Vectors and the Geometry of Space

- Three dimensional co-ordinate systems.

- Distance and spheres in space:

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

- The standard Equation for a sphere of radius a & center (x_0, y_0, z_0) is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2.$$

Problem: Find the center and radius of the sphere:

$$x^2+y^2+z^2+3x-4z+1=0.$$

Problem: What are the geometric interpretations of inequalities and equations involving spheres

(i) $x^2+y^2+z^2 < 4$

(ii) $x^2+y^2+z^2 \leq 4$

(iii) $x^2+y^2+z^2 > 4$

(iv) $x^2+y^2+z^2 = 4,\ z \leq 0$.

Problem: Give a geometric description of the followings:

(i) $x^2+y^2=4$, $z=0$

(ii) $x^2+y^2+z^2=1$, $x=0$

(iii) $y=x^2$; $z=0$.

(iv) $x^2+y^2=4$, $y=z$.

(v) $x \geq 0$, $y \leq 0$, $z=0$

(vi) $1 \leq x^2+y^2+z^2 \leq 4$.

(vii) $x^2+y^2+z^2 \leq 1$, $z \geq 0$.

Problem: Find the centers and radius of the following spheres:

(i) $x^2+y^2+z^2+14x-14z=0$

(ii) $3x^2+3y^2+3z^2+x+y+z=14$

(iii) $x^2+y^2+z^2-8x+10y-10z=11$

(iv) $(x-1)^2+(y-2)^2+(z+1)^2=103+2x+4y-2z$.

Problem: Find the equations for the spheres whose centers and radii are given below

	Center	Radius
(i)	$(2,3,4)$	$\sqrt{15}$
(ii)	$(0,-1,5)$	2
(iii)	$(-4, 5/2, -3/4)$	$7/10$

Vectors:

Defn: A vector is a quantity that is determined by both its magnitude and its direction. (Directed line segment).

- A scalar is a quantity that is determined by its magnitude.

Example: scalar: Length, Temperature, Height, Voltage etc.
vector: Force, Velocity, Acceleration.

Notation: We denote vectors by lower case or upper case letters with arrows. e.g.

$$\vec{a}, \vec{b}, \vec{v}, \vec{u} \ldots$$

- A vector has a tail, called its initial point, and a tip, called its terminal point.
- Distance between the initial point and the terminal point is called the length/magnitude of a vector.
- A vector of length 1 is called a unit vector.

Component of a vector: Let us consider a 3D cartesian co-ordinate system. Let $\vec{a}$ be a given vector with initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$.

Then $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$ are called the components of the vector $\vec{a}$ w.r.to the co-ordinate system and we write $\vec{a}$ as

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

- Length of $\vec{a} = |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

- If we choose the origin $(0,0,0)$ as initial pt of a vector $\vec{a}$ and (x,y,z) as terminal point then the co-ordinates equals the components of $\vec{a}$. This suggests that we can determine each point in space by a vector, called the position vector of the points.

- In cartesian coordinate system, the position vector $\vec{r}$ of a point (x,y,z) is a vector with the origin $(0,0,0)$ as the initial pt and (x,y,z) as the terminal point.

  Thus $\vec{r} = \langle x, y, z \rangle$.

- Two vectors $\vec{a}$ & $\vec{b}$ are called equal if they are same in both magnitude and direction.

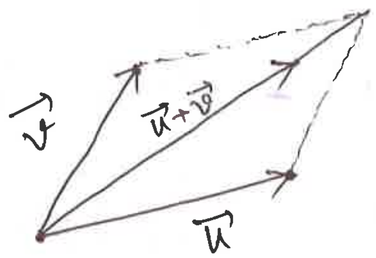
Vector Algebra

Vector Algebra operations:

Vector Addition: The sum $\vec{u}+\vec{v}$ of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is obtained by

$$\vec{u}+\vec{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$$

Remark:



Basic Properties of vector addition:

(i) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

(ii) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

(iii) $\vec{u}+\vec{0} = \vec{u} = \vec{0}+\vec{u}$

(iv) $\vec{u}+(-\vec{u}) = \vec{0}$

Scalar Multiplication of a vector:

c = scalar (Real Number), $\vec{u} = \langle u_1, u_2, u_3 \rangle$.

$$c\vec{u} = \langle cu_1, cu_2, cu_3 \rangle$$

Basic Properties of scalar multiplication of a vector:

(i) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

(ii) $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

(iii) $c(d\vec{u}) = (cd)\vec{u}$.

(iv) $1\cdot\vec{u} = \vec{u}$ (v) $0\cdot\vec{u} = \vec{0}$.

Examples: Let $\vec{u} = \langle -1, 3, 1\rangle$ & $\vec{v} = \langle 4, 7, 0\rangle$.

Find (i) $2\vec{u} + 3\vec{v}$ (ii) $\vec{u} - \vec{v}$ & (iii) $\left|\frac{1}{2}\vec{u}\right|$.

Solution: (i) $2\vec{u} + 3\vec{v} = 2\langle -1, 3, 1\rangle + 3\langle 4, 7, 0\rangle$

$$= \langle -2, 6, 2\rangle + \langle 12, 21, 0\rangle$$

$$= \langle 10, 27, 2\rangle$$

(ii) $\vec{u} - \vec{v} = \langle -5, -4, 1\rangle$

(iii) $\frac{1}{2}\vec{u} = \langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\rangle$

$$\left|\frac{1}{2}\vec{u}\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$$

Unit vectors:

- A vector $\vec{v}$ of length 1 is called a unit vector.
- The standard unit vectors are

$$\hat{i} = \langle 1, 0, 0\rangle, \quad \hat{j} = \langle 0, 1, 0\rangle, \quad \hat{k} = \langle 0, 0, 1\rangle.$$

- Any vector $\vec{v} = \langle v_1, v_2, v_3\rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\vec{v} = \langle v_1, v_2, v_3\rangle = \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle$$

$$= v_1\langle 1, 0, 0\rangle + v_2\langle 0, 1, 0\rangle + v_3\langle 0, 0, 1\rangle$$

$$= v_1\hat{i} + v_2\hat{j} + v_3\hat{k}.$$

- We call the scalar v_1 — the $\hat{i}$-component of $\vec{v}$

v_2 — the $\hat{j}$ - - - of $\vec{v}$

v_3 — the $\hat{k}$ - - - of $\vec{v}$.

- When $P_1 \equiv (x_1, y_1, z_1)$ & $P_2 \equiv (x_2, y_2, z_2)$.

then $\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

- If $\vec{v} \neq \vec{0}$, then $|\vec{v}| \neq 0$.

& $\left|\frac{1}{|\vec{v}|}\vec{v}\right| = \frac{1}{|\vec{v}|}|\vec{v}| = 1$.

— The vector $\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the direction of $\vec{v}$.
(called the direction of the non-zero vector $\vec{v}$).

Problem: Find a unit vector $\vec{u}$ in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution: $\overrightarrow{P_1P_2} = (3-1)\hat{i} + (2-0)\hat{j} + (0-1)\hat{k}$

$$= 2\hat{i} + 2\hat{j} - \hat{k}.$$

$$|\overrightarrow{P_1P_2}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3.$$

The unit vector in the direction of $\overrightarrow{P_1P_2}$ is

$$\vec{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

=

Problem: If $\vec{v} = 3\hat{i} - 4\hat{j}$ is a velocity vector, express $\vec{v}$ as a product of its speed times its direction of motion.

Solution: speed is the magnitude of velocity $\vec{v}$

$$\text{speed} = |\vec{v}| = \sqrt{(3)^2 + (-4)^2} = 5.$$

The unit vector $\frac{\vec{v}}{|\vec{v}|}$ is the direction of $\vec{v}$:

$$\frac{\vec{v}}{|\vec{v}|} = \frac{3\hat{i} - 4\hat{j}}{5} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

So, $$\vec{v} = \underbrace{5}_{\text{speed}} \cdot \underbrace{\left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)}_{\text{Direction of motion}}.$$

Problem: A force of 6 unit is applied in the direction of the vector $\vec{v} = 2\hat{i} + 2\hat{j} - \hat{k}$. Express the force $\vec{F}$ as a product of its magnitude and direction.

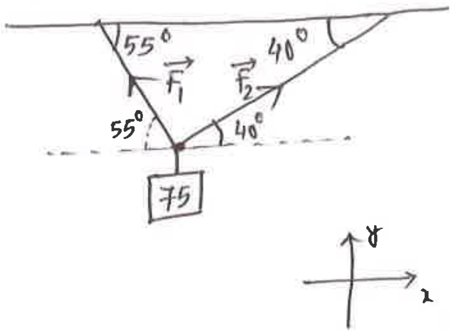
Solution: The force vector has magnitude 6 and direction $\frac{\vec{v}}{|\vec{v}|}$,

So, $$\vec{F} = 6\frac{\vec{v}}{|\vec{v}|} = 6\,\frac{2\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{2^2 + 2^2 + (-1)^2}}$$

$$= 6 \cdot \left(\frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}\right) = 6\left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right).$$

Application:

Problem: A 75-unit weight is suspended by two wires, as shown in following figure. Find the forces $\vec{F_1}$ and $\vec{F_2}$ acting in both wires.



Solution:

$$\vec{F_1} = \left\langle -|\vec{F_1}|\cos 55^\circ, |\vec{F_1}|\sin 55^\circ \right\rangle$$

$$\& \quad \vec{F_2} = \left\langle |\vec{F_2}|\cos 40^\circ, |\vec{F_2}|\sin 40^\circ \right\rangle .$$

Total Force $\quad \vec{F} = \langle 0, 75 \rangle$.

Hence
$$-|\vec{F_1}|\cos 55^\circ + |\vec{F_2}|\cos 40^\circ = 0 .$$
$$|\vec{F_1}|\sin 55^\circ + |\vec{F_2}|\sin 40^\circ = 75 .$$

Solving,
$$|\vec{F_1}| = \frac{75}{\sin 55^\circ + \cos 55^\circ \tan 40^\circ} \approx 57.67$$

$$\& \quad |\vec{F_2}| = \frac{75\cos 55^\circ}{\sin 55^\circ \cos 40^\circ + \cos 55^\circ \sin 40^\circ} \approx 43.18$$

Then
$$\vec{F_1} = \left\langle -|\vec{F_1}|\cos 55^\circ, |\vec{F_1}|\sin 55^\circ \right\rangle \approx \langle -33.08, 47.24 \rangle .$$

$$\& \quad \vec{F_2} = \left\langle |\vec{F_2}|\cos 40^\circ, |\vec{F_2}|\sin 40^\circ \right\rangle \approx \langle 33.08, 27.76 \rangle .$$

The Dot Product :

Definition: The dot product $\vec{u}\cdot\vec{v}$ of vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is the scalar $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Examples: (i) $\vec{u} = \langle 1, -2, -1\rangle$, $\vec{v} = \langle -6, 2, -3\rangle$.

$$\vec{u}\cdot\vec{v} = -6 - 4 + 3 = -7.$$

(ii) $\vec{u} = \frac{1}{2}\hat{i} + 3\hat{j} + \hat{k}$ & $\vec{v} = 4\hat{i} - \hat{j} + 2\hat{k}$

$$\vec{u}\cdot\vec{v} = \frac{1}{2}\cdot 4 - 3 + 2 = 1.$$

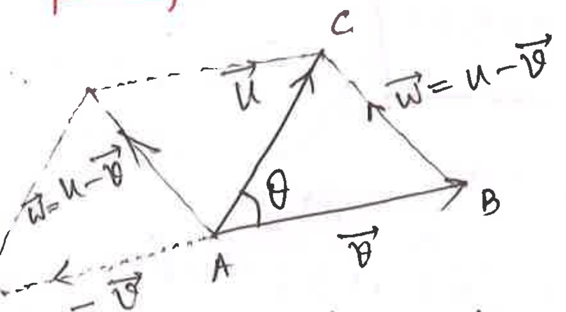
Angle between two vectors:

Theorem: The angle θ between two non-zero vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}|\,|\vec{v}|}\right).$$

When two nonzero vectors $\vec{u}$ & $\vec{v}$ are placed so their initial pts coincide, they form an angle θ, s.t $0 \le \theta \le \pi$.

proof:



From the figure:

For $\triangle ABC$, $AB = |\vec{v}|$, $BC = |\vec{w}|$ & $AC = |\vec{u}|$

Applying Law of cosines

$$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta.$$

$$\Rightarrow 2|\vec{u}||\vec{v}|\cos\theta = |\vec{u}|^2 + |\vec{v}|^2 - |\vec{w}|^2. \quad \text{--- (1)}$$

Now $|\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2$

$|\vec{v}|^2 = v_1^2 + v_2^2 + v_3^2$

& $|\vec{w}|^2 = |\vec{u} - \vec{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2.$

Now from (i).

$$2|\vec{u}||\vec{v}| \cos\theta = |\vec{u}|^2 + |\vec{v}|^2 - |\vec{w}|^2.$$

$$= u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 - (u_3 - v_3)^2$$

$$= 2(u_1 v_1 + u_2 v_2 + u_3 v_3).$$

$$\Rightarrow \cos\theta = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\vec{u}||\vec{v}|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\vec{u}||\vec{v}|}\right).$$

Remark: (i) Angle between two non-zero vectors $\vec{u}$ & $\vec{v}$ is $\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$

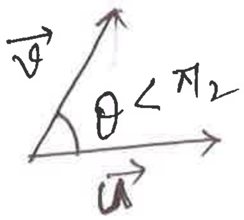
(ii) $\boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta.}$

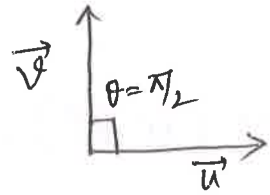
Problem: Find the angle between $\vec{u} = \hat{i} - 2\hat{j} - 2\hat{k}$ & $\vec{v} = 6\hat{i} + 3\hat{j} + 2\hat{k}$.

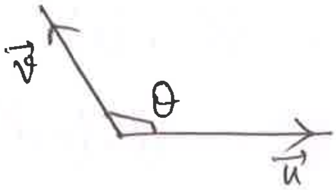
Solution: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{6 - 6 - 4}{\sqrt{9}\cdot\sqrt{49}} = \frac{-4}{21}$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{4}{21}\right).$

Note:



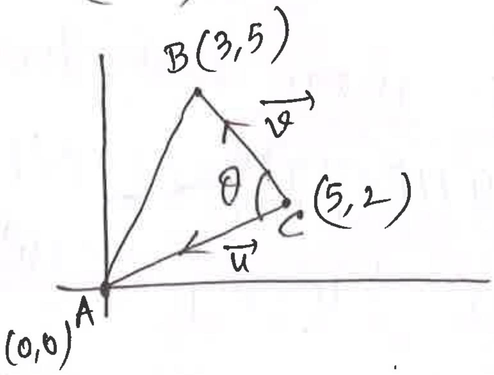




$\vec{u}\cdot\vec{v} > 0$ $\vec{u}\cdot\vec{v} = 0$ $\vec{u}\cdot\vec{v} < 0$.

Problem: Find the angle θ in the triangle ABC determined by the vertices

$A = (0,0)$, $B = (3,5)$ & $C = (5,2)$.



Solution:

$\vec{u} = \overrightarrow{CA} = \langle -5, -2 \rangle$

$\vec{v} = \overrightarrow{CB} = \langle -2, 3 \rangle$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}$$

$$= \frac{10-6}{\sqrt{29}\,\sqrt{13}} = \frac{4}{\sqrt{29\times 13}}$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{29\times 13}}\right).$$

Problem: Find the angles between

(i) $\vec{u} = \sqrt{3}\,\hat{i} + \hat{j} - \sqrt{3}\,\hat{k}$ & $\vec{v} = -3\hat{i} + 3\hat{j} - 3\hat{k}$

(ii) $u = \sqrt{3}\,\hat{i} - 7\hat{j}$ & $\vec{v} = \sqrt{3}\,\hat{i} + \hat{j} - 2\hat{k}$.

Orthogonal vectors: Vectors $\vec{u}$ & $\vec{v}$ are orthogonal iff $\vec{u}\cdot\vec{v} = 0$. [clearly, this happens when $\theta = \pi/2$].

Examples: $\vec{u} = \langle 3, -2 \rangle$ & $\vec{v} = \langle 4, 6 \rangle$ are orthogonal since $\vec{u}\cdot\vec{v} = 12 - 12 = 0$.

Properties of the Dot Product

(i) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

(ii) $(c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v})$.

(iii) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

(iv) $\vec{u}\cdot\vec{u} = |\vec{u}|^2$.

(v) $\vec{0}\cdot\vec{u} = 0$

proof: Easy.

Problem: $\vec{u}=\langle 2,1,4\rangle$, $\vec{v}=\langle -4,0,3\rangle$, $\vec{w}=\langle 3,-2,1\rangle$

Find

(i) $\vec{u}\cdot(\vec{v}+\vec{w})$

(ii) $(\vec{u}\cdot\vec{v})\,c$.

(iii) $\vec{u}(\vec{v}\cdot\vec{c})$

(iv) $4\vec{u}\cdot 3\vec{w}$

(v) $\vec{u}\cdot(\vec{w}-\vec{v})$

(vi)

Some more properties:

(i) $|\vec{u}\cdot\vec{v}| \le |\vec{u}|\,|\vec{v}|$ (Cauchy-Schwarz Inequality) (proof!!).

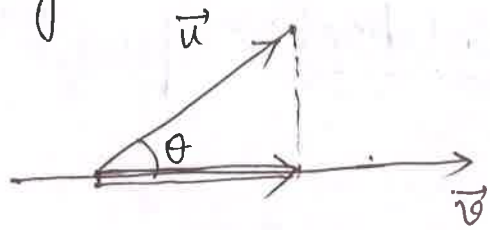
(ii) $|\vec{u}+\vec{v}| \le |\vec{u}|+|\vec{v}|$ (Triangle Inequality). (proof!!).

(iii) $|\vec{u}+\vec{v}|^2+|\vec{u}-\vec{v}|^2 = 2\left(|\vec{u}|^2+|\vec{v}|^2\right)$. (proof!!).
(Parallelogram Equality).

Projection of a vector $\vec{u}$ onto a nonzero vector $\vec{v}$

Projection of a vector $\vec{u}$ onto a vector $\vec{v}$ is denoted by $\text{proj}_{\vec{v}}\vec{u}$ and is defined by

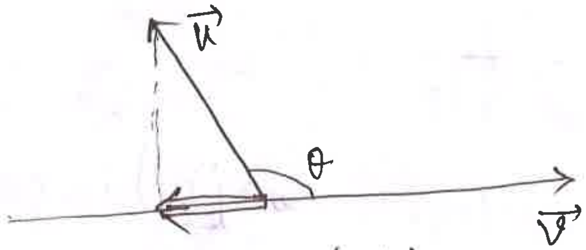
$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$



$$\text{proj}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta\,\frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{|\vec{u}|\,|\vec{v}|\cos\theta}{|\vec{v}|^2}\,\vec{v}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$



$$\text{proj}_{\vec{v}}\vec{u} = -|\vec{u}|\cos\theta\left(\frac{-\vec{v}}{|\vec{v}|}\right) = |\vec{u}|\cos\theta\,\frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}.$$

(iii) Scalar component of $\vec{u}$ in the direction of $\vec{v}$ is the scalar $|\vec{u}|\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$.

Problem: Find the vector projection of $\vec{u} = 6\hat{i}+3\hat{j}+2\hat{k}$ onto $\vec{v} = \hat{i}-2\hat{j}-2\hat{k}$ and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

Solution:

- $$\text{proj}_{\vec{v}}\vec{u} = \frac{(\vec{u}\cdot\vec{v})}{|\vec{v}|^2}\,\vec{v} = \frac{6-6-4}{1+4+4}(\hat{i}-2\hat{j}-2\hat{k})$$

$$= -\frac{4}{9}\hat{i}+\frac{8}{9}\hat{j}+\frac{8}{9}\hat{k}.$$

- Scalar component $= |\vec{u}|\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

$$= -4/3.$$

Problem: Find the vector projection of a force $\vec{F} = 5\hat{i} + 2\hat{j}$ onto $\vec{v} = \hat{i} - 3\hat{j}$. and the scalar component of $\vec{F}$ in the direction of $\vec{v}$.

Solution: $$\text{proj}_{\vec{v}} \vec{F} = \left(\frac{\vec{F}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

$$= -\frac{1}{10}\hat{i} + \frac{3}{10}\hat{j}$$

& Scalar component of $\vec{F}$ in the direction of $\vec{v}$ is

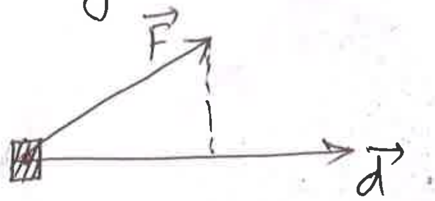
$$= \frac{\vec{F}\cdot\vec{v}}{|\vec{v}|} = \frac{5-6}{\sqrt{1+9}} = -\frac{1}{\sqrt{10}}.$$

Problem: Verify that the vector $(\vec{u} - \text{proj}_{\vec{v}}\vec{u})$ is orthogonal to the projection vector $\text{proj}_{\vec{v}}\vec{u}$.

Solution:

Application of Dot product:

Let a constant force $\vec{F}$ acts on a body. Let the body be given a displacement $\vec{d}$.
Then the work done by $\vec{F}$ is defined as.



$$W = |\vec{F}|\,|\vec{d}|\cos\theta$$

$$= |\vec{F}|\cos\theta\,|\vec{d}|$$

$$= \left(\begin{matrix}\text{Scalar component of } F \\ \text{in the direction of } \vec{d}\end{matrix}\right)(\text{length of } \vec{d}).$$

$$= \vec{F}\cdot\vec{d}$$

ie $\boxed{W = \vec{F}\cdot\vec{d}}$

Problem: Let $|\vec{F}| = 40$ unit. $|\vec{d}| = 3$ unit & $\theta = 60^0$.
What is the work done?

Solution: Work done $= \vec{F}\cdot\vec{d}$

$$= |\vec{F}|\,|\vec{d}|\cos\theta$$

$$= (40)(3)\cos 60^0 = 60 \text{ unit.}$$

The Cross Product : (vector Product) :

The cross product $\vec{u} \times \vec{v}$ of two vectors $\vec{u}$ and $\vec{v}$ is the vector as follows:

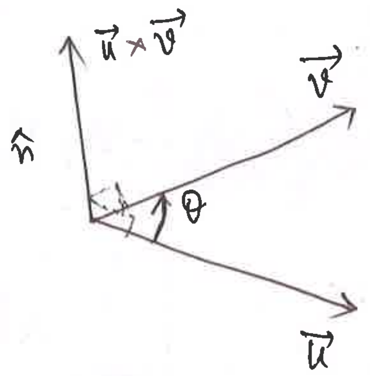
(i) If $\vec{u}$ & $\vec{v}$ have the same or opposite direction, then $\vec{w} = \vec{u} \times \vec{v} = \vec{0}$.

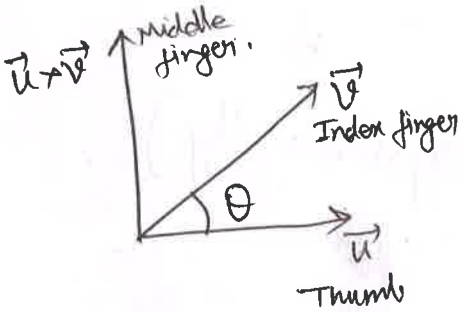
(ii) If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then $\vec{w} = \vec{u} \times \vec{v} = \vec{0}$.

(iii) In other case,

$$\vec{w} = |\vec{u}|\,|\vec{v}|\sin\theta\ \hat{n}.$$

where θ is the angle between $\vec{u}$ & $\vec{v}$ & $\hat{n}$ is the unit vector in the direction of $\vec{w} = \vec{u} \times \vec{v}$ which is perpendicular to both $\vec{u}$ and $\vec{v}$, and that points the way your right thumb points when your fingers curl through the angle θ from $\vec{u}$ to $\vec{v}$





(iv) Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$

Cross product of the standard basis vectors:

$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$, $\hat{i} \times \hat{i} = \vec{0}$

$\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$, $\hat{j} \times \hat{j} = \vec{0}$

$\hat{k} \times \hat{k} = \vec{0}$

Determinant formula for $\vec{u} \times \vec{v}$

Let $\vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$ & $\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$.

$$\vec{u} \times \vec{v} = (u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}) \times (v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k})$$

$$= \cancel{u_1 v_1 \hat{i} \times \hat{i}} + u_1 v_2 \hat{i} \times \hat{j} + u_1 v_3 \hat{i} \times \hat{k}$$
$$+ u_2 v_1 \hat{j} \times \hat{i} + \cancel{u_2 v_2 \hat{j} \times \hat{j}} + u_2 v_3 \hat{j} \times \hat{k}$$
$$+ u_3 v_1 \hat{k} \times \hat{i} + u_3 v_2 \hat{k} \times \hat{j} + \cancel{u_3 v_3 \hat{k} \times \hat{k}}$$

$$= u_1 v_2 \hat{k} - u_1 v_3 \hat{j}$$
$$- u_2 v_1 \hat{k} + u_2 v_3 \hat{i} + u_3 v_1 \hat{j} - u_3 v_2 \hat{i}$$

$$= (u_2 v_3 - u_3 v_2) \hat{i} + (u_3 v_1 - u_1 v_3) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}.$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \text{(symbolically)}.$$

Problem: Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$

if $\vec{u} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{v} = -4\hat{i} + 3\hat{j} + \hat{k}$.

Problem: Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution: The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is $\perp^{r}$ to the plane because it is $\perp^{r}$ to both vectors.

Now $\overrightarrow{PQ} =$

$\overrightarrow{PR} =$

& $\overrightarrow{PQ} \times \overrightarrow{PR} =$

$= 6\hat{i} + 6\hat{k}$

Problem: Find a unit vector $\perp^{r}$ to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ & $R(-1,1,2)$.

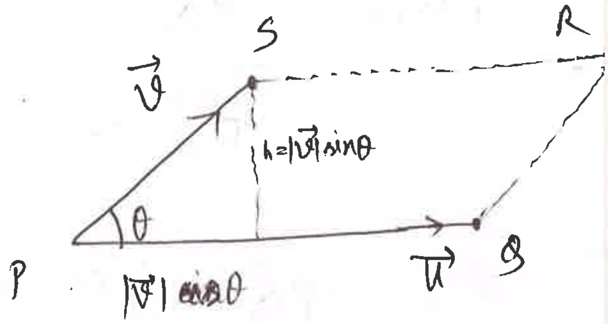
Solution:
$$\hat{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} =$$

$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}.$$

$|\vec{u} \times \vec{v}|$ as the area of a Parallelogram:

$$\vec{u} \times \vec{v} = |\vec{u}|\,|\vec{v}| \sin\theta \;\hat{n}.$$

$$\Rightarrow |u \times \vec{v}| = |\vec{u}|\,|\vec{v}|\,|\sin\theta|\,|\hat{n}|$$
$$= |\vec{u}|\,|\vec{v}|\,|\sin\theta|$$
$$= |\vec{u}|\,|\vec{v}|\,|\sin\theta|$$



PQRS is the parallelogram determined by $\vec{u}$ & $\vec{v}$.

$$\text{Area of } PQRS = |\vec{u}|\,|\vec{v}| \sin\theta.$$
$$= |\vec{u} \times \vec{v}|.$$

Problem: Find the area of the triangle with vertices $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Problem: Find the areas of the triangles whose vertices are

(i) $A(0,0)$, $B(-2,3)$, $C(3,1)$.

(ii) $A(1,-1,1)$, $B(0,1,1)$, $C(1,0,-1)$

(iii) $A(1,1,1)$, $B(-2,2,-2)$, $C(5,0,5)$.

Properties of the cross product

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are any vectors and r, s are scalars, then

(i) $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$

(ii) $u \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(iii) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

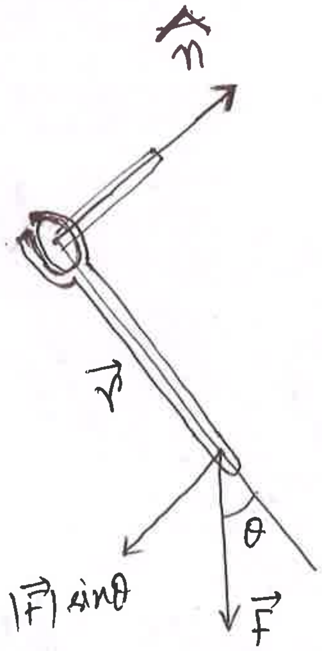
(iv) $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

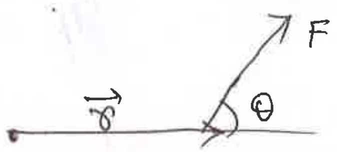
(v) $\vec{0} \times \vec{u} = \vec{0}$

(vi) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\,\vec{v} - (\vec{u} \cdot \vec{v})\,\vec{w}$

(vii) $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

Application of Cross Product





Torque vector $= \vec{r} \times \vec{F}$

When we turn a bolt by applying a force $\vec{F}$ to a wrench, we produce a torque that causes the bolt to rotate.

The torque vector points in the direction of the axis of the bolt according to the right hand rule (rotation is counterclockwise when viewed from the tip of the vector).

Magnitude of torque vector

$$|\vec{r}|\,|\vec{F}| \sin\theta = |\vec{r} \times \vec{F}|$$

$$= |\vec{r}|\,|\vec{F}| \sin\theta$$

Torque vector $= |\vec{r}|\,|\vec{F}| \sin\theta\, \hat{n}$

$$= \vec{r} \times \vec{F}$$

Scalar Triple Product / Box product

Scalar triple product of three vectors $\vec{u}, \vec{v}, \vec{w}$ is denoted by $(\vec{u}\ \vec{v}\ \vec{w})$ and is defined by

$$(\vec{u}\ \vec{v}\ \vec{w}) = \vec{u}\cdot(\vec{v}\times\vec{w})$$

(III) when $\vec{u} = \langle u_1, u_2, u_3\rangle$

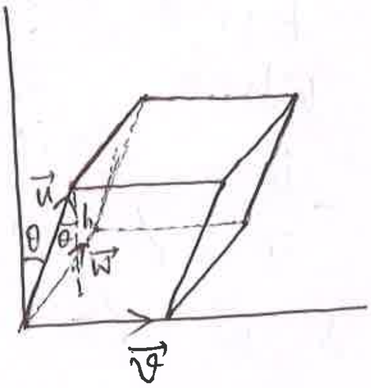
$\vec{v} = \langle v_1, v_2, v_3\rangle$

& $\vec{w} = \langle w_1, w_2, w_3\rangle$

$$(\vec{u}\ \vec{v}\ \vec{w}) = \vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Property: (i) $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{w}\times\vec{u})\cdot\vec{v}$.

(ii) The absolute value $|(\vec{u}\ \vec{v}\ \vec{w})|$ is the volume of a parallelopiped with $\vec{u}, \vec{v}, \vec{w}$ as edge vectors.



Area of the base parallelogram

$= |\vec{v}\times\vec{w}|$

Height of the parallelopiped

$= h = |\vec{u}|\cos\theta$

Volume of the parallelopiped = Area of base × Height

$= |\vec{u}|\,|\vec{v}\times\vec{w}|\cos\theta$

$= |\vec{u}\cdot(\vec{v}\times\vec{w})| = |(\vec{u}\ \vec{v}\ \vec{w})|$

Problem: Find the volume of the parallelopiped determined by $\vec{u} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{v} = -2\hat{i} + 3\hat{k}$ & $w = 7\hat{j} - 4\hat{k}$.

Solution: Volume $= \left| \vec{u} \cdot (\vec{v} \times \vec{w}) \right|$

$= 23 \text{ unit}^3$.

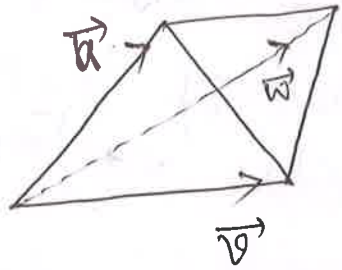
Problem: A tetrahedron is determined by three edge vectors $\vec{u}, \vec{v}, \vec{w}$ as the following figure.

Find its volume when

$\vec{u} = \langle 2, 0, 3 \rangle$

$\vec{v} = \langle 0, 4, 1 \rangle$

& $\vec{w} = \langle 5, 6, 0 \rangle$



Solution: Volume of the parallelopiped

$= \left| (\vec{u}\ \vec{v}\ \vec{w}) \right| =$

$= 72$

Volume of the tetrahedron

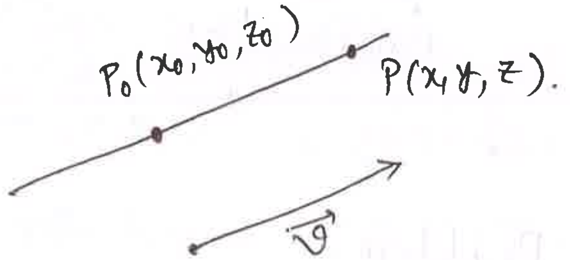
$= \frac{1}{6} \times$ Volume of the parallelopiped

$= \frac{1}{6} \times 72 = \underline{\underline{12}}$

Lines and Planes in space

- Use of scalar and vector products to write equations for lines, line segments, and planes in space.
- In plane, a line is determined by a point and a number giving the slope of the line.
- In space, a line is determined by a point and a vector giving the direction of the line

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$. Then L is the set of points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\vec{v}$.



Thus
$$\overrightarrow{P_0P} = t\,\vec{v}$$
(for some scalar parameter t).

$$\Rightarrow (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k} = t(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}).$$

$$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k} + t(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}).$$

$$\Rightarrow \vec{r}(t) = \vec{r}_0 + t\vec{v}$$

If $\vec{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line & $\vec{r}_0$ is position vector of the point $P_0(x_0, y_0, z_0)$.

A vector equation for the line L through $P(x_0, y_0, z_0)$ parallel to $\vec{v}$ is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty.$$

where $\vec{r}$ is position vector of $P(x, y, z)$ on L & $\vec{r}_0$ is p. v. of $P_0(x_0, y_0, z_0)$.

Parametric Equations for a line:

- The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is

$$x = x_0 + tv_1,\ y = y_0 + tv_2,\ z = z_0 + tv_3\ ,\ -\infty < t < \infty .$$

Problem: Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\vec{v} = 2\hat{i} + 4\hat{j} - 2\hat{k}$.

Solution: $(x_0, y_0, z_0) = (-2, 0, 4)$, $\langle v_1, v_2, v_3\rangle = \langle 2, 4, -2\rangle$

Parametric eq^{n}s

$$x = -2 + 2t,\ y = 4t,\ z = 4 - 2t,\ -\infty < t < \infty .$$

Problem: Find the parametric eq^{n}s for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution: $\vec{v} = \vec{PQ} = (1-(-3))\hat{i} + (-1-2)\hat{j} + (4-(-3))\hat{k}$

$$= 4\hat{i} - 3\hat{j} + 7\hat{k} .$$

$(x_0, y_0, z_0) = (-3, 2, -3)$.

$x = -3 + 4t$.
$y = 2 - 3t,\ -\infty < t < \infty$
$z = -3 + 7t,$

or $(x_0, y_0, z_0) = (1, -1, 4)$.

$x = 1 + 4t$
$y = -1 - 3t$, $-\infty < t < \infty$.
$z = 4 + 7t$

Problem: Parametrize the line segment joining the points $P(-3,2,-3)$ and $Q(1,-1,4)$.

Solution: Line passing through P & Q is.

$$x = -3+4t\ ,\ y = 2-3t\ ,\ z = -3+7t\ .$$

We see that

$x=-3,\ y=2$ & $z=-3$ at $t=0$.

& $x=1,\ y=-1,\ z=4$ at $t=1$.

So we add the restriction $0 \le t \le 1$ to parametrize the segment:

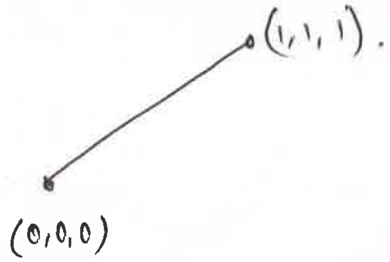
$$x = -3+4t\ ,\ y = 2-3t\ ,\ z = -3+7t\ ,\ 0 \le t \le 1.$$

Remark:

$$\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$$

$$= \underbrace{\vec{r}_0}_{\text{Initial Position}} + \underbrace{t}_{\text{Time}}\ \underbrace{|\vec{v}|}_{\text{speed}}\ \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{Direction}}$$

Problem: A helicopter is to fly directly from a helipad at origin in the direction of the pt $(1,1,1)$ at a speed 60 ft/sec. What is the position of the helicopter after 10 sec.



Solution: $\vec{r}(t) = \vec{r}_0 + t(\text{speed})(\text{unit vector})$

$\vec{r}_0 = \langle 0,0,0\rangle$

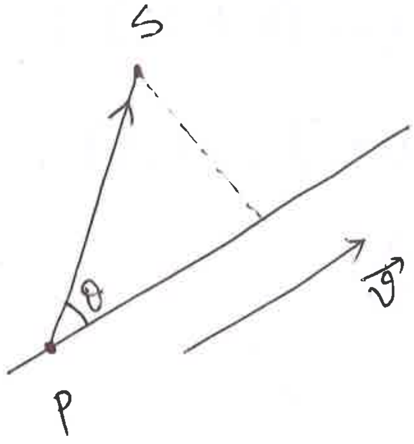
speed = 60 ft/sec.

unit vector $= \dfrac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$.

$$\vec{r}(10) = \vec{0} + (10)(60)\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right) = 200\sqrt{3}\,\hat{i} + 200\sqrt{3}\,\hat{j} + 200\sqrt{3}\,\hat{k}.$$

The distance from a point to a line in space:

- A line passing through P, parallel to a vector $\vec{v}$.
- S be an arbitrary point



Distance from S to the line passing through P

$$= |\vec{PS}| \sin\theta$$

$$= \frac{|\vec{PS}|\,|\vec{v}| \sin\theta}{|\vec{v}|} = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}.$$

> Distance from a point S to a line through P parallel to $\vec{v}$
>
> $$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

Problem: Find the distance from the point S(1,1,5) to the line

$$L: \; x = 1+t, \; y = 3-t, \; z = 2t$$

Solution: From the eqn.s for L, we see that L passes through P(1,3,0) & is parallel to $\vec{v} = \hat{i} - \hat{j} + 2\hat{k}$.

$$\vec{PS} = 0\hat{i} - 2\hat{j} + 5\hat{k}, \quad \vec{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i} + 5\hat{j} + 2\hat{k}.$$

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \sqrt{5}.$$

Problem: Find the distance from the point to the line:

(i) $(0,0,12)$; $x = 4t$, $y = -2t$, $z = 2t$.

(ii) $(4,6,2)$; $x = 3t+2$, $y = 2t+2$, $z = t+2$.

(iii) $(3,-1,4)$; $x = 4-t$, $y = 3+2t$, $z = -5+3t$.

Solution: H.W.