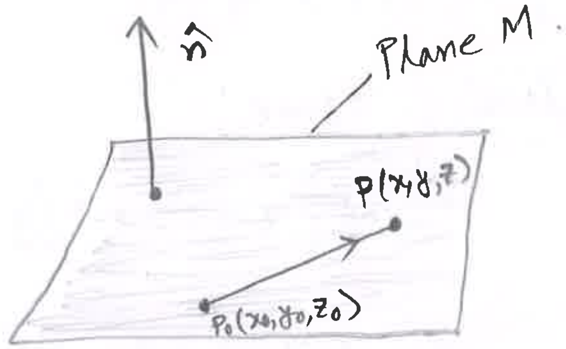
An equation for a plane in Space



Suppose that the plane M passes through a point $P_0(x_0, y_0, z_0)$ & is normal to the nonzero vector

$$\hat{n} = A\hat{i} + B\hat{j} + C\hat{k}.$$

- A vector from P_0 to any pt. P on the plane is orthogonal to $\hat{n}$.
- Then M is the set of points $P(x,y,z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\hat{n}$.
- Thus $\hat{n} \cdot \overrightarrow{P_0P} = 0$

$$\Rightarrow \left(A\hat{i} + B\hat{j} + C\hat{k}\right) \cdot \left((x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}\right) = 0$$

$$\Rightarrow A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

Equation for a plane:	
The plane through $P_0(x_0,y_0,z_0)$, normal to $\hat{n} = A\hat{i}+B\hat{j}+C\hat{k}$	
Vector Eq^n:	$\hat{n} \cdot \overrightarrow{P_0P} = 0$
Component Eq^n:	$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$
component Eq^n. simplified:	$Ax+By+Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

Problem: Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\hat{n} = 5\hat{i} + 2\hat{j} - \hat{k}$.

Solution: Plane:

$$5(x+3) + 2(y-0) - 1(z-7) = 0$$

$$\Rightarrow \quad 5x + 2y - z = -22.$$

Problem: Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

Solution:
$$\vec{AB} = 2\hat{i} + 0\hat{j} - \hat{k}$$
$$\vec{AC} = 0\hat{i} + 3\hat{j} - \hat{k}$$

Now $\vec{AB} \times \vec{AC}$ is normal to the plane

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\hat{i} + 2\hat{j} + 6\hat{k}.$$

Plane:
$$3(x-0) + 2(y-0) + 6(z-1) = 0$$

$$\Rightarrow \quad 3x + 2y + 6z = 6.$$

Problem: Find the eq^n. of the plane through $P_0(0, 2, -1)$ normal to $\hat{n} = 3\hat{i} - 2\hat{j} - \hat{k}$.

Problem: Find the eq^n of the plane through the point $(2, -3, 1)$ & $\perp^r$ to the line joining the points $(3, 4, -1)$ & $(2, -1, 5)$.

Lines of Intersection:

- Lines are parallel if and only if they have same direction.
- Two planes are parallel if and only if their normals are parallel,
  ie $\vec{n}_1 = k\vec{n}_2$, for some scalar k.
- Two planes that are not parallel intersect in a line.

Problem: (i) Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$ & $2x+y-2z=5$.

(ii) Find parametric eqⁿs for the line in which the planes $3x-6y-2z=15$ & $2x+y-2z=5$ intersect.

Solution:

(i) $\vec{n}_1 = 3\hat{i} - 6\hat{j} - 2\hat{k}$

$\vec{n}_2 = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{n}_1 \times \vec{n}_2 = 14\hat{i} + 2\hat{j} + 15\hat{k}$.

Any nonzero multiple of $\vec{n}_1 \times \vec{n}_2$ is parallel to the line of intersection.

(ii) The line is parallel to $\vec{v} = 14\hat{i} + 2\hat{j} + 15\hat{k}$.

- Find a pt. on the line:

Put $z=0$. $3x-6y=15$
$2x+y=5$.

$x=3, y=-1$.

Hence $(3,-1,0)$ is a pt. on the line.

Parametric eqⁿ of the line is.

$x = 3+14t, y = -1+2t, z = 15t$.

Problem: Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1+t$, intersects the plane $3x + 2y + 6z = 6$.

Solution:

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6.$$

$$\Rightarrow \quad t = -1.$$

Pt. of intersection is $(x, y, z)\big|_{t=-1} = \left(\frac{2}{3}, 2, 0\right)$.

The distance from a point to a Plane

- If P is a point on a plane with normal $\vec{n}$, then the distance from any point S to the plane is the length of the vector projection of $\vec{PS}$ onto $\vec{n}$.

$$\boxed{d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|}$$

Problem: Find the distance from $S(1,1,3)$ to the plane $3x+2y+6z=6$.

Solution: $\vec{n} = 3\hat{i} + 2\hat{j} + 6\hat{k}$.

A pt. on the plane: $P(0,3,0)$ [some other choices $(2,0,0), (0,0,1) \cdots$]

$$\vec{PS} = (1-0)\hat{i} + (1-3)\hat{j} + 3\hat{k}$$
$$= \hat{i} - 2\hat{j} + 3\hat{k}.$$

$$d = \left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right| = \qquad = 17/7.$$

Problem: Find the distance from the point to the plane.

(i) $(2,-3,4)$; $x+2y+2z=13$ H.W.

(ii) $(1,3,4)$; $2x-y+z=0$.

Angles between Planes:

① $\boxed{\text{The acute angle between their normal vectors.}}$

Problem: Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution: $\vec{n_1} = 3\hat{i} - 6\hat{j} - 2\hat{k}$, $\vec{n_2} = 2\hat{i} + \hat{j} - 2\hat{k}$

are normals to the planes

Angle between them.

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

$$= \frac{4}{21}.$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Problem: Find the angles between the planes:

(i) $x + y = 1$, $2x + y - 2z = 2$ (H.W).

(ii) $x + 2y + z - 1 = 0$, $2x + y - z + 1 = 0$.

4)