Homogeneous Dynamical System (Spring 2022, Runlin Zhang)

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Contents

1 Introduction of Homogeneous Dynamics

§1.1 Motivations and applications

§1.1.i Horocycles on constant negative curvature surfaces

Equip $\mathbb{H}^2 := \{x + iy \in \mathbb{C}, y > 0\}$ with the metric $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$. Let $\Gamma \leqslant \mathrm{Isom}(\mathbb{H}^2)$ be a discrete (torsion free) subgroup such that $\Gamma \setminus \mathbb{H}^2$ is compact (such a subgroup is called a uniform lattice). Then $\Gamma \setminus \mathbb{H}^2$ is a compact surface of constant negative curvature.

Let $\pi : \mathbb{H}^2 \to \Gamma \setminus \mathbb{H}^2 = M$ be the quotient map. Consider a horocycle $\mathcal{H} \subset \mathbb{H}^2$.

Theorem 1.1.1
For every $\mathcal{H}$, $\pi(\mathcal{H})$ is dense in M.

Theorem 1.1.2
If $M = \Gamma \setminus \mathbb{H}^2$ ($\Gamma \leqslant \mathrm{Isom}(\mathbb{H}^2)$ still discrete) is just of finite volume, then:

1. $\pi(\mathcal{H})$ is either closed or dense in M.
2. Consider a sequence of closed horocycles $\pi(\mathcal{H}_i)$ with length $\to \infty$, then $\pi(\mathcal{H}_i)$ becomes dense in $\Gamma \setminus \mathbb{H}^2$.

§1.1.ii Isometric immersion of hyperbolic spaces

Let $\mathbb{H}^3$ be the three dimensional hyperbolic space $\{(x+iy, z) \in \mathbb{C}\times\mathbb{R}, z > 0\}$ equipped with the metric $\frac{1}{z^2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$. Let $\Gamma \leqslant \mathbb{H}^3$ be a discrete (torsion free) subgroup, such that $\mathbb{H}^3$ is compact (finite volume suffices). Consider an isometric embedding $\iota : \mathbb{H}^2 \to \mathbb{H}^3$. The image of ι can be explicitly described.

Theorem 1.1.3
The following holds:

1. $\pi(\iota(\mathbb{H}^2))$ is either closed or dense in M;
2. Given an infinite sequence of distinct closed $\pi(\iota_i(\mathbb{H}^2))$, then $\lim_i \pi(\iota_i(\mathbb{H}^2))$ is dense in M.

§1.1.iii Oppenheim conjecture/Margulis theorem

Let Q be a real quadratic form in 3 variables, indefinite and non-degenerated. Consider Q as a function $\mathbb{R}^3 \to \mathbb{R}$.

Theorem 1.1.4

Assume Q is **not** proportional to a quadratic form with $\mathbb{Q}$-coefficients. Then $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.

Remark 1.1.5 — It is true for $k \geqslant 3$ variables. But it is false for Q only has two variables.

Theorem 1.1.6 (Eskin-Margulis-Mozes)

Further assume Q has at least signature $(3, 1)$, then for every $a < b \in \mathbb{R}$,

$$\begin{aligned}
&\# \left\{ v \in \mathbb{Z}^4 : \|v\| \leqslant T, Q(v) \in (a, b) \right\} \\
\sim\ & \mathrm{Vol} \left\{ v \in \mathbb{R}^4 : \|v\| \leqslant T, Q(v) \in (a, b) \right\} \\
\sim\ & C_Q (b - a) T^2.
\end{aligned}$$

§1.1.iv Littlewood conjecture

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we have $\inf \{ n \langle n\alpha \rangle : n \in \mathbb{Z}_+ \} < 1$.

Fact 1.1.7. There exists α such that $\inf \{ n \langle n\alpha \rangle : n \in \mathbb{Z}_+ \} > 0$.

Conjecture 1.1.8

For all $(\alpha, \beta) \in \mathbb{R}^2,\ \alpha, \beta \notin \mathbb{Q}$,

$$\inf \{ n \langle n\alpha \rangle \langle n\beta \rangle : n \in \mathbb{Z}_+ \} = 0.$$

Remark 1.1.9 — The conjecture is reasonable in some sense:

1. $\forall \delta > 0$, $\inf \left\{ n^{1-\delta} \langle n\alpha \rangle \langle n\beta \rangle : n \in \mathbb{Z}_+ \right\} = 0$.
2. $\forall \delta > 0, \exists (\alpha, \beta)$, such that $\inf \left\{ n^{1+\delta} \langle n\alpha \rangle \langle n\beta \rangle : n \in \mathbb{Z}_+ \right\} > 0$.

§1.1.v Quantum unique ergodicity

Consider $M^2 = \Gamma \setminus \mathbb{H}^2$ is a closed hyperbolic surface. Consider $\Delta \coloneqq -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ acts on $C^\infty(M)$. Then:

1. $\exists \lambda_0 = 0 < \lambda_1 < \cdots,\ \lambda_i \to \infty$,
2. Let $E_{\lambda_i} \coloneqq \{ f \in C^\infty(M) : \Delta f = \lambda_i f \}$, then $E_{\lambda_i} \neq \varnothing$ and $\dim E_{\lambda_i} < \infty$.

For each i, choose $f_i \in E_{\lambda_i}$. Consider $(|f_i|^2 \mathrm{Vol})_i$ a sequence of measure on M, normalized to be probability measure.

Conjecture 1.1.10

$|f_i|^2 \mathrm{Vol}$ tends to $\frac{1}{\mathrm{Vol}(M)} \mathrm{Vol}$ in the weak* topology.

Further assume Γ is a "congruence subgroup". In this situation, there is an additional supply of operators, called Hecke operators, that commute with the Laplacian. Let $f_i \in E_{\lambda_i}$ which is also an eigenfunction of Hecke operator.

Theorem 1.1.11 (Lindenstrauss-Bourgain)
In such settings, the conjecture holds.

§1.2 Measure rigidity

§1.2.i Unipotent rigidity

Let $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma \leqslant G$ a discrete subgroup. G has a right G-invariant Riemannian metric. It induces a volume measure Vol on G/Γ.

Fact 1.2.1. Vol is left G-invariant.

Let $U = \left\{ u_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R} \right\}$.

Theorem 1.2.2
If G/Γ is compact, then Vol is the unique U-invariant finite measure(up to a scalar).

Theorem 1.2.3
If Vol is finite (normalized to be probability measure). Then every U-invariant probability measure is a "convex combination" of:

(i) the U-invariant measure supported on a closed(and compact) orbit.

(ii) Vol.

Theorem 1.2.4 (Measure Rigidity Theorem)
Let G be a (conneted) Lie group, let $U = \{u_s : s \in \mathbb{R}\}$ be an Ad-unipotent one-parameter subgroup of G. Let $\Gamma \leqslant G$ be a closed subgroup. Then every U-invariant ergodic probability measure on G/Γ is "homogeneous".

Theorem 1.2.5 (Equidistribution and Topological Rigidity)
Assume Γ is a lattice in G, then for any $x \in G/\Gamma$:

1. There exists a probability "homogeneous" measure μ such that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_s x) \mathrm{d}s = \int f(x) \mathrm{d}\mu(x), \quad \forall f \in C_c(G/\Gamma).$$

2. The closure of the orbit Ux is "homogeneous", which means $\exists H \leqslant G$ closed such that $\overline{Ux} = Hx$.

§1.2.ii Higher rank diagonalizable flow

Let $G = \mathrm{SL}(2,\mathbb{R}),\ \Gamma \leqslant G$ lattice. Consider $A = \left\{\begin{bmatrix} e^t & 0 \\ 0 & e^{-t}\end{bmatrix}\right\}$ acts on G/Γ.

Conjecture 1.2.6

$G = \mathrm{SL}(3,\mathbb{R}),\ \Gamma = \mathrm{SL}(3,\mathbb{Z})$. Consider

$$\mathbb{R}^2 \cong A := \left\{\begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3}\end{bmatrix} : t_1 + t_2 + t_3 = 0\right\}$$

acts on G/Γ.

1. Every A-ergodic probability measure is homogeneous.
2. Every bounded A-orbit is closed.

Theorem 1.2.7

A, G, Γ as in the conjecture, then:

1. Every A-invariant ergodic probability measure with "positive entropy" is homogeneous.
2. The Hausdorff dimension of $\{x \in G/\Gamma : Ax \text{ is bounded}\}$ is equal to 2.

2 Oppenheim Conjecture

§2.1 22.2.25: The unipotent flow is minimal on compact space

- Let $G = \mathrm{SL}(2, \mathbb{R})$, let $\Gamma \leqslant G$ be a discrete subgroup.
- Assume for today $X = G/\Gamma$: is compact.
- $U^+ = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \geqslant 0 \right\}$.

Theorem 2.1.1
For all $x \in X$, $U^+ x$ is dense in X.

Definition 2.1.2. Let A be a semigroup acting on a topological space Z :

1. We say the action is **minimal** if every A-orbit is dense in Z.
2. We say the subset $W \subset Z$ is **A-minimal** if W is A-stable, closed and $A \curvearrowright W$ is minimal.

Theorem 2.1.3
Let Y be a U^+-minimal subset of X. Then $Y = \varnothing$ or $Y = X$.

Claim 2.1.4. Theorem 2.1.3 implies Theorem 2.1.1

Proof. Zorn's lemma + compactness of X. We can always find a nonempty U^+-minimal subset of X, which must be X. $\square$

Fact 2.1.5. $\mathrm{SL}(2, \mathbb{R})$ admits a right-invariant metric compatible with its topology.

Now we fix such a metric $d : G \times G \to \mathbb{R}$. It induces a "quotient" metric $d_X : X \times X \to \mathbb{R}$ by

$$d_X(g\Gamma, h\Gamma) := \inf_{\gamma_1, \gamma_2 \in \Gamma} d(g\gamma_1, h\gamma_2) = \inf_{\gamma \in \Gamma} d(g\gamma, h).$$

For $x \in X = G/\Gamma$, define the **injective radius** of x as

$$\mathrm{InjRad}(x) := \sup \{\delta > 0 : \text{such that } g \mapsto g.x \text{ is injective on } g \in B(\mathrm{Id}, \delta) \subseteq G\}.$$

Exercise 2.1.6. For all $x \in X$, $\mathrm{InjRad}(x) > 0$.

Proof. By Γ is discrete. $\square$

Exercise 2.1.7. $\exists r_X > 0$, such that $\forall x \in X$, $\mathrm{InjRad}(x) > r_X$.

Proof. By the compactness of X. Because Γ is cocompact, there exists $C \subseteq G$ compact, such that $\forall x \in X, \exists g_x \in C$, $x = g_x \Gamma$. $\square$

> **Lemma 2.1.8**
> $U^+ \curvearrowright X = G/\Gamma$ has no closed (compact) orbit.

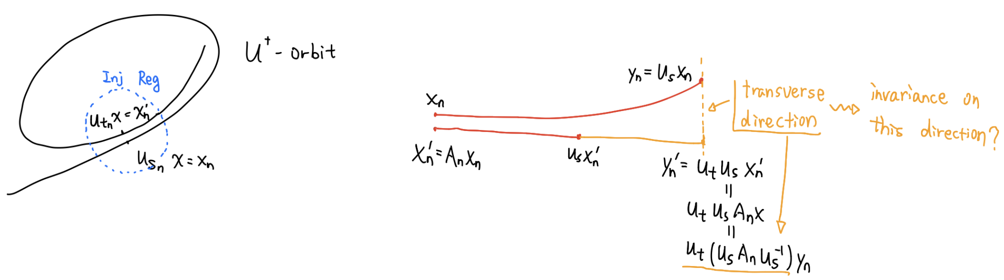
Proof. Say: we have a compact orbit $\{u_s.x : s \geqslant 0\}$. Define $s_0 = \inf\{s > 0 : u_s.x = x\}$, then

$$\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} u_{s_0} \begin{bmatrix} e^{t} & \\ & e^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x = \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x.$$

This shows that $\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix}.x$ is invariant under $\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} u_{s_0} \begin{bmatrix} e^{t} & \\ & e^{-t} \end{bmatrix} = u_{e^{-2t}s_0}$. ☐

> **Corollary 2.1.9**
> Γ contains no nontrivial unipotent matrix.

> **Corollary 2.1.10**
> The following holds:
>
> 1. $\forall x \in X$, the map $s \mapsto u_s.x$ is injective.
> 2. $\forall x, \exists s_n, t_n \to \infty$ with $|s_n - t_n| \to \infty$, such that $d_X(u_{s_n}.x, u_{t_n}.x) \to 0$.



Proof of Theorem 2.1.3. By corollary 2.1.10, we can find $A_n \in G \backslash U$ and $x_n, x_n' \in U^+x \subseteq X$ with $d_X(x_n, x_n') \to 0$ and $x_n' = A_n.x_n$. Moreover, we can choose $A_n \to \mathrm{Id}$ (use the fact that injective radius is larger than r_X).

Say $A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$, where $a_n, d_n \to 1, b_n, c_n \to 0$. We have

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} A_n \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_n + sc_n & b_n - sa_n + sd_n - s^2 c_n \\ c_n & d_n - sc_n \end{bmatrix}.$$

We want to choose $t = t_s$ such that

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_n + sc_n & b_n - sa_n + sd_n - s^2 c_n \\ c_n & d_n - sc_n \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Take $t = t_s = \frac{-(b_n - sa_n + sd_n - s^2 c_n)}{d_n - sc_n}$. Then

$$u_t u_s A_n u_s^{-1} = \begin{bmatrix} \frac{1}{d_n - sc_n} & 0 \\ c_n & d_n - sc_n \end{bmatrix}.$$

Fix $\delta > 0$, choose $s = s_{\delta,n} \geqslant 0$ such that $d_n - sc_n = 1 - \delta$ or $1 + \delta$. Let $y_n = u_s.x_n$, $y_n' = u_t u_s A_n.x_n = \begin{bmatrix} (1+\delta)^{-1} & 0 \\ c_n & (1+\delta) \end{bmatrix}.y_n$. By passing to a subsequence, assume that $y_n \to y_\infty$ and $y_n' \to y_\infty'$ both in Y, where $y_\infty' = \begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix}.y_\infty$. Then

$$Y = \overline{U^+ y_\infty'} = \begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix} \overline{U^+ y_\infty} = \begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix} Y.$$

Let $B^+ = \{a_t u_s : s \in \mathbb{R}_+, t \in \mathbb{R}\}$, where $a_t = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}$, then Y is B^+ invariant. The theorem is immediate by the following lemma. □

Lemma 2.1.11

We have:

1. $B \curvearrowright \mathrm{SL}(2, \mathbb{R})/\Gamma$ is minimal.
2. $B^+ \curvearrowright \mathrm{SL}(2, \mathbb{R})/\Gamma$ is minimal.

§2.2 22.3.4: Weak Oppenheim conjecture I

Theorem 2.2.1 (Weak Version of Oppenheim Conjecture)

Let Q be a nondegenerate, indefinite, quadratic form with real coefficients in 3 variables. Assume Q is **not** proportional to a quadratic form with $\mathbb{Q}$-coefficients. Then $\overline{Q(\mathbb{Z}^3 \setminus (0))}$ contains 0.

Example 2.2.2

$Q(x, y, z) = xy - \sqrt{2}z^2$, the statement is trivial for Q because $Q(1, 0, 0) = 0$.

Definition 2.2.3. Define the special orthogonal group of Q as

$$\mathrm{SO}(Q, \mathbb{R}) := \{g \in \mathrm{SL}(3, \mathbb{R}), Q \circ g = Q\}, \quad \mathrm{SO}(Q, \mathbb{Z}) := \{g \in \mathrm{SL}(3, \mathbb{Z}), Q \circ g = Q\}.$$

Definition 2.2.4. A subgroup $\Lambda \leqslant \mathbb{R}^N$ is a **lattice** if Γ is discrete and cocompact.

Definition 2.2.5. $\Lambda \leqslant \mathbb{R}^n$ is a **unimodular lattice** if Λ is a lattice and $\mathrm{Vol}(\mathbb{R}^N/\Lambda) = 1$.

Definition 2.2.6. Let $X_N := \{\text{unimodular lattice in } \mathbb{R}^N\}$ equipped with the **Chabauty topology**.

Remark 2.2.7 — A sequence $\{\Lambda_n\} \subseteq X_N$ converges to $\Lambda_\infty \in X_N$ iff we can find a basis $\{v_1^n, v_2^n, \cdots, v_N^n\}$ of Λ_n such that for every $i = 1, 2, \cdots, N$, $v_i^n \to v_i^\infty \in \mathbb{R}^N$, and $\Lambda_\infty = \mathbb{Z}v_1^\infty \oplus \mathbb{Z}v_2^\infty \oplus \cdots \oplus \mathbb{Z}v_N^\infty$.

Remark 2.2.8 — $\mathrm{SL}(N, \mathbb{R})$ naturally acts on X_N.

Lemma 2.2.9

The map $g \mapsto g \cdot \mathbb{Z}^N$, induces a homeomorphism $\mathrm{SL}(N, \mathbb{R})/\mathrm{SL}(N, \mathbb{Z}) \cong X_N$.

Definition 2.2.10. For a discrete subgroup $\Lambda \leqslant \mathbb{R}^N$, define $\delta(\Lambda) := \inf \{\|v\| : v \neq 0 \in \Lambda\}$.

Fact 2.2.11. $\delta : X_N \to \mathbb{R}_{>0}$ is continuous.

Lemma 2.2.12 (Mahler's Criterion)

$\delta : X_N \to \mathbb{R}_{>0}$ is proper, i.e. $(x_n) \subseteq X_N$ diverges iff $\delta(x_n) \to 0$.

Remark 2.2.13 — (x_n) diverges iff for every compact $K \subseteq X_N$, (x_n) will eventually out of K. This is equivalent to (x_n) has no convergent subsequence.

Proof. **The "if" part:** If $\delta(x_n) \to 0$, we need to show (x_n) is divergent. This is immediate by (x_n) has a convergence subsequence.

The "only if" part: By passing to a subsequence, $\exists \varepsilon > 0$ such that $\delta(x_n) \geqslant \varepsilon > 0$. The statement follows by the following claim. □

Claim 2.2.14. $\exists C = C(N, \varepsilon) > 0$, such that every Λ with $\delta(\Lambda) > \varepsilon$ has a basis $(v_1, v_2, \cdots, v_N)$ with $\|v_i\| \leqslant C(N, \varepsilon), i = 1, 2, \cdots, N$.

Proof. Consider the projection $p : \mathbb{R}^N \to \mathbb{R}^N/\Lambda$. Then p is not injective restricted to $[-1, 1]^N$. There will be $v \neq w \in [-1, 1]^N$ such that $v - w \in \Lambda$ and $\|v - w\| \leqslant 2\sqrt{N}$. Now we pick $w_1 \in \Lambda$ that minimize $\{\|v\| : v \neq 0 \in \Lambda\}$, then $\|w_1\| \leqslant 2\sqrt{N}$.

Let $\pi_1^\perp : \mathbb{R}^N \to w_1^\perp$ be the orthogonal projection. Consider $\pi_1^\perp(\Lambda) \leqslant w_1^\perp \cong \mathbb{R}^{N-1}$. Then:

1. $\pi_1^\perp(\Lambda)$ is discrete and is a lattice in $w_1^\perp$.
2. $1 = \|\Lambda\| = \|w_1\| \, \|\pi_1^\perp(\Lambda)\| \geqslant \varepsilon \|\pi_1^\perp(\Lambda)\|$.

Then $\|\pi_1^\perp(\Lambda)\| \leqslant \varepsilon^{-1}$ and $\delta(\pi_1^\perp(\Lambda))$ is controlled by a function of ε. We can reduce to the situation of dimensional $N - 1$. □

Lemma 2.2.15

Let Q be a nondegenerate quadratic form in N variables with real coefficients, then the followings are equivalent:

(i) $\overline{Q(\mathbb{Z}^N \setminus \{0\})}$ contains 0.

(ii) $\mathrm{SO}(Q, \mathbb{R}) \cdot \mathbb{Z}^N$ is unbounded in X_N.

Proof. **(ii) $\implies$ (i):** By assumption, $\exists g_n \in \mathrm{SO}(Q, \mathbb{R})$ such that $(g_n \cdot \mathbb{Z}^N)_n$ diverges in X_N. By Mahler's Criterion 2.2.12, $\delta(g_n \cdot \mathbb{Z}^N) \to 0$, hence $\exists v_n \neq 0 \in \mathbb{Z}^N$ such that $g_n v_n \to 0$. □

Consider $N = 3$, Q indefinite.

Fact 2.2.16. $\exists g_Q \in \mathrm{SL}(3,\mathbb{R})$ such that $Q = \lambda(Q_0 \circ g_Q)$, where $\lambda \in \mathbb{R}\setminus\{0\}$ and $Q_0 = 2xz - y^2$.

Then $\mathrm{SO}(Q,\mathbb{R}) = g_Q^{-1}\mathrm{SO}_{Q_0}(\mathbb{R})g_Q$, hence $\mathrm{SO}(Q,\mathbb{R})\cdot\mathbb{Z}^3$ is unbounded iff $\mathrm{SO}(Q_0,\mathbb{Z})g_Q\cdot\mathbb{Z}^3$ is unbounded.

Theorem 2.2.17

Every orbit of $\mathrm{SO}(Q_0,\mathbb{R})$ on $X_3 \cong \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ either unbounded or is closed.

Proof of Theorem 2.2.1 assuming Theorem 2.2.17. Otherwise, $\mathrm{SO}(Q,\mathbb{R})\cdot\mathbb{Z}^3$ is compact. Then $\mathrm{SO}(Q,\mathbb{Z}) := \mathrm{SO}(Q,\mathbb{R})\cap\mathrm{SL}(3,\mathbb{Z})$ is cocompact in $\mathrm{SO}(Q,\mathbb{R})$. We want to show that Q is proportional to a $\mathbb{Q}$-coefficient quadratic form. Otherwise, $\exists\alpha,\beta$ coefficients of Q such that $\frac{\alpha}{\beta}\notin\mathbb{Q}$. Then $\exists\sigma\in\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ such that $\sigma(Q)$ is not proportional to Q.

Step 1: $\mathrm{SO}(Q,\mathbb{R})^0 = \mathrm{SO}(\sigma(Q),\mathbb{R})^0 = \sigma(\mathrm{SO}(Q,\mathbb{R}))^0$.
$\mathrm{SO}(Q,\mathbb{R})^0 \supseteq \mathrm{SO}(Q,\mathbb{Z})\cap\mathrm{SO}(Q,\mathbb{R})^0 = \Gamma \subseteq \sigma(\mathrm{SO}(Q,\mathbb{R}))^0$. Consider

$$\mathrm{SL}(3,\mathbb{R}) \curvearrowright \mathrm{Sym} := \{\mathbb{R}-\text{Symmetric matrices}\}\,,\quad g.M = gMg^t.$$

Let $\psi : \mathrm{SO}(Q,\mathbb{R}) \to \mathrm{Sym}, g \mapsto g.\sigma(Q)$, then ψ factors through $\mathrm{SO}(Q,\mathbb{R})/\mathrm{SO}(Q,\mathbb{Z}) \to \mathrm{Sym}$. Hence, the image of ψ is compact. The following two facts shows that $\mathrm{SO}(Q,\mathbb{R})^0$ fixes $\sigma(Q)$ and the statement follows immediately:

1. $\mathrm{SO}(Q,\mathbb{R})^0$ is generated by one-parameter unipotent flows.
2. For every unipotent flow $\{u_t\}$ and $M\in\mathrm{Sym}$, either $\{u_t.M\}$ is unbounded or M is fixed by $\{u_t\}$.

Step 2: A direct compute shows that $\mathrm{SO}(Q,\mathbb{R})^0 = \mathrm{SO}(\sigma(Q),\mathbb{R})^0$ implies $\sigma(Q)$ is proportional to Q. □

§2.3 22.3.8: Weak Oppenheim conjecture II

Theorem 2.3.1

An orbit of $H = \mathrm{SO}(Q_0,\mathbb{R})$ on X_3 is either:

(i) unbounded.

(ii) compact.

(iii) its closure contains a $\{v_s\}_{s\geqslant 0}$-orbit or a $\{v_s\}_{s\leqslant 0}$-orbit, where $v_s = \begin{bmatrix}1&0&s\\0&1&0\\0&0&1\end{bmatrix}$.

Fact 2.3.2. Theorem 2.3.1 $\implies$ Theorem 2.2.17.

Now, we calculate H. Let $\mathfrak{h}$ be the Lie algebra of H, then

$$\mathfrak{h} = \left\{X\in\mathfrak{gl}(3,\mathbb{R}) : M_0X = -X^tM_0, M_0 = \begin{bmatrix} & & 1\\ & -1 & \\ 1 & & \end{bmatrix}\right\}.$$

After some tough work, we get

$$\mathfrak{h} = \left\{\begin{bmatrix}x_1 & x_2 & 0\\ x_4 & 0 & x_2\\ 0 & x_4 & -x_1\end{bmatrix} : x_1,x_2,x_4\in\mathbb{R}\right\}.$$

In particular,

$$u_t := \exp\left(t\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}\right) = \begin{bmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{bmatrix}, a_t = \exp\left(t\begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix}\right) = \begin{bmatrix} e^t & & \\ & 1 & \\ & & e^{-t} \end{bmatrix} \in H.$$

Proof of Theorem 2.3.1. Take $x_0 \in X_3$ such that $Y_0 = \overline{H.x_0} \neq H.x_0$ and $H.x_0$ is bounded. Let $\Omega := \{y \in Y_0 : Hy \text{ is open in } Y_0\}$. We need the following lemma.

Lemma 2.3.3

$\Omega \neq Y_0$.

Proof. Otherwise, every orbit of H in Y_0 is closed, in particular Hx_0 is closed. Contradiction. □

Continued proof of Theorem 2.3.1. Let Y_1 be a nonempty U-minimal nonempty subset of $Y_0 \setminus \Omega$, where $U = \{u_t\}$. If $y \in Y_0 \setminus \Omega$, then $H.y$ is not open in Y_0, hence $\exists y_n \in Y_0$ such that $y_n \notin H.y, y_n \to y$.

Case 1: Y_1 is closed U-orbit. Impossible.

Case 2: Y_1 is **not** a closed U-orbit but Y_1 is A-stable, where $A = \{a_t\}$. We want to find a $\{v_s\}_{s\geqslant 0}$-orbit or a $\{v_s\}_{s\leqslant 0}$-orbit inside Y_0.

Fact 2.3.4. The map $\mathfrak{h} \oplus \mathfrak{h}^\perp \to X_3, (h, w) \mapsto \exp(h)\exp(w).x_1$ is a local diffeomorphism.

$$\mathfrak{h} = \left\{ X \in \mathfrak{gl}(3, \mathbb{R}) : M_0 X = -X^t M_0, M_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \right\}.$$

$$\mathfrak{h}^\perp = \left\{ X \in \mathfrak{gl}(3, \mathbb{R}) : \operatorname{tr} X = 0, M_0 X = X M_0, M_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \right\}.$$

Fact 2.3.5. $\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{h} \oplus \mathfrak{h}^\perp$, moreover $\mathfrak{h}^\perp$ is invariant under $\operatorname{Ad}(H)$.

In this case, there exists $x_1 \in Y_1, A_n \to \operatorname{Id}, A_n.x_1 \in Y_0$ where $A_n \notin H$. Write $A_n = \exp(h_n)\exp(w_n), h_n \in \mathfrak{h}, w_n \neq 0 \in \mathfrak{h}^\perp$. Let $x_n = \exp(w_n)x_1 \in Y_0$, $\|w_n\| \to 0$.

Lemma 2.3.6

For δ sufficiently small, n sufficiently large, there exists $t_{n,\delta} \in \mathbb{R}$ such that:

(i) $\left\|\operatorname{Ad}(u_{t_{n,\delta}})w_n\right\| \in \left[10^{-10}\delta, 10^{10}\delta\right]$.

(ii) Every limit of $\operatorname{Ad}(u_{t_{n,\delta}})w_n$ is in Lie algebra of $\{v_s\}$.

Let $y_{n,\delta} = u_{t_{n,\delta}}.x_1$, $z_{n,\delta} = u_{t_{n,\delta}}.x_n$. As $x_n = \exp(w_n)x_1$, hence $z_{n,\delta} = \exp(\operatorname{Ad}(u_{t_{n,\delta}})w_n)y_{n,\delta}$. By passing to a subsequence, we assume that

$$z_{n,\delta} \to z_{\infty,\delta}, \quad \operatorname{Ad}(u_{t_{n,\delta}})w_n \to w_{\infty,\delta}, \quad y_{n,\delta} \to y_{\infty,\delta}.$$

Then $z_{n,\delta} \in Y_0, y_{\infty,\delta} \in Y_1$ and $w_{\infty,\delta}$ is in Lie algebra of $\{v_s\}$. Note that v_s commutes with u_t, we get $\exp(w_{\infty,\delta})Y_1 \subseteq Y_0$. By assumption, Y_1 is A-stable, after some calculation, $a_t\exp(w_{n,\delta})a_t^{-1}$ can go through ever v_s for $s \geqslant 0$ or $s \leqslant 0$.

Case 3: Y_1 is **not** A-stable.

Take $x \in Y_1$, because Ux is not closed, a same argument of the proof 2.1, we can find $y_n = \exp(h_n)\exp(w_n)x \in Y_1$ with $h_n \in \mathfrak{h}, w_n \in \mathfrak{h}^\perp$, such that $w_n, h_n \to 0, w_n + h_n$ is not in the Lie algebra of U.

Lemma 2.3.7

For δ sufficiently small, for n sufficiently large. There exists $s_{n,\delta}, t_{n,\delta} \in \mathbb{R}$, $h_{n,\delta} \oplus w_{n,\delta} \in \mathfrak{h} \oplus \mathfrak{h}^{\perp}$, such that:

(i) $u_{s_{n,\delta}} \exp(\mathrm{Ad}(u_t)h_n) \exp(\mathrm{Ad}(u_t)w_n) = \exp(h_{n,\delta}) \exp(w_{n,\delta})$.

(ii) $\max\{\|h_{n,\delta}\|, \|w_{n,\delta}\|\} \in [10^{-100}\delta, 10^{100}\delta]$.

(iii) Every limit of $h_{n,\delta}$ is in Lie algebra of $\{a_t\}$, every limit of $w_{n,\delta}$ is in Lie algebra of $\{v_s\}$.

Let $h_{\infty,\delta}, w_{\infty,\delta}$ be a limit of $(h_{n,\delta} \oplus w_{n,\delta})$. Write $g_\delta := \exp(h_{n,\delta}) \exp(w_{n,\delta})$, then g_δ normalize U, i.e. $g_\delta U g_\delta^{-1} = U$. We have

$$y_{\infty,\delta} = g_\delta . x_{\infty,\delta} \in Y_1, \quad x_{\infty,\delta} \in Y_1,$$

hence Y_1 is g_δ invariant. Let $g_\delta = \exp(\nu_\delta)$ and take a limit point ν of ν_δ as $\delta \to 0$. Then Y_1 is $\exp(s\nu)$ invariant for all $s \in \mathbb{R}$. Where ν is in Lie algebra of $\{a_t v_s\}$ and Y_1 is not A-stable, hence ν has a nonzero $\mathrm{Lie}(\{v_s\})$ component. □

§2.4 22.3.11: Completion of some gaps

Fact 2.4.1. If Q is "irrational", then $\mathrm{SO}(Q_0, \mathbb{R}) g_Q \mathbb{Z}^3$ is **not** compact.

Proof of Theorem 2.2.1 assuming Theorem 2.3.1. If suffices to show that $\mathrm{SO}(Q_0, \mathbb{R}) g_Q \mathbb{Z}^3$ is unbounded. So if $\mathrm{SO}(Q_0, \mathbb{R}) g_Q \mathbb{Z}^3$ is not unbounded, then (WLOG) $\overline{\mathrm{SO}(Q_0, \mathbb{R}) g_Q \mathbb{Z}^3}$ contains a $\{v_s\}_{s \leqslant 0}$-orbit.

Let $h \in \mathrm{SL}(3, \mathbb{R})$ such that $\overline{\mathrm{SO}(Q_0, \mathbb{R}) g_Q \mathbb{Z}^3} \supseteq \{v_s . h\mathbb{Z}^3 : s \leqslant 0\}$. Then

$$\overline{Q(\mathbb{Z}^3)} = \overline{Q_0(g_Q \mathbb{Z}^3)} \supseteq Q_0(\{v_s h \mathbb{Z}^3 : s \leqslant 0\}).$$

We want to find $s_n \leqslant 0, x_n \in h\mathbb{Z}^3$ such that $Q_0(v_{s_n} x_n) \to 0$. After some specific calculation, it suffices to find $x \in h\mathbb{Z}^3$ such that $2x_1x_3 - x_2^2 > 0$. The lattice and this cone always intersect. □

Proof of Lemma 2.3.6. We have

$$\mathfrak{h}^{\perp} = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & -2x_{11} & -x_{12} \\ x_{31} & -x_{21} & x_{11} \end{bmatrix} \right\}.$$

For $x \in \mathfrak{h}^{\perp}$, we can calculate $u_t x u_t^{-1}$ explicitly. We have

$$u_t x u_t^{-1} = \begin{bmatrix} * & * & P_x(t) = \frac{t^4}{4!}x_{31} + \frac{t^3}{3!}x_{21} + \frac{t^2}{2!}x_{11} + \frac{t}{3}(-x_{21}) + \frac{x_{13}}{6} \\ * & * & * \\ * & * & * \end{bmatrix}$$

Let $M_t := \max\left\{ \left|\frac{t^4}{4!}x_{31}\right|, \left|\frac{t^3}{3!}x_{21}\right|, \left|\frac{t^2}{2!}x_{11}\right|, \left|\frac{t}{3}x_{21}\right|, \left|\frac{x_{13}}{6}\right| \right\}$, then we can prove that

$$\max\{|P_x(t)|, |P_x(2t)|, |P_x(3t)|, |P_x(4t)|, |P_x(5t)|\} \geqslant 10^{-10} M_t.$$

For x_n, choose t such that $M_t = \delta$, choose $t_{n,\delta} \in \{t, 2t, 3t, 4t, 5t\}$ such that $|P_{x_n}(t_{n,\delta})| \geqslant 10^{-10}\delta$. Then the statement follows. □

A dynamics exposition of the case $N = 2$

Recall lemma 2.2.15, it suffices to find an indefinite "irrational" Q such that $\mathrm{SO}(Q,\mathbb{R})\mathbb{Z}^2$ is bounded. Let $Q_1 = xy$, then $\exists g_Q \in \mathrm{SL}(2,\mathbb{R})$ such that $Q = \lambda(Q_1 \circ g_Q)$ where $\lambda \neq 0 \in \mathbb{R}$. We want to find $g \in \mathrm{SL}(2,\mathbb{R})$ such that:

(i) $Q_1 \circ g$ is "irrational".

(ii) $\mathrm{SO}(Q_1,\mathbb{R})g\mathbb{Z}^2$ is bounded.

We can calculate that $\mathrm{SO}(Q_1,\mathbb{R}) = \left\{a_t = \begin{bmatrix} t & 0 \\ 0 & t^{-1}\end{bmatrix}\right\}$.

Example 2.4.2

Let $\Lambda := \mathbb{Z}\begin{bmatrix}1\\1\end{bmatrix} \oplus \mathbb{Z}\begin{bmatrix}\sqrt{2}\\-\sqrt{2}\end{bmatrix}$, let $\Lambda' = \frac{\Lambda}{\sqrt{2\sqrt{2}}}$, then $\Lambda' \in X_2$. Consider $t_0 = 3 + 2\sqrt{2}$, we can prove $a_{t_0}\Lambda \subseteq \Lambda$ hence $a_{t_0}\Lambda' \subseteq \Lambda'$. Note that a_{t_0} preserve the volume of lattice, hence $a_{t_0}\Lambda' = \Lambda'$ which shows that $\{a_t.\Lambda\}$ is compact.

Fact 2.4.3. If $\mathrm{SO}(Q_1,\mathbb{R})g\mathbb{Z}^2$ is **not** closed, then $Q_1 \circ g$ is "irrational".

So it suffices to construct an orbit of $\mathrm{SO}(Q_1,\mathbb{R}) = \{a_t\}$ that is not compact and is bounded.

Fact 2.4.4. The union of all compact a_t-orbits are dense.

Proof. Firstly, there exists at least one compact a_t-orbit, say $a_t\Lambda$. Then we can prove that $\{\Lambda' \in X_2 : \Lambda' \text{ is commensurable with } \Lambda\}$ is dense in X_2 and those Λ' are with compact a_t-orbit. The statement follows by the following lemma 2.4.6. □

Definition 2.4.5. We say two lattice Λ_1 and Λ_2 is **commensurable**, denoted by $\Lambda_1 \sim \Lambda_2$, iff $\Lambda_1 \cap \Lambda_2$ is of finite index in Λ_1 and Λ_2.

Lemma 2.4.6

If $a_t\Lambda$ is compact and $\Lambda' \sim \Lambda$, then $a_t\Lambda'$ is also compact.

For the final construction, we want to find $x, y, z \in X$ such that $\{a_t.x\}, \{a_t.y\}$ both closed and

$$a_t.z \to a_t.x(t \to 0), \quad a_t.z \to a_t.y(t \to \infty).$$

Then $\{a_t.z\}$ is not closed but bounded. Given x with closed a_t-orbit, we can choose z as $\begin{bmatrix}1 & s\\0 & 1\end{bmatrix}a_t.x$ and choose y as $\begin{bmatrix}1 & 0\\s' & 1\end{bmatrix}.z$, then the choice of y contains an open set in X_2. Hence, there is a suitable y with closed a_t-orbit.

Remark 2.4.7 — In the case of $N = 2$, the orthogonal group of Q_0 corresponding to the diagonal flow. But for $N \geqslant 3$, the orthogonal group is semisimple, which brings more rigidity.

§2.5 22.3.18: Unipotent flows on X_2

Let $X_2 := \left\{\text{unimodular lattices in } \mathbb{R}^2\right\} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. Let $U = \left\{u_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, s \in \mathbb{R}\right\}$.

Theorem 2.5.1

We have the following dichotomy regarding orbits of U in X_2 :

(1) the orbit is compact.

(2) the orbit is dense in X_2.

Say the orbit is $U.\Lambda$, case (1) happens exactly when Λ contains a horizontal vector, i.e., $\Lambda \cap \mathbb{R}e_1 = \mathbb{R}e_1$.

Example 2.5.2

$\Lambda = \mathbb{Z}^2$, we check that $U.\mathbb{Z}^2$ is compact. Because $u_1.\mathbb{Z}^2 = \mathbb{Z}^2$.

> **Question 2.5.3.** Given $x \in X_2$, could the U-orbit $U.x$ diverge? Or could $s \mapsto u_s.x$ be a proper map? The answer is **NO**.

For $\Lambda \in X_2$, define $\mathrm{Sys}(\Lambda) := \inf\{\|v\| : v \neq 0, v \in \Lambda\}$. Recall Mahler's criterion.

Proposition 2.5.4 (Mahler's criterion)

The following holds:

1. For any $\varepsilon > 0, \mathscr{C}_\varepsilon := \{\Lambda \in X_2 : \mathrm{Sys}(\Lambda) \geqslant \varepsilon\}$ is compact.
2. $\forall K \subseteq X_2$ compact, $\exists \varepsilon > 0$ such that $K \subseteq \mathscr{C}_\varepsilon$.

Theorem 2.5.5

For any $K \subseteq X_2$ compact, $\forall \varepsilon > 0$, $\exists \delta = \delta(K, \varepsilon) > 0$, such that the following holds. For every interval (a, b) and $\Lambda_0 \in X_2$, satisfying $u_{s_0}\Lambda_0 \in K$ for some $s_0 \in (a, b)$, then

$$\frac{1}{b-a}\mathrm{Leb}\left\{s \in (a,b) : u_s.\Lambda_0 \notin \mathscr{C}_\delta\right\} \leqslant \varepsilon.$$

Corollary 2.5.6

$\forall \varepsilon > 0$, $\exists \delta > 0$, for any $x \in X_2$ does not have compact U-orbit, then

$$\limsup_{T \to \infty} \frac{1}{T}\mathrm{Leb}\left\{s \in [0, T] : u_s.x \notin \mathscr{C}_\delta\right\} \leqslant \varepsilon.$$

Observation 2.5.7. It is impossible for a unimodular lattice Λ to contain two linearly independent vectors of length < 1.

Proof of Corollary assuming Theorem 2.5.5. Let $K := \mathscr{C}_1$, we want to find some $s \geqslant 0$ such that $u_s.x \in K := \mathscr{C}_1$. Otherwise, for any $s \geqslant 0$, $\exists v_s \neq 0 \in \Lambda_x = x$, such that $\|u_s v_s\| < 1$. Let v_s be primitive, i.e., $\mathbb{R}v \cap \Lambda = \mathbb{Z}v$, then v_s is unique up to a sign. For any primitive $v \in \Lambda_x$, consider $I_v = \{s > 0 : \|u_s v\| < 1\}$. Moreover, for $v \neq \pm w$, we have $I_v \cap I_w = \varnothing$. Then $\{I_v\}$ could not be an open cover of $(0, \infty)$ otherwise $I_v = (0, \infty)$ for some v. This shows that v is a horizontal vector, hence $U.x$ is compact.

Therefore, if $x \in X_2$ such that $U.x$ is not compact, then $\exists s \in (0, \infty)$ such that $u_s.x \in \mathscr{C}_1$. For any $\varepsilon > 0$, let $K = \mathscr{C}_1$, there is $\delta = \delta(\varepsilon, K)$ such that

$$\frac{1}{T}\mathrm{Leb}\left\{t \in [0, T] : u_t.x \notin \mathscr{C}_\delta\right\} \leqslant \varepsilon$$

for any $T > s$, by Theorem 2.5.5. Let $T \to \infty$ and the statement follows. □

Remark 2.5.8 — This corollary can give another view of showing that X_2 is of finite volume.

Lemma 2.5.9

$\exists C_1, \alpha_1 > 0$ such that for every interval (a, b), every vector $v \in \mathbb{R}^2$, every $\rho \in (0, 1)$,

$$\frac{1}{b-a}\mathrm{Leb}\left\{s \in (a, b) : \|u_s v\| \leqslant \rho M_0\right\} \leqslant C_1 \rho^{\alpha_1},$$

where $M_0 := \sup_{s \in (a,b)} \|u_s v\|$.

Proof. Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then $u_s v = \begin{bmatrix} v_1 + s v_2 \\ v_2 \end{bmatrix}$, let $M_0 = u_{s_0} v = \begin{bmatrix} v_1 + s_0 v_2 \\ v_2 \end{bmatrix}$. Take $C_1 = 100$ and $\alpha_1 = 1$. Consider each case of $|v_2| > \frac{1}{4}$ and $|v_2| \leqslant \frac{1}{4}$, both easy to verify. □

Proof of Theorem 2.5.5. K compact implies that $\exists \delta_1 < 1$ such that $K \subseteq \mathscr{C}_{\delta_1}$. Hence, there is $s_0 \in (a, b)$ such that $\forall v \neq 0 \in \Lambda_0$, $\|u_{s_0} v\| \geqslant \delta_1$. Let

$$I(\delta_1) := \{s \in (a, b) : \mathrm{Sys}(u_s.\Lambda_0) < \delta_1\} = \coprod_{\alpha \in \mathscr{A}} I_\alpha = \coprod_{\alpha \in \mathscr{A}} (a_\alpha, b_\alpha).$$

For every $\alpha \in \mathscr{A}$, there exists $v_\alpha \in \Lambda_0$ primitive such that $\forall s \in I_\alpha$, $\|u_s v_\alpha\| < \delta_1$. Take ρ such that $C_1 \rho^{\alpha_1} < \varepsilon$, take $\delta = \rho \delta_1$. Apply the lemma to each I_α, the conclusion follows. □

Proof of Theorem 2.5.1. Fix $x_0 \in X_2$ such that $U.x_0$ is not compact. Choose a minimal element from $\left\{\overline{U.y} : y \in \overline{U.x_0}, U.y \text{ is not compact}\right\}$. Consider $Y_0 = \overline{U.y_0}$, there are two cases.

Case 1: Y_0 does not contain any compact U-orbit.

Applying the argument in proof 2.1, we choose $x_n, x_n' \in \mathscr{C}_1$ by Theorem 2.5.5 such that $d(x_n', x_n) \to 0$, then $x_n' = A_n x_n$ for some $A_n \to \mathrm{Id}$. Let $y_n = u_s x_n$ and $y_n' = u_{s+t} x_n'$ for some $s = s_n, t = t_n$. But for fixed δ, we should allow $s_{n,\delta}$ to vary in some interval to guarantee that y_n lives a fixed compact set. The range of $s_{n,\delta}$ is controlled by Theorem 2.5.5. Then there are $y_{\infty,\delta}$ and $y_{\infty,\delta}'$ differ from each other by a diagonal matrix. The diagonal element is also dominated by δ. Finally, we can show that Y_0 is invariant under positive diagonal matrices.

Case 2: Y_0 contains some compact U-orbits.

Same as case 1, but easier to show that Y_0 is invariant under positive diagonal matrices. □

§2.6 22.3.22: Strong Oppenheim conjecture

Notation 2.6.1. $\mathrm{Prim}(\mathbb{Z}^3)$ denotes $\left\{v \in \mathbb{Z}^3 : \mathbb{R}v \cap \mathbb{Z}^3 = \mathbb{Z}v\right\}$.

Theorem 2.6.2 (Strong Oppenheim Conjecture)

Let Q be a nondegenerate, indefinite, quadratic form with real coefficients in 3 variables. Assume Q is **not** proportional to a quadratic form with $\mathbb{Q}$-coefficients. Then $Q(\mathbb{Z}^3)$ or $Q(\mathrm{Prim}(\mathbb{Z}^3))$ is dense in $\mathbb{R}^3$.

Theorem 2.6.3

Let $\mathrm{SO}(Q,\mathbb{R}) := \{g \in \mathrm{SL}(3,\mathbb{R}) : Q \circ g = Q\}$. If Q is as in the theorem above, then $\overline{\mathrm{SO}(Q,\mathbb{R})\mathbb{Z}^3}$ in X_3 contains a $\{v_s\}_{s\geqslant 0}$ or $\{v_s\}_{s\leqslant 0}$ orbit.

Claim 2.6.4. Theorem 2.6.3 $\implies$ Theorem 2.6.2.

Recall $Q_0(x_1,x_2,x_3) = 2x_1x_3 - x_2^2$.

Theorem 2.6.5

Let $H := \mathrm{SO}(Q_0,\mathbb{R})$, then every orbit of H on X_3 is either closed or the orbit closure contains a $\{v_s\}_{s\geqslant 0}$ or $\{v_s\}_{s\leqslant 0}$ orbit.

Theorem 2.6.6

If Q is as in Theorem 2.6.2, then $\mathrm{SO}(Q,\mathbb{Z}^3)\mathbb{Z}^3 = \mathrm{SO}(Q_0)g_Q\mathbb{Z}^3$ is **not** closed.

Claim 2.6.7. Theorem 2.6.5 + Theorem 2.6.6 $\implies$ Theorem 2.6.3.

Theorem 2.6.8

$\forall \varepsilon > 0$, $\exists$ a compact $C \subseteq X_3$ such that for every $\Lambda \in X_3$, at least one of the following holds:

(1) $\limsup_{T\to\infty} \frac{1}{T}\mathrm{Leb}\,\{t \in [0,T] : u_t.\Lambda \notin C\} \leqslant \varepsilon$.

(2) $\Lambda \cap \mathbb{R}e_1$ is a lattice in $\mathbb{R}e_1$ and $\|\Lambda \cap \mathbb{R}e_1\|_{\mathbb{R}e_1} < \varepsilon$.

(3) $\Lambda \cap \mathbb{R}e_1 \oplus \mathbb{R}e_2$ is a lattice in $\mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $\|\Lambda \cap \mathbb{R}e_1 \oplus \mathbb{R}e_2\|_{\mathbb{R}e_1\oplus\mathbb{R}e_2} < \varepsilon$.

Claim 2.6.9. Theorem 2.6.8 + some arguments in Section 2.3 $\implies$ Theorem 2.6.5 and Theorem 2.6.6.

Recall what happens for X_2. Assume $\Lambda \in X_2$ contains no horizontal vector. Then

1. $\forall v \neq 0 \in \lambda$, $\|u_t v\| \to \infty (t \to \pm\infty)$.

2. if $\|u_t.v\| \geqslant M_0$ for some $t \in (a,b)$, then for most $t \in (a,b)$, $\|u_t.v\| \geqslant \frac{M_0}{10^{10}}$.

Notation 2.6.10. $\mathrm{Prim}^1(\Lambda)$ denotes $\{\Delta \subseteq \Lambda : \operatorname{rank}\Delta = 1, \operatorname{span}_{\mathbb{R}} \Delta \cap \Lambda = \Delta\}$. $\mathrm{Prim}^2(\Lambda)$ denotes $\{\Delta \subseteq \Lambda : \operatorname{rank}\Delta = 2, \operatorname{span}_{\mathbb{R}} \Delta \cap \Lambda = \Delta\}$.

Definition 2.6.11. $\varepsilon, \rho \in (0,1)$, Λ is said to be **(ε, ρ)-protected** (with respect to $\{0\} \subseteq \mathbb{Z}v \subseteq \Delta \subseteq \Lambda$) if exist $\mathbb{Z}v \in \mathrm{Prim}^1(\Lambda)$ and $\Delta \in \mathrm{Prim}^2(\Lambda)$ such that

(i) $\mathbb{Z}v \subseteq \Delta$.

(ii) $\|\mathbb{Z}v\|, \|\Delta\| \in (\rho\varepsilon, \varepsilon)$.

Lemma 2.6.12

If Λ is (ε, ρ)-protected with respect to $\{0\} \subseteq \mathbb{Z}v \subseteq \Delta \subseteq \Lambda$, then $\mathrm{Sys}(\Lambda) \geqslant \rho\varepsilon$.

Proof. Take $w \neq 0 \in \Lambda$, then

(1) if $w \in \Lambda \setminus \Delta$, then $\|w\| \geqslant \frac{1}{\varepsilon} > 1$,

(2) if $w \in \Delta \setminus \mathbb{Z}v$, then $\|w\| \geqslant \rho$.

(3) if $w \in \mathbb{Z}v$, then $\|w\| \geqslant \rho\varepsilon$.

□

Lemma 2.6.13

$\exists C_2, \alpha_2 > 0$, such that for every $v \in \mathbb{R}^3 \oplus \wedge^2(\mathbb{R}^3)$, for every $a < b$ in $\mathbb{R}$,

$$\frac{1}{b-a}\mathrm{Leb}\{t \in (a,b) : \|u_t v\| \leqslant \rho M_0\} \leqslant C_2 \rho^{\alpha_2},$$

where $M_0 := \sup_{t \in (a,b)} \|u_t v\|$.

Exercise 2.6.14. Proof this lemma.

Observation 2.6.15. $\Lambda \in X_3$, if $\mathbb{Z}v \in \mathrm{Prim}^1(\Lambda)$ and $\Delta \in \mathrm{Prim}^2(\Lambda)$ such that $\|\mathbb{Z}v\| \leqslant 1$ and $\|\Delta\| \leqslant 1$, then $\mathbb{Z}v \subseteq \Delta$.

Proof of Theorem 2.6.8. Assume $\Lambda \in X_3$ which does not satisfy (2) or (3). The parameters $\varepsilon', \delta, \rho$ will be determined later. Consider

$$I_1 = \left\{t \in [0,T] : \mathrm{Sys}(u_t\Lambda) < \varepsilon', \nexists \mathbb{Z}v \in \mathrm{Prim}^1(\Lambda), \rho\delta < |u_t v| < \delta\right\},$$

$$I_1 = \left\{t \in [0,T] : \mathrm{Sys}(u_t\Lambda) < \varepsilon', \nexists \Delta \in \mathrm{Prim}^2(\Lambda), \rho\delta < |u_t \Delta| < \delta\right\},$$

then $I_1 \cup I_2$ is the set of t such that $u_t\Lambda \notin C$ for some compact C. We will choose $\varepsilon', \delta, \rho$ such that for T large enough, $|I_1| \leqslant \varepsilon T$, the proof of I_2 is the same.

Let $\varepsilon' = \delta/2$, let

$$I = \left\{t \in (0,T) : \mathrm{Sys}(u_t\Lambda) < \varepsilon'\right\}.$$

Then I is open, write $I = \coprod_\alpha (a_\alpha, b_\alpha)$. Fix α, for every $t \in (a,b)$, there is $v \in \mathrm{Prim}^1(\Lambda)$ such that $\|u_t v\| < \varepsilon' = \delta/2$. Let $I(t,v)$ be the maximal interval containing t such that $\|u_s v\| < \delta$ for every $s \in I(t,v)$. Then $\bigcup I(t,v) \supseteq [a,b]$. By passing to a sub-covering, we can assume the cover is of multiplicity at most 2.

Choose T_0 large enough, we assume $\sup_{t\in[0,T]} \mathrm{Sys}(u_t\Lambda) \geqslant \delta$ for every $T \geqslant T_0$. Then $\sup_{s\in I(t,v)} \|u_s v\| \geqslant \varepsilon' = \delta/2$. By lemma, we can choose ρ smaller enough such that

$$\mathrm{Leb}\left\{s \in I(t,v) : \|u_s v\| \leqslant 2\rho\frac{\delta'}{2}\right\} \leqslant C_2|I(t,v)|(2\rho)^{\alpha_2} \leqslant \frac{1}{2}\varepsilon|I(t,v)|,$$

then the conclusion follows. □

§2.7 22.3.25: General dimension

Theorem 2.7.1

Let $X := \{\text{unimodular lattice in } \mathbb{R}^N\}$, let $u \in \mathfrak{sl}(N,\mathbb{R})$ be a nilpotent matrix, let $\phi_s := \exp(su)$. For every $\varepsilon,\delta \in (0,1)$, $\exists \mathscr{C} \subseteq X_N$ compact, such that for all interval $I = (a,b) \subseteq \mathbb{R}, \Lambda \in X_N$, such that if

$$\sup_{s\in I} \|\phi_s\Delta\| \geqslant \delta, \quad \forall \Delta \in \mathrm{Prim}(\Lambda).$$

Then we have

$$\frac{1}{b-a}\mathrm{Leb}\{s \in (a,b) : \phi_s\Lambda \notin \mathscr{C}\} \leqslant \varepsilon.$$

Definition 2.7.2. For $\Lambda \in X_N$, for every $k \in \{0,\cdots,N\}$, let

$$\mathrm{Prim}^k(\Lambda) := \{\Delta \leqslant \Lambda : \mathrm{rank}\,\Delta = k, \Delta_{\mathbb{R}}(=\mathrm{span}_{\mathbb{R}}\,\Delta) \cap \Lambda = \Delta\}.$$

Let $\|\Delta\| := \mathrm{Vol}(\Delta_{\mathbb{R}}/\Delta)$, $\|0\| := 1$. Let $\mathrm{Prim}(\Lambda) := \bigcup_{k=0}^N \mathrm{Prim}^k(\Lambda)$.

Definition 2.7.3. Let I be a interval in $\mathbb{R}$, a continuous map $\phi : I \to \mathrm{SL}(N,\mathbb{R})$ is said to be **(C,α)-good** at $\Lambda \in X_N$ if for every $\Delta \in \mathrm{Prim}(\Lambda)$, the map

$$s \mapsto \|\phi_s\Delta\|$$

is **(C,α)-good** in the sense that $\forall J \subseteq I$ interval, for every $\rho \in (0,1)$,

$$\frac{1}{|J|}\mathrm{Leb}\left\{s \in J : \|\phi_s\Delta\| < \rho\sup_{s\in J}\|\phi_s\Delta\|\right\} \leqslant C\rho^\alpha.$$

Lemma 2.7.4

$\exists C_N, \alpha_N > 0$, such that for every unipotent matrix $u \in \mathfrak{sl}(N,\mathbb{R})$, for every interval $I \subseteq \mathbb{R}$, for every $\Lambda \in X_N$, the map $s \mapsto \exp(su) \in \mathrm{SL}(N,\mathbb{R})$ is (C,α)-good on I at Λ.

Now, we can restate the theorem.

Theorem 2.7.5

Let $\Lambda \in X_N$, let $X := \{\text{unimodular lattice in } \mathbb{R}^N\}$, let $I \subseteq \mathbb{R}$ be a interval, let $\phi : I \to \mathrm{SL}(N,\mathbb{R})$ be (C,α)-good. For every $\varepsilon,\delta \in (0,1)$, $\exists \kappa = \kappa(\varepsilon,\delta,C,\alpha)$ such that if

$$\sup_{s\in I} \|\phi_s\Delta\| \geqslant \delta, \quad \forall \Delta \in \mathrm{Prim}(\Lambda),$$

then

$$\frac{1}{b-a}\mathrm{Leb}\{s \in (a,b) : \phi_s\Lambda \notin \mathscr{C}_\kappa\} \leqslant \varepsilon.$$

We will prove for $N = 3$ as an example.

Proof. Let $\mathrm{Sys}'(\Lambda) := \inf \{\|\Delta\| : \Delta \in \mathrm{Prim}(\Lambda)\}$, let

$$I' := \left\{s \in I : \mathrm{Sys}'(\phi_s) < 0.9\delta\right\} = \coprod_{\alpha \in \mathscr{I}_0} I_\alpha.$$

Take some $\alpha \in \mathscr{I}_0$, for every $x \in I_\alpha, \Delta \in \mathrm{Prim}(\Lambda)$, consider

$$I(x, \Delta) := \text{the connected component of } \{s \in I_\alpha : \|\phi_s \Delta\| < \delta\} \text{ containing } x.$$

Take a maximal element from $\{I(x, \Delta) : \Delta \in \mathrm{Prim}(\Lambda)\}$, denoted by $I_x = I(x, \Delta_x)$. Then I_x is an open interval satisfying:

(i) $\sup_{s \in I_x} \|\phi_s \Delta_x\| \leqslant \delta$.

(ii) $\forall \Delta \in \mathrm{Prim}(\Lambda)$, $\sup_{s \in I_x} \|\phi_s \Delta\| \geqslant 0.9\delta$.

(iii) $\{I_x\}_{x \in I_\alpha}$ forms an open cover of I_α which admits a finite sub-cover $\{I_x\}_{x \in \mathscr{I}_\alpha}$ of I_α with multiplicity $\leqslant 2$.

Definition 2.7.6. Let $\delta, \rho \in (0, 1)$, we say $\Lambda \in X_N$ is **(δ, ρ)-protected** by a flag $\mathscr{F} = \{\Delta_1 \leqslant \Delta_2 \leqslant \cdots \leqslant \Delta_l\}$ in $\mathrm{Prim}(\Lambda)$, if

(i) $\rho\delta \leqslant \|\Delta_i\| \leqslant \delta$, $\forall i = 1, 2, \cdots, l$.

(ii) if $\Delta \in \mathrm{Prim}(\Lambda)$ is such that $\Delta \notin \mathscr{F}$ and $\{\Delta\} \cup \mathscr{F}$ is also a flag, then $\|\Delta\| \geqslant 0.5\delta$.

Remark 2.7.7 — $\mathrm{rank}\, \Delta_1 < \mathrm{rank}\, \Delta_2 < \cdots < \mathrm{rank}\, \Delta_l$, hence $l \leqslant N + 1$.

Definition 2.7.8. We say a $\mathbb{R}$ linear subspace W of $\mathbb{R}^N$ is **Λ-rational** iff $W \cap \Lambda$ is lattice in W.

Lemma 2.7.9

$\Delta \mapsto \Delta_{\mathbb{R}}$ gives a bijection between $\mathrm{Prim}(\Lambda) \cong \{\Lambda\text{-rational subspaces}\}$.

Lemma 2.7.10

$\delta, \rho \in (0, 1), \rho < 0.5$. If Λ is (δ, ρ)-protected by $\mathscr{F} = \{\Delta_1 \leqslant \Delta_2 \leqslant \cdots \leqslant \Delta_l\}$, then $\mathrm{Sys}(\Delta) \geqslant \rho\delta$.

Remark 2.7.11 — It suffices to find (δ', ρ') take place of κ.

Continued proof of Theorem 2.7.5. Let

$$\mathscr{P}_x := \{\Delta \in \mathrm{Prim}(\Lambda) : \Delta \neq \Delta_x, \{\Delta, \Delta_x\} \text{ is a flag}\},$$

let

$$I'_x = \{s \in I_x : \forall \Delta \in \mathscr{P}_x, \|\phi_s \Delta\| < 0.8\delta\} = \coprod_{b \in \mathscr{J}_x} I_\beta.$$

Then for every $y \in I_\beta, \Delta \in \mathscr{P}_x$, let

$$I(y,\Delta) := \text{the connected component of } \{s \in I_\alpha : \|\phi_s \Delta\| < 0.9\delta\} \text{ containing } y.$$

For every $y \in I'_x$, take a maximal element, denoted by $I_y = I(y, \Delta_y)$. Take a sub-cover as before. We have

$$I_\alpha \supseteq I_x \supseteq I'_x \supseteq I_y.$$

Let

$$I_y(\text{bad}) = \left\{s \in I_y : \|\phi_s \Delta_y\| < \rho'\delta\right\}, \quad I_x(\text{bad}) = \left\{s \in I_x : \|\phi_s \Delta_x\| < \rho'\delta\right\}.$$

By (C,α)-good, we can choose ρ' sufficiently small such that $|I_y(\text{bad})| \leqslant 0.01\varepsilon |I_y|$ and $|I_x(\text{bad})| \leqslant 0.01\varepsilon|I_x|$. Consider the complement of all bad sets, denoted by $I(\text{good})$, which is of at least $(1-\varepsilon)$ density. It suffices to check for every $s \in I(\text{good})$, $\phi_s\Lambda$ is (δ, ρ')-protected.

(1) $s \in I \setminus I'$, then $\phi_s\Lambda$ is (δ,ρ')-protected by $\varnothing$.

(2) $s \in I', s \notin I'_x$, then $\phi_s\Lambda$ is (δ,ρ')-protected by $\{\Delta_x\}$.

(3) $s \in I', s \in I'_x$, then $s \in I(y,\Delta_y)$, then $\phi_s\Lambda$ is (δ,ρ')-protected by $\{\Delta_x, \Delta_y\}$.

□

Remark 2.7.12 — This proof is different with the proof in last section. It is not hard to extend this proof to general dimension $N \geqslant 3$. We just need to choose $I_x \supseteq I_y \supseteq I_z \supseteq \cdots$ repeatedly. Where in the case of $N = 3$, twice is enough.

3 Measure Rigidity

§3.1 22.4.8: Ergodicity and mixing

Exercise 3.1.1. Let

$$B = \left\{ \begin{bmatrix} t & s \\ 0 & t^{-1} \end{bmatrix} : t > 0, s \in \mathbb{R} \right\}, \quad A = \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t > 0 \right\}, \quad U = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R} \right\}.$$

Does there exist a probability measure space $(x, \mathcal{B}, \mu)$ such that

(i) X is a locally compact metrizable space, $\mathcal{B}$ is the Borel σ-algebra.

(ii) $B \curvearrowright X$ continuously.

(iii) B preserves μ.

(iv) μ is "totally ergodic", i.e., μ is ergodic with respect to A and U.

(v) μ is **not** mixing with respect to U.

Basic notions

- X is a compact metrizable space.
- H is a Lie group.
- H acts on X continuously, i.e., $H \times X \to X$ is continuous and some compatibility conditions.
- $\mathcal{B}_X$ is the Borel σ-algebra on X.
- $\mathrm{Prob}(X)$ denotes all probability measures on $(X, \mathcal{B}_X)$.
- $\mathrm{Prob}(X)^H$ denotes all elements μ in $\mathrm{Prob}(X)$ that is H-invariant, i.e.,

$$h_*\mu = \mu(h^{-1} \cdot) = \mu, \quad \forall h \in H.$$

Definition 3.1.2. An H-invariant probability measure μ is said to be **ergodic** with respect to H if every H-invariant measurable set E is either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Fact 3.1.3. If μ is ergodic, then for every "almost H-invariant" measurable set E, i.e., $\mu(hE \triangle E) = 0, \forall h \in H$, then there is either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

If $\mu \in \mathrm{Prob}(X)^H$, consider a natural action $H \curvearrowright L^2(X, \mu)$. Then this action gives a homomorphism

$$\pi : H \to \mathcal{U}(L^2(X, \mu))$$

where $\mathcal{U}(L^2(X, \mu))$ is the family of unitary operators on $L^2(X, \mu)$.

Proposition 3.1.4

π is continuous with respect to SOT (**strong operator norm**), i.e., for every convergent sequence $(h_n) \subseteq H$, assuming $h_n \to h \in H$, then for every $f \in L^2(X, \mu)$,

$$h_n.f \to h.f \text{ in } L^2.$$

Remark 3.1.5 — Generally, π is not continuous with respect to operator norm topology.

Lemma 3.1.6

$H \curvearrowright (X, \mathcal{B}_X, \mu)$ continuously, $\mu \in \mathrm{Prob}(X)^H$, then the followings are equivalent

(1) μ is ergodic with respect to H.

(2) the associated unitary representation has no fixed vector other than constants.

Example 3.1.7

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let R_α be the rotation on $\mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{T}$ defined by $x \mapsto x + \alpha \bmod \mathbb{Z}$. Then R_α preserves the Haar measure m on $\mathbb{T}$ and m is ergodic with respect to R_α.

Example 3.1.8

1. Let $M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ acting on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then M preserves the Haar measure m and m is ergodic with respect to $\{M^n : n \in \mathbb{Z}\}$.
2. $M = \exp(W)$ for some matrix M. Consider
$$\mathbb{R} \cong \{W_t = \exp(tW) : t \in \mathbb{R}\} \curvearrowright W_t.\mathbb{Z}^2 \subseteq X_2 = \left\{\text{unimodular lattices in } \mathbb{R}^2\right\},$$
then this induces an action $W_t : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/W_t.\mathbb{Z}^2$. Note that $W_1.\mathbb{Z}^2 = \mathbb{Z}^2$, we consider an action
$$W_t \curvearrowright T = \text{“a torus bundle” over } \mathbb{S}^1.$$
Then W_t preserves the natural measure on T, is ergodic but **not** mixing.

Definition 3.1.9. Assume $\mu \in \mathrm{Prob}(X)^H$, we say that μ is **mixing** with respect to H if for every $(h_n) \subseteq H$ that diverges, for every $\varphi, \psi \in L^2(X, \mu)$,
$$\int \varphi(h_n^{-1}x)\overline{\psi(x)}\mathrm{d}\mu(x) \to \int \varphi \mathrm{d}\mu \int \overline{\psi}\mathrm{d}\mu.$$

Lemma 3.1.10

$\mu \in \mathrm{Prob}(X)^H$, if μ is mixing, then μ is ergodic.

Theorem 3.1.11

Assume $\pi : \mathrm{SL}(2, \mathbb{R}) \to \mathcal{U}(\mathcal{H})$ is a unitary representation continuous with respect to SOT, where $\mathcal{H}$ is a separable Hilbert space. Assume π has no fixed vectors, then π is mixing, i.e., for every (h_n) divergent in $\mathrm{SL}(2, \mathbb{R})$, for every $\varphi, \psi \in \mathcal{H}$,
$$\langle h_n.\varphi, \psi \rangle \to 0.$$

Proof. We assume $(h_n) \subseteq A$, let $h_n = \begin{bmatrix} e^{t_n} & \\ & e^{-t_n} \end{bmatrix}$, assume $t_n \to \infty$. By the separability, there is a subsequence (h_{n_k}) such that

$$\langle h_{n_k}\varphi, \psi \rangle \text{ exists}, \quad \forall \varphi, \psi \in \mathcal{H}.$$

Fixed ψ, there exists $E\varphi \in \mathcal{H}$ such that

$$\langle E\varphi, \psi \rangle = \lim_{k\to\infty} \langle h_{n_k}\varphi, \psi \rangle .$$

Then $E : \mathcal{H} \to \mathcal{H}$ is linear, bounded. We will show that $\operatorname{Im} E$ is fixed by $\mathrm{SL}(2, \mathbb{R})$.

For every $v = \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$, we have $h_{n_k} v h_{n_k}^{-1} \to \mathrm{Id}$. Hence

$$\langle E(v\varphi), \psi \rangle = \lim_{k\to\infty} \left\langle h_{n_k} v h_{n_k}^{-1} h_{n_k}\varphi, \psi \right\rangle = \lim_{k\to\infty} \langle h_{n_k}\varphi, \psi \rangle = \langle E\varphi, \psi \rangle .$$

Similarly, we can show that $\langle uE\varphi, \psi \rangle = \langle E\varphi, \psi \rangle$ for every $u = \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix}$. Hence we have $u \circ E = E$ and $E \circ v = E$, or, $v^* \circ E^* = E^*$.

Notice that $E^* = \lim_k h_{n_k}^{-1}$ in the weak operator topology, and we can prove that $\ker E = \ker E^*$. Then

$$\operatorname{Im}(\mathrm{Id} - v) \subseteq \ker E = \ker E^* \implies v^* \circ E = E.$$

$v^* = v^{-1} \in V$, hence U, V both fix elements in $\operatorname{Im} E$. Because U, V generates G, it follows $\operatorname{Im} E = \{0\}$, we are done. □

§3.2 22.4.15: Classification of finite invariant measures under unipotent flows in $\mathrm{SL}(2, \mathbb{R})$

- G "nice" topological group.
- X "nice" topological group.
- $G \curvearrowright X$ continuously $\rightsquigarrow G \curvearrowright (X, \mathcal{B}_X)$.
- $\mathrm{Prob}(X) \coloneqq \{\text{probability measures on } (X, \mathcal{B}_X)\}$.
- $\mathrm{Prob}(X)^G \coloneqq \{\mu \in \mathrm{Prob}(X) : g_*\mu = \mu, \forall g \in G\}$.

Lemma 3.2.1
$\mathrm{Prob}(X)^G$ has a convex structure and the extremal points in $\mathrm{Prob}(X)^G$ is exactly the measures in $\mathrm{Prob}(X)^{G,\mathrm{erg}}$.

Theorem 3.2.2 (Choquet, Ergodic Decomposition)
$\forall \mu \in \mathrm{Prob}(X), \exists_1 \lambda \in \mathrm{Prob}(\mathrm{Prob}(X)^G)$, such that

(i) $\mu = \int_{\mathrm{Prob}(X)^G} \nu \mathrm{d}\lambda(\nu)$,

(ii) $\lambda(\mathrm{Prob}(X)^{G,\mathrm{erg}}) = 1$.

Remark 3.2.3 — In general, $\mathrm{Prob}(X)^{G,\mathrm{erg}}$ is **not** closed in $\mathrm{Prob}(X)^G$, hence we can **not** say $\operatorname{supp}\lambda = \mathrm{Prob}(X)^{G,\mathrm{erg}}$.

Assume we have an $\mathbb{R}$-action on X (flow), $\mathbb{R} \times X \to X, (t, x) \mapsto T_t(x)$. Take some $x \in X$, consider a limit point μ of

$$\left\{ \frac{1}{T} \int_{t=0}^{T} (T_t)_* \delta_x \mathrm{d}t : T \geqslant 0 \right\},$$

is $(T_t)_{t\geqslant 0}$-invariant.

Lemma 3.2.4

If further assume X is compact, then $\mathrm{Prob}(X)^{(T_t)_{t\geqslant 0}} \neq \varnothing$.

Example 3.2.5

If X is not compact, let $(T_t)_{t\geqslant 0}$ be translations on $\mathbb{R}$, then $\mathrm{Prob}(\mathbb{R})^{(T_t)_{t\geqslant 0}} = \varnothing$.

Example 3.2.6

If G is not $\mathbb{R}$, X is compact, consider $\mathrm{SL}(2, \mathbb{R}) \curvearrowright \mathbb{RP}^1$ linearly, then $\mathrm{Prob}(X)^G = \varnothing$.

Theorem 3.2.7 (Pointwise Ergodic Theorem)

Assume we have a flow $T_t : X \to X$ on a nice X. Let μ be a (T_t)-invariant, ergodic, probability Borel measure. Then for every $f \in \mathrm{L}^1(X, \mathcal{B}_X, \mu)$, there exists $E_f \in \mathcal{B}_X, \mu(E_f) = 1$ such that

$$\lim_{T\to\pm\infty} \frac{1}{T} \int_0^T f(T_t x_0)\mathrm{d}t = \int f(x)\mathrm{d}\mu(x), \quad \forall x_0 \in E_f.$$

Corollary 3.2.8

Assumption as above, then there exists a set $E \in \mathcal{B}_X$ with μ full measure such that

$$\lim_{T\to\pm\infty} \frac{1}{T} \int_0^T (T_t)_* \delta_x \mathrm{d}t \to \mu, \quad \forall x \in E,$$

in the weak* topology.

Definition 3.2.9. $G \curvearrowright X$, we say this action is **uniquely ergodic** if there exists a unique G-invariant probability measure on X.

Lemma 3.2.10

If $G = \mathbb{R}$, X is compact and $G \curvearrowright X$ is uniquely ergodic. Assume $\mathrm{Prob}(X)^G = \{\mu\}$, then for every $x \in X$,

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T (T_t)_* \delta_x \mathrm{d}t = \mu.$$

Example 3.2.11

Consider $\mathbb{R} \curvearrowright \{\mathrm{pt}\} \coprod \mathbb{R}$ and $\mathrm{SL}(2, \mathbb{R}) \curvearrowright \{\mathrm{pt}\} \coprod \mathbb{RP}^1$ as examples above. They both uniquely ergodic. It shows that the condition of $X = \mathbb{R}$ and the compactness of X are both necessary.

Example 3.2.12

$\mathbb{R} \curvearrowright \mathbb{T} = \mathbb{R}/\mathbb{Z}$ by $T_t(x) := x + t \bmod \mathbb{Z}$ is uniquely ergodic.

Example 3.2.13

$\mathrm{SL}(2, \mathbb{R}) \curvearrowright \mathrm{SL}(2, \mathbb{R})/\Gamma$ where $\Gamma \leqslant \mathrm{SL}(2, \mathbb{R})$ is discrete and cocompact, is uniquely ergodic.

Example 3.2.14

$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \curvearrowright \mathrm{SL}(2, \mathbb{R})/\Gamma$ where $\Gamma \leqslant \mathrm{SL}(2, \mathbb{R})$ is discrete and cocompact, is uniquely ergodic.

Theorem 3.2.15

$U := \left\{ u_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R} \right\}$, $\Gamma \leqslant G = \mathrm{SL}(2, \mathbb{R})$, consider $G \curvearrowright X = G/\Gamma$. Then every $\mu \in \mathrm{Prob}(X)^{U,\mathrm{erg}}$ is

(i) either supported on a compact U-orbit.

(ii) or is the unique $\mathrm{SL}(2, \mathbb{R})$-invariant measure (up to a scalar).

Fact 3.2.16. For every discrete $\Gamma \leqslant G = \mathrm{SL}(2, \mathbb{R})$, there exists a unique (up to a scalar) G-invariant locally finite measure m_X on $X = G/\Gamma$.

Lemma 3.2.17

Assumptions as above. Then

(i) either μ is supported on a compact U-orbit.

(ii) or μ is $B := \left\{ \begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix} : t, s \in \mathbb{R} \right\}$-invariant.

Proof. Recall the argument in Section 2.1, we want to mimic the proof. There are some analogies between topology and measure theory.

- compact space $\rightsquigarrow$ invariant probability measure
- minimal set $\rightsquigarrow$ "generic points" and "ergodicity"

Let E be the set of generic points of μ, then $\mu(E)$. Take $E' \subseteq E$ compact such that $\mu(E') > 0.8$. Then $\exists F', \mu(F') = 1, \forall x \in F'$ we have

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T \mathbb{1}_{E'}(u_s x)\mathrm{d}s = \mu(E') > 0.8.$$

We can find a set $F \subseteq F', \mu(F) > 0.9$ such that the convergence is uniform for $x \in F$. Then $\exists T_0$, such that $\forall x \in F, T > T_0$, we have

$$\frac{1}{T} \int_0^T \mathbb{1}_{E'}(u_s x)\mathrm{d}s > 0.5.$$

Claim $\forall \varepsilon > 0, \exists x \neq y \in F$ such that $d(x,y) < \varepsilon$ and $y \notin \{u_s x : s \in (-1,1)\}$.

Argue by contradiction, then $\exists \varepsilon > 0$, such that for every $x \neq y \in F, d(x,y) < \varepsilon$ implies $y \in u_{(-1,1)}x$. Cover F by countable boxes with diameter $< \varepsilon$. Then there is a local u-orbit with positive μ-measure. Assume $y \in F$ such that $\mu(u_{(-1,1)}y) > 0$. Then we can choose $s \in (-1,1)$ such that $u_s y$ is generic, hence

$$\frac{1}{T} \int_0^T \mathbb{1}_{u_{(-1,1)}y}(u_t(u_s y)) \to \mu(u_{(-1,1)}y) > 0.$$

Then $\exists t > 1$, such that $u_t y' \in u_{(-1,1)}y$. Then Uy is compact and μ supported on it. This is case (i).

By the claim, recall the notation in Section 2.1, we can replace $s_{n,\delta}$ by $s'_{n,\delta} \in [\frac{1}{2}s_{n,\delta}, \frac{3}{2}s_{n,\delta}]$ such that

(i) $u_{s'_{n,\delta}} x_n \in E' \subseteq E$,

(ii) $u_{s'_{n,\delta}} y_n \in E' \subseteq E$.

Then $u_{s'_{n,\delta}} x_n, u_{s'_{n,\delta}} y_n$ are both in a compact set and take limit points $x = x_{\infty,\delta}, y = y_{\infty,\delta} \in E'$. Then x, y are different by some a_t where $t \in [\delta/C, C\delta]$ for some absolute constant C. Then

$$(a_t)_*\mu = \lim_{T \to \infty} \int_0^T (a_t)_*(u_s)_*\delta_x \mathrm{d}s = \lim_{T \to \infty} \int_0^T (u_{\lambda s})_*\delta_{a_t x}\mathrm{d}s = \mu,$$

it follows that μ is B-invariant. □