

# **ODE: Qualitative Theory (Spring 2022, Shaobo Gan)**

Ajorda Jiao

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# 1 Basic Concepts

## §1.1 Basic notions and results

Assume  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, (t, x) \mapsto f(t, x)$  continuous, consider the **equation** (or **system**)

$$\dot{x} = \frac{dx}{dt} = f(t, x).$$

A differentiable function  $\gamma : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}^n$  is said to be a **solution** (or **solution curve**), if for every  $t \in (a, b)$ ,

$$\frac{d\gamma(t)}{dt} = f(t, \gamma(t)).$$

The **graph** of  $\gamma$  is

$$\{(t, \gamma(t)) : t \in (a, b)\} \subset \mathbb{R} \times \mathbb{R}^n.$$

For  $t_0 \in (a, b)$ , let  $x_0 = \gamma(t_0)$ , then  $\gamma$  is called the solution of the **initial value problem**

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

**The initial value problem has a unique solution:** Let  $\gamma_i : (a_i, b_i) \rightarrow \mathbb{R}^n$  be two solutions of the initial value problem. Then there exists  $\delta > 0$ ,  $(t_0 - \delta, t_0 + \delta) \subset (a_1, b_1) \cap (a_2, b_2)$ , such that  $\gamma_1(t) = \gamma_2(t), \forall t \in (t_0 - \delta, t_0 + \delta)$ ,

### Theorem 1.1.1 (Existence and Uniqueness Theorem)

$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, f(t, x)$  continuous, given  $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, a > 0, b > 0$ , consider the region

$$R = R(t_0, x_0, a, b) = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}.$$

If  $f$  is Lipchitz in  $x$  on  $R$ , i.e.  $\exists L > 0, \forall (t, x_1), (t, x_2) \in R$ ,

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|,$$

then the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution on  $[t_0 - h, t_0 + h]$ , where  $h = \min \left\{ a, \frac{b}{M} \right\}$ ,  $M = \max_{(t, x) \in R} |f(t, x)|$ .

**Remark 1.1.2** — The solution is denoted as  $\varphi(t; t_0, x_0)$ .

**Corollary 1.1.3**

When  $f \in C^1$ , the existence and uniqueness theorem holds.

Denotes the **maximal interval** of  $\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$  as  $I(t_0, x_0)$ , it is an open interval.

**Corollary 1.1.4**

Assume  $f \in C^1$  and  $|f(x)| \leq A(t)|x| + B(t)$ , then the maximal interval of the initial value problem is  $(-\infty, +\infty)$ .

**§1.2 Flows**

Now we consider the **autonomous equation**

$$\dot{x} = f(x).$$

$\mathbb{R}^n$  is called the **phase space** and  $\mathbb{R} \times \mathbb{R}^n$  is called the **generalized phase space**.

The solution of the initial value problem  $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$  is denoted as  $\varphi(t, x_0)$ , the set

$$\text{Orb}(x_0) := \{\varphi(t, x_0) : t \in I(x_0)\} \subset \mathbb{R}^n$$

is called the **orbit** pass by  $x_0$ .

**Corollary 1.2.1** (Continuous Dependence on the Initial Value)

Assume  $f \in C^1$ , then  $U = \{(t, x) : t \in I(x)\}$  is open and  $\varphi : U \rightarrow \mathbb{R}^n, (t, x) \mapsto \varphi(t, x)$  is continuous.

**Theorem 1.2.2**

$f(x) \in C^1$ , then:

1.  $\varphi_0(x) = x$  for every  $x \in \mathbb{R}^n$ .
2.  $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$  for every  $s \in I(x), t \in I(\varphi(s, x))$ .

**Definition 1.2.3.**  $\psi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , continuous, is said to be a **(continuous) flow** if:

- (i)  $\psi(0, x) = x$ ,
- (ii)  $\psi(t, \psi(s, x)) = \psi(t + s, x)$ .

**Remark 1.2.4** — The solution of an autonomous equation is a **local flow**.

**Corollary 1.2.5**

Let  $\psi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a flow, then  $\psi_t := \psi(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are homeomorphisms.

**Remark 1.2.6** — Consider the group of self-homeomorphisms of  $\mathbb{R}^n$ , denoted as  $\text{Homeo}(\mathbb{R}^n)$ , then  $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$  is a group homomorphism. More generally, we can consider  $G \rightarrow \text{Homeo}(\mathbb{R}^n)$  for some group  $G$ .

**Proposition 1.2.7**

Assume  $f$  is a  $C^1$  vector field, then the orbits of the flow generated by  $f$  are either coincide or disjoint.

$\bigcup_{x \in \mathbb{R}^n} \text{Orb}(x)$  forms a partition of  $\mathbb{R}^n$ , is called the **orbit space**. For each orbit, orient it to indicate the direction of motion, the family of the oriented orbit  $\varphi(t, x)/f(x)$  is called the **phase portrait**.

A point  $x_0 \in \mathbb{R}^n$  with  $f(x_0) = 0$  is called a **critical point** (or a **singularity**, **equilibrium**). The orbit  $\text{Orb}(x_0)$  is a single point  $\{x_0\}$ .

**Example 1.2.8**

$$\begin{cases} \frac{dx}{dt} = x \\ x(0) = x_0 \end{cases},$$

the solutions are  $\varphi(t, x_0) = x_0 e^t$ . There are three orbits  $\mathbb{R}_+, \mathbb{R}_-, \{0\}$ .

**Example 1.2.9**

$$\begin{cases} \frac{dx}{dt} = x^2 \\ x(0) = x_0 \end{cases},$$

the solutions are  $\varphi(t, x_0) = \frac{x_0}{1 - x_0 t}$ . There are three orbits  $\mathbb{R}_+, \mathbb{R}_-, \{0\}$ . But the phase portrait is different from the former examples, because the orientations on  $\mathbb{R}_-$  are different.

**Theorem 1.2.10**

Assume  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$  vector field,  $\beta(x) : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1$  and  $\beta(x) > 0$ . Then the equations  $\dot{x} = f(x)$  and  $\dot{x} = \beta(x)f(x)$  have the same phase portraits.

*Proof.*  $\varphi : I \rightarrow \mathbb{R}^n$  a solution of  $f$ . Find a  $C^1$  diffeomorphism  $h : J \rightarrow I$  such that  $\varphi \circ h$  is the solution of  $\dot{x} = \beta(x)f(x)$ . It suffices that

$$\frac{d}{dt} \Big|_{t=h(s)} \varphi(t) \cdot \frac{dh(s)}{ds} = \beta(\varphi \circ h(s))f(\varphi \circ h(s)),$$

i.e.  $\frac{dh(s)}{ds} = \beta(\varphi \circ h(s)) > 0$ , it is an initial value problem. It shows that the maximal solution curve of  $f$  is contained in some solution curve of  $\beta f$ .  $\square$

**Theorem 1.2.11** (Differentiable Dependence on the Initial Value)

Assume  $f \in C^1$ , it generates the flow  $\phi_t$ , then  $\phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^1$ .

**Exercise 1.2.12.**

$$\frac{\partial}{\partial t} \frac{\partial \phi(t, x)}{\partial x} = \frac{\partial}{\partial x} \frac{\partial \phi(t, x)}{\partial t}.$$

Let  $\Phi(t, x) = \Phi_t(x) = \frac{\partial \phi(t, x)}{\partial t}$ , then  $\Phi$  is the solution of the equation

$$\begin{cases} \frac{dy(t)}{dt} = A(t)y(t), A(t) = Df(\phi_t(x)) \\ y(0) = \text{Id} \end{cases}.$$

The equation is called the **variation equation** of  $f(x)$  along  $\phi_t(x)$ .

**Lemma 1.2.13**

$f \in C^1$ ,  $\Phi(t, x)$ , then

$$\Phi_t(\phi_s(x))\Phi_s(x) = \Phi_{t+s}(x).$$

**Remark 1.2.14** — This property is called the **cocycle** condition.

We already know that  $\phi_t$  are self-homeomorphisms of  $\mathbb{R}^n$ , and lemma 1.2.13 shows that the differential is invertible, hence  $\phi_t$  are diffeomorphisms. Define

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (t, x, v) &\mapsto (\phi_t(x), \Phi_t(x)v). \end{aligned}$$

**Proposition 1.2.15**

$\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  is a flow.

**Remark 1.2.16** — We call  $\Phi_t$  is a skew product flow of  $\phi_t$ .

**Theorem 1.2.17**

$$\Phi_t(x)f(x) = f(\phi_t(x)).$$

If  $\psi$  is a  $C^1$  flow, let

$$g(x) = \left. \frac{\partial \psi(t, x)}{\partial t} \right|_{t=0},$$

then  $\psi(t, x_0)$  solve the initial value problem  $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$ . Because

$$\frac{\partial \psi(t, x_0)}{\partial t} = \left. \frac{\partial \psi(t+s, x_0)}{\partial s} \right|_{s=0} = \left. \frac{\partial \psi(s, \psi(t, x_0))}{\partial s} \right|_{s=0} = g(\psi(t, x_0)).$$

### §1.3 Equations on manifolds

Let  $M$  be a closed smooth manifold,  $X$  is a  $C^1$  vector field on  $M$ . Then  $X$  is bounded, hence the maximal intervals are  $(-\infty, +\infty)$ . Consider the equation

$$\begin{cases} \frac{dx}{dt} = X(x) \\ x(0) = x_0 \end{cases},$$

then the solution  $\varphi(t, x)$  generates a flow.

# 2 Linear Systems

## §2.1 Plane linear singularities

Consider the equation

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}, \quad (x, y) \in \mathbb{R}^2.$$

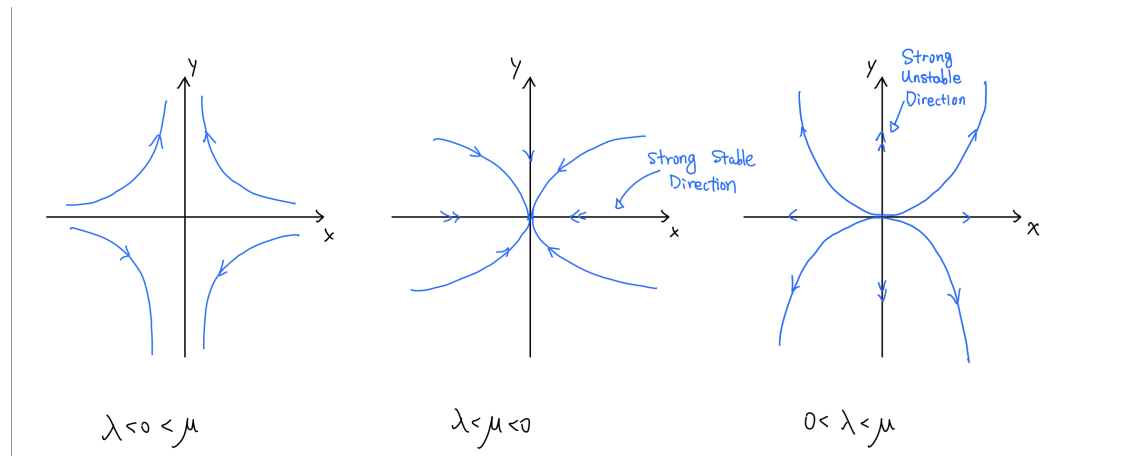
It is said to be a **plane linear system** if  $f, g$  both linear functions of  $x, y$ , i.e.

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad a, b, c, d \in \mathbb{R}.$$

If  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ , then  $(0, 0)$  is the only singularity of the vector field, elementary singularity.

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , consider the Jordan form of  $A$ . There are four cases:

- I. Two different real eigenvalues:  $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$ .
  - i.  $\lambda < 0 < \mu$ : the origin is called a **saddle point**.
  - ii.  $\lambda < \mu < 0$ : the origin is called a **stable node**.
  - iii.  $0 < \lambda < \mu$ : the origin is called a **unstable node**.



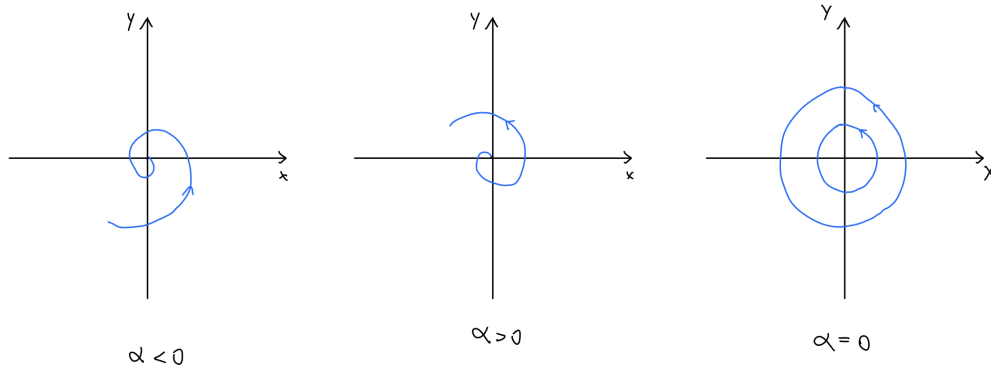
- II. Conjugated imaginary eigenvalues:  $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \beta > 0$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ .

If we consider this equation in the polar coordinates, it turns  $\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$ .

- i.  $\alpha < 0$ , the origin is called a **stable focus**.
- ii.  $\alpha > 0$ , the origin is called a **unstable focus**.
- iii.  $\alpha = 0$ , the origin is called a **center**.

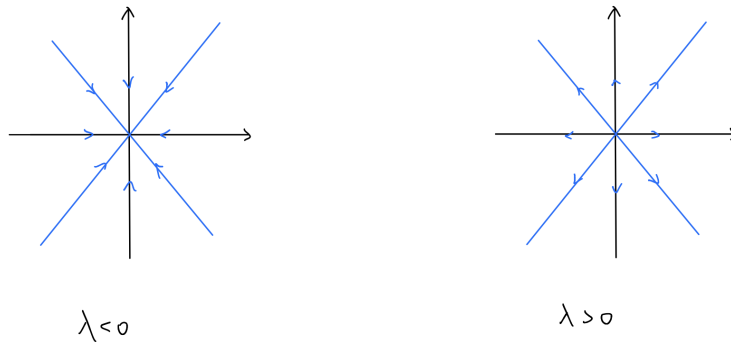
**Definition 2.1.1.**  $\varphi_t$  a flow. If  $p$  is not a singularity and  $\exists T > 0$ , such that  $\varphi_T(p) = p$ . Then  $p$  is called a **periodic point**,  $\text{Orb}(p)$  is called a **periodic orbit**. If  $p$  is a periodic point, the smallest  $T > 0$  is called the **minimum positive period**.





III. Two same real eigenvalues, diagonalizable:  $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$ .

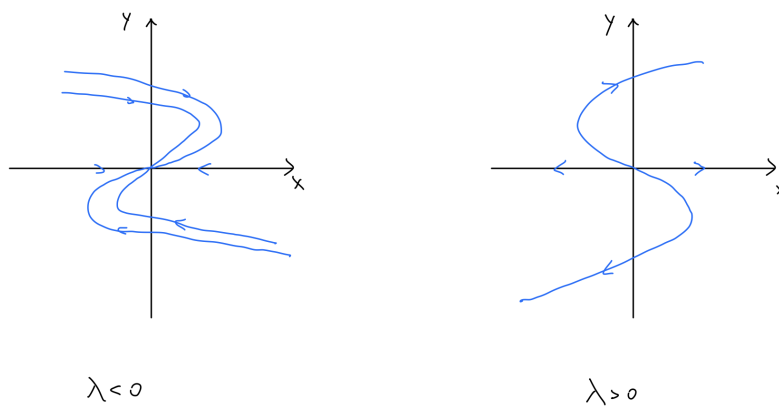
- i.  $\lambda < 0$ , the origin is called a **stable critical node**.
- ii.  $\lambda > 0$ , the origin is called a **unstable critical node**.



IV. Two same real eigenvalues, not diagonalizable:  $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}(x_0 + ty_0) \\ e^{\lambda t} \end{bmatrix} y_0$ ,

or  $x(t) = \frac{x_0}{y_0} y(t) + \frac{y(t)}{\lambda} \ln \frac{y(t)}{y_0}$ .

- i.  $\lambda < 0$ , the origin is called a **stable unidirectional node**.
- ii.  $\lambda > 0$ , the origin is called a **unstable unidirectional node**.



**Exercise 2.1.2.** Draw the phase portraits of non-elementary plane systems (i.e. the determinant is 0).

## §2.2 Topological conjugacies between linear systems

**Definition 2.2.1.** Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  homeomorphisms.  $f$  and  $g$  are said to be **topologically conjugate** if there exists  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $h \circ f = g \circ h$ .

**Remark 2.2.2** — Conjugacy is an equivalence relation.

**Definition 2.2.3.** Let  $\varphi_t, \psi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be two flows, we call  $\varphi_t$  and  $\psi_t$  are conjugate if there is a homeomorphism  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $h \circ \varphi_t = \psi_t \circ h$ . Let  $X, Y$  be two  $C^1$  vector fields on  $\mathbb{R}^n$ , we call  $X, Y$  are conjugate if the flows generated by them, respectively, are conjugate.

### Example 2.2.4

$A, B \in M(n, \mathbb{R})$  are similar, then  $\dot{x} = Ax$  and  $\dot{y} = By$  are conjugate.

$f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$   $C^1$  vector fields, generate flows  $\phi_t, \psi_t$ . Let  $x = h(y) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  diffeomorphism gives the conjugate, i.e.,  $h\psi_t(y) = \phi_t h(y)$ . Then

$$\frac{d}{dt}h(y) = f(h(y)) \implies D_{h(y)}g(y) = D_{h(y)}\frac{dy}{dt} = f(h(y)).$$

If there exists a  $C^1$  diffeomorphism conjugate  $e^{Bt}y$  to  $e^{At}x$  via  $x = h(y)$ , i.e.  $h(e^{Bt}y) = e^{At}h(y)$ . Then  $Dh_0 e^{Bt} = e^{At}Dh_0$ , hence  $Dh_0 B = A Dh_0$ . It shows that  $C^1$  conjugate generically not hold even if topologically conjugate.

### Proposition 2.2.5

Assume  $f, g$   $C^1$  vector fields generate  $\phi_t, \psi_t$ , let  $h$  be a conjugate between  $\phi_t$  and  $\psi_t$ . Then:

1.  $h(\text{Orb}(x, \phi)) = \text{Orb}(hx, \psi)$ .
2.  $h$  maps the singularities of  $f$  to the singularities of  $g$ .
3.  $h$  maps the periodic orbits of  $f$  to the periodic orbits of  $g$ . Moreover, it preserves the minimum positive period.

### Example 2.2.6

$\dot{x} = -2x$  and  $\dot{y} = -4y$  are conjugate.

Let  $h : \mathbb{R} \rightarrow \mathbb{R}, h(0) = 0$ . Take  $x_0, y_0 > 0$ , let  $h(x_0) = y_0$ , then  $h(e^{-2t}x_0) = e^{-4t}y_0$  or  $h(x) = \left(\frac{x}{x_0}\right)^2 y_0$ . The construction for the negative part is similar.

**Exercise 2.2.7.**  $\lambda\mu \neq 0$ , show that  $\dot{x} = \lambda x$  is conjugate to  $\dot{y} = \mu y$  if and only if  $\lambda\mu > 0$ .

**Proposition 2.2.8**

$\phi_t^i, \psi_t^i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$  are topologically conjugate by  $h_i, i = 1, 2$ . Then  $\phi_t^1 \times \phi_t^2$  and  $\psi_t^1 \times \psi_t^2$  are topologically conjugate by  $h_1 \times h_2$ .

**Example 2.2.9**

$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$  and  $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$  are conjugate.

*Proof.*  $\phi_t(x, y) = e^{-t}(x, y)$  and  $\psi_t(x, y) = e^{-t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ . For every  $(x, y) \neq (0, 0)$ , there exists unique  $t = t(x, y)$  such that  $\phi_t(x, y) \in \mathbb{S}^1$ . Let  $h(x, y) := \psi_{-t}\phi_t(x, y)$ , where  $t = t(x, y)$ , then  $h$  gives the conjugate.  $\square$

**Exercise 2.2.10.** Show that  $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$  and  $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -y \end{cases}$  are conjugate.

Classification of elementary plane linear systems:

- (I) Stable: node, critical node, unidirectional node, focus.
- (II) Unstable: node, critical node, unidirectional node, focus.
- (III) Saddle point.
- (IV) Center.

**Definition 2.2.11.** The linear system  $\dot{x} = Ax$  in  $\mathbb{R}^n$  is called **hyperbolic** if the real parts of eigenvalues of  $A$  are nonzero. The **(stable) index** of  $A$  is the number of eigenvalues with negative real parts, denoted by  $\text{Ind } A$ .

**Theorem 2.2.12**

Two plane hyperbolic linear system  $\dot{x} = Ax, \dot{y} = By$  are topologically conjugate if and only if  $\text{Ind } A = \text{Ind } B$ .

*Proof.* “ $\implies$ ”: Let  $W_A^s = \{x : e^{tA}x \rightarrow 0, t \rightarrow \infty\}$ ,  $W_B^s = \{x : e^{tB}x \rightarrow 0, t \rightarrow \infty\}$ , then  $h$  and  $h^{-1}$  preserves the stable manifolds. Then  $\text{Ind } A = \dim W_A^s = \dim W_B^s = \text{Ind } B$ .  $\square$

**Example 2.2.13**

Consider  $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$  and  $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$  with the same phase portraits are not topologically conjugate. Because the topologically conjugate preserves the minimum positive orbits.

**Definition 2.2.14.**  $\phi_t, \psi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  flows,  $h$  is a homeomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  maps the orbit of  $\phi$  to the orbit of  $\psi$  preserves the orientation. Then  $\phi$  and  $\psi$  is called **topologically equivalent** or **flow equivalent**.

**Theorem 2.2.15** (Grobman-Hartman)

If  $x_0$  is a hyperbolic singularity of  $f(x)$ , then the flows generated by  $\dot{x} = f(x)$  and  $\dot{y} = Ay$  where  $y = Df(x_0)$  are topologically conjugate near 0.

## §2.3 Non-autonomous linear systems

$A : \mathbb{R} \rightarrow M(n, \mathbb{R})$  continuous, consider the equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n$$

a non-autonomous linear system.

**Theorem 2.3.1**

The followings hold:

1. The initial problem of the equation exist the unique solution.
2. The maximal interval of any solution is  $(-\infty, \infty)$ .
3. All solutions of the equation form an  $n$ -dimensional linear space  $S$ .

**Theorem 2.3.2** (Liouville's Formular)

Assume  $X(t)$  is a solution of  $\dot{x} = A(t)x$ , then

$$\frac{d}{dt} \det X(t) = \text{tr } A(t) \det X(t),$$

hence  $\det X(t) = \det X(t_0) \exp \int_{t_0}^t \text{tr } A(s) ds$ .

Let  $X_1(t), X_2(t), \dots, X_n(t)$  be a basis of  $S$ , let

$$X(t) := [X_1(t), X_2(t), \dots, X_n(t)] \in \text{GL}(n, \mathbb{R}),$$

it called a **fundamental solution** of the equation. The fundamental solution of

$$\begin{cases} \frac{dX}{dt} = A(t)X \\ X(t_0) = I_n \in \text{GL}(n, \mathbb{R}) \end{cases}$$

is called the **standard fundamental solution**.

If  $X(t), Y(t)$  are two fundamental solutions, suppose  $Y(0) = X(0)C$ , then

$$\frac{dX(t)C}{dt} = \frac{dX(t)}{dt}C = A(t)X(t)C,$$

is a non-degenerate solution of  $\frac{dX}{dt} = AX$ . By the uniqueness, we get  $Y(t) = X(t)C$ .

**Example 2.3.3**

$A(t) \equiv A$ , the fundamental solution of  $\dot{x} = Ax$  is

$$e^{tA} = \text{Id} + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{k!}t^kA^k + \cdots.$$

**Example 2.3.4**

$\dot{x} = f(x)$ ,  $x \in \mathbb{R}^n$ , where  $f \in C^1$ , generates the flow  $\varphi_t(x)$ . Consider  $\Phi_t(x) = \frac{\partial}{\partial t}\varphi_t(x)$  and the variation equation

$$\frac{d}{dt}\Phi_t(x) = Df_{\varphi_t(x)}\Phi_t(x).$$

Given  $x \in \mathbb{R}^n$ , let  $A(t) := Df_{\varphi_t(x)}$ , then  $\Phi_t(x)$  is the standard fundamental solution ( $t_0 = 0$ ) of  $\dot{x} = A(t)x$ . Consider two special types of orbits:

- $x$  is a singularity, denoted by  $\sigma$ . Then  $\varphi_t(\sigma) = \sigma$ ,  $\dot{x} = Ax$  where  $A = Df(\sigma)$ .
- $x$  is a periodic point, denoted by  $p$ , the minimum period  $T > 0$ . Then  $A$  is  $T$ -periodic.

**§2.4 Periodic linear systems**

**Definition 2.4.1.** The equation  $\dot{x} = A(t)x$  satisfies  $A(t+T) = A(t)$  for some  $T > 0$  is called a **periodic linear systems**.

**Theorem 2.4.2 (Floquet)**

Assume  $\dot{x} = A(t)x$  is a  $T$ -periodic linear system, if  $X$  is a fundamental solution, then  $X(t+T)$  is a fundamental solution, i.e.  $\exists C \in \text{GL}(n, \mathbb{R})$  such that  $X(t+T) = X(t)C$ . Moreover, there exists a  $T$ -periodic map  $P : \mathbb{R} \rightarrow \text{GL}(n, \mathbb{C})$  and a constant matrix  $B \in M(n, \mathbb{C})$  such that  $X(t) = P(t)e^{tB}$ .

**Lemma 2.4.3**

$\forall C \in \text{GL}(n, \mathbb{R})$ ,  $\exists B \in M(n, \mathbb{C})$  such that  $C = e^B$ .

*Proof.* It suffices to show for Jordan block. This follows by the matrix series

$$\ln(I + N) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} N^k$$

is convergence for nilpotent matrix  $N$ . □

**Lemma 2.4.4**

$\forall C \in \text{GL}(n, \mathbb{R})$ ,  $\exists B \in M(n, \mathbb{R})$  such that  $C^2 = e^B$ .

*Proof.* Note that the Jordan block of  $C^2$  is either:

$$(i) \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & \lambda & 1 \\ 0 & \cdots & & \lambda \end{bmatrix}, \text{ where } \lambda > 0, \text{ or}$$

$$(ii) \begin{bmatrix} J & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & J & I_2 \\ 0 & \cdots & & J \end{bmatrix}, \text{ where } J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a, b \in \mathbb{R}, b > 0.$$

And  $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  have a real matrix logarithm because  $\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\} \cong \mathbb{C} = \{a + bi\}$ .  $\square$

#### Theorem 2.4.5 (Real Form of Floquet Theorem)

Assume  $\dot{x} = A(t)x$  is a  $T$ -periodic linear system, if  $X$  is a fundamental solution. Then there exists a  **$2T$ -periodic** map  $P : \mathbb{R} \rightarrow \text{GL}(n, \mathbb{R})$  and a constant matrix  $B \in M(n, \mathbb{R})$  such that  $X(t) = P(t)e^{tB}$ .

#### Example 2.4.6 ( $2T$ is necessary)

Let  $\Phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \exp \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t \right) \exp \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right)$ . Let

$$A(t) = \dot{\Phi}(t)\Phi(t)^{-1} = \begin{bmatrix} -\cos t \sin t & -\sin^2 t \\ \cos^2 t & \cos t \sin t \end{bmatrix},$$

then  $A(t)$  is  $\pi$ -periodic. Then  $\Phi(t)$  is a standard fundamental solution of  $\dot{x} = A(t)x$ , hence  $\exists \pi$ -periodic  $P(t)$  and  $B$  such that  $\Phi(t) = P(t)e^{tB}$ . Then  $e^{\pi B} = \begin{bmatrix} -1 & -\pi \\ 0 & -1 \end{bmatrix}$ , there is no real matrix  $B$  satisfying this equation.

**Definition 2.4.7.** In Floquet theorem,  $X(t+T) = X(t)C$ . We call  $C$  is a **monodromy matrix**. The eigenvalues of  $C$  are called **Floquet multipliers**. If  $\rho$  is a Floquet multiplier with  $\rho = e^{\lambda T}$ , then  $\lambda$  is called a **Floquet exponent**.

#### Corollary 2.4.8

Consider a  $T$ -periodic linear system  $\dot{x} = A(t)x$ . Then there exists a linear transformation (non-autonomous)  $x = P(t)y$  such that  $\dot{y} = By$ .

*Proof.* Let  $X(t) = P(t)e^{tB}$  be a fundamental solution, then

$$AX = \dot{X} \implies \dot{P}e^{tB} + PB e^{tB} = AP e^{tB},$$

hence  $\dot{P} + PB = AP$ . Then  $APy = \frac{d}{dt}(Py) = \dot{P}y + P\dot{y}$ , hence  $\dot{y} = By$ .  $\square$

**Remark 2.4.9** — This type of equation is called reducible, which means after some reduction, the equation can become independent with time  $t$ .

**Corollary 2.4.10**

Let  $\lambda$  be a Floquet multiplier of  $\dot{x} = A(t)x$ . Then there exists a  $T$ -periodic function  $p(t)$  such that  $e^{\lambda t}p(t)$  is a solution of the equation  $\dot{x} = A(t)x$ .

*Proof.*  $e^{\lambda T}$  is an eigenvalue of  $C$ , then  $\exists x_0$  such that  $Cx_0 = e^{\lambda T}x_0$ . Then  $X(t)x_0$  is a solution. Let  $p(t) = e^{-\lambda t}X(t)x_0$  is  $T$ -periodic and  $e^{\lambda t}p(t)$  is a solution.  $\square$

**Corollary 2.4.11**

The equation admits a nonzero  $T$ -periodic solution if and only if 1 is a Floquet multiplier.

**Corollary 2.4.12**

Assume  $\rho_1, \rho_2, \dots, \rho_n$  are all Floquet multipliers of  $\dot{x} = A(t)x$ , then

$$\rho_1 \rho_2 \cdots \rho_n = \det \Phi(T) = \exp \int_0^T \operatorname{tr} A(t) \, dt.$$

**Example 2.4.13**

The equation  $\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^2 t & \frac{1}{2} \sin 2t - 1 \\ \frac{1}{2} \sin 2t + 1 & \sin^2 t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$  has an unbounded solution. Because the product of two multipliers is  $\exp \int_0^\pi 1 \, dt = e^\pi > 1$ .

Consider **Hill equation**

$$\ddot{x} + p(t)x = 0,$$

where  $p(t)$  is  $\pi$ -periodic. This is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -p(t)x \end{cases},$$

then  $\rho_1 \rho_2 = \exp \int_0^\pi \operatorname{tr} A(t) \, dt = 0$ , where  $A(t) = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}$ .

**Lemma 2.4.14**

If  $\rho_1, \rho_2$  both are imaginary numbers, then every solution of Hill equation is bounded.

*Proof.* Because  $\rho_1, \rho_2$  are conjugate imaginary numbers, hence  $\Phi(\pi)$  is similar to a rotation. Then  $\Phi(\pi)^n$  is bounded independent of  $n$  and  $\Phi(s)$  is bounded for  $s \in [0, \pi]$ .  $\square$

**Definition 2.4.15.** A particular Hill equation with  $p(t) = a + \varepsilon \cos 2t$  is called **Mathieu equation**.

**Exercise 2.4.16.** Consider Mathieu equation

$$\ddot{x} + (a + \varepsilon \cos 2t)x = 0.$$

- (1)  $U = \{(a, \varepsilon) \in [0, 10] \times [-1, 1] : \text{every solution is bounded}\}$ . Draw the figure of  $U$  by some calculation.
- (2) Guess some conclusions by the figure of  $U$ .

**Example 2.4.17**

Let  $p(t)$  be a  $\pi$ -periodic continuous function satisfying

- (i)  $p(t) \not\equiv 0$ .
- (ii)  $\int_0^\pi p(t) dt \geq 0$ .
- (iii)  $\pi \int_0^\pi |p(t)| dt \leq 4$ .

Then every solution of  $\ddot{x} + p(t)x = 0$  is bounded.

*Proof.* If Floquet multipliers are conjugate imaginary numbers, the statement follows. Otherwise there is a real Floquet multiplier  $\rho \neq 0$ . There is a solution  $x(t) \not\equiv 0$  such that  $x(t+T) = \rho x(t)$ . If  $x(t)$  has no zeros, assume  $x(t) > 0$ , we have  $\frac{\dot{x}}{x}(\pi) = \frac{\dot{x}}{x}(0)$ . Note that

$$0 = \frac{\ddot{x}}{x} + p(t) = \left(\frac{\dot{x}}{x}\right)' + \left(\frac{\dot{x}}{x}\right)^2 + p(t) = 0,$$

take the integral and we get a contradiction. Then there must be some zeros, let  $a, b$  be two successive zeros, WLOG,  $0 < a < b < \pi$ . Assume  $x(t) > 0$  in  $(a, b)$  and  $x(c)$  takes the maximum. Then  $\exists \alpha \in (a, c), \beta \in (c, b)$  such that  $\dot{x}(\alpha) = \frac{x(c)}{c-a}, \dot{x}(\beta) = \frac{-x(c)}{b-c}$ . We have

$$\frac{4}{\pi} \geq \int_0^\pi |p(t)| dt > \int_a^b \left| \frac{\ddot{x}}{x}(t) \right| dt \geq \frac{\int_\alpha^\beta |\ddot{x}(t)| dt}{x(c)} \geq \frac{1}{c-a} + \frac{1}{b-c} \geq \frac{4}{a-b},$$

the identity holds if and only if  $x \equiv 0$ , contradiction.  $\square$

Back to Mathieu equation, consider

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0, \quad \omega > 0, \varepsilon < \omega^2.$$

We apply the conclusion of the example, for  $\omega < \frac{2}{\pi}$ ,

$$\int_0^\pi (\omega^2 + \varepsilon \cos 2t) dt = \omega^2 \pi \leq \frac{4}{\pi}.$$

Consider  $\varepsilon = 0$ , then

$$\Phi(t) = \begin{bmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

is a standard fundamental solution. The monodromy matrix for  $(\omega, \varepsilon)$  where  $\omega > 0$  is a perturbation of

$$C = \Phi(\pi) = \begin{bmatrix} \cos \omega \pi & \frac{1}{\omega} \sin \omega \pi \\ -\omega \sin \omega \pi & \cos \omega \pi \end{bmatrix}.$$

Note that  $|\text{tr } \Phi(\pi)| = |2 \cos \omega \pi| < 2$  for  $\omega \notin \mathbb{Z}$ . Then there is a small neighborhood  $U$  of  $(\omega, 0)$  such that every solution is bounded.



**Definition 2.4.18.** Let  $A : \mathbb{R} \rightarrow M(n, \mathbb{R})$  continuous, bounded, assume that

$$\sup \{|A(t)| : t \in \mathbb{R}\} < \infty.$$

Let  $\Phi(t)$  be a standard fundamental solution of the equation  $\dot{x} = A(t)x$ . For every  $v \neq 0 \in \mathbb{R}^n$ , define **Lyapunov exponent** of  $v$

$$\chi(v) := \limsup_{t \rightarrow \infty} \frac{\ln \|\Phi(t)v\|}{t}.$$

**Exercise 2.4.19.** For every  $v \neq 0$ , show that  $\chi(v) \neq \pm\infty$ .

Then  $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the following properties

1.  $\chi(\alpha v) = \chi(v)$  for every  $\alpha \neq 0$ .
2.  $\chi(v + w) \leq \max \{\chi(v), \chi(w)\}$ .
3. If  $\chi(v) < \chi(w)$ , then  $\chi(v + w) = \chi(w)$ .

**Fact 2.4.20.** The number of different Lyapunov exponents  $\leq n$ .

**Example 2.4.21**

$\dot{X} = AX$ , where  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $A$  is a constant matrix. Regard as a  $T$ -periodic system, then the eigenvalues  $\lambda_1, \lambda_2$  of  $A$  are Floquet exponents. Lyapunov exponents are

- (1)  $\lambda_1, \lambda_2$ , if  $\lambda_1 \neq \lambda_2$  real.
- (2)  $\lambda = \lambda_1 = \lambda_2$ , if  $\lambda_1 = \lambda_2$ .
- (3)  $\alpha$ , if  $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$ .

For the  $T$ -periodic system, assume that  $\lambda$  is a Floquet exponent, then  $\chi = \text{Re}(\lambda)$  is a Lyapunov exponent. For  $n = 2$ ,  $T$ -periodic system, we always have

$$\chi_1 + \chi_2 = \text{Re}(\lambda_1 + \lambda_2) = \frac{1}{T} \int_0^T \text{tr } A(t) dt.$$

**Example 2.4.22**

Consider

$$\begin{cases} \dot{x} = (-\mu - (\sin \ln t + \cos \ln t))x \\ \dot{y} = (-\mu + (\sin \ln t + \cos \ln t))y \end{cases},$$

then the solution

$$\begin{cases} x = C_1 e^{-\mu t - t \sin \ln t} \\ y = C_2 e^{-\mu t + t \sin \ln t} \end{cases}.$$

Then  $\chi(v) = -\mu + 1$  for every  $v \neq 0$ . But  $\chi_1 + \chi_2 = -2\mu + 2 \neq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{tr } A(t) dt = -2\mu$ . This example is called non-regular.

# 3 Stability

## §3.1 Lyapunov stability

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n, 0 \in \mathbb{R}^n, f(0) = 0$ , generates a (local) flow  $\varphi_t(x)$ .

**Definition 3.1.1.** 1.  $\sigma$  is called **(forward Lyapunov) stable**, if  $\forall \varepsilon > 0, \exists \delta > 0$ , such that if  $|x - \sigma| < \delta$ , then  $|\varphi_t(x) - \sigma| < \varepsilon$  for  $t \geq 0$ . Otherwise, we call  $\sigma$  is **unstable**.

2.  $\sigma$  is called **asymptotically stable**, if

(i)  $\sigma$  is stable,

(ii) there exists  $\delta_0 > 0$ , such that if  $|x - \sigma| < \delta$ , then  $\lim_{t \rightarrow \infty} \varphi_t(x) = \sigma$ .

3.  $\sigma$  is called **exponentially stable**, if exists  $\delta_0 > 0, C \geq 1, \lambda > 0$ , such that if  $|x - \sigma| < \delta$ , then  $|\varphi_t(x) - \sigma| \leq Ce^{-\lambda t}|x - \sigma|$  for  $t \geq 0$ .

Similarly, we can define backward stable, backward asymptotically stable, backward exponentially stable.

**Remark 3.1.2** — If we replace the condition of stability by **given  $t \geq 0$** , then it always holds by the continuous independence of solutions with respect to initial value.

### Example 3.1.3

For the equation in polar coordinates

$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2(\theta/2) \end{cases}.$$

Then the fixed point  $(1, 0)$  satisfy the second condition of asymptotically stable but it is **not** stable.

In general, we can prove that if  $\varphi_t(x) \not\equiv \sigma$  and  $\lim_{t \rightarrow \pm\infty} \varphi_t(x) = \sigma$ , then  $\sigma$  is not stable.

### Example 3.1.4

Consider the linear elementary singularities, recall the classification, then

1. Stable type: forward stable.
2. Unstable type: unstable, but backward stable.
3. Saddle point: unstable.
4. Center: forward and backward stable.

**Theorem 3.1.5**

Let  $A \in M(n, \mathbb{R})$ , consider the equation  $\dot{X} = AX$ , 0 is a singularity, then

1. 0 is stable iff each eigenvalue of  $A$  is with non-positive real part and Jordan block are trivial for every eigenvalue with zero real part.
2. 0 is asymptotically stable iff 0 is exponentially stable iff every eigenvalue of  $A$  is with negative real part.

**Lemma 3.1.6** (Gronwall's Inequality)

Let  $u : [0, T] \rightarrow \mathbb{R}$  non-negative, continuous. If  $C \geq 0, K > 0$  such that for every  $t \in [0, T]$ ,

$$u(t) \leq C + K \int_0^t u(s) ds,$$

then  $u(t) \leq Ce^{Kt}$  for  $t \in [0, T]$ .

**Theorem 3.1.7**

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $C^1$ ,  $f(\sigma) = 0$ . Assume that every eigenvalue of  $A = Df(0)$  is with negative real part, then  $\sigma$  is exponentially stable.

*Proof.* There  $\exists C \geq 1, \mu > 0$ , such that  $|e^{At}| \leq Ce^{-\mu t}$  for  $t \geq 0$ . WOLG,  $\sigma = 0$ . Let  $f(x) = Ax + g(x)$  where  $g(x) = o(|x|)$ , let  $\varphi_t(x)$  be a maximal solution of the initial value problem. Then

$$e^{-tA}(\dot{\varphi}_t(x) - A\varphi_t(x)) = e^{-tA}g(\varphi_t(x)),$$

hence

$$\varphi_t(x) = e^{tA}x + \int_0^t e^{(t-s)A}g(\varphi_s(x))ds.$$

Fix  $\varepsilon_0 > 0$  to be determined later,  $\exists \delta_0 > 0$  such that  $|g(x)| \leq \varepsilon_0|x|$  if  $|x| \leq \delta_0$ . Assume the right maximal interval of  $\varphi_t$  is  $[0, \beta), \beta > 0$ . Let

$$T^* = T^*(x) = \sup \left\{ t < \beta : \varphi_{[0,t]}(x) \subseteq \overline{B(\delta_0, \sigma)} \right\}.$$

Then, for every  $|x| \leq \delta_0, 0 \leq t \leq T^*$ , we have

$$e^{\mu t}|\varphi_t(x)| \leq C|x| + C\varepsilon_0 \int_0^t e^{s\mu}|\varphi_s(x)|ds.$$

By Gronwall's inequality, we have  $|\varphi_t(x)| \leq C|x|e^{-(\mu - C\varepsilon_0)t}, \forall t < T^*$ . Let  $C\varepsilon_0 = \frac{\mu}{2}$  is enough. For all  $|x| \leq \frac{\delta_0}{2C}$ , then  $|\varphi_t(x)| \leq \frac{\delta_0}{2}e^{-\mu t}$  for every  $t < T^*$ . Then we can show that  $T^* = \beta = \infty$  and  $\varphi_t$  is exponentially stable.  $\square$

**Proposition 3.1.8**

$f, g, C^1$  vector fields. Assume  $f, g$  are topologically conjugate, i.e.,  $h \circ \varphi_t = \psi_t \circ h$  where  $\varphi_t, \psi_t$  are flows generated by  $f, g$ , respectively. Let  $\sigma, h\sigma$  be singularities of  $f, g$ , respectively, then  $\sigma$  is stable if and only if  $h\sigma$  is stable.

Now, we state a celebrated theorem, Hartman-Grobman Theorem. But we will not give a proof here.

**Theorem 3.1.9 (Hartman-Grobman)**

Let  $\sigma$  be a hyperbolic singularity of  $f$ . Then there exists a neighborhood  $V \ni \sigma$  and a homeomorphism  $h : V \rightarrow \mathbb{R}^n$  onto its image,  $h(\sigma) = 0$ , such that  $h \circ \varphi_t(x) = Df(\sigma) \circ h(x)$  for every  $x, \varphi_t(x) \in V$ .

### §3.2 Lyapunov functions

**Definition 3.2.1.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , be a  $C^1$  vector field,  $f(0) = 0$ . A  $C^1$  function  $V : D \rightarrow \mathbb{R}$  where  $D$  is a neighborhood of  $\sigma$  is called a **Lyapunov function** of  $f$  (for  $\sigma$ ) if

- (i)  $V(\sigma) = 0, V(x) > 0, \forall x \in D \setminus \{\sigma\}$ .
- (ii)  $\forall x \in D \setminus \{\sigma\}, \dot{V}(x) \leq 0$ , where

$$\dot{V}(x) = \left. \frac{\partial}{\partial t} \right|_{t=0} V(\varphi_t(x)) = DV(x)f(x).$$

$V$  is called a **strict Lyapunov function** if  $\dot{V}(x) \leq 0$  is replaced by  $\dot{V}(x) < 0$ .

**Theorem 3.2.2**

Assume  $\sigma$  is a singularity of  $f$ , if there is a Lyapunov function for  $\sigma$ , then  $\sigma$  is stable. If there is a strict Lyapunov function for  $\sigma$ , then  $\sigma$  is asymptotically stable.

*Proof.* Let  $V$  be a Lyapunov function, for every  $\varepsilon > 0$ , assume  $B_\varepsilon(\sigma) = \{x : |x - \sigma| \leq \varepsilon\} \subseteq D$ . Let  $m = \min \{V(x) : x \in \partial B_\varepsilon(\sigma)\} > 0$ , take  $\delta > 0$  such that  $V(x) < m, \forall x \in B_\delta(\sigma)$ . By  $\dot{V}(x) \leq 0$ , we have that every solution curve start at  $x \in B_\delta(\sigma)$  can not reach  $\partial B_\varepsilon(\sigma)$ .

If  $\dot{V}(x) < 0$  for every  $x \in D \setminus \{\sigma\}$ , it suffices to show that each convergent subsequence of  $\varphi_t(x)$  converges to  $\sigma$ . Otherwise, assume converges to  $y \neq \sigma$ , but  $\dot{V}(y) < 0$ , there is some  $s > 0$  such that  $V(\varphi_s(y)) < V(y)$ . Contradiction.  $\square$

**Example 3.2.3**

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}.$$

Let  $V(x, y) = x^2 + y^2$ , then  $\dot{V}(x, y) = 0$ , hence 0 is stable.

**Example 3.2.4**

Consider the equation

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - y^3 \end{cases}.$$

Let  $V(x, y) = x^2 + y^2$ , then  $\dot{V}(x, y) = -2x^2 - 2y^4 < 0$ , hence 0 is asymptotically stable.

**Example 3.2.5**

Consider the equation

$$\begin{cases} \dot{x} = -x - y + x^2 \\ \dot{y} = x \end{cases}.$$

Let  $V(x, y) = x^2 + y^2$ , then  $\dot{V}(x, y) = -2x^2(1 - x) \leq 0$ , hence 0 is stable. In fact, 0 is asymptotically stable, but we need to consider another Lyapunov function  $Q(x, y) = x^2 + y^2 + xy$ .

**Theorem 3.2.6**

If  $V$  is a Lyapunov function of  $f$ , assume

$$K = \{x \in D \setminus \{\sigma\}, \dot{V}(x) = 0\}$$

does not contain any complete positive orbit  $\varphi_{[0, \infty)}(x)$ , then  $\sigma$  is asymptotically stable.

**Example 3.2.7**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}, C^1, f(0) = 0$ , satisfying  $xf(x) > 0, \forall x \neq 0$ . Consider the stability of  $\ddot{x} + f(x) = 0$ , or

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}.$$

Let

$$E(x, y) = \frac{1}{2}y^2 + \int_0^x f(z)dz$$

be an energy function, then  $\dot{E}(x, y) \equiv 0$ .

**Example 3.2.8**

Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}, C^2$ , the gradient of  $V$  is

$$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}.$$

The system  $\dot{x} = -V(x)$  is called the **gradient system** generated by  $V$ . Then,

1.  $\dot{V}(x) \leq 0$ .
2.  $\sigma$  is a singularity if and only if  $\dot{V}(\sigma) = 0$ .
3. If  $\sigma$  is a minimum point of  $V(x)$ , then  $\sigma$  is stable.

**Theorem 3.2.9**

Let  $\sigma$  be a singularity of  $C^1$  vector field  $f$ , a  $C^1$  function  $V : D \rightarrow \mathbb{R}$  satisfies

- (i)  $V(\sigma) = 0$ , and  $V$  can take positive value on any neighborhood of  $\sigma$ .
- (ii)  $\dot{V}(x) > 0, \forall x \in D \setminus \{0\}$ .

Then  $\sigma$  is unstable.

**Example 3.2.10**

Consider the equation

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}.$$

Let  $V(x, y) = x^2 - y^2$ , then  $\dot{V}(x, y) = 2x^2 + 2y^2 > 0$ , hence 0 is unstable.

**Theorem 3.2.11**

Let  $f$  be a  $C^1$  vector field,  $f(\sigma) = 0$ . If  $\sigma$  is stable, then every eigenvalue of  $Df(\sigma)$  is with non-negative real part.

*Proof.* Prove for  $n = 2$ . Assume  $\sigma = 0$ , the equation is

$$\begin{cases} \dot{x} = \lambda x + \alpha(x, y) \\ \dot{y} = \mu y + \beta(x, y) \end{cases},$$

where  $\lambda < \mu, \mu > 0, |\alpha|, |\beta| = o(r)$ . Let  $V(x, y) = -x^2 + y^2$ , then

$$\dot{V}(x, y) = -2\lambda x^2 + 2\mu y^2 - 2x\alpha + 2y\beta.$$

If  $\lambda < 0$ , then  $\dot{V} > 0$  in a neighborhood of 0, then 0 is unstable. If  $\lambda \geq 0$ , consider

$$C = \{(x, y) : V(x, y) \geq 0\}.$$

We can show that for some  $\varepsilon_0 \geq 0$ ,  $\dot{V}(x, y) > 0$  on  $C \cap B(0, \varepsilon_0) \setminus \{0\}$ . Let  $H(x, y) = x^2 + y^2$ , then  $\dot{H}(x, y) \geq \frac{\mu}{2}H(x, y)$  on some neighborhood of 0. Hence

$$H(\varphi_t(x, y)) \geq H(x, y)e^{\frac{\mu}{2}t}$$

will be out of  $C \cap B_\varepsilon(x, y)$ . □

**Remark 3.2.12** — In fact, there exists  $(x, y) \in B(0, \varepsilon_0) \setminus \{0\}$ , such that

$$\lim_{t \rightarrow -\infty} \varphi_t(x, y) = 0, \quad \frac{f(\varphi_t(x, y))}{|f(\varphi_t(x, y))|} \rightarrow (0, 1).$$

$\varphi_t(x, y)$  is called the unstable manifold.

**Exercise 3.2.13.** Prove the theorem for general dimension  $n$ .

Now, we consider a perturbation of a singularity of center type. Consider the system

$$\begin{cases} \dot{x} = -y + \alpha(x, y) \\ \dot{y} = x + \beta(x, y) \end{cases},$$

then

$$\dot{\theta} = 1 + \frac{x\beta - y\alpha}{x^2 + y^2},$$

$$\dot{r} = \frac{x\alpha + y\beta}{r} = \alpha \cos \theta + \beta \sin \theta = R_2(\theta)r^2 + R_3(\theta)r^3 + \dots.$$

### Example 3.2.14

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + (x^2y + x^3) \end{cases}.$$

Then

$$\dot{r} = \sin \theta (x^2y + x^3) = r^3 (\cos^2 \theta \sin^2 \theta + \cos^3 \theta \sin \theta),$$

we calculate

$$\overline{R}_3 = \int_0^{2\pi} R_3(\theta) d\theta = \frac{\pi}{4} > 0.$$

Let  $g(\theta) = \int_0^\theta R_3(\theta) d\theta$ , then

$$\varphi_3(\theta) = g(\theta) - \frac{\theta}{2\pi} \int_0^{2\pi} R_3(\theta) d\theta$$

is  $2\pi$ -periodic. Let  $r = \rho + \varphi_3(\theta)\rho^3$ , then

$$\frac{d\rho}{d\theta} = \overline{R}_3\rho^3 + \dots,$$

hence  $\rho$  is increasing. Therefore, 0 is unstable.

**Example 3.2.15**

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We want to construct a Lyapunov function of the form  $V(x, y) = x^2 + y^2 + F(x, y)$ , where  $F(x, y)$  is a homogeneous polynomial of  $\deg = 3$ . Then

$$\dot{V}(x, y) = -yF_x + xF_y + 2y^3 + y^2F_y,$$

we want  $-yF_x + xF_y + 2y^3 = 0$ . Consider  $L : H_k \rightarrow H_k$ , where  $H_k$  is the family of homogeneous polynomials of  $\deg = k$ ,  $L(F) = -yF_x + xF_y$ . After repetition, we can let

$$V(x, y) = \lambda(x^2 + y^2)^k + \dots.$$

Then 0 is stable if  $\lambda < 0$ , 0 is unstable if  $\lambda > 0$ . Or we can find  $V$  such that  $\dot{V}(x, y) = 0$ , then 0 is still a center.

**Example 3.2.16**

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We can solve this equation,

$$y^2 = -x + \frac{1}{2}(1 - e^{-2x}) + Ce^{-2x},$$

hence  $e^{2x}(x^2 + y^2) = C + \dots$ . 0 is still a center.

**Example 3.2.17**

Consider the equation

$$\begin{cases} \dot{x} = -y & = X(x, y) \\ \dot{y} = x + y^2 & = Y(x, y) \end{cases}.$$

Notice that  $X(x, -y) = -X(x, y)$ ,  $Y(x, -y) = Y(x, y)$ , hence the solution curve is symmetric with respect to  $x$ -axis. We can prove this fact by showing  $(x(-t), -y(-t))$  is a solution if  $(x(t), y(t))$  is a solution. Then we can show 0 is a center.

**§3.3 Stability under perturbations**

**Definition 3.3.1.** Consider an autonomous system  $\dot{x} = f(x)$ , generating a flow  $\varphi_t$ . For every  $x_0 \in \mathbb{R}^n$ , the orbit  $\varphi_t(x_0)$  is said to be **stable** if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|\varphi_t(x) - \varphi_t(x_0)| < \varepsilon, \quad \forall t \geq 0, x \in B(x_0, \delta).$$



**Example 3.3.2**

Consider the equation

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = r^2 \end{cases}.$$

Then the orbit of  $(r_0, \theta_0) = (1, 0)$  is **not** stable.

**Definition 3.3.3.** Consider a non-autonomous system  $\dot{x} = f(x, t)$ , let  $\varphi(t; t_0, x_0)$  be the solution of the initial value problem  $x(t_0) = x_0$ . The orbit  $x(t; t_0, x_0)$  is said to be **stable**, if, for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|\varphi(t; t_0, x) - \varphi(t; t_0, x_0)| < \varepsilon, \quad \forall t \geq t_0, x \in B(x_0, \delta).$$

Similarly, we can define the asymptotically stable and the exponentially stable for general orbits of autonomous or non-autonomous systems.

**Theorem 3.3.4**

$A : \mathbb{R} \rightarrow M(n, \mathbb{R})$ , consider a non-autonomous system  $\dot{x} = A(t)x$ . Then

1. Every solution is stable iff 0 is stable.
2. 0 is stable iff  $\sup_{t \geq 0} |X(t)| < \infty$ , where  $X(t)$  is a fundamental solution.
3. 0 is asymptotically stable iff  $\lim_{t \rightarrow \infty} |X(t)| = 0$ .

**Theorem 3.3.5**

Consider a  $T$ -periodic system  $\dot{x} = A(t)x$ . Then

2. 0 is stable iff the Floquet exponents are of non-positive real parts and Jordan block are trivial for every Floquet exponent with zero real part.
2. 0 is asymptotically stable iff Floquet exponents are of negative real parts iff 0 is exponentially stable.

For an autonomous system, let  $f(0) = 0, f(x) = Ax + \varphi(x)$ , where  $\varphi(0) = 0, D\varphi(0) = 0$ . Rewrite the system as  $\dot{x} = Ax + \varphi(x)$ , if every eigenvalue of  $A$  is with negative real parts, then 0 is stable.

For a non-autonomous system, assume

$$\dot{x} = Ax + \varphi(t, x), \quad \varphi(t, 0) = 0, D\varphi(t, 0) = 0,$$

if every eigenvalue of  $A$  is with negative real parts, then 0 is stable. In general,

$$\dot{x} = A(t)x + \varphi(t, x),$$

but the negativeness of Lyapunov exponents do **not** imply the stableness. See the following example.

**Example 3.3.6**

Consider

$$\begin{cases} \dot{x} = (-\mu - (\sin \ln t + \cos \ln t))x \\ \dot{y} = (-\mu + (\sin \ln t + \cos \ln t))y + x^2 \end{cases},$$

let  $a(t) = t \sin \ln t$ , the solutions are

$$\begin{cases} x = C_1 e^{-\mu t - a(t)} \\ y = C_2 e^{-\mu t + a(t)} + C_1^2 e^{-\mu t + a(t)} \int_1^t e^{-\mu s - 3a(s)} ds \end{cases}.$$

For  $\mu = 1 + \sigma$ ,  $\sigma$  is sufficiently small, then 0 is not stable.

For this case, we need a stronger condition. Let  $\Phi(t)$  be a fundamental solution of the linear part, if  $\exists \mu > 0$ ,

$$|\Phi(t)\Phi(-s)| \leq C e^{-\mu(t-s)}, \quad \forall t \geq s \geq 0,$$

then 0 is also stable under the perturbation .

# 4 Poincaré-Bendixson Theory

## §4.1 Basic notions

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  vector field, generating a flow  $\varphi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

**Definition 4.1.1.**  $A \subseteq \mathbb{R}^n$  is said to be  $f(\varphi_t)$  **invariant** if for every  $t \in \mathbb{R}$ ,  $\varphi_t(A) = A$ .

For every  $x \in \mathbb{R}^n$ , the orbit  $\text{Orb}(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$  is an invariant set. In general, if  $A$  is invariant, then

$$A = \text{Orb}_{x \in A} \text{Orb}(x).$$

**Definition 4.1.2.** Let  $A$  be a compact invariant set,  $A$  is said to be **Lyapunov orbit stable** if for every neighborhood  $U \supseteq A$ , there exists a neighborhood  $V \supseteq A$  such that

$$\varphi_t(x) \in U, \quad \forall x \in V, t \geq 0.$$

Let

$$\text{Orb}^+ := \{\varphi_t(x) : t \geq 0\}, \quad \text{Orb}^- := \{\varphi_t(x) : t \leq 0\}$$

be the **positive semi-orbit** and the **negative semi-orbit**.

**Definition 4.1.3.** Given  $p \in \mathbb{R}^n$ ,  $x$  is called a **positive limit point** if  $\exists t_n \rightarrow +\infty$ ,  $\varphi_{t_n} \rightarrow x$ . The set of all positive limit points is called the  **$\alpha$ -limit set** of  $p$ , denoted by  $\alpha(p)$ . Similarly, we can define the **negative limit points**, they form a set is called  **$\omega$ -limit set**, denoted by  $\omega(p)$ .

**Remark 4.1.4** — In the Greek alphabet,  $\alpha$  is the first letter and  $\omega$  is the last letter, it is very graphic that the orbit of  $p$  ran from  $\alpha$  to  $\omega$ .

### Example 4.1.5

Consider the equation

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases}.$$

Then  $\omega(0) = \alpha(0) = 0$ . For every  $p \in \mathbb{S}^1$ , we have  $\omega(p) = \alpha(p) = \mathbb{S}^1$ . Otherwise, let  $p = (x, y)$ , we have

- (1) If  $0 < x^2 + y^2 < 1$ , then  $\omega(p) = \mathbb{S}^1, \alpha(p) = \{0\}$ .
- (2) If  $x^2 + y^2 > 1$ , then  $\omega(p) = \mathbb{S}^1, \alpha(p) = \emptyset$ .

### Proposition 4.1.6

$\forall p \in \mathbb{R}^n$ , we have

$$\omega(p) = \bigcap_{t \geq 0} \overline{\text{Orb}^+(\varphi_t(p))} = \bigcap_{k \in \mathbb{Z}_+} \overline{\text{Orb}^+(\varphi_k(p))}.$$

**Proposition 4.1.7**

Assume  $\text{Orb}^+(p)$  is bounded, then

1.  $\omega(p)$  is non-empty, compact, invariant, connected.
2.  $\lim_{t \rightarrow \infty} d(\varphi_t(p), \omega(p)) = 0$ .

*Proof.* 1. Non-empty, compact, invariant are trivial. The connected follows by the fact that  $A_k = \text{Orb}^+(\varphi_k(p))$  are connected and  $A_k \supseteq A_{k+1} \supseteq \dots$ .

2. For every  $\varepsilon > 0$ ,  $A_k \subseteq B(\omega(p), \varepsilon)$  for every  $k$  sufficiently large. □

**Definition 4.1.8.**  $p$  is said to be **positively recurrent** if  $p \in \omega(p)$ ,

The singularities and periodic points are called trivial recurrent points, other recurrent points are said to be non-trivial.

**Definition 4.1.9.** Let  $\Lambda$  be a non-empty, compact, invariant set.  $\Lambda$  is called a **minimal set** of  $\varphi_t$  if it does not contain a proper, nonempty, compact invariant set.

**Theorem 4.1.10 (Flow Box Theorem)**

Let  $f$  be a  $C^1$  vector field,  $p \in \mathbb{R}^n$ ,  $f(p) \neq 0$ . Then there is a neighborhood  $U \ni p$  and a  $C^1$  diffeomorphism  $h : U \rightarrow h(U)$  on to its image, such that  $Dh(x)f(x) = (1, 0, \dots, 0)^t$ .

*Proof.* WLOG,  $p = 0$ ,  $f(p) = (1, 0, \dots, 0)^t$ . We construct  $g : (-\varepsilon_0, \varepsilon_0) \times L \rightarrow U$  some neighborhood of  $p$ . Let

$$x = g(y) = g(y_1, y_2, \dots, y_n) := \varphi_{y_1}(0, y_2, y_3, \dots, y_n).$$

Then

$$\left. \frac{\partial}{\partial t} \varphi_t(x) \right|_{(t,x)=(y_1,0,y_2,\dots,y_n)} = f(\varphi_{y_1}(0, y_2, \dots, y_n)) = f(g(y)),$$

let  $(y_1, y_2, \dots, y_n) = (0, 0, \dots, 0)$ , then  $\frac{\partial g}{\partial y_1}(y) = f(g(y))$ . Moreover,

$$\text{Id} = \left. \frac{\partial \varphi_t(x_1, \dots, x_n)}{\partial (x_1, \dots, x_n)} \right|_{t=0} \implies \left. \frac{\partial g}{\partial y} \right|_{y=0} = \text{Id}.$$

Hence,  $g$  gives a local diffeomorphism. Let  $h = g^{-1}$ , the statement follows. □

**Remark 4.1.11** — Let  $L_{\varepsilon_0} = \{(y_2, \dots, y_n) : y_2^2 + \dots + y_n^2 \leq \varepsilon_0^2\}$ , let

$$U = h^{-1}((-\varepsilon_0, \varepsilon_0) \times L_{\varepsilon_0}),$$

then  $U$  is called a **tubular neighborhood** near  $p$ , or a **flow box** near  $p$ .

## §4.2 The Poincaré-Bendixson Theorem

**Definition 4.2.1.**  $C \subseteq \mathbb{R}^2$  is called a **Jordan curve** if it is homeomorphism to  $\mathbb{S}^1$ .

### Theorem 4.2.2 (Jordan Separation Theorem)

Let  $C \subseteq \mathbb{R}^2$  be a Jordan curve. Then  $\mathbb{R}^2 \setminus C$  has exactly two connected components. One of them is bounded, which is called the interior of  $C$ . Another one is bounded, which is called the exterior of  $C$ . Both of them are with bound  $C$ .

### Theorem 4.2.3 (Jordan-Schoenflies)

Let  $C$  be a Jordan curve, then there is a homeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , such that  $h(C) = \mathbb{S}^1$ .

**Definition 4.2.4.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be  $C^1$ ,  $L \subseteq \mathbb{R}^2$  is a line segment.  $L$  is called **transverse** to  $f$  if  $\forall x \in L$ ,  $f(x)$  and the direction of  $L$  generates  $\mathbb{R}^2$ . We then say  $L$  is a **transversal** to  $f$ .

### Lemma 4.2.5

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be  $C^1$ ,  $L$  is a transversal to  $f$ . Assume there are three points  $P_1, P_2, P_3 \in L$  and  $x \in \mathbb{R}^2$  such that

$$\varphi_{t_i}(x) = P_i, \quad t_1 < t_2 < t_3,$$

$$\varphi_{(t_1, t_2)}(x) \cap L = \emptyset, \quad \varphi_{(t_2, t_3)}(x) \cap L = \emptyset,$$

then  $P_2 \in (P_1, P_3)$ .

*Proof.* Assume  $A, B$  are extreme points of  $L$ . Consider a Jordan curve

$$C = \varphi_{[t_1, t_2]}(x) \cup (P_1, P_2),$$

let  $D$  be the interior of  $C$ . Assume  $B \in D$ , we show that  $P_3 \in D$ . By the Flow Box Theorem, there exists  $\varepsilon > 0$  such that  $\varphi_{(t_2, t_2+\varepsilon]}(x) \subseteq D$ . Let  $\tau = \inf \{t > t_2 : \varphi_t(x) \notin D\} > t_2 + \varepsilon$  if exists. Then  $\varphi_\tau(x) \in C$ , but it can not on  $\varphi_{(t_1, t_2)}(x)$  or  $P_1, P_2$ . So  $\varphi_\tau(x) \in (P_1, P_2)$ , but this contradict with  $L$  is a transversal to  $f$ .  $\square$

**Remark 4.2.6** — Assume  $\varphi_t(x)$  intersect with a transversal  $L$  at  $P_i = \varphi_{t_i}(x), i = 1, 2, \dots$  in chronological order, i.e.,  $0 < t_1 < t_2 < \dots$ , then

$$P_1 < P_2 < \dots \quad \text{or} \quad P_1 > P_2 > \dots \quad \text{or} \quad P_1 = P_2 = \dots.$$

### Proposition 4.2.7

Assume  $L$  is a transversal of  $f$ , then for every  $x \in \mathbb{R}^2$ ,

$$\sharp(\omega(x) \cap L) \leq 1.$$

*Proof.* Assume for a contradiction. Let  $q \neq q' \in \omega(x) \cap L$ , then  $\exists t_n \rightarrow \infty, t'_n \rightarrow \infty$  such that  $\varphi_{t_n}(x) \rightarrow q, \varphi_{t'_n}(x) \rightarrow q'$ . WLOG, assume  $t_1 < t'_1 < t_2 < t'_2 < \dots$ . By the Flow Box Theorem, for  $k$  sufficiently large, there exists  $\tau_k, \tau'_k$  such that

$$|\tau_k - t_k| \rightarrow 0, |\tau'_k - t'_k| \rightarrow 0, \quad \varphi_{\tau_k}(x), \varphi_{\tau'_k}(x) \in L, \quad \varphi_{\tau_k}(x) \rightarrow q, \varphi_{\tau'_k}(x) \rightarrow q'.$$

We can also assume that  $\tau_k < \tau'_k < \tau_{k+1} < \dots$ , then this contradicts with the monotonicity of  $\varphi_t(x)$  intersecting the transversal.  $\square$

#### Theorem 4.2.8 (Poincaré-Bendixson Theorem)

Assume  $\text{Orb}^+(x)$  is bounded and  $\omega(x)$  contains no singularities, then  $\omega(x)$  is a periodic orbit.

*Proof.* Because  $\text{Orb}^+(x)$  is bounded,  $\omega(x) \neq \emptyset$ . For every  $p \in \omega(x)$ , take  $q \in \omega(p) \subseteq \omega(x)$  arbitrarily. Take a transversal  $L_q$  of  $f$  through  $q$ , then  $\exists t_n \rightarrow \infty, \varphi_{t_n}(p) \rightarrow q$ . WLOG,  $\varphi_{t_n}(x) \in L_q$ . Because  $\varphi_{t_n}(p) \in \omega(x)$  and  $\sharp \omega(x) \cap L_q = 1$ , then  $\varphi_{t_n}(p) = \varphi_{t_{n+1}}(p)$ , hence  $p$  is a periodic point.

Take  $p \in \omega(x)$ , it is a periodic point. If  $\omega(x) \neq \text{Orb}(x)$ , take a transversal  $L_p$  of  $f$  through  $p$ . Because  $\omega(x)$  is connected, hence  $\text{Orb}(p)$  is not isolated in  $\omega(x)$ . Take  $q_n \in \omega(x) \setminus \text{Orb}(p), q_n \rightarrow \text{Orb}(p)$ . WLOG,  $q_n \rightarrow p$  and  $q_n \in L_p$ , this contradicts with  $\sharp \omega(x) \cap L_p \leq 1$ .  $\square$

#### Theorem 4.2.9 (P-B Annular Region Theorem)

Assume  $A$  is an annular region and  $\partial A$  is two  $C^1$  curves. If for every  $x \in \partial A$ ,  $f(x)$  is pointing inside of  $A$ , and  $A$  contains no singularities. Then there is a periodic orbit in  $A$ .

#### Example 4.2.10

The system

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 3x - 1) - y \\ \dot{y} = y(x^2 + y^2 - 3x - 1) + x \end{cases}$$

contains a non-trivial periodic orbit.

*Proof.* 0 is the only singularity. Let

$$A = \{(x, y) \in \mathbb{R}^2, r^2 \leq x^2 + y^2 \leq R^2\}, \quad r < R,$$

let  $V(x, y) = x^2 + y^2$ . Then for  $r$  small enough  $\dot{V} < 0$ , for  $R$  large enough  $\dot{V} > 0$ . Hence  $f(x)$  is pointing outside of  $A$  on  $\partial A$ , consider the  $\alpha$ -limit set.  $\square$

The **Liénard equation**

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

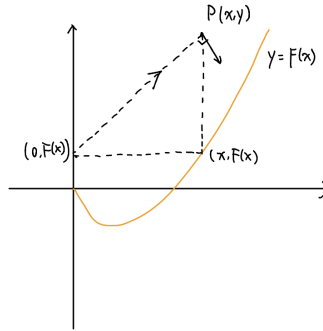
Let  $F(x) = \int_0^x f(t)dt$ , then the equation is equivalent to

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}.$$

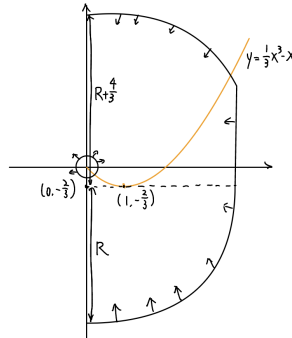
Consider a particular Liénard equation, which is called **van der Pol equation**:  $f(x) = x^2 - 1, g(x) = x$ . Then this equation is equivalent to

$$\begin{cases} \dot{x} = y - (\frac{1}{3}x^3 - x) \\ \dot{y} = -x \end{cases}.$$

We introduce the **Liénard graphing method**. Draw the graph of  $y = F(x)$ , for  $P = (x, y) \in \mathbb{R}^2$ , draw  $(x, F(x))$  and  $Q = (0, F(x))$ . Hence the slope of  $QP = \frac{F(x)-y}{-x}$ , then rotate it  $90^\circ$  clockwise, we get  $f(x, y)$ .



Consider  $V = x^2 + y^2$ , then  $\dot{V} = 2x^2(1 - \frac{1}{3}x^2) \geq 0$  when  $|x| < 1$ . Take  $x = 1$ , then  $F(x) = -\frac{2}{3}$  attain the minimum. For  $R$  sufficiently large, we can construct a curve such that the poin, see the figure. For  $r$  small enough, vectors on  $B_r$  are pointing outside. This gives an annular region and hence there is a periodic orbit in it. Furthermore, we can prove that the periodic orbit is unique.



#### Theorem 4.2.11 (P-B)

$f$  is a  $C^1$  vector field, assume there are only finite singularities. If  $\text{Orb}^+$  is bounded, then there is a trichotomy:

- (1)  $\omega(x)$  is a periodic orbit.
- (2)  $\omega(x)$  is a singularity.
- (3)  $\omega(x)$  contains regular points and singularities and  $\forall y \in \omega(x), \alpha(y), \omega(y)$  are both a singularity.

*Proof.* If  $\omega(x)$  contains no singularities, by P-B,  $\omega(x)$  is a periodic point. If  $\omega(x)$  contains no regular points, then  $\omega(x)$  is finite singularities, but  $\omega(x)$  is connected, hence  $\omega(x)$  is a singularity.

If  $\omega(x)$  contains regular points and singularities both, for  $y \in \omega(x)$ , assume for a contradiction that  $\omega(y)$  contains some regular points. Take  $z \in \omega(y)$  regular and a transversal  $L_z$  contains  $z$ . Take  $t_n \rightarrow +\infty, \varphi_{t_n}(y) \rightarrow z, \varphi_{t_n}(y) \in L_z$ . Same argument shows that  $y$  is a periodic point.

Take a transversal  $L_y$  contains  $y$ , take  $\tau_n \rightarrow +\infty, \varphi_{\tau_n}(x) \rightarrow y, \varphi_{\tau_n}(x) \in L_y$ . Because  $\omega(x)$  contains some singularities, then  $x$  is not a periodic point, hence  $\varphi_{\tau_n}(x) \neq y$ . Let

$$P_n = \varphi_{\tau_n}(x), \quad C_n = \varphi_{[\tau_n, \tau_{n+1}]}(x) \cap [P_n, P_{n+1}],$$

then  $C_n \rightarrow \text{Orb}(y)$ , hence we can show that  $\omega(x) = \text{Orb}(y)$ . A contradiction.  $\square$

### §4.3 Poincaré recurrence and limit cycle

#### Lemma 4.3.1

Let  $f$  be a  $C^1$  vector field, generating a flow  $\varphi_t$ . Let  $p$  be a regular point,  $L_t$  be a transversal of  $f$  at  $\varphi_t(p)$ . Then there is a neighborhood  $U \ni p$  and a  $C^1$  map:  $U \rightarrow \mathbb{R}$  such that

$$\tau(p) = t, \quad \varphi_{\tau(x)}(x) \in L_t, \forall x \in U.$$

*Proof.* WLOG, assume  $L_t \perp f(\varphi_t(x))$ . Let  $Q = \varphi_t(p)$ , consider

$$F(\tau, x) : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (\tau, x) \mapsto (\varphi_\tau(x) - Q) \cdot f(Q).$$

Then  $\varphi_\tau(x) \in L_t$  iff  $F(\tau, x) = 0$ . We have

$$\left. \frac{\partial F}{\partial \tau} \right|_{(\tau, x) = (t, p)} = f(\varphi_\tau(x))f(Q)|_{(\tau, x) = (t, p)} = |f(Q)|^2 > 0,$$

The conclusion follows by implicit function theorem.  $\square$

#### Corollary 4.3.2

Let  $f$  be a  $C^1$  vector field, generating a flow  $\varphi_t$ . For every regular point  $p$  and two transversals  $L_0, L_t$  at  $p$  and  $\varphi_t(p)$ , respectively. There exists an neighborhood  $I_0$  of  $p$  in  $L_0$  and a  $C^1$  map  $\tau : I_0 \rightarrow \mathbb{R}$  such that

$$\tau(p) = t, \quad \varphi_{\tau(x)}(x) \in L_t, \forall x \in I_0.$$

Furthermore, let  $P : I_0 \rightarrow L_t, x \mapsto \varphi_{\tau(x)}(x)$ , then  $P$  is a diffeomorphism onto image.

*Proof.* Assume  $L_t : (X - Q) \cdot f(Q) = 0$ , then  $\tau$  is given by  $f(Q)^t(\varphi_\tau(x) - Q)$ . Hence

$$D\tau = -\frac{f(Q)^t D\varphi_\tau(x)}{f(Q)^t f(\varphi_t(x))}.$$

Let  $I_0$  be a connected component of  $U \cap L_0$  containing  $x$ , define the **Poincaré map**

$$P : I_0 \rightarrow L_t, \quad x \mapsto \varphi_{\tau(x)}(x).$$



Then, for  $v \in T_p L_0$ .

$$DP(p)v = \frac{\partial \varphi}{\partial \tau} \frac{\partial \tau}{\partial x} v + D\varphi_{\tau(x)}(x)v|_{x=p} = -f(Q) \frac{f(Q)^t D\varphi_t(x)}{f(Q)^t f(Q)} + D\varphi_t(p)v,$$

it is the orthogonal projection of  $D\varphi_t(x)v$  to  $L_t$ . It suffices to show  $DP(p) \neq 0$ . Otherwise,  $D\varphi_t(p)v = \eta f(Q)$  for some  $v$ , but  $D\varphi_t(p)f(p) = f(Q)$ , a contradiction with  $D\varphi_t(p)$  is an isomorphism.  $\square$

Let  $p$  be a  $T$ -periodic point, then  $D\varphi_T f(p) = f(p)$ . Assume  $f(p) \parallel \mathbf{e}_1$ , then

$$D\varphi_T(p) = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}.$$

Take  $L_0 = L_T \parallel \mathbf{e}_2$ , then  $DP(p)e_2 = d \cdot \mathbf{e}_2$ . We choose a coordinate on  $L_0$ , assume  $p = 0$ , then  $P : I_0 \rightarrow L_0$  is a real-valued function on  $\mathbb{R}$  such that  $P'(0) = d$ .

### Proposition 4.3.3

Assume  $p$  is a  $T$ -periodic point of  $C^1$  vector field, then  $P'(0) = \det D\varphi_T(p) > 0$ .

*Proof.*  $t \mapsto \det D\varphi_t(p)$  is continuous, nonzero and  $0 \mapsto 1$ , hence is positive.  $\square$

**Definition 4.3.4.** An isolated closed orbit is called a **limit cycle**.

### Lemma 4.3.5

Let  $\gamma$  be a periodic point,  $L$  is a transversal of some point on  $\gamma$ . Let  $P : I \rightarrow L$  be the Poincaré recurrent map,  $0 \in I$  correspond to the given periodic point. Then  $x \in I$  is a fixed point of  $P$  if and only if  $x$  is a periodic point.

*Proof.* Let  $x$  be a periodic point of  $\varphi$ . If  $P(x) \neq x$ , then  $P(x) \in \text{Orb}(x) = \omega(x)$ , but  $\sharp(\omega(x) \cap L_0) \leq 1$ , a contradiction.  $\square$

**Definition 4.3.6.** Let  $\gamma$  be a limit cycle. Then  $\gamma$  is said:

- (1) **stable**, if there exists a neighborhood  $U$  of  $\gamma$ , such that  $\forall x \in U, \omega(x) = \gamma$ .
- (2) **unstable**, if there exists a neighborhood  $U$  of  $\gamma$ , such that  $\forall x \in U, \alpha(x) = \gamma$ .
- (3) **semi-stable**, if there exists a neighborhood  $U$  of  $\gamma$  and a splitting  $U \setminus \gamma = U_+ \cup U_-$ , such that  $\forall x \in U_+, \omega(x) = \gamma$  and  $\forall x \in U_-, \alpha(x) = \gamma$ .

### Proposition 4.3.7

Let  $\gamma$  be a limit cycle and  $p \in \gamma$ . If  $P'(p) < 1$ , then  $\gamma$  is stable. If  $P'(p) > 1$ , then  $\gamma$  is unstable.

Compare with the definition of Lyapunov asymptotically stable, this definition does not guarantee the Lyapunov stability. But under the condition  $P'(p) < 1$ , we can show that  $\forall \varepsilon > 0, \exists \delta > 0$  such that if  $|x - p| < \delta$ , then  $|\varphi_t(x) - \varphi_t(p)| < \varepsilon$  for every  $t \geq 0$ . It suffices to show for  $x \in L$ . Note that

$$P^n(x) = \varphi_{\tau(x) + \tau(P(x)) + \dots + \tau(P^{n-1}(x))}(x),$$

because  $|P^n(x) - P^n(p)| \leq \lambda^n |x - p|$  for some  $\lambda < 1$ , we can bound the time  $|\tau(x) + \tau(P(x)) + \dots + \tau(P^{n-1}(x)) - nT|$  by some  $\frac{C}{1-\lambda} |x - p|$ .

**Definition 4.3.8.** Let  $p$  be a periodic point, if the every eigenvalue of  $DP(p)$  is not on the unit circle, then  $p$  is called **hyperbolic**.

**Exercise 4.3.9.**  $n \geq 2$ , let  $p$  be a  $T$ -periodic point, then  $p$  is hyperbolic if and only if there are  $n - 1$  eigenvalues of  $D\varphi_T(p)$  are with absolute value  $\neq 1$ .

**Example 4.3.10**

Let  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ , be a  $C^2$  function. Consider the **plane Hamilton system**

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases},$$

then there is not limit cycle in this system.

*Proof.* Let  $\Phi_t(x, y) = D\varphi_t(x, y)$ , then  $\det \Phi_t(x, y) = 1$ . Hence  $\varphi_t$  is area preserving. This contradicts with the existence of limit cycle.  $\square$

**Hilbert's 16-th problem** Consider the plane polynomial system

$$\begin{cases} \dot{x} = P_n(x, y) = \sum_{i+j=0}^n a_{ij}x^i y^j, \\ \dot{y} = Q_n(x, y) = \sum_{i+j=0}^n b_{ij}x^i y^j. \end{cases}$$

The second part of this problem is considering about the number of limit cycles in this system. For a given system, Dulac-Ilyashenko-Écalle proved that the number of limit cycles is finite.

Furthermore, let  $H(n)$  be the upper bound of all systems with degree at most  $n$ . A further problem is whether  $H(n)$  is finite. It trivial that  $H(1) = 0$ , but the existence of  $H(2)$  is still unknown.

# 5 Bifurcations

## §5.1 Structural stability

Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be two  $C^1$  vector fields. The  $C^1$  distance of  $f, g$  is defined as

$$|f - g|_{C^1} := \sup \{ |f(x) - g(x)|, |Df(x) - Dg(x)| : x \in \mathbb{R}^n \}.$$

**Definition 5.1.1.** Given a  $C^1$  vector field  $f$  on  $\mathbb{R}^n$ ,  $f$  is called  **$C^1$  structurally stable** if there exists  $\varepsilon > 0$  such that for every  $C^1$  vector field  $g$ ,  $|g - f|_{C^1} < \varepsilon$ , then  $g$  is topologically equivalent with  $f$ .

**Remark 5.1.2** — The condition is just “topologically equivalent” is reasonable. A  $C^1$ -conjugate preserves the derivation at a singularity. A  $C^0$ -conjugate preserves the time on a periodic orbit. Both conditions are too much for a general vector field.

### Example 5.1.3

$f(x) = x, x \in \mathbb{R}$  is  $C^1$  structurally stable.

**Exercise 5.1.4.** Show that the plane system

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases}$$

is  $C^1$  structurally stable.

**Exercise 5.1.5.** Show that a center is not structurally stable.

Let  $\overline{\mathbb{D}}$  be the closed unit disk  $\{x^2 + y^2 \leq 1\}$ , let  $\mathcal{X}^1(\overline{\mathbb{D}})$  be the set of  $C^1$  vector fields on  $\overline{\mathbb{D}}$  which are not tangent to  $\partial\mathbb{D}$ .

### Theorem 5.1.6 (Andronov-Pontryagin)

Given  $f \in \mathcal{X}^1(\overline{\mathbb{D}})$ , then  $f$  is  $C^1$  structurally stable if and only if the followings hold:

- all singularities and periodic orbits are hyperbolic,
- there are no saddle connections.

### Corollary 5.1.7

Under the same condition, given  $x \in \overline{\mathbb{D}}$ , if  $\text{Orb}(x) \subset \overline{\mathbb{D}}$ , then  $\omega(x)$  is a periodic point or a singularity.

**Theorem 5.1.8** (Peixoto-Pugh)

Let  $M$  be a 2-dimensional closed Riemannian manifold. Given  $f \in \mathcal{X}^1(M)$ , then  $f$  is  $C^1$  structurally stable if and only if the followings hold:

- all singularities and periodic orbits are hyperbolic,
- there are no saddle connections.
- for every  $x \in M$ ,  $\omega(x)$  is a periodic point or a singularity.

**Example 5.1.9**

Consider a system on  $\mathbb{T}^2$ ,

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha, \end{cases}$$

if  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , then this system satisfying the first two conditions. But this system is not  $C^1$  structurally stable. Hence for a general Riemannian manifold, the third condition is necessary.

**Remark 5.1.10** — Peixoto has proved  $C^r$  structural stability for orientable  $M$ . For general surface  $M$ , Pugh applied  $C^1$ -closed theorem to show the statement. But  $C^r$ -closed theorem is still a big open problem.

## §5.2 Bifurcations

**Definition 5.2.1.** Let  $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$  be a  $C^1$  map, which is a family of vector fields with  $k$ -parameters. The parameter  $\epsilon_0 \in \mathbb{R}^k$  is called a **bifurcation** if  $\forall \delta > 0, \exists \epsilon_1 \in \mathbb{R}^k, |\epsilon_1 - \epsilon_0| < \delta$ , such that  $f_{\epsilon_1}$  is not topologically equivalent with  $f_{\epsilon_0}$ .

**Example 5.2.2**

Consider  $f_\beta(x, y) = (-\beta y, \beta x)$ , then  $f_\beta$  is not  $C^1$  structurally stable, but is also not a bifurcation.

**Remark 5.2.3** — A general problem is how to embed a non structurally stable vector field into a vector field family to be a bifurcation. These embeddings are called an **unfolding**. Some topics are to find a **universal unfolding**.

**Example 5.2.4**

Consider  $f_\epsilon(x) = x(\epsilon - x)$ , then  $\epsilon = 0$  is a bifurcation. This bifurcation is called a **transcritical bifurcation**: there are two hyperbolic singularities before and after the collision, and after the collision, the stability of two singularities exchange.

**Example 5.2.5**

Consider  $f_\epsilon(x) = \epsilon - x^2$ , then  $\epsilon = 0$  is a bifurcation. This bifurcation is called a **saddle-node bifurcation**: it can be regarded as a collision of a saddle and a node, there is a non-hyperbolic singularity at the bifurcation which is called a saddle-node.

It comes from the plane vector fields  $f_\epsilon(x, y) = (\epsilon - x^2, y)$ , when  $\epsilon = 0$ , the singularity seems like a node on one side and saddle on another side.

**Example 5.2.6**

Consider  $f_\epsilon(x) = x(\epsilon - x^2)$ , then  $\epsilon = 0$  is a bifurcation. This bifurcation is called a **pitchfork bifurcation**:  $x = 0$  is always a singularity, it is non-hyperbolic only if  $\epsilon = 0$ , after that, it create two hyperbolic singularities, one stable and one unstable.

**Example 5.2.7**

Consider the system

$$\begin{cases} \dot{x} = \epsilon x - y - x(x^2 + y^2) \\ \dot{y} = x + \epsilon y - y(x^2 + y^2) \end{cases} \quad \begin{cases} \dot{r} = \epsilon r - r^3 \\ \dot{\theta} = 1 \end{cases}.$$

It is called a **Hopf bifurcation** and corresponds to a creation of periodic orbit (at  $r = \sqrt{\epsilon}$  for  $\epsilon > 0$ ).

# 6 Index Theory and Dynamics on $\mathbb{S}^1$

## §6.1 Index

**Definition 6.1.1.** A **path** in  $\mathbb{R}^2$  is a continuous map  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ .

Let  $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$  be the covering map,  $\pi(\theta) = (\cos \theta, \sin \theta)$ .

### Theorem 6.1.2

Let  $\gamma : [a, b] \rightarrow \mathbb{S}^1$  be a continuous map, then for every  $\theta_0 \in \pi^{-1}(\gamma(a))$ , there exists a unique continuous map  $\theta : [a, b] \rightarrow \mathbb{R}$ ,  $\theta(a) = \theta_0$  such that  $\gamma = \pi \circ \theta$ .

### Lemma 6.1.3

Given a path  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  and a continuous vector field  $f = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . If  $f|_\gamma \neq 0$ , then there exists  $\theta : [a, b] \rightarrow \mathbb{R}$ , such that

$$\begin{cases} \cos \theta(t) = \frac{P(\gamma(t))}{\sqrt{P^2(\gamma(t)) + Q^2(\gamma(t))}} \\ \sin \theta(t) = \frac{Q(\gamma(t))}{\sqrt{P^2(\gamma(t)) + Q^2(\gamma(t))}} \end{cases},$$

i.e.  $\pi \circ \theta = \frac{f \circ \gamma}{|f \circ \gamma|}$ . If there exists another continuous map  $\tilde{\theta}$  such that  $\pi \circ \tilde{\theta} = \frac{f \circ \gamma}{|f \circ \gamma|}$ , then  $\exists k \in \mathbb{Z}$  such that  $\forall t \in [a, b], \tilde{\theta}(t) - \theta(t) = 2k\pi$ . In particular,  $\theta(b) - \theta(a) = \tilde{\theta}(b) - \tilde{\theta}(a)$ .

**Definition 6.1.4.** A path  $\gamma : [a, b]$  is said to be **closed** if  $\gamma(a) = \gamma(b)$ .

**Definition 6.1.5.** Let  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a closed path,  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a continuous vector field,  $f|_\gamma \neq 0$ . Then

$$\frac{\theta(b) - \theta(a)}{2\pi}$$

is called the **index** of  $\gamma$  with respect to  $f$ , denoted by  $\text{Ind}_f(\gamma)$ . Where  $\theta : [a, b] \rightarrow \mathbb{R}$  satisfying  $\pi \circ \theta = \frac{f \circ \gamma}{|f \circ \gamma|}$ .

### Lemma 6.1.6

$\text{Ind}_f(\gamma) \in \mathbb{Z}$  and the index is independent with the choice of  $\theta$ .

### Theorem 6.1.7 (Hopf)

Let  $f$  be a  $C^1$  vector field, let  $\gamma : [0, T] \rightarrow \mathbb{R}^2$  be a periodic orbit of  $f$ , i.e.,  $\dot{\gamma}(t) = f(\gamma(t))$ ,  $\gamma(0) = \gamma(T)$ , then  $\text{Ind}_f(\gamma) = \pm 1$ .

*Proof.* WLOG, we can assume that  $\gamma(0) = 0$  is the lowest point on  $\gamma([0, T])$ , and  $\dot{\gamma}(t)$  is pointing the positive direction of  $x$ -axis (this might take the index to its opposite). Let  $\Delta = \{(s, t) : 0 \leq s \leq t \leq T\}$ , we define the map  $\eta : \Delta \rightarrow \mathbb{S}^1$  as

$$\eta(s, t) = \begin{cases} \frac{f(\gamma(t))}{|f(\gamma(t))|}, & s = t, \\ -\eta(0, 0), & (s, t) = (0, T), \\ \frac{\gamma(t) - \gamma(s)}{|\gamma(t) - \gamma(s)|}, & s \neq t, (s, t) \neq (0, T). \end{cases}$$

Then  $\eta : \Delta \rightarrow \mathbb{S}^1$  is continuous. There exists a unique continuous map  $\phi : \Delta \rightarrow \mathbb{R}$ ,  $\phi(0, 0) = 0$  and  $\pi \circ \phi = \eta$ . By definition, we have

$$\text{Ind}_f(\gamma) = \frac{\phi(T, T) - \phi(0, 0)}{2\pi}.$$

On the other hand, note that  $\phi(0, t) \in [0, \pi]$  and  $\phi(t, T) \in [\pi, 2\pi]$ , we have

$$\phi(0, T) - \phi(0, 0) = \pi - 0 = \pi \quad \text{and} \quad \phi(T, T) - \phi(0, T) = 2\pi - \pi = \pi.$$

Hence  $\text{Ind}_f(\gamma) = 1$ . □

**Definition 6.1.8.** Let  $\gamma_0, \gamma_1 : [a, b] \rightarrow \mathbb{R}^2$  be two paths,  $\gamma_0$  and  $\gamma_1$  are called **homotopic** if  $\exists H : [a, b] \times [0, 1] \rightarrow \mathbb{R}^2$  continuous such that  $H(\cdot, 0) = \gamma_0, H(\cdot, 1) = \gamma_1$ .  $H$  is called a **homotopy** of  $\gamma_0$  and  $\gamma_1$ .

#### Theorem 6.1.9

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a continuous vector field, let  $H : [a, b] \times [0, 1] \rightarrow \mathbb{R}^2$  be a homotopy of closed paths  $\gamma_0$  and  $\gamma_1$ . If  $f \circ H \neq 0$ , then  $\text{Ind}_f(\gamma_0) = \text{Ind}_f(\gamma_1)$ .

#### Theorem 6.1.10

Let  $f : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^2$  be a continuous family of vector field,  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a closed path such that  $f_s(\gamma(t)) \neq 0$  for every  $s, t$ . Then  $\text{Ind}_{f_s}(\gamma)$  is a constant with respect to  $s$ .

#### Corollary 6.1.11

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a continuous vector field, if  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  is a Jordan curve and there is no singularity of  $f$  on  $\gamma$  or inside  $\gamma$ , then  $\text{Ind}_\gamma = 0$ .

#### Corollary 6.1.12

Let  $f$  be a  $C^1$  vector field, if  $\gamma : [0, T] \rightarrow \mathbb{R}^2$  is a periodic orbit, then there exists a singularity inside  $\gamma$ .

**Theorem 6.1.13**

Let  $f$  be a continuous vector field,  $\gamma$  be a  $C^1$  Jordan curve. If  $f$  is transverse to  $\gamma$ , then  $\text{Ind}_f(\gamma) = \pm 1$ .

*Proof.* Let

$$f_s(\gamma(t)) = sf(\gamma(t)) + (1-s)\dot{\gamma}(t),$$

then  $f_s(\gamma(t)) \neq 0$ , hence

$$\text{Ind}_f(\gamma) = \text{Ind}_{f_1}(\gamma) = \text{Ind}_{f_0}(\gamma) = \text{Ind}_{\dot{\gamma}}(\gamma) = \pm 1.$$

□

**Corollary 6.1.14**

Let  $f$  be a continuous vector field,  $\gamma$  be a  $C^1$  Jordan curve. If  $f$  is transverse to  $\gamma$ , then there exists a singularity inside  $\gamma$ .

**§6.2 Index of a singularity**

**Definition 6.2.1.** Let  $f$  be a continuous vector field,  $\sigma$  is a singularity.  $\sigma$  is called an **isolated singularity** if  $\exists \varepsilon_0 > 0$  such that  $\sigma$  is the unique singularity in  $B(\sigma, \varepsilon_0)$ .

**Lemma 6.2.2**

Let  $\sigma$  be an isolated singularity, then for every  $\varepsilon \in (0, \varepsilon_0]$ ,  $\text{Ind}_f(\gamma_\varepsilon) = \text{Ind}_f(\gamma_{\varepsilon_0})$ , where

$$\gamma_\varepsilon : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad \gamma_\varepsilon(t) = \sigma + \varepsilon(\cos t, \sin t).$$

**Definition 6.2.3.** Let  $f$  be a continuous vector field and  $\sigma$  is an isolated singularity of  $f$ . There exists  $\varepsilon_0 > 0$  such that  $\sigma$  is the unique singularity in  $B(\sigma, \varepsilon_0)$ . Then  $\text{Ind}_f(\gamma_{\varepsilon_0})$  is called the **index of isolated singularity**  $\sigma$ , denoted by  $\text{Ind}_f(\sigma)$ .

**Lemma 6.2.4**

Let  $f = (P, Q)$  be a  $C^1$  vector field,  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a path such that  $f|_\gamma \neq 0$ . Let  $\theta : [a, b] \rightarrow \mathbb{R}$  be a lifting of  $\frac{f \circ \gamma}{|f \circ \gamma|}$ , then

$$\theta(b) - \theta(a) = \int_\gamma \frac{P dQ - Q dP}{P^2 + Q^2}.$$

In particular, if  $\gamma$  is a closed path, then

$$\text{Ind}_f(\gamma) = \frac{1}{2\pi} \oint_\gamma \frac{P dQ - Q dP}{P^2 + Q^2}.$$



**Theorem 6.2.5**

Let  $f$  be a  $C^2$  vector field and  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a  $C^1$  Jordan curve with positive direction,  $f \circ \gamma \neq 0$ . If there are only finite singularities inside  $\gamma$ , denoted by  $\sigma_1, \dots, \sigma_n$ , then

$$\text{Ind}_f(\gamma) = \text{Ind}_f(\sigma_1) + \text{Ind}_f(\sigma_2) + \dots + \text{Ind}_f(\sigma_n).$$

**Remark 6.2.6** — This theorem also is also valid for just continuous settings.

**Example 6.2.7**

Consider

$$f_{\alpha, \beta} = (\alpha x - \beta y, \beta x + \alpha y), \quad (\alpha, \beta) \neq (0, 0).$$

Then  $O$  is the unique singularity, consider  $\gamma = \gamma(t) = (\cos t, \sin t)$ . Then  $\text{Ind}_{f_{\alpha, \beta}}(\gamma)$  is the constant 1.

**Example 6.2.8**

Consider

$$g_{\lambda, \mu}(x, y) = (\lambda x, \mu y), \quad \lambda \mu \neq 0.$$

Then for  $\lambda \mu > 0$ ,  $\text{Ind}_{g_{\lambda, \mu}} = 1$ ; for  $\lambda \mu < 0$ ,  $\text{Ind}_{g_{\lambda, \mu}} = -1$ .

**Exercise 6.2.9.** Let  $z = (x, y) = x + iy$ ,  $f_n(z) = z^n$ ,  $n \in \mathbb{N}$ . Regard  $f_n$  as a vector field on  $\mathbb{R}^2$ , then  $O$  is the unique singularity. Calculate  $\text{Ind}_{f_n}(O)$ .

**Exercise 6.2.10.** Let  $P, Q : \mathbb{R}^2 \rightarrow \mathbb{R}$  be two bounded continuous functions. Show that there exists singularities in the vector field  $f(x, y) = (P(x, y) - y, x + Q(x, y))$ .

## §6.3 Rotation numbers on the torus

Let  $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$  be the torus, then  $\mathbb{T}^2$  is an abelian group and a Riemannian manifold with a flat metric. Let  $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$  be the covering map, then for every map  $f : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ , the map  $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\tilde{f} = f \circ \pi$  is a doubly periodic function.

**Lemma 6.3.1**

Let  $f : \mathbb{R}^2 \rightarrow X$  be a map. Then  $f$  is doubly periodic if and only if there exists  $\bar{f} : \mathbb{T}^2 \rightarrow X$  such that  $f = \bar{f} \circ \pi$ .

In particular, a doubly periodic vector field on  $\mathbb{R}^2$  can be regarded as a vector field on  $\mathbb{T}^2$ .

**Lemma 6.3.2**

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^1$  doubly periodic vector field, let  $\phi_t$  be the flow generated by  $f$ , then

$$\phi_t(p + k) = \phi_t(p) + k, \quad \forall k \in \mathbb{Z}^2, p \in \mathbb{R}^2.$$

Therefore,  $\pi \circ : \mathbb{R}^2 \rightarrow \mathbb{T}^2$  is a doubly periodic map. There exists  $\bar{\phi}_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  such that  $\pi \circ \phi_t = \bar{\phi}_t \circ \pi$ . Then  $\bar{\phi}_0 = \text{Id}$ ,  $\bar{\phi}_{t+s} = \bar{\phi}_t \circ \bar{\phi}_s$ , hence  $\{\bar{\phi}_t\}$  is a flow on  $\mathbb{T}^2$ , which is generated by  $\bar{f}$ .

**Example 6.3.3**

Consider the constant vector field  $f(x, y) = (1, \alpha)$  on  $\mathbb{R}^2$ . It induces a flow on  $\mathbb{T}^2$ .

(1) If  $\alpha = \frac{p}{q}$ ,  $p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1$ . Then  $\pi(x(q), y(q)) = \pi(x(0), y(0))$ , hence every orbit is a periodic orbit with a common period  $q$ .

(2) If  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , let  $K = \{(\bar{x}, \bar{y}) \in \mathbb{T}^2 : \bar{x} = 0\}$ . Then the orbit intersects with  $K$  at

$$\{(0, \overline{n\alpha + y_0 - x_0\alpha}) : n \in \mathbb{Z}\}.$$

**Theorem 6.3.4**

Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , then every orbit of the flow generated by  $(1, \alpha)$  is dense on  $\mathbb{T}^2$ .

The flow generated by  $(1, \alpha)$  on  $\mathbb{T}^2$  is called the **irrational flow** on  $\mathbb{T}^2$ .

**Definition 6.3.5.** Let  $X$ , be a compact metric space, a flow on  $X$  is called **minimal** if every orbit is dense in  $X$ .

$K = \{(\bar{x}, \bar{y}) \in \mathbb{T}^2 : \bar{x} = 0\}$  is a transversal of  $f = (1, \alpha)$ , then the orbit recurs to  $K$  at times one. It induces a map  $K \rightarrow K$ , which is also called **Poincaré recurrent map**.

**Lemma 6.3.6**

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^1$ , doubly periodic vector field without singularities. Then there exists an closed curve  $\Gamma$  on  $\mathbb{T}^2$  such that  $\Gamma$  is a transversal of  $\bar{f}$  and  $\Gamma$  is not null-homotopic.

*Proof.* Let  $g$  be the vector orthogonal complement vector field of  $f$ . If  $\Gamma$  has some periodic orbit, take it. Otherwise, take some  $q \in \omega(p, g)$ , take a flow box near  $q$ . the the orbit of  $p$  will recurs to the flow box. We can closed the orbit by a  $C^1$  curve transverse to  $f$  in the flow box.

For every transversal closed curve  $\Gamma$  of  $f$ , if  $\Gamma$  is null-homotopic, then  $\Gamma$  can be lifted to a closed curve  $\tilde{\Gamma}$  in  $\mathbb{R}^2$ . By index theory,  $\text{Ind}_f(\tilde{\Gamma}) \neq 0$ , which contradicts with  $f$  is without singularities.  $\square$

**Exercise 6.3.7.** Let  $f$  be a  $C^1$ , non-singularity vector field on  $\mathbb{T}^2$ .  $K \subset \mathbb{T}^2$  be a  $C^1$  Jordan curve, transverse to  $f$ . If there exists  $p_0 \in K$  and  $t > 0$  such that  $\phi_t(p_0) \in K$ , then for every  $p \in K$ , there exists  $t > 0$  such that  $\phi_t(p) \in K$ .

Under the condition of the exercise, we can define the Poincaré recurrent map  $P : K \rightarrow K$  as  $P(x) = \phi_{t(x)}(x)$ . Then  $P$  is a orientation preserving diffeomorphism of  $K$ .

**Exercise 6.3.8.** Let  $(P, Q)$  be a  $C^1$ , doubly periodic vector field on  $\mathbb{R}^2$ ,  $P > 0$ . Let  $\phi(x; x_0, y_0)$  be the solution of the following initial problem

$$\begin{cases} \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}, \\ y(x_0) = y_0. \end{cases}$$

Show that:

1.  $\phi(x; x_0, y_0 + 1) = \phi(x; x_0, y_0) + 1$ ;
2.  $\phi(x + 1; x_0 + 1, y_0) = \phi(x; x_0, y_0)$ ;
3.  $f(y_0) = \phi(1; 0, y_0)$ .

### Lemma 6.3.9

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly increasing continuous function, assume  $f(y+1) = f(y) + 1$ . Then there exists  $\alpha \in \mathbb{R}$  and  $C > 0$  such that

$$|f^n(y) - y - n\alpha| < C, \quad \forall y \in \mathbb{R}, n \in \mathbb{Z}. \quad (*)$$

In particular,

$$\alpha = \lim_{n \rightarrow \infty} \frac{f^n(y) - y}{n} = \alpha, \quad \forall y \in \mathbb{R}.$$

**Definition 6.3.10.** The constant  $\alpha$  is called the **rotation number** of  $f$ , denoted by  $\rho(f)$ . We call  $f$  a **bounded mean motion** if  $f$  satisfies (\*).

*Proof.* Let  $\phi(y) = f(y) - y$ , then  $|\phi(y) - \phi(z)| \leq 1$  for every  $y, z \in \mathbb{R}$ . Then for every  $n$ , we have  $|f^n(y) - y - f^n(z) + z| \leq 1$ . In particular, we consider  $y = x$  and  $z = f^m(x)$ , let  $a_n = f^n(x) - x$ , then  $|a_{m+n} - a_n - a_m| \leq 1$ . Hence,

$$\frac{a_n}{n} - \frac{1}{n} \leq \frac{a_{kn}}{kn} - \frac{1}{kn} \leq \frac{a_{kn}}{kn} \leq \frac{a_{kn}}{kn} + \frac{1}{kn} \leq \frac{a_n}{n} + \frac{1}{n}.$$

Let  $I_n = [(a_n - 1)/n, (a_n + 1)/n]$ , then  $J = \bigcap_{n \geq 1} I_n$  is a one-point set. Assume  $J = \{\alpha\}$ , and we can show that  $\alpha$  is independent with the choice of  $x$ .  $\square$

**Exercise 6.3.11.** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is increasing but not strictly increasing,  $f(y+1) = f(y) + 1$ , does the conclusion still hold?

### Proposition 6.3.12

$\rho(f)$  is continuous with respect to  $f$ , i.e. for every  $\varepsilon > 0$ ,  $\exists \delta > 0$ , such that for every  $g$  satisfying the same condition,  $\|g - f\|_\infty < \delta$ , we have  $|\rho(g) - \rho(f)| < \varepsilon$ .

*Proof.* By the previous lemma, we have  $|f^n(x) - x - n\alpha| < 1$ , take  $N$  such that  $\frac{2}{N} < \varepsilon$ . Take  $\delta > 0$ , such that for  $\|g - f\|_\infty < \delta$ , we have  $|g^N(x) - x - (f^N(x) - x)| < 1$ .  $\square$

**Theorem 6.3.13** (Poincaré)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly increasing continuous function, assume  $f(y+1) = f(y)+1$ . Then  $\rho(f) \in \mathbb{Q}$  if and only if  $\bar{f} : \mathbb{T} \rightarrow \mathbb{T}$  has some periodic points.

*Proof.* We prove that if  $\rho(f) = p/q$ , then there exists  $x \in \mathbb{R}$  such that  $f^q(x) = x + p$ . Let  $\psi(y) = f^q(y) - y - p$ , let  $\alpha = \min \psi(y)$  and  $\beta = \max \psi(y)$ , we want to show  $\alpha \leq 0$  and  $\beta \geq 0$ . If  $\alpha > 0$ , then  $\rho(f) \geq (p + \alpha)/q$ .  $\square$

**§6.4 Denjoy's theorem**

**Definition 6.4.1.** Let  $(X, d)$  be a metric space,  $f : X \rightarrow X$  continuous. For  $x \in X$ ,  $y \in X$  is called a **positive limit point** of  $x$  if  $\exists n_i \rightarrow \infty$ ,  $f^{n_i}x \rightarrow y$ . The **positive limit set**  $\omega(x)$  is the set of all positive limit points of  $x$ .

**Lemma 6.4.2**

Let  $(X, d)$  be a metric space,  $f : X \rightarrow X$  continuous. Then

1.  $\forall x \in X$ ,  $\omega(x) = \bigcap_{n=0}^{\infty} \overline{\text{Orb}^+(f^n x)}$ .
2.  $\forall x \in X$ ,  $\omega(x)$  is closed and  $f(\omega(x)) = \omega(x)$ .
3. If  $X$  is compact, then  $\omega(x) \neq \emptyset$ .

If  $f : X \rightarrow X$  is a homeomorphism, then we can define negative limit points and the negative limit set  $\alpha(x)$  similarly.

**Definition 6.4.3.** Let  $(X, d)$  be a metric space,  $f : X \rightarrow X$  continuous.  $f$  is called **minimal** if  $\forall x \in X$ ,  $\overline{\text{Orb}^+(x)} = X$ . Let  $\Lambda \subset X$  be a closed subset such that  $f(\Lambda) = \Lambda$  and  $\forall x \in \Lambda$ ,  $\overline{\text{Orb}^+(x)} = \Lambda$ , then  $\Lambda$  is called a **minimal set**.

**Theorem 6.4.4** (Poincaré)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly increasing continuous function, assume  $f(y+1) = f(y)+1$ . If  $\rho(f) \notin \mathbb{Q}$ , then for every  $\bar{x} \in \mathbb{T}$ ,  $\omega(\bar{x})$  is a minimal set. Moreover,  $\forall \bar{y} \in \mathbb{T}$ ,  $\omega(\bar{y}) = \omega(\bar{x})$ .

**Remark 6.4.5** — At this time,  $\partial\omega(\bar{x}) = \emptyset$  or  $\omega(\bar{x})$ , hence  $\omega(\bar{x})$  is either  $\mathbb{T}$  or a Cantor set.