

# **Harmonic Analysis**

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Ajorda Jiao

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# 1 Fourier Series and Integrals

## §1.1 Fourier series

For  $f \in L^1(\mathbb{T})$ , define the **Fourier coefficients**

$$\widehat{f}(k) := \int_0^1 f(x) e^{-2\pi i k x} dx.$$

Let

$$\sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{2\pi i k x}$$

be the **Fourier series** of  $f$ . When we discuss the convergence of Fourier series, we consider two types of sum:

$$S_N f = \sum_{k=-N}^N \widehat{f}(k) e^{2\pi i k x}, \quad \sigma_N f = \frac{1}{N+1} \sum_{k=0}^N S_k f.$$

We concern about the following questions:

**Question 1.1.1.** The pointwise convergence of  $S_N f$ .

**Question 1.1.2.** The  $L^p$  convergence of  $S_N f$ .

**Question 1.1.3.** The almost everywhere convergence of  $S_N f$ .

**Question 1.1.4.** The convergence of  $\sigma_N f$ .

## §1.2 The pointwise convergence

**Definition 1.2.1.** The **Dirichlet kernel**  $D_N$  is given by

$$D_N(t) := \sum_{k=-N}^N e^{2\pi i k t} = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}.$$

It satisfies

$$\int_0^1 D_N(t) dt = 1.$$

### **Theorem 1.2.2** (Dini's Criterion)

For  $x \in \mathbb{T}$ , if  $\exists \delta > 0$ , such that

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty,$$

then  $S_N f(x) \rightarrow f(x)$ .

**Theorem 1.2.3**

If  $f$  is bounded variation on a neighborhood of  $x$ , then

$$S_N f(x) \rightarrow \frac{f(x+) + f(x-)}{2}.$$

**Example 1.2.4**

$f_1(t) = |t|^{-\alpha} \mathbb{1}_{(0,1/2)}$ ,  $f_2(t) = t^\alpha \sin \frac{1}{t} \mathbb{1}_{(0,1/2)}$ , where  $\alpha \in (0, 1)$ .

**Theorem 1.2.5** (Riemann Localization Principle)

If  $f$  is zero in a neighborhood of  $x$ , then  $S_N f(x) \rightarrow 0$ .

**Theorem 1.2.6** (Riemann-Lebesgue)

If  $f \in L^1(\mathbb{T})$ , then  $\widehat{f}(k) \rightarrow 0 (|k| \rightarrow \infty)$ .

**§1.3 Fourier series of continuous functions****Theorem 1.3.1**

There exists  $f \in C(\mathbb{T})$  such that  $S_N f(0)$  diverges.

*Proof.* Consider  $T_N : C(\mathbb{T}) \rightarrow \mathbb{C}, f \mapsto S_N f(0)$ . By theorem 1.3.2, it suffices to show  $\sup \|T_N\| = \infty$ . Suppose  $L_N = \|D_N\|_1$ , we can prove that  $\|T_N\| = L_N$ . Consider the functions  $f_n(t) = \frac{n D_N(t)}{1 + n |D_N(t)|}$  is enough. The statement follows by lemma 1.3.3.  $\square$

**Theorem 1.3.2** (Uniform Boundedness Principle)

$X, Y$ , Banach Spaces.  $\{T_a\}_{a \in A}$  is a family of bounded linear operators from  $X$  to  $Y$ . Then one of the following holds:

1.  $\sup_{a \in A} \|T_a\| < \infty$ .
2.  $\exists x \in X$ , such that  $\sup_{a \in A} \|T_a x\| = \infty$ .

**Lemma 1.3.3**

$$L_N = \frac{4}{\pi^2} \ln N + O(1).$$

## §1.4 Convergence in norm

**Question 1.4.1.** We can ask:

1. Does  $\|S_N f - f\|_p \rightarrow 0$  for  $f \in L^p(\mathbb{T})$ ?
2. Does  $S_N f \rightarrow f$  a.e. for  $f \in L^p(\mathbb{T})$ ?

### Lemma 1.4.2

$S_N f$  convergence to  $f$  in  $L^p$  norm,  $1 \leq p < \infty$ , iff exists  $C_p$  such that

$$\|S_N f\|_p \leq C_p \|f\|_p.$$

## §1.5 Summability method

**Definition 1.5.1.** The **Fejér kernel** is given by

$$F_N(t) := \frac{1}{N+1} \sum_{k=0}^N D_k(t) = \frac{1}{N+1} \left( \frac{\sin(\pi(N+1)t)}{\sin(\pi t)} \right)^2$$

It satisfies

$$\int_0^1 F_N(t) dt = 1 \text{ and } F_N(t) \geq 0.$$

### Theorem 1.5.2

If  $f \in L^p(\mathbb{T})$ ,  $1 \leq p < \infty$ , or  $f \in C(\mathbb{T})$  and  $p = \infty$ , then

$$\|\sigma_N f - f\|_p \rightarrow 0.$$

*Proof.* Applying Minkowski's inequality and it follows by Fejér kernel is a good kernel.  $\square$

### Corollary 1.5.3

The following holds:

1. The trigonometric polynomials  $V = \left\{ \sum_{k=-N}^N c_k e^{2\pi i k x} : c_k \in \mathbb{C}, N \in \mathbb{Z}_+ \right\}$  is dense in  $L^p(\mathbb{T})$ .
2. If  $f \in L^1(\mathbb{T})$  and  $\hat{f}(k) = 0$  for every  $k \in \mathbb{Z}$ , then  $f = 0$  a.e. .

### Theorem 1.5.4

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \text{ and } \|S_N f\|_2 \leq \|f\|_2.$$

Define the **Poisson kernel**

$$P_r(t) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k t} = \frac{1 - r^2}{1 - 2r \cos(2\pi t) + r^2} = \frac{1 - |z|^2}{|1 - z|^2}, \quad z = r e^{2\pi i t}.$$

Let

$$u(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k + \sum_{k=-\infty}^{-1} \widehat{f}(k) \bar{z}^{|k|}$$

be the Poisson sum, then  $u(r e^{2\pi i \theta}) = P_r * f(\theta)$ .

### Theorem 1.5.5

If  $f \in L^p(\mathbb{T})$ ,  $1 \leq p < \infty$ , or  $f \in C(\mathbb{T})$  and  $p = \infty$ , then

$$\|P_r * f - f\|_p \rightarrow 0 (r \rightarrow 1^-).$$

**Remark 1.5.6** —  $\Delta u = 0$  in  $D = \{z \in \mathbb{C} : |z| < 1\}$ , and  $\mathbb{T} \cong \partial D = \mathbb{S}^1$ . If  $f \in C(\mathbb{T})$ , then  $u \in C(\overline{D})$  and  $u = f$  on  $\partial D$ .

**Fact 1.5.7.**  $\sigma_N f \rightarrow f$  a.e. and  $P_r * f \rightarrow f$  a.e. . We will prove these in the next chapter.

## §1.6 The Fourier transform of $L^1$ functions

For  $f \in L^1(\mathbb{R}^n)$ , let

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \xi \cdot x} dx = (\mathcal{F}f)(\xi).$$

### Proposition 1.6.1

The following holds:

1.  $\widehat{\alpha f + \beta g} = \alpha \widehat{f} + \beta \widehat{g}$ .
2.  $\|\widehat{f}\|_{\infty} \leq \|f\|_1$  and  $\widehat{f} \in C(\mathbb{R}^n)$ .
3.  $\lim_{|\xi| \rightarrow \infty} \widehat{f}(\xi) = 0$ .
4.  $\widehat{f * g} = \widehat{f} \widehat{g}$ .
5.  $\widehat{\tau_h f} = \widehat{f}(\xi) e^{2\pi i h \cdot \xi}$  where  $\tau_h f = f(\cdot + h)$ .  $\widehat{f e^{2\pi i h \cdot x}}(\xi) = \widehat{f}(\xi - h)$ .
6.  $\rho \in O_n$ , then  $\widehat{f(\rho \cdot)}(\xi) = \widehat{f}(\rho \xi)$ .
7. If  $g(x) = \lambda^{-n} f(\lambda^{-1} x)$ , then  $\widehat{g}(\xi) = \widehat{f}(\lambda \xi)$  for every  $\lambda > 0$ .
8.  $\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$ , if  $\frac{\partial f}{\partial x_j} \in L^1$ .
9.  $\widehat{(-2\pi i x_j f)}(\xi) = \frac{\partial \widehat{f}}{\partial \xi_j}(\xi)$ , if  $x_j f \in L^1$ .

## §1.7 The Schwartz class and tempered distributions

Define the **Schwartz class**

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : p_{\alpha\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f|, \forall \alpha, \beta \in \mathbb{N}^n \right\}.$$

Then  $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ ,  $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$ . Moreover  $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$  and is dense in  $L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ). The topology on  $\mathcal{S}$  is defined as

$$f_k \rightarrow f \text{ in } \mathcal{S} \iff \lim_{k \rightarrow \infty} p_{\alpha,\beta}(f_k - f) = 0, \forall \alpha, \beta \in \mathbb{N}^n.$$

We can give a family of semi-norms on  $\mathcal{S}(\mathbb{R}^n)$  as

$$\|f\|_{(k)} = \sup \{ p_{\alpha,\beta}(f) : \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq k \}$$

and a quasi-norm on  $\mathcal{S}(\mathbb{R}^n)$  as

$$\|f\|_{(*)} = \sum_{k=0}^{\infty} \min \{ \|f\|_{(k)}, 2^{-k} \}.$$

Let  $d(f, g) := \|f - g\|_{(*)}$ , which makes  $\mathcal{S}$  a metric space  $(\mathcal{S}, d)$  and the topology is identified.

### Theorem 1.7.1

The following holds:

1.  $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$  and  $\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S}$  is continuous.

$$2. \int_{\mathbb{R}^n} f \widehat{g} = \int_{\mathbb{R}^n} \widehat{f} g.$$

### Lemma 1.7.2

If  $f(x) = e^{-\pi|x|^2}$ , then  $\widehat{f}(\xi) = e^{-\pi|\xi|^2}$ .

**Remark 1.7.3** —  $\widehat{e^{-\pi\lambda|x|^2}} = \lambda^{-\frac{n}{2}} e^{-\pi|\xi|^2/\lambda}$  for every  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > 0$ .

### Theorem 1.7.4

The following holds:

1. If  $f \in \mathcal{S}$  (or  $f \in L^1$  and  $\widehat{f} \in L^1$ ), then  $f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$ .

$$2. \forall f, g \in \mathcal{S}, \int_{\mathbb{R}^n} \widehat{f} \widehat{g} = \int_{\mathbb{R}^n} f \bar{g}.$$

*Proof.* For  $f \in \mathcal{S}$ , let  $g_\lambda(x) = e^{-\pi\lambda|x|^2}$ , by DCT and the identity

$$\int_{\mathbb{R}^n} \widehat{f}(x) g(\lambda x) dx = \int_{\mathbb{R}^n} f(\lambda x) \widehat{g}(x) dx.$$

□

Let  $\overline{\mathcal{F}}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{2\pi i\xi \cdot x} dx$ ,  $\sigma f(x) = \tilde{f}(x) = f(-x)$ ,  $Cf(x) = \overline{f(x)}$ . Then  $\overline{\mathcal{F}} = C\mathcal{F}C$ ,  $\overline{\mathcal{F}} = \mathcal{F}^{-1}$ ,  $\mathcal{F}^4 = \text{Id}$ .

**Corollary 1.7.5** (Plancherel)

$$\|f\|_2 = \|\mathcal{F}f\|_2, \forall f \in \mathcal{S}.$$

We define the family of **tempered distributions**  $\mathcal{S}'$  as the continuous linear function on  $\mathcal{S}$ . Then  $T \in \mathcal{S}'$  if and only if  $\exists m \in \mathbb{N}$ , such that  $|\langle T, f \rangle| \leq C \|f\|_{(m)}$  for every  $f \in \mathcal{S}$ . For every  $1 \leq p \leq \infty$ , we have a natural embedding  $j_p : L^p \hookrightarrow \mathcal{S}'$ .

**Definition 1.7.6.**  $\forall T \in \mathcal{S}'$ , define  $\hat{T}(f) = T(\hat{f}), \forall f \in \mathcal{S}$ .

Let  $\mathcal{F}_1 : T \mapsto \hat{T}$ . Then  $\mathcal{F}_1$  maps  $\mathcal{S}'$  to  $\mathcal{S}'$  is continuous. Moreover,  $\mathcal{F}_1 \circ j_1 = j_\infty \circ \mathcal{F}$ .

**Proposition 1.7.7**

If  $T \in \mathcal{S}'$ ,  $\hat{T} \in L^1$ , then  $T(x) = \int_{\mathbb{R}^n} \hat{T}(\xi) e^{2\pi i\xi \cdot x} d\xi$  a.e. .

## §1.8 The Fourier transform on $L^p, 1 < p \leq 2$

**Theorem 1.8.1**

For  $\forall f \in L^2(\mathbb{R}^n)$ , then  $\hat{f} \in L^2$  and  $\|\hat{f}\|_2 = \|f\|_2$ .

**Theorem 1.8.2**

It holds  $\hat{f}(\xi) = \lim_{R \rightarrow \infty} \int_{|x| < R} f(x) e^{-2\pi i\xi \cdot x} dx$ ,  $f(x) = \lim_{R \rightarrow \infty} \int_{|\xi| < R} \hat{f}(\xi) e^{-2\pi i\xi \cdot x} d\xi$ , both convergences is in the sense of  $L^2$  norm.

Because  $\mathcal{F} : L^1 \rightarrow L^\infty, L^2 \rightarrow L^2$ , then by  $L^p \subset L^1 + L^2$  for  $1 < p < 2$ , we have  $\mathcal{F} : L^p \rightarrow L^1 + L^\infty$ .

**Theorem 1.8.3** (Riesz-Thorin Interpolation Theorem)

$p_0, p_1, q_0, q_1 \in [1, \infty], 0 < \theta < 1$ , let  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ . If  $T : L^{p_0} + L^{p_1} \rightarrow L^{q_0} + L^{q_1}$  such that  $\|Tf\|_{q_0} \leq M_0 \|f\|_{p_0}, \|Tf\|_{q_1} \leq M_1 \|f\|_{p_1}$ , then  $\|Tf\|_q \leq M_0^{1-\theta} M_1^\theta \|f\|_p$ .

**Corollary 1.8.4**

If  $f \in L^p, 1 \leq p \leq 2$ , then  $\mathcal{F}f \in L^{p'}$  and  $\|\mathcal{F}f\|_{p'} \leq \|f\|_p$ .



**Corollary 1.8.5**

$f \in L^p, g \in L^q, p, q, r \in [1, \infty]$  with  $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ , then  $\|f * g\|_r \leq \|f\|_p \|g\|_q$ .

**§1.9 The convergence and summability of Fourier integral**

Let  $B_R = R \cdot B$  where  $B$  is a neighborhood of origin.

**Question 1.9.1.**  $f(x) = \lim_{R \rightarrow \infty} \int_{B_R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$ ?

Let  $\widehat{S_R f} = \chi_{B_R} \widehat{f}$ , then  $\lim_{R \rightarrow \infty} \|S_R f - f\|_p = 0$  iff  $\|S_R f\|_p \leq C_p \|f\|_p$ .

**Fact 1.9.2.**  $S_R : L^p \rightarrow L^p$  bounded iff  $n = 1, 1 < p < \infty$  or  $n = 1, p = 2 (B = B(0, 1))$  or  $n > 1, 1 < p < \infty (B = Q(0, 1))$ .

$n = 1, B = (-1, 1)$ , then  $S_R f = D_R * f$ , where  $D_R$  is the Dirichlet kernel

$$D_R(x) = \int_{-R}^R e^{2\pi i \xi \cdot x} d\xi = \frac{\sin(2\pi R x)}{R x}.$$

Then  $D_R \notin L^1$  but  $D_R \in L^q (1 < q \leq \infty)$ .

**Almost everywhere convergence** Now we consider the almost everywhere convergence, an argument (Carleson-Hunt) shows that

$$\left\| \sup_R |S_R f|_p \right\| \leq C_p \|f\|_p \implies \lim_{R \rightarrow \infty} S_R f(x) = f(x) \text{ a.e. }, \forall f \in L^p, 1 < p < \infty.$$

**Cesàro sum** Let  $\sigma_R f(x) = \frac{1}{R} \int_0^R S_t f(x) dt = F_R * f(x)$ , where  $F_R$  is the Fejér kernel

$$F_R(x) = \frac{1}{R} \int_0^R D_t(x) dt = \frac{\sin^2(\pi R x)}{R(\pi x)^2}.$$

Then  $F_R \in L^1$  and  $F_R \geq 0$ . We have  $\lim_{R \rightarrow \infty} \|\sigma_R f - f\|_p = 0 \forall p \in [1, \infty)$  and  $\sigma_R f \rightarrow f$  a.e. .

**Abel-Poisson sum** Let  $u(x, t) = \int_{\mathbb{R}^n} e^{-2\pi t |\xi|} \widehat{f}(\xi) e^{-2\pi i \xi \cdot x} d\xi = P_t * f(x)$ , where

$$P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}, \quad \widehat{P}_t(\xi) = e^{-2\pi t |\xi|}.$$

We have  $\Delta_{t,x} P_t(x) = 0$ , then  $\Delta u = 0$  on  $\mathbb{R}_+ \times \mathbb{R}^n$ . We also have  $\lim_{t \rightarrow 0+} u(x, t) = f(x)$  a.e. ,  $\forall f \in L^p(\mathbb{R}^n)$ .

Conversely, if  $\Delta u = 0$  in  $\mathbb{R}_+^{n+1}$ ,  $\sup_{t>0} \int_{\mathbb{R}^n} |u(x, t)|^p dx < \infty, 1 < p \leq \infty$ . Then  $\exists f \in L^p(\mathbb{R}^n)$  such that  $u(x, t) = P_t * f(x)$ .

**Gauss sum** Let  $w(x, t) = \int_{\mathbb{R}^n} e^{-2\pi t^2 |\xi|^2} \widehat{f}(\xi) e^{-2\pi i \xi \cdot x} d\xi = W_t * f(x)$ , where  $W_t$  is the

**Gauss kernel**  $W_t := \mathcal{F}(e^{-\pi t^2 |\xi|^2}) = E^n e^{-\pi |x|^2/t}$ . Let  $\widetilde{W}(x, t) = W(x, \sqrt{4\pi t})$ , then

$\frac{\partial \widetilde{W}}{\partial t} - \Delta \widetilde{W} = 0$  in  $\mathbb{R}_+^{n+1}$ . We have  $\lim_{t \rightarrow 0+} \widetilde{W}(x, t) = \lim_{t \rightarrow 0+} W(x, t) = f(x)$  a.e.,  $\forall f \in L^p(\mathbb{R}^n)$ .

# 2 The Hardy-Littlewood Maximal Function

## §2.1 Approximations of the identity

$\phi \in L^1(\mathbb{R}^n)$ ,  $\int \phi = 1$ . For  $t > 0$ , let  $\phi_t = t^{-n}\phi(t^{-1}x)$ . Then  $\phi_t \rightarrow \delta(t \rightarrow 0)$  in  $\mathcal{S}'$ , hence  $\phi_t * g \rightarrow g(t \rightarrow 0)$ .

**Example 2.1.1** (Cesàro sum)

$$\phi = F_1 = \frac{\sin^2 \pi x}{(\pi x)^2}, \text{ then } F_R = \phi_{1/R}.$$

**Example 2.1.2** (Poisson kernel)

$$\phi = P_1 = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{1}{(1 + |x|^2)^{\frac{n+1}{2}}}, \text{ then } P_t = \phi_t.$$

**Example 2.1.3** (Gauss kernel)

$$\phi = W_1 = e^{-\pi|x|^2}, \text{ then } W_t = \phi_t.$$

**Theorem 2.1.4**

$\int_{\mathbb{R}^n} \phi = A$ ,  $f \in L^p$ ,  $1 \leq p < \infty$  or  $p = \infty$ ,  $f \in C_0(\mathbb{R}^n)$ , then  $\lim_{t \rightarrow 0+} \|\phi_t * f - Af\|_p \rightarrow 0$ .

**Remark 2.1.5** — Then  $\exists \{t_k\} \rightarrow 0$  such that  $\phi_{t_k} * f \rightarrow f$  a.e. . Hence,

$$\left| \left\{ x : \lim_{t \rightarrow 0} \phi_t * f(x) \text{ exists but not equal to } f(x) \right\} \right| = 0.$$