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CALCULUS

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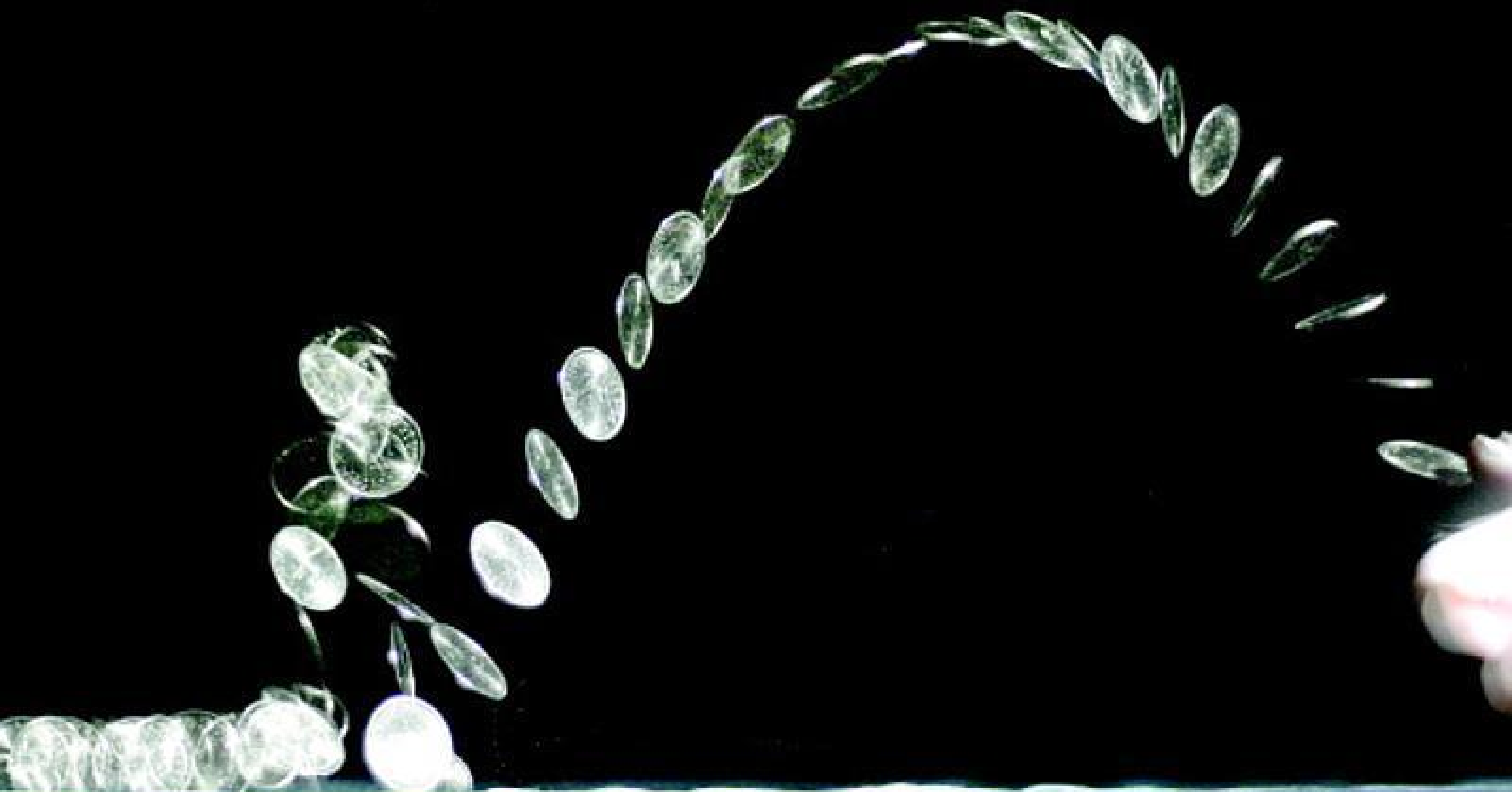

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CHAPTER 1 BASIC PROPERTIES OF NUMBERS

The title of this chapter expresses in a few words the mathematical knowledge required to read this book. In fact, this short chapter is simply an explanation of what is meant by the "basic properties of numbers," all of which—addition and multiplication, subtraction and division, solutions of equations and inequalities, factoring and other algebraic manipulations—are already familiar to us. Nevertheless, this chapter is not a review. Despite the familiarity of the subject, the survey we are about to undertake will probably seem quite novel; it does not aim to present an extended review of old material, but to condense this knowledge into a few simple and obvious properties of numbers. Some may even seem too obvious to mention, but a surprising number of diverse and important facts turn out to be consequences of the ones we shall emphasize.

Of the twelve properties which we shall study in this chapter, the first nine are concerned with the fundamental operations of addition and multiplication. For the moment we consider only addition: this operation is performed on a pair of numbers—the sum $a + b$ exists for any two given numbers a and b (which may possibly be the same number, of course). It might seem reasonable to regard addition as an operation which can be performed on several numbers at once, and consider the sum $a_1 + \cdots + a_n$ of n numbers $a_1, \ldots, a_n$ as a basic concept. It is more convenient, however, to consider addition of pairs of numbers only, and to define other sums in terms of sums of this type. For the sum of three numbers a, b, and c, this may be done in two different ways. One can first add b and c, obtaining $b + c$, and then add a to this number, obtaining $a + (b + c)$; or one can first add a and b, and then add the sum $a + b$ to c, obtaining $(a + b) + c$. Of course, the two compound sums obtained are equal, and this fact is the very first property we shall list:

(P1) If a, b, and c are any numbers, then

$$a + (b + c) = (a + b) + c.$$

The statement of this property clearly renders a separate concept of the sum of three numbers superfluous; we simply agree that $a + b + c$ denotes the number $a + (b + c) = (a + b) + c$. Addition of four numbers requires similar, though slightly more involved, considerations. The symbol $a + b + c + d$ is defined to mean

$$\begin{array}{lll} & (1) & ((a + b) + c) + d, \\ \text{or} & (2) & (a + (b + c)) + d, \\ \text{or} & (3) & a + ((b + c) + d), \\ \text{or} & (4) & a + (b + (c + d)), \\ \text{or} & (5) & (a + b) + (c + d). \end{array}$$

This definition is unambiguous since these numbers are all equal. Fortunately, *this* fact need not be listed separately, since it follows from the property P1 already listed. For example, we know from P1 that

$$(a + b) + c = a + (b + c),$$

and it follows immediately that (1) and (2) are equal. The equality of (2) and (3) is a direct consequence of P1, although this may not be apparent at first sight (one must let $b + c$ play the role of b in P1, and d the role of c). The equalities (3) = (4) = (5) are also simple to prove.

It is probably obvious that an appeal to P1 will also suffice to prove the equality of the 14 possible ways of summing five numbers, but it may not be so clear how we can reasonably arrange a proof that this is so without actually listing these 14 sums. Such a procedure is feasible, but would soon cease to be if we considered collections of six, seven, or more numbers; it would be totally inadequate to prove the equality of all possible sums of an arbitrary finite collection of numbers $a_1, \dots, a_n$. This fact may be taken for granted, but for those who would like to worry about the proof (and it is worth worrying about once) a reasonable approach is outlined in Problem 24. Henceforth, we shall usually make a tacit appeal to the results of this problem and write sums $a_1 + \cdots + a_n$ with a blithe disregard for the arrangement of parentheses.

The number 0 has one property so important that we list it next:

(P2) If a is any number, then

$$a + 0 = 0 + a = a.$$

An important role is also played by 0 in the third property of our list:

(P3) For every number a, there is a number $-a$ such that

$$a + (-a) = (-a) + a = 0.$$

Property P2 ought to represent a distinguishing characteristic of the number 0, and it is comforting to note that we are already in a position to prove this. Indeed, if a number x satisfies

$$a + x = a$$

for any one number a, then $x = 0$ (and consequently this equation also holds for all numbers a). The proof of this assertion involves nothing more than subtracting a from both sides of the equation, in other words, adding $-a$ to both sides; as the following detailed proof shows, all three properties P1–P3 must be used to justify this operation.

$$\begin{array}{ll}
\text{If} & a + x = a, \\
\text{then} & (-a) + (a + x) = (-a) + a = 0; \\
\text{hence} & ((-a) + a) + x = 0; \\
\text{hence} & 0 + x = 0; \\
\text{hence} & x = 0.
\end{array}$$

As we have just hinted, it is convenient to regard subtraction as an operation derived from addition: we consider $a - b$ to be an abbreviation for $a + (-b)$. It is then possible to find the solution of certain simple equations by a series of steps (each justified by P1, P2, or P3) similar to the ones just presented for the equation $a + x = a$. For example:

$$\begin{array}{ll} \text{If} & x + 3 = 5, \\ \text{then} & (x+3) + (-3) = 5 + (-3); \\ \text{hence} & x + (3 + (-3)) = 5 - 3 = 2; \\ \text{hence} & x + 0 = 2; \\ \text{hence} & x = 2. \end{array}$$

Naturally, such elaborate solutions are of interest only until you become convinced that they can always be supplied. In practice, it is usually just a waste of time to solve an equation by indicating so explicitly the reliance on properties P1, P2, and P3 (or any of the further properties we shall list).

Only one other property of addition remains to be listed. When considering the sums of three numbers a, b, and c, only two sums were mentioned: $(a + b) + c$ and $a + (b + c)$. Actually, several other arrangements are obtained if the order of a, b, and c is changed. That these sums are all equal depends on

(P4) If a and b are any numbers, then

$$a + b = b + a.$$

The statement of P4 is meant to emphasize that although the operation of addition of pairs of numbers might conceivably depend on the order of the two numbers, in fact it does not. It is helpful to remember that not all operations are so well behaved. For example, subtraction does not have this property: usually $a - b \neq b - a$. In passing we might ask just when $a - b$ does equal $b - a$, and it is amusing to discover how powerless we are if we rely only on properties P1–P4 to justify our manipulations. Algebra of the most elementary variety shows that $a - b = b - a$ only when $a = b$. Nevertheless, it is impossible to derive this fact from properties P1–P4; it is instructive to examine the elementary algebra carefully and determine which step(s) cannot be justified by P1–P4. We will indeed be able to justify all steps in detail when a few more properties are listed. Oddly enough, however, the crucial property involves multiplication.

The basic properties of multiplication are fortunately so similar to those for addition that little comment will be needed; both the meaning and the consequences should be clear. (As in elementary algebra, the product of a and b will be denoted by $a \cdot b$, or simply ab.)

(P5) If a, b, and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

(P6) If a is any number, then

$$a \cdot 1 = 1 \cdot a = a.$$

Moreover, $1 \neq 0$.

(The assertion that $1 \neq 0$ may seem a strange fact to list, but we have to list it, because there is no way it could possibly be proved on the basis of the other properties listed—these properties would all hold if there were only one number, namely, 0.)

(P7) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

(P8) If a and b are any numbers, then

$$a \cdot b = b \cdot a.$$

One detail which deserves emphasis is the appearance of the condition $a \neq 0$ in P7. This condition is quite necessary; since $0 \cdot b = 0$ for all numbers b, there is *no* number 0^{-1} satisfying $0 \cdot 0^{-1} = 1$. This restriction has an important consequence for division. Just as subtraction was defined in terms of addition, so division is defined in terms of multiplication: The symbol a/b means $a \cdot b^{-1}$. Since 0^{-1} is meaningless, $a/0$ is also meaningless—division by 0 is *always* undefined.

Property P7 has two important consequences. If $a \cdot b = a \cdot c$, it does not necessarily follow that $b = c$; for if $a = 0$, then both $a \cdot b$ and $a \cdot c$ are 0, no matter what b and c are. However, if $a \neq 0$, then $b = c$; this can be deduced from P7 as follows:

$$\begin{array}{ll} \text{If} & a \cdot b = a \cdot c \text{ and } a \neq 0, \\ \text{then} & a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c); \\ \text{hence} & (a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c; \\ \text{hence} & 1 \cdot b = 1 \cdot c; \\ \text{hence} & b = c. \end{array}$$

It is also a consequence of P7 that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. In fact,

$$\begin{array}{ll} \text{if} & a \cdot b = 0 \text{ and } a \neq 0, \\ \text{then} & a^{-1} \cdot (a \cdot b) = 0; \\ \text{hence} & (a^{-1} \cdot a) \cdot b = 0; \\ \text{hence} & 1 \cdot b = 0; \\ \text{hence} & b = 0. \end{array}$$

(It may happen that $a = 0$ and $b = 0$ are both true; this possibility is not excluded when we say "either $a = 0$ or $b = 0$"; in mathematics "or" is always used in the sense of "one or the other, or both.")

This latter consequence of P7 is constantly used in the solution of equations. Suppose, for example, that a number x is known to satisfy

$$(x - 1)(x - 2) = 0.$$

Then it follows that either $x - 1 = 0$ or $x - 2 = 0$; hence $x = 1$ or $x = 2$.

The properties P1–P9 have descriptive names which are not essential to remember, but which are often convenient for reference. We will take this opportunity to list properties P1–P9 together and indicate the names by which they are commonly designated.

(P1)	(Associative law for addition)	$a + (b + c) = (a + b) + c.$
(P2)	(Existence of an additive identity)	$a + 0 = 0 + a = a.$
(P3)	(Existence of additive inverses)	$a + (-a) = (-a) + a = 0.$
(P4)	(Commutative law for addition)	$a + b = b + a.$
(P5)	(Associative law for multiplication)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
(P6)	(Existence of a multiplicative identity)	$a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
(P7)	(Existence of multiplicative inverses)	$a \cdot a^{-1} = a^{-1} \cdot a = 1, \text{ for } a \neq 0.$
(P8)	(Commutative law for multiplication)	$a \cdot b = b \cdot a.$
(P9)	(Distributive law)	$a \cdot (b + c) = a \cdot b + a \cdot c.$

The three basic properties of numbers which remain to be listed are concerned with inequalities. Although inequalities occur rarely in elementary mathematics, they play a prominent role in calculus. The two notions of inequality, $a < b$ (a is less than b) and $a > b$ (a is greater than b), are intimately related: $a < b$ means the same as $b > a$ (thus $1 < 3$ and $3 > 1$ are merely two ways of writing the same assertion). The numbers a satisfying $a > 0$ are called **positive**, while those numbers a satisfying $a < 0$ are called **negative**. While positivity can thus be defined in terms of $<$, it is possible to reverse the procedure: $a < b$ can be defined to mean that $b - a$ is positive. In fact, it is convenient to consider the collection of all positive numbers, denoted by P, as the basic concept, and state all properties in terms of P:

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

- (i) $a = 0$,
- (ii) a is in the collection P,
- (iii) $-a$ is in the collection P.

(P11) (Closure under addition) If a and b are in P, then $a + b$ is in P.

(P12) (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

These three properties should be complemented with the following definitions:

$$\begin{array}{lll} a > b & \text{if} & a - b \text{ is in } P; \\ a < b & \text{if} & b > a; \\ a \geq b & \text{if} & a > b \text{ or } a = b; \\ a \leq b & \text{if} & a < b \text{ or } a = b.^* \end{array}$$

Note, in particular, that $a > 0$ if and only if a is in P.

All the familiar facts about inequalities, however elementary they may seem, are consequences of P10–P12. For example, if a and b are any two numbers, then precisely one of the following holds:

(i) $a - b = 0$,
(ii) $a - b$ is in the collection P,
(iii) $-(a - b) = b - a$ is in the collection P.

Using the definitions just made, it follows that precisely one of the following holds:

(i) $a = b$,
(ii) $a > b$,
(iii) $b > a$.

A slightly more interesting fact results from the following manipulations. If $a < b$, so that $b - a$ is in P, then surely $(b + c) - (a + c)$ is in P; thus, if $a < b$, then $a + c < b + c$. Similarly, suppose $a < b$ and $b < c$. Then

$$\begin{array}{ll} & b - a \text{ is in } P, \\ \text{and} & c - b \text{ is in } P, \\ \text{so} & c - a = (c - b) + (b - a) \text{ is in } P. \end{array}$$

This shows that if $a < b$ and $b < c$, then $a < c$. (The two inequalities $a < b$ and $b < c$ are usually written in the abbreviated form $a < b < c$, which has the third inequality $a < c$ almost built in.)

The following assertion is somewhat less obvious: If $a < 0$ and $b < 0$, then $ab > 0$. The only difficulty presented by the proof is the unraveling of definitions. The symbol $a < 0$ means, by definition, $0 > a$, which means $0 - a = -a$ is in P. Similarly $-b$ is in P, and consequently, by P12, $(-a)(-b) = ab$ is in P. Thus $ab > 0$.

The fact that $ab > 0$ if $a > 0$, $b > 0$ and also if $a < 0$, $b < 0$ has one special consequence: $a^2 > 0$ if $a \neq 0$. Thus squares of nonzero numbers are always positive, and in particular we have proved a result which might have seemed sufficiently elementary to be included in our list of properties: $1 > 0$ (since $1 = 1^2$).

*There is one slightly perplexing feature of the symbols $\geq$ and $\leq$. The statements

$$\begin{array}{l} 1 + 1 \leq 3 \\ 1 + 1 \leq 2 \end{array}$$

are both true, even though we know that $\leq$ could be replaced by $<$ in the first, and by $=$ in the second. This sort of thing is bound to occur when $\leq$ is used with specific numbers; the usefulness of the symbol is revealed by a statement like Theorem 1—here equality holds for some values of a and b, while inequality holds for other values.

The fact that $-a > 0$ if $a < 0$ is the basis of a concept which will play an extremely important role in this book. For any number a, we define the **absolute value** $|a|$ of a as follows:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0. \end{cases}$$

Note that $|a|$ is always positive, except when $a = 0$. For example, we have $|-3| = 3$, $|7| = 7$, $|1 + \sqrt{2} - \sqrt{3}| = 1 + \sqrt{2} - \sqrt{3}$, and $|1 + \sqrt{2} - \sqrt{10}| = \sqrt{10} - \sqrt{2} - 1$. In general, the most straightforward approach to any problem involving absolute values requires treating several cases separately, since absolute values are defined by cases to begin with. This approach may be used to prove the following very important fact about absolute values.

THEOREM 1 For all numbers a and b, we have

$$|a + b| \leq |a| + |b|.$$

PROOF We will consider 4 cases:

$$\begin{array}{lll} (1) & a \geq 0, & b \geq 0; \\ (2) & a \geq 0, & b \leq 0; \\ (3) & a \leq 0, & b \geq 0; \\ (4) & a \leq 0, & b \leq 0. \end{array}$$

In case (1) we also have $a + b \geq 0$, and the theorem is obvious; in fact,

$$|a + b| = a + b = |a| + |b|,$$

so that in this case equality holds.

In case (4) we have $a + b \leq 0$, and again equality holds:

$$|a + b| = -(a + b) = -a + (-b) = |a| + |b|.$$

In case (2), when $a \geq 0$ and $b \leq 0$, we must prove that

$$|a + b| \leq a - b.$$

This case may therefore be divided into two subcases. If $a + b \geq 0$, then we must prove that

$$\begin{aligned} a + b &\leq a - b, \\ \text{i.e.,} \qquad b &\leq -b, \end{aligned}$$

which is certainly true since $b \leq 0$ and hence $-b \geq 0$. On the other hand, if $a + b \leq 0$, we must prove that

$$\begin{aligned} -a - b &\leq a - b, \\ \text{i.e.,} \qquad -a &\leq a, \end{aligned}$$

which is certainly true since $a \geq 0$ and hence $-a \leq 0$.

Finally, note that case (3) may be disposed of with no additional work, by applying case (2) with a and b interchanged. ▮

Although this method of treating absolute values (separate consideration of various cases) is sometimes the only approach available, there are often simpler methods which may be used. In fact, it is possible to give a much shorter proof of Theorem 1; this proof is motivated by the observation that

$$|a| = \sqrt{a^2}.$$

(Here, and throughout the book, $\sqrt{x}$ denotes the *positive* square root of x; this symbol is defined only when $x \geq 0$.) We may now observe that

$$\begin{aligned}(|a+b|)^2 = (a+b)^2 &= a^2 + 2ab + b^2 \\ &\leq a^2 + 2|a| \cdot |b| + b^2 \\ &= |a|^2 + 2|a| \cdot |b| + |b|^2 \\ &= (|a| + |b|)^2.\end{aligned}$$

From this we can conclude that $|a+b| \leq |a| + |b|$ because $x^2 < y^2$ implies $x < y$, provided that x and y are both nonnegative; a proof of *this* fact is left to the reader (Problem 5).

One final observation may be made about the theorem we have just proved: a close examination of either proof offered shows that

$$|a+b| = |a| + |b|$$

if a and b have the same sign (i.e., are both positive or both negative), or if one of the two is 0, while

$$|a+b| < |a| + |b|$$

if a and b are of opposite signs.

We will conclude this chapter with a subtle point, neglected until now, whose inclusion is required in a conscientious survey of the properties of numbers. After stating property P9, we proved that $a - b = b - a$ implies $a = b$. The proof began by establishing that

$$a \cdot (1+1) = b \cdot (1+1),$$

from which we concluded that $a = b$. This result is obtained from the equation $a \cdot (1+1) = b \cdot (1+1)$ by dividing both sides by $1+1$. Division by 0 should be avoided scrupulously, and it must therefore be admitted that the validity of the argument depends on knowing that $1+1 \neq 0$. Problem 25 is designed to convince you that this fact cannot possibly be proved from properties P1–P9 alone! Once P10, P11, and P12 are available, however, the proof is very simple: We have already seen that $1 > 0$; it follows that $1+1 > 0$, and in particular $1+1 \neq 0$.

This last demonstration has perhaps only strengthened your feeling that it is absurd to bother proving such obvious facts, but an honest assessment of our present situation will help justify serious consideration of such details. In this chapter we have assumed that numbers are familiar objects, and that P1–P12 are merely explicit statements of obvious, well-known properties of numbers. It would be difficult, however, to justify this assumption. Although one learns how to "work with" numbers in school, just what numbers *are*, remains rather vague. A great deal of this book is devoted to elucidating the concept of numbers, and by the end of the book we will have become quite well acquainted with them. But it will be necessary to work with numbers throughout the book. It is therefore reasonable to admit frankly that we do not yet thoroughly understand numbers; we may still say that, in whatever way numbers are finally defined, they should certainly have properties P1–P12.

Most of this chapter has been an attempt to present convincing evidence that P1–P12 are indeed basic properties which we should assume in order to deduce other familiar properties of numbers. Some of the problems (which indicate the derivation of other facts about numbers from P1–P12) are offered as further evidence. It is still a crucial question whether P1–P12 actually account for *all* properties of numbers. As a matter of fact, we shall soon see that they do *not*. In the next chapter the deficiencies of properties P1–P12 will become quite clear, but the proper means for correcting these deficiencies is not so easily discovered. The crucial additional basic property of numbers which we are seeking is profound and subtle, quite unlike P1–P12. The discovery of this crucial property will require all the work of Part II of this book. In the remainder of Part I we will begin to see why some additional property is required; in order to investigate this we will have to consider a little more carefully what we mean by "numbers."

PROBLEMS

1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.
(ii) $x^2 - y^2 = (x - y)(x + y)$.
(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.
(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.
(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. (There is a particularly easy way to do this, using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

2. What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned} x^2 &= xy, \\ x^2 - y^2 &= xy - y^2, \\ (x + y)(x - y) &= y(x - y), \\ x + y &= y, \\ 2y &= y, \\ 2 &= 1. \end{aligned}$$

3. Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, if $b, d \neq 0$.

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(b) Using the fact that

$$x^2 + 2xy + y^2 = (x + y)^2 \geq 0,$$

show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.

(c) Use part (b) to find out when $(x + y)^4 = x^4 + y^4$.

(d) Find out when $(x+y)^5 = x^5+y^5$. Hint: From the assumption $(x+y)^5 = x^5+y^5$ you should be able to derive the equation $x^3+2x^2y+2xy^2+y^3 = 0$, if $xy \neq 0$. This implies that $(x + y)^3 = x^2y + xy^2 = xy(x + y)$.

You should now be able to make a good guess as to when $(x+y)^n = x^n + y^n$; the proof is contained in Problem 11-57.

17. (a) Find the smallest possible value of $2x^2 - 3x + 4$. Hint: "Complete the square," i.e., write $2x^2 - 3x + 4 = 2(x - 3/4)^2 + ?$

(b) Find the smallest possible value of $x^2 - 3x + 2y^2 + 4y + 2$.

(c) Find the smallest possible value of $x^2 + 4xy + 5y^2 - 4x - 6y + 7$.

18. (a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \qquad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

(b) Suppose that $b^2 - 4c < 0$. Show that there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x. Hint: Complete the square.

(c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

(d) For which numbers α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?

(e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

19. The fact that $a^2 \geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}.$$

(A more general form occurs in Problem 2-21.) The three proofs of the Schwarz inequality outlined below have only one thing in common—their reliance on the fact that $a^2 \geq 0$ for all a.

(a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number λ, then equality holds in the Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that y_1 and y_2 are not both 0, and that there is no number

λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$\begin{aligned} 0 &< (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 \\ &= \lambda^2({y_1}^2 + {y_2}^2) - 2\lambda(x_1 y_1 + x_2 y_2) + ({x_1}^2 + {x_2}^2). \end{aligned}$$

Using Problem 18, complete the proof of the Schwarz inequality.

(b) Prove the Schwarz inequality by using $2xy \le x^2 + y^2$ (how is this derived?) with

$$x = \frac{x_i}{\sqrt{{x_1}^2 + {x_2}^2}}, \qquad y = \frac{y_i}{\sqrt{{y_1}^2 + {y_2}^2}},$$

first for $i = 1$ and then for $i = 2$.

(c) Prove the Schwarz inequality by first proving that

$$({x_1}^2 + {x_2}^2)({y_1}^2 + {y_2}^2) = (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2.$$

(d) Deduce, from each of these three proofs, that equality holds only when $y_1 = y_2 = 0$ or when there is a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

In our later work, three facts about inequalities will be crucial. Although proofs will be supplied at the appropriate point in the text, a personal assault on these problems is infinitely more enlightening than a perusal of a completely worked-out proof. The statements of these propositions involve some weird numbers, but their basic message is very simple: if x is close enough to x_0, and y is close enough to y_0, then $x + y$ will be close to $x_0 + y_0$, and xy will be close to $x_0 y_0$, and $1/y$ will be close to $1/y_0$. The symbol "ε" which appears in these propositions is the fifth letter of the Greek alphabet ("epsilon"), and could just as well be replaced by a less intimidating Roman letter; however, tradition has made the use of ε almost sacrosanct in the contexts to which these theorems apply.

20. Prove that if

$$|x - x_0| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2},$$

then

$$\begin{aligned} |(x + y) - (x_0 + y_0)| &< \varepsilon, \\ |(x - y) - (x_0 - y_0)| &< \varepsilon. \end{aligned}$$

***21.** Prove that if

$$|x - x_0| < \min\left(\frac{\varepsilon}{2(|y_0| + 1)}, 1\right) \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2(|x_0| + 1)},$$

then $|xy - x_0 y_0| < \varepsilon$.

(The notation "min" was defined in Problem 13, but the formula provided by that problem is irrelevant at the moment; the first inequality in the hypothesis just means that

$$|x - x_0| < \frac{\varepsilon}{2(|y_0| + 1)} \quad \text{and} \quad |x - x_0| < 1;$$

at one point in the argument you will need the first inequality, and at another point you will need the second. One more word of advice: since the hypotheses only provide information about $x - x_0$ and $y - y_0$, it is almost a foregone conclusion that the proof will depend upon writing $xy - x_0y_0$ in a way that involves $x - x_0$ and $y - y_0$.)

***22.** Prove that if $y_0 \neq 0$ and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right),$$

then $y \neq 0$ and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon.$$

***23.** Replace the question marks in the following statement by expressions involving ε, x_0, and y_0 so that the conclusion will be true:

If $y_0 \neq 0$ and

$$|y - y_0| < ? \quad \text{and} \quad |x - x_0| < ?$$

then $y \neq 0$ and

$$\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon.$$

This problem is trivial in the sense that its solution follows from Problems 21 and 22 with almost no work at all (notice that $x/y = x \cdot 1/y$). The crucial point is not to become confused; decide which of the two problems should be used first, and don't panic if your answer looks unlikely.

***24.** This problem shows that the actual placement of parentheses in a sum is irrelevant. The proofs involve "mathematical induction"; if you are not familiar with such proofs, but still want to tackle this problem, it can be saved until after Chapter 2, where proofs by induction are explained.

Let us agree, for definiteness, that $a_1 + \cdots + a_n$ will denote

$$a_1 + (a_2 + (a_3 + \cdots + (a_{n-2} + (a_{n-1} + a_n))) \cdots).$$

Thus $a_1 + a_2 + a_3$ denotes $a_1 + (a_2 + a_3)$, and $a_1 + a_2 + a_3 + a_4$ denotes $a_1 + (a_2 + (a_3 + a_4))$, etc.

(a) Prove that

$$(a_1 + \cdots + a_k) + a_{k+1} = a_1 + \cdots + a_{k+1}.$$

Hint: Use induction on k.

(b) Prove that if $n \geq k$, then

$$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n.$$

Hint: Use part (a) to give a proof by induction on k.

(c) Let $s(a_1, \ldots, a_k)$ be some sum formed from $a_1, \ldots, a_k$. Show that

$$s(a_1, \ldots, a_k) = a_1 + \cdots + a_k.$$

Hint: There must be two sums $s'(a_1, \ldots, a_l)$ and $s''(a_{l+1}, \ldots, a_k)$ such that

$$s(a_1, \ldots, a_k) = s'(a_1, \ldots, a_l) + s''(a_{l+1}, \ldots, a_k).$$

25. Suppose that we interpret "number" to mean either 0 or 1, and + and · to be the operations defined by the following two tables.

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Check that properties P1–P9 all hold, even though $1 + 1 = 0$.

CHAPTER 2 NUMBERS OF VARIOUS SORTS

In Chapter 1 we used the word "number" very loosely, despite our concern with the basic properties of numbers. It will now be necessary to distinguish carefully various kinds of numbers.

The simplest numbers are the "counting numbers"

$$1, 2, 3, \ldots .$$

The fundamental significance of this collection of numbers is emphasized by its symbol $\mathbf{N}$ (for **natural numbers**). A brief glance at P1–P12 will show that our basic properties of "numbers" do not apply to $\mathbf{N}$—for example, P2 and P3 do not make sense for $\mathbf{N}$. From this point of view the system $\mathbf{N}$ has many deficiencies. Nevertheless, $\mathbf{N}$ is sufficiently important to deserve several comments before we consider larger collections of numbers.

The most basic property of $\mathbf{N}$ is the principle of "mathematical induction." Suppose $P(x)$ means that the property P holds for the number x. Then the principle of mathematical induction states that $P(x)$ is true for all natural numbers x provided that

(1) $P(1)$ is true.

(2) Whenever $P(k)$ is true, $P(k+1)$ is true.

Note that condition (2) merely asserts the truth of $P(k+1)$ under the assumption that $P(k)$ is true; this suffices to ensure the truth of $P(x)$ for all x, if condition (1) also holds. In fact, if $P(1)$ is true, then it follows that $P(2)$ is true (by using (2) in the special case $k = 1$). Now, since $P(2)$ is true it follows that $P(3)$ is true (using (2) in the special case $k = 2$). It is clear that each number will eventually be reached by a series of steps of this sort, so that $P(k)$ is true for all numbers k.

A favorite illustration of the reasoning behind mathematical induction envisions an infinite line of people,

$$\text{person number 1, person number 2, person number 3, } \ldots .$$

If each person has been instructed to tell any secret he hears to the person behind him (the one with the next largest number) and a secret is told to person number 1, then clearly every person will eventually learn the secret. If $P(x)$ is the assertion that person number x will learn the secret, then the instructions given (to tell all secrets learned to the next person) assures that condition (2) is true, and telling the secret to person number 1 makes (1) true. The following example is a less facetious use of mathematical induction. There is a useful and striking formula which expresses the sum of the first n numbers in a simple way:

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$$\sum_{\substack{i=1\\ i\neq 4}}^{n} a_i,$$

for example, is an obvious way of writing

$$a_1 + a_2 + a_3 + a_5 + a_6 + \cdots + a_n,$$

or more precisely,

$$\sum_{i=1}^{3} a_i + \sum_{i=5}^{n} a_i.$$

The deficiencies of the natural numbers which we discovered at the beginning of this chapter may be partially remedied by extending this system to the set of **integers**

$$\ldots, -2, -1, 0, 1, 2, \ldots.$$

This set is denoted by **Z** (from German "Zahl," number). Of properties P1–P12, only P7 fails for **Z**.

A still larger system of numbers is obtained by taking quotients m/n of integers (with $n \neq 0$). These numbers are called **rational numbers**, and the set of all rational numbers is denoted by **Q** (for "quotients"). In this system of numbers all of P1–P12 are true. It is tempting to conclude that the "properties of numbers," which we studied in some detail in Chapter 1, refer to just one set of numbers, namely, **Q**. There is, however, a still larger collection of numbers to which properties P1–P12 apply—the set of all **real numbers**, denoted by **R**. The real numbers include not only the rational numbers, but other numbers as well (the **irrational numbers**) which can be represented by infinite decimals; π and $\sqrt{2}$ are both examples of irrational numbers. The proof that π is irrational is not easy—we shall devote all of Chapter 16 of Part III to a proof of this fact. The irrationality of $\sqrt{2}$, on the other hand, is quite simple, and was known to the Greeks. (Since the Pythagorean theorem shows that an isosceles right triangle, with sides of length 1, has a hypotenuse of length $\sqrt{2}$, it is not surprising that the Greeks should have investigated this question.) The proof depends on a few observations about the natural numbers. Every natural number n can be written either in the form $2k$ for some integer k, or else in the form $2k+1$ for some integer k (this "obvious" fact has a simple proof by induction (Problem 8)). Those natural numbers of the form $2k$ are called **even**; those of the form $2k+1$ are called **odd**. Note that even numbers have even squares, and odd numbers have odd squares:

$$\begin{aligned}(2k)^2 &= 4k^2 = 2\cdot(2k^2),\\ (2k+1)^2 &= 4k^2+4k+1 = 2\cdot(2k^2+2k)+1.\end{aligned}$$

In particular it follows that the converse must also hold: if n^2 is even, then n is even; if n^2 is odd, then n is odd. The proof that $\sqrt{2}$ is irrational is now quite simple. Suppose that $\sqrt{2}$ were rational; that is, suppose there were natural numbers p

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5. (a) Prove by induction on n that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$ (if $r = 1$, evaluating the sum certainly presents no problem).

(b) Derive this result by setting $S = 1 + r + \cdots + r^n$, multiplying this equation by r, and solving the two equations for S.

6. The formula for $1^2 + \cdots + n^2$ may be derived as follows. We begin with the formula

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Writing this formula for $k = 1, \ldots, n$ and adding, we obtain

$$\begin{aligned}
2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\
3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\
&\;\;\vdots \\
(n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1 \\
\hline
(n+1)^3 - 1 &= 3[1^2 + \cdots + n^2] + 3[1 + \cdots + n] + n.
\end{aligned}$$

Thus we can find $\sum_{k=1}^{n} k^2$ if we already know $\sum_{k=1}^{n} k$ (which could have been found in a similar way). Use this method to find

(i) $1^3 + \cdots + n^3$.

(ii) $1^4 + \cdots + n^4$.

(iii) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)}$.

(iv) $\dfrac{3}{1^2 \cdot 2^2} + \dfrac{5}{2^2 \cdot 3^2} + \cdots + \dfrac{2n+1}{n^2(n+1)^2}$.

***7.** Use the method of Problem 6 to show that $\sum_{i=1}^{n} k^p$ can always be written in the form

$$\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + Cn^{p-2} + \cdots .$$

(The first 10 such expressions are

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21. The Schwarz inequality (Problem 1-19) actually has a more general form:

$$\sum_{i=1}^{n} x_i y_i \le \sqrt{\sum_{i=1}^{n} {x_i}^2}\sqrt{\sum_{i=1}^{n} {y_i}^2}.$$

Give three proofs of this, analogous to the three proofs in Problem 1-19.

22. The result in Problem 1-7 has an important generalization: If $a_1, \dots, a_n \ge 0$, then the "arithmetic mean"

$$A_n = \frac{a_1 + \cdots + a_n}{n}$$

and "geometric mean"

$$G_n = \sqrt[n]{a_1 \dots a_n}$$

satisfy

$$G_n \le A_n.$$

(a) Suppose that $a_1 < A_n$. Then some a_i satisfies $a_i > A_n$; for convenience, say $a_2 > A_n$. Let $\bar{a}_1 = A_n$ and let $\bar{a}_2 = a_1 + a_2 - \bar{a}_1$. Show that

$$\bar{a}_1\bar{a}_2 \ge a_1 a_2.$$

Why does repeating this process enough times eventually prove that $G_n \le A_n$? (This is another place where it is a good exercise to provide a formal proof by induction, as well as an informal reason.) When does equality hold in the formula $G_n \le A_n$?

The reasoning in this proof is related to another interesting proof.

(b) Using the fact that $G_n \le A_n$ when $n = 2$, prove, by induction on k, that $G_n \le A_n$ for $n = 2^k$.

(c) For a general n, let $2^m > n$. Apply part (b) to the 2^m numbers

$$a_1, \dots, a_n, \underbrace{A_n, \dots, A_n}_{2^m - n \text{ times}}$$

to prove that $G_n \le a_n$.

23. The following is a recursive definition of a^n:

$$\begin{aligned} a^1 &= a, \\ a^{n+1} &= a^n \cdot a. \end{aligned}$$

Prove, by induction, that

$$\begin{aligned} a^{n+m} &= a^n \cdot a^m, \\ (a^n)^m &= a^{nm}. \end{aligned}$$

(Don't try to be fancy: use either induction on n or induction on m, not both at once.)

PART 2
FOUNDATIONS

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(4) $r(x) = x^2$ for all x such that $-17 \leq x \leq \pi/3$.

(5) $s(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational.} \end{cases}$

(6) $\theta(x) = \begin{cases} 5, & x = 2 \\ \dfrac{36}{\pi}, & x = 17 \\ 28, & x = \dfrac{\pi^2}{17} \\ 28, & x = \dfrac{36}{\pi} \\ 16, & x \neq 2,\ 17,\ \dfrac{\pi^2}{17}, \text{ or } \dfrac{36}{\pi}, \text{ and } x = a + b\sqrt{2} \text{ for } a, b \text{ in } \mathbf{Q}. \end{cases}$

(7) $\alpha_x(t) = t^3 + x$ for all numbers t.

(8) $y(x) = \begin{cases} n, & \text{exactly } n \text{ 7's appear in the decimal expansion of } x \\ -\pi, & \text{infinitely many 7's appear in the decimal expansion of } x. \end{cases}$

These definitions illustrate the common procedure adopted for defining a function f—indicating what $f(x)$ is for every number x in the domain of f. (Notice that this is exactly the same as indicating $f(a)$ for every number a, or $f(b)$ for every number b, etc.) In practice, certain abbreviations are tolerated. Definition (1) could be written simple

$$(1) \quad f(x) = x^2$$

the qualifying phrase "for all x" being understood. Of course, for definition (4) the only possible abbreviation is

$$(4) \quad r(x) = x^2, \qquad -17 \leq x \leq \pi/3.$$

It is usually understood that a definition such as

$$k(x) = \frac{1}{x} + \frac{1}{x-1}, \quad x \neq 0, 1$$

can be shortened to

$$k(x) = \frac{1}{x} + \frac{1}{x-1};$$

in other words, *unless the domain is explicitly restricted further, it is understood to consist of all numbers for which the definition makes any sense at all.*

You should have little difficulty checking the following assertions about the functions defined above:

$$f(x+1) = f(x) + 2x + 1;$$
$$g(x) = h(x) \text{ if } x^3 + 3x + 5 = 0;$$
$$r(x+1) = r(x) + 2x + 1 \text{ if } -17 \leq x \leq \frac{\pi}{3} - 1;$$

these must *not* be confused, since they are not usually equal; in fact, almost any f and g chosen at random will illustrate this point (try $f = I \cdot I$ and $g = \sin$, for example). Lest you become too apprehensive about the operation of composition, let us hasten to point out that composition *is* associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

(and the proof is a triviality); this function is denoted by $f \circ g \circ h$. We can now write the functions (10), (11), (12) as

(10) $\quad f = \sin \circ (I \cdot I)$,

(11) $\quad f = \sin \circ \sin \circ (I \cdot I)$,

(12) $\quad f = (\sin \cdot \sin) \circ \sin \circ (\sin \cdot \sin) \circ (I \cdot [(\sin \cdot \sin) \circ (I \cdot I)]) \cdot$

$$\sin \circ \left(\frac{I + \sin \circ (I \cdot \sin)}{I + \sin} \right).$$

One fact has probably already become clear. Although this method of writing functions reveals their "structure" very clearly, it is hardly short or convenient. The shortest name for the function f such that $f(x) = \sin(x^2)$ for all x unfortunately seems to be "the function f such that $f(x) = \sin(x^2)$ for all x." The need for abbreviating this clumsy description has been clear for two hundred years, but no reasonable abbreviation has received universal acclaim. At present the strongest contender for this honor is something like

$$x \to \sin(x^2)$$

(read "x goes to $\sin(x^2)$" or just "x arrow $\sin(x^2)$"), but it is hardly popular among writers of calculus textbooks. In this book we will tolerate a certain amount of ellipsis, and speak of "the function $f(x) = \sin(x^2)$." Even more popular is the quite drastic abbreviation: "the function $\sin(x^2)$." For the sake of precision we will never use this description, which, strictly speaking, confuses a number and a function, but it is so convenient that you will probably end up adopting it for personal use. As with any convention, utility is the motivating factor, and this criterion is reasonable so long as the slight logical deficiencies cause no confusion. On occasion, confusion *will* arise unless a more precise description is used. For example, "the function $x + t^3$" is an ambiguous phrase; it could mean either

$$x \to x + t^3, \text{ i.e., the function } f \text{ such that } f(x) = x + t^3 \text{ for all } x$$

or

$$t \to x + t^3, \text{ i.e., the function } f \text{ such that } f(t) = x + t^3 \text{ for all } t.$$

As we shall see, however, for many important concepts associated with functions, calculus has a notation which contains the "$x \to$" built in.

By now we have made a sufficiently extensive investigation of functions to warrant reconsidering our definition. We have defined a function as a "rule," but it is hardly clear what this means. If we ask "What happens if you break this rule?" it is not easy to say whether this question is merely facetious or actually profound.

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answer should be a *function*.

(i) $f(x) = 2^{\sin x}$.
(ii) $f(x) = \sin 2^x$.
(iii) $f(x) = \sin x^2$.
(iv) $f(x) = \sin^2 x$ (remember that $\sin^2 x$ is an abbreviation for $(\sin x)^2$).
(v) $f(t) = 2^{2^t}$. (Note: a^{b^c} *always* means $a^{(b^c)}$; this convention is adopted because $(a^b)^c$ can be written more simply as a^{bc}.)
(vi) $f(u) = \sin(2^u + 2^{u^2})$.
(vii) $f(y) = \sin\left(\sin\left(\sin\left(2^{2^{2^{\sin y}}}\right)\right)\right)$.
(viii) $f(a) = 2^{\sin^2 a} + \sin(a^2) + 2^{\sin(a^2 + \sin a)}$.

Polynomial functions, because they are simple, yet flexible, occupy a favored role in most investigations of functions. The following two problems illustrate their flexibility, and guide you through a derivation of their most important elementary properties.

6. (a) If $x_1, \ldots, x_n$ are distinct numbers, find a polynomial function f_i of degree $n-1$ which is 1 at x_i and 0 at x_j for $j \neq i$. Hint: the product of all $(x - x_j)$ for $j \neq i$, is 0 at x_j if $j \neq i$. (This product is usually denoted by

$$\prod_{\substack{j=1 \\ j \neq i}}^{n} (x - x_j),$$

the symbol Π (capital pi) playing the same role for products that Σ plays for sums.)

(b) Now find a polynomial function f of degree $n-1$ such that $f(x_i) = a_i$, where $a_1, \ldots, a_n$ are given numbers. (You should use the functions f_i from part (a). The formula you will obtain is called the "Lagrange interpolation formula.")

7. (a) Prove that for any polynomial function f, and any number a, there is a polynomial function g, and a number b, such that $f(x) = (x-a)g(x) + b$ for all x. (The idea is simply to divide $(x-a)$ into $f(x)$ by long division, until a constant remainder is left. For example, the calculation

$$\begin{array}{r|llll}
 & x^2 & +x & & -2 \\ \hline
x-1 & x^3 & & & -3x+1 \\
 & x^3 & -x^2 & & \\ \cline{3-4}
 & & x^2 & -3x & \\
 & & x^2 & -x & \\ \cline{3-4}
 & & & & -2x+1 \\
 & & & & -2x+2 \\ \cline{5-5}
 & & & & -1
\end{array}$$

23. Suppose that $f \circ g = I$, where $I(x) = x$. Prove that

(a) if $x \neq y$, then $g(x) \neq g(y)$;
(b) every number b can be written $b = f(a)$ for some number a.

***24.** (a) Suppose g is a function with the property that $g(x) \neq g(y)$ if $x \neq y$. Prove that there is a function f such that $f \circ g = I$.
(b) Suppose that f is a function such that every number b can be written $b = f(a)$ for some number a. Prove that there is a function g such that $f \circ g = I$.

***25.** Find a function f such that $g \circ f = I$ for some g, but such that there is no function h with $f \circ h = I$.

***26.** Suppose $f \circ g = I$ and $h \circ f = I$. Prove that $g = h$. Hint: Use the fact that composition is associative.

27. (a) Suppose $f(x) = x + 1$. Are there any functions g such that $f \circ g = g \circ f$?
(b) Suppose f is a constant function. For which functions g does $f \circ g = g \circ f$?
(c) Suppose that $f \circ g = g \circ f$ for *all* functions g. Show that f is the identity function, $f(x) = x$.

28. (a) Let F be the set of all functions whose domain is **R**. Prove that, using $+$ and $\cdot$ as defined in this chapter, all of properties P1–P9 except P7 hold for F, provided 0 and 1 are interpreted as constant functions.
(b) Show that P7 does not hold.
*(c) Show that P10–P12 cannot hold. In other words, show that there is no collection P of functions in F, such that P10–P12 hold for P. (It is sufficient, and will simplify things, to consider only functions which are 0 except at two points x_0 and x_1.)
(d) Suppose we define $f < g$ to mean that $f(x) < g(x)$ for all x. Which of P′10–P′13 (in Problem 1-8) now hold?
(e) If $f < g$, is $h \circ f < h \circ g$? Is $f \circ h < g \circ h$?

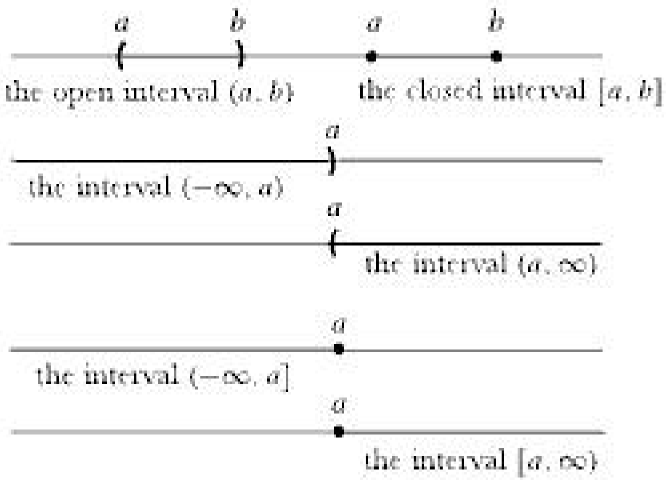

FIGURE 3

tice, however, it is almost always assumed (explicitly if one has been careful, and implicitly otherwise), that whenever an interval (a, b) is mentioned, the number a is less than b.

The set $\{x : a \le x \le b\}$ is denoted by $[a, b]$ and is called the **closed interval** from a to b. This symbol is usually reserved for the case $a < b$, but it is sometimes used for $a = b$, also. The usual pictures for the intervals (a, b) and $[a, b]$ are shown in Figure 3; since no reasonably accurate picture could ever indicate the difference between the two intervals, various conventions have been adopted. Figure 3 also shows certain "infinite" intervals. The set $\{x : x > a\}$ is denoted by (a, ∞), while the set $\{x : x \ge a\}$ is denoted by $[a, \infty)$; the sets $(-\infty, a)$ and $(-\infty, a]$ are defined similarly. At this point a standard warning must be issued: the symbols ∞ and $-\infty$, though usually read "infinity" and "minus infinity," are *purely* suggestive; there is no number "∞" which satisfies $\infty \ge a$ for all numbers a. While the symbols ∞ and $-\infty$ will appear in many contexts, it is always necessary to define these uses in ways that refer only to numbers. The set $\mathbf{R}$ of all real numbers is also considered to be an "interval," and is sometimes denoted by $(-\infty, \infty)$.

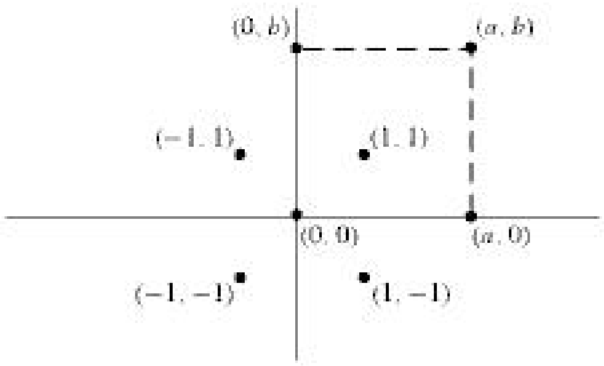

FIGURE 4

Of even greater interest to us than the method of drawing numbers is a method of drawing pairs of numbers. This procedure, probably also familiar to you, requires a "coordinate system," two straight lines intersecting at right angles. To distinguish these straight lines, we call one the *horizontal axis*, and one the *vertical axis*. (More prosaic terminology, such as the "first" and "second" axes, is probably preferable from a logical point of view, but most people hold their books, or at least their blackboards, in the same way, so that "horizontal" and "vertical" are more descriptive.) Each of the two axes could be labeled with real numbers, but we can also label points on the horizontal axis with pairs $(a, 0)$ and points on the vertical axis with pairs $(0, b)$, so that the intersection of the two axes, the "origin" of the coordinate system, is labeled $(0, 0)$. Any pair (a, b) can now be drawn as in Figure 4, lying at the vertex of the rectangle whose other three vertices are labeled $(0, 0)$, $(a, 0)$, and $(0, b)$. The numbers a and b are called the *first* and *second coordinates*, respectively, of the point determined in this way.

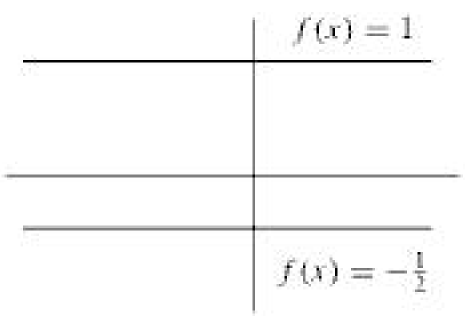

FIGURE 5

Our real concern, let us recall, is a method of drawing functions. Since a function is just a collection of pairs of numbers, we can draw a function by drawing each of the pairs in the function. The drawing obtained in this way is called the **graph** of the function. In other words, the graph of f contains all the points corresponding to pairs $(x, f(x))$. Since most functions contain infinitely many pairs, drawing the graph promises to be a laborious undertaking, but, in fact, many functions have graphs which are quite easy to draw.

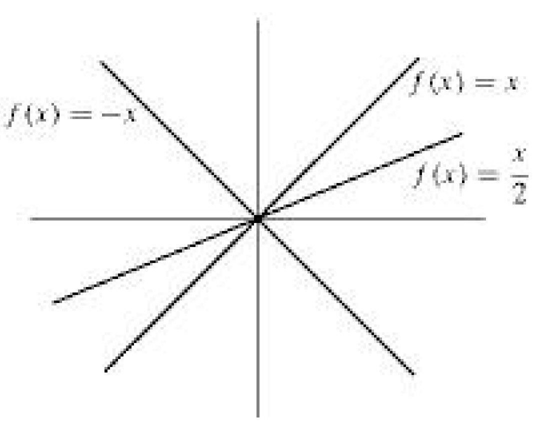

FIGURE 6

Not surprisingly, the simplest functions of all, the constant functions $f(x) = c$, have the simplest graphs. It is easy to see that the graph of the function $f(x) = c$ is a straight line parallel to the horizontal axis, at distance c from it (Figure 5).

The functions $f(x) = cx$ also have particularly simple graphs—straight lines through $(0, 0)$, as in Figure 6. A proof of this fact is indicated in Figure 7:

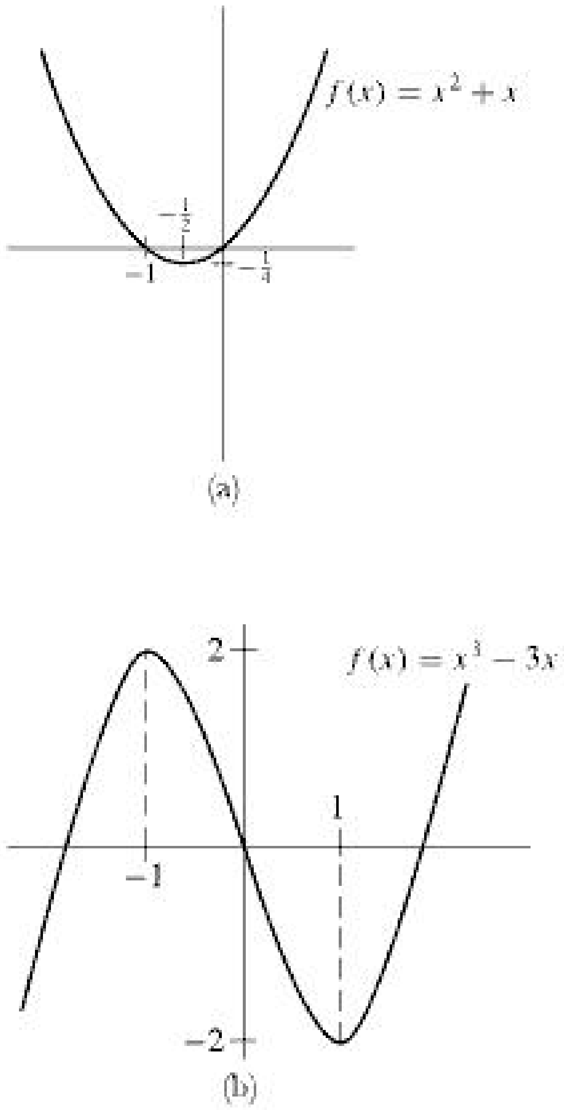

FIGURE 15

Figure 15, while Figure 16 is meant to give a general idea of the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

in the case $a_n > 0$.

In general, the graph of f will have at most $n-1$ "peaks" or "valleys" (a "peak" is a point like $(x, f(x))$ in Figure 16, while a "valley" is a point like $(y, f(y))$. The number of peaks and valleys may actually be much smaller (the power functions, for example, have at most one valley). Although these assertions are easy to make, we will not even contemplate giving proofs until Part III (once the powerful methods of Part III are available, the proofs will be very easy).

Figure 17 illustrates the graphs of several rational functions. The rational functions exhibit even greater variety than the polynomial functions, but their behavior will also be easy to analyze once we can use the derivative, the basic tool of Part III.

Many interesting graphs can be constructed by "piecing together" the graphs of functions already studied. The graph in Figure 18 is made up entirely of straight lines. The function f with this graph satisfies

$$f\left(\frac{1}{n}\right) = (-1)^{n+1},$$

$$f\left(\frac{-1}{n}\right) = (-1)^{n+1},$$

$$f(x) = 1, \qquad |x| \geq 1,$$

and is a linear function on each interval $[1/(n+1), 1/n]$ and $[-1/n, -1/(n+1)]$. (The number 0 is not in the domain of f.) Of course, one can write out an explicit formula for $f(x)$, when x is in $[1/(n+1), 1/n]$; this is a good exercise in the use of linear functions, and will also convince you that a picture is worth a thousand words.

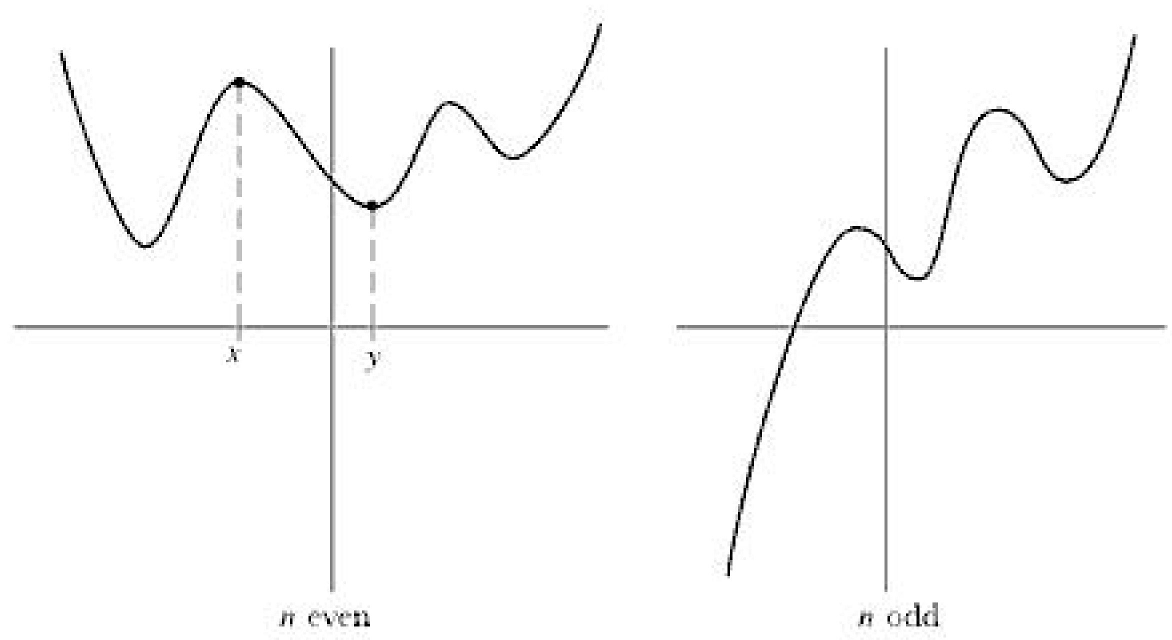

FIGURE 16

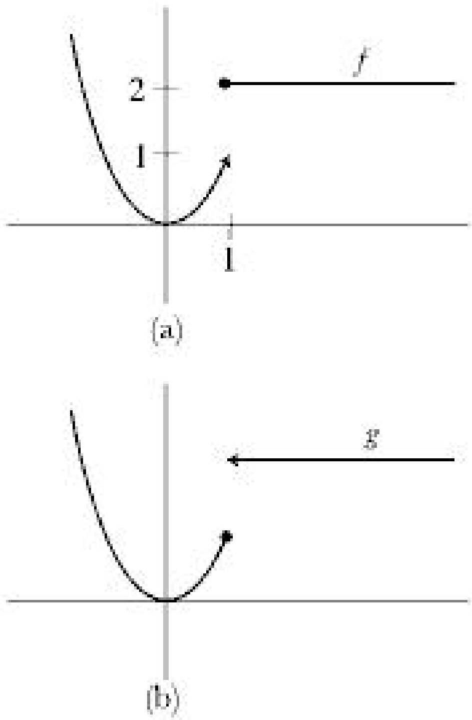

FIGURE 23

analyze. Since $\sin 1/x$ is getting close to 0, while x is getting larger and larger, there seems to be no telling what the product will do. It *is* possible to decide, but this is another question that is best deferred to Part III. The graph of $f(x) = x^2 \sin 1/x$ has also been illustrated (Figure 22).

For these infinitely oscillating functions, it is clear that the graph cannot hope to be really "accurate." The best we can do is to show part of it, and leave out the part near 0 (which is the interesting part). Actually, it is easy to find much simpler functions whose graphs cannot be "accurately" drawn. The graphs of

$$f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x \geq 1 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x^2, & x \leq 1 \\ 2, & x > 1 \end{cases}$$

can only be distinguished by some convention similar to that used for open and closed intervals (Figure 23).

Out last example is a function whose graph is spectacularly nondrawable:

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational.} \end{cases}$$

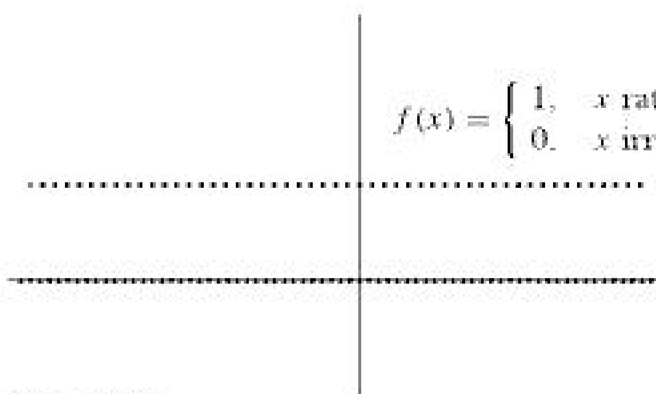

FIGURE 24

The graph of f must contain infinitely many points on the horizontal axis and also infinitely many points on a line parallel to the horizontal axis, but it must not contain either of these lines entirely. Figure 24 shows the usual textbook picture of the graph. To distinguish the two parts of the graph, the dots are placed closer together on the line corresponding to irrational x. (There is actually a mathematical reason behind this convention, but it depends on some sophisticated ideas, introduced in Problems 21-5 and 21-6.)

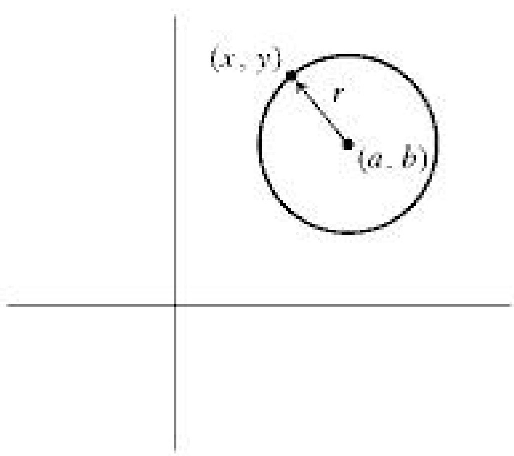

FIGURE 25

The peculiarities exhibited by some functions are so engrossing that it is easy to forget some of the simplest, and most important, subsets of the plane, which are not the graphs of functions. The most important example of all is the **circle**. A circle with center (a, b) and radius $r > 0$ contains, by definition, all the points (x, y) whose distance from (a, b) is equal to r. The circle thus consists (Figure 25) of all points (x, y) with

$$\sqrt{(x-a)^2 + (y-b)^2} = r$$

or

$$(x-a)^2 + (y-b)^2 = r^2.$$

(v) $\dfrac{1}{1+x^2} \geq \frac{1}{5}$.

(vi) $\dfrac{1}{1+x^2} \leq a$ (give an answer in terms of a, distinguishing various cases).

(vii) $x^2 + 1 \geq 2$.

(viii) $(x+1)(x-1)(x-2) > 0$.

2. There is a very useful way of describing the points of the closed interval $[a, b]$ (where we assume, as usual, that $a < b$).

(a) First consider the interval $[0, b]$, for $b > 0$. Prove that if x is in $[0, b]$, then $x = tb$ for some t with $0 \leq t \leq 1$. What is the significance of the number t? What is the mid-point of the interval $[0, b]$?

(b) Now prove that if x is in $[a, b]$, then $x = (1-t)a + tb$ for some t with $0 \leq t \leq 1$. Hint: This expression can also be written as $a + t(b-a)$. What is the midpoint of the interval $[a, b]$? What is the point $1/3$ of the way from a to b?

(c) Prove, conversely, that if $0 \leq t \leq 1$, then $(1-t)a + tb$ is in $[a, b]$.

(d) The points of the *open* interval (a, b) are those of the form $(1-t)a + tb$ for $0 < t < 1$.

3. Draw the set of all points (x, y) satisfying the following conditions. (In most cases your picture will be a sizable portion of a plane, not just a line or curve.)

(i) $x > y$.

(ii) $x + a > y + b$.

(iii) $y < x^2$.

(iv) $y \leq x^2$.

(v) $|x - y| < 1$.

(vi) $|x + y| < 1$.

(vii) $x + y$ is an integer.

(viii) $\dfrac{1}{x+y}$ is an integer.

(ix) $(x-1)^2 + (y-2)^2 < 1$.

(x) $x^2 < y < x^4$.

4. Draw the set of all points (x, y) satisfying the following conditions:

(i) $|x| + |y| = 1$.

(ii) $|x| - |y| = 1$.

(iii) $|x - 1| = |y - 1|$.

(iv) $|1 - x| = |y - 1|$.

(v) $x^2 + y^2 = 0$.

(vi) $xy = 0$.

(vii) $x^2 - 2x + y^2 = 4$.

(viii) $x^2 = y^2$.

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Many functions may be described in terms of the decimal expansion of a number. Although we will not be in a position to describe infinite decimals rigorously until Chapter 23, your intuitive notion of infinite decimals should suffice to carry you through the following problem, and others which occur before Chapter 23. There is one ambiguity about infinite decimals which must be eliminated: Every decimal ending in a string of 9's is equal to another ending in a string of 0's (e.g., $1.23999\ldots = 1.24000\ldots$). We will always use the one ending in 9's.

***19.** Describe as best you can the graphs of the following functions (a complete picture is usually out of the question).

(i) $f(x) =$ the 1st number in the decimal expansion of x.
(ii) $f(x) =$ the 2nd number in the decimal expansion of x.
(iii) $f(x) =$ the number of 7's in the decimal expansion of x if this number is finite, and 0 otherwise.
(iv) $f(x) = 0$ if the number of 7's in the decimal expansion of x is finite, and 1 otherwise.
(v) $f(x) =$ the number obtained by replacing all digits in the decimal expansion of x which come after the first 7 (if any) by 0.
(vi) $f(x) = 0$ if 1 never appears in the decimal expansion of x, and n if 1 first appears in the nth place.

***20.** Let

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ rational in lowest terms.} \end{cases}$$

(A number p/q is in **lowest terms** if p and q are integers with no common factor, and $q > 0$). Draw the graph of f as well as you can (don't sprinkle points randomly on the paper; consider first the rational numbers with $q = 2$, then those with $q = 3$, etc.).

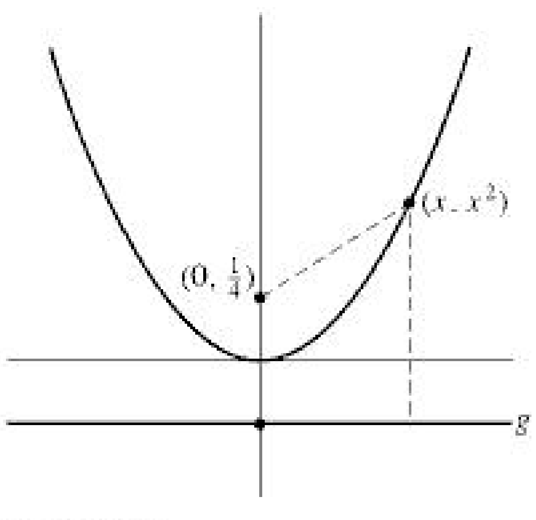

FIGURE 30

21. (a) The points on the graph of $f(x) = x^2$ are the ones of the form (x, x^2). Prove that each such point is equidistant from the point $(0, \frac{1}{4})$ and the graph of $g(x) = -\frac{1}{4}$. (See Figure 30.)

(b) Given a point $P = (\alpha, \beta)$ and a horizontal line L, the graph of $g(x) = \gamma$, show that the set of all points (x, y) equidistant from P and L is the graph of a function of the form $f(x) = ax^2 + bx + c$.

***22.** (a) Show that the square of the distance from (c, d) to (x, mx) is

$$x^2(m^2 + 1) + x(-2md - 2c) + d^2 + c^2.$$

Using Problem 1-18 to find the minimum of these numbers, show that the distance from (c, d) to the graph of $f(x) = mx$ is

$$|cm - d|/\sqrt{m^2 + 1}.$$

(b) Find the distance from (c, d) to the graph of $f(x) = mx + b$. (Reduce this case to part (a).)

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Just as with numbers, our definition of $w - v$ simply means that it satisfies

$$v + (w - v) = w.$$

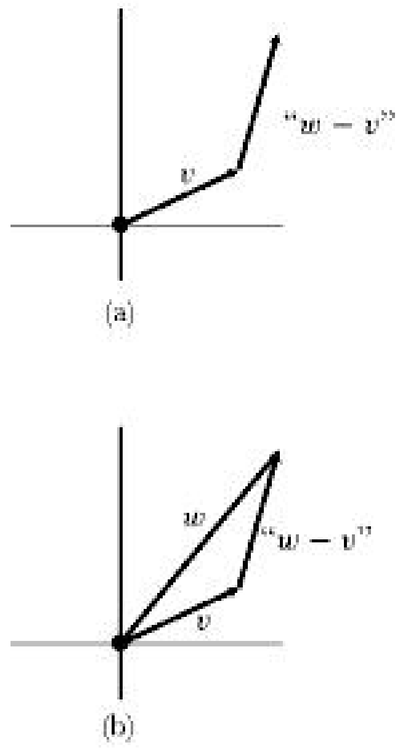

FIGURE 6

Figure 6(a) shows v and an arrow "$w - v$" that is parallel to $w - v$ but that starts at the endpoint of v. As we established with Figure 4, the vector from the origin to the endpoint of this arrow is just $v + (w - v) = w$ (Figure 6(b)). In other words, we can picture $w - v$ geometrically as the arrow that goes from v to w (except that it must then be moved back to the origin).

There is also a way of multiplying a number by a vector: For a number a and a vector $v = (v_1, v_2)$, we define

$$a \cdot v = (av_1, av_2)$$

(We sometimes simply write av instead of $a \cdot v$; of course, it is then especially important to remember that v denotes a vector, rather than a number.) The vector $a \cdot v$ points in the same direction as v when $a > 0$ and in the opposite direction when $a < 0$ (Figure 7).

You can easily check the following formulas:

$$\begin{aligned} a \cdot (b \cdot v) &= (ab) \cdot v, \\ 1 \cdot v &= v, \\ 0 \cdot v &= O, \\ -1 \cdot v &= -v. \end{aligned}$$

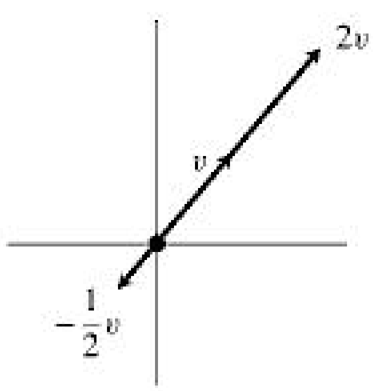

FIGURE 7

Notice that we have only defined a product of a number and a vector, we have not defined a way of 'multiplying' two vectors to get another vector.* However, there are various ways of 'multiplying' vectors to get numbers, which are explored in the following problems.

PROBLEMS

1. Given a point v of the plane, let $R_\theta(v)$ be the result of rotating v around the origin through an angle of θ (Figure 8). The aim of this problem is to obtain a formula for R_θ, with minimal calculation.

(a) Show that

$$\begin{aligned} R_\theta(1, 0) &= (\cos\theta, \sin\theta), \\ R_\theta(0, 1) &= (-\sin\theta, \cos\theta). \end{aligned}$$

(b) Explain why we have

$$\begin{aligned} R_\theta(v + w) &= R_\theta(v) + R_\theta(w), \\ R_\theta(a \cdot w) &= a \cdot R_\theta(w). \end{aligned}$$

(c) Now show that for any point (x, y) we have

$$R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

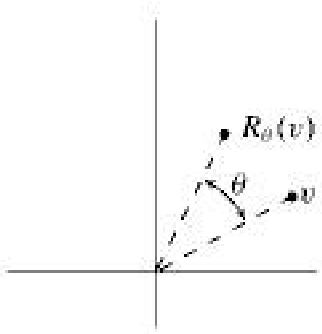

FIGURE 8

* If you jump to Chapter 25, you'll find that there is an important way of defining a product, but this is something very special for the plane—it doesn't work for vectors in 3-space, for example, even though the other constructions do.

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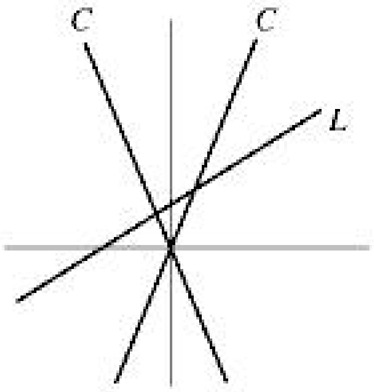

FIGURE 4

in a straight line. We can make things a lot simpler for ourselves if we rotate everything so that this intersection line points straight out from the plane of the paper, while the first axis is in the usual position that we are familiar with. The plane P is thus viewed "straight on," so that all we see (Figure 4) is its intersection L with the plane of the first and third axes; from this view-point the cone itself simply appears as two straight lines.

In the plane of the first and third axes, the line L can be described as the collection of all points of the form

$$(x, Mx + B),$$

where M is the slope of L. For an arbitrary point (x, y, z) it follows that

$$(2) \qquad (x, y, z) \text{ is in the plane } P \text{ if and only if } z = Mx + B.$$

Combining (1) and (2), we see that (x, y, z) is in the intersection of the cone and the plane if and only if

$$(*) \qquad Mx + B = \pm C\sqrt{x^2 + y^2}.$$

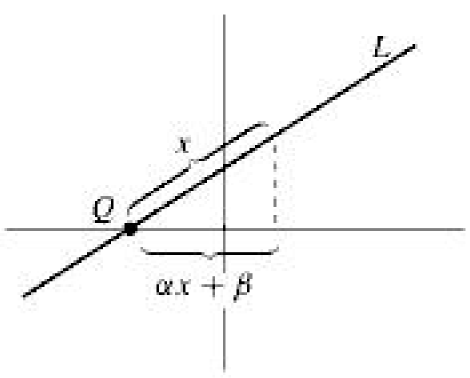

FIGURE 5

Now we have to choose coordinate axes in the plane P. We can choose L as the first axis, measuring distances from the intersection Q with the horizontal plane (Figure 5); for the second axis we just choose the line through Q parallel to our original second axis. If the first coordinate of a point in P with respect to these axes is x, then the first coordinate of this point with respect to the original axes can be written in the form

$$\alpha x + \beta$$

for some α and β. On the other hand, if the second coordinate of the point with respect to these axes is y, then y is also the second coordinate with respect to the original axes.

Consequently, $(*)$ says that the point lies on the intersection of the plane and the cone if and only if

$$M(\alpha x + \beta) + B = \pm C\sqrt{(\alpha x + \beta)^2 + y^2}.$$

Although this looks fairly complicated, after squaring we can write this as

$$\alpha^2 C^2 y^2 + \alpha^2 (M^2 - A^2)x^2 + Ex + F = 0$$

for some E and F that we won't bother writing out. Dividing by α^2 simplifies this to

$$C^2 y^2 + (C^2 - M^2)x^2 + Gx + H = 0.$$

Now Problem 4-16 indicates that this is either a parabola, an ellipse, or an hyperbola. In fact, looking a little more closely at the solution (and interchanging

Conversely, if a point has cartesian coordinates (x, y), then (any of) its polar coordinates (r, θ) satisfy

$$r = \pm\sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x} \qquad \text{if } x \neq 0.$$

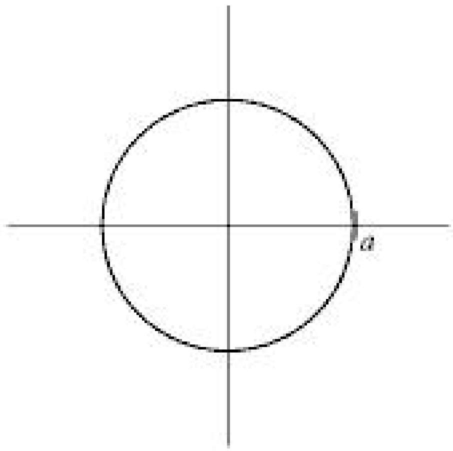

FIGURE 3

Now suppose that f is a function. Then by the **graph of f in polar coordinates** we mean the collection of all points P with polar coordinates (r, θ) satisfying $r = f(\theta)$. In other words, the graph of f in polar coordinates is the collection of all points with polar coordinates $(f(\theta), \theta)$. No special significance should be attached to the fact that we are considering pairs $(f(\theta), \theta)$, with $f(\theta)$ first, as opposed to pairs $(x, f(x))$ in the usual graph of f; it is purely a matter of convention that r is considered the first polar coordinate and θ is considered the second.

The graph of f in polar coordinates is often described as "the graph of the equation $r = f(\theta)$." For example, suppose that f is a constant function, $f(\theta) = a$ for all θ. The graph of the equation $r = a$ is simply a circle with center O and radius a (Figure 3). This example illustrates, in a rather blatant way, that polar coordinates are likely to make things simpler in situations that involve symmetry with respect to the origin O.

The graph of the equation $r = \theta$ is shown in Figure 4. The solid line corresponds to all values of $\theta \geq 0$, while the dashed line corresponds to values of $\theta \leq 0$.

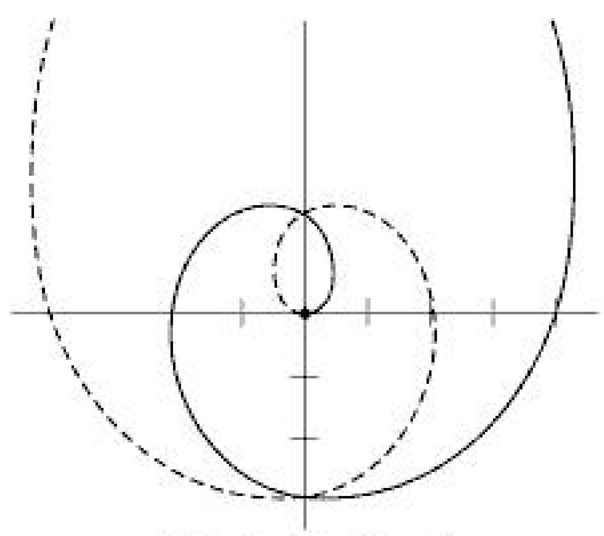

FIGURE 4 Spiral of Archimedes

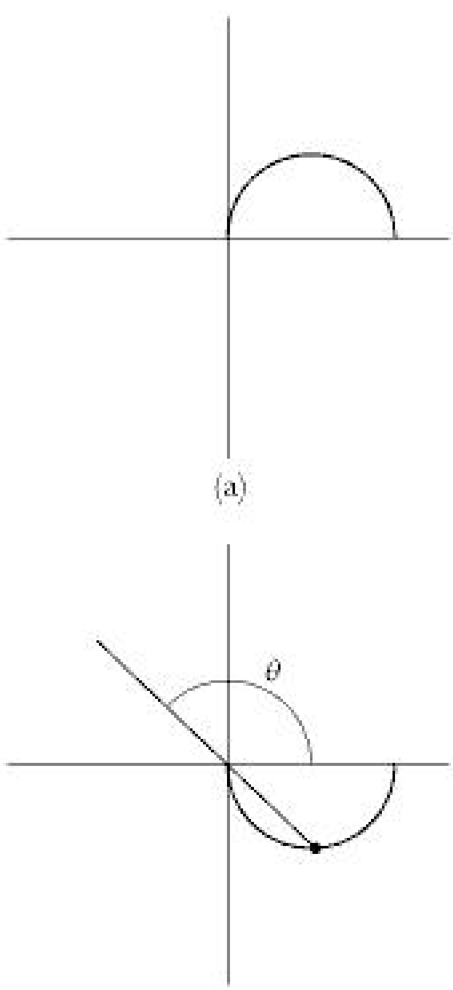

FIGURE 5

As another example involving both positive and negative r, consider the graph of the equation $r = \cos\theta$. Figure 5(a) shows the part that corresponds to $0 \leq \theta \leq 90$ [with θ in degrees]. Figure 5(b) shows the part corresponding to $90 \leq \theta \leq 180$; here $r < 0$. You can check that no new points are added for $\theta > 180$ or $\theta < 0$. It is easy to describe this same graph in terms of the cartesian coordinates of its points. Since the polar coordinates of any point on the graph satisfy

$$r = \cos\theta,$$

8. (a) Sketch the graph of the *cardioid* $r = 1 - \sin\theta$.
(b) Show that it is also the graph of $r = -1 - \sin\theta$.
(c) Show that it can be described by the equation

$$x^2 + y^2 = \sqrt{x^2 + y^2} - y,$$

and conclude that it can be described by the equation

$$(x^2 + y^2 + y)^2 = x^2 + y^2$$

9. Sketch the graphs of the following equations.

(i) $r = 1 - \frac{1}{2}\sin\theta$.
(ii) $r = 1 - 2\sin\theta$.
(iii) $r = 2 + \cos\theta$.

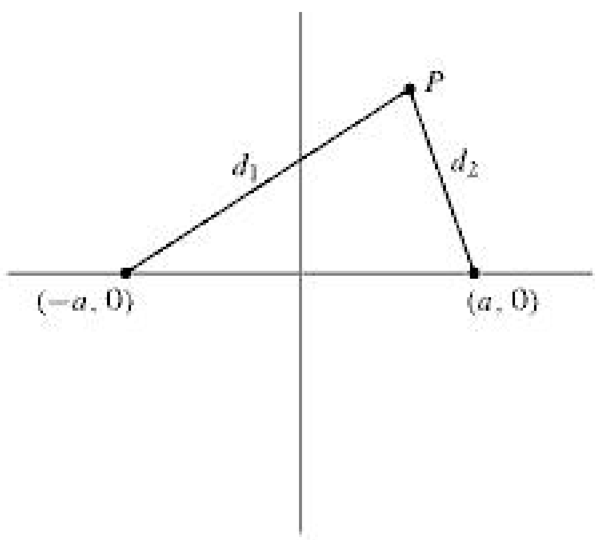

FIGURE 8

10. (a) Sketch the graph of the *lemniscate*

$$r^2 = 2a^2\cos 2\theta.$$

(b) Find an equation for its cartesian coordinates.
(c) Show that it is the collection of all points P in Figure 8 satisfying $d_1 d_2 = a^2$.
(d) Make a guess about the shape of the curves formed by the set of all P satisfying $d_1 d_2 = b$, when $b > a^2$ and when $b < a^2$.

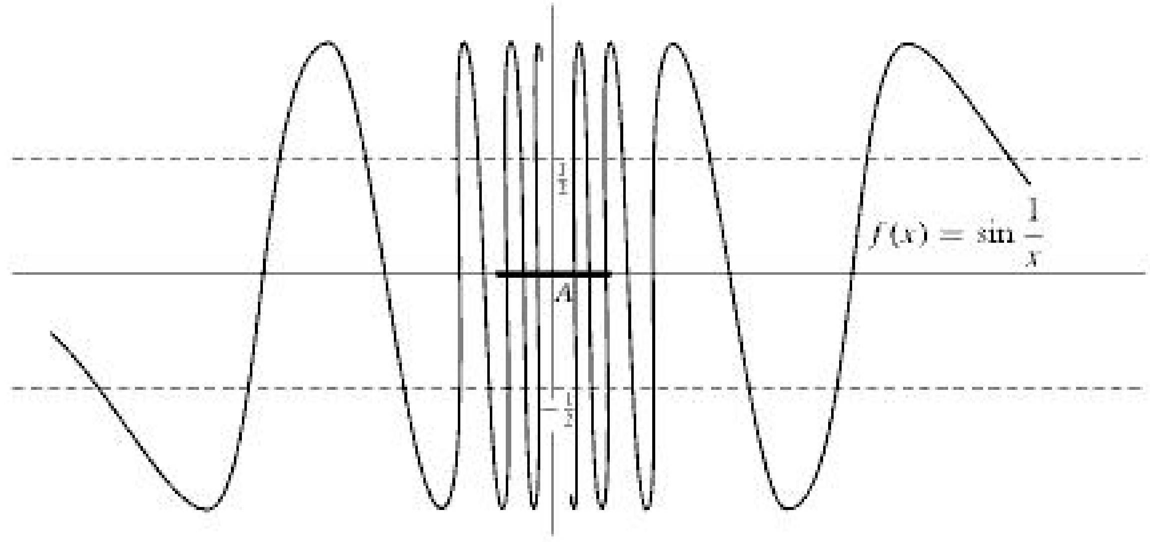

FIGURE 9

provided that $\varepsilon \le 1$. If we are given an ε which is greater than 1 (it might be, even thought it is "small" ε's which are of interest), then it does not suffice to require that $|x| < \varepsilon$, but it certainly suffices to require that $|x| < 1$ and $x \neq 0$.

As a third example, consider the function $f(x) = \sqrt{|x|} \sin 1/x$ (Figure 8). In order to make $|\sqrt{|x|} \sin 1/x| < \varepsilon$ we can require that

$$|x| < \varepsilon^2 \quad \text{and} \quad x \neq 0$$

(the algebra is left to you).

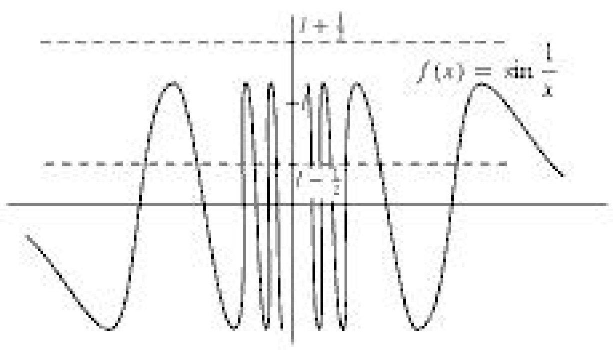

FIGURE 10

Finally, let us consider the function $f(x) = \sin 1/x$ (Figure 9). For this function it is *false* that f approaches 0 near 0. This amounts to saying that it is not true for every number $\varepsilon > 0$ that we can get $|f(x) - 0| < \varepsilon$ by choosing x sufficiently small, and $\neq 0$. To show this we simply have to find *one* $\varepsilon > 0$ for which the condition $|f(x) - 0| < \varepsilon$ cannot be guaranteed, no matter how small we require $|x|$ to be. In fact, $\varepsilon = \frac{1}{2}$ will do: it is impossible to ensure that $|f(x)| < \frac{1}{2}$ no matter how small we require $|x|$ to be; for if A is any interval containing 0, there is some number $x = 1/(90 + 360n)$ which is in this interval, and for this x we have $f(x) = 1$.

This same argument can be used (Figure 10) to show that f does not approach *any* number near 0. To show this we must again find, for any particular number l, some number $\varepsilon > 0$ so that $|f(x) - l| < \varepsilon$ is *not* true, no matter how small x is required to be. The choice $\varepsilon = \frac{1}{2}$ works for any number l; that is, no matter how small we require $|x|$ to be, we cannot ensure that $|f(x) - l| < \frac{1}{2}$. The reason is, that for any interval A containing 0 there is some $x_1 = 1/(90 + 360n)$ in this interval, so that

$$f(x_1) = 1,$$

and also some $x_2 = 1/(270 + 360m)$ in this interval, so that

$$f(x_2) = -1.$$

This definition is so important (*everything* we do from now on depends on it) that proceeding any further without knowing it is hopeless. If necessary memorize it, like a poem! That, at least, is better than stating it incorrectly; if you do this you are doomed to give incorrect proofs. A good exercise in giving correct proofs is to review every fact already demonstrated about functions approaching limits, giving real proofs of each. This requires writing down the correct definition of what you are proving, but not much more—all the algebraic work has been done already. When proving that f does *not* approach l at a, be sure to negate the definition correctly:

If it is *not* true that

> for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$,

then

> there is *some* $\varepsilon > 0$ such that for *every* $\delta > 0$ there is *some* x which satisfies $0 < |x - a| < \delta$ but not $|f(x) - l| < \varepsilon$.

Thus, to show that the function $f(x) = \sin 1/x$ does not approach 0 near 0, we consider $\varepsilon = \frac{1}{2}$ and note that for every $\delta > 0$ there is some x with $0 < |x - 0| < \delta$ but not $|\sin 1/x - 0| < \frac{1}{2}$—namely, an x of the form $1/(90 + 360n)$, where n is so large that $1/(90 + 360n) < \delta$.

As an illustration of the use of the definition of a function approaching a limit, we have reserved the function shown in Figure 14, a standard example, but one of the most complicated:

$$f(x) = \begin{cases} 0, & x \text{ irrational}, 0 < x < 1 \\ 1/q, & x = p/q \text{ in lowest terms}, 0 < x < 1. \end{cases}$$

(Recall that p/q is in lowest terms if p and q are integers with no common factor and $q > 0$.)

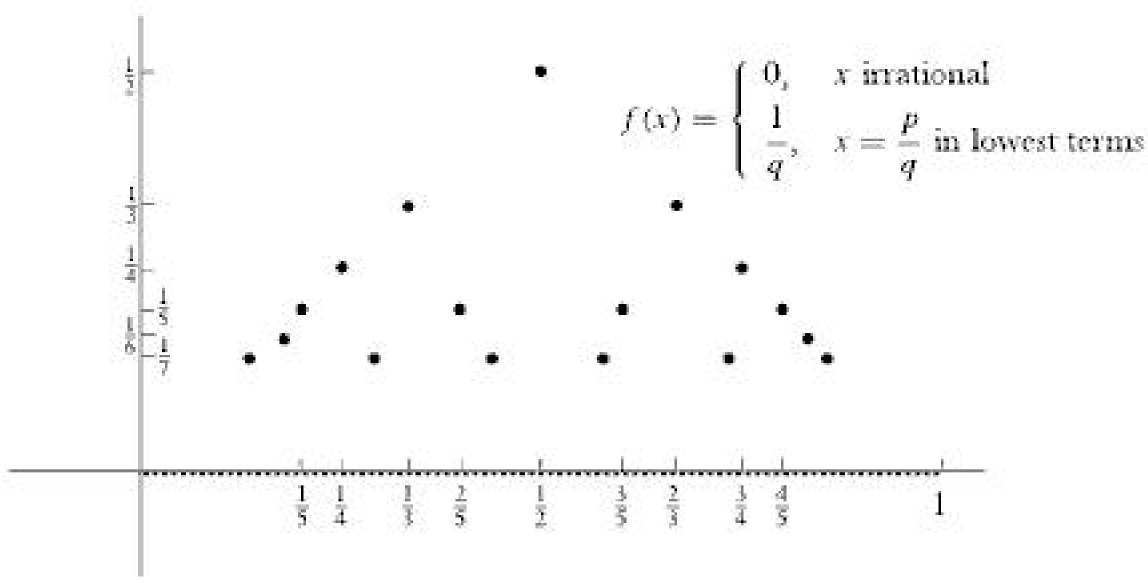

FIGURE 14

LEMMA (1) If

$$|x - x_0| < \frac{\varepsilon}{2} \quad\text{and}\quad |y - y_0| < \frac{\varepsilon}{2},$$

then

$$|(x + y) - (x_0 + y_0)| < \varepsilon.$$

(2) If

$$|x - x_0| < \min\left(1, \frac{\varepsilon}{2(|y_0| + 1)}\right) \quad\text{and}\quad |y - y_0| < \frac{\varepsilon}{2(|x_0| + 1)},$$

then

$$|xy - x_0y_0| < \varepsilon.$$

(3) If $y_0 \neq 0$ and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right),$$

then $y \neq 0$ and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon.$$

PROOF (1)

$$\begin{aligned}|(x + y) - (x_0 + y_0)| &= |(x - x_0) + (y - y_0)| \\ &\leq |x - x_0| + |y - y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

(2) Since $|x - x_0| < 1$ we have

$$|x| - |x_0| \leq |x - x_0| < 1,$$

so that

$$|x| < 1 + |x_0|.$$

Thus

$$\begin{aligned}|xy - x_0y_0| &= |x(y - y_0) + y_0(x - x_0)| \\ &\leq |x| \cdot |y - y_0| + |y_0| \cdot |x - x_0| \\ &< (1 + |x_0|) \cdot \frac{\varepsilon}{2(|x_0| + 1)} + |y_0| \cdot \frac{\varepsilon}{2(|y_0| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

(3) We have

$$|y_0| - |y| \leq |y - y_0| < \frac{y_0}{2},$$

so $|y| > |y_0|/2$. In particular, $y \neq 0$, and

$$\frac{1}{|y|} < \frac{2}{|y_0|}.$$

Thus

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| = \frac{|y_0 - y|}{|y| \cdot |y_0|} < \frac{2}{|y_0|} \cdot \frac{1}{|y_0|} \cdot \frac{\varepsilon|y_0|^2}{2} = \varepsilon. \blacksquare$$

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$\lim\limits_{x \uparrow a} f(x) = l$) means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x,

$$\text{if } 0 < a - x < \delta, \text{ then } |f(x) - l| < \varepsilon.$$

It is quite possible to consider limits from above and below even if f is defined for numbers both greater and less than a. Thus, for the function f of Figure 13, we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1.$$

It is an easy exercise (Problem 29) to show that $\lim\limits_{x \to a} f(x)$ exists if and only if $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ both exist and are equal.

Like the definitions of limits from above and below, which have been smuggled into the text informally, there are other modifications of the limit concept which will be found useful. In Chapter 4 it was claimed that if x is large, then $\sin 1/x$ is close to 0. This assertion is usually written

$$\lim_{x \to \infty} \sin 1/x = 0.$$

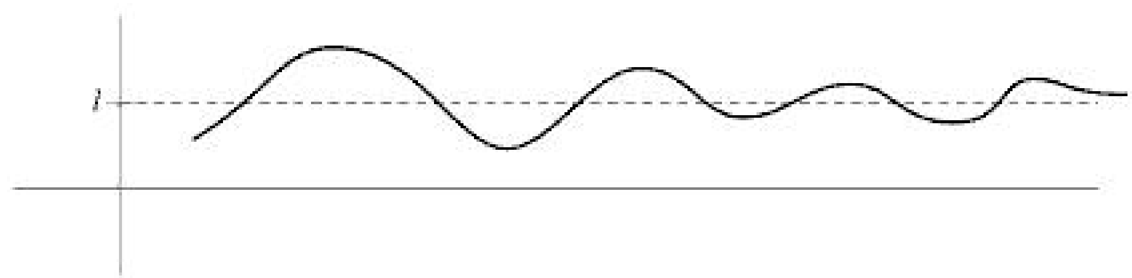

FIGURE 18

The symbol $\lim\limits_{x \to \infty} f(x)$ is read "the limit of $f(x)$ as x approaches ∞," or "as x becomes infinite," and a limit of the form $\lim\limits_{x \to \infty} f(x)$ is often called a limit at infinity. Figure 18 illustrates a general situation where $\lim\limits_{x \to \infty} f(x) = l$. Formally, $\lim\limits_{x \to \infty} f(x) = l$ means that for every $\varepsilon > 0$ there is a number N such that, for all x,

$$\text{if } x > N, \text{ then } |f(x) - l| < \varepsilon.$$

The analogy with the definition of ordinary limits should be clear: whereas the condition "$0 < |x - a| < \delta$" expresses the fact that x is close to a, the condition "$x > N$" expresses the fact that x is large.

We have spent so little time on limits from above and below, and at infinity, because the general philosophy behind the definitions should be clear if you understand the definition of ordinary limits (which are by far the most important). Many exercises on these definitions are provided in the Problems, which also contain several other types of limits which are occasionally useful.

17. (a) Prove that $\lim_{x \to 0} 1/x$ does not exist, i.e., show that $\lim_{x \to 0} 1/x = l$ is false for every number l.

(b) Prove that $\lim_{x \to 1} 1/(x-1)$ does not exist.

18. Prove that if $\lim_{x \to a} f(x) = l$, then there is a number $\delta > 0$ and a number M such that $|f(x)| < M$ if $0 < |x - a| < \delta$. (What does this mean pictorially?) Hint: Why does it suffice to prove that $l - 1 < f(x) < l + 1$ for $0 < |x - a| < \delta$?

19. Prove that if $f(x) = 0$ for irrational x and $f(x) = 1$ for rational x, then $\lim_{x \to a} f(x)$ does not exist for any a.

***20.** Prove that if $f(x) = x$ for rational x, and $f(x) = -x$ for irrational x, then $\lim_{x \to a} f(x)$ does not exist if $a \neq 0$.

21. (a) Prove that if $\lim_{x \to 0} g(x) = 0$, then $\lim_{x \to 0} g(x) \sin 1/x = 0$.

(b) Generalize this fact as follows: If $\lim_{x \to 0} g(x) = 0$ and $|h(x)| \le M$ for all x, then $\lim_{x \to 0} g(x)h(x) = 0$. (Naturally it is unnecessary to do part (a) if you succeed in doing part (b); actually the statement of part (b) may make it easier than (a)—that's one of the values of generalization.)

22. Consider a function f with the following property: if g is any function for which $\lim_{x \to 0} g(x)$ does not exist, then $\lim_{x \to 0} [f(x) + g(x)]$ also does not exist. Prove that this happens if and only if $\lim_{x \to 0} f(x)$ *does* exist. Hint: This is actually very easy: the assumption that $\lim_{x \to 0} f(x)$ does not exist leads to an immediate contradiction if you consider the right g.

****23.** This problem is the analogue of Problem 22 when $f + g$ is replaced by $f \cdot g$. In this case the situation is considerably more complex, and the analysis requires several steps (those in search of an especially challenging problem can attempt an independent solution).

(a) Suppose that $\lim_{x \to 0} f(x)$ exists and is $\neq 0$. Prove that if $\lim_{x \to 0} g(x)$ does not exist, then $\lim_{x \to 0} f(x)g(x)$ also does not exist.

(b) Prove the same result if $\lim_{x \to 0} |f(x)| = \infty$. (The precise definition of this sort of limit is given in Problem 37.)

(c) Prove that if neither of these two conditions holds, then there is a function g such that $\lim_{x \to 0} g(x)$ does not exist, but $\lim_{x \to 0} f(x)g(x)$ does exist. Hint: Consider separately the following two cases: (1) for some $\varepsilon > 0$ we have $|f(x)| > \varepsilon$ for all sufficiently small x. (2) For every $\varepsilon > 0$, there are arbitrarily small x with $|f(x)| < \varepsilon$. In the second case, begin by choosing points x_n with $|x_n| < 1/n$ and $|f(x_n)| < 1/n$.

***24.** Suppose that A_n is, for each natural number n, some *finite* set of numbers in $[0, 1]$, and that A_n and A_m have no members in common if $m \neq n$. Define

CHAPTER 6 CONTINUOUS FUNCTIONS

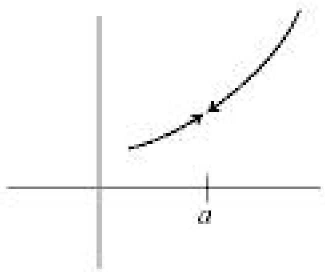

FIGURE 1

If f is an arbitrary function, it is not necessarily true that

$$\lim_{x \to a} f(x) = f(a).$$

In fact, there are many ways this can fail to be true. For example, f might not even be defined at a, in which case the equation makes no sense (Figure 1).

Again, $\lim_{x \to a} f(x)$ might not exist (Figure 2). Finally, as illustrated in Figure 3, even if f is defined at a and $\lim_{x \to a} f(x)$ exists, the limit might not equal $f(a)$.

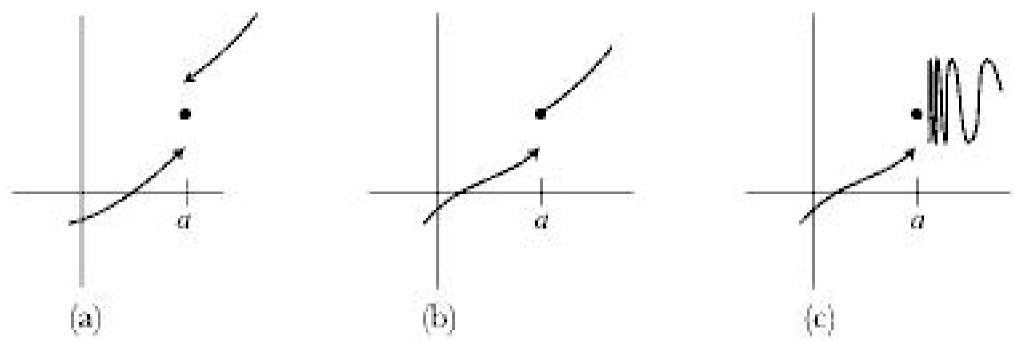

FIGURE 2

We would like to regard all behavior of this type as abnormal and honor, with some complimentary designation, functions which do not exhibit such peculiarities. The term which has been adopted is "continuous." Intuitively, a function f is continuous if the graph contains no breaks, jumps, or wild oscillations. Although this description will usually enable you to decide whether a function is continuous simply by looking at its graph (a skill well worth cultivating) it is easy to be fooled, and the precise definition is *very* important.

DEFINITION

The function f is **continuous at *a*** if

$$\lim_{x \to a} f(x) = f(a).$$

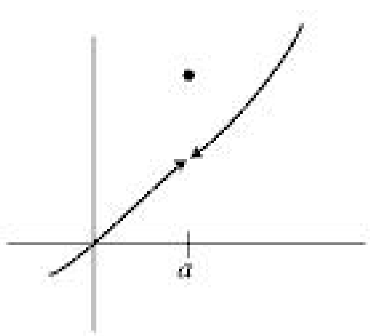

FIGURE 3

We will have no difficulty finding many examples of functions which are, or are not, continuous at some number a—every example involving limits provides an example about continuity, and Chapter 5 certainly provides enough of these.

The function $f(x) = \sin 1/x$ is not continuous at 0, because it is not even defined at 0, and the same is true of the function $g(x) = x \sin 1/x$. On the other hand, if we are willing to extend the second of these functions, that is, if we wish to define

but there is a simple theorem which forms a bridge between the two kinds of results. The hypothesis of this theorem requires continuity at only a single point, but the conclusion describes the behavior of the function on some interval containing the point. Although this theorem is really a lemma for later arguments, it is included here as a preview of things to come.

THEOREM 3 Suppose f is continuous at a, and $f(a) > 0$. Then there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$. Similarly, if $f(a) < 0$, then there is a number $\delta > 0$ such that $f(x) < 0$ for all x satisfying $|x - a| < \delta$.

PROOF Consider the case $f(a) > 0$. Since f is continuous at a, if $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x,

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \varepsilon.$$

Since $f(a) > 0$ we can take $f(a)$ as the ε. Thus there is $\delta > 0$ so that for all x,

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < f(a),$$

and this last inequality implies $f(x) > 0$.

A similar proof can be given in the case $f(a) < 0$; take $\varepsilon = -f(a)$. Or one can apply the first case to the function $-f$. ▌

PROBLEMS

1. For which of the following functions f is there a continuous function F with domain $\mathbf{R}$ such that $F(x) = f(x)$ for all x in the domain of f?

(i) $f(x) = \dfrac{x^2 - 4}{x - 2}$.

(ii) $f(x) = \dfrac{|x|}{x}$.

(iii) $f(x) = 0$, x irrational.

(iv) $f(x) = 1/q$, $x = p/q$ rational in lowest terms.

2. At which points are the functions of Problems 4-17 and 4-19 continuous?

3. (a) Suppose that f is a function satisfying $|f(x)| \le |x|$ for all x. Show that f is continuous at 0. (Notice that $f(0)$ must equal 0.)

(b) Give an example of such a function f which is not continuous at any $a \neq 0$.

(c) Suppose that g is continuous at 0 and $g(0) = 0$, and $|f(x)| \le |g(x)|$. Prove that f is continuous at 0.

4. Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere.

5. For each number a, find a function which is continuous at a, but not at any other points.

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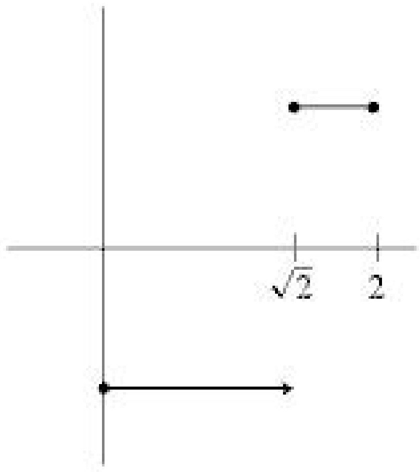

FIGURE 4

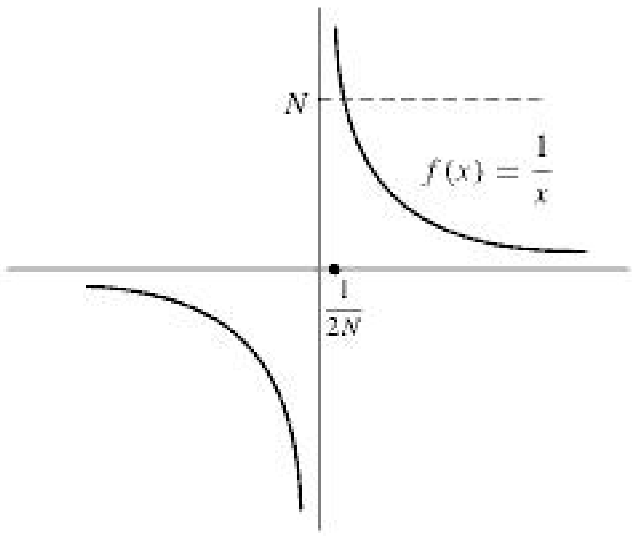

FIGURE 5

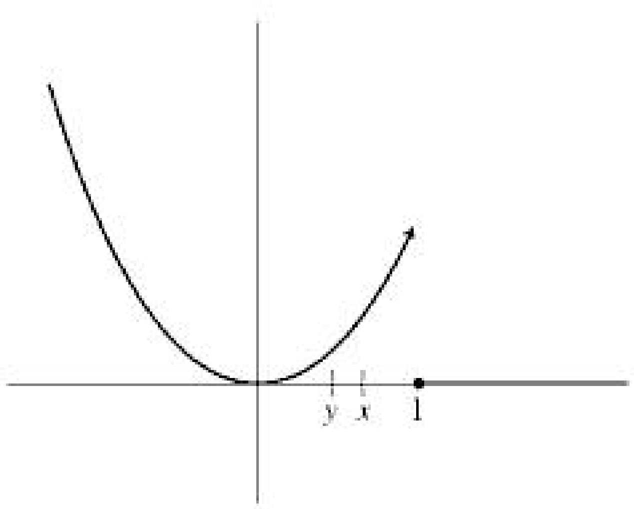

FIGURE 6

the hypotheses of the present theorems require continuity on a whole interval $[a, b]$—if continuity fails to hold at a single point, the conclusions may fail. For example, let f be the function shown in Figure 4,

$$f(x) = \begin{cases} -1, & 0 \le x < \sqrt{2} \\ 1, & \sqrt{2} \le x \le 2. \end{cases}$$

Then f is continuous at every point of $[0, 2]$ except $\sqrt{2}$, and $f(0) < 0 < f(2)$, but there is no point x in $[0, 2]$ such that $f(x) = 0$; the discontinuity at the single point $\sqrt{2}$ is sufficient to destroy the conclusion of Theorem 1.

Similarly, suppose that f is the function shown in Figure 5,

$$f(x) = \begin{cases} 1/x, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

Then f is continuous at every point of $[0, 1]$ except 0, but f is not bounded above on $[0, 1]$. In fact, for any number $N > 0$ we have $f(1/2N) = 2N > N$.

This example also shows that the closed interval $[a, b]$ in Theorem 2 cannot be replaced by the open interval (a, b), for the function f is continuous on $(0, 1)$, but is not bounded there.

Finally, consider the function shown in Figure 6,

$$f(x) = \begin{cases} x^2, & x < 1 \\ 0, & x \ge 1. \end{cases}$$

On the interval $[0, 1]$ the function f is bounded above, so f does satisfy the conclusion of Theorem 2, even though f is not continuous on $[0, 1]$. But f does not satisfy the conclusion of Theorem 3—there is no y in $[0, 1]$ such that $f(y) \ge f(x)$ for all x in $[0, 1]$; in fact, it is certainly not true that $f(1) \ge f(x)$ for all x in $[0, 1]$ so we cannot choose $y = 1$, nor can we choose $0 \le y < 1$ because $f(y) < f(x)$ if x is any number with $y < x < 1$.

This example shows that Theorem 3 is considerably stronger than Theorem 2. Theorem 3 is often paraphrased by saying that a continuous function on a closed interval "takes on its maximum value" on that interval.

As a compensation for the stringency of the hypotheses of our three theorems, the conclusions are of a totally different order than those of previous theorems. They describe the behavior of a function, not just near a point, but on a whole interval; such "global" properties of a function are always significantly more difficult to prove than "local" properties, and are correspondingly of much greater power. To illustrate the usefulness of Theorems 1, 2, and 3, we will soon deduce some important consequences, but it will help to first mention some simple generalizations of these theorems.

THEOREM 4 If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there is some x in $[a, b]$ such that $f(x) = c$.

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theorem the entire point of the proof is to choose an interval $[a, b]$ in such a way that this cannot happen.

THEOREM 10 If n is even and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then there is a number y such that $f(y) \le f(x)$ for all x.

PROOF As in the proof of Theorem 9, if

$$M = \max(1, 2n|a_{n-1}|, \ldots, 2n|a_0|),$$

then for all x with $|x| \ge M$, we have

$$\frac{1}{2} \le 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}.$$

Since n is even, $x^n \ge 0$ for all x, so

$$\frac{x^n}{2} \le x^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) = f(x),$$

provided that $|x| \ge M$. Now consider the number $f(0)$. Let $b > 0$ be a number such that $b^n \ge 2f(0)$ and also $b > M$. Then, if $x \ge b$, we have (Figure 9)

$$f(x) \ge \frac{x^n}{2} \ge \frac{b^n}{2} \ge f(0).$$

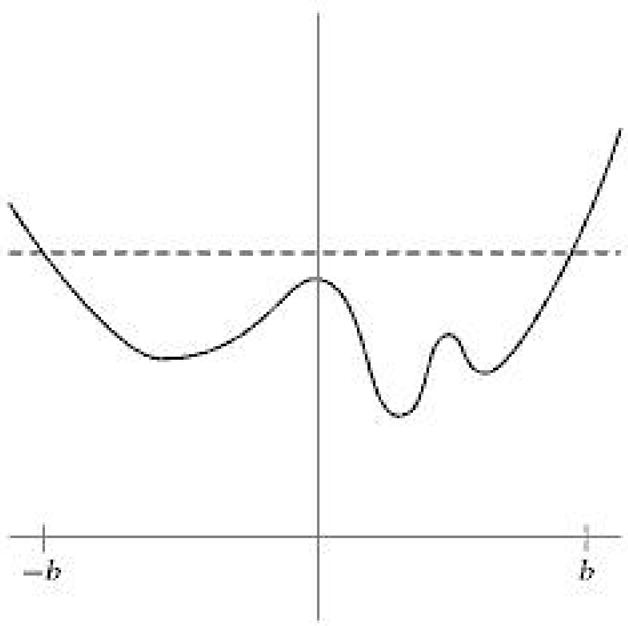

FIGURE 9

Similarly, if $x \le -b$, then

$$f(x) \ge \frac{x^n}{2} \ge \frac{(-b)^n}{2} = \frac{b^n}{2} \ge f(0).$$

Summarizing:

$$\text{if } x \ge b \text{ or } x \le -b, \text{ then } f(x) \ge f(0).$$

Now apply Theorem 7 to the function f on the interval $[-b, b]$. We conclude that there is a number y such that

$$(1) \qquad \text{if } -b \le x \le b, \text{ then } f(y) \le f(x).$$

In particular, $f(y) \le f(0)$. Thus

$$(2) \qquad \text{if } x \le -b \text{ or } x \ge b, \text{ then } f(x) \ge f(0) \ge f(y).$$

Combining (1) and (2) we see that $f(y) \le f(x)$ for all x. ▮

Theorem 10 now allows us to prove the following result.

THEOREM 11 Consider the equation

$$(*) \qquad x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c,$$

(b) Again assume that f is continuous and that $f(x) = 0$ only for $x = a$, but suppose, instead, that $f(x) > 0$ for some $x > a$ and $f(x) < 0$ for some $x < a$. Now what can be said about $f(x)$ for $x \neq a$?

*(c) Discuss the sign of $x^3 + x^2y + xy^2 + y^3$ when x and y are not both 0.

10. Suppose f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some x in $[a, b]$. (If your proof isn't very short, it's not the right one.)

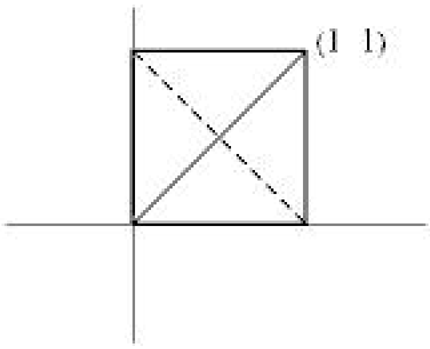

FIGURE 14

11. Suppose that f is a continuous function on $[0, 1]$ and that $f(x)$ is in $[0, 1]$ for each x (draw a picture). Prove that $f(x) = x$ for some number x.

12. (a) Problem 11 shows that f intersects the diagonal of the square in Figure 14 (solid line). Show that f must also intersect the other (dashed) diagonal.

(b) Prove the following more general fact: If g is continuous on $[0, 1]$ and $g(0) = 0$, $g(1) = 1$ or $g(0) = 1$, $g(1) = 0$, then $f(x) = g(x)$ for some x.

13. (a) Let $f(x) = \sin 1/x$ for $x \neq 0$ and let $f(0) = 0$. Is f continuous on $[-1, 1]$? Show that f satisfies the conclusion of the Intermediate Value Theorem on $[-1, 1]$; in other words, if f takes on two values somewhere on $[-1, 1]$, it also takes on every value in between.

*(b) Suppose that f satisfies the conclusion of the Intermediate Value Theorem, and that f takes on each value *only once*. Prove that f is continuous.

*(c) Generalize to the case where f takes on each value only finitely many times.

14. If f is a continuous function on $[0, 1]$, let $\|f\|$ be the maximum value of $|f|$ on $[0, 1]$.

(a) Prove that for any number c we have $\|cf\| = |c| \cdot \|f\|$.

*(b) Prove that $\|f + g\| \leq \|f\| + \|g\|$. Give an example where $\|f + g\| \neq \|f\| + \|g\|$.

(c) Prove that $\|h - f\| \leq \|h - g\| + \|g - f\|$.

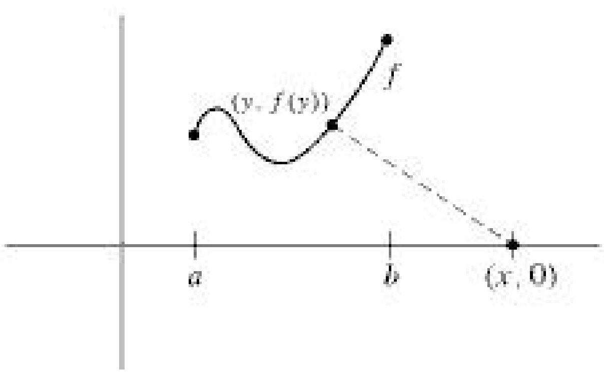

FIGURE 15

***15.** Suppose that ϕ is continuous and $\lim_{x \to \infty} \phi(x)/x^n = 0 = \lim_{x \to -\infty} \phi(x)/x^n$.

(a) Prove that if n is odd, then there is a number x such that $x^n + \phi(x) = 0$.

(b) Prove that if n is even, then there is a number y such that $y^n + \phi(y) \leq x^n + \phi(x)$ for all x.

Hint: Of which proofs does this problem test your understanding?

***16.** Let f be any polynomial function. Prove that there is some number y such that $|f(y)| \leq |f(x)|$ for all x.

***17.** Suppose that f is a continuous function with $f(x) > 0$ for all x, and $\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$. (Draw a picture.) Prove that there is some number y such that $f(y) \geq f(x)$ for all x.

our assertion is true—and very important, definitely important enough to warrant consideration of details. We are finally ready to state the last property of the real numbers which we need.

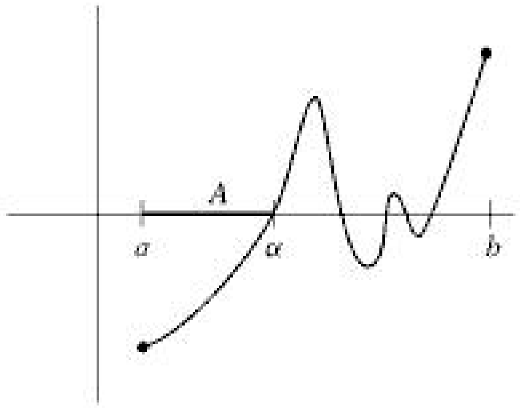

FIGURE 1

(P13) (The least upper bound property) If A is a set of real numbers, $A \neq \emptyset$, and A is bounded above, then A has a least upper bound.

Property P13 may strike you as anticlimactic, but that is actually one of its virtues. To complete our list of basic properties for the real numbers we require no particularly abstruse proposition, but only a property so simple that we might feel foolish for having overlooked it. Of course, the least upper bound property is not really so innocent as all that; after all, it does *not* hold for the rational numbers $\mathbf{Q}$. For example, if A is the set of all rational numbers x satisfying $x^2 < 2$, then there is no *rational* number y which is an upper bound for A and which is less than or equal to every other *rational* number which is an upper bond for A. It will become clear only gradually how significant P13 is, but we are already in a position to demonstrate its power, by supplying the proofs which were omitted in Chapter 7.

THEOREM 7-1 If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some number x in $[a, b]$ such that $f(x) = 0$.

PROOF Our proof is merely a rigorous version of the outline developed at the end of Chapter 7—we will locate the smallest number x in $[a, b]$ with $f(x) = 0$.

Define the set A, shown in Figure 1, as follows:

$$A = \{x : a \le x \le b, \text{ and } f \text{ is negative on the interval } [a, x]\}.$$

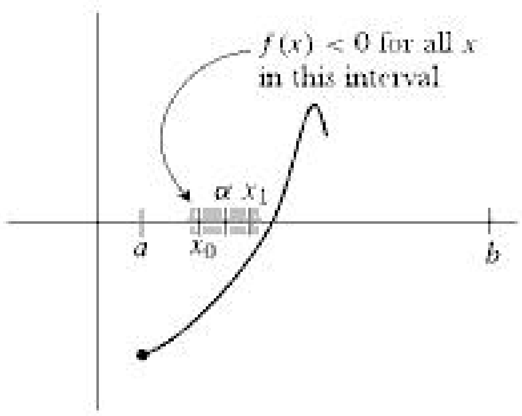

FIGURE 2

Clearly $A \neq \emptyset$, since a is in A; in fact, there is some $\delta > 0$ such that A contains all points x satisfying $a \le x < a + \delta$; this follows from Problem 6-15, since f is continuous on $[a, b]$ and $f(a) < 0$. Similarly, b is an upper bound for A and, in fact, there is a $\delta > 0$ such that all points x satisfying $b - \delta < x \le b$ are upper bounds for A; this also follows from Problem 6-15, since $f(b) > 0$.

From these remarks it follows that A has a least upper bound α and that $a < \alpha < b$. We now wish to show that $f(\alpha) = 0$, by eliminating the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

Suppose first that $f(\alpha) < 0$. By Theorem 6-3, there is a $\delta > 0$ such that $f(x) < 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 2). Now there is some number x_0 in A which satisfies $\alpha - \delta < x_0 < \alpha$ (because otherwise α would not be the *least* upper bound of A). This means that f is negative on the whole interval $[a, x_0]$. But if x_1 is a number between α and $\alpha + \delta$, then f is also negative on the whole interval $[x_0, x_1]$. Therefore f is negative on the interval $[a, x_1]$, so x_1 is in A. But this contradicts the fact that α is an upper bound for A; our original assumption that $f(\alpha) < 0$ must be false.

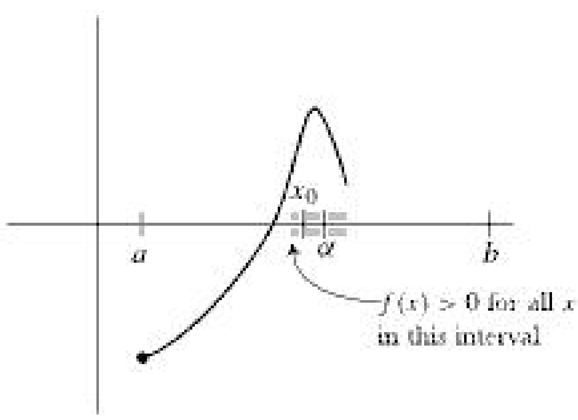

FIGURE 3

Suppose, on the other hand, that $f(\alpha) > 0$. Then there is a number $\delta > 0$ such that $f(x) > 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 3). Once again we know that there is an x_0 in A satisfying $\alpha - \delta < x_0 < \alpha$; but this means that f is negative on $[a, x_0]$,

There is a consequence of Theorem 2 (actually an equivalent formulation) which we have very often assumed implicitly.

THEOREM 3 For any $\varepsilon > 0$ there is a natural number n with $1/n < \varepsilon$.

PROOF Suppose not; then $1/n \geq \varepsilon$ for all n in $\mathbf{N}$. Thus $n \leq 1/\varepsilon$ for all n in $\mathbf{N}$. But this means that $1/\varepsilon$ is an upper bound for $\mathbf{N}$, contradicting Theorem 2. ▌

A brief glance through Chapter 6 will show you that the result of Theorem 3 was used in the discussion of many examples. Of course, Theorem 3 was not available at the time, but the examples were so important that in order to give them some cheating was tolerated. As partial justification for this dishonesty we can claim that this result was never used in the proof of a *theorem*, but if your faith has been shaken, a review of all the proofs given so far is in order. Fortunately, such deception will not be necessary again. We have now stated every property of the real numbers that we will ever need. Henceforth, no more lies.

PROBLEMS

1. Find the least upper bound and the greatest lower bound (if they exist) of the following sets. Also decide which sets have greatest and least elements (i.e., decide when the least upper bound and greatest lower bound happens to belong to the set).

(i) $\left\{\frac{1}{n} : n \text{ in } \mathbf{N}\right\}$.

(ii) $\left\{\frac{1}{n} : n \text{ in } \mathbf{Z} \text{ and } n \neq 0.\right\}$.

(iii) $\{x : x = 0 \text{ or } x = 1/n \text{ for some } n \text{ in } \mathbf{N}\}$.

(iv) $\{x : 0 \leq x \leq \sqrt{2} \text{ and } x \text{ is rational}\}$.

(v) $\{x : x^2 + x + 1 \geq 0\}$.

(vi) $\{x : x^2 + x - 1 < 0\}$.

(vii) $\{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$.

(viii) $\left\{\frac{1}{n} + (-1)^n : n \text{ in } \mathbf{N}\right\}$.

2. (a) Suppose $A \neq \emptyset$ is bounded below. Let $-A$ denote the set of all $-x$ for x in A. Prove that $-A \neq \emptyset$, that $-A$ is bounded above, and that $-\sup(-A)$ is the greatest lower bound of A.

(b) If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A. Show that $B \neq \emptyset$, that B is bounded above, and that $\sup B$ is the greatest lower bound of A.

3. Let f be a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$.

(a) The proof of Theorem 1 showed that there is a smallest x in $[a, b]$ with $f(x) = 0$. Is there necessarily a second smallest x in $[a, b]$ with

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(c) It follows from part (b) that inf B exists; this number is called the **limit superior** of A, and denoted by $\overline{\lim}\, A$ or lim sup A. Find $\overline{\lim}\, A$ for each set A in Problem 1.

(d) Define $\underline{\lim}\, A$, and find it for all A in Problem 1.

***19.** If A is a bounded infinite set prove

(a) $\underline{\lim}\, A \le \overline{\lim}\, A$.

(b) $\overline{\lim}\, A \le \sup A$.

(c) If $\overline{\lim}\, A < \sup A$, then A contains a largest element.

(d) The analogues of parts (b) and (c) for $\underline{\lim}$.

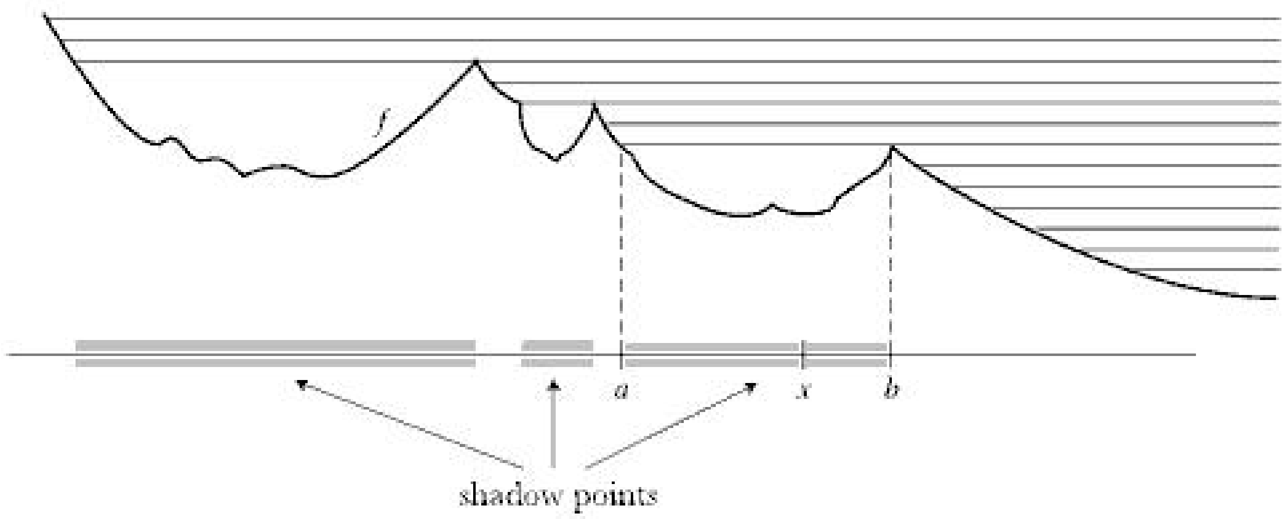

FIGURE 9

***20.** Let f be a continuous function on $\mathbf{R}$. A point x is called a **shadow point** of f if there is a number $y > x$ with $f(y) > f(x)$. The rationale for this terminology is indicated in Figure 9; the parallel lines are the rays of the sun rising in the east (you are facing north). Suppose that all points of (a, b) are shadow points, but that a and b are not shadow points.

(a) For x in (a, b), prove that $f(x) \le f(b)$. Hint: Let $A = \{y : x \le y \le b$ and $f(x) \le f(y)\}$. If $\sup A$ were less than b, then $\sup A$ would be a shadow point. Use this fact to obtain a contradiction to the fact that b is not a shadow point.

(b) Now prove that $f(a) \le f(b)$. (This is a simple consequence of continuity.)

(c) Finally, using the fact that a is not a shadow point, prove that $f(a) = f(b)$.

This result is known as the Rising Sun Lemma. Aside from serving as a good illustration of the use of least upper bounds, it is instrumental in proving several beautiful theorems that do not appear in this book; see page 443.

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PART 3
DERIVATIVES AND INTEGRALS

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calculations, by considering a few simple examples. Simplest of all is a constant function, $f(x) = c$. In this case

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

Thus f is differentiable at a for every number a, and $f'(a) = 0$. This means that the tangent line to the graph of f always has slope 0, so the tangent line always coincides with the graph.

Constant functions are not the only ones whose graphs coincide with their tangent lines—this happens for any linear function $f(x) = cx + d$. Indeed

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{c(a+h) + d - [ca+d]}{h} \\ &= \lim_{h\to 0} \frac{ch}{h} = c; \end{aligned}$$

the slope of the tangent line is c, the same as the slope of the graph of f.

A refreshing difference occurs for $f(x) = x^2$. Here

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= \lim_{h\to 0} 2a + h \\ &= 2a. \end{aligned}$$

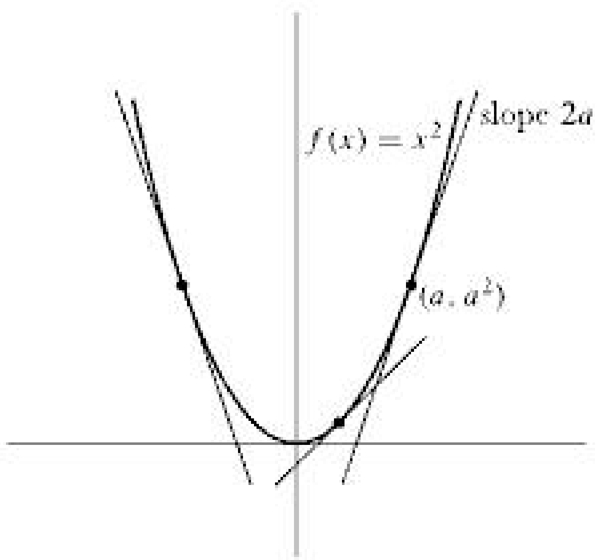

FIGURE 9

Some of the tangent lines to the graph of f are shown in Figure 9. In this picture each tangent line appears to intersect the graph only once, and this fact can be checked fairly easily: Since the tangent line through (a, a^2) has slope $2a$, it is the graph of the function

$$\begin{aligned} g(x) &= 2a(x-a) + a^2 \\ &= 2ax - a^2. \end{aligned}$$

Now, if the graphs of f and g intersect at a point $(x, f(x)) = (x, g(x))$, then

$$\begin{aligned} & x^2 = 2ax - a^2 \\ \text{or} \quad & x^2 - 2ax + a^2 = 0; \\ \text{so} \quad & (x-a)^2 = 0 \\ \text{or} \quad & x = a. \end{aligned}$$

In other words, (a, a^2) is the only point of intersection.

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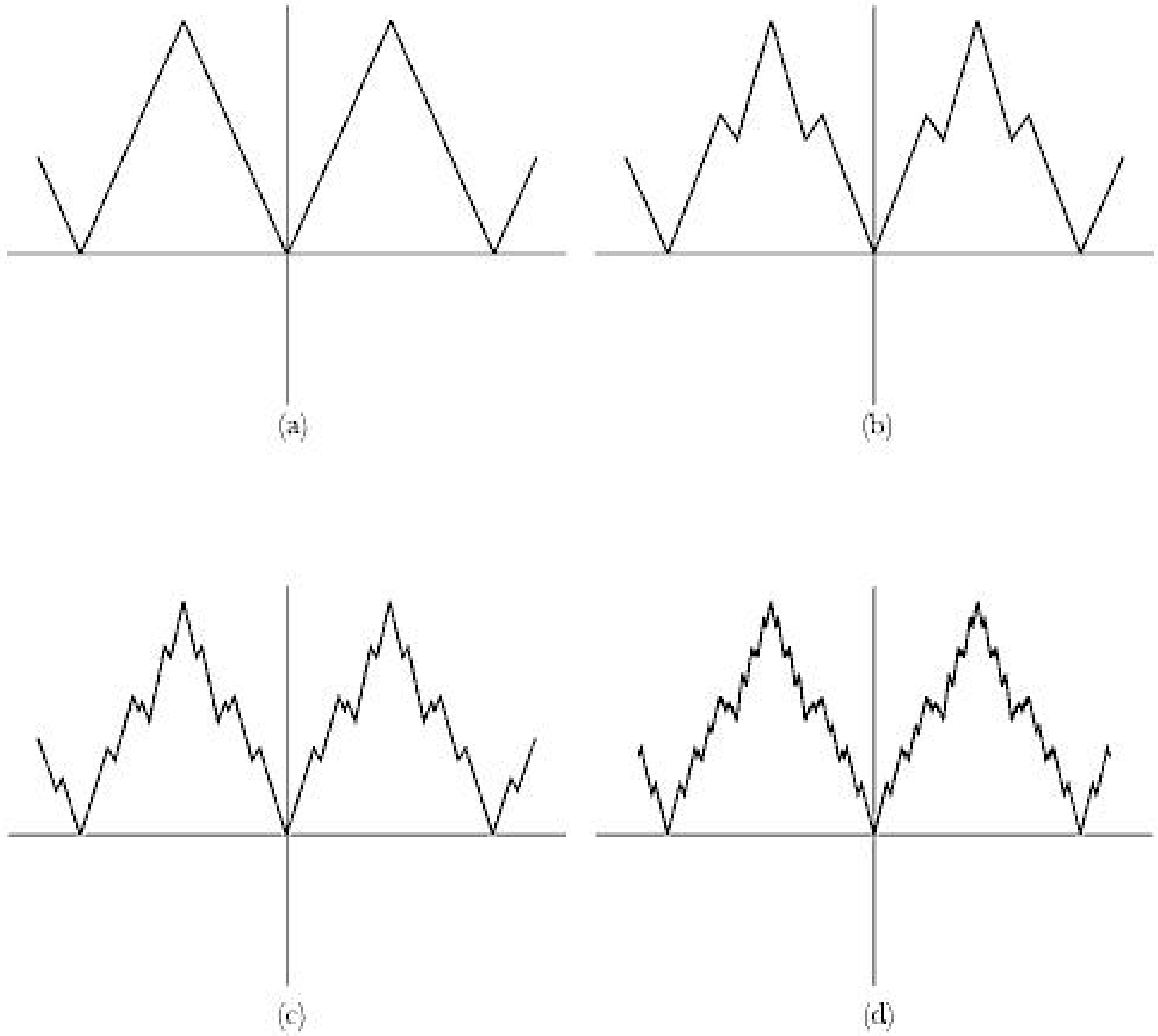

FIGURE 16

everywhere and *differentiable nowhere!* Unfortunately, the definition of this function will be inaccessible to us until Chapter 24, and I have been unable to persuade the artist to draw it (consider carefully what the graph should look like and you will sympathize with her point of view). It is possible to draw some rough approximations to the graph, however; several successively better approximations are shown in Figure 16.

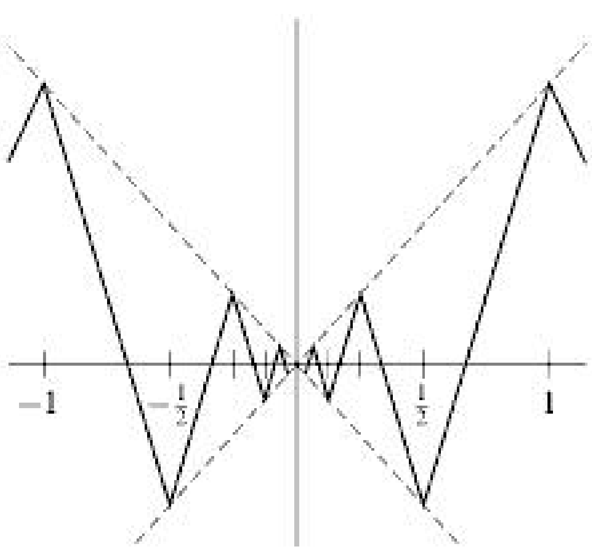

FIGURE 17

Although such spectacular examples of nondifferentiability must be postponed, we can, with a little ingenuity, find a continuous function which is not differentiable at infinitely many points, *all of which are in* $[0, 1]$. One such function is illustrated in Figure 17. The reader is given the problem of defining it precisely; it is a straight line version of the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

This particular function f is itself quite sensitive to the question of differentiability. Indeed, for $h \neq 0$ we have

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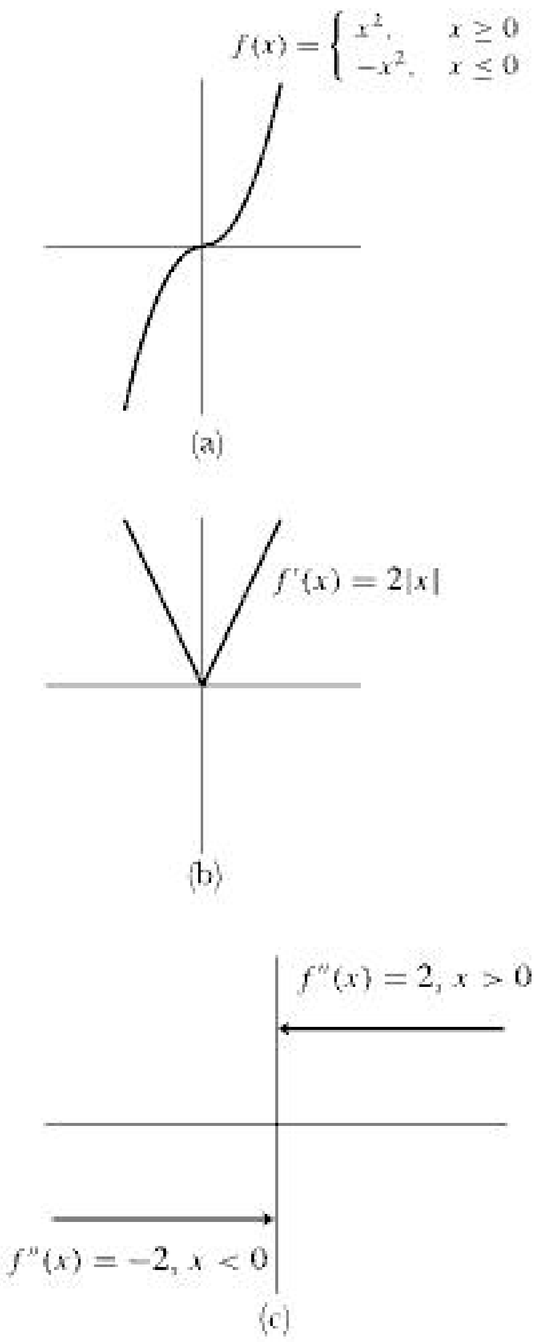

FIGURE 21

It follows that $f''(0)$ does not exist! Existence of the second derivative is thus a rather strong criterion for a function to satisfy. Even a "smooth looking" function like f reveals some irregularity when examined with the second derivative. This suggests that the irregular behavior of the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

might also be revealed by the second derivative. At the moment we know that $g'(0) = 0$, but we do not know $g'(a)$ for any $a \neq 0$, so it is hopeless to begin computing $g''(0)$. We will return to this question at the end of the next chapter, after we have perfected the technique of finding derivatives.

PROBLEMS

1. (a) Prove, working directly from the definition, that if $f(x) = 1/x$, then $f'(a) = -1/a^2$, for $a \neq 0$.
(b) Prove that the tangent line to the graph of f at $(a, 1/a)$ does not intersect the graph of f, except at $(a, 1/a)$.

2. (a) Prove that if $f(x) = 1/x^2$, then $f'(a) = -2/a^3$ for $a \neq 0$.
(b) Prove that the tangent line to f at $(a, 1/a^2)$ intersects f at one other point, which lies on the opposite side of the vertical axis.

3. Prove that if $f(x) = \sqrt{x}$, then $f'(a) = 1/(2\sqrt{a})$, for $a > 0$. (The expression you obtain for $[f(a+h) - f(a)]/h$ will require some algebraic face lifting, but the answer should suggest the right trick.)

4. For each natural number n, let $S_n(x) = x^n$. Remembering that $S_1{}'(x) = 1$, $S_2{}'(x) = 2x$, and $S_3{}'(x) = 3x^2$, conjecture a formula for $S_n{}'(x)$. Prove your conjecture. (The expression $(x+h)^n$ may be expanded by the binomial theorem.)

5. Find f' if $f(x) = [x]$.

6. Prove, starting from the definition (and drawing a picture to illustrate):
(a) if $g(x) = f(x) + c$, then $g'(x) = f'(x)$;
(b) if $g(x) = cf(x)$, then $g'(x) = cf'(x)$.

7. Suppose that $f(x) = x^3$.
(a) What is $f'(9)$, $f'(25)$, $f'(36)$?
(b) What is $f'(3^2)$, $f'(5^2)$, $f'(6^2)$?
(c) What is $f'(a^2)$, $f'(x^2)$?

If you do not find this problem silly, you are missing a very important point: $f'(x^2)$ means the derivative of f at the number which we happen to be calling x^2; it is *not* the derivative at x of the function $g(x) = f(x^2)$. Just to drive the point home:

(d) For $f(x) = x^3$, compare $f'(x^2)$ and $g'(x)$ where $g(x) = f(x^2)$.

30. Interpret the following specimens of Leibnizian notation; each is a restatement of some fact occurring in a previous problem.

(i) $$\frac{dx^n}{dx} = nx^{n-1}$$

(ii) $$\frac{dz}{dy} = -\frac{1}{y^2} \quad \text{if} \quad z = \frac{1}{y}.$$

(iii) $$\frac{d[f(x)+c]}{dx} = \frac{df(x)}{dx}.$$

(iv) $$\frac{d[cf(x)]}{dx} = c\frac{df(x)}{dx}.$$

(v) $$\frac{dz}{dx} = \frac{dy}{dx} \quad \text{if} \quad z = y + c.$$

(vi) $$\left.\frac{dx^3}{dx}\right|_{x=a^2} = 3a^4.$$

(vii) $$\left.\frac{df(x+a)}{dx}\right|_{x=b} = \left.\frac{df(x)}{dx}\right|_{x=b+a}.$$

(viii) $$\left.\frac{df(cx)}{dx}\right|_{x=b} = c \cdot \left.\frac{df(x)}{dx}\right|_{x=cb}.$$

(ix) $$\frac{df(cx)}{dx} = c \cdot \left.\frac{df(y)}{dy}\right|_{y=cx}.$$

(x) $$\frac{d^k x^n}{dx^k} = k!\binom{n}{k}x^{n-k}.$$

You might even try to prove (by induction) the general formula:

$$(f_1 \cdot \cdots \cdot f_n)'(x) = \sum_{i=1}^{n} f_1(x) \cdot \cdots \cdot f_{i-1}(x) f_i'(x) f_{i+1}(x) \cdot \cdots \cdot f_n(x).$$

Differentiating the most interesting functions obviously requires a formula for $(f \circ g)'(x)$ in terms of f' and g'. To ensure that $f \circ g$ be differentiable at a, one reasonable hypothesis would seem to be that g be differentiable at a. Since the behavior of $f \circ g$ near a depends on the behavior of f near $g(a)$ (not near a), it also seems reasonable to assume that f is differentiable at $g(a)$. Indeed we shall prove that if g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

This extremely important formula is called the *Chain Rule*, presumable because a composition of functions might be called a "chain" of functions. Notice that $(f \circ g)'$ is practically the product of f' and g', but not quite: f' must be evaluated at $g(a)$ and g' at a. Before attempting to prove this theorem we will try a few applications. Suppose

$$f(x) = \sin x^2.$$

Let us, temporarily, use S to denote the ("squaring") function $S(x) = x^2$. Then

$$f = \sin \circ S.$$

Therefore we have

$$\begin{aligned} f'(x) &= \sin'(S(x)) \cdot S'(x) \\ &= \cos x^2 \cdot 2x. \end{aligned}$$

Quite a different result is obtained if

$$f(x) = \sin^2 x.$$

In this case

$$f = S \circ \sin,$$

so

$$\begin{aligned} f'(x) &= S'(\sin x) \cdot \sin'(x) \\ &= 2 \sin x \cdot \cos x. \end{aligned}$$

Notice that this agrees (as it should) with the result obtained by writing $f = \sin \cdot \sin$ and using the product formula.

Although we have invented a special symbol, S, to name the "squaring" function, it does not take much practice to do problems like this without bothering to write down special symbols for functions, and without even bothering to write down the particular composition which f is—one soon becomes accustomed to taking f apart in one's head. The following differentiations may be used as practice for such mental gymnastics—if you find it necessary to work a few out on paper, by all means do so, but try to develop the knack of writing f' immediately after seeing

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function, $g(x) = c$. Then $g(a+h) - g(a) = 0$ for all h. In this case, $f \circ g$ is also a constant function, $(f \circ g)(x) = f(c)$, so the Chain Rule does indeed hold:

$$(f \circ g)'(a) = 0 = f'(g(a)) \cdot g'(a).$$

However, there are also nonconstant functions g for which $g(a+h) - g(a) = 0$ for arbitrarily small h. For example, if $a = 0$, the function g might be

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

In this case, $g'(0) = 0$, as we showed in Chapter 9. If the Chain Rule is correct, we must have $(f \circ g)'(0) = 0$ for any differentiable f, and this is not exactly obvious. A proof of the Chain Rule can be found by considering such recalcitrant functions separately, but it is easier simply to abandon this approach, and use a trick.

THEOREM 9 (THE CHAIN RULE) If g is differentiable at a, and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

PROOF Define a function ϕ as follows:

$$\phi(h) = \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}, & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)), & \text{if } g(a+h) - g(a) = 0. \end{cases}$$

It should be intuitively clear that ϕ is continuous at 0: When h is small, $g(a+h) - g(a)$ is also small, so if $g(a+h) - g(a)$ is not zero, then $\phi(h)$ will be close to $f'(g(a))$; and if it is zero, then $\phi(h)$ actually equals $f'(g(a))$, which is even better. Since the continuity of ϕ is the crux of the whole proof we will provide a careful translation of this intuitive argument.

We know that f is differentiable at $g(a)$. This means that

$$\lim_{k \to 0} \frac{f(g(a) + k) - f(g(a))}{k} = f'(g(a)).$$

Thus, if $\varepsilon > 0$ there is some number $\delta' > 0$ such that, for all k,

$$(1) \quad \text{if } 0 < |k| < \delta', \text{ then } \left| \frac{f(g(a)+k) - f(g(a))}{k} - f'(g(a)) \right| < \varepsilon.$$

Now g is differentiable at a, hence continuous at a, so there is a $\delta > 0$ such that, for all h,

$$(2) \quad \text{if } |h| < \delta, \text{ then } |g(a+h) - g(a)| < \delta'.$$

Consider now any h with $|h| < \delta$. If $k = g(a+h) - g(a) \neq 0$, then

$$\phi(h) = \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} = \frac{f(g(a)+k) - f(g(a))}{k};$$

it follows from (2) that $|k| < \delta'$, and hence from (1) that

$$|\phi(h) - f'(g(a))| < \varepsilon.$$

(xv) $f(x) = \dfrac{\sin x^2 \sin^2 x}{1 + \sin x}$.

(xvi) $f(x) = \dfrac{1}{x - \dfrac{2}{x + \sin x}}$.

(xvii) $f(x) = \sin\left(\dfrac{x^3}{\sin\left(\dfrac{x^3}{\sin x}\right)}\right)$.

(xviii) $f(x) = \sin\left(\dfrac{x}{x - \sin\left(\dfrac{x}{x - \sin x}\right)}\right)$.

3. Find the derivatives of the functions tan, cotan, sec, cosec. (You don't have to memorize these formulas, although they will be needed once in a while; if you express your answers in the right way, they will be simple and somewhat symmetrical.)

4. For each of the following functions f, find $f'(f(x))$ (*not* $(f \circ f)'(x)$).

(i) $f(x) = \dfrac{1}{1+x}$.
(ii) $f(x) = \sin x$.
(iii) $f(x) = x^2$.
(iv) $f(x) = 17$.

5. For each of the following functions f, find $f(f'(x))$.

(i) $f(x) = \dfrac{1}{x}$.
(ii) $f(x) = x^2$.
(iii) $f(x) = 17$.
(iv) $f(x) = 17x$.

6. Find f' in terms of g' if

(i) $f(x) = g(x + g(a))$.
(ii) $f(x) = g(x \cdot g(a))$.
(iii) $f(x) = g(x + g(x))$.
(iv) $f(x) = g(x)(x - a)$.
(v) $f(x) = g(a)(x - a)$.
(vi) $f(x + 3) = g(x^2)$.

7. (a) A circular object is increasing in size in some unspecified manner, but it is known that when the radius is 6, the rate of change of the radius is 4. Find the rate of change of the area when the radius is 6. (If $r(t)$ and $A(t)$ represent the radius and the area at time t, then the functions r and A satisfy $A = \pi r^2$; a straightforward use of the Chain Rule is called for.)

***27.** Suppose f is differentiable at 0, and that $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g which is continuous at 0. Hint: What happens if you try to write $g(x) = f(x)/x$?

28. If $f(x) = x^{-n}$ for n in $\mathbf{N}$, prove that

$$\begin{aligned} f^{(k)}(x) &= (-1)^k \frac{(n+k-1)!}{(n-1)!} x^{-n-k} \\ &= (-1)^k k! \binom{n+k-1}{k-1} x^{-n-k}, \qquad \text{for } x \neq 0. \end{aligned}$$

***29.** Prove that it is impossible to write $x = f(x)g(x)$ where f and g are differentiable and $f(0) = g(0) = 0$. Hint: Differentiate.

30. What is $f^{(k)}(x)$ if

(a) $f(x) = 1/(x-a)^n$?
*(b) $f(x) = 1/(x^2 - 1)$?

***31.** Let $f(x) = x^{2n} \sin 1/x$ if $x \neq 0$, and let $f(0) = 0$. Prove that $f'(0), \dots, f^{(n)}(0)$ exist, and that $f^{(n)}$ is not continuous at 0. (You will encounter the same basic difficulty as that in Problem 19.)

***32.** Let $f(x) = x^{2n+1} \sin 1/x$ if $x \neq 0$, and let $f(0) = 0$. Prove that $f'(0), \dots, f^{(n)}(0)$ exist, that $f^{(n)}$ is continuous at 0, and that $f^{(n)}$ is not differentiable at 0.

33. In Leibnizian notation the Chain Rule ought to read:

$$\frac{df(g(x))}{dx} = \left.\frac{df(y)}{dy}\right|_{y=g(x)} \cdot \frac{dg(x)}{dx}.$$

Instead, one usually finds the following statement: "Let $y = g(x)$ and $z = f(y)$. Then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}."$$

Notice that the z in dz/dx denotes the composite function $f \circ g$, while the z in dz/dy denotes the function f; it is also understood that dz/dy will be "an expression involving y," and that in the final answer $g(x)$ must be substituted for y. In each of the following cases, find dz/dx by using this formula; then compare with Problem 1.

(i) $z = \sin y, \quad y = x + x^2.$
(ii) $z = \sin y, \quad y = \cos x.$
(iii) $z = \cos u, \quad u = \sin x.$
(iv) $z = \sin v, \quad v = \cos u, \quad u = \sin x.$

Suppose we wish to find the maximum and minimum value of the function

$$f(x) = x^3 - x$$

on the interval $[-1, 2]$. To begin with, we have

$$f'(x) = 3x^2 - 1,$$

so $f'(x) = 0$ when $3x^2 - 1 = 0$, that is, when

$$x = \sqrt{1/3} \quad \text{or} \quad -\sqrt{1/3}.$$

The numbers $\sqrt{1/3}$ and $-\sqrt{1/3}$ both lie in $[-1, 2]$, so the first group of candidates for the location of the maximum and the minimum is

$$(1) \quad \sqrt{1/3}, \ -\sqrt{1/3}.$$

The second group contains the end points of the interval,

$$(2) \quad -1, \ 2.$$

The third group is empty, since f is differentiable everywhere. The final step is to compute

$$\begin{aligned}
f(\sqrt{1/3}) &= (\sqrt{1/3})^3 - \sqrt{1/3} = \tfrac{1}{3}\sqrt{1/3} - \sqrt{1/3} = -\tfrac{2}{3}\sqrt{1/3},\\
f(-\sqrt{1/3}) &= (-\sqrt{1/3})^3 - (-\sqrt{1/3}) = -\tfrac{1}{3}\sqrt{1/3} + \sqrt{1/3} = \tfrac{2}{3}\sqrt{1/3},\\
f(-1) &= 0,\\
f(2) &= 6.
\end{aligned}$$

Clearly the minimum value is $-\frac{2}{3}\sqrt{1/3}$, occurring at $\sqrt{1/3}$, and the maximum value is 6, occurring at 2.

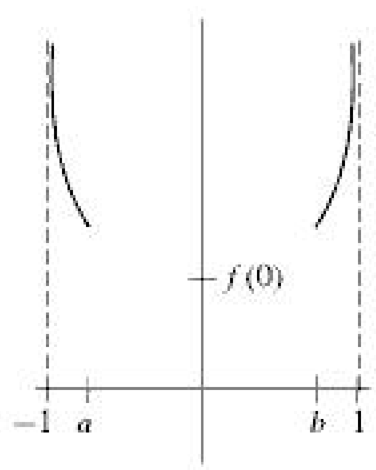

FIGURE 5

This sort of procedure, if feasible, will always locate the maximum and minimum value of a continuous function on a closed interval. If the function we are dealing with is not continuous, however, or if we are seeking the maximum or minimum on an open interval or the whole line, then we cannot even be sure beforehand that the maximum and minimum values exist, so all the information obtained by this procedure may say nothing. Nevertheless, a little ingenuity will often reveal the nature of things. In Chapter 7 we solved just such a problem when we showed that if n is even, then the function

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has a minimum value on the whole line. This proves that the minimum value must occur at some number x satisfying

$$0 = f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

If we can solve this equation, and compare the values of $f(x)$ for such x, we can actually find the minimum of f. One more example may be helpful. Suppose we wish to find the maximum and minimum, if they exist, of the function

$$f(x) = \frac{1}{1-x^2}$$

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result already mentioned in Problem 3-7: If

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

then f has at most n "roots," i.e., there are at most n numbers x such that $f(x) = 0$. Although this is really an algebraic theorem, calculus can be used to give an easy proof. Notice that if x_1 and x_2 are roots of f (Figure 17), so that $f(x_1) = f(x_2) = 0$, then by Rolle's Theorem there is a number x between x_1 and x_2 such that $f'(x) = 0$. This means that if f has k different roots $x_1 < x_2 < \cdots < x_k$, then f' has at least $k - 1$ different roots: one between x_1 and x_2, one between x_2 and x_3, etc. It is now easy to prove by induction that a polynomial function

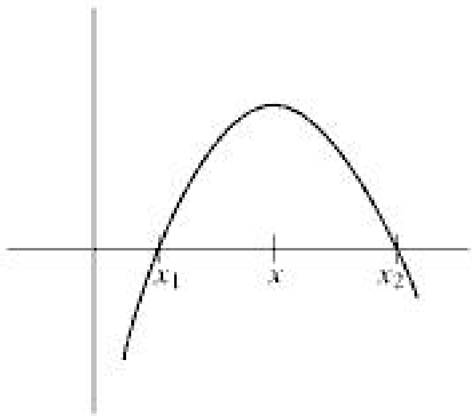

FIGURE 17

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has at most n roots: The statement is surely true for $n = 1$, and if we assume that it is true for n, then the polynomial

$$g(x) = b_{n+1}x^{n+1} + b_n x^n + \cdots + b_0$$

could not have more than $n + 1$ roots, since if it did, g' would have more than n roots.

With this information it is not hard to describe the graph of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

The derivative, being a polynomial function of degree $n - 1$, has at most $n - 1$ roots. Therefore f has at most $n - 1$ critical points. Of course, a critical point is not necessarily a local maximum or minimum point, but at any rate, if a and b are adjacent critical points of f, then f' will remain either positive or negative on (a, b), since f' is continuous; consequently, f will be either increasing or decreasing on (a, b). Thus f has at most n regions of decrease or increase.

As a specific example, consider the function

$$f(x) = x^4 - 2x^2.$$

Since

$$f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1),$$

the critical points of f are -1, 0, and 1, and

$$\begin{aligned} f(-1) &= -1, \\ f(0) &= 0, \\ f(1) &= -1. \end{aligned}$$

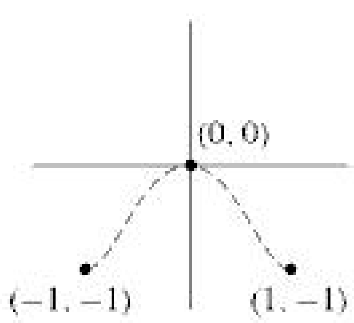

FIGURE 18

The behavior of f on the intervals between the critical points can be determined by one of the methods mentioned before. In particular, we could determine the sign of f' on these intervals simply be examining the formula for $f'(x)$. On the other hand, from the three critical values alone we can see (Figure 18) that f increases on $(-1, 0)$ and decreases on $(0, 1)$. To determine the sign of f' on

Suppose now that $f''(a) > 0$. Then $f'(a+h)/h$ must be positive for sufficiently small h. Therefore:

$$f'(a+h) \text{ must be positive for sufficiently small } h > 0$$
$$\text{and } f'(a+h) \text{ must be negative for sufficiently small } h < 0.$$

This means (Corollary 3) that f is increasing in some interval to the right of a and f is decreasing in some interval to the left of a. Consequently, f has a local minimum at a.

The proof for the case $f''(a) < 0$ is similar. ▌

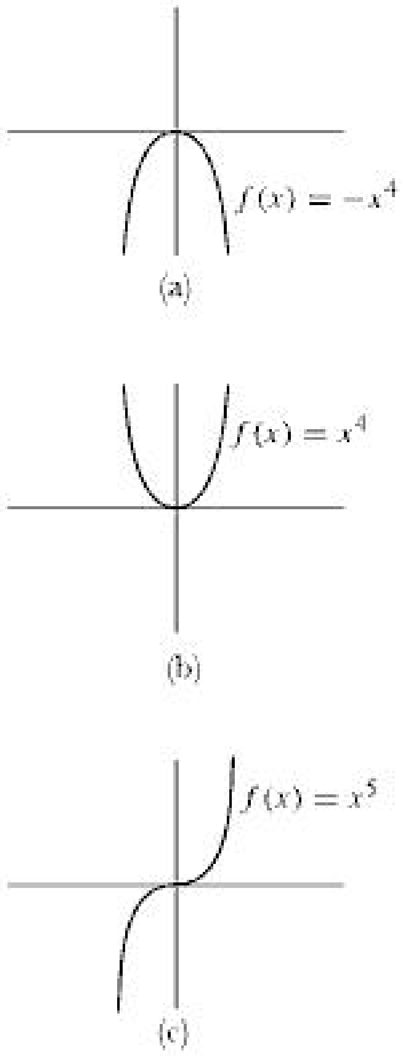

FIGURE 21

Theorem 5 may be applied to the function $f(x) = x^3 - x$, which has already been considered. We have

$$f'(x) = 3x^2 - 1$$
$$f''(x) = 6x.$$

At the critical points, $-\sqrt{1/3}$ and $\sqrt{1/3}$, we have

$$f''(-\sqrt{1/3}) = -6\sqrt{1/3} < 0,$$
$$f''(\sqrt{1/3}) = 6\sqrt{1/3} > 0.$$

Consequently, $-\sqrt{1/3}$ is a local maximum point and $\sqrt{1/3}$ is a local minimum point.

Although Theorem 5 will be found quite useful for polynomial functions, for many functions the second derivative is so complicated that it is easier to consider the sign of the first derivative. Moreover, if a is a critical point of f it may happen that $f''(a) = 0$. In this case, Theorem 5 provides no information: it is possible that a is a local maximum point, a local minimum point, or neither, as shown (Figure 21) by the functions

$$f(x) = -x^4, \quad f(x) = x^4, \quad f(x) = x^5;$$

in each case $f'(0) = f''(0) = 0$, but 0 is a local maximum point for the first, a local minimum point for the second, and neither a local maximum nor minimum point for the third. This point will be pursued further in Part IV.

It is interesting to note that Theorem 5 automatically proves a partial converse of itself.

THEOREM 6 Suppose $f''(a)$ exists. If f has a local minimum at a, then $f''(a) \geq 0$; if f has a local maximum at a, then $f''(a) \leq 0$.

PROOF Suppose f has local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a, by Theorem 5. Thus f would be constant in some interval containing a, so that $f''(a) = 0$, a contradiction. Thus we must have $f''(a) \geq 0$.

The case of a local maximum is handled similarly. ▌

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2. Now sketch the graph of each of the functions in Problem 1, and find all local maximum and minimum points.

3. Sketch the graphs of the following functions.

(i) $f(x) = x + \dfrac{1}{x}$.

(ii) $f(x) = x + \dfrac{3}{x^2}$.

(iii) $f(x) = \dfrac{x^2}{x^2 - 1}$.

(iv) $f(x) = \dfrac{1}{1 + x^2}$.

4. (a) If $a_1 < \cdots < a_n$, find the minimum value of $f(x) = \displaystyle\sum_{i=1}^{n}(x - a_i)^2$.

*(b) Now find the minimum value of $f(x) = \displaystyle\sum_{i=1}^{n}|x - a_i|$. This is a problem where calculus won't help at all: on the intervals between the a_i's the function f is linear, so that the minimum clearly occurs at one of the a_i, and these are precisely the points where f is not differentiable. However, the answer is easy to find if you consider how $f(x)$ changes as you pass from one such interval to another.

*(c) Let $a > 0$. Show that the maximum value of

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - a|}$$

is $(2 + a)/(1 + a)$. (The derivative can be found on each of the intervals $(-\infty, 0)$, $(0, a)$, and (a, ∞) separately.)

5. For each of the following functions, find all local maximum and minimum points.

(i) $f(x) = \begin{cases} x, & x \neq 3, 5, 7, 9 \\ 5, & x = 3 \\ -3, & x = 5 \\ 9, & x = 7 \\ 7, & x = 9. \end{cases}$

(ii) $f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/q, & x = p/q \text{ in lowest terms.} \end{cases}$

(iii) $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$

(iv) $f(x) = \begin{cases} 1, & x = 1/n \text{ for some } n \text{ in } \mathbf{N} \\ 0, & \text{otherwise.} \end{cases}$

(v) $f(x) = \begin{cases} 1, & \text{if the decimal expansion of } x \text{ contains a 5} \\ 0, & \text{otherwise.} \end{cases}$

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30. A cannon ball is shot from the ground with velocity v at an angle α (Figure 31) so that it has a vertical component of velocity $v \sin \alpha$ and a horizontal component $v \cos \alpha$. Its distance $s(t)$ above the ground obeys the law $s(t) = -16t^2 + (v \sin \alpha)t$, while its horizontal velocity remains constantly $v \cos \alpha$.

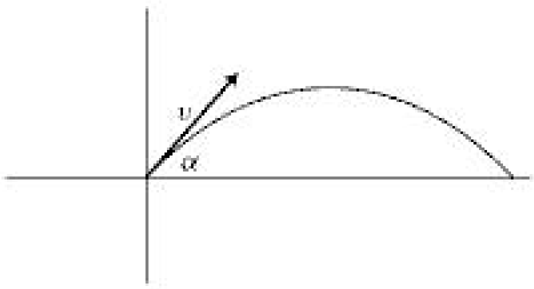

FIGURE 31

(a) Show that the path of the cannon ball is a parabola (find the position at each time t, and show that these points lie on a parabola).

(b) Find the angle α which will maximize the horizontal distance traveled by the cannon ball before striking the ground.

31. (a) Give an example of a function f for which $\lim_{x\to\infty} f(x)$ exists, but $\lim_{x\to\infty} f'(x)$ does not exist.

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

(c) Prove that if $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f''(x)$ exists, then $\lim_{x\to\infty} f''(x) = 0$. (See also Problem 20-15.)

32. Suppose that f and g are two differentiable functions which satisfy $fg' - f'g = 0$. Prove that if a and b are adjacent zeros of f, and $g(a)$ and $g(b)$ are not both 0, then $g(x) = 0$ for some x between a and b. (Naturally the same result holds with f and g interchanged; thus, the zeros of f and g separate each other.) Hint: Derive a contradiction from the assumption that $g(x) \neq 0$ for all x between a and b: if a number is not 0, there is a natural thing to do with it.

33. Suppose that $|f(x) - f(y)| \leq |x - y|^n$ for $n > 1$. Prove that f is constant by considering f'. Compare with Problem 3-20.

34. A function f is *Lipschitz of order* α at x if there is a constant C such that

$$(*) \qquad |f(x) - f(y)| \leq C|x - y|^{\alpha}$$

for all y in an interval around x. The function f is *Lipschitz of order* α *on an interval* if $(*)$ holds for all x and y in the interval.

(a) If f is Lipschitz of order $\alpha > 0$ at x, then f is continuous at x.

(b) If f is Lipschitz of order $\alpha > 0$ on an interval, then f is uniformly continuous on this interval (see Chapter 8, Appendix).

(c) If f is differentiable at x, then f is Lipschitz of order 1 at x. Is the converse true?

(d) If f is differentiable on $[a, b]$, is f Lipschitz of order 1 on $[a, b]$?

(e) If f is Lipschitz of order $\alpha > 1$ on $[a, b]$, then f is constant on $[a, b]$.

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(b) Now write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} \cdot \frac{f(x)}{f(x)-f(a)} \cdot \frac{g(x)-g(a)}{g(x)}$$

(why can we assume that $f(x) - f(a) \neq 0$ for large x?) and conclude that

$$\left| \frac{f(x)}{g(x)} - l \right| < 2\varepsilon \quad \text{for sufficiently large } x.$$

53. To complete the orgy of variations on l'Hôpital's Rule, use Problem 52 to prove a few more cases of the following general statement (there are so many possibilities that you should select just a few, if any, that interest you):

If $\lim_{x\to[\ ]} f(x) = \lim_{x\to[\ ]} g(x) = \{\quad\}$ and $\lim_{x\to[\ ]} f'(x)/g'(x) = (\quad)$, then $\lim_{x\to[\ ]} f(x)/g(x) = (\quad)$. Here [ ] can be a or a^+ or a^- or ∞ or $-\infty$, and $\{\quad\}$ can be 0 or ∞ or $-\infty$, and ( ) can be l or ∞ or $-\infty$.

***54.** (a) Suppose that f is differentiable on $[a, b]$. Prove that if the minimum of f on $[a, b]$ is at a, then $f'(a) \geq 0$, and if it is at b, then $f'(b) \leq 0$. (One half of the proof of Theorem 1 will go through.)

(b) Suppose that $f'(a) < 0$ and $f'(b) > 0$. Show that $f'(x) = 0$ for some x in (a, b). Hint: Consider the minimum of f on $[a, b]$; why must it be somewhere in (a, b)?

(c) Prove that if $f'(a) < c < f'(b)$, then $f'(x) = c$ for some x in (a, b). (This result is known as Darboux's Theorem.) Hint: Cook up an appropriate function to which part (b) may be applied.

55. Suppose that f is differentiable in some interval containing a, but that f' is discontinuous at a.

(a) The one-sided limits $\lim_{x\to a^+} f'(x)$ and $\lim_{x\to a^-} f'(x)$ cannot both exist. (This is just a minor variation on Theorem 7.)

(b) Neither of these one-sided limits can exist even in the sense of being $+\infty$ or $-\infty$. Hint: Use Darboux's Theorem (Problem 54).

***56.** It is easy to find a function f such that $|f|$ is differentiable but f is not. For example, we can choose $f(x) = 1$ for x rational and $f(x) = -1$ for x irrational. In this example f is not even continuous, nor is this a mere coincidence: Prove that if $|f|$ is differentiable at a, and f is continuous at a, then f is also differentiable at a. Hint: It suffices to consider only a with $f(a) = 0$. Why? In this case, what must $|f|'(a)$ be?

***57.** (a) Let $y \neq 0$ and let n be even. Prove that $x^n + y^n = (x+y)^n$ only when $x = 0$. Hint: If $x_0{}^n + y^n = (x_0 + y)^n$, apply Rolle's Theorem to $f(x) = x^n + y^n - (x+y)^n$ on $[0, x_0]$.

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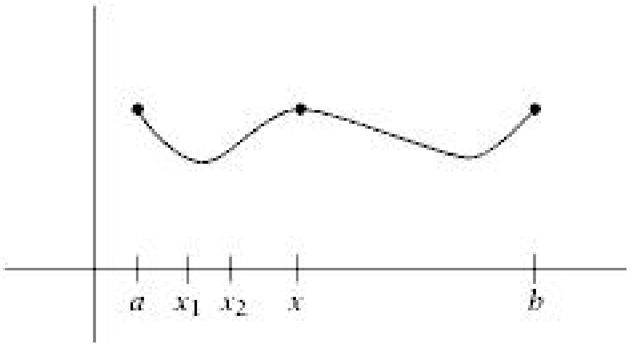

FIGURE 8

contradicting the fact that f' is increasing. This proves that $f(x) \le f(a) = f(b)$ for $a < x < b$, and it only remains to prove that $f(x) = f(a)$ is also impossible for x in (a, b).

Suppose $f(x) = f(a)$ for some x in (a, b). We know that f is not constant on $[a, x]$ (if it were, f' would not be increasing on $[a, x]$) so there is (Figure 8) some x_1 with $a < x_1 < x$ and $f(x_1) < f(a)$. Applying the Mean Value Theorem to $[x_1, x]$ we conclude that there is x_2 with $x_1 < x_2 < x$ and

$$f'(x_2) = \frac{f(x) - f(x_1)}{x - x_1} > 0.$$

On the other hand, $f'(x) = 0$, since a local maximum occurs at x. Again this contradicts the hypothesis that f' is increasing. ∎

We now attack the general case by the same sort of algebraic machinations that we used in the proof of the Mean Value Theorem.

THEOREM 2 If f is differentiable and f' is increasing, then f is convex.

PROOF Let $a < b$. Define g by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

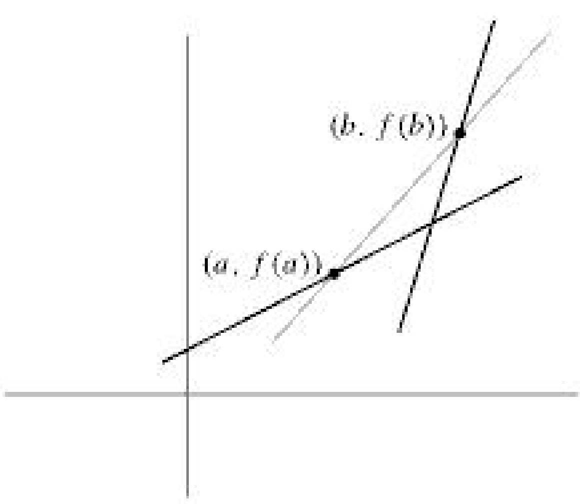

FIGURE 9

It is easy to see that g' is also increasing; moreover, $g(a) = g(b) = f(a)$. Applying the lemma to g we conclude that

$$g(x) < f(a) \quad \text{if} \quad a < x < b.$$

In other words, if $a < x < b$, then

$$f(x) - \frac{f(b) - f(a)}{b - a}(x - a) < f(a)$$

or

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}.$$

Hence f is convex. ∎

THEOREM 3 If f is differentiable and the graph of f lies above each tangent line except at the point of contact, then f is convex.

PROOF Let $a < b$. It is clear from Figure 9 that if $(b, f(b))$ lies above the tangent line at $(a, f(a))$, and $(a, f(a))$ lies above the tangent line at $(b, f(b))$, then the slope of the tangent line at $(b, f(b))$ must be larger than the slope of the tangent line at $(a, f(a))$. The following argument just says this with equations.

Since the tangent line at $(a, f(a))$ is the graph of the function

$$g(x) = f'(a)(x - a) + f(a),$$

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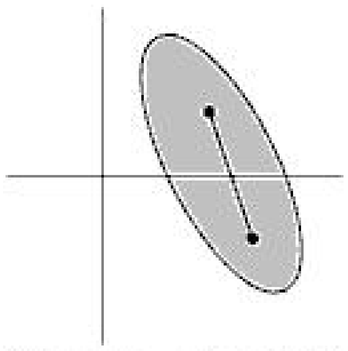
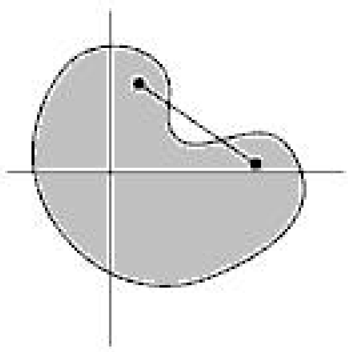


(a) a convex subset of the plane



(b) a non-convex subset of the plane

FIGURE 14

12. Call a function f *weakly convex* on an interval if for $a < b < c$ in this interval we have

$$\frac{f(x) - f(a)}{x - a} \le \frac{f(b) - f(a)}{b - a}.$$

(a) Show that a weakly convex function is convex if and only if its graph contains no straight line segments. (Sometimes a weakly convex function is simply called "convex," while convex functions in our sense are called "strictly convex".)

(b) Reformulate the theorems of this section for weakly convex functions.

13. A set A of points in the plane is called *convex* if A contains the line segment joining any two points in it (Figure 14). For a function f, let A_f be the set of points (x, y) with $y \ge f(x)$, so that A_f is the set of points on or above the graph of f. Show that A is convex if and only if f is weakly convex, in the terminology of the previous problem. Further information on convex sets will be found in reference [19] of the Suggested Reading.

Theorem it would also by 0 somewhere in $[0, 1]$). But $g(0) > 0$ by (i). So also $g(1) > 0$, which means that (i) also holds for a_1, b_1. ▌

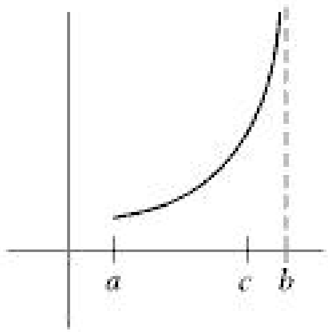


FIGURE 8

Henceforth we shall be concerned almost exclusively with continuous increasing or decreasing functions which are defined on an interval. If f is such a function, it is possible to say quite precisely what the domain of f^{-1} will be like.

Suppose first that f is a continuous increasing function on the closed interval $[a, b]$. Then, by the Intermediate Value Theorem, f takes on every value between $f(a)$ and $f(b)$. Therefore, the domain of f^{-1} is the closed interval $[f(a), f(b)]$. Similarly, if f is continuous and decreasing on $[a, b]$, then the domain of f^{-1} is $[f(b), f(a)]$.

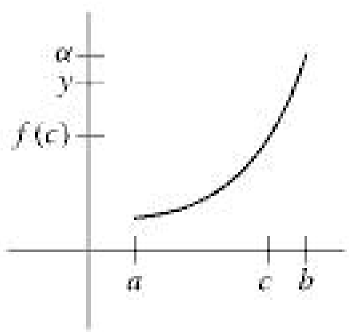


FIGURE 9

If f is a continuous increasing function on an *open* interval (a, b) the analysis becomes a bit more difficult. To begin with, let us choose some point c in (a, b). We will first decide which values $> f(c)$ are taken on by f. One possibility is that f takes on arbitrarily large values (Figure 8). In this case f takes on *all* values $> f(c)$, by the Intermediate Value Theorem. If, on the other hand, f does not take on arbitrarily large values, then $A = \{f(x) : c \leq x < b\}$ is bounded above, so A has a least upper bound α (Figure 9). Now suppose y is any number with $f(c) < y < \alpha$. Then f takes on some value $f(x) > y$ (because α is the least upper bound of A). By the Intermediate Value Theorem, f actually takes on the value y. Notice that f cannot take on the value α itself; for if $\alpha = f(x)$ for $a < x < b$ and we choose t with $x < t < b$, then $f(t) > \alpha$, which is impossible.

Precisely the same arguments work for values less than $f(c)$: either f takes on all values less than $f(c)$ or there is a number $\beta < f(c)$ such that f takes on all values between β and $f(c)$, but not β itself.

This entire argument can be repeated if f is decreasing, and if the domain of f is $\mathbf{R}$ or (a, ∞) or $(-\infty, a)$. Summarizing: if f is a continuous increasing, or decreasing, function whose domain is an interval having one of the forms

$$(a, b), \quad (-\infty, b), \quad (a, \infty), \quad \text{or } \mathbf{R},$$

then the domain of f^{-1} is also an interval which has one of these four forms.

Now that we have completed this preliminary analysis of continuous one-one functions, it is possible to begin asking which important properties of a one-one function are inherited by its inverse. For continuity there is no problem.

THEOREM 3 If f is continuous and one-one on an interval, then f^{-1} is also continuous.

PROOF We know by Theorem 2 that f is either increasing or decreasing. We might as well assume that f is increasing, since we can then take care of the other case by applying the usual trick of considering $-f$.

We must show that

$$\lim_{x \to b} f^{-1}(x) = f^{-1}(b)$$

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By Theorem 5 we have, for $x \neq 0$,

$$\begin{aligned} g_n{}'(x) &= \frac{1}{f_n{}'(f_n{}^{-1}(x))} \\ &= \frac{1}{n(f_n{}^{-1}(x))^{n-1}} \\ &= \frac{1}{n(x^{1/n})^{n-1}} \\ &= \frac{1}{n} \cdot \frac{1}{x^{1-(1/n)}} \\ &= \frac{1}{n} \cdot x^{(1/n)-1}. \end{aligned}$$

Thus, if $f(x) = x^a$, and a is an integer or the reciprocal of a natural number, then $f'(x) = ax^{a-1}$. It is now easy to check that this formula is true if a is any rational number: Let $a = m/n$, where m is an integer, and n is a natural number; if

$$f(x) = x^{m/n} = \left(x^{1/n}\right)^m,$$

then, by the Chain Rule,

$$\begin{aligned} f'(x) &= m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n} \cdot x^{(1/n)-1} \\ &= \frac{m}{n} \cdot x^{[(m/n)-(1/n)]+[(1/n)-1]} \\ &= \frac{m}{n} x^{(m/n)-1}. \end{aligned}$$

Although we now have a formula for $f'(x)$ when $f(x) = x^a$ and a is rational, the treatment of the function $f(x) = x^a$ for irrational a will have to be saved for later—at the moment we do not even know the *meaning* of a symbol like $x^{\sqrt{2}}$. Actually, inverse functions will be involved crucially in the definition of x^a for irrational a. Indeed, in the next few chapters several important functions will be defined in terms of their inverse functions.

PROBLEMS

1. Find f^{-1} for each of the following f.

(i) $f(x) = x^3 + 1$.

(ii) $f(x) = (x-1)^3$.

(iii) $f(x) = \begin{cases} x, & x \text{ rational} \\ -x, & x \text{ irrational.} \end{cases}$

(iv) $f(x) = \begin{cases} -x^2 & x \geq 0 \\ 1 - x^3, & x < 0. \end{cases}$

(v) $f(x) = \begin{cases} x, & x \neq a_1, \dots, a_n \\ a_{i+1} & x = a_i, \quad i = 1, \dots, n-1 \\ a_1, & x = a_n. \end{cases}$

(vi) $f(x) = x + [x]$.

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25. A function f is **nondecreasing** if $f(x) \leq f(y)$ whenever $x < y$. (To be more precise we should stipulate that the domain of f be an interval.) A **nonincreasing** function is defined similarly. Caution: Some writers use "increasing" instead of "nondecreasing," and "strictly increasing" for our "increasing."

(a) Prove that if f is nondecreasing, but not increasing, then f is constant on some interval. (Beware of unintentional puns: "not increasing" is not the same as "nonincreasing.")

(b) Prove that if f is differentiable and nondecreasing, then $f'(x) \geq 0$ for all x.

(c) Prove that if $f'(x) \geq 0$ for all x, then f is nondecreasing.

***26.** (a) Suppose that $f(x) > 0$ for all x, and that f is decreasing. Prove that there is a *continuous* decreasing function g such that $0 < g(x) \leq f(x)$ for all x.

(b) Show that we can even arrange that g will satisfy $\lim_{x \to \infty} g(x)/f(x) = 0$.

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On the first interval $[t_0, t_1]$ the function f has the minimum value m_1 and the maximum value M_1; similarly, on the ith interval $[t_{i-1}, t_i]$ let the minimum value of f be m_i and let the maximum value be M_i. The sum

$$s = m_1(t_1 - t_0) + m_2(t_2 - t_1) + m_3(t_3 - t_2) + m_4(t_4 - t_3)$$

represents the total area of rectangles lying inside the region $R(f, a, b)$, while the sum

$$S = M_1(t_1 - t_0) + M_2(t_2 - t_1) + M_3(t_3 - t_2) + M_4(t_4 - t_3)$$

represents the total area of rectangles containing the region $R(f, a, b)$. The guiding principle of our attempt to define the area A of $R(f, a, b)$ is the observation that A should satisfy

$$s \leq A \quad \text{and} \quad A \leq S,$$

and that this should be true, *no matter how the interval* $[a, b]$ *is subdivided.* It is to be hoped that these requirements will determine A. The following definitions begin to formalize, and eliminate some of the implicit assumptions in, this discussion.

DEFINITION

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

The points in a partition can be numbered $t_0, \dots, t_n$ so that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b;$$

we shall always assume that such a numbering has been assigned.

DEFINITION

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$. Let

$$\begin{aligned} m_i &= \inf\{f(x) : t_{i-1} \leq x \leq t_i\}, \\ M_i &= \sup\{f(x) : t_{i-1} \leq x \leq t_i\}. \end{aligned}$$

The **lower sum** of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The **upper sum** of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

The lower and upper sums correspond to the sums s and S in the previous example; they are supposed to represent the total areas of rectangles lying below and above the graph of f. Notice, however, that despite the geometric motivation, these sums have been defined precisely without any appeal to a concept of "area."

Let P be a partition which contains both P' and P''. Then, according to the lemma,

$$U(f, P) \le U(f, P''),$$
$$L(f, P) \ge L(f, P');$$

consequently,

$$U(f, P) - L(f, P) \le U(f, P'') - L(f, P') < \varepsilon. \blacksquare$$

Although the mechanics of the proof take up a little space, it should be clear that Theorem 2 amounts to nothing more than a restatement of the definition of integrability. Nevertheless, it is a very convenient restatement because there is no mention of sup's and inf's, which are often difficult to work with. The next example illustrates this point, and also serves as a good introduction to the type of reasoning which the complicated definition of the integral necessitates, even in very simple situations.

Let f be defined on $[0, 2]$ by

$$f(x) = \begin{cases} 0, & x \ne 1 \\ 1, & x = 1. \end{cases}$$

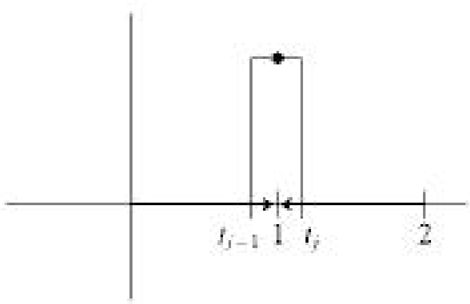

FIGURE 8

Suppose $P = \{t_0, \dots, t_n\}$ is a partition of $[0, 2]$ with

$$t_{j-1} < 1 < t_j$$

(see Figure 8). Then

$$m_i = M_i = 0 \quad \text{if} \quad i \ne j,$$

but

$$m_j = 0 \qquad \text{and} \qquad M_j = 1.$$

Since

$$L(f, P) = \sum_{i=1}^{j-1} m_i(t_i - t_{i-1}) + m_j(t_j - t_{j-1}) + \sum_{i=j+1}^{n} m_i(t_i - t_{i-1}),$$
$$U(f, P) = \sum_{i=1}^{j-1} M_i(t_i - t_{i-1}) + M_j(t_j - t_{j-1}) + \sum_{i=j+1}^{n} M_i(t_i - t_{i-1}),$$

we have

$$U(f, P) - L(f, P) = t_j - t_{j-1}.$$

This certainly shows that f is integrable: to obtain a partition P with

$$U(f, P) - L(f, P) < \varepsilon,$$

it is only necessary to choose a partition with

$$t_{j-1} < 1 < t_j \quad \text{and} \quad t_j - t_{j-1} < \varepsilon.$$

Moreover, it is clear that

$$L(f, P) \le 0 \le U(f, P) \quad \text{for all partitions } P.$$

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