegible]

Soln:

Here $n = 10$

$$\bar{x} = \frac{\Sigma x}{n} = 68$$

$$\bar{y} = \frac{\Sigma y}{n} = 24$$

$$X = x - \bar{x} = x - 68$$

$Y = y - \bar{y} = y - 24$

x	y	$X = x - 68$	$Y = y - 24$	XY	X^2	Y^2
60	21	-8	-3	24	64	9
60	23	-8	-1	8	64	1
65	24	-3	0	0	9	0
70	26	2	2	4	4	4
75	26	7	2	14	49	4
75	25	7	1	7	49	1
75	25	7	1	7	49	1
70	24	2	0	0	0	0
70	23	2	-1	-2	4	1
80	23	-8	-1	8	64	1
60		$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma XY = 70$	$\Sigma X^2 = 360$	$\Sigma Y^2 = 22$

∴ Coefficient of correlation is given by

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \Sigma Y^2}}$$

$$= \frac{70}{\sqrt{360 \times 20}}$$

$$= \frac{70}{12\sqrt{55}}$$

$$= 0.7866$$

Lines of regression:

line of regression of y on x is:

$$y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

line of regression of x on y is

$$x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

$a_1 = r\frac{dy}{dx}$ is regression coefficient of y on x

$b_1 = r\frac{dx}{dy}$ is regression coefficient of x on y

$$\boxed{r = \sqrt{a_1 b_1}}$$

Q2. Find coefficient of correlation between the variable x and y hence find the regression lines.

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Here $n = 8$

$\bar{x} = \frac{\Sigma x}{n} = 7$

$\bar{y} = \frac{\Sigma y}{n} = 5$

$X = x - \bar{x} = x - 7$
$Y = y - \bar{y} = y - 5$

x	y	$X = x-7$	$Y = y-5$	XY	X^2	Y^2
1	1	-6	-4	24	36	16
3	2	-4	-3	12	16	9
4	4	-3	-1	3	9	1
6	4	-1	-1	1	1	1
8	5	1	0	0	1	0
9	7	2	2	4	4	4
11	8	4	3	12	16	9
14	9	7	4	28	49	16
		$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma XY = 84$	$\Sigma X^2 = 132$	$\Sigma Y^2 = 56$

$\therefore$ Coefficient of correlation is given by

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \, \Sigma Y^2}}$$

$$= \frac{84}{\sqrt{132 \times 56}}$$

$$= 0.9770$$

$$\sigma_x = \sqrt{\frac{\Sigma X^2}{n}} = \sqrt{\frac{132}{8}} = 4.06$$

$$\sigma_y = \sqrt{\frac{\Sigma Y^2}{n}} = \sqrt{\frac{56}{7}} = 2.6457$$

Regression line of y on x is given by

$$y-\bar{y} = r \frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

$$y-5 = (0.977) \times \frac{2.6457}{4.06}(x-7)$$

$$y-5 = 0.6367x - 4.4567$$

$$y = 0.6367x + 0.5433$$

Regression line of y x on y is given by

$$x-\bar{x} = r \frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

$$x-7 = (0.977) \times \frac{4.06}{2.6457}(y-5)$$

$$x-7 = 1.4992y - 7.4963$$

$$x = 1.4992y + 0.4963$$

Q3 Calculate the coefficient of correlation & hence eqn of line of regression for the following data.

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	15

Soln:

Here, $n=9$

$$\bar{x} = \frac{\Sigma x}{x} = 5$$

$$\bar{y} = \frac{\Sigma y}{n} = 12$$

$$X = x - \bar{x} = x - 5$$

$$Y = y - \bar{y} = y - 12$$

x	y	$X = x-5$	$Y = y-12$	XY	X^2	Y^2
1	9	-4	-3	12	16	9
2	8	-3	-4	12	9	16
3	10	-2	-2	4	4	4
4	12	-1	0	0	1	0
5	11	0	-1	0	0	1
6	13	1	1	1	1	1
7	14	2	2	4	4	4
8	16	3	4	12	9	16
9	15	4	3	12	16	9
		$\Sigma X=0$	$\Sigma Y=0$	$\Sigma XY=57$	$\Sigma X^2=60$	$\Sigma Y^2=60$

$\therefore$ Coefficient of correlation is given by

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \Sigma Y^2}} = \frac{57}{\sqrt{60 \times 60}} = \frac{57}{60} = 0.95$$

$$\sigma x = \sqrt{\frac{\Sigma x^2}{n}} = \sqrt{\frac{60}{9}} = 2.5819$$

$$\sigma y = \sqrt{\frac{\Sigma y^2}{n}} = \sqrt{\frac{60}{9}} = 2.5819$$

Regression line of y on x is given by

$$y - \bar{y} = r \frac{\sigma y}{\sigma x} (x - \bar{x})$$

$$y - 12 = (0.95) \frac{2.5819}{2.5819} (x - 5)$$

$$y - 12 = 0.95x - 4.75$$

$$y = 0.95x + 7.25$$

Regression line of x on y is given by

$$x - \bar{x} = r \frac{\sigma x}{\sigma y} (y - \bar{y})$$

$$x - 5 = (0.95) \frac{2.5819}{2.5819} (y - 12)$$

$$x - 5 = 0.95y - 11.4$$

$$x = 0.95y - 6.4$$

Q4 Calculate the coefficient of correlation for the following data. Also find equation to lines of regression.

x	24	13	27	12	31	42	13	29	17	11
y	24	25	21	25	22	19	24	20	25	26

Here $n = 10$

$$\bar{x} = \frac{\Sigma x}{x} =$$

$$\bar{y} = \frac{\Sigma y}{y} =$$

Q5 Two lines of regression are given by $x+2y-5=0$, $2x+3y-8=0$. If $\sigma_x^2=12$ find i) mean values of x & y
ii) S.D. of y
iii) coefficient of correlation between x & y

Soln: Since both the lines of regression pass through the point $\bar{x}$ and $\bar{y}$

We have

$$\bar{x}+2\bar{y}-5=0$$
$$2\bar{x}+3\bar{y}-8=0$$
$$\bar{x}=1 \quad , \quad \bar{y}=2$$

$\therefore$ Mean values of x & y are $\bar{x}=1$ & $\bar{y}=2$

Given that $\sigma_x^2=12$

$$\sigma_x=\sqrt{12}=3.46$$

The equation of lines of regression can be written as

$$x+2y-5=0$$
$$x=-2y+5$$
$$r\frac{\sigma_x}{\sigma_y}=-2 \quad \text{——— (1)}$$
$$\boxed{\therefore b_1=-2}$$

$$2x+3y-8=0$$
$$3y=8-2x$$
$$y=-\frac{2}{3}x+\frac{8}{3}$$
$$a_1=-\frac{2}{3}$$

$$\gamma = \sqrt{a_1 b_1}$$

$$\gamma = \sqrt{\left(-\frac{2}{3}\right)(-2)}$$

$$\gamma = \sqrt{\frac{4}{3}}$$

$$\gamma = 1.1547$$

$\therefore$ Coefficient of correlation between x & y is 1.1547

eqⁿ ① $\Rightarrow$ $(1.1547)\dfrac{(3.46)}{6y} = -2$

$$\sigma y = -1.999$$

Q6 Two lines of regression is given by $5y - 8x + 17 = 0$, $2y - 5x + 14 = 0$. If $\sigma_y^2 = 16$, find

i) mean values of x & y

ii) σ_x^2

iii) Coefficient of correlation between x & y

Soln: Since both the lines of regression pass through the point $\bar{x}$ & $\bar{y}$

We have

$$5\bar{y} - 8\bar{x} + 17 = 0$$
$$2\bar{y} - 5\bar{x} + 14 = 0$$
$$\bar{x} = 4 \qquad \bar{y} = 3$$

$\therefore$ Mean values of x & y are $\bar{x} = 1$ & $\bar{y} = 2$

given that $\sigma_y^2 = \sqrt{16}$

$$\sigma_y = \sqrt{16} = 4$$

The equation of lines of regression can be written as

$$5y - 8x + 17 = 0$$
$$5y + 17 = 8x \Rightarrow \frac{5y}{8} + \frac{17}{8} = x$$

$$r\frac{\sigma_x}{\sigma_y} = \cancel{1/2}\ \frac{5}{2} \quad \text{——— (1)}$$

$$\therefore\ b_1 = \cancel{1/2}\ \frac{5}{2}$$

$$2y - 5x + 14 = 0$$
$$2y = 5x - 14$$
$$y = \frac{5}{2}x - \frac{14}{2} = \frac{5}{2}x - 7$$

$$a_1 = \frac{5}{2}$$

$$r = \sqrt{a_1 b_1}$$

$$r =$$

$$r = 1$$

$\therefore$ Coefficient of correlation between x & y is 1

$$eq^n \ (1) \Rightarrow 1 \times \frac{\sigma x}{4} = \frac{5}{2}$$

$$\sigma x = \frac{5}{2} \times 4$$

$$\sigma x = 10$$

$$\sigma^2 x = 100$$

Unit 4: Linear Algebra

Vector Space:

Let V be a non empty set of certain objects which may be vectors, matrices, functions or some other objects

Algebraic Operations:

i] Vector Addition:

$$a+b = (a_1, a_2, a_3, a_4, \ldots \ldots a_n) + (b_1 + b_2 + b_3 + \ldots b_n)$$
$$= (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + \ldots \ldots + a_n b_n)$$

ii] Scalar Multiplication:

$$\alpha a = \alpha(a_1, a_2, a_3, \ldots \ldots, a_n)$$
$$\alpha a = (\alpha a_1, \alpha a_2, \alpha a_3, \ldots \ldots \alpha a_n)$$

where α is scalar

a & b are vectors.

Defination: The set V defines a vector space if for any element a, b, c in V and any scalars α, β the following properties are satisfied

1. $a + b = b + a$
2. $a + (b+c) = (a+b) + c$
3. $a + 0 = 0 + a = a$

4. $a + (-a) = 0$

5. $(\alpha + \beta) a = \alpha a + \beta a$

6. $(\alpha\beta) a = \alpha(\beta a)$

7. $\alpha(a+b) = \alpha a + \alpha b$

8. $1 \cdot a = a$

Q1 Let Mmn be the collection of all $m \times n$ matrices S.T. show that Mmn is a vector space using matrix addition and scalar multiplication.

Soln: Let $A, B, C \in M_{mn}$

1) $A + B = (a_{ij}) + (b_{ij})$
$= (a_{ij} + b_{ij})$
$= (b_{ij} + a_{ij})$
$= (b_{ij}) + (a_{ij})$
$= B + A$

Since addition of scalar is commutative

ii) Similarlly addition of scalar is associative
$\therefore A + (B+C) = (A+B) + C$

iii) $A + 0 = (a_{ij}) + 0$
$= (a_{ij} + 0)$
$= (a_{ij})$
$= A$

iv) $A + (-A) = (a_{ij}) + (-a_{ij})$

$= (a_{ij} - a_{ij})$

$= 0$

v) $(\alpha+\beta)A = (\alpha+\beta)(a_{ij})$

$= (\alpha(a_{ij}) + (\beta(a_{ij}))$

$= \alpha a + \beta a$

vi) $(\alpha\beta)A = (\alpha\beta)(a_{ij})$

$= (\alpha(\beta(a_{ij}))$

$= \alpha(\beta A)$

vii) $\alpha(A+B) = \alpha((a_{ij}) + (b_{ij}))$

$= \alpha(a_{ij}) + \alpha(b_{ij})$

$= \alpha A + \alpha B$

viii) $1.A = 1.(a_{ij})$

$= a_{ij}$

$= A$

Q2 Let $V = \{(x,y) : x \geq 0, y \geq 0\}$ shown that the set V fails to be a vector space under the standard operations on $\mathbb{R}^2$

Soln: For any $x, y \in V$
with $x, y > 0$
We have
$$-(x,y) \notin V$$
$\therefore$ inverse element does not exists (prop 4 is not proved)
hence V is not a vector space.

Q3 Let V be the set of all ordered pairs ~~polynomials~~ ~~with~~ (x,y) where x, y are real numbers. Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$ be two elements in V. Define the addition as
$$a+b = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$
and the scalar multiplication as
$$\alpha(x_1, y_1) = (\alpha x_1/3, \alpha y_1/3)$$
Show that V is not a vector space.

Soln: Given that
$a+b = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$
Now,
$b+a = (x_2, y_2) + (x_1, y_1) = (2x_2 - 3x_1, y_2 - y_1)$
$\therefore a+b \neq b+a$
Similarly, we can prove that
$$a+(b+c) \neq (a+b)+c$$
$\therefore$ V is not a vector space.

Let V be the set of all ordered pair (x, y) where x, y are real numbers

Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$ be two elements in V.

Define the addition as

$$a + b = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

and the scalar multiplication as $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$

Show that V is not a vector space.

Soln: Given vector addition is

$$a+b = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

Now we have,

$$(x_1, y_1) + (1,1) = (x_1, y_1)$$

$\therefore$ (1,1) is the zero element in V.

There exist additive inverse element $\left(\frac{1}{x_1}, \frac{1}{y_1}\right)$

S.T. $(x_1, y_1) + \left(\frac{1}{x_1}, \frac{1}{y_1}\right) = (1,1)$

let $\alpha = 1, \ \beta = 2$

Now,

$$(\alpha+\beta)a = 3(x_1, y_1) = (3x_1, 3y_1)$$

and

$$\alpha a + \beta a = 1(x_1, y_1) + 2(x_1, y_1)$$

$$= (x_1, y_1) + (2x_1, 2y_1)$$

$$\alpha a + \beta a = (2x_1^2 + y_1^2)$$

$$\therefore (\alpha+\beta)a \neq \alpha a + \beta a$$

Subspaces

A subset W of a vector space 'V' is called a subspace of V if the following properties are satisfied

i) If u, v are in W then u + v is also in W

ii) If α is a scalar and u is in W then αu is also in W

Note:

Every vector space V as atleast 2 subspace; V itself and the subspace consisting of the 0 vector of V these are called the trivial subspaces of V

Q Show that W is a subspace of V if and only if $\alpha u + v \in W$ for all $u, v \in W$ and $\alpha \in \mathbb{R}$

Ans: Suppose that W is a subspace of V

If $u, v \in W$ then $u + v \in W$

α is scalar as $\alpha \in \mathbb{R}$ then

$\alpha u \in W$

$\therefore \alpha u + v \in W$

Conversely,

Suppose that $\alpha u + v \in W$

$\alpha = 1$

$u + v \in W$

$v = 0$

$\alpha u \in W$

$\therefore$ W is a subspace.

Conversely,

Suppose that $\alpha u + v \in W$

for all $u, v \in W$ and $\alpha \in R$

In particular if if. $\alpha = 1$ then we have $u + v \in W$

If $v = 0$ then

$\alpha u + 0 \in W$

$\therefore \alpha u \in W$

Hence, W is a subspace.

Q. Let M_{22} be the collection of 2×2 matrices. Show that the set W of all 2×2 matrices having zeroes on the main diagonal is a subspace of M_{22}.

Soln: The set W is defined as

$$W = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} ; a, b \in \mathbb{R} \right\}$$

Method I
$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+c \\ b+d & 0 \end{bmatrix}$$

$A, B \in W$

$A + B \in W$

$$\alpha \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a \\ \alpha b & 0 \end{bmatrix} \in W$$

Method II

Clearly, the 2×2 zero matrix belong to W.

Also,
$$\alpha \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a + c \\ \alpha b + d & 0 \end{bmatrix} \in W$$

$\therefore$ W is a subspace of M_{22}.

Q. let F and G be subspace of a vector space V such that $F \cap G = \{0\}$. The sum of F and G is written as F + G and is defined by

$$F + G = \{f + g : f \in F, g \in G\}$$

Show that F + G is a subspace of V assuming the usual definition of vector addition and scalar multiplication

Soln: let $W = F + G$ and $f \in F$, $g \in G$

$0 \in F$, $0 \in G$ we have

$0 + 0 = 0 \in F + G$

let $u, v \in F + G$

where,

$u = F_1 + G_1 \quad v = F_2 + G_2$

$u + v = (F_1 + G_1) + (F_2 + G_2)$

$= (F_1 + F_2) + (G_1 + G_2)$

$F_1 + F_2 \in F$ as $F_1, F_2 \in F$

$G_1 + G_2 \in G$ as $G_1, G_2 \in G$

Linear independence of vector:

Let V be a vector space a finite space $V_1, V_2, \ldots V_n$ of the elements of V is a set to be linearly dependent if their scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ not all zero.

If eqn ① is satisfied only for $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n =$ then the set of vectors is said to be linearly independe

Q Let $V_1 = (1,-1,0)$, $V_2 = (0,1,-1)$ and $V_3 = (0,0,1)$ are elements of $\mathbb{R}^3$. Show that the set of vectors $\{V_1, V_2, ~~and~~ V_3\}$ is linearly independent.

Soln: We consider the vector equation

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0 \quad \text{——— ①}$$

Substituting the value of $(\alpha_1, \alpha_2, \alpha_3)$ we get

$$\alpha_1(1,-1,0) + \alpha_2(0,1,-1) + \alpha_3(0,0,1) = 0$$

$$(\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3) = 0$$

Comparing both sides, we get,

$$\alpha_1 = 0, \quad -\alpha_1 + \alpha_2 = 0, \quad -\alpha_2 + \alpha_3 = 0$$

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0$$

$\therefore$ The given set of vectors is linearly independent.

Q Let $V_1 = (1,-1,0)$, $V_2 = (0,1,-1)$, $V_3(0,2,1)$ & $V_4(1,0,3)$ be elements of $\mathbb{R}^3$. Show that $\{V_1, V_2, V_3, V_4\}$ is linearly dependent.

Ans: The given set of elements will be linearly dependent if their exist scalar

$\alpha_1, \alpha_2, \alpha_3, \& \alpha_4$ not all zero

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0 \quad \text{——— (1)}$$

Given that:

$V_1 = (1,-1,0)$, $V_2 = (0,1,-1)$, $V_3 = (0,2,1)$, $V_4 = (1,0,3)$

Substituting eqn (1) $\Rightarrow$

$$\alpha_1(1,-1,0) + \alpha_2(0,1,-1) + \alpha_3(0,2,1) + \alpha_4(1,0,3) = 0$$

$$(\alpha_1 + \alpha_4, -\alpha_1 + \alpha_2 + 2\alpha_3, -\alpha_2 + \alpha_3 + 3\alpha_4) = 0$$

$$\alpha_1 + \alpha_4 = 0 \quad \text{——— (2)}$$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \quad \text{——— (3)}$$

$$-\alpha_2 + \alpha_3 + 3\alpha_4 = 0 \quad \text{——— (4)}$$

Let $\alpha_4 = k$ then

eqn (2) $\Rightarrow$ $\alpha_1 + k = 0$

$\alpha_1 = -k$

eqn (3) $\Rightarrow$ $k + \cancel{\alpha_2} + 2\alpha_3 = 0$

eqn (4) $\Rightarrow$ $-\cancel{\alpha_2} + \alpha_3 + 3k = 0$

$4k + 3\alpha_3 = 0$

$$\alpha_3 = \frac{-4k}{3}$$

$$\alpha_2 = \frac{5k}{3}$$

$$\alpha_1 = -k, \quad \alpha_2 = \frac{5k}{3}, \quad \alpha_3 = -\frac{4k}{3}; \quad \alpha_4 = k$$

eqⁿ ① ⇒

$$-KV_1 + \frac{5}{3}KV_2 - \frac{4}{3}KV_3 + KV_4 = 0$$

$$V_1 - \frac{5}{3}V_2 + \frac{4}{3}V_3 - V_4 = 0$$

$$3V_1 - 5V_2 + 4V_3 - 3V_4 = 0$$

Hence their exist scalar not all zero such that eqnⁿ ① is satisfied therefore the set of vectors is linearly dependent.

Dimension and Basis:

Let V be a vector space. If for some positive integer n their exist S of n linearly independe elements of V and if every set of (n+1) or more elements in V is linearly dependent, then V is said to have dimension n.

It is denoted by dim(V)=n, Thus the maximum no. of linearly independent element of V is the dimension of V.

The set S of n linearly independent vectors is called bases of V.

A vector space whose any element is zero has dimension zero.

Q Determine weather the following set of vectors $\{u, v, w\}$ forms basis in $\mathbb{R}^3$

i) $u = (2,2,0)$, $v = (3,0,2)$, $w = (2,-2,2)$

ii) $u = (0,1,-1)$, $v = (-1,0,-1)$, $w = (3,1,3)$

Sol^n: If the set $\{u, v, w\}$ forms a basis in $\mathbb{R}^3$, $\{u, v, w\}$ must be linearly independent. Let, $\alpha_1\ \alpha_2\ \alpha_3$ be scalars then the only solution of eq^n $\alpha_1 u + \alpha_2 v + \alpha_3 w = 0$ must be $\alpha_1 = \alpha_2 = \alpha_3 = 0$

i) Given that

$u = (2,2,0)$, $v(3,0,2)$, $w = (2,-2,2)$

substituting eq^n 1 $\Rightarrow$

$$\alpha_1(2,2,0) + \alpha_2(3,0,2) + \alpha_3(2,-2,2) = 0$$

$$(2\alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 - 2\alpha_3, 2\alpha_2 + 2\alpha_3) = 0$$

$$2\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0 \quad \text{--- (2)}$$

$$2\alpha_1 - 2\alpha_3 = 0 \quad \text{--- (3)}$$

$$2\alpha_2 + 2\alpha_3 = 0 \quad \text{--- (4)}$$

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3$$

$\therefore$ $\{u, v, w\}$ are linearly independent and they form a basis in $\mathbb{R}^3$

$S \rightarrow$ set of n linearly independent elements

$$\dim(V) = n$$

Thus, the maximum number of lineally independent element of 'V' is the dimension of 'V'.

Bases: The set s of linearly independent vectors is called the basics of 'V'.

Note: A vector space whose only element is zero has dimension zero

Q. Find the dimension of the subsapce of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$ Hence find its basis.

Solⁿ: The dimension of the subspace is ≤ 4.
If it is 4 then the vector eqⁿ

$$\alpha_1(1,0,0,0) + \alpha_2(0,1,0,0) + \alpha_3(1,2,0,1) + \alpha_4(0,0,0,1) = 0 \quad \text{--- (1)}$$

has solⁿ $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$\therefore$ eqⁿ (1) $\Rightarrow$

$$(\alpha_1 + \alpha_3, \alpha_2 + 2\alpha_3 \cancel{+}, \alpha_3 + \alpha_4) = 0$$

$$\alpha_1 + \alpha_3 = 0 \quad \text{--- (2)}$$

$$\alpha_2 + 2\alpha_3 = 0 \quad \text{--- (3)}$$

$$\alpha_3 + \alpha_4 = 0 \quad \text{--- (4)}$$

$\therefore$ The solⁿ of the system is given: eqⁿ (2) $\Rightarrow \alpha_1 = -\alpha_3$

eqⁿ (3) $\Rightarrow \alpha_2 = -2\alpha_3$

eqⁿ (4) $\Rightarrow \alpha_4 = -\alpha_3$

$\therefore$ The vectors are linearly dependent

$\therefore$ The dimension of the set is < 4.

Now consider any three elements say
$(1,0,0,0), (0,1,0,0)$ and $(1,2,0,1)$
Consider the vector eq^n
$\alpha_1 (1,0,0,0) + \alpha_2 (0,1,0,0) + \alpha_3 (1,2,0,1) = 0$
$(\alpha_1 + \alpha_3 , \alpha_2 + 2\alpha_3 , \alpha_3) = 0$
$\alpha_1 + \alpha_3 = 0, \quad \alpha_2 + 2\alpha_3 = 0, \quad \alpha_3 = 0$
$\therefore$ Sol^n is $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$
$\therefore$ These three vectors are linearly independent.
$\therefore$ We can say that, dimension of the subspace 3.

The basis is the set of vectors
$\{(1,0,0,0), (0,1,0,0), (1,2,0,1)\}$

Linear Transformation:
A linear transformation 'T' from a vector space 'V' to a vector space 'W' is a function $T: V \to W$ that satisfies the following two conditions that

i) $T(u+v) = T(u) + T(v)$
for all u, v in V.

ii) $T(\alpha u) = \alpha T(u)$
for all u in V and scalar α.

Q. Show that $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ y \end{bmatrix}$ is a linear transformation.

Soln: Given that:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ y \end{bmatrix} \quad \text{—— (1)}$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + y_1 + y_2 \\ y_1 + y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_1 + y_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 + y_2 \\ y_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_1 \\ y_2 \end{bmatrix}\right) \quad \text{—— (2)}$$

$$T\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \alpha x_1 \\ \alpha x_1 + \alpha y_1 \\ \alpha y_1 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x_1 \\ x_1 + y_1 \\ y_1 \end{bmatrix}$$

$$T\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$$

Q. Show that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ x-y \end{bmatrix}$$

is a linear transformation

Soln: let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ & $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + y_1 + y_2 \\ x_1 + x_2 - (y_1 + y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + y_1 + y_2 \\ x_1 - y_1 + \{x_2 - y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 + y_2 \\ x_2 - y_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$T\left(\alpha\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix}\right) = \begin{bmatrix} \alpha x \\ \alpha x + \alpha y \\ \alpha x - \alpha y \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{bmatrix}$$

$$= \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$$

$\therefore$ T is a linear transformation

Q Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Show that T is not linear.

Sol^n: Let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ & $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

$$\neq T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

T is not linear.

Q Show that the function $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-2y \\ 3x \end{bmatrix}$$

is a linear transformation

Sol^n: Let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ & $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ x_1 + x_2 - 2y_1 - 2y_2 \\ 3x_1 + 3x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 \\ x_1 - 2y_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_1 \\ x_2 - 2y_2 \\ 3x_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$T\left(\alpha\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \alpha x_1 + \alpha y_1 \\ \alpha x_1 - 2\alpha y_1 \\ 3\alpha x_1 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x+y \\ x-2y \\ 3x \end{bmatrix}$$

$$= \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$$

$\therefore$ T is a linear Transformation

dimension of range

Rank Nullity Theorem

If T has rank R and the dimension of V is n then the nullity of T is n-r that is rank(T) + nullity = n = dim(V)

Unit 5 : Order Pair

Ordered pair :

$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

Here $X \times Y$ is cartesian product & (x, y) is ordered pair.

Relation :

$R = \{(x, y) / x, y \in \mathbb{R}\}$

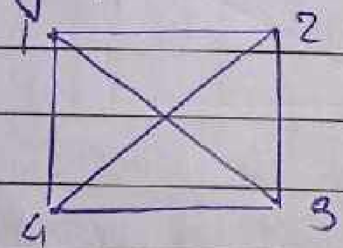
Ex :

$x - y$ is divisible by 3

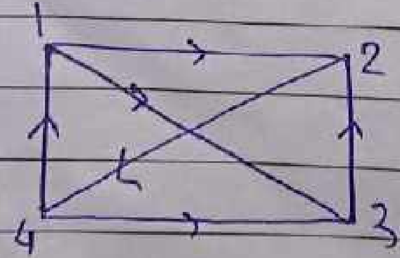
$R = \{ \ldots, (3,3), (0,0), (1,4), \ldots \}$

Graph :

a) Undirected graph



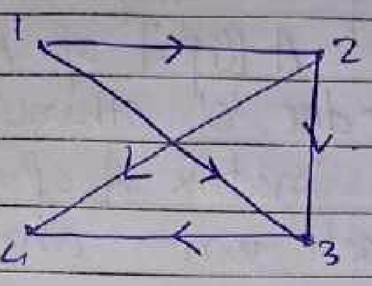
b) Directed Graph (Di-Graph)



Ex: Draw the di-graph for following relation:

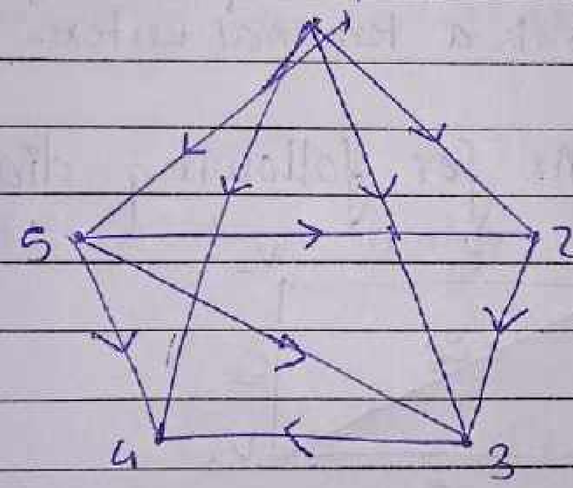
$R = \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$

Soln: Di-graph for the given relation is a follows:



Q. Draw the graph corresponding to following relation

$R = \{(1,2), (1,3) (1,4), (1,5), (2,3), (3,4), (5,2), (5,3), (5,4)\}$



Incidence Matrix:

An incidence matrix can be defined for a network in the following way.

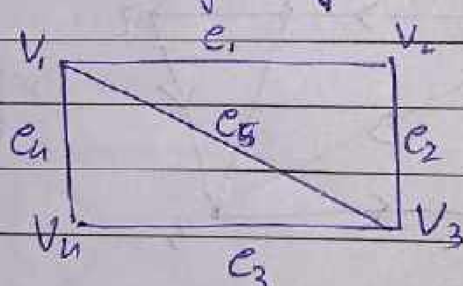
The incidence matrix A of an undirected graph has a row for each vertex and a column for each edge of the graph.

The element A[(ij)] of A is '1' if the ith vertex is of jth edge, '0' otherwise.

The incidence matrix A of a directed graph has a row for each vertex and a column for each edge of the graph.

The element A[(ij)] of A is '-1' if the ith vector is an initial vertex of the jth edge, '1' if the ith vertex is a terminal vertex.

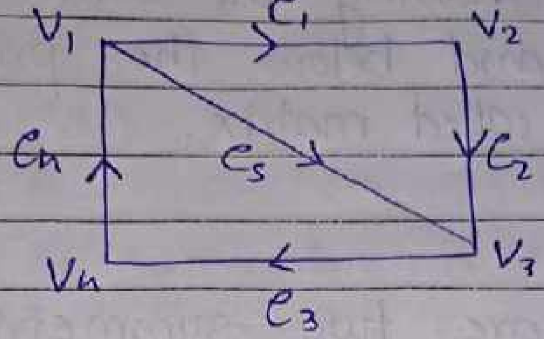
Ex. Write An incident matrix for following dia-graph.



Sol^n: incidence matrix for the given graph is

$$\begin{array}{c c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Q Write an incidence matrix for following dia-graph

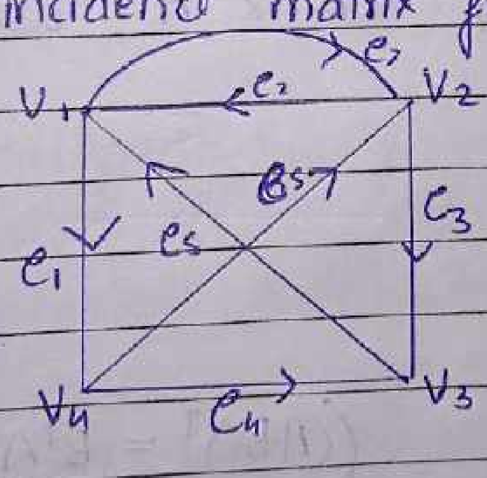


Soln: The incidence matrix for given graph is

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}
\end{array}
$$

~~# Symmetric~~

Q Draw the incidence matrix for the given dia-graph



Symmetric matrix:
A square matrix having the same corresponding element above and below the principle diagonal is called matrix. ($A^T = A$)
i.e. $a_{ij} = a_{ji}$

# Properties:
If A and B are two symmetric matrices then
1] $A+B$ & $A-B$ are symmetric
2] If $AB = BA$ then product of A & B is symmetric
3] A^n is also symmetric
4] A^{-1} is also symmetric

Theorem:- Let A be any vector matrix. Show that AA^T and A^TA are symmetric matrices.

To show that AA^T and A^TA are symmetric matrices for any matrix A, we have to show that
1] $(AA^T)^T = AA^T$
2] $(A^TA)^T = A^TA$

1] $L.H.S = (AA^T)^T$
$= (A^T)^T A^T \qquad ((AB)^T = B^TA^T)$
$= AA^T$
$= R.H.S$

$(AA^T)^T = AA^T$

$\therefore$ AA^T is symmetric matrix

2] $L.H.S = (A^TA)^T$

$= (A^T)^T (A^T)^T \qquad ((AB)^T = B^TA^T)$

$= A^TA$

$= R.H.S$

$(A^TA)^T = A^TA$

$\therefore$ A^TA is symmmetric matrix

Eigen spaces:

Let V_λ denote the set of eigen vectors of a matrix corresponding to an eigen value λ. The set $V^\lambda = V_\lambda \cup \{0\}$ is a subspace of $\mathbb{R}^n$. This subspace is called a eigen space of A corresponding to λ.

Algebaric multiplicity:

Algebaric multiplicity of an eigen value λ of a matric is the multiplicity of λ as a root of characteristic polynomial.

Geometric Multiplicity:

The dimension of the eigen space corresponding to λ is called geometric multiplicity of λ.

Q Find bases ~~for eigen~~ for eigen spaces of matrix

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Soln: Characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$0 = (3-\lambda)[15-5\lambda-3\lambda+\lambda^2-0]+2[-2(5-\lambda)-0]+0$$

$$0 = [(3-\lambda)^2-4](5-\lambda)=0$$

$$0 = (9-6\lambda+\lambda^2-4)(5-\lambda)$$

$$0 = (\lambda^2-6\lambda+5)(5-\lambda)$$

$$0 = (\lambda-1)(\lambda-5)(\lambda-5)$$

$$\lambda = 1, 5, 5$$

A vector $x=[x_1, x_2, x_3]^T$ corresponding to ideal value λ if and any x is a non trivial solution to the homogeneous

$$[A-\lambda I]x = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$(3-\lambda)x_1 - 2x_2 = 0$ —— ①

$-2x_1 + (3-\lambda)x_2 = 0$ —— ②

$(5-\lambda)x_3 = 0$ —— ③

For $\lambda=1$,

$$\left.\begin{aligned} eq^n\ ① &\Rightarrow 2x_1-2x_2=0 \\ eq^n\ ② &\Rightarrow -2x_1+2x_2=0 \end{aligned}\right\} x_1-x_2=0$$

eq^n ③ $\Rightarrow$ $4x_3=0$

$$x_1=5,\ x_2=5,\ x_3=0$$

$$\cancel{V^1 = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}; s \in \mathbb{R} \} = \text{span}\{[} \quad V^1 = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix}; s \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\dim(V^1) = 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is basis

For $\lambda = 5$,

eqn ① $\Rightarrow -2x_1 - 2x_2 = 0$

eqn ② $\Rightarrow -2x_1 - 2x_2 = 0$ $\Big\} \; x_1 + x_2 = 0$

eqn ③ $\Rightarrow 0x_3 = 0$

$x_1 = -t, \quad x_2 = t, \quad x_3 = s$

The eigen space corresponding to $\lambda = 5$ is

$$V^5 = \left\{ \begin{bmatrix} t \\ t \\ s \end{bmatrix}; s, t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; s, t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim(V^5) = 2$

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis of V^5

Q Find the bases for eigen spaces of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Characteristic eqn matric of A is

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

$\lambda = 1, 2, 2$

To find eigen space

$$\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-\lambda x_1 - 2x_3 = 0$ —— ①

$x_1 + (2-\lambda)x_2 + x_3 = 0$ —— ②

$x_1 + (3-\lambda)x_3 = 0$ —— ③

For $\lambda = 1$

eqn ① $\Rightarrow$ $-x_1 - 2x_3 = 0$

eqn ② $\Rightarrow$ $x_1 + x_2 + x_3 = 0$

eqn ③ $\Rightarrow$ $x_1 + 2x_3 = 0$

$x_1 = -2s, \ x_2 = s, \ x_3 = s$

$\therefore$ Eigen space corresponding to $\lambda=1$ is

$$V' = \left\{ \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\therefore \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is basis of V' and $\dim(V')=1$

For: $\lambda=2$

$$-2x_1 - 2x_3 = 0$$
$$x_1 + x_3 = 0$$
$$x_1 + x_3 = 0$$

$x_1 = s$, $x_2 = t$, $x_3 = -s$

$\therefore$ Eigen space corresponding to $\lambda=2$ is

$$V^2 = \left\{ \begin{bmatrix} s \\ t \\ -s \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; s, t \in \mathbb{R} \right\}$$

$$V^2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is basis of V^2 and $\dim(V^2)=2$

Q. Find algebaric multiplicity and geometric multiplicity of the eigen values of the

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Characteristic eqn for A is

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

$$(-\lambda)(-\lambda^2-1) - 1(-\lambda-1) + 1(-\lambda-1) = 0$$

$$(-\lambda)(\lambda+1)(\lambda-1) + (\lambda+1) - (\lambda+1) = 0$$

$$(\lambda+1)(\lambda^2+\lambda+2) = 0$$

$$(\lambda+1)(\lambda-2)(\lambda+1) = 0$$

$$\lambda = -1, -1, 2$$

Algebaric multiplicity of eigen value $\lambda = -1$ is 2 & multiplicity of $\lambda = 2$ is 1.

To find eigen space we consider

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-\lambda x_1 + x_2 + x_3 = 0$$

$$x_1 - \lambda x_2 + x_3 = 0$$

$$x_1 + x_2 - \lambda x_3 = 0$$

For $\lambda = -1$

eqⁿ ① $\Rightarrow$ $x_1 + x_2 + x_3 = 0$

eqⁿ ② $\Rightarrow$ $x_1 + x_2 + x_3 = 0$

eqⁿ ③ $\Rightarrow$ $x_1 + x_2 + x_3 = 0$

$x_1 = s, \quad x_2 = t, \quad x_3 = -s-t$

$\therefore$ Eigen space for $\lambda = -1$ is

$$V^{-1} = \left\{ \begin{bmatrix} s \\ t \\ -s-t \end{bmatrix} ; s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} ; s, t \in \mathbb{R} \right\}$$

$$V^{-1} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is basis of V^{-1}.

$\therefore$ Geometric multiplicity for $\lambda = -1$ is $\dim(V^{-1}) = 2$

For $\lambda = -2$

eqⁿ ① $\Rightarrow$ $-2x_1 + x_2 + x_3 = 0$

eqⁿ ② $\Rightarrow$ $x_1 - 2x_2 + x_3 = 0$

eqⁿ ③ $\Rightarrow$ $x_1 + x_2 - 2x_3 = 0$

By Crammer's rule, we have

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{3} = \frac{-x_2}{-3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ Eigen space for $\lambda = 2$ is

$$V^2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ; 1 \in \mathbb{R} \right\}$$

$$V^2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is basis of V^2

$\therefore$ Geometric multiplicity for $\lambda = 2$ is $\dim(V^2) = 1$

Network

A network is simple, connected weighted directed graph.

$\longrightarrow$ $N(V, E)$ satisfying the following condition

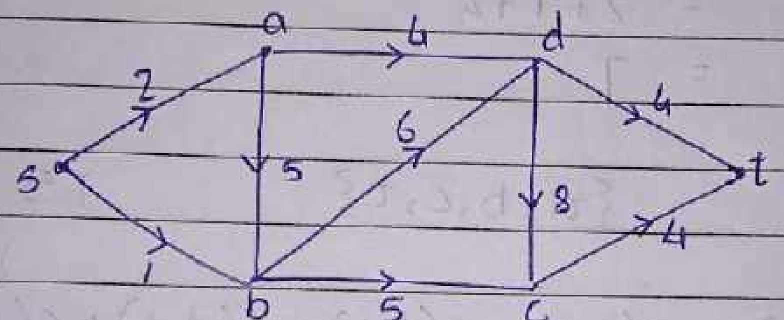
1] their is a uniqu vertex $s \in V$ if it in degree 0. This is called source.

2] Their exist a unique vertex $t \in V$ if it has out degree 0. This vertex is called the sink.

3] Every directed edge $e = (v, w) \in E$ has been asign a non-negative number called the capacity of e.

4] If $N(V, E)$ and C is a ~~network~~ cut set for the undirected graph associated with N, then 'C' is called a cut or ~~st~~ s-t cut if the romoval of edges in 'C' from the network separates the ~~source~~ sink 'T'.

Ex: list all s-t cuts for the following figures



Sol^n: The following table gives all s-t cuts and its capacities

	P	$\bar{P}$	Capacity $(P, \bar{P})$
1]	$\{s\}$	$\{a, b, c, d, t\}$	3

$$c(P, \bar{P}) = c(s, a) + c(s, b)$$
$$= 2 + 1$$
$$= 3$$

2]	$\{s, a\}$	$\{b, c, d, t\}$	8

$$c(P; \bar{P}) = c(s, b) + c(a, b) + c(a, b)$$
$$= 1 + 3 + 4 = 8$$

	P	$\bar{P}$	Capacity $(P,\bar{P})$
8]	$\{s, a, d\}$	$\{c, b, t\}$	16

$$C(P,\bar{P}) = c(s,b) + c(a,b) + c(d,t) + c(d,c)$$
$$= 1 + 3 + 4 + 8$$
$$= 16$$

9] $\{s, b, c\}$ $\{a, d, t\}$ 12

$$c(P,\bar{P}) = c(s,a) + c(b,d) + c(c,t)$$
$$= 2 + 6 + 4$$
$$= 12$$

10] $\{s, b, d\}$ $\{a, c, t\}$ 19

$$c(P,\bar{P}) = c(b,c) + c(s,a) + c(d,t) + c(d,c)$$
$$= 5 + 2 + 4 + 8$$
$$= 19$$

11] $\{s, c, d\}$ $\{a, b, t\}$

$$c(P,\bar{P}) = c(s,a) + c(s,b) + c(c,t) + c(d,t)$$
$$= 2 + 1 + 4 + 4$$
$$= 11$$

12] $\{s, a, b, c\}$ $\{d, t\}$ 14

$$c(P,\bar{P}) = c(a,d) + c(b,d) + c(c,t)$$
$$= 4 + 6 + 4$$
$$= 14$$

13] $\{s, a, b, d\}$ $\{c, t\}$ 17

$$c(P,\bar{P}) = c(b,c) + c(d,c) + c(d,t)$$
$$= 5 + 8 + 4 = 17$$

	P	$\bar{P}$	Capacity $(P, \bar{P})$
14]	$\{s, a, c, d\}$	$\{b, t\}$	12

$C(P, \bar{P}) = c(a,b) + c(s,b) + c(c,t) + c(d,t)$
$= 3 + 1 + 4 + 4$
$= 12$

15]	$\{s, b, c, d\}$	$\{a, t\}$	10

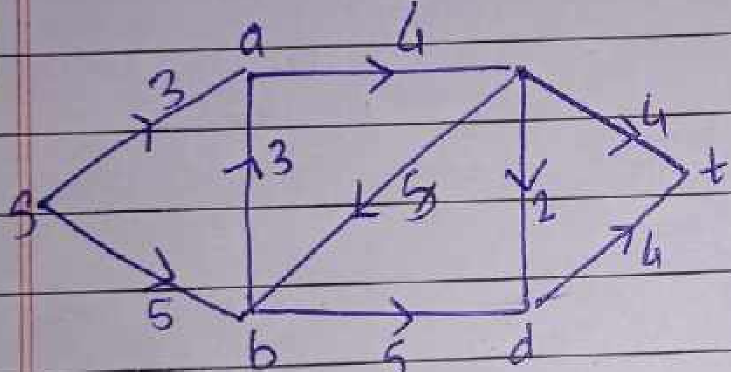
$C(P, \bar{P}) = \cancel{c(a,d) + c(a,b)}$
$= c(s,a) + c(c,t) + c(d,t) = 2+4+4 = 10$

16]	$\{s, a, b, c, d\}$	$\{t\}$	8

$C(P, \bar{P}) = c(d,t) + c(c,t)$
$= 4 + 4$
$= 8$

Find all s-t cuts

i)



ii]