Properties of Relations:

Reflexive Relation: A relation R on a set A is reflexive if aRa for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. This simply means that each element a of A is related to itself.

For example,

(a) If $R_1 = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$ be a relation on $A = \{1, 2, 3\}$, then R_1 is reflexive relation since for every $a \in A$, $(a, a) \in R_1$.

(b) If $R_2 = \{(1,1), (1,2), (2,3), (3,3)\}$ be a relation on $A = \{1, 2, 3\}$, then R_2 is not a reflexive relation since for $2 \in A$, $(2,2) \notin R_2$.

(c) $R_3 = \{(x,y) \in R^2 : x \leq y\}$ is a reflexive relation since $x \leq x$ for any $x \in R$ (a set of real numbers).

Irreflexive Relation: A relation R on a set A is irreflexive if, for every $a \in A$, $(a, a) \notin R$. In other words, there is no $a \in A$ such that aRa. The terms reflexive and irreflexive are extreme cases. Reflexive means that aRa is true for all a, and irreflexive means that aRa is true for no a.

For example,

(a) The relation $R_1 = \{(1,2), (1,3), (2,1), (2,3)\}$ on $A = \{1,2,3\}$ is irreflexive relation since $(x,x) \notin R_1$ for every $x \in R_1$.

(b) The relation $R_2 = \{(x,y) \in R^2 : x < y\}$ is an irreflexive relation since $x < x$ for no $x \in R$.

Non-reflexive Relation: A relation R on a set A is non-reflexive if R is neither reflexive nor irreflexive i.e., if aRa is true for some a and false for others.

For example,

$R = \{(1,2), (2,2), (2,3), (3,1)\}$ on $A = \{1,2,3\}$ is a non-reflexive relation since $2R2$ is true but $1R1$ and $3R3$ are false.

Symmetric Relation: A relation R on a set A is symmetric if whenever $(a,b) \in R$ then $(b,a) \in R$, i.e., if $aRb \Rightarrow bRa$. This means if any one element is related to any other element, then the second element is related to the first.

For example,

(a) $R_1 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,1)\}$ on $A = \{1,2,3\}$ is a symmetric relation.

(b) $R_2 = \{(x,y) \in R^2 : x^2+y^2=1\}$ is a symmetric relation on R since if $x^2+y^2=1$ then $y^2+x^2=1$ too, i.e.. if $(x,y) \in R_2$ then $(y,x) \in R_2$.

Assymmetric Relation: A relation R on a set A is asymmetric if whenever $(a,b) \in R$ then $(b,a) \notin R$ for $a \neq b$ i.e., if $aRb \Rightarrow b \not R a$. These means that the presence of (a,b) in R excludes the possibility of presence of (b,a) in R.

For example,

The relation $R_1 = \{(1,1), (2,1), (2,3), (3,1)\}$ on $A = \{1,2,3\}$ is an asymmetric relation.

<u>Antisymmetric Relation</u>: A relation R on a set A is antisymmetric if aRb and $bRa \Rightarrow a=b$ for all $a,b \in A$.

<u>For example</u>,

(a) $R_1 = \{(1,2), (2,2), (2,3)\}$ on $A = \{1,2,3\}$ is an antisymmetric relation.

(b) $R_2 = \{(x,y) \in R^2 \mid x \leq y\}$ is an antisymmetric relation on R since $x \leq y$ and $y \leq x$ implies $x=y$, then $(x,y) \in R$ and $(y,x) \in R$ implies $x=y$.

(c) $R = \{(x,y) \in N \mid x$ is a divisor of $y\}$ is an antisymmetric relation since x divides y and y divides x implies $x=y$.

<u>Note</u>:

Antisymmetric is not the same as not symmetric. A relation may be symmetric as well as antisymmetric at the same time.

<u>For example</u>,

The relation $R = \{(1,1), (3,3)\}$ is both symmetric and antisymmetric on $A = \{1,2,3\}$.

<u>Transitive Relation</u>: A relation R on a set A is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, i.e.,

$$aRb \text{ and } bRc \Rightarrow aRc.$$

<u>For example</u>:

(a) The relation 'is parallel to' on the set of lines in a plane is transitive, because if a line x is parallel to the line y and if y is parallel to line z, then x is parallel to z.

(b) The relation 'is less than' and 'is greater than' are transitive relations on the set of real numbers. If $a<b$ and $b<c$ implies $a<c$ and if $a>b$ and $b>c \Rightarrow a>c$ $\forall$ real numbers a,b,c.

The following table summarizes the above properties.

Property	Meaning
1. Reflexivity	$(a,a) \in R$, i.e. $aRa \ \forall a \in A$
2. Irreflexivity	$(a,a) \notin R$, i.e. $a \not R a \ \forall a \in A$
3. Symmetry	$(a,b) \in R \Rightarrow (b,a) \in R$, i.e. $aRb \Rightarrow bRa \ \forall a,b \in A$
4. Asymmetry	$(a,b) \in R \Rightarrow (b,a) \notin R$, i.e. $aRb \Rightarrow b \not R a \ \forall a,b \in A$
5. Antisymmetry	$(a,b) \in R \wedge (b,a) \in R \Rightarrow a=b$, i.e. aRb and $bRa \Rightarrow a=b \ \forall a,b \in A$
6. Transitivity	$(a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R$, i.e aRb and $bRc \Rightarrow aRc \ \forall a,b,c \in A$

Equivalence Relation :

A relation on a set A is called an equivalence relation or RST relation if it is reflexive, symmetric and transitive. i.e., R is an equivalence relation on A if it has the following three properties:

1. $(a,a) \in R \ \forall a \in A$ (reflexive)
2. $(a,b) \in R$ implies $(b,a) \in R$ (symmetric)
3. (a,b) and $(b,c) \in R \Rightarrow (a,c) \in R$ (transitive)

Example 1. If R be a relation in the set of integers Z defined by $R = \{(x,y) : x \in Z, y \in Z, (x-y) \text{ is divisible by } 6\}$
Then prove that R is an equivalence relation.

Sol. Let $x \in Z$. Then $x-x=0$ and 0 is divisible by 6.
Therefore, $xRx \ \forall x \in Z$.
Hence, R is reflexive.

Again, $xRy \Rightarrow (x-y)$ is divisible by 6

$\Rightarrow -(x-y)$ is divisible by 6

$\Rightarrow (y-x)$ is divisible by 6

$\Rightarrow yRx$.

Hence, R is symmetric.

xRy and $yRz \Rightarrow (x-y)$ is divisible by 6 and

$(y-z)$ is divisible by 6

$\Rightarrow [(x-y)+(y-z)]$ is divisible by 6

$\Rightarrow (x-z)$ is divisible by 6

$\Rightarrow xRz$.

Hence R is transitive.

Thus R is an equivalence relation.

Example 2. For any two real numbers θ and ϕ, we define $\theta R \phi$ if and only if $\sec^2\theta - \tan^2\phi = 1$. Show that R is an equivalence relation.

Sol. Reflexive

$\theta R \theta$ if and only if $\sec^2\theta - \tan^2\theta = 1$ which is true for all $\theta \in R$ hence R is reflexive.

Symmetric

$$\theta R \phi \Rightarrow \sec^2\theta - \tan^2\phi = 1$$
$$\Rightarrow 1+\tan^2\theta - (\sec^2\phi - 1) = 1$$
$$\Rightarrow 1+\tan^2\theta - \sec^2\phi + 1 = 1$$
$$\Rightarrow \sec^2\phi - \tan^2\theta = 1$$
$$\Rightarrow \phi R \theta$$

Hence R is symmetric.

Transitive

$\theta R \phi$ and $\phi R \psi \Rightarrow \sec^2\theta - \tan^2\phi = 1$ and $\sec^2\phi - \tan^2\psi = 1$

$$\Rightarrow \sec^2\theta - \tan^2\phi + \sec^2\phi - \tan^2\psi = 2$$

$$\Rightarrow \sec^2\theta + (\sec^2\phi - \tan^2\phi) - \tan^2\psi = 2$$

$$\Rightarrow \sec^2\theta + 1 - \tan^2\psi = 2$$

$$\Rightarrow \sec^2\theta - \tan^2\psi = 1$$

$$\Rightarrow \theta R \psi$$

Hence R is transitive.

Thus R is are equivalence relation.

Partial Order Relation:

A relation R on a set S is called a partial order if it is reflexive, antisymmetric and transitive. That is.

1. Reflexive: $aRa \ \forall \ a \in S$
2. Antisymmetric: aRb and $bRa \Rightarrow a = b$
3. Transitive: aRb and $bRc \Rightarrow aRc$.

A set S together with a partial order R is called a partial order set or a poset and is denoted by (S, R).

For example,

The greater or equal $(\geq)$ relation is a partial ordering on Z, the set of integers.

Reflexive: Since $a \geq a$ for every integer a, $\geq$ is reflexive.

Antisymmetric: Since $a \geq b$ & $b \geq a \Rightarrow a = b$, $\geq$ is antisymmetric.

Transitive: Since $a \geq b$ & $b \geq c \Rightarrow a \geq c$, $\geq$ is transitive.

Hence, $\geq$ is a partial ordering on Z, and $(Z, \geq)$ is a poset.