Higher Rank Abelian Smooth Action with Hyperbolicity
(Spring 2022, Disheng Xu)

Ajorda Jiao

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§1 May 13

For classical dynamical systems, we consider about discrete dynamics or flows. It consists of a space X and a family of maps $X \to X$,

$$\{f^{(n)} : n \in \mathbb{Z}\} \quad \text{or} \quad \{f^t : t \in \mathbb{R}\},$$

satisfying the group conditions.

We will consider a more general settings: **an abelian group action on** X. The settings are

- X a manifold.
- A family of maps $\{f^{\boldsymbol{t}} \in \mathrm{Homeo}(X) : \boldsymbol{t} \in \mathbb{Z}^l\}$, satisfies $f^{\boldsymbol{t}} \circ f^{\boldsymbol{s}} = f^{\boldsymbol{t}+\boldsymbol{s}}$.

Or, we can rewrite the second condition as a group homomorphism

$$\alpha : \mathbb{Z}^l \to \mathrm{Homeo}(X).$$

Example 1.1

A non-invertible example, i.e. α is just a semi-group homomorphism

$$\alpha : \mathbb{N}^2 \to C^0(\mathbb{T}, \mathbb{T}), \quad (m, n) \mapsto (\times 2)^m (\times 3)^n.$$

Furstenberg showed that the orbit of this action is either finite or dense. This is an example of a hyperbolic setting.

Example 1.2

Let R_α be the α-rotation on $\mathbb{T}$. We can consider the action

$$\alpha : (m,n) \to \mathrm{Homeo}(\mathbb{T}), \quad (m,n) \to R_\alpha^m R_\beta^n.$$

This is an example of a non hyperbolic setting.

Remark 1.3 — Fayad-Kanin showed that if $f, g : \mathbb{T} \to \mathbb{T}$, $R(f) = \alpha$, $R(g) = \beta$ and (α, β) satisfies some number-theoretic conditions, then $\exists \varphi \in C^\infty(\mathbb{T}, \mathbb{T})$ such that $\varphi \circ f \circ \varphi^{-1} = R_\alpha$ and $\varphi \circ g \circ \varphi^{-1} = R_\beta$.

For a hyperbolic setting, we consider a baby case. Let $A \in \mathrm{SL}(n, \mathbb{C})$ be a diagonalizable matrix, assume

$$A \sim \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

and $|\lambda_j| \neq 1$ for every j, then we call A to be a **hyperbolic matrix**. Let $\sigma_j = \log |\lambda_j|$, then "hyperbolicity" means $\sigma_j \neq 0$.

Example 1.4

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \curvearrowright \mathbb{T}^2$, a classical Anosov map.

Proposition 1.5 (C^0-Rigidity of Anosov Map)

For an Anosov map $f \in \mathrm{Diff}^\infty(X)$, if another map $g \in \mathrm{Diff}^\infty(X)$ is C^1-closed to f, then $\exists h \in \mathrm{Homeo}(X)$ such that $h \circ g \circ h^{-1} = f$.

Remark 1.6 — In general, the regularity of h cannot be C^1. Because a C^1 conjugacy preserves the derivates of fixed points.

Question 1.7. If we have higher rank action with at least one Anosov element, can we have the similar result?

Example 1.8

A baby case: for f_1, f_2 commutes with each other, consider the action

$$\alpha : \mathbb{Z}^2 \to \mathrm{Diff}^\infty(\mathbb{T}^2), \quad (m,n) \to f_1^m f_2^n.$$

Assume there exists $(m,n) \in \mathbb{Z}^2$ such that $f_1^m f_2^n$ is Anosov. Then we perturb (f_1, f_2) to $(\widetilde{f}_1, \widetilde{f}_2)$ a little bit such that $\widetilde{f}_1 \widetilde{f}_2 = \widetilde{f}_2 \widetilde{f}_1$ still holds. Then there exists h such that $h\widetilde{f}_1 h^{-1} = f_1$ and $h\widetilde{f}_2 h^{-1} = f_2$.

Question 1.9. Can the conjugate h be more regular?

It is easy to construct a counter example such that h could not be C^1. For example, we can regard a $\mathbb{Z}^1$-action as a "degenerated" $\mathbb{Z}^2$-action.

Example 1.10

Let $T_A, T_B : \mathbb{T}^2 \to \mathbb{T}^2$ be two hyperbolic matrices. We consider

$$\alpha : (m,n) \to T^m_{A\times \mathrm{Id}} T^n_{\mathrm{Id}\times B},$$

a $\mathbb{Z}^2$-action on $\mathbb{T}^2$. The conjugate h in general still cannot be C^1.

This counter example is a non degenerated $\mathbb{Z}^2$-action, but is somehow not "genuinely higher rank". So, we need a "**genuinely higher rank assumption**".

Question 1.11. Let $\alpha_0 : \mathbb{Z}^2 \to \mathrm{SL}(d,\mathbb{Z}) \subset \mathrm{Diff}^\infty(\mathbb{T}^d)$ be an action such that there exists (m,n), $\alpha_0(m,n)$ is Anosov (i.e. a hyperbolic matrix). Then for a C^1-perturbation α of α_0, $\alpha : \mathbb{Z}^2 \to \mathrm{Diff}^\infty(\mathbb{T}^d)$, can we show that $\exists h \in \mathrm{Diff}^\infty(\mathbb{T}^d)$ such that $h \circ \alpha \circ h^{-1} = \alpha_0$?

To avoid the rank-one case, we need an additional assumption.
"Totally ergodic ergodic assumption": $\forall (m,n) \neq (0,0)$, $\alpha_0(m,n)$ is ergodic with respect to the Lebesgue measure on $\mathbb{T}^d$.

§2 May 20

Conjecture 2.1 ($\mathbb{Z}^l$ version of Karok-Spatzier's Conjecture)

Let $\alpha : \mathbb{Z}^l \to \mathrm{Diff}^\infty(M)$ be an action such that there exists $a \neq 0 \in \mathbb{Z}^l$, $\alpha(a)$ is Anosov. Then under some suitable "higher rank" assumption (no rank-one factor), α is C^∞ conjugate to an "algebraic-defined" model.

As a contrast, we consider a famous conjecture of Smale and Borel in the case of rank-one.

Conjecture 2.2 (Smale-Borel)

If f is Anosov, then f is C^0-conjugate to a $\mathbb{T}^d$ automorphism.

This conjecture in general is **False**, for Borel have constructed an Anosov diffeomorphism on a nil-manifold. Later, there has been constructed an example of Anosov diffeomorphism on an infra-nil-manifold.

Theorem 2.3 (Franks-Manning)

Suppose $f \in \mathrm{Diff}^1(M)$ is Anosov, where M is a nil-manifold. Then f is C^0-conjugate to an affine map on M.

Corollary 2.4

Assume $f, g \in \mathrm{Diff}^1(M)$ are Anosov, where M is a nil-manifold. Then there exists $h \in \mathrm{Homeo}(M)$ such that $hfh^{-1} = f_0, hgh^{-1} = g_0$ where f_0, g_0 are affine maps on M.

Theorem 2.5 (Hertz-Z.Wang, 2014)

Consider the action $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ which is homotopic to $\alpha_0 : \mathbb{Z}^k \to \mathrm{GL}(d, \mathbb{Z})$, if α is Anosov (in the sense that $\exists a \in \mathbb{Z}^k \setminus \{0\}$, $\alpha(a)$ is Anosov). Assume that $\exists \mathbb{Z}^2 \subseteq \mathbb{Z}^k$ such that $\alpha_0|_{\mathbb{Z}^2}$ is totally ergodic, then α is C^∞-conjugate to an affine action.

Theorem 2.6 (Fisher-Kalinin-Spatzier,2013)

The same result (as Theorem 2.5) holds under a stronger assumption that α has "many" Anosov elements.

Weyl Chamber picture

The Lyapunov exponent for a matrix is $\sigma_i = \log |\lambda_i|$, where λ_i is an eigenvalue of A. Then

$$A \sim \begin{bmatrix} \square & & & \\ & \square & & \\ & & \ddots & \\ & & & \square \end{bmatrix},$$

where each $\square$ is a block with all the same eigenvalues. Then we can get a coarse decomposition of $\mathbb{R}^d$ corresponding to different Lyapunov exponents. Denotes this splitting by

$$\mathbb{R}^d = V_1 \oplus V_2 \oplus \cdots \oplus V_r,$$

then V_i is A-invariant. Moreover, for every B commutes with A, B also preserves each V_i. Hence for a $\mathbb{Z}^k$ action of $\mathrm{GL}(d, \mathbb{Z})$, we can split $\mathbb{R}^d$ into a direct sum of finite many subspaces $\{V_i\}$. Such that, for every $A \in \alpha(\mathbb{Z}^k)$, $A|_{V_i}$ has a constant Lyapunov exponent. Then we can define the **Lyapunov functionals** $\lambda_i : A \mapsto \sigma(A|_{V_i})$, these functionals will induce linear functionals $\mathbb{Z}^k \to \mathbb{R}$.

§3 June 3

Today, we are going to show the idea of the proof of Theorem 2.6. For a references, for example, see here.

Consider actions

$$\alpha_0 : \mathbb{Z}^2 \to \mathrm{Aut}(\mathbb{T}^d), \quad \alpha : \mathbb{Z}^2 \to \mathrm{Diff}^1(\mathbb{T}^d)$$

which are homotopic. Assume that for every $a \neq \mathrm{Id}$, $\alpha_0(a)$ is ergodic. We want to show under some conditions ("many Anosov elements"), α is C^∞-conjugate to α_0.

> **Aim 3.1.** Find a way to state the **"many Anosov elements"** condition formally.

Recall the Lyapunov functionals introduced in last lecture. It corresponds to a Lyapunov (or Oseledec) decomposition $\mathbb{R}^d = \bigoplus_{i=1}^k V^i$, such that for every $v \neq 0 \in V^i$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\alpha_0(na)v\| = \lambda_i(a).$$

Now we introduce the **Weyl chamber picture**. Consider the kernel of each λ_i, which is a line in $\mathbb{Z}^2$. Then these lines divide the plane into several connected components. Each connected component is called a **Weyl chamber**. Let $\mathscr{C}$ be a Weyl chamber, then the signs of $\lambda_i(a)$'s are the same for every $a \in \mathscr{C}$. Hence for each Weyl chamber $\mathscr{C}$, we can use k signs $(\operatorname{sgn}\lambda_1(a), \cdots, \operatorname{sgn}\lambda_k(a)), a \in \mathscr{C}$, to denote it.

Weyl chamber picture for non-linear settings

Recall Oseledec's theorem. For the case of $\mathbb{Z}^1$-action, let

$$f : (X, \mu) \to (X, \mu)$$

be a ergodic maps. Let

$$F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d, \quad (x, v) \mapsto (f(x), F_x v)$$

be a linear cocycle over f, where $F_x \in \mathrm{GL}(d, \mathbb{R})$ for every $x \in X$ and

$$\int_X \|F_x\| \mathrm{d}\mu(x) < \infty.$$

Oseledec's theorem tells us there exists an (μ-a.e.) F-invariant splitting of $X \times \mathbb{R}^d = \bigoplus V_x^i$ and k Lyapunov exponents $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|F_x^n v\| = \lambda_i, \quad \forall v \in V_x^i \setminus \{0\}.$$

For a $\mathbb{Z}^2$-action case, let

$$f_{(m,n)} : (X, \mu) \to (X, \mu), \quad (m, n) \in \mathbb{Z}^2$$

be ergodic maps satisfying group conditions. Let

$$\{F_{(m,n,x)} \in \mathrm{GL}(d, \mathbb{R}) : (m, n, x) \in \mathbb{Z}^2 \times X\}$$

be a family of linear maps satisfying the cocycle condition

$$F_{(m+m', n+n', x)} = F_{(m', n', f_{m,n}(x))} \circ F_{(m,n,x)}.$$

Then there exists linear functions $\lambda_i : \mathbb{Z}^2 \to \mathbb{R}$ and a splitting

$$\mathbb{R}^d = \bigoplus V^i_{(m,n,x)}, \quad (m, n, x) \in \mathbb{Z}^2 \times X.$$

which is (μ-a.e.) cocycle-invariant. Such that for μ- a.e. $x \in X$, for every $(m, n) \in \mathbb{Z}^2$,

$$\lim_{l\to\infty} \frac{1}{l} \log \|F_{(lm,ln,x)} v\| = \lambda_i(m, n), \quad \forall v \in V^i_{(m,n,x)} \setminus \{0\}.$$

In our case, we will consider the derivative cocycle, i.e. $f_{(m,n)} = \alpha(m, n)$ and $F_{(m,n,x)} = \mathrm{D}_x f_{(m,n)}$. Then there is a μ-a.e. defined α-invariant measurable splitting $T\mathbb{T}^d = \bigoplus V^i$, and each V^i corresponds to a Lyapunov functional $\lambda_i : \mathbb{Z}^2 \to \mathbb{R}$. Besides, we can define the Weyl chamber picture similarly.

By Franks-Manning's theorem, α conjugates to an affine action. By passing to a finite index subgroup if necessary, we can assume that it conjugates an action on $\mathrm{Aut}(\mathbb{T}^d)$. That is, there exists a bi-Hölder map h such that $h \circ \alpha_0 \circ h^{-1} = \alpha$ (α_0 is the linear model of α). Then $\nu = h_*(\mathrm{Leb}_{\mathbb{T}^d})$ is an α-invariant ergodic measure on $\mathbb{T}^d$.

Now, we need the **"many Anosov" condition**: for every Weyl chamber $\mathscr{C}$ of linear action α_0, there exists $a \in \mathscr{C}$ such that $\alpha(a)$ is Anosov.

Proposition 3.2

For (α, ν), it has exactly the same Weyl chamber picture as the linear model (α_0, Leb).

Note that $\alpha_0(a)$ is a Anosov iff $a \notin \bigcup \ker \lambda_i$. Moreover, we can show that

Proposition 3.3

$\alpha(a)$ is Anosov iff $\alpha_0(a)$ is Anosov.

Explanation. We have found sufficiently many Anosov elements $a_1, a_2, \cdots, a_s$ (each Weyl chamber of α_0 has at least one). We consider the stable/unstable foliation of each $\alpha(a_i)$.

Fact 3.4. If a_i, a_j in the same Weyl chamber, then they share the same $\mathcal{W}^{u/s}$.

Fact 3.5. Each $\mathcal{W}^{u/s}_{a_i}$ is invariant under $\mathbb{Z}^2$-action α.

Fact 3.6. Any non-trivial intersection $\bigcap_{i \in \mathscr{I}} \mathcal{W}^{u/s}_{a_i}$ is α-invariant.

Let $\mathcal{W}^1, \mathcal{W}^2, \cdots, \mathcal{W}^n$ be (the finest) non-trivial intersections of these stable/unstable manifolds. Let $E^i = T\mathcal{W}^i$. Then $T\mathbb{T}^d = \bigoplus E^i$, which is a splitting possibly coarser than the Oseledec's splitting. Moreover, we can show that

Fact 3.7. Each E^i has the form $\bigoplus_{\lambda_j : \exists c > 0, \lambda_j = c\lambda} V^j$ for a fixed Lyapunov functional λ.

Roughly speaking, this splitting is the finest splitting such that: for every E^i, for every $a \in \mathbb{Z}^2 \setminus \bigcup \ker \lambda_j$, $\alpha(a)$ contracts or expands E^i simultaneously. We call E^i the **coarse Lyapunov distribution** and $\mathcal{W}^i$ the **coarse Lyapunov foliation**.

Fact 3.8. $h : \mathcal{W}^i_{\alpha_0} \to \mathcal{W}^i_{\alpha}$.

Then for every $a \notin \bigcup \ker \lambda_j$, let ν be an $\alpha(a)$-invariant ergodic measure. Note that the Lyapunov exponents of $\alpha_0(a)$ on $\mathcal{W}^i_{\alpha_0}$ are bounded away from zero. Applying Pesin theory and the conjugacy is bi-Hölder, we can show that the Lyapunov exponents of $(\alpha(a), \nu)$ have the same sign and are bounded away from zero on each $\mathcal{W}^i_{\alpha}$. It follows that $\alpha(a)$ is uniformly contracting or expanding along each $\mathcal{W}^i_{\alpha}$, hence $\alpha(a)$ is Anosov. □

Aim 3.9. To show h is C^∞.

Idea Try to apply the philosophy of Journé lemma (see here).

Proposition 3.10 (A regularity result)

Let $u \in L^1(\mathbb{T}^d)$ and $\mathcal{W}^1, \cdots, \mathcal{W}^k$ be strongly absolutely continuous foliations with C^∞ leaves such that $T\mathcal{W}_1 \oplus \cdots \oplus T\mathcal{W}_k = T\mathbb{T}^d$. If for every $\varepsilon > 0$ small enough,

$$|\left\langle D^\beta_{\mathcal{W}^i} u, \varphi \right\rangle| \leqslant C(\alpha, \varepsilon) \|\varphi\|_\varepsilon, \quad \forall i, \forall \varphi \in C^\infty(\mathbb{T}^d),$$

where $\|\cdot\|_\varepsilon$ is the ε-Hölder norm $\|\varphi\|_\varepsilon := \sup \frac{|\varphi(x)-\varphi(y)|}{d(x,y)^\varepsilon} + \|\varphi\|_\infty$. Then u is C^∞.

Aim 3.11. To estimate $|\left\langle D^\alpha_{\mathcal{W}^i}(h - \mathrm{Id}), \varphi \right\rangle|$.

We need a dynamical view to show this fact. More precisely, α-action is exponentially mixing with respect to $h_*(\mathrm{Leb}_{\mathbb{T}^d})$.

§4 June 9

Recall the last lecture:

1. We can define Weyl chamber picture for linear action α_0.
2. Use Weyl chamber picture of α_0, we can define "many" Anosov elements of α.
3. Under the "many Anosov" assumption, we can show that $\alpha_0(m,n)$ is Anosov iff $\alpha(m,n)$ is Anosov. See Proposition 4.3.

For our proof, we assume acting manifold is standard torus $\mathbb{T}^d$. In FKS's original proof, a priori α can act on exotic $\mathbb{T}^d$, but they show that this case cannot happen because of the following two facts.

Fact 4.1. Exotic $\mathbb{T}^d$ $(d > 4)$ is finitely covered by standard $\mathbb{T}^d$.

Fact 4.2. $d = 3, 4$, by a measurable normal form theorem.

Proposition 4.3

Let M be a C^∞ closed manifold and $f_1, f_2 \in \mathrm{Diff}^2(M)$. Let $\mathcal{F}_i$ $(i = 1, 2)$ be an f_i-invariant topological foliation with C^2-leaves. If there exists a bi-Hölder homeomorphism h such that $hf_1h^{-1} = f_2$, and $f_1|_{\mathcal{F}_1}$ is expanding, then $f_2|_{\mathcal{F}_2}$ is expanding.

Remark 4.4 — (f_1, M) is uniformly hyperbolic does **not** imply (f_2, M) is uniformly hyperbolic. There exists an example that (f_1, M) is Anosov but not (f_2, M).

Now, we back to our aim 3.11, where the pair $\langle \cdot, \cdot \rangle$ means integral with respect to a smooth volume. Let $h \circ \alpha_0 \circ h^{-1} = \alpha$, we want to show h is C^∞. Let $\mu = h_*(\mathrm{Leb}_{\mathbb{T}_d})$, then α preserves μ.

Proposition 4.5 (Journé Lemma)

Let $\mathcal{F}_1, \mathcal{F}_2$ be two topological leaves with uniformly C^r-leaves of M such that $T\mathcal{F}_1 \oplus T\mathcal{F}_2 = TM$. Let $1 \leqslant s \leqslant r$ and $h : M \to \mathbb{R}$ be a function which is uniformly C^s along $\mathcal{F}_i$. Then h is a C^{s-} function. Moreover, if s is not an integer or $s = \infty$, then h is C^s.

Proposition 4.6

μ is a smooth measure on $\mathbb{T}^d$.

Proof. Let $\mathrm{J}_f x$ be the Jacobian of f at x. Because h is Hölder, then $\log \mathrm{J}_{\alpha(a)} h(x)$ is a Hölder cocycle over α_0. By Katok-Spatzier's rigidity theorem, a higher rank Hölder cocycle is homologous to a constant. That is, there exists $c : \mathbb{Z}^2 \to \mathbb{R}$ linear such that

$$\log \mathrm{J}_{\alpha(a)} h(x) = c(a) + \Phi(\alpha_0(a)x) - \Phi(x).$$

Then $\log \mathrm{J}_{\alpha(a)} x = c(a) + \Psi(\alpha(a)x) - \Psi(x)$. Hence the normalized measure of $e^{-\Psi}\mathrm{Leb}$ is α-invariant. Denote this measure by μ', take an Anosov element a, then μ' is the unique SRB measure of $\alpha(a)$. Again by the rigidity of cocycle, we know that the equilibrium state of $\log \mathrm{J}^u$ coincides with the equilibrium state of constants which is the measure of maximal entropy. Then the SRB measure μ' is also MME, hence $\mu' = \mu$. Besides, μ is a invariant measure of an smooth Anosov diffeomorphism and absolutely continuous, hence the density of μ is C^∞. $\square$

Proposition 4.7 (Exponential mixing)

α with respect to μ has exponential decay of matrix coefficients. More precisely, for every γ-Hölder functions φ_1, φ_2, we have

$$\begin{aligned}&\left|\langle \alpha(m,n)\varphi_1, \varphi_2\rangle - \int \varphi_1 \mathrm{d}\mu \int \varphi_2 \mathrm{d}\mu\right| \\ &\leqslant C(\alpha,\gamma)e^{-C(\alpha,\gamma)(m+n)}(\|\varphi_1\|_\gamma \|\varphi_2\|_2 + \|\varphi_1\|_2 \|\varphi_2\|_\gamma).\end{aligned}$$

§5 June 22

Recall the last lecture. We give some comments to the proof of proposition 4.6.

In the case of rank-one, recall the Livsic theorem. Let $X \to X$ be a Anosov diffeomorphism and $\varphi : X \to \mathbb{C}$ be a Hölder continuous function. We want to know when there exists $\psi : X \to \mathbb{C}$ such that

$$\varphi(x) = \psi(f(x)) - \psi(x) + c.$$

The Livsic theorem says that, such ψ exists if and only if for every k-periodic point p,

$$\frac{1}{k}\sum_{l=1}^{k} \varphi(f^l(p)) = c.$$

In summary, if we want to show φ is cohomologous to a constant, it suffices to check it at all periodic points.

But on a higher rank case, Katok-Spatzier shows that any Hölder continuous cocycle is cohomologous to a constant if the original action is "higher rank" with a uniform hyperbolicity.

If we want to upgrade the regularity of solution ψ, the result was demonstrated by Livsic in 1970s. Later, Amie Wilkinson showed a similar result for a partially hyperbolic case in 2010s.

Now back to our goal: to proof proposition 4.7. For a torus case, the result can be shown by Fourier analysis on torus. For a more general case on nil-manifolds, it was shown in *Exponential Mixing of Nilmanifold Automorphisms* by Gorodnik-Spatzier in 2015.

Idea To use trigonometric polynomials to approximate general Hölder functions.

Firstly, we consider the case of both φ_1, φ_2 are trigonometric polynomials. We can show that for (m,n) sufficiently large, we have

$$\int \alpha_0(m,n)\varphi_1 \cdot \varphi_2 \mathrm{dLeb} = \int \varphi_1 \mathrm{dLeb} \int \varphi_2 \mathrm{dLeb}.$$

Lemma 5.1

For α_0, there exists $r_0 > 1$ such that: for every $1 < r < r_0$ and $\forall l$ large enough, consider the cube

$$H_l := \left\{ z \in \mathbb{Z}^d : \|z\|_\infty \leqslant r^l \right\}$$

and $a \in \mathbb{Z}^2$ such that $\|a\| > l$. We have $\tau(a)H_l \cap H_l = \{0\}$, where $\tau(a)$ is the induced action of $\alpha_0(a)$ on $\pi_1(\mathbb{T}^d)$.

Then we will approximate the Hölder functions by applying Fejér kernel

$$K_l(t) = \sum_{i=-l}^{l} \left(1 - \frac{|j|}{l+1}\right) e^{2\pi i j t}$$

on $\mathbb{T}^1$, and

$$F_l(t_1, \cdots, t_d) = K_l(t_1) \cdots K_l(t_d)$$

on $\mathbb{T}^d$. For a θ-Hölder function φ on $\mathbb{T}^d$, we have an estimate of the convergence speed as

$$\|F_l * \varphi - \varphi\|_\infty \leqslant C(\theta) \|\varphi\|_\theta m^{-\theta}.$$

Then we can show an exponentially mixing for α_0 for Hölder functions. By the conjugacy is Hölder regular, it also follows that α is exponentially mixing.

Now we begin to show the conjugate is C^∞. Recall that we assume α and α_0 preserves a common fixed point at $0 \in \mathbb{T}^d$ and h is bi-Hölder regular such that $h \circ \alpha \circ h^{-1} = \alpha_0$. Then we can lifts both $\alpha(a), \alpha_0(a), h$ to $\mathbb{R}^n$, we abbreviate these maps on $\mathbb{R}^d$ to $\widetilde{a}, A, \widetilde{h}$, respectively. We can assume that $\widetilde{h}(0) = 0$ and it holds

$$\widetilde{h} \circ \widetilde{a} = A \circ \widetilde{h}, \quad \forall a \in \mathbb{Z}^2.$$

Let $\widetilde{h} = \mathrm{Id} + \phi$, this identity can be written as

$$(\mathrm{Id} + \phi) \circ \widetilde{a} = A \circ (\mathrm{Id} + \phi),$$

or,

$$\phi(x) = A^{-1}(\widetilde{a}(x) - A(x)) + A^{-1}(\phi(\widetilde{a}(x))) = Q(x) + A^{-1}(\phi(\widetilde{a}(x))).$$

For every coarse Lyapunov foliation $\mathcal{V}$ of α, let V be the corresponding Lyapunov foliation (indeed a linear subspace) of α_0. Let $\phi_V : \pi_V \circ \phi$, where π_V is the projection of $\mathbb{R}^d$ to V along an α_0 invariant direction. Then

$$\phi_V(x) = Q_V(x) + A_V^{-1}(\phi_V(\widetilde{a}(x))),$$

where Q_V is a smooth function. All we want is to find a well-chosen $a \in \mathbb{Z}^2$ to upgrade the regularity of ϕ_V.

Idea To show that the derivative of ϕ_V along each $\mathcal{V}'$ is a distribution of Hölder functions.

A priori, we don't know ϕ_V is differentiable. Hence we consider it in $\mathcal{D}$, the space of distributions of smooth functions. Then the generalized derivative of ϕ_V is meaningful. And we will do some estimate to show that each derivative is indeed a distribution of Hölder functions.

A key method is write ϕ_V as a summation of series. By the former identity, we have

$$\phi_V(x) = \sum_{m=0}^{N-1} A_V^{-m}(\phi_V(\widetilde{a}^m(x))) + A_V^{-N}(\phi_V(\widetilde{a}^N(x))).$$

We can choose a such that A_V expands slowly, then $\left\| A_V^{-N} \right\|$ grows at most polynomial in N. Hence by the exponentially mixing,

$$\left\langle A_V^{-N}(\phi_V(\widetilde{a}^N(x))), f \right\rangle \to 0$$

for every Hölder function f with zero mean. Then

$$\phi_V = \sum_{m=0}^{\infty} A_V^{-m}(\phi_V(\widetilde{a}^m(x)))$$

in $\mathcal{D}_0$, the space of distributions on Hölder functions. In particular, the convergence is in $\mathcal{D}$.

> **Remark 5.2** — Something subtle here is that, although the series is convergence in $\mathcal{D}_0$, but we cannot do a derivative on $\mathcal{D}_0$. Hence we need to consider these series in $\mathcal{D}$, do the derivatives and estimate it.

For another coarse Lyapunov foliation $\mathcal{V}'$, let $(\phi_V)^{k,\mathcal{V}'}$ be the k-th generalized derivative of ϕ_V along $\mathcal{V}'$. For a smooth test function f, by the exponentially mixing, we can bound

$$\left\langle (A_V^{-m}(\phi_V(\widetilde{a}^m)))^{k,\mathcal{V}'}, f \right\rangle = \left\langle A_V^{-m}(\phi_V(\widetilde{a}^m)), f^{k,\mathcal{V}'} \right\rangle$$

by some $Cr^{-m} \|f\|_\theta$, where $r > 1$. In which we can always choose some a well to help us do the estimation. (In practice, we need to approximate a Hölder function by smooth functions. This need a little more delicate estimate.)

> **Remark 5.3** — Our notations is a little bit different with the original paper, in which ϕ and h are exchanged.

At the end, by the regularity result (proposition 3.10), the conclusion follows.