Central Limit Theorem and Confidence Intervals

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Learning Objectives

By the end of the day you should be able to...

— **Define** the normal distribution and the concept of normality.

— **Describe** the uses of the 68-95-99.7 rule and the z-score.

— **Describe** and apply the Central Limit Theorem.

Learning Objectives

By the end of the day you should be able to...

— **Describe** the relationship between the mean, the standard deviation, and the standard error of the mean.

— **Calculate** the standard error of the mean.

— **Define** and **calculate** a confidence interval.

Normality

and

The Normal Distribution

The Normal Distribution

— aka "bell curve" or Gaussian distribution
— very commonly used in analysis
— often an assumption in modeling

Turn and Talk (6 minutes, 3 volunteers)

Assume that IQ is normally distributed with a mean (μ) of 100 and a standard deviation (σ) of 15, i.e. $\text{IQ} \sim N(100, 15)$.

1. How much of the population is within two standard deviations (2σ) of the mean?
2. If I pick a random person, what is the probability that their IQ will be somewhere between 85 and 115?
3. What is the Z score associated with an IQ of 140? 145?

Python Time

go to notebook `central-limit-theorem.ipynb`

Central Limit Theorem

Motivation

I want to know the mean height of male Ewoks on Endor.

- I can't measure them all (the whole population)...
- So I will select a sample of 100
- Let's say that the mean height of all Ewoks is 1.02 meters

Calculating the mean height

— Collect a sample of 100 heights

— Calculate the mean height, $\bar{h}$

```
import numpy as np

heights = [0.95, 1.1, 1.02, 0.88, 0.87, ...]
mean_height = np.mean(heights)
```

Calculating the mean height

Q: Is the mean height, $\bar{h}$, calculated in this manner, the sample or the population mean height?

— Raise one arm for sample

— Raise two arms for population

Calculating the mean height

Q: Is the mean height, $\bar{h}$, calculated in this manner, the sample or the population mean height?

- Raise one arm for sample
- Raise two arms for population

What would happen if I repeated this "experiment" and collected a different sample?

Multiple samples

Sample #	Mean Height
1	1.01
2	0.97
3	1.04
4	0.91
...	...

Multiple samples

Sample #	Mean Height
1	1.01
2	0.97
3	1.04
4	0.91
...	...

We need to make sense of the randomness in mean height!

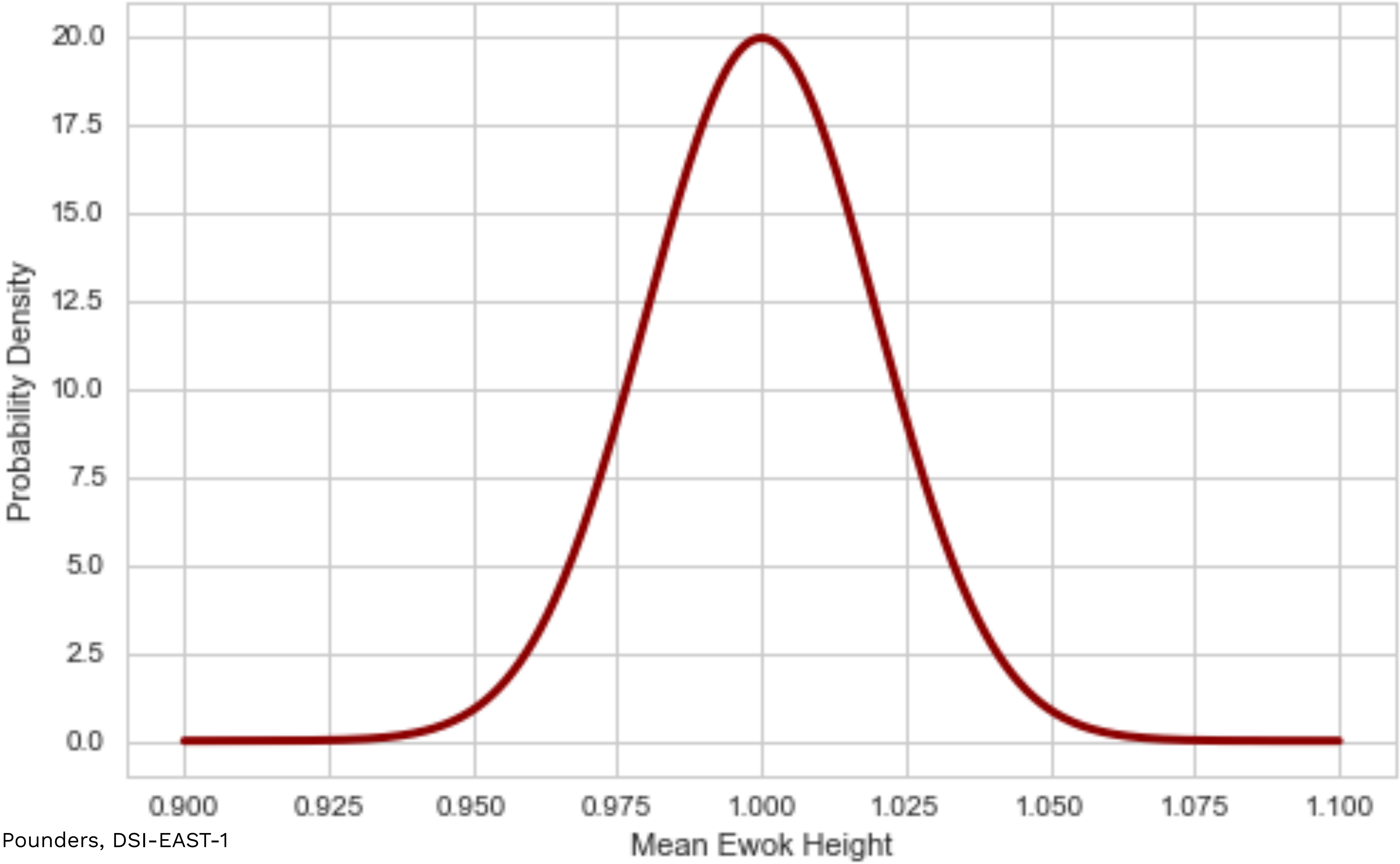
Understanding multiple samples

- An individual Ewok's height is a **random variable**
- The collection of mean heights from multiple samples is a **random variable**

Think of the sample mean as a data point from a known probability distribution.

Understanding multiple samples

- An individual Ewok's height is a **random variable**
  - Unknown probability distribution
- The collection of mean heights from multiple samples is a **random variable**
  - Normal probability distribution!!! (approximately)



20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
Probability Density
0.900
0.925
0.950
0.975
1.000
1.025
1.050
1.075
1.100
Mean Ewok Height

Summary of Central Limit Theorem

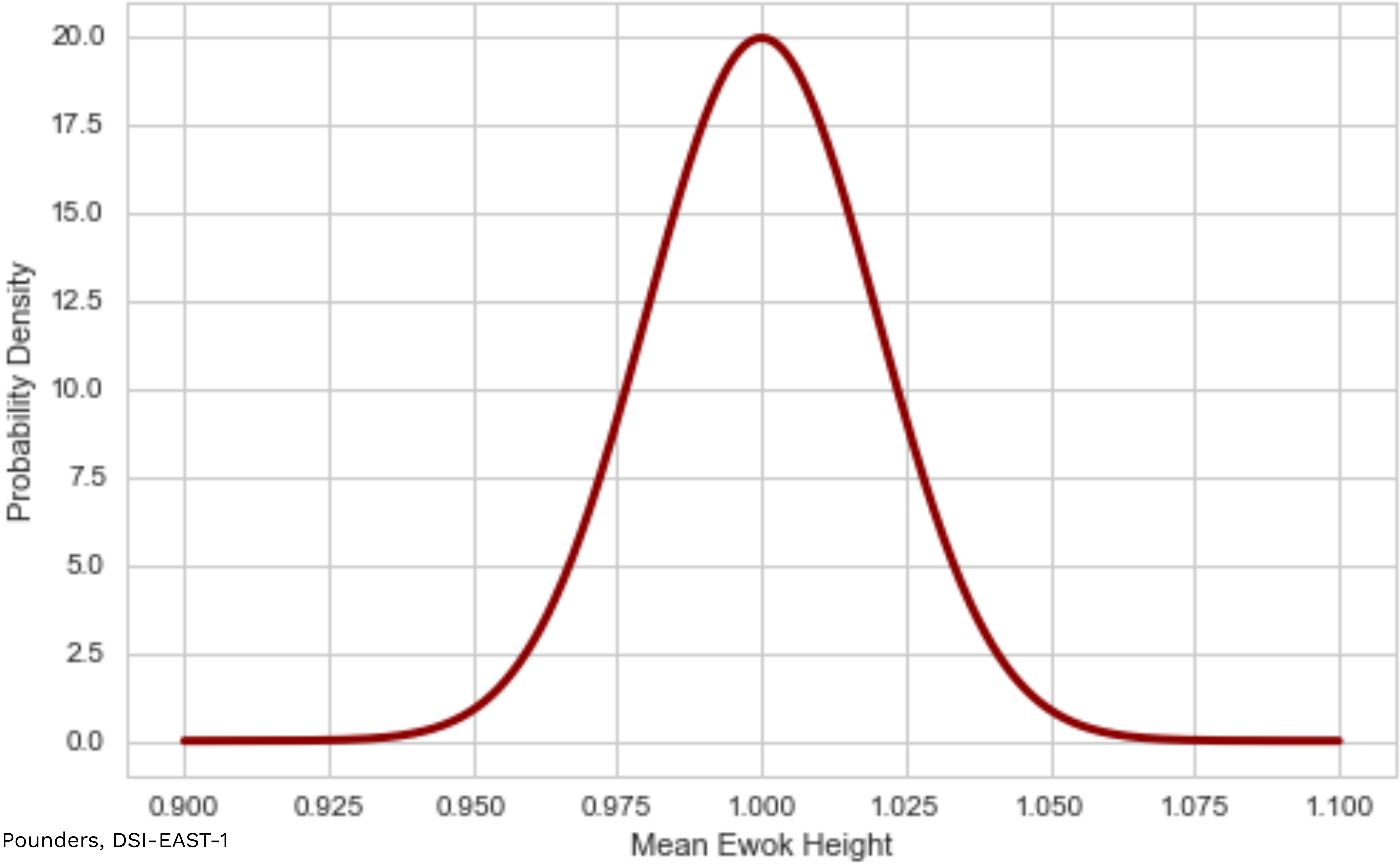
For samples of independent random variables (fixed sample size), the sample means are random and normally distributed.

Main implications and takeaways

- The distribution of the **sample mean** is called the sampling distribution
- The sampling distribution is approximately the normal distribution
- The standard deviation of the sampling distribution is called the standard error

$$\sigma_{\bar{h}} = \frac{\sigma}{\sqrt{n}}$$

The standard error represents the uncertainty of the sample mean relative to the true population mean.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Mean Ewok Height

CLT: Properties

— If $X \sim N(\mu, \sigma)$, then $\bar{X}$ is exactly $N(\mu, \frac{\sigma}{\sqrt{n}})$

i,e,

— If X is a normal random variable, the mean of a sample of X is exactly normally distributed

CLT: Properties

— If X is not normally distributed, then $\bar{X}$ is approximately $N(\mu, \sigma/\sqrt{n})$ if the sample size n is at least 30.

i.e.,

— If X is not a normal random variable, then the mean of a sample of X is approximately normally distributed as long as the sample is size is more than 30.

CLT: Practical Issue

$\bar{X} \sim N(\mu, \sigma/\sqrt{n})$, but μ and σ typically have to replaced with sample estimates.

$$N(\mu, \sigma) \approx N(\bar{X}, s/\sqrt{n})$$

Python Time

`go to notebook central-limit-theorem.ipynb`

Reflect

Q: What is the most confusing part about central limit theorem or standard error?

Respond on slack.

Confidence Interval

What we got with CLT:
- sample mean is normally distributed...
- with mean μ and...
- standard deviation σ/n (aka standard error)

What we want next:
- give me a range of values that has a...
- 95% chance of including the true (population mean)

A confidence interval describes a **range of possible values for the true mean** based on a sample mean.

Confidence Interval

$$\text{CI} = \bar{x} \pm z \cdot \frac{s}{\sqrt{n}}$$

where
- $\bar{x}$ is the sample mean
- z is a set z score
- s is the sample standard deviation
- n is the number of observations in sample

How to pick z?

Confidence Interval

Pick z to establish level of confidence:

- 90% confidence... $z = 1.645$
- 95% confidence... $z = 1.96$
- 99% confidence... $z = 2.575$

Confidence Interval

Pick z to establish level of confidence:

- 90% confidence... $z = 1.645$
- 95% confidence... $z = 1.96$
- 99% confidence... $z = 2.575$

What do we mean by "confidence"?

Confidence Interval

Suppose the 95% confidence interval for my average morning commute is 35 to 55 minutes.

Correct Interpretation

If I generated lots of samples from my commute history, and calculated a confidence interval for each, those intervals would contain the "true mean" of my commute 95% of the time.

Confidence Interval

Suppose the 95% confidence interval for my average morning commute is 35 to 55 minutes.

Not Quite Correct Interpretation

There is a 95% probability that my commute time each day will be between 35 and 55 minutes.

Python Time

`go to notebook confidence-intervals.ipynb`