Euler's Formulae :- The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \quad \text{--- (1)}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx. \end{aligned}\right\} \text{--- (2)}$$

Case (1) for $\alpha = 0$, interval becomes $0 < x < 2\pi$ and (2) becomes

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx. \quad \text{--- (3)}$$

Case (2) for $\alpha = -\pi$, interval becomes $-\pi < x < \pi$ and (2) becomes

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{--- (4)}$$

Case (3) for any interval of length $2l$, i.e $x \in (\alpha, \alpha + 2l)$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}} \quad \text{--- (5)}$$

$$\left.\begin{aligned} a_0 &= \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos \frac{n\pi x}{l}\,dx \\ b_n &= \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin \frac{n\pi x}{l}\,dx \end{aligned}\right\} \text{--- (6)}$$

Note :- for $l = \pi$ or $2l = 2\pi$ interval length (6) & (2) are identical.

Ex ① Find the Fourier series expression for $f(x) = x + x^2$ in $[-\pi, \pi]$ and hence deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Solⁿ Let $x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — ①

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\underbrace{\cancel{\frac{\sin nx}{n}}}_{0} - (2x+1)\left(\frac{-\cos nx}{n^2}\right) + (2)\underbrace{\left(-\cancel{\frac{\sin nx}{n^3}}\right)}_{0}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$\therefore a_1 = -4,\ a_2 = \frac{4}{2^2} = 1,\ a_3 = \frac{-4}{3^2},\ a_4 = -\frac{4}{4^2}, \cdots$

Some Results

$\because \sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n$

$\cos\left(n+\frac{1}{2}\right)\pi = 0$

Similarly
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3} + (-\pi+\pi^2)\frac{\cos n\pi}{n} - 2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

$\therefore b_1 = 2,\ b_2 = -\frac{2}{2},\ b_3 = \frac{2}{3},\ b_4 = -\frac{2}{4}, \cdots$

Substituting a_0, a_n and b_n in ① we have

$$x+x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$
$$-2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right] \quad \text{— ②}$$

put $x=\pi$ in ②

$$\Rightarrow \pi+\pi^2 = \frac{\pi^2}{3} + 4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right] \quad \text{— ③}$$

and put $x=-\pi$ in ②

$$\Rightarrow -\pi+\pi^2 = \frac{\pi^2}{3} + 4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right] \quad \text{— ④}$$

Adding ③ and ④

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right]$$

$$\Rightarrow \frac{4\pi^2}{3} = 8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

Exercise Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}, \ -\pi<x<\pi$

Hence show that

(i) $\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2/6$ [Hint put $x=\pi$]

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}$ [put $x=0$]

(iii) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ [Add (i) & (ii)]

(iv) $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

Conditions for a Fourier Series Expansion:-

OR Dirichlet's Conditions

Any function $f(x)$ can be developed as a Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are constants provided:

(i) $f(x)$ is periodic, single-valued and finite.

(ii) $f(x)$ has a finite number of discontinuities in any one period.

(iii) $f(x)$ has at the most a finite number of maxima & minima.

— The infinite series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ converges to $f(x)$ as $n \to \infty$ at all values of x for which $f(x)$ is continuous and the sum of the series is equal to $\frac{1}{2}[f(x-0) + f(x+0)]$ at point of discontinuity.

— Expressing $f(x)$ as Fourier series depends upon the evaluation of integrals.

$$\frac{1}{\pi}\int f(x) \cos nx \, dx \text{ and } \frac{1}{\pi}\int f(x) \sin nx \, dx \text{ for interval of } 2\pi,$$

If the intervals are $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha+2\pi)$, $f(x)$ must be defined for all values of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha+2\pi)$.

Problem:- (1) Give reason why the following functions can not be expanded in Fourier Series in $[-\pi, \pi]$

(i) $f(x) = \operatorname{cosec} x$ (ii) $f(x) = \sin 1/x$

(iii) $f(x) = \frac{1}{3-x}$ in the interval $[0, 2\pi]$

Complex Form of Fourier Series:-

The Fourier series of a periodic function $f(x)$ of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$\therefore$ (1) $\Rightarrow$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_0\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right\}$$

$$= C_0 + \sum_{n=1}^{\infty}\left\{C_n e^{in\pi x/l} + C_{-n} e^{-in\pi x/l}\right\} \quad \text{--- (2)}$$

where $C_0 = \frac{a_0}{2}$, $C_n = \frac{a_n - i b_n}{2}$, $C_{-n} = \frac{a_n + i b_n}{2}$

Now by definition of a_n & b_n

$$C_n = \frac{1}{2l}\left\{\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx - i\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\,dx$$

Similarly $C_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{in\pi x/l}\,dx.$

Combining C_n and C_{-n} we have.

$$\boxed{C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\,dx} \quad \text{--- (3)}$$

where $n = 0, \pm1, \pm2, \pm3, \ldots$

and the series (2) can be compactly written as:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{in\pi x/l}} \quad \text{--- (4)}$$

which is called as complex form of Fourier series & coefficients given by (3)

So $$\boxed{f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}} \quad \text{--- (4)}$$

to get the formula for Fourier coefficients c_n

multiply (4) by $e^{-\frac{in\pi x}{l}}$ both sides and integrate from

α to $\alpha+2l$ for interval $[\alpha, \alpha+2l]$ of length $2l$.

$$\Rightarrow \int_{\alpha}^{\alpha+2l} f(x) e^{-\frac{in\pi x}{l}} dx = c_n \int_{\alpha}^{\alpha+2l} e^{\frac{in\pi x}{l}} \cdot e^{-\frac{in\pi x}{l}} dx$$

$$= c_n \int_{\alpha}^{\alpha+2l} dx = c_n [x]_{\alpha}^{\alpha+2l} = c_n(2l)$$

$$\Rightarrow \boxed{c_n = \frac{1}{2l} \int_{\alpha}^{\alpha+2l} f(x) e^{-\frac{in\pi x}{l}} dx}$$

In particular if interval is $[-\pi, \pi]$ of length 2π

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

and $$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} \quad //.$$

Ex (1) Find complex form of Fourier series of the function $f(x) = x^2$ in the interval $[-1, 1]$.

Sol^n Here $l = 1$ the coefficient c_0 is given by

$$c_0 = \frac{1}{2l} \int_{-l}^{l} f(x) dx = \frac{1}{2} \int_{-1}^{1} x^2 dx = \frac{1}{2} \left[\left| \frac{x^3}{3} \right|_{-1}^{1} \right]$$

$$= \frac{1}{6} [1^3 - (-1)^3] = \frac{1}{3}.$$

for $n \neq 0$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^2 e^{-in\pi x}\, dx$$

integrating by parts twice, we obtain.

$$c_n = \frac{1}{2}\left[\left(\frac{x^2 e^{-in\pi x}}{-in\pi}\right)\Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{2x\, e^{-in\pi x}}{-in\pi}\, dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{x^2 e^{-in\pi x}}{-in\pi}\right)\Bigg|_{-1}^{1} + \frac{2}{in\pi}\int_{-1}^{1} x\, e^{-in\pi x}\, dx\right]$$

$$= \cancel{\frac{1}{2in\pi}\left[e^{in\pi} \quad e^{in\pi}\right.}$$

$$= \frac{1}{2in\pi}\left[e^{in\pi} - e^{-in\pi} - \frac{2}{in\pi}\left(e^{in\pi} - e^{-in\pi}\right) + \frac{2}{(in\pi)^2}\left(e^{in\pi} - e^{-in\pi}\right)\right]$$

$$= \left[\frac{1}{n\pi}\, \frac{e^{in\pi} - e^{-in\pi}}{2i} + \frac{2}{n^2\pi^2}\, \frac{e^{in\pi} - e^{-in\pi}}{2} - \frac{2}{n^3\pi^3}\, \frac{e^{in\pi} - e^{-in\pi}}{2i}\right]$$

$$= \frac{1}{n\pi}\sin n\pi + \frac{2}{n^2\pi^2}\cos n\pi - \frac{2}{n^3\pi^3}\sin n\pi \qquad \left[\because \begin{array}{l} \sin n\pi = 0 \\ \cos n\pi = (-1)^n \end{array}\right.$$

$$\therefore\ c_n = \frac{2}{n^2\pi^2}(-1)^n$$

thus fourier series in complex form is given by

$$f(x) = x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}(-1)^n e^{in\pi x} + \sum_{n=1}^{\infty} \frac{2}{(-n)^2\pi^2}(-1)^{-n} e^{-in\pi x}$$

As $(-1)^{-n} = (-1)^n$

$$f(x) = x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\, \frac{e^{in\pi x} + e^{-in\pi x}}{2} = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

//.

Ex ① obtain Complex form of Fourier Series of $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$

Sol^n $C_0 = \cancel{\frac{1}{2\pi}\int_{-\pi}^{\pi} dx} = \frac{1}{2}\left(\because a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = 1\right)$ and $C_0 = \frac{a_0}{2}$

$$\boxed{C_0 = \frac{1}{2}}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx. \quad (\because l = \pi)$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot e^{-inx}\,dx + \int_0^{\pi} 1\cdot e^{-inx}\,dx\right] = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$= \frac{-1}{2n\pi i}\left[e^{-in\pi} - 1\right] = \frac{-1}{2n\pi i}\left[\cos n\pi - i\sin n\pi - 1\right]$$

$$= \frac{-1}{2n\pi i}\left((-1)^n - 0 - 1\right) = \begin{cases} \frac{1}{in\pi}, & n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$$

$$\therefore f(x) = \frac{1}{2} + \frac{1}{i\pi}\left[\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots\right] + \frac{1}{i\pi}\left[\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \frac{e^{-5ix}}{-5} + \cdots\right]$$

$$= \frac{1}{2} - \frac{1}{i\pi}\left[\left(e^{ix} - e^{-ix}\right) + \frac{1}{3}\left(e^{3ix} - e^{-3ix}\right) + \frac{1}{5}\left(e^{5ix} - e^{-5ix}\right) + \cdots\right]$$

$$= \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right].$$

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Exercise find complex form of Fourier series and hence show that

(i) $f(x) = e^{-x}$ in $-1 \leq x \leq 1$

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} \sinh 1 \cdot e^{in\pi x}$$

(ii) $f(x) = e^{ax}$ in $-l \leq x < l$

$$e^{ax} = \frac{2}{\pi} - \frac{2}{\pi}\left[\frac{e^{2ix} + e^{-2ix}}{1 \cdot 3} + \frac{e^{4ix} + e^{-4ix}}{3 \cdot 5} + \frac{e^{6ix} - e^{-6ix}}{5 \cdot 7} + \cdots \right]$$

(iii) $f(x) = \cos ax$ in $-\pi < x < \pi$

$$\cos ax = \frac{a}{\pi} \sin a\pi \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2} \frac{e^{inx}}{-n^2}.$$