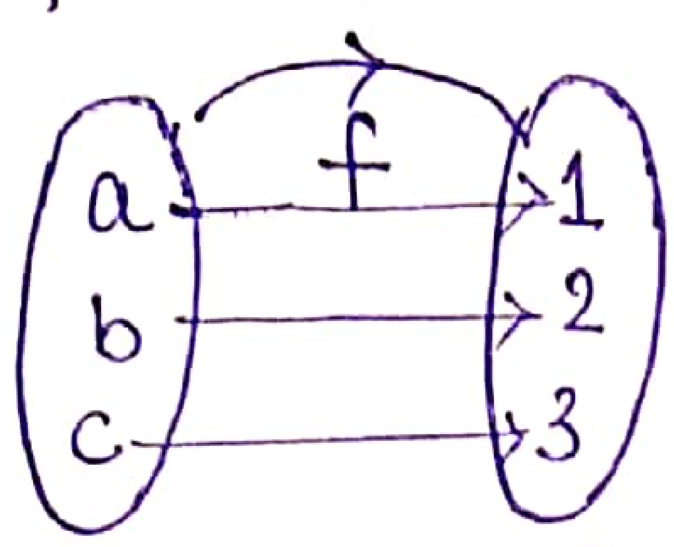
Functions

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assign--ments is called a Function from A into B.

The set A is called domain of the function, and the set B is called the co-domain.

$$f : A \to B$$



A domain   B co-domain

$a, b, c \in A$

$1, 2, 3 \in B$

i.e.

$f(a) = 1$ (Images of a,

$f(b) = 2$ b, c under

$f(c) = 3$ f).

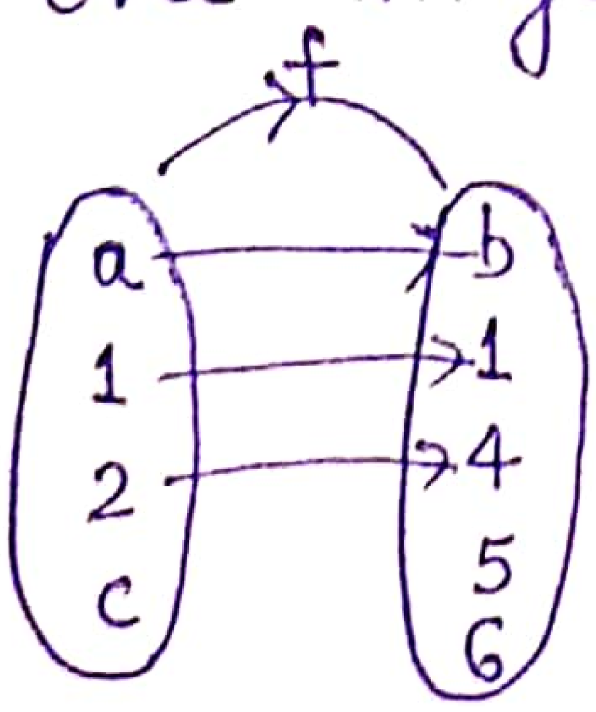
Range

The set of all image values is called the range or image of f. It is denoted by Ran(f), Im(f) or f(A).

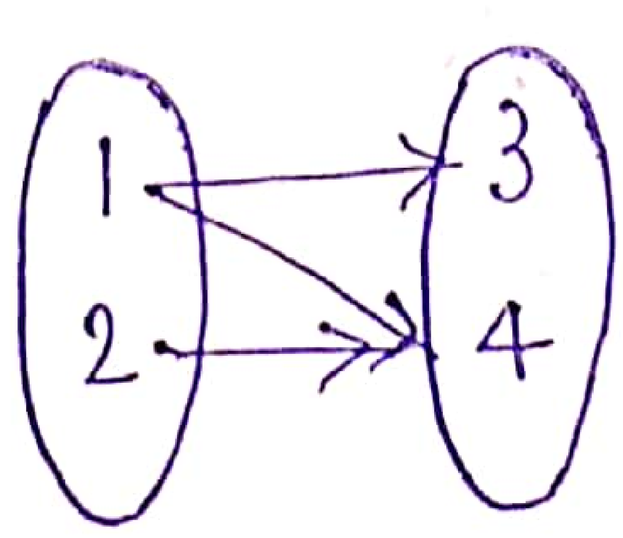
$$f : A \to B.$$

$$\text{Ran}(f) : \{ y : y \in B \text{ and } y = f(x),\ x \in A \}.$$

Note :- It is not necessary that each element of B shall be the image of some element of set A but it is necessary that every element of A must have one and only one image in B.



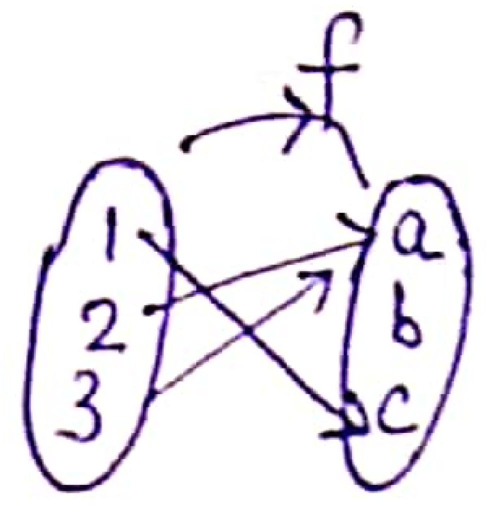
Not function



Not function

ex 1 : Let $f: \{1,2,3\} \to \{a,b,c\}$ defined by

$f(1) = c$, $f(2) = a$, $f(3) = a$.



function

domain $= \{1,2,3\}$

co-domain $= \{a,b,c\}$

$Ran(f) = \{a,c\}$.

ex 2 : $f: Z \to Z$ defined by

$f(n) = 3n$

domain $= Z$

Co-domain $= Z$

$Ran(f) = 3Z$.

ex 3 : $f: N \to N$ defined by

$f(n) = \frac{n}{2}$.

This is not a function because every input doesn't have an output in N. {since $\frac{n}{2} \notin N$ for some $n \in N$}.

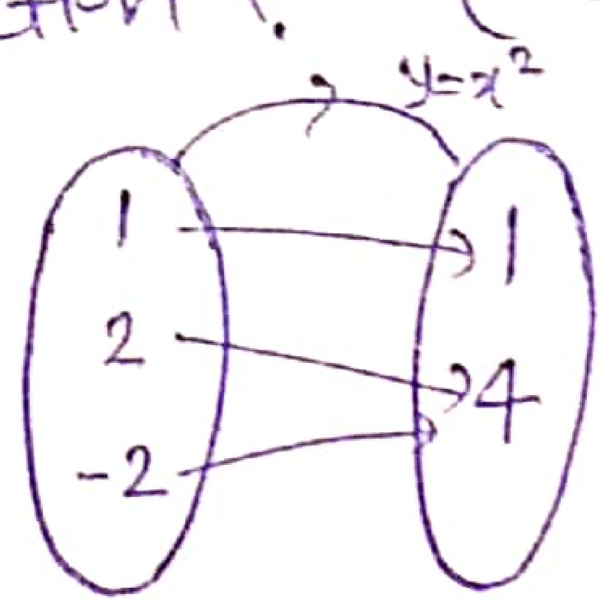
$f(3) = \frac{3}{2}$ by rule.

as $\frac{3}{2} \notin N$ for $3 \in N$,

ex 4 : $y = x^2$, Is it a function? $(x \in R)$

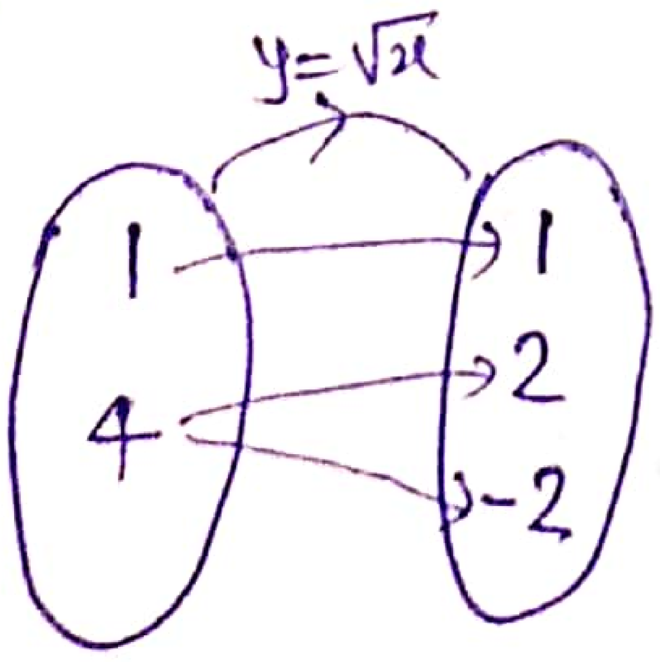
for $x = 1, 2, -2$

$y = 1, 4, 4$



Yes

ex 5 : $y = \sqrt{x}$, Is it a func.?



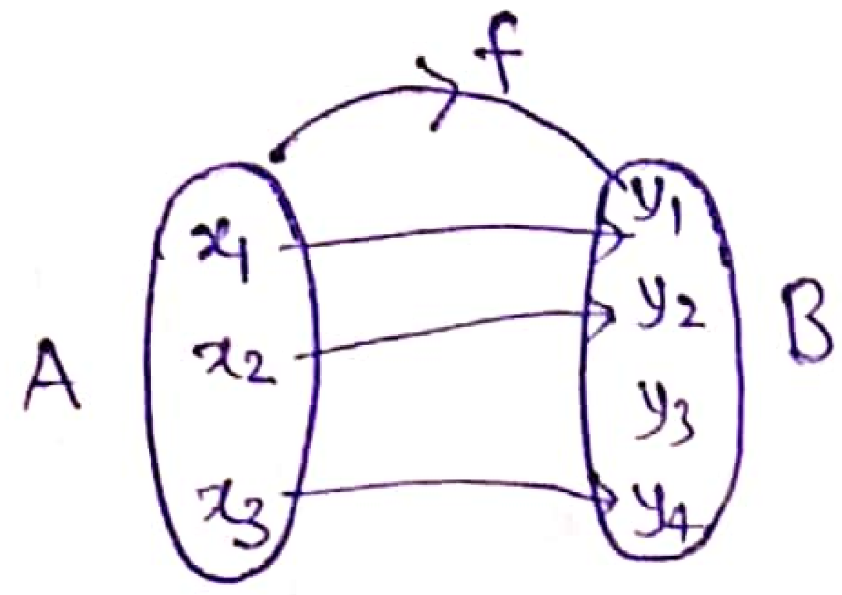
No

Types of Functions

ONE-ONE OR INJECTIVE :→ A mapping or function f from $A \to B$ is said to be

injective, if for each distinct element of set A (domain), there exists a distinct image in set B (Co-domain).

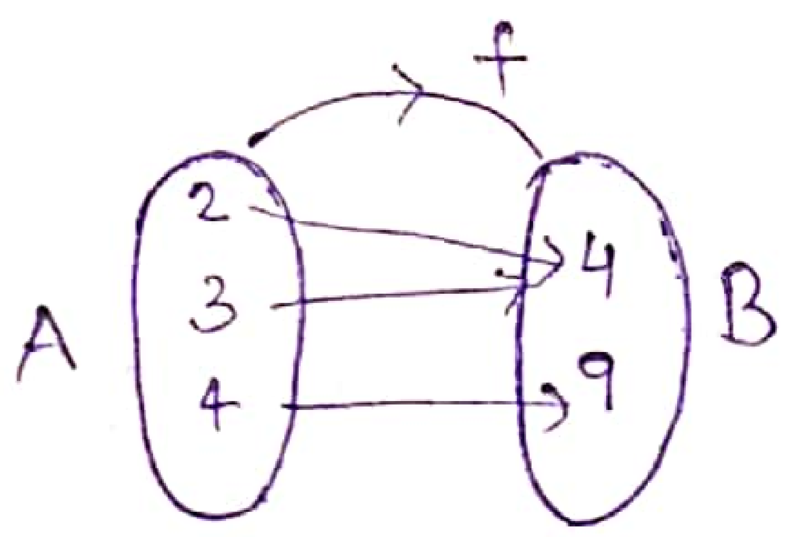
ex→



Mathematically, if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ OR
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Note :- A mapping f from $A \to B$ is many one if two or more elements in A have same image in B.

ex→

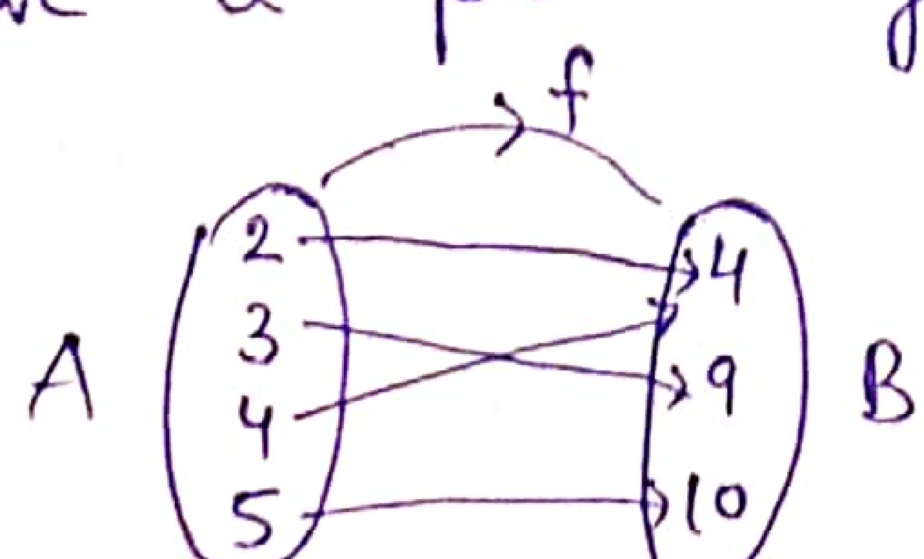


Mathematically, $x_1, x_2 \in A$ and $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

Surjective OR ONTO :→ A mapping is said to be surjective if every element

of B have a pre image in A.

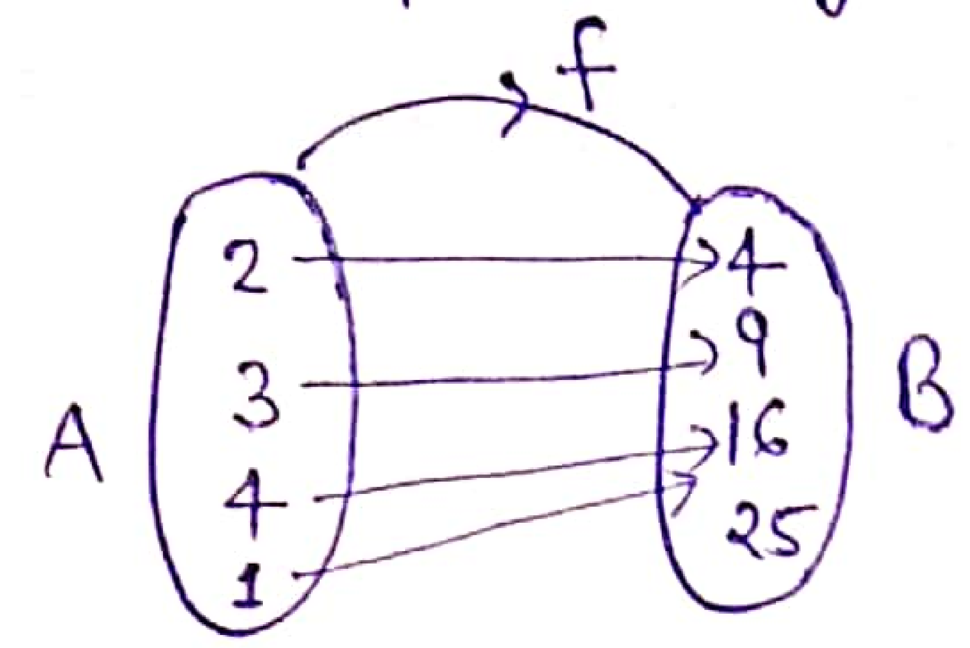
ex→



Mathematically, $\forall\ y \in B$ there exists $x \in A$ such that $y = f(x)$.

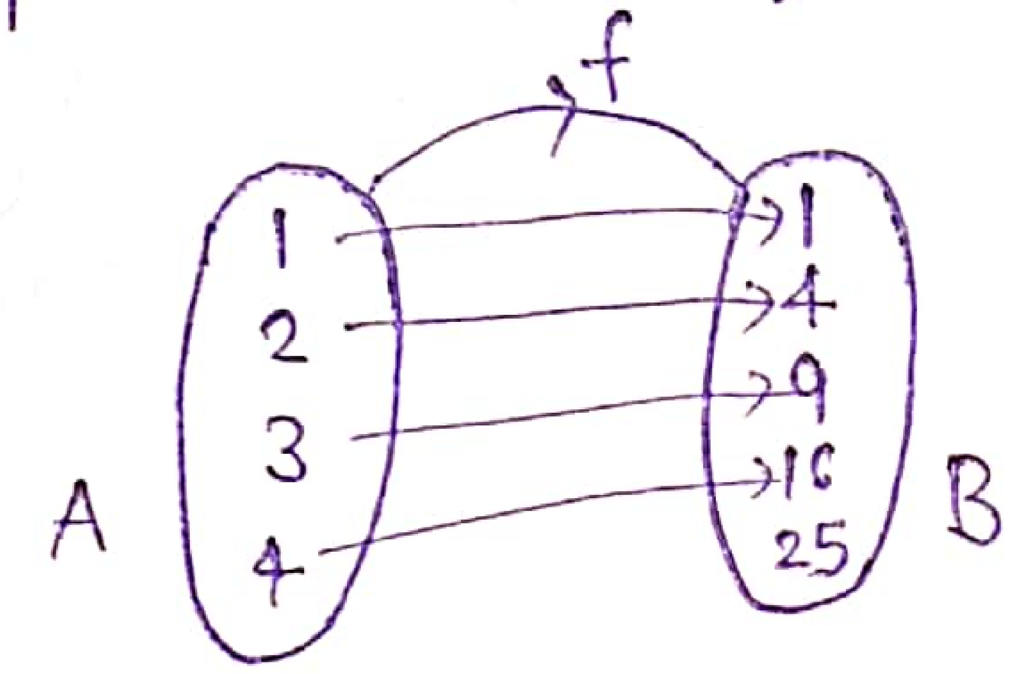
INTO :→ A mapping f is said to be into if there exists atleast one element in B having no pre-image in A.

ex→



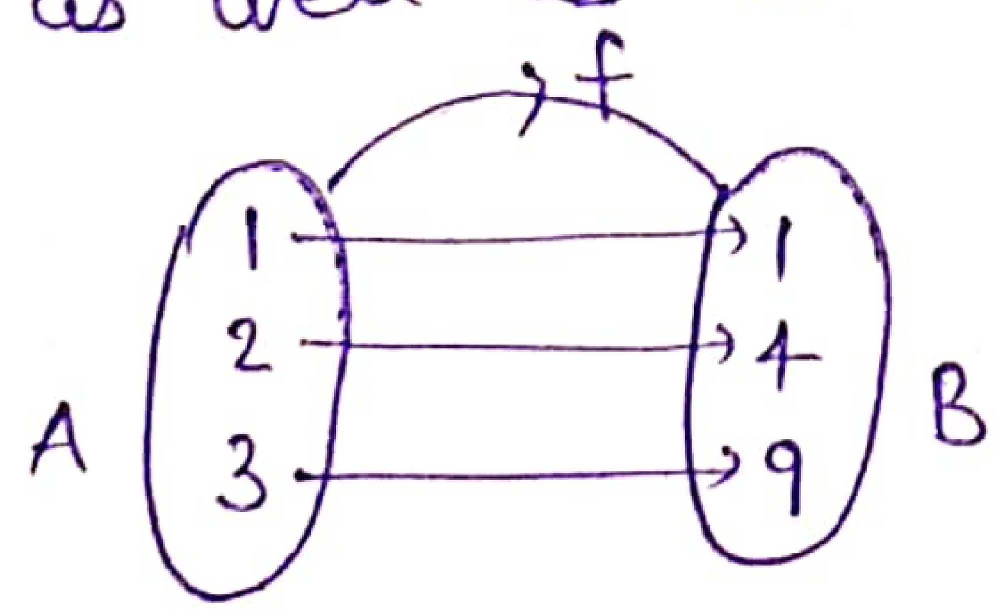
ONE-ONE INTO :- A function is said to be one-one into if it has the properties of one-one function and also into function.

ex→



ONE-ONE ONTO OR BIJECTIVE FUNCTION :- A function is said to be bijective function if the function holds the properties of one-one as well as onto function.

ex→



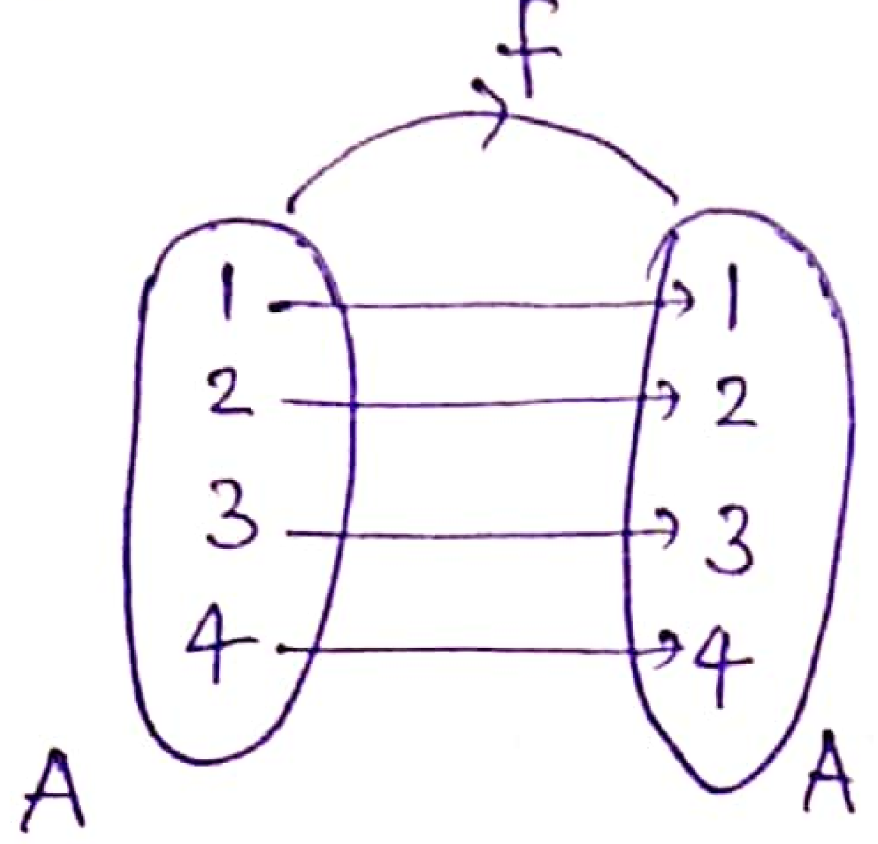
Mathematically, for every ($\forall$) $y \in B$, there exists one and only one x in A such that $y = f(x)$.

Equal Functions :→ Consider two functions f and g from the set A to B. The two functions f and g are said to be equal if and only if $f(a) = g(a)$ $\forall$ $a \in A$.

Note :→ We say two functions f and g are equal if they have the same domain and the same co-domain.

Unequal Functions :→ If there exists at least one element a in A such that $f(a) \neq g(a)$, then f and g are called unequal functions.

IDENTITY FUNCTION :→ Let us consider a set A. A mapping $f: A \to A$ is called the identity function, i.e. each element of A is mapped to itself.



$f: A \to B$

Note :- 1 Method to check one-one function

let $f(x_1) = f(x_2)$, $x_1, x_2 \in A$

and show that $x_1 = x_2$

2 Method to check onto function

$\forall\ y \in B$ there must exist some x in A

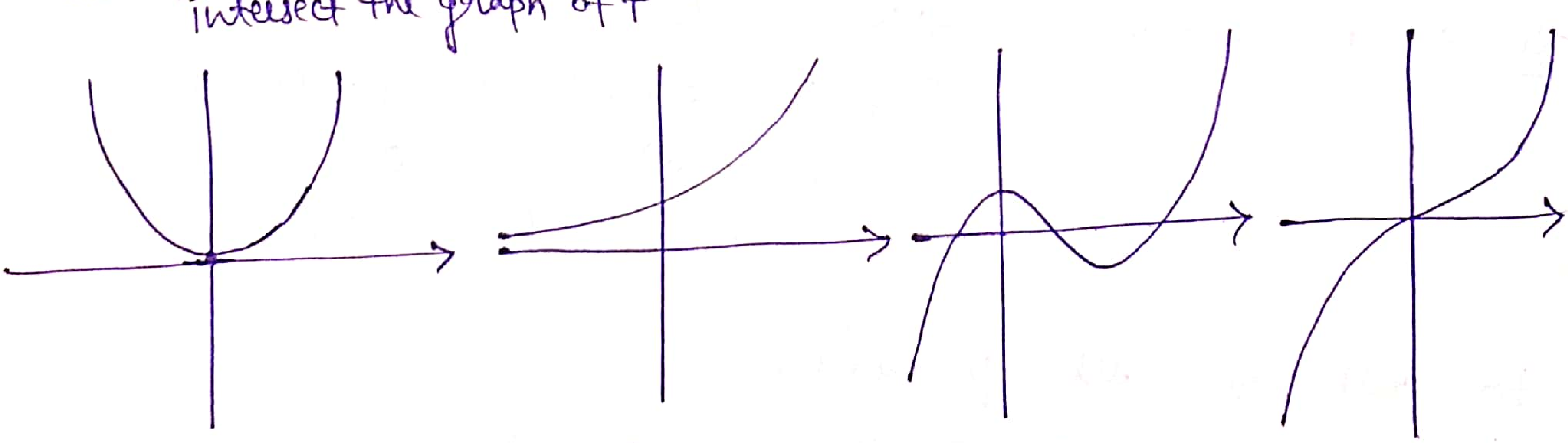
such that $y = f(x)$

Remark : Whenever a function is given by a formula in terms of a variable x, we assume, unless it is otherwise stated, that the domain of the function is $\mathbb{R}$ and also the codomain as $\mathbb{R}$.

3 Graphically we can say that a function is one-one if each horizontal line intersects the graph of f in at most one point.

4 Similarly, a function is onto if each horizontal line intersect the graph of f at one or more points.

ex →



$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
Neither one-one Nor onto	one-one Not onto	Not one-one onto	one-one onto

Q. Ex. Let $A = \{-1, 1, -3, 3, 4\}$, $B = \{1, 9, 16\}$

$f: A \to B$ is defined by

$= \{(x, y),\ y = x^2,\ x \in A,\ y \in B\}$.

Prove that the function is many one onto.

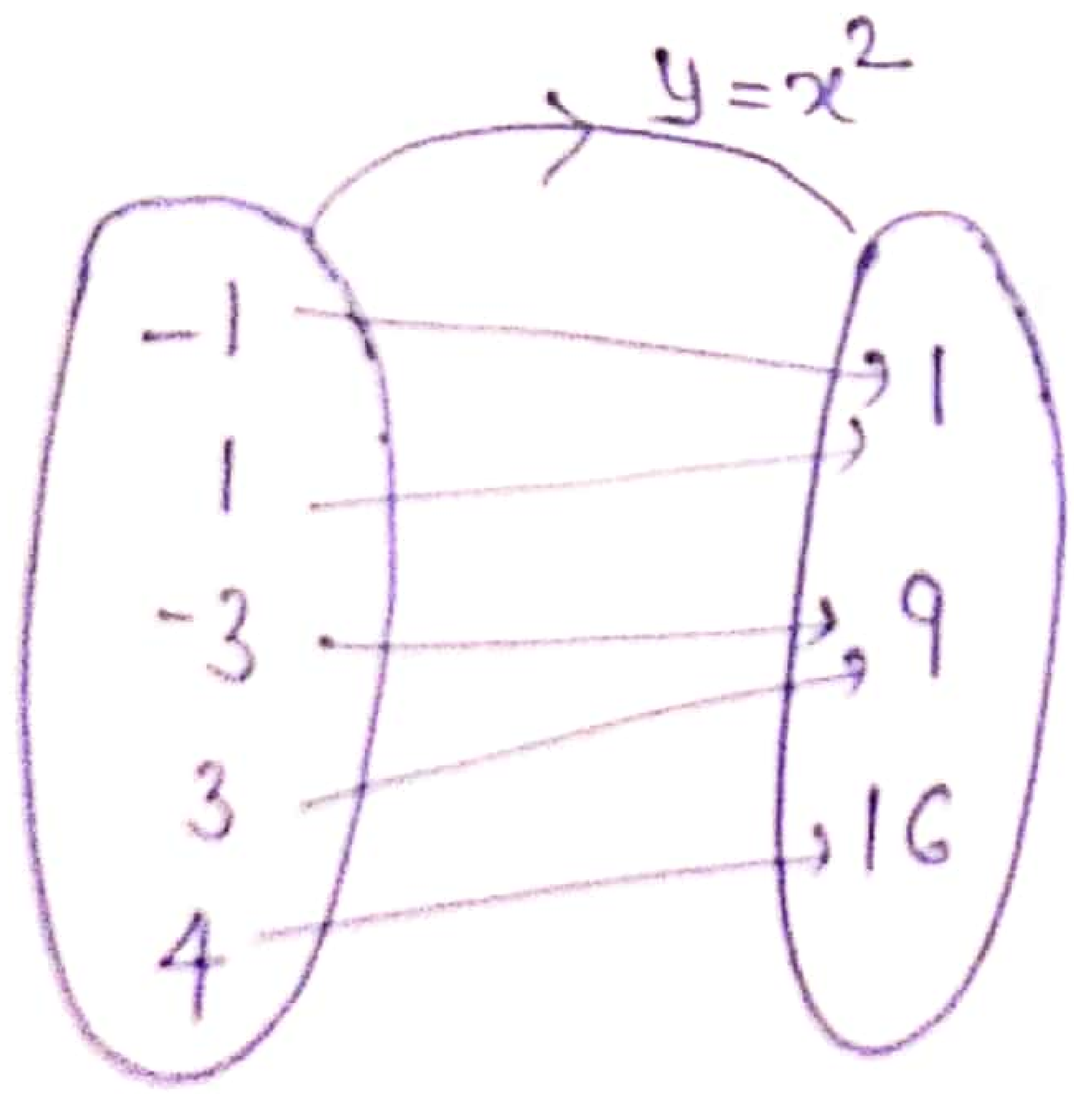
Sol → Given $y = x^2 \Rightarrow f(x) = x^2$

$$f(-1) = (-1)^2 = 1$$
$$f(1) = (1)^2 = 1$$
$$f(-3) = (-3)^2 = 9$$
$$f(3) = 9$$
$$f(4) = 16$$

Two elements of the domain A have the same image in B

$\Rightarrow$ It is many-one.

As every element of co-domain B has some pre-image in A $\Rightarrow$ It is onto mapping.



Many-one and onto func.

Ex1. Let $f: R \to R$ such that

$f(x) = 4x + 5$ for all $x \in R$.

Show that it is one-one onto.

Sol → For one-one ⇒ let $f(x_1) = 4x_1 + 5$

$f(x_2) = 4x_2 + 5$

Now $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + 5 = 4x_2 + 5$

$\Rightarrow x_1 = x_2$ ⇒ Hence one-one.

For onto ⇒ let $y \in R$ (co-domain) s.t.

$y = f(x)$

$y = 4x + 5$

$\Rightarrow x = \frac{y-5}{4} \in R$

It implies that for all y in R (co-domain) there will be x in R (domain).

⇒ Hence onto.

Ex2. Let $f: R \to R$ such that $f(x) = 2x^2 + 3$.

Check for one-one and onto.

Sol → For one-one ⇒ $f(x_1) = f(x_2)$

$2x_1^2 + 3 = 2x_2^2 + 3 \Rightarrow x_1^2 - x_2^2 = 0$

$(x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow x_1 = x_2$ & $x_1 = -x_2$ ⇒ Not one-one i.e. Many one

For onto ⇒ let $y = 2x^2 + 3$

$x^2 = \frac{y-3}{2} \Rightarrow x = \sqrt{\frac{y-3}{2}}$

once $y < 3$, we get imaginary values of x, Not real. Hence not onto.

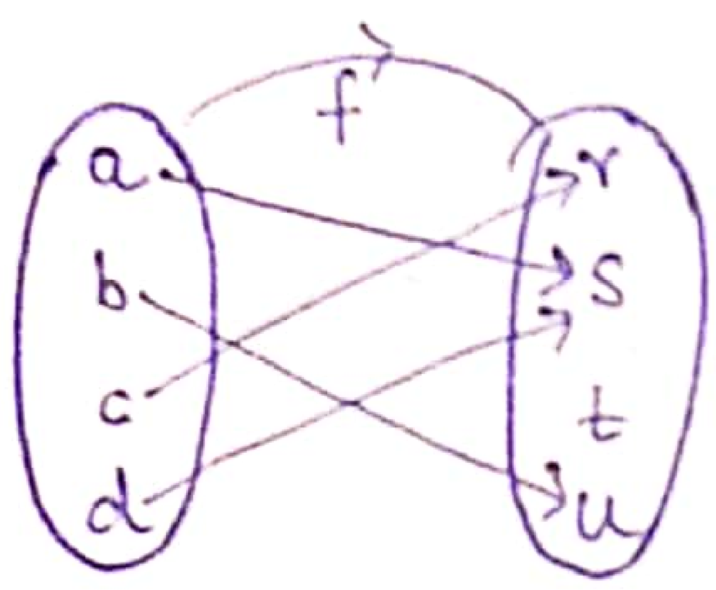
Functions as Relations

A function $f : A \to B$ is a relation from A to B (ie. a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Hence every function $f : A \to B$ can be represented as (called graph of f)

Graph of f $= \{(a,b) \mid a \in A, b = f(a)\}$

Ex→ Let $f : A \to B$, where $A = \{a, b, c, d\}$, $B = \{r, s, t, u\}$.



Sol→ Hence $f(a) = s$, $f(b) = u$, $f(c) = r$, $f(d) = s$

then Graph of f or $f = \{(a,s), (b,u), (c,r), (d,s)\}$

Ex→ On the set $A = \{1, 2, 3\}$, consider three relations on A^2.

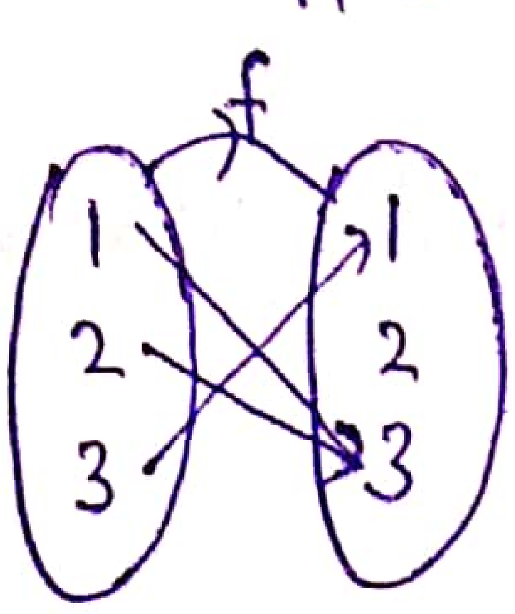
$f = \{(1,3), (2,3), (3,1)\}$

$g = \{(1,2), (3,1)\}$

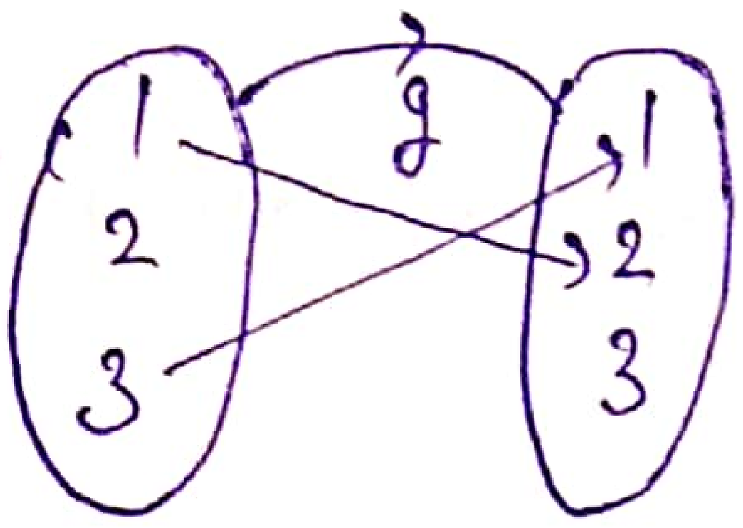
$h = \{(1,3), (2,1), (1,2), (3,1)\}$

check for the function.

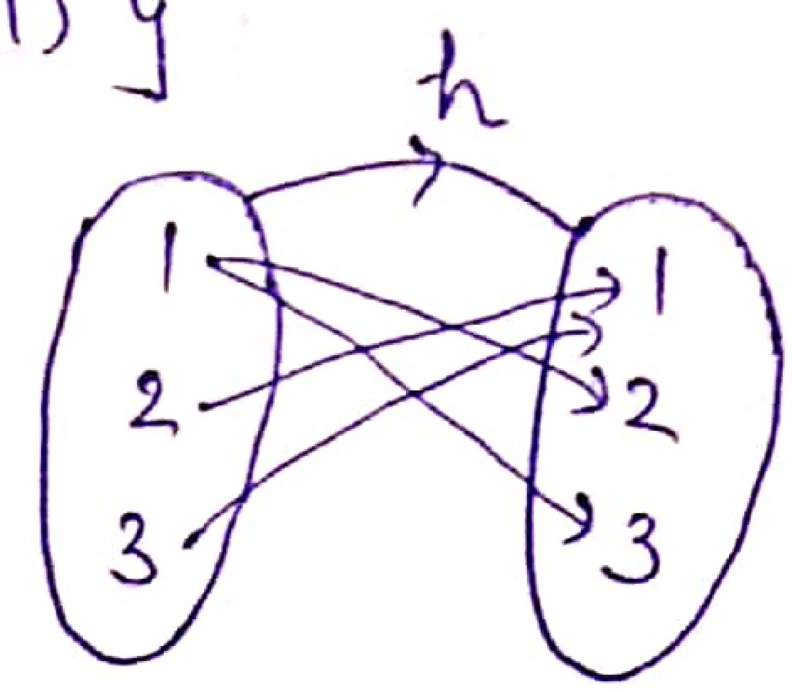
Sol→



Function ?
Yes



Function ?
No



Function ?
No

Difference of Relation and Function

* In function, no element of domain (suppose A) can have more than one image in co-domain (say B).
But in relation, any element in set A can have more than one image in set B.

* In function, also every element of set must have an image in set B.

Ex → Let $A = \{a, b, c\}$, $B = \{2, 3, 4, 5\}$.

Given $F_1 = \{(a,2), (b,2), (c,3)\}$

$F_2 = \{(a,2), (a,4), (b,3), (c,5)\}$

$F_3 = \{(a,2), (b,3)\}$

Sol → For F_1, every element of A has distinct Image in B.
⇒ F_1 is a function. (relation also)

For F_2, element a has two images in B.
⇒ F_2 is not a function, just a relation.

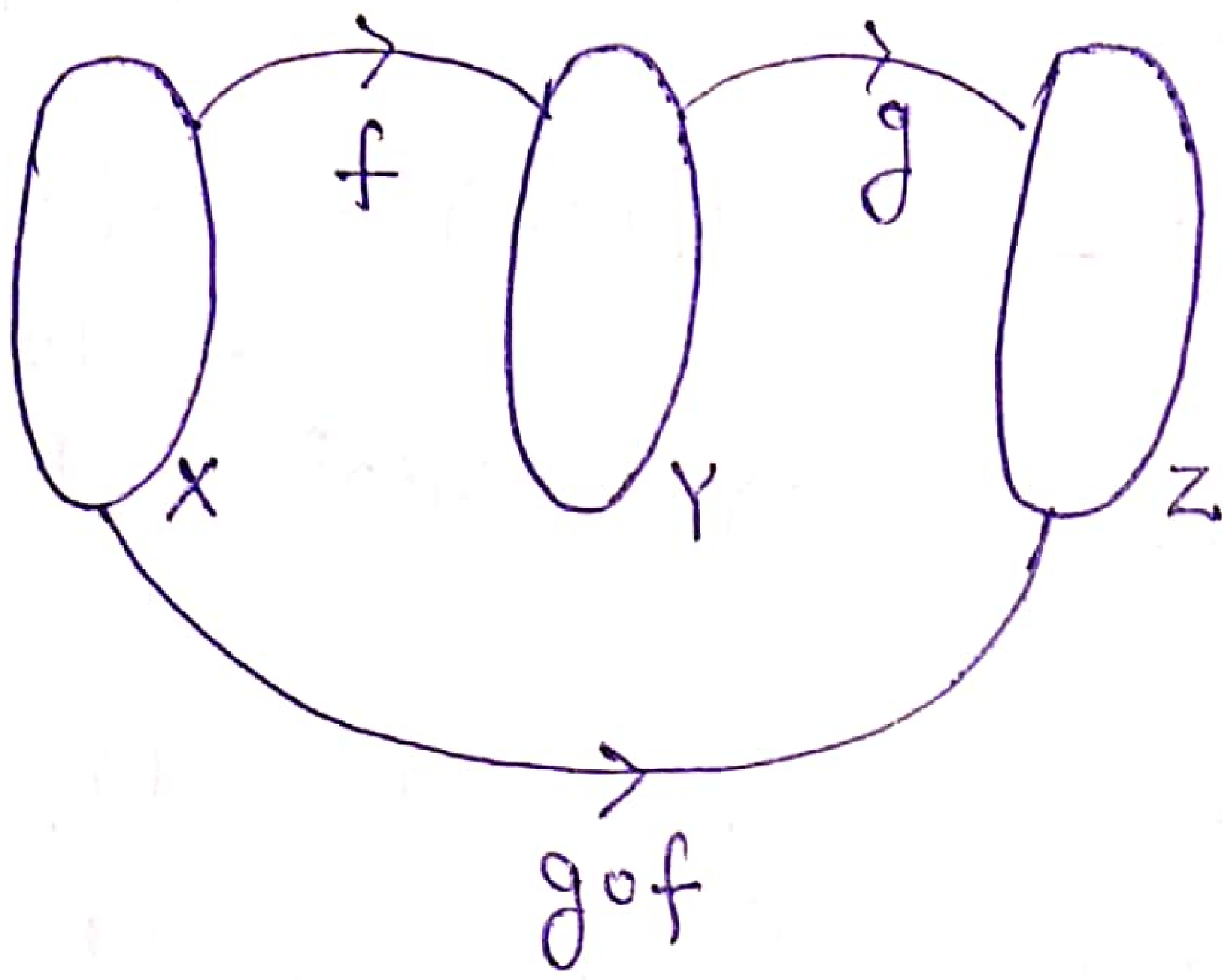
For F_3, element c doesn't have image in B.
⇒ f_3 is not a function. just a relation.

Note :→ Every function is a relation but not vice-versa.

Composition of Functions

Let X, Y and Z be three non-empty set. Let us consider two functions

$$f : X \to Y \quad \text{and} \quad g : Y \to Z$$



$(gof)(x)$ is said to be the composite (product) of the two functions f and g such that $gof : X \to Z$

$$(gof)(x) = g(f(x)) \quad , \quad x \in X, \; f(x) \in Y$$

Similarly we can define $(fog)(x) = f(g(x))$.

ex 1. Let $X = \{1,2,3,4\}$, $Y = \{a,b,c,d,e\}$, $Z = \{u,v,w\}$
The two functions $f : X \to Y$ and $g : Y \to Z$ are given by

$f = \{(1,b), (2,a), (3,d), (4,e)\}$
$g = \{(a,u), (b,u), (c,v), (d,w), (e,v)\}$

Find $(gof)(x)$

Sol^n. $(gof)(1) = g(f(1)) = g(b) = u$
$(gof)(2) = g(f(2)) = g(a) = u$
$(gof)(3) = g(f(3)) = g(d) = w$
$(gof)(4) = g(f(4)) = g(e) = v$

Hence $gof = \{(1,u), (2,u), (3,w), (4,v)\}$.

Ex2. Let $f: R \to R$ defined by $f(x) = 3x+4$ and $g: R \to R$ defined by $g(x) = x^2$. Then find $(gof)(x)$, $(fog)(x)$, $(fof)(x)$, $(gog)(x)$.

Sol^n. $(gof)(x) = g(f(x)) = g(3x+4) = (3x+4)^2$

$(fog)(x) = f(g(x)) = f(x^2) = 3x^2+4$

$(fof)(x) = f(f(x)) = f(3x+4) = 3(3x+4)+4$
$= 9x+16$

$(gog)(x) = g(g(x)) = g(x^2) = (x^2)^2 = x^4$.

Ex3. Let $A = \{a, b, c\}$, $B = \{x, y, z\}$, $C = \{r, s, t\}$. Let f and g be two func. such that $f: A \to B$ and $g: B \to C$ are defined by :

$f = \{(a,y), (b,x), (c,y)\}$, $g = \{(x,s), (y,t), (z,r)\}$

Find (i) $gof: A \to C$

(ii) Im(f) or Range(f), Im(g), Im(gof)

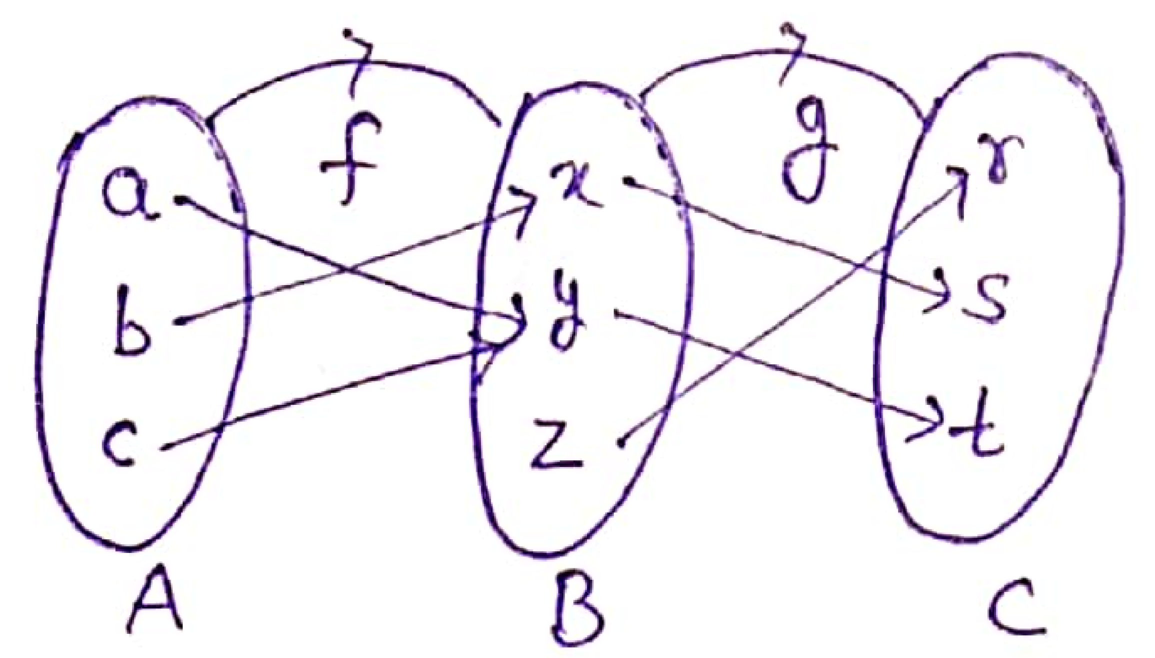
Sol^n (i) $(gof)(a) = g(f(a)) = g(y) = t$

$(gof)(b) = g(f(b)) = g(x) = s$

$(gof)(c) = g(f(c)) = g(y) = t$

$gof = \{(a,t), (b,s), (c,t)\}$.

(ii)



$Ran(f) = \{x, y\}$

$Ran(g) = \{r, s, t\}$

$Im(gof) = \{s, t\}$

<u>Note</u> :- ① $(f \circ g)(x) \neq (g \circ f)(x)$

② If f and g are one-one function $\Rightarrow$ $(g \circ f)$ is one-one.

③ If f and g are onto then $\Rightarrow$ $(g \circ f)$ is also onto.

INVERTIBLE OR INVERSE MAPPING

If a function f from $A \to B$ is one-one and onto then inverse function can be defined as $f^{-1} : B \to A$ such that $f^{-1} = \{ (y, x) ; y \in B, x \in A \text{ and } (x, y) \in f \}$

ex →

$f = \{ (1, a), (2, b), (3, c) \}$, $f^{-1} = \{ (a, 1), (b, 2), (c, 3) \}$

Remark :– f^{-1} is possible only when f is one-one onto.

Ex → check whether the function $f : R \to R$, $f(x) = x + 5$ is investible. If yes write down the inverse function.

Sol → Yes the function is invertible as it is one-one and onto function. (check by graph or Mathematically)

Given $f(x) = x+5$.

$\Rightarrow \quad y = x+5$

$\Rightarrow \quad x = y-5 \qquad \Rightarrow f^{-1}(x) = x-5 \quad [\because f^{-1}(y) = x]$

hence $f^{-1}(x) = x-5$.

Ex → If $f: R \to R$, is given by $f(x) = 3x-5$. Find $f^{-1}(x)$.

Sol → Given $f(x) = 3x-5$

$y = 3x-5$

$\Rightarrow x = \dfrac{y+5}{3}$

hence. $f^{-1}(x) = \dfrac{x+5}{3}$

Ex → If $f: R \to R$ given by $f(x) = x^2$. find $f^{-1}(x)$.

Sol → Given $f(x) = x^2$

$y = x^2$

$\Rightarrow x = \pm\sqrt{y}$ ↙ Not onto.

hence $f^{-1}(x)$ doesn't exist.