

# **Homogeneous Dynamical System**

## **(Spring 2022, Runlin Zhang)**

Ajorda Jiao

# Contents

<b>1</b>	<b>Introduction of Homogeneous Dynamics</b>	<b>3</b>
1.1	Motivations and applications . . . . .	3
1.2	Measure rigidity . . . . .	5
<b>2</b>	<b>Oppenheim Conjecture</b>	<b>7</b>
2.1	22.2.25: The unipotent flow is minimal on compact space . . . . .	7
2.2	22.3.4: Weak Oppenheim conjecture I . . . . .	9
2.3	22.3.8: Weak Oppenheim conjecture II . . . . .	11
2.4	22.3.11: Completion of some gaps . . . . .	13

# 1 Introduction of Homogeneous Dynamics

## §1.1 Motivations and applications

### §1.1.i Horocycles on constant negative curvature surfaces

Equip  $\mathbb{H}^2 := \{x + iy \in \mathbb{C}, y > 0\}$  with the metric  $\frac{dx^2 + dy^2}{y^2}$ . Let  $\Gamma \leq \text{Isom}(\mathbb{H}^2)$  be a discrete (torsion free) subgroup such that  $\Gamma \backslash \mathbb{H}^2$  is compact (such a subgroup is called a uniform lattice). Then  $\Gamma \backslash \mathbb{H}^2$  is a compact surface of constant negative curvature.

Let  $\pi : \mathbb{H}^2 \rightarrow \Gamma \backslash \mathbb{H}^2 = M$  be the quotient map. Consider a horocycle  $\mathcal{H} \subset \mathbb{H}^2$ .

#### Theorem 1.1.1

For every  $\mathcal{H}$ ,  $\pi(\mathcal{H})$  is dense in  $M$ .

#### Theorem 1.1.2

If  $M = \Gamma \backslash \mathbb{H}^2$  ( $\Gamma \leq \text{Isom}(\mathbb{H}^2)$  still discrete) is just of finite volume, then:

1.  $\pi(\mathcal{H})$  is either closed or dense in  $M$ .
2. Consider a sequence of closed horocycles  $\pi(\mathcal{H}_i)$  with length  $\rightarrow \infty$ , then  $\pi(\mathcal{H}_i)$  becomes dense in  $\Gamma \backslash \mathbb{H}^2$ .

### §1.1.ii Isometric immersion of hyperbolic spaces

Let  $\mathbb{H}^3$  be the three dimensional hyperbolic space  $\{(x + iy, z) \in \mathbb{C} \times \mathbb{R}, z > 0\}$  equipped with the metric  $\frac{1}{z^2}(dx^2 + dy^2 + dz^2)$ . Let  $\Gamma \leq \text{Isom}(\mathbb{H}^3)$  be a discrete (torsion free) subgroup, such that  $\Gamma \backslash \mathbb{H}^3$  is compact (finite volume suffices). Consider an isometric embedding  $\iota : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ . The image of  $\iota$  can be explicitly described.

#### Theorem 1.1.3

The following holds:

1.  $\pi(\iota(\mathbb{H}^2))$  is either closed or dense in  $M$ ;
2. Given an infinite sequence of distinct closed  $\pi(\iota_i(\mathbb{H}^2))$ , then  $\lim_i \pi(\iota_i(\mathbb{H}^2))$  is dense in  $M$ .

### §1.1.iii Oppenheim conjecture/Margulis theorem

Let  $Q$  be a real quadratic form in 3 variables, indefinite and non-degenerated. Consider  $Q$  as a function  $\mathbb{R}^3 \rightarrow \mathbb{R}$ .

**Theorem 1.1.4**

Assume  $Q$  is NOT proportional to a quadratic form with  $\mathbb{Q}$ -coefficients. Then  $Q(\mathbb{Z}^3)$  is dense in  $\mathbb{R}$ .

**Remark 1.1.5** — It is true for  $k \geq 3$  variables. But it is false for  $Q$  only has two variables.

**Theorem 1.1.6** (Eskin-Margulis-Mozes)

Further assume  $Q$  has at least signature  $(3, 1)$ , then for every  $a < b \in \mathbb{R}$ ,

$$\begin{aligned} & \# \{v \in \mathbb{Z}^4 : \|v\| \leq T, Q(v) \in (a, b)\} \\ & \sim \text{Vol} \{v \in \mathbb{R}^4 : \|v\| \leq T, Q(v) \in (a, b)\} \\ & \sim C_Q(b - a)T^2. \end{aligned}$$

**§1.1.iv Littlewood conjecture**

For  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , we have  $\inf \{n \langle n\alpha \rangle : n \in \mathbb{Z}_+\} < 1$ .

**Fact 1.1.7.** There exists  $\alpha$  such that  $\inf \{n \langle n\alpha \rangle : n \in \mathbb{Z}_+\} > 0$ .

**Conjecture 1.1.8**

For all  $(\alpha, \beta) \in \mathbb{R}^2$ ,  $\alpha, \beta \notin \mathbb{Q}$ ,

$$\inf \{n \langle n\alpha \rangle \langle n\beta \rangle : n \in \mathbb{Z}_+\} = 0.$$

**Remark 1.1.9** — The conjecture is reasonable in some sense:

1.  $\forall \delta > 0$ ,  $\inf \{n^{1-\delta} \langle n\alpha \rangle \langle n\beta \rangle : n \in \mathbb{Z}_+\} = 0$ .
2.  $\forall \delta > 0$ ,  $\exists (\alpha, \beta)$ , such that  $\inf \{n^{1+\delta} \langle n\alpha \rangle \langle n\beta \rangle : n \in \mathbb{Z}_+\} > 0$ .

**§1.1.v Quantum unique ergodicity**

Consider  $M^2 = \Gamma \setminus \mathbb{H}^2$  is a closed hyperbolic surface. Consider  $\Delta := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$  acts on  $C^\infty(M)$ . Then:

1.  $\exists \lambda_0 = 0 < \lambda_1 < \dots, \lambda_i \rightarrow \infty$ ,
2. Let  $E_{\lambda_i} := \{f \in C^\infty(M) : \Delta f = \lambda_i f\}$ , then  $E_{\lambda_i} \neq \emptyset$  and  $\dim E_{\lambda_i} < \infty$ .

For each  $i$ , choose  $f_i \in E_{\lambda_i}$ . Consider  $(|f_i|^2 \text{Vol})_i$  a sequence of measure on  $M$ , normalized to be probability measure.

**Conjecture 1.1.10**

$|f_i|^2 \text{Vol}$  tends to  $\frac{1}{\text{Vol}(M)} \text{Vol}$  in the weak\* topology.

Further assume  $\Gamma$  is a “congruence subgroup”. In this situation, there is an additional supply of operators, called Hecke operators, that commute with the Laplacian. Let  $f_i \in E_{\lambda_i}$  which is also an eigenfunction of Hecke operator.

**Theorem 1.1.11 (Lindenstrauss-Bourgain)**

In such settings, the conjecture holds.

**§1.2 Measure rigidity****§1.2.i Unipotent rigidity**

Let  $G = \text{SL}(2, \mathbb{R})$ ,  $\Gamma \leq G$  a discrete subgroup.  $G$  has a right  $G$ -invariant Riemannian metric. It induces a volume measure  $\text{Vol}$  on  $G/\Gamma$ .

**Fact 1.2.1.**  $\text{Vol}$  is left  $G$ -invariant.

Let  $U = \left\{ u_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R} \right\}$ .

**Theorem 1.2.2**

If  $G/\Gamma$  is compact, then  $\text{Vol}$  is the unique  $U$ -invariant finite measure (up to a scalar).

**Theorem 1.2.3**

If  $\text{Vol}$  is finite (normalized to be probability measure). Then every  $U$ -invariant probability measure is a “convex combination” of:

- (i) the  $U$ -invariant measure supported on a closed (and compact) orbit.
- (ii)  $\text{Vol}$ .

**Theorem 1.2.4 (Measure Rigidity Theorem)**

Let  $G$  be a (connected) Lie group, let  $U = \{u_s : s \in \mathbb{R}\}$  be an  $\text{Ad}$ -unipotent one-parameter subgroup of  $G$ . Let  $\Gamma \leq G$  be a closed subgroup. Then every  $U$ -invariant ergodic probability measure on  $G/\Gamma$  is “homogeneous”.

**Theorem 1.2.5** (Equidistribution and Topological Rigidity)

Assume  $\Gamma$  is a lattice in  $G$ , then for any  $x \in G/\Gamma$  :

1. There exists a probability “homogeneous” measure  $\mu$  such that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(u_s x) ds = \int f(x) d\mu(x), \quad \forall f \in C_c(G/\Gamma).$$

2. The closure of the orbit  $Ux$  is “homogeneous”, which means  $\exists H \leq G$  closed such that  $\overline{Ux} = Hx$ .

**§1.2.ii Higher rank diagonalizable flow**

Let  $G = \mathrm{SL}(2, \mathbb{R})$ ,  $\Gamma \leq G$  lattice. Consider  $A = \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \right\}$  acts on  $G/\Gamma$ .

**Conjecture 1.2.6**

$G = \mathrm{SL}(3, \mathbb{R})$ ,  $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ . Consider

$$\mathbb{R}^2 \cong A := \left\{ \begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{bmatrix} : t_1 + t_2 + t_3 = 0 \right\}$$

acts on  $G/\Gamma$ .

1. Every  $A$ -ergodic probability measure is homogeneous.
2. Every bounded  $A$ -orbit is closed.

**Theorem 1.2.7**

$A, G, \Gamma$  as in the conjecture, then:

1. Every  $A$ -invariant ergodic probability measure with “positive entropy” is homogeneous.
2. The Hausdorff dimension of  $\{x \in G/\Gamma : Ax \text{ is bounded}\}$  is equal to 2.

# 2 Oppenheim Conjecture

## §2.1 22.2.25: The unipotent flow is minimal on compact space

- Let  $G = \mathrm{SL}(2, \mathbb{R})$ , let  $\Gamma \leq G$  be a discrete subgroup.
- Assume for today  $X = G/\Gamma$  : is compact.
- $U^+ = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \geq 0 \right\}$ .

### Theorem 2.1.1

For all  $x \in X$ ,  $U^+x$  is dense in  $X$ .

**Definition 2.1.2.** Let  $A$  be a semigroup acting on a topological space  $Z$  :

1. We say the action is **minimal** if every  $A$ -orbit is dense in  $Z$ .
2. We say the subset  $W \subset Z$  is **A-minimal** if  $W$  is  $A$ -stable, closed and  $A \cap W$  is minimal.

### Theorem 2.1.3

Let  $Y$  be a  $U^+$ -minimal subset of  $X$ . Then  $Y = \emptyset$  or  $Y = X$ .

**Claim 2.1.4.** Theorem 2.1.3 implies Theorem 2.1.1

*Proof.* Zorn's lemma + compactness of  $X$ . We can always find a nonempty  $U^+$ -minimal subset of  $X$ , which must be  $X$ .  $\square$

**Fact 2.1.5.**  $\mathrm{SL}(2, \mathbb{R})$  admits a right-invariant metric compatible with its topology.

Now we fix such a metric  $d : G \times G \rightarrow \mathbb{R}$ . It induces a “quotient” metric  $d_X : X \times X \rightarrow \mathbb{R}$  by

$$d_X(g\Gamma, h\Gamma) := \inf_{\gamma_1, \gamma_2 \in \Gamma} d(g\gamma_1, h\gamma_2) = \inf_{\gamma \in \Gamma} d(g\gamma, h).$$

For  $x \in X = G/\Gamma$ , define the **injective radius** of  $x$  as

$$\mathrm{InjRad}(x) := \sup \{ \delta > 0 : \text{such that } g \mapsto g.x \text{ is injective on } g \in B(\mathrm{Id}, \delta) \subseteq G \}.$$

**Exercise 2.1.6.** For all  $x \in X$ ,  $\mathrm{InjRad}(x) > 0$ .

*Proof.* By  $\Gamma$  is discrete.  $\square$

**Exercise 2.1.7.**  $\exists r_X > 0$ , such that  $\forall x \in X$ ,  $\mathrm{InjRad}(x) > r_X$ .

*Proof.* By the compactness of  $X$ . Because  $\Gamma$  is cocompact, there exists  $C \subseteq G$  compact, then  $\forall x \in X$ ,  $\exists g_x \in C$ ,  $x = g_x\Gamma$ .  $\square$

**Lemma 2.1.8**

$U^+ \curvearrowright X = G/\Gamma$  has no closed(compact) orbit.

*Proof.* Say: we have a compact orbit  $\{u_s.x : s \geq 0\}$ . Define  $s_0 = \inf \{s > 0 : u_s.x = x\}$ , then

$$\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} u_{s_0} \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x = \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x.$$

This shows that  $\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x$  is invariant under  $\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} u_{s_0} \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} = u_{e^{-2t}s_0}$ .  $\square$

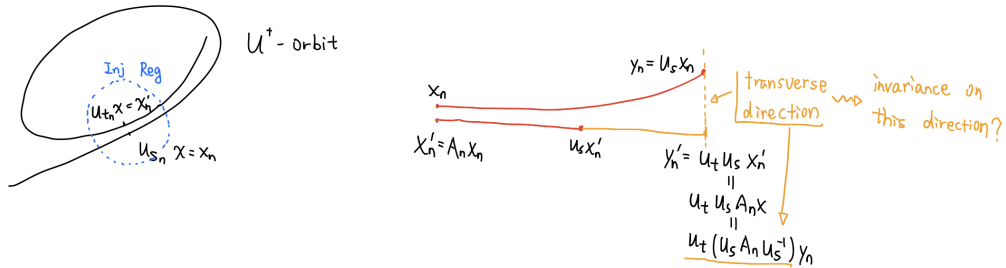
**Corollary 2.1.9**

$\Gamma$  contains no nontrivial unipotent matrix.

**Corollary 2.1.10**

The following holds:

1.  $\forall x \in X$ , the map  $s \mapsto u_s.x$  is injective.
2.  $\forall x, \exists s_n, t_n \rightarrow \infty$  with  $|s_n - t_n| \rightarrow \infty$ , such that  $d_X(u_{s_n}.x, u_{t_n}.x) \rightarrow 0$ .



*Proof of Theorem 2.1.3.* By corollary 2.1.10, we can find  $A_n \in G \setminus U$  and  $x_n, x'_n \in U^+x \subseteq X$  with  $d_X(x_n, x'_n) \rightarrow 0$  and  $x'_n = A_n.x_n$ . Moreover, we can choose  $A_n \rightarrow \text{Id}$  (use the fact that injective radius is larger than  $r_X$ ).

Say  $A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ , where  $a_n, d_n \rightarrow 1, b_n, c_n \rightarrow 0$ . We have

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} A_n \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_n + sc_n & b_n - sa_n + sd_n - s^2 c_n \\ c_n & d_n - sc_n \end{bmatrix}.$$

We want to choose  $t = t_s$  such that

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_n + sc_n & b_n - sa_n + sd_n - s^2 c_n \\ c_n & d_n - sc_n \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Take  $t = t_s = \frac{-(b_n - sa_n + sd_n - s^2 c_n)}{d_n - sc_n}$ . Then

$$u_t u_s A_n u_s^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{c_n}{d_n - sc_n} & d_n - sc_n \end{bmatrix}.$$



Fix  $\delta > 0$ , choose  $s = s_{\delta,n} \geq 0$  such that  $d_n - sc_n = 1 - \delta$  or  $1 + \delta$ . Let  $y_n = u_s \cdot x_n$ ,  $y'_n = u_t u_s A_n \cdot x_n = \begin{bmatrix} (1+\delta)^{-1} & 0 \\ c_n & (1+\delta) \end{bmatrix} \cdot y_n$ . By passing to a subsequence, assume that  $y_n \rightarrow y_\infty$  and  $y'_n \rightarrow y'_\infty$  both in  $Y$ , where  $y'_\infty = \begin{bmatrix} (1+\delta)^{-1} & 0 \\ c_\infty & (1+\delta) \end{bmatrix} \cdot y_\infty$ . Then

$$Y = \overline{U^+ y'_\infty} = \begin{bmatrix} (1+\delta)^{-1} & 0 \\ c_\infty & (1+\delta) \end{bmatrix} \overline{U^+ y_\infty} = \begin{bmatrix} (1+\delta)^{-1} & 0 \\ c_\infty & (1+\delta) \end{bmatrix} Y.$$

Let  $B^+ = \{a_t u_s : s \in \mathbb{R}_+, t \in \mathbb{R}\}$ , where  $a_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ , then  $Y$  is  $B^+$  invariant. The theorem is immediate by the following lemma.  $\square$

### Lemma 2.1.11

We have:

1.  $B \curvearrowright \mathrm{SL}(2, \mathbb{R})/\Gamma$  is minimal.
2.  $B^+ \curvearrowright \mathrm{SL}(2, \mathbb{R})/\Gamma$  is minimal.

## §2.2 22.3.4: Weak Oppenheim conjecture I

### Theorem 2.2.1 (Weak Version of Oppenheim Conjecture)

Let  $Q$  be a nondegenerate, indefinite, quadratic form with real coefficients in 3 variables. Assume  $Q$  is not proportional to a quadratic form with  $\mathbb{Q}$ -coefficients. Then  $\overline{Q(\mathbb{Z}^3 \setminus (0))}$  contains 0.

### Example 2.2.2

$Q(x, y, z) = xy - \sqrt{2}z^2$ , the statement is trivial for  $Q$  because  $Q(1, 0, 0) = 0$ .

**Definition 2.2.3.** Define the special orthogonal group of  $Q$  as

$$\mathrm{SO}(Q, \mathbb{R}) := \{g \in \mathrm{SL}(3, \mathbb{R}), Q \circ g = Q\}, \quad \mathrm{SO}(Q, \mathbb{Z}) := \{g \in \mathrm{SL}(3, \mathbb{Z}), Q \circ g = Q\}.$$

**Definition 2.2.4.** A subgroup  $\Lambda \leq \mathbb{R}^N$  is a **lattice** if  $\Gamma$  is discrete and cocompact.

**Definition 2.2.5.**  $\Lambda \leq \mathbb{R}^n$  is a **unimodular lattice** if  $\Lambda$  is a lattice and  $\mathrm{Vol}(\mathbb{R}^N/\Lambda) = 1$ .

**Definition 2.2.6.** Let  $X_N := \{\text{unimodular lattice in } \mathbb{R}^N\}$  equipped with the **Chabauty topology**.

**Remark 2.2.7 —** A sequence  $\{\Lambda_N\} \subseteq X_N$  converges to  $\Lambda_\infty \in X_N$  iff we can find a basis  $\{v_1^n, v_2^n, \dots, v_N^n\}$  of  $\Lambda_n$  such that for every  $i = 1, 2, \dots, N$ ,  $v_i^n \rightarrow v_i^\infty \in \mathbb{R}^N$ , and  $\Lambda_\infty = \mathbb{Z}v_1^\infty \oplus \mathbb{Z}v_2^\infty \oplus \dots \oplus \mathbb{Z}v_N^\infty$ .

**Remark 2.2.8** —  $\mathrm{SL}(N, \mathbb{R})$  naturally acts on  $X_N$ .

**Lemma 2.2.9**

The map  $g \mapsto g \cdot \mathbb{Z}^N$ , induces a homeomorphism  $\mathrm{SL}(N, \mathbb{R})/\mathrm{SL}(N, \mathbb{Z}) \cong X_N$ .

**Definition 2.2.10.** For a discrete subgroup  $\Lambda \leq \mathbb{R}^N$ , define  $\delta(\Lambda) := \inf \{\|v\| : v \neq 0 \in \Lambda\}$ .

**Fact 2.2.11.**  $\delta : X_N \rightarrow \mathbb{R}_{>0}$  is continuous.

**Lemma 2.2.12 (Mahler's Criterion)**

$\delta : X_N \rightarrow \mathbb{R}_{>0}$  is proper, i.e.  $(x_n) \subseteq X_N$  diverges iff  $\delta(x_n) \rightarrow 0$ .

**Remark 2.2.13** —  $(x_n)$  diverges iff for every compact  $K \subseteq X_N$ ,  $(x_n)$  will eventually out of  $K$ . This is equivalent to  $(x_n)$  has no convergent subsequence.

*Proof. The “if” part:* If  $\delta(x_n) \rightarrow 0$ , we need to show  $(x_n)$  is divergent. This is immediate by  $(x_n)$  has a convergence subsequence.

*The “only if” part:* By passing to a subsequence,  $\exists \varepsilon > 0$  such that  $\delta(x_n) \geq \varepsilon > 0$ . The statement follows by the following claim.  $\square$

**Claim 2.2.14.**  $\exists C = C(N, \varepsilon) > 0$ , such that every  $\Lambda$  with  $\delta(\Lambda) > \varepsilon$  has a basis  $(v_1, v_2, \dots, v_N)$  with  $\|v_i\| \leq C(N, \varepsilon), i = 1, 2, \dots, N$ .

*Proof.* Consider the projection  $p : \mathbb{R}^N \rightarrow \mathbb{R}^N/\Lambda$ . Then  $p$  is not injective restricted to  $[-1, 1]^N$ . There will be  $v \neq w \in [-1, 1]^N$  such that  $v - w \in \Lambda$  and  $\|v - w\| \leq 2\sqrt{N}$ . Now we pick  $w_1 \in \Lambda$  that minimize  $\{\|v\| : v \neq 0 \in \Lambda\}$ , then  $\|w_1\| \leq 2\sqrt{N}$ .

Let  $\pi_1^\perp : \mathbb{R}^N \rightarrow w_1^\perp$  be the orthogonal projection. Consider  $\pi_1^\perp(\Lambda) \leq w_1^\perp \cong \mathbb{R}^{N-1}$ . Then:

1.  $\pi_1^\perp(\Lambda)$  is discrete and is a lattice in  $w_1^\perp$ .
2.  $1 = \|\Lambda\| = \|w_1\| \|\pi_1^\perp(\Lambda)\| \geq \varepsilon \|\pi_1^\perp(\Lambda)\|$ .

Then  $\|\pi_1^\perp(\Lambda)\| \leq \varepsilon^{-1}$  and  $\delta(\pi_1^\perp(\Lambda))$  is controlled by a function of  $\varepsilon$ . We can reduce to the situation of dimensional  $N - 1$ .  $\square$

**Lemma 2.2.15**

Let  $Q$  be a nondegenerate quadratic form in  $N$  variables with real coefficients, then the followings are equivalent:

- (i)  $\overline{Q(\mathbb{Z}^N \setminus \{0\})}$  contains 0.
- (ii)  $\mathrm{SO}(Q, \mathbb{R}) \cdot \mathbb{Z}^N$  is unbounded in  $X_N$ .

*Proof. (ii)  $\implies$  (i):* By assumption,  $\exists g_n \in \mathrm{SO}(Q, \mathbb{R})$  such that  $(g_n \cdot \mathbb{Z}^N)_n$  diverges in  $X_N$ . By Mahler's Criterion 2.2.12,  $\delta(g_n \cdot \mathbb{Z}^N) \rightarrow 0$ , hence  $\exists v_n \neq 0 \in \mathbb{Z}^N$  such that  $g_n v_n \rightarrow 0$ .  $\square$

Consider  $N = 3$ ,  $Q$  indefinite.

**Fact 2.2.16.**  $\exists g_Q \in \mathrm{SL}(3, \mathbb{R})$  such that  $Q = \lambda(Q_0 \circ g_Q)$ , where  $\lambda \in \mathbb{R} \setminus \{0\}$  and  $Q_0 = 2xz - y^2$ .

Then  $\mathrm{SO}(Q, \mathbb{R}) = g_Q^{-1} \mathrm{SO}_{Q_0}(\mathbb{R}) g_Q$ , hence  $\mathrm{SO}(Q, \mathbb{R}) \cdot \mathbb{Z}^3$  is unbounded iff  $\mathrm{SO}(Q_0, \mathbb{Z}) g_Q \cdot \mathbb{Z}^3$  is unbounded.

### Theorem 2.2.17

Every orbit of  $\mathrm{SO}(Q_0, \mathbb{R})$  on  $X_3 \cong \mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$  either unbounded or is closed.

*Proof of Theorem 2.2.1 assuming Theorem 2.2.17.* Otherwise,  $\mathrm{SO}(Q, \mathbb{R}) \cdot \mathbb{Z}^3$  is compact. Then  $\mathrm{SO}(Q, \mathbb{Z}) := \mathrm{SO}(Q, \mathbb{R}) \cap \mathrm{SL}(3, \mathbb{Z})$  is cocompact in  $\mathrm{SO}(Q, \mathbb{R})$ . We want to show that  $Q$  is proportional to a  $\mathbb{Q}$ -coefficient quadratic form. Otherwise,  $\exists \alpha, \beta$  coefficients of  $Q$  such that  $\frac{\alpha}{\beta} \notin \mathbb{Q}$ . Then  $\exists \sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$  such that  $\sigma(Q)$  is not proportional to  $Q$ .

**Step 1:**  $\mathrm{SO}(Q, \mathbb{R})^0 = \mathrm{SO}(\sigma(Q), \mathbb{R})^0 = \sigma(\mathrm{SO}(Q, \mathbb{R}))^0$ .

$\mathrm{SO}(Q, \mathbb{R})^0 \supseteq \mathrm{SO}(Q, \mathbb{Z}) \cap \mathrm{SO}(Q, \mathbb{R})^0 = \Gamma \subseteq \sigma(\mathrm{SO}(Q, \mathbb{R}))^0$ . Consider

$$\mathrm{SL}(3, \mathbb{R}) \curvearrowright \mathrm{Sym} := \{\mathbb{R} - \text{Symmetric matrices}\}, \quad g.M = g M g^t.$$

Let  $\psi : \mathrm{SO}(Q, \mathbb{R}) \rightarrow \mathrm{Sym}, g \mapsto g.\sigma(Q)$ , then  $\psi$  factors through  $\mathrm{SO}(Q, \mathbb{R})/\mathrm{SO}(Q, \mathbb{Z}) \rightarrow \mathrm{Sym}$ . Hence, the image of  $\psi$  is compact. The following two facts shows that  $\mathrm{SO}(Q, \mathbb{R})^0$  fixes  $\sigma(Q)$  and the statement follows immediately:

1.  $\mathrm{SO}(Q, \mathbb{R})^0$  is generated by one-parameter unipotent flows.
2. For every unipotent flow  $\{u_t\}$  and  $M \in \mathrm{Sym}$ , either  $\{u_t.M\}$  is unbounded or  $M$  is fixed by  $\{u_t\}$ .

**Step 2:** A direct compute shows that  $\mathrm{SO}(Q, \mathbb{R})^0 = \mathrm{SO}(\sigma(Q), \mathbb{R})^0$  implies  $\sigma(Q)$  is proportional to  $Q$ .  $\square$

## §2.3 22.3.8: Weak Oppenheim conjecture II

### Theorem 2.3.1

An orbit of  $H = \mathrm{SO}(Q_0, \mathbb{R})$  on  $X_3$  is either:

- (i) unbounded.
- (ii) compact.
- (iii) its closure contains a  $\{v_s\}_{s \geq 0}$ -orbit or a  $\{v_s\}_{s \leq 0}$ -orbit, where  $v_s = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**Fact 2.3.2.** Theorem 2.3.1  $\implies$  Theorem 2.2.17.

Now, we calculate  $H$ . Let  $\mathfrak{h}$  be the Lie algebra of  $H$ , then

$$\mathfrak{h} = \left\{ X \in \mathfrak{gl}(3, \mathbb{R}) : M_0 X = -X^t M_0, M_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \right\}.$$

After some tough work, we get

$$\mathfrak{h} = \left\{ \begin{bmatrix} x_1 & x_2 & 0 \\ x_4 & 0 & x_2 \\ 0 & x_4 & -x_1 \end{bmatrix} : x_1, x_2, x_4 \in \mathbb{R} \right\}.$$

In particular,

$$u_t := \exp \left( t \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{bmatrix}, a_t = \exp \left( t \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix} \right) = \begin{bmatrix} e^t & & \\ & 1 & \\ & & e^{-t} \end{bmatrix} \in H.$$

*Proof of Theorem 2.3.1.* Take  $x_0 \in X_3$  such that  $Y_0 = \overline{H.x_0} \neq H.x_0$  and  $H.x_0$  is bounded. Let  $\Omega := \{y \in Y_0 : Hy \text{ is open in } Y_0\}$ . We need the following lemma.

**Lemma 2.3.3**

$\Omega \neq Y_0$ .

*Proof.* Otherwise, every orbit of  $H$  in  $Y_0$  is closed, in particular  $Hx_0$  is closed. Contradiction.  $\square$

*Continued proof of Theorem 2.3.1.* Let  $Y_1$  be a nonempty  $U$ -minimal nonempty subset of  $Y_0 \setminus \Omega$ , where  $U = \{u_t\}$ . If  $y \in Y_0 \setminus \Omega$ , then  $H.y$  is NOT open in  $Y_0$ , hence  $\exists y_n \in Y_0$  such that  $y_n \notin H.y, y_n \rightarrow y$ .

**Case 1:**  $Y_1$  is closed  $U$ -orbit. Impossible.

**Case 2:**  $Y_1$  is NOT a closed  $U$ -orbit but  $Y_1$  is  $A$ -stable, where  $A = \{a_t\}$ . We want to find a  $\{v_s\}_{s \geq 0}$ -orbit or a  $\{v_s\}_{s \leq 0}$ -orbit inside  $Y_0$ .

**Fact 2.3.4.** The map  $\mathfrak{h} \oplus \mathfrak{h}^\perp \rightarrow X_3, (h, w) \mapsto \exp(h) \exp(w).x_1$  is a local diffeomorphism.

$$\mathfrak{h} = \left\{ X \in \mathfrak{gl}(3, \mathbb{R}) : M_0 X = -X^t M_0, M_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \right\}.$$

$$\mathfrak{h}^\perp = \left\{ X \in \mathfrak{gl}(3, \mathbb{R}) : \text{tr } X = 0, M_0 X = X M_0, M_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \right\}.$$

**Fact 2.3.5.**  $\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ , moreover  $\mathfrak{h}^\perp$  is invariant under  $\text{Ad}(H)$ .

In this case, there exists  $x_1 \in Y_1, A_n \rightarrow \text{Id}, A_n.x_1 \in Y_0$  where  $A_n \notin H$ . Write  $A_n = \exp(h_n) \exp(w_n), h_n \in \mathfrak{h}, w_n \neq 0 \in \mathfrak{h}^\perp$ . Let  $x_n = \exp(w_n)x_1 \in Y_0, \|w_n\| \rightarrow 0$ .

**Lemma 2.3.6**

For  $\delta$  sufficiently small,  $n$  sufficiently large, there exists  $t_{n,\delta} \in \mathbb{R}$  such that:

- (i)  $\|\text{Ad}(u_{t_{n,\delta}})w_n\| \in [10^{-10}\delta, 10^{10}\delta]$ .
- (ii) Every limit of  $\text{Ad}(u_{t_{n,\delta}})w_n$  is in Lie algebra of  $\{v_s\}$ .

Let  $y_{n,\delta} = u_{t_{n,\delta}}.x_1, z_{n,\delta} = u_{t_{n,\delta}}.x_n$ . As  $x_n = \exp(w_n)x_1$ , hence  $z_{n,\delta} = \exp(\text{Ad}(u_{t_{n,\delta}})w_n)y_{n,\delta}$ . By passing to a subsequence, we assume that

$$z_{n,\delta} \rightarrow z_{\infty,\delta}, \quad \text{Ad}(u_{t_{n,\delta}})w_n \rightarrow w_{\infty,\delta}, \quad y_{n,\delta} \rightarrow y_{\infty,\delta}.$$

Then  $z_{n,\delta} \in Y_0, y_{\infty,\delta} \in Y_1$  and  $w_{\infty,\delta}$  is in Lie algebra of  $\{v_s\}$ . Note that  $v_s$  commutes with  $u_t$ , we get  $\exp(w_{\infty,\delta})Y_1 \subseteq Y_0$ . By assumption,  $Y_1$  is  $A$ -stable, after some calculation,  $a_t \exp(w_{n,\delta})a_t^{-1}$  can go through ever  $v_s$  for  $s \geq 0$  or  $s \leq 0$ .

**Case 3:**  $Y_1$  is NOT  $A$ -stable.

**Lemma 2.3.7**

For  $\delta$  sufficiently small, for  $n$  sufficiently large. There exists  $s_{n,\delta}, t_{n,\delta} \in \mathbb{R}, h_{n,\delta} \oplus w_{n,\delta} \in \mathfrak{h} \oplus \mathfrak{h}^\perp$ , such that:

- (i)  $u_{s_{n,\delta}} \exp(\text{Ad}(u_t)h_n) \exp(\text{Ad}(u_t)w_n) = \exp(h_{n,\delta}) \exp(w_{n,\delta})$ .
- (ii)  $\max \{\|h_{n,\delta}\|, \|w_{n,\delta}\|\} \in [10^{-100}\delta, 10^{100}\delta]$ .
- (iii) Every limit of  $h_{n,\delta}$  is in Lie algebra of  $\{a_t\}$ , every limit of  $w_{n,\delta}$  is in Lie algebra of  $\{v_s\}$ .

**§2.4 22.3.11: Completion of some gaps**

**Fact 2.4.1.** If  $Q$  is “irrational”, then  $\text{SO}(Q_0, \mathbb{R})g_Q\mathbb{Z}^3$  is not compact.

*Proof of Theorem 2.2.1 assuming Theorem 2.3.1.* It suffices to show that  $\text{SO}(Q_0, \mathbb{R})g_Q\mathbb{Z}^3$  is unbounded. So if  $\text{SO}(Q_0, \mathbb{R})g_Q\mathbb{Z}^3$  is not unbounded, then (WLOG)  $\overline{\text{SO}(Q_0, \mathbb{R})g_Q\mathbb{Z}^3}$  contains a  $\{v_s\}_{s \leq 0}$ -orbit.

Let  $h \in \text{SL}(3, \mathbb{R})$  such that  $\overline{\text{SO}(Q_0, \mathbb{R})g_Q\mathbb{Z}^3} \supseteq \{v_s.h\mathbb{Z}^3 : s \leq 0\}$ . Then

$$\overline{Q(\mathbb{Z}^3)} = \overline{Q_0(g_Q\mathbb{Z}^3)} \supseteq Q_0(\{v_s h\mathbb{Z}^3 : s \leq 0\}).$$

We want to find  $s_n \leq 0, x_n \in h\mathbb{Z}^3$  such that  $Q_0(v_{s_n}x_n) \rightarrow 0$ . After some specific calculation, it suffices to find  $x \in h\mathbb{Z}^3$  such that  $2x_1x_3 - x_2^2 > 0$ . The lattice and this cone always intersect.  $\square$

*Proof of Lemma 2.3.6.* We have

$$\mathfrak{h}^\perp = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & -2x_{11} & -x_{12} \\ x_{31} & -x_{21} & x_{11} \end{bmatrix} \right\}.$$

For  $x \in \mathfrak{h}^\perp$ , we can calculate  $u_t x u_t^{-1}$  explicitly. We have

$$u_t x u_t^{-1} = \begin{bmatrix} * & * & P_x(t) = \frac{t^4}{4!}x_{31} + \frac{t^3}{3!}x_{21} + \frac{t^2}{2!}x_{11} + \frac{t}{3}(-x_{21}) + \frac{x_{13}}{6} \\ * & * & * \\ * & * & * \end{bmatrix}$$

Let  $M_t := \max \left\{ \left| \frac{t^4}{4!}x_{31} \right|, \left| \frac{t^3}{3!}x_{21} \right|, \left| \frac{t^2}{2!}x_{11} \right|, \left| \frac{t}{3}x_{21} \right|, \left| \frac{x_{13}}{6} \right| \right\}$ , then we can prove that

$$\max \{|P_x(t)|, |P_x(2t)|, |P_x(3t)|, |P_x(4t)|, |P_x(5t)|\} \geq 10^{-10} M_t.$$

For  $x_n$ , choose  $t$  such that  $M_t = \delta$ , choose  $t_{n,\delta} \in \{t, 2t, 3t, 4t, 5t\}$  such that  $|P_{x_n}(t_{n,\delta})| \geq 10^{-10}\delta$ . Then the statement follows.  $\square$

**A dynamics exposition of the case  $N = 2$** 

Recall lemma 2.2.15, it suffices to find an indefinite “irrational”  $Q$  such that  $\text{SO}(Q, \mathbb{R})\mathbb{Z}^2$  is bounded. Let  $Q_1 = xy$ , then  $\exists g_Q \in \text{SL}(2, \mathbb{R})$  such that  $Q = \lambda(Q_1 \circ g_Q)$  where  $\lambda \neq 0 \in \mathbb{R}$ . We want to find  $g \in \text{SL}(2, \mathbb{R})$  such that:

- (i)  $Q_1 \circ g$  is “irrational”.

(ii)  $\mathrm{SO}(Q_1, \mathbb{R})g\mathbb{Z}^2$  is bounded.

We can calculate that  $\mathrm{SO}(Q_1, \mathbb{R}) = \left\{ a_t = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \right\}$ .

**Example 2.4.2**

Let  $\Lambda := \mathbb{Z} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$ , let  $\Lambda' = \frac{\Lambda}{\sqrt{2}\sqrt{2}}$ , then  $\Lambda' \in X_2$ . Consider  $t_0 = 3 + 2\sqrt{2}$ , we can prove  $a_{t_0}\Lambda \subseteq \Lambda$  hence  $a_{t_0}\Lambda' \subseteq \Lambda'$ . Note that  $a_{t_0}$  preserve the volume of lattice, hence  $a_{t_0}\Lambda' = \Lambda'$  with shows that  $\{a_t.\Lambda\}$  is compact.

**Fact 2.4.3.** If  $\mathrm{SO}(Q_1, \mathbb{R})g\mathbb{Z}^2$  is not closed, then  $Q_1 \circ g$  is “irrational”.

So it suffices to construct an orbit of  $\mathrm{SO}(Q_1, \mathbb{R}) = \{a_t\}$  that is not compact and is bounded.

**Fact 2.4.4.** The union of all compact  $a_t$ -orbits are dense.

*Proof.* Firstly, there exists at least one compact  $a_t$ -orbit, say  $a_t\Lambda$ . Then we can prove that  $\{\Lambda' \in X_2 : \Lambda' \text{ is commensurable with } \Lambda\}$  is dense in  $X_2$  and those  $\Lambda'$  are with compact  $a_t$ -orbit.  $\square$

**Definition 2.4.5.** We say two lattice  $\Lambda_1$  and  $\Lambda_2$  is **commensurable**, denoted by  $\Lambda_1 \sim \Lambda_2$ , iff  $\Lambda_1 \cap \Lambda_2$  is of finite index in  $\Lambda_1$  and  $\Lambda_2$ .

**Lemma 2.4.6**

If  $a_t\Lambda$  is compact and  $\Lambda' \sim \Lambda$ , then  $a_t\Lambda'$  is also compact.