Differentiable Dynamical Systems (Spring 2022, Shaobo Gan)

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Contents

1 Hyperbolic Fixed Points

§1.1 Hyperbolic linear isomorphisms

E finite dimensional linear space.

Definition 1.1.1. $A : E \to E$ linear isomorphism, we say A is **hyperbolic** if E splits into a direct sum

$$E = E^s \oplus E^u,$$

invariant under A, i.e., $A(E^s) = E^s, A(E^u) = E^u$. And there is a norm $|\cdot|$ on E with constants $C > 0, \lambda \in (0,1)$ such that:

(i) $|A^n v| \leqslant C\lambda^n |v|, \forall v \in E^s, n \geqslant 0.$

(ii) $|A^{-n} v| \leqslant C\lambda^n |v|, \forall v \in E^u, n \geqslant 0.$

Remark 1.1.2 — The definition of hyperbolic is independent with the choice of norm, as all norms on a given finite dimensional linear space are equivalent.

$E = E^s \oplus E^u$ is called the **hyperbolic splitting**, E^s is called the **contracting subspace**, E^u is called the **expanding subspace**. $\dim E^s$ is called the **index** of A, denoted by $\operatorname{Ind} A$.

If $E^s = \{0\}$, we call A of **source** type. If $E^u = \{0\}$, we call A of **sink** type. Otherwise, A is said to be of **saddle** type.

Theorem 1.1.3

A is hyperbolic if and only if $\sigma(A) \cap \mathbb{S}^1 = \varnothing$.

For $\gamma > 0$, let

$$C_\gamma(E^s) := \{v \in E : |v_u| \leqslant \gamma |v_s|\}$$

be the γ-cone about E^s. Similarly, we can define $C_\gamma(E^u)$ the γ-cone about E^u.

Theorem 1.1.4

Assume $A : E \to E$ hyperbolic with the splitting $E^s \oplus E^u$, then

$$\begin{aligned} E^s &= \{v \in E : |A^n v| \to 0, n \to \infty\} \\ &= \{v \in E : \exists r > 0, \text{ such that } |A^n v| \leqslant r, \forall n \geqslant 0\} \\ &= \{v \in E : \exists \gamma > 0, \text{ such that } A^n v \in C_\gamma(E^s), \forall n \geqslant 0\}. \end{aligned}$$

Corollary 1.1.5

The hyperbolic splitting $E = E^s \oplus E^u$ is unique.

Theorem 1.1.6

Let $A : E \to E$ hyperbolic, E splits into $E^s \oplus E^u$, then there exists a norm $\|\cdot\|$ on E and a constant $\tau \in (0,1)$ such that:

(i) $\|Av\| \leqslant \tau \|v\|, \forall v \in E^s$.

(ii) $\|A^{-1}v\| \leqslant \tau \|v\|, \forall v \in E^u$.

Proof. Take N such that $C\lambda^N < 1$, let $\|v\| := \sum_{n=0}^{N-1} |A^n v|$. Let $a = 1 + C\sum_{n=1}^{N-1} \lambda^n \geqslant 1$, then $\|Av\| \leqslant \left(1 - \frac{1 - C\lambda^N}{a}\right) \|v\|$ for all $v \in E^s$. □

Remark 1.1.7 — The norm $\|\cdot\|$ in this theorem is said to be **adapted** to A.

Remark 1.1.8 — The minimum constant $\tau = \tau(A, \|\cdot\|)$ is called the **skewness** of A with respect to the adapted norm $\|\cdot\|$.

Definition 1.1.9. A norm $|\cdot|$ on E is called of **box type** with respect to $E_1 \oplus E_2$ if $\|v\| = \max\{\|v_1\|, \|v_2\|\}$ where v_1, v_2 are components of v with respect to $E_1 \oplus E_2$.

For a norm $|\cdot|$ on E, the **box-adjusted** norm $\|\cdot\|$ of $|\cdot|$ with respect to $E_1 \oplus E_2$ is constructed by

$$\|v\| := \max\{|v_1|, |v_2|\}.$$

§1.2 Persistence of hyperbolic fixed points

Let $O \subseteq E$ be an open set, $f : O \to E$ is C^1. Assume p is a fixed point of f, it is called a **hyperbolic fixed point** if $A = Df(p) : E \to E$ is a hyperbolic linear isomorphism.

Let p be a hyperbolic fixed point, because $Df(p)$ is a linear isomorphism, there exists a neighborhood U of p such that $f : U \to f(U)$ is a diffeomorphism.

Definition 1.2.1. For $f, g : U \to E$, we define the C^1 distance between f and g as

$$d^1(f,g) := \sup_{x\in U} \{|f(x) - g(x)|, |Df(x) - Dg(x)|\}.$$

The closed ball in the C^1 topology is as

$$\mathscr{B}^1(f,\delta) := \left\{g \in C^1(U,E) : d^1(f,g) \leqslant \delta\right\}.$$

The "**persistence**": if δ sufficiently small, $\forall g \in \mathscr{B}^1(f,\delta)$ has a hyperbolic fixed point.

Recall $\phi : E \to E$ is called Lipschitz if there is a constant $k \geqslant 0$ such that

$$|\phi(x) - \phi(y)| \leqslant k|x - y|, \quad \forall x, y \in E.$$

The minimum k is called the **Lipschitz constant** of ϕ, denoted $\operatorname{Lip}\phi$.

Lemma 1.2.2

Assume $A : E \to E$ hyperbolic isomorphism with a splitting $E^s \oplus E^u$. Let $|\cdot|$ be a norm adapted to and of box type to A. Let τ be the skewness with respect to $|\cdot|$. Let $r > 0$, if $\varphi : E(r) = \{v \in E : |v| \leqslant r\} \to E$ is Lipschitz with

$$\operatorname{Lip} \varphi < 1 - \tau.$$

Then $A+\varphi$ has at most one fixed point in $E(r)$. If, in addition, $|\varphi(0)| \leqslant (1-\tau-\operatorname{Lip}\phi)r$, then $A+\varphi$ has a unique fixed point p_φ in $E(r)$ with $|p_\varphi| \leqslant \frac{|\varphi(0)|}{1-\tau-\operatorname{Lip}\varphi}$.

Proof. Let $A_{ss} \coloneqq A|_{E_s}, A_{uu} \coloneqq A|_{E_u}$, then $A_{ss} : E_s \to E_s$ and $A_{uu} : E_u \to E_u$. Let $\varphi_u = \pi_u \varphi$ and $\varphi_s = \pi_s \varphi$. Then we have the equation

$$A_{ss} x_s + \varphi_s(x) = x_s, \quad A_{uu} x_u + \varphi_u(x) = x_u,$$

or

$$A_{ss} x_s + \varphi_s(x) = x_s, \quad A_{uu}^{-1} x_u - A_{uu}^{-1} \varphi_u(x) = x_u.$$

Let $T : E(r) \to E, (x_s, x_u) \mapsto (A_{ss} x_s + \varphi_s(x), A_{uu}^{-1} x_u - A_{uu}^{-1}\varphi_u(x))$, then the fixed point of T corresponding to the fixed point of $A + \varphi$. Since

$$|T_s(x) - T_s(x')| \leqslant (\tau + \operatorname{Lip}\varphi)|x - x'|, \quad |T_u(x) - T_u(x')| \leqslant (\tau + \operatorname{Lip}\varphi)|x - x'|,$$

hence $|T(x) - T(x')| \leqslant (\tau + \operatorname{Lip}\varphi)|x - x'|$. This proves that T has at most one fixed point in $E(r)$. If $|\varphi(0)| \leqslant (1 - \tau - \operatorname{Lip}\phi)r$, then for every $x \in E(r)$, we have $Tx \in E(r)$. Hence there exists a unique fixed point in $E(r)$ and the estimate is trivial. $\square$

Theorem 1.2.3

Let $p \in U$ be a hyperbolic fixed point of f. Then $\exists \delta_0 > 0, \exists \varepsilon_0 > 0$, such that any $g \in \mathscr{B}^1(f, \delta_0)$, there at most one fixed point of g in $B(p, \varepsilon_0)$. Moreover, for every $\varepsilon \in (0, \varepsilon_0]$, there is $\delta \in (0, \delta_0]$, such that any $g \in \mathscr{B}^1(f, \delta)$ has a unique fixed point in $B(p, \varepsilon)$.

Proof. WLOG, assume $p = 0$. Let $A = Df(0)$ with hyperbolic splitting $E^s \oplus E^u$. Let $|\cdot|$ be a norm adapted to and of box type to A. Let τ be the skewness with respect to $|\cdot|$. Take $\lambda \in (\tau, 1)$, then $\exists \delta_0 > 0, \exists \varepsilon_0 > 0$ such that $\forall g \in \mathscr{B}^1(f, \delta_0)$ with $g = A(x) + \varphi(x)$, $\operatorname{Lip}\varphi|_{E(\varepsilon_0)} < \lambda - \tau < 1 - \tau$. Then g has at most one fixed point in $E(\varepsilon_0)$.

For any $\varepsilon \in (0, \varepsilon_0]$, take δ sufficiently small, such that $|g(0)| \leqslant (1 - \lambda)\varepsilon$ for every $g \in \mathscr{B}^1(f, \delta_0)$. Hence there exists a unique fixed point p_g with

$$|p_g| \leqslant \frac{|\varphi(0)|}{1 - \tau - \operatorname{Lip}\varphi} < \frac{(1 - \lambda)\varepsilon}{1 - \lambda} = \varepsilon,$$

which means $p_g \in B(0, \varepsilon)$. $\square$

Remark 1.2.4 — This theorem shows that $p : \mathscr{B}^1(f, \delta_0) \to B(p, \varepsilon_0), g \mapsto p_g$ is well-defined and continuous at f. Moreover, p is continuous on $\mathscr{B}^1(f, \delta_0)$. Because if $g_n \to g$ in $\mathscr{B}^1(f, \delta_0)$ with $p_{g_n} \to p \neq p_g$, then p is also a fixed point of g which contradicts with the uniqueness of the fixed point.

Remark 1.2.5 — The unique fixed point p_g of g in $B(p, \varepsilon_0)$ is called the **continuation** of p under g.

§1.3 Persistence of hyperbolicity

We want to show that under the hyperbolicity is persistent under perturbations, that is, $Dg(p_g)$ is still hyperbolic.

Lemma 1.3.1

Assume linear isomorphism $B : E \to E$ represents as $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ under the decomposition $E = E_1 \oplus E_2$, where $B_{ij} = \pi_i B|_{E_j}$. Let $\lambda \in (0,1), \varepsilon > 0$ satisfying $\lambda + \varepsilon < 1$. If there exists a norm $|\cdot|$ such that $\left|B_{11}^{-1}\right|, |B_{22}| < \lambda, |B_{21}|, |B_{12}| < \varepsilon$. Then there exists unique linear map $P_B : E_1 \mapsto E_2, |P_B| < 1$ such that $\mathrm{gr}(P_B)$ is invariant under B and P_B is continuous with respect to B. Where $\mathrm{gr}(P_B) := \{(v, P_B v) : v \in E_1\}$ is the graph of P_B.

Remark 1.3.2 — Under the norm of box type, $\mathrm{gr}(P_B)$ is indeed the expanding subspace.

Remark 1.3.3 — The argument of this lemma is very important, which is known as **graph transformation**.

Remark 1.3.4 — More often, we will regard the continuous dependence as the variation of $\mathrm{gr}(P_B)$ with respect to B.

Proof. For all $P : E_1 \to E_2, |P| \leqslant 1$. Consider

$$B \begin{bmatrix} v \\ Pv \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} v \\ pv \end{bmatrix} = \begin{bmatrix} w \\ Qw \end{bmatrix},$$

where $Q = (B_{21} + B_{22}P)(B_{11} + B_{12}P)^{-1}$. We need another lemma for the invertibility.

Definition 1.3.5. For linear map $A : E \to E$, we define the **mininorm** of A as $m(A) = \inf_{|v|=1} |Av|$. Then $m(A) = \left|A^{-1}\right|^{-1}$.

Lemma 1.3.6

$A : E \to E$ isomorphism, if $|B| < m(A)$, then $A + B$ is invertible and $\left|(A+B)^{-1}\right| \leqslant \frac{1}{m(A) - |B|}$.

Proof. Write $A + B = A(I + A^{-1}B)$. □

Notation 1.3.7. $L(E_1, E_2)$ denotes the set of all linear map from E_1 to E_2, $L(E_1, E_2)(1)$ denotes the unit ball in $L(E_1, E_2)$.

Continued proof of Theorem 1.3.1. Define the graph transform $T : L(E_1, E_2)(1) \to L(E_1, E_2), P \mapsto Q$, then $|Q| < 1$ shows that T maps to $L(E_1, E_2)(1)$. For $P, P' \in L(E_1, E_2)(1)$, let $Q = T(P), Q' = T(P')$, then

$$Q - Q' = (B_{11} - Q'B_{12})(P - P')(B_{11} + B_{12}P)^{-1}.$$

Hence $|Q - Q'| \leqslant (\lambda + \varepsilon)(\lambda^{-1} - \varepsilon)^{-1}|P - P'| = \alpha|P - P'|$ where $\alpha < 1$. Then there exists unique $P = P_B$ such that $T(P) = P$. Therefore, $B\mathrm{gr}(P) \subseteq \mathrm{gr}(P)$ and by the finite dimension, $\mathrm{gr}(P)$ is invariant under B.

Take the norm of box type, then $\left|B\begin{bmatrix} v \\ Pv \end{bmatrix}\right| = |(B_{11} + B_{12}P)v| \geqslant (\lambda^{-1} - \varepsilon)\left|\begin{bmatrix} v \\ Pv \end{bmatrix}\right|$.

The continuous dependence of P with respect to B follows by the following theorem. □

Theorem 1.3.8 (Contracting Map Principle with Parameters)

A, X metric spaces, X complete, $T : A \times X \to X$, $\lambda \in (0, 1)$. Satisfying $\forall a \in A, x_1, x_2 \in X$,

$$d(T(a, x_1), T(a, x_2)) \leqslant \lambda d(x_1, x_2).$$

Then for every $a \in A$, there exists unique $p(a) \in X$ such that $T(a, p(a)) = p(a)$. Moreover $p : A \to X$ is:

1. continuous if T is continuous.

2. Lipschitz if T is Lipschitz.

Theorem 1.3.9

Assume $A : E \to E$ is a hyperbolic isomorphism, then $\exists\delta_0 > 0$ such that B is a hyperbolic isomorphism for every B of $|B - A| < \delta_0$. Moreover, the hyperbolic splitting $E^s_B \oplus E^u_B$ vary continuously with respect to B.

Proof. Let $E^u \oplus E^s$ be the hyperbolic splitting of A. Take a norm $|\cdot|$ adapted to and of box type to A. Let τ be the skewness. Take $\lambda \in (\tau, 1)$ and $\varepsilon > 0$ such that $\lambda + \varepsilon < 1$. Then, there exists $\delta_0 > 0$ such that B satisfying the condition of lemma whenever $|B - A| < \delta_0$. Then $\exists P : E^u \to E^s$ such that $\mathrm{gr}(P)$ is invariant and expanding under B. Then $E^u_B = \mathrm{gr}(P)$ is the expanding subspace. For constructing the contracting subspace, consider B^{-1} and A^{-1} and apply the same argument, adjust δ_0 if necessary. □

Theorem 1.3.10

Let $p \in U$ be a hyperbolic fixed point of f, then there exists $\delta_0 > 0, \varepsilon_0 > 0$, such that $\forall g \in \mathscr{B}^1(f, \delta_0)$, g has a unique fixed point p_g in $B_(p, \varepsilon_0)$ and p_g is a hyperbolic fixed point.

Definition 1.3.11. $A : E \to E$ isomorphism. We say A is **quasi-hyperbolic** if there exists a splitting $E = E_1 \oplus E_2$ invariant under A. And there exists $C \geqslant 1, \mu \in (0, 1)$ such that

$$\frac{|Av_2|}{|Av_1|} \leqslant C\mu^n \frac{|v_2|}{|v_1|}, \quad \forall v_1 \in E^1, v_2 \in E^2, n \geqslant 0.$$

The splitting $E = E_1 \oplus E_2$ is called a **dominated splitting** of A.

Remark 1.3.12 — The dominated splitting is **not** unique.

Remark 1.3.13 — If f admits a "quasi-hyperbolic" fixed point, then the perturbation of f may **not** have fixed point. But Theorem 1.3.9 still holds for a quasi-hyperbolic version.