Higher Rank Abelian Smooth Action with Hyperbolicity (Spring 2022, Disheng Xu)

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§1 22.5.13

For classical dynamical systems, we consider about discrete dynamics or flows. It consists of a space X and a family of maps $X \to X$,

$$\{f^{(n)} : n \in \mathbb{Z}\} \quad \text{or} \quad \{f^t : t \in \mathbb{R}\},$$

satisfying the group conditions.

We will consider a more general settings: **an abelian group action on** X. The settings are

- X a manifold.
- A family of maps $\{f^{\boldsymbol{t}} \in \mathrm{Homeo}(X) : \boldsymbol{t} \in \mathbb{Z}^l\}$, satisfies $f^{\boldsymbol{t}} \circ f^{\boldsymbol{s}} = f^{\boldsymbol{t}+\boldsymbol{s}}$.

Or, we can rewrite the second condition as a group homomorphism

$$\alpha : \mathbb{Z}^l \to \mathrm{Homeo}(X).$$

Example 1.1

A non-invertible example, i.e. α is just a semi-group homomorphism

$$\alpha : \mathbb{N}^2 \to C^0(\mathbb{T}, \mathbb{T}), \quad (m, n) \mapsto (\times 2)^m (\times 3)^n.$$

Furstenberg showed that the orbit of this action is either finite or dense. This is an example of a hyperbolic setting.

Example 1.2

Let R_α be the α-rotation on $\mathbb{T}$. We can consider the action

$$\alpha : (m,n) \to \mathrm{Homeo}(\mathbb{T}), \quad (m,n) \to R_\alpha^m R_\beta^n.$$

This is an example of a non hyperbolic setting.

Remark 1.3 — Fayad-Kanin showed that if $f, g : \mathbb{T} \to \mathbb{T}$, $R(f) = \alpha, R(g) = \beta$ and (α, β) satisfies some number-theoretic conditions, then $\exists \varphi \in C^\infty(\mathbb{T}, \mathbb{T})$ such that $\varphi \circ f \circ \varphi^{-1} = R_\alpha$ and $\varphi \circ g \circ \varphi^{-1} = R_\beta$.

For a hyperbolic setting, we consider a baby case. Let $A \in \mathrm{SL}(n, \mathbb{C})$ be a diagonalizable matrix, assume

$$A \sim \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

and $|\lambda_j| \neq 1$ for every j, then we call A to be a **hyperbolic matrix**. Let $\sigma_j = \log|\lambda_j|$, then "hyperbolicity" means $\sigma_j \neq 0$.

Example 1.4

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \curvearrowright \mathbb{T}^2$, a classical Anosov map.

Proposition 1.5 (C^0-Rigidity of Anosov Map)

For an Anosov map $f \in \mathrm{Diff}^\infty(X)$, if another map $g \in \mathrm{Diff}^\infty(X)$ is C^1-closed to f, then $\exists h \in \mathrm{Homeo}(X)$ such that $h \circ g \circ h^{-1} = f$.

Remark 1.6 — In general, the regularity of h cannot be C^1. Because a C^1 conjugacy preserves the derivates of fixed points.

Question 1.7. If we have higher rank action with at least one Anosov element, can we have the similar result?

Example 1.8

A baby case: for f_1, f_2 commutes with each other, consider the action

$$\alpha : \mathbb{Z}^2 \to \mathrm{Diff}^\infty(\mathbb{T}^2), \quad (m,n) \to f_1^m f_2^n.$$

Assume there exists $(m,n) \in \mathbb{Z}^2$ such that $f_1^m f_2^n$ is Anosov. Then we perturb (f_1, f_2) to $(\widetilde{f}_1, \widetilde{f}_2)$ a little bit such that $\widetilde{f}_1 \widetilde{f}_2 = \widetilde{f}_2 \widetilde{f}_1$ still holds. Then there exists h such that $h\widetilde{f}_1 h^{-1} = f_1$ and $h\widetilde{f}_2 h^{-1} = f_2$.

Question 1.9. Can the conjugate h be more regular?

It is easy to construct a counter example such that h could not be C^1. For example, we can regard a $\mathbb{Z}^1$-action as a "degenerated" $\mathbb{Z}^2$-action.

Example 1.10

Let $T_A, T_B : \mathbb{T}^2 \to \mathbb{T}^2$ be two hyperbolic matrices. We consider

$$\alpha : (m, n) \to T^m_{A\times \mathrm{Id}} T^n_{\mathrm{Id}\times B},$$

a $\mathbb{Z}^2$-action on $\mathbb{T}^2$. The conjugate h in general still cannot be C^1.

This counter example is a non degenerated $\mathbb{Z}^2$-action, but is somehow not "genuinely higher rank". So, we need a "**genuinely higher rank assumption**".

Question 1.11. Let $\alpha_0 : \mathbb{Z}^2 \to \mathrm{SL}(d, \mathbb{Z}) \subset \mathrm{Diff}^\infty(\mathbb{T}^d)$ be an action such that there exists (m, n), $\alpha_0(m, n)$ is Anosov (i.e. a hyperbolic matrix). Then for a C^1-perturbation α of α_0, $\alpha : \mathbb{Z}^2 \to \mathrm{Diff}^\infty(\mathbb{T}^d)$, can we show that $\exists h \in \mathrm{Diff}^\infty(\mathbb{T}^d)$ such that $h \circ \alpha \circ h^{-1} = \alpha_0$?

To avoid the rank-one case, we need an additional assumption.

"Totally ergodic ergodic assumption": $\forall (m, n) \neq (0, 0)$, $\alpha_0(m, n)$ is ergodic with respect to the Lebesgue measure on $\mathbb{T}^d$.

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Conjecture 2.1 ($\mathbb{Z}^l$ version of Karok-Spatzier's Conjecture)

Let $\alpha : \mathbb{Z}^l \to \mathrm{Diff}^\infty(M)$ be an action such that there exists $a \neq 0 \in \mathbb{Z}^l$, $\alpha(a)$ is Anosov. Then under some suitable "higher rank" assumption (no rank-one factor), α is C^∞ conjugate to an "algebraic-defined" model.

As a contrast, we consider a famous conjecture of Smale and Borel in the case of rank-one.

Conjecture 2.2 (Smale-Borel)

If f is Anosov, then f is C^0-conjugate to a $\mathbb{T}^d$ automorphism.

This conjecture in general is **False**, for Borel have constructed an Anosov diffeomorphism on a nil-manifold. Later, there has been constructed an example of Anosov diffeomorphism on an infra-nil-manifold.

Theorem 2.3 (Franks-Manning)

Suppose $f \in \mathrm{Diff}^1(M)$ is Anosov, where M is a nil-manifold. Then f is C^0-conjugate to an affine map on M.

Corollary 2.4

Assume $f, g \in \mathrm{Diff}^1(M)$ are Anosov, where M is a nil-manifold. Then there exists $h \in \mathrm{Homeo}(M)$ such that $hfh^{-1} = f_0, hgh^{-1} = g_0$ where f_0, g_0 are affine maps on M.

Theorem 2.5 (Hertz-Z.Wang, 2014)

Consider the action $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ which is homotopic to $\alpha_0 : \mathbb{Z}^k \to \mathrm{GL}(d, \mathbb{Z})$, if α is Anosov (in the sense that $\exists a \in \mathbb{Z}^k \setminus \{0\}$, $\alpha(a)$ is Anosov). Assume that $\exists \mathbb{Z}^2 \subseteq \mathbb{Z}^k$ such that $\alpha_0|_{\mathbb{Z}^2}$ is totally ergodic, then α is C^∞-conjugate to an affine action.

Theorem 2.6 (Fisher-Kalinin-Spatzier,2013)

The same result (as Theorem 2.5) holds under a stronger assumption that α has "many" Anosov element.

Weyl Chamber picture

The Lyapunov exponent for a matrix is $\sigma_i = \log|\lambda|_i$, where λ_i is an eigenvalue of A. Then

$$A \sim \begin{bmatrix} \square & & & \\ & \square & & \\ & & \ddots & \\ & & & \square \end{bmatrix},$$

where each $\square$ is a block with all the same eigenvalues. Then we can get a coarse decomposition of $\mathbb{R}^d$ corresponding to different Lyapunov exponents. Denotes this splitting by

$$\mathbb{R}^d = V_1 \oplus V_2 \oplus \cdots \oplus V_r,$$

then V_i is A-invariant. Moreover, for every B commutes with A, B also preserves each V_i. Hence for a $\mathbb{Z}^k$ action of $\mathrm{GL}(d, \mathbb{Z})$, we can split $\mathbb{R}^d$ into a direct sum of finite many subspaces $\{V_i\}$. Such that, for every $A \in \alpha(\mathbb{Z}^k)$, $A|_{V_i}$ has a constant Lyapunov exponent. Then we can define the **Lyapunov functionals** $\lambda_i : A \mapsto \sigma(A|_{V_i})$, these functionals will induce linear functionals $\mathbb{Z}^k \to \mathbb{R}$.