# Localization of AUV with Kalman filter

In this document, consider the AUV in XY plane (later it can be extended into XYZ plane).

Initially AUV at (0, 0) position. After 't' seconds it's position is (Sx,Sy). We need to find this coordinates.

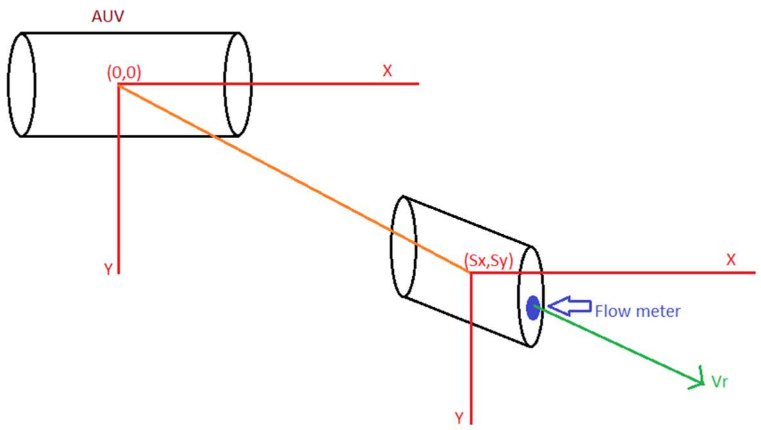


**Figure-1**

Flow meter shown in this figure-1 measures the flow rate of the water passing through it. We can fix this sensor on the top of the front side. The flow rate can be used to find velocity of AUV. For that we need to have the relation between flow rate and velocity value.

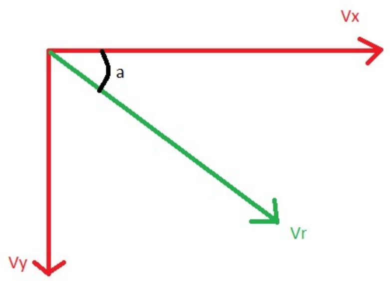


**Figure -2**

**Gyroscope** can be used to find the angle 'a' given in the figure-2. So we can obtain velocity in X and Y direction.

In $k^{th}$ instance if we have position, velocity and acceleration in X direction, then, we can predict the position and velocity at $k+1^{th}$ instance.

$$Position_{k+1} = Position_k + velocity_k \times dt + \frac{1}{2} \times acceleration_k \times dt^2 \tag{1.1}$$

$$Velocity_{k+1} = velocity_k + acceleration_k \times dt \tag{1.2}$$

Position and velocity changes because the thrust (force) applied to the water. So force is the applied quantity.

$F = ma$

So, acceleration is the input applied to the system. We can obtain the relation between PWM signal applied to the thrusters and acceleration measurement from accelerometer.

**State space representation**

Eqn(1.1) and (1.2) can be written as:

$$\begin{bmatrix} Velocity_{k+1} \\ Position_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & dt \end{bmatrix} \begin{bmatrix} Velocity_k \\ Position_k \end{bmatrix} + \begin{bmatrix} dt \\ \frac{dt^2}{2} \end{bmatrix} \times acceleration_k \tag{1.3}$$

$X_k = \begin{bmatrix} Velocity_k \\ Position_k \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 1 & dt \end{bmatrix}$ $B = \begin{bmatrix} dt \\ \frac{dt^2}{2} \end{bmatrix}$ , u=acceleration

$$X_{k+1} = AX_k + Bu_k \tag{1.4}$$

$X_k$ is known as the state of the system.

**Kalman filter algorithm**

We are trying to get true state of the system (AUV) that is unknown for us in real time situation (means actual position and actual velocity are unknown) .With mathematical modelling and sensor measurement we can estimate the state of the system. The Kalman filter try to minimize the difference between true state and estimated state of the system. Kalman filter algorithm having two steps: prediction and updation.

Eqn(1.4) represents the state prediction step. Last estimated state can be used to find predicted state in next instance. Predicted state is represented as $\overset{|}{X}_k$ .

$$\overset{|}{X}_k = A\hat{X}_{k-1} + Bu_{k-1} \tag{1.5}$$

Last estimated state or the updated state is represented as $\hat{X}_k$ .

State updation equation is represented as:

$$\hat{X}_k = \overset{|}{X}_k + K_k\left(Z_k - \overset{|}{V}_k\right) \tag{1.6}$$

Here $K_k$ is the Kalman gain. $Z_k$ is the velocity measurement based on flow meter. $\overset{|}{V}_k$ is the velocity prediction( from eqn(1.5), the first element of the $\overset{|}{X}_k$ ).

Let $C = \begin{bmatrix}1 & 0\end{bmatrix}$ ,then

$$\overset{|}{V}_k = C\overset{|}{X}_k \tag{1.7}$$

$\hat{X}_k$ is the estimated state of the system. Let $X_k$ be the true state of the system.

$$e_k = X_k - \hat{X}_k \tag{1.8}$$

$e_k$ is the error in estimation. Kalman filter try to minimize this error by finding appropriate Kalman gain value ( $K_k$ ).

$$e_k = \begin{bmatrix} Err(Velocity) \\ Err(Position) \end{bmatrix} \tag{1.9}$$

Mean squared error of $e_k$ is known as the covariance matrix of estimation- $P_k$ .

$$P_k = E[e_k e_k^T] \tag{1.10}$$

$$P_k = \begin{bmatrix} E[Err^2(Velocity)] & E[Err(velocity)] \times E[Err(Position)] \\ E[Err(Position)] \times E[Err(velocity)] & E[Err^2(Position)] \end{bmatrix} \tag{1.11}$$

Since errors are uncorrelated only diagonal elements are non-zeros.

In similar way error in prediction can be represented as:

$$\overset{|}{e}_k = X_k - \overset{|}{X}_k \tag{1.12}$$

Covariance matrix of prediction is

$$\overset{|}{P}_k = E[\overset{|}{e}_k(\overset{|}{e}_k)^T] \tag{1.13}$$

From eqn(1.12),

$$P_k = E[(X_k - \hat{X}_k)(X_k - \hat{X}_k)^T] \tag{1.14}$$

$Z_k$ is the velocity measurement using flow meter. We know that $X_k$ is the true state which contains true position and true velocity that is unknown for us.

True velocity $= CX_k$

There will be some error in the $Z_k$ value compared to the true velocity that is taken as 'v'.

$$Z_k = CX_k + v_k \tag{1.15}$$

Here 'v' is the measurement noise of the system. The covariance of the measurement noise(mean squared error) is given as:

$$R = E[v_k v_k^T] \tag{1.16}$$

If we know true state $X_k$ and input applied to the system at $k^{th}$ instance, then we can determine the true state at $k+1^{th}$ instance using state space model. But there will be some error, given as 'w'.

$$X_{k+1} = AX_k + Bu_k + w_k \tag{1.17}$$

Mean squared error of 'w' is given as:

$$Q = E[w_k w_k^T] \tag{1.18}$$

State updation equation at $k^{th}$ instance can be written as:

$$\hat{X}_k = \overset{|}{X}_k + K_k(C\,X_k + v_k - C\,\overset{|}{X}_k) \tag{1.19}$$

From eqn(1.14),

$$P_k = E[(X_k - \overset{|}{X}_k - K_k(C\,X_k + v_k - C\,\overset{|}{X}_k))(X_k - \overset{|}{X}_k - K_k(C\,X_k + v_k - C\,\overset{|}{X}_k))^T] \tag{1.20}$$

$$P_k = E[((I - K_kC\,)X_k - (I - K_kC\,)\overset{|}{X}_k - K_kv_k)((I - K_kC\,)X_k - (I - K_kC\,)\overset{|}{X}_k - K_kv_k)^T] \tag{1.21}$$

$$P_k = E[((I - K_kC\,)(X_k - \overset{|}{X}_k) - K_kv_k)((I - K_kC\,)(X_k - \overset{|}{X}_k) - K_kv_k)^T] \tag{1.22}$$

While expanding(1.23), expectation of few terms become zero as they are uncorrelated.

$$P_k = (I - K_kC\,)E[(X_k - \overset{|}{X}_k)(X_k - \overset{|}{X}_k)^T](I - K_kC\,)^T + K_kE[v_kv_k^T]K_k^T \tag{1.24}$$

$$P_k = (I - K_kC\,)\overset{|}{P}_k(I - K_kC\,)^T + K_kRK_k{}^T \tag{1.25}$$

$$P_k = \overset{|}{P}_k - \overset{|}{P}_k K_k^T C^T - K_k C \overset{|}{P}_k + K_k C \overset{|}{P}_k C^T K^T + K_k R K_k^T \tag{1.26}$$

$$P_k = \overset{|}{P}_k - \overset{|}{P}_k K_k^T C^T - K_k C \overset{|}{P}_k + K_k (C \overset{|}{P}_k C^T + R) K_k^T \tag{1.27}$$

The value of Kalman gain is determined in a way so that it minimizes the error in state estimation. This in turn means that $P_k$ is minimum.

Since errors are uncorrelated, trace of the matrix is the sum of the diagonal elements. Trace of a matrix and trace of its transpose matrix are same.

Trace of the matrix $P_k$ is represented as" $T[P_k]$".

Trace of both side of eqn(1.27),

$$T[P_k] = T[\overset{|}{P}_k] - 2T[\overset{|}{P}_k K_k C] + T[K_k (C \overset{|}{P}_k C^T + R) K_k^T] \tag{1.28}$$

In order to minimize error with respective '$K_k$'

$$\frac{dT[P_k]}{dK_k} = -2(C \overset{|}{P}_k)^T + 2K_k (C \overset{|}{P}_k C^T + R) = 0 \tag{1.29}$$

Therefore, Kalman gain $K_k$ can be determined as:

$$K_k = (\overset{|}{P}_k C^T)(C \overset{|}{P}_k C^T + R)^{-1} \tag{1.30}$$

Substitute value of Kalman gain in (1.27),

$$P_k = \overset{|}{P}_k - \overset{|}{P}_k \left( \frac{(\overset{|}{P}_k)^T C}{(C \overset{|}{P}_k C^T + R)^T} \right) C^T - \left( \frac{\overset{|}{P}_k C^T}{C \overset{|}{P}_k C^T + R} \right) C \overset{|}{P}_k + \left( \frac{\overset{|}{P}_k C^T}{(C \overset{|}{P}_k C^T + R)^T} \right) (\overset{|}{P}_k)^T C \tag{1.31}$$

$$P_k = \overset{|}{P}_k - \left( \frac{\overset{|}{P}_k C^T}{C \overset{|}{P}_k C^T + R} \right) C \overset{|}{P}_k \tag{1.32}$$

Use $K_k$ instead of the term in the bracket in eqn(1.32),

$$P_k = \overset{|}{P}_k - K_k C \overset{|}{P}_k \tag{1.33}$$

$$P_k = \left( I - K_k C \right) \overset{|}{P}_k \tag{1.34}$$

Now, all the parameters for $k^{th}$ instance, i.e. $\hat{X}_k, P_k, K_k$ have been obtained. All the parameters at $k^{th}$ instance have to be projected to the $k+1^{th}$ instance.

Error in state prediction at $k+1^{th}$ instance is given as:

$$\overset{|}{e}_{k+1} = X_{k+1} - \overset{|}{X}_{k+1} \tag{1.35}$$

From the state space model,

$$\overset{|}{X}_{k+1} = A\hat{X}_k + Bu_k \tag{1.36}$$

Substitute eqn(1.36) in (1.35),

$$\overset{|}{e}_{k+1} = (AX_k + Bu_k + w_k) - (A\hat{X}_k + Bu_k) \tag{1.37}$$

$$\overset{|}{e}_{k+1} = A(X_k - \hat{X}_k) + w_k \tag{1.38}$$

$$\overset{|}{e}_{k+1} = Ae_k + w_k \tag{1.39}$$

Covariance matrix prediction at $k+1^{th}$ instance can be represented as:

$$\overset{|}{P}_{k+1} = E[\overset{|}{e}_{k+1}(\overset{|}{e}_{k+1})^T] \tag{1.40}$$

$$\overset{|}{P}_{k+1} = E[(Ae_k + w_k)(Ae_k + w_k)^T] \tag{1.41}$$

$$\overset{|}{P}_{k+1} = E[Ae_k(Ae_k)^T] + E[w_k {w_k}^T] \tag{1.42}$$

$$\overset{|}{P}_{k+1} = AE[e_k(e_k)^T]A^T + Q \tag{1.43}$$

$$\overset{|}{P}_{k+1} = AP_kA^T + Q \tag{1.44}$$

Replacing $k$ by $k-1$ in (1.44), we get,

$$\overset{|}{P}_k = AP_{k-1}A^T + Q \tag{1.45}$$

**<u>Important steps in Kalman filter algorithm</u>**

State prediction: $\overset{|}{X}_k = A\hat{X}_{k-1} + Bu_{k-1}$

Covariance matrix prediction: $\overset{|}{P}_k = AP_{k-1}A^T + Q$

Kalman gain calculation: $K_k = (\overset{|}{P}_k\, C^{T})(C\ \overset{|}{P}_k\, C^{T} + R)^{-1}$

State updation: $\hat{X}_k = \overset{|}{X}_k + K_k\left(Z_k - \overset{|}{V}_k\right)$

Covariance matrix updation: $P_k = \left(I - K_k C\right)\overset{|}{P}_k$

**<u>Implementation of Kalman filter in matlab</u>** (just for understanding)

Initialization:

%parameters are initialized with typical values not exact.

```
A=[1 0;1 0.01];
B=[0.01;(0.01^2)*0.5];
X=[0;0];  %initial position and velocity are taken as 0,0 as reference
C=[1 0];
P=[0.01 0;0 0.01];
Q=0.001;  % need some kind of tuning for this value
R=5;   % it is depends on the flow meter which we use.
I=[1 0;0 1];
```

In loop:

```
X=A*X+B*acc(i-1);
P=A*P*A'+Q;
K=(P*C')/(C*P*C'+R);
y=C*X;
X=X+K*(Z(i)-y);
P=(I-K*C)*P;
```

Here ‘acc’ is the acceleration input and ‘Z’ is the velocity measurement with flow meter. We can convert it into python code as use it.