

# **ODE: Qualitative Theory (2022, Spring, Shaobo Gan)**

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# 1 Basic Concepts

## §1.1 Basic notions and results

Assume  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $(t, x) \mapsto f(t, x)$  continuous, consider the **equation** (or **system**)

$$\dot{x} = \frac{dx}{dt} = f(t, x).$$

A differentiable function  $\gamma : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}^n$  is said to be a **solution** (or **solution curve**), if for every  $t \in (a, b)$ ,

$$\frac{d\gamma(t)}{dt} = f(t, \gamma(t)).$$

The **graph** of  $\gamma$  is

$$\{(t, \gamma(t)) : t \in (a, b)\} \subset \mathbb{R} \times \mathbb{R}^n.$$

For  $t_0 \in (a, b)$ , let  $x_0 = \gamma(t_0)$ , then  $\gamma$  is called the solution of the **initial value problem**

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

**The initial value problem has a unique solution:** Let  $\gamma_i : (a_i, b_i) \rightarrow \mathbb{R}^n$  be two solutions of the initial value problem. Then there exists  $\delta > 0$ ,  $(t_0 - \delta, t_0 + \delta) \subset (a_1, b_1) \cap (a_2, b_2)$ , such that  $\gamma_1(t) = \gamma_2(t), \forall t \in (t_0 - \delta, t_0 + \delta)$ ,

### Theorem 1.1.1 (Existence and Uniqueness Theorem)

$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $f(t, x)$  continuous, given  $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, a > 0, b > 0$ , consider the region

$$R = R(t_0, x_0, a, b) = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}.$$

If  $f$  is Lipchitz in  $x$  on  $R$ , i.e.  $\exists L > 0, \forall (t, x_1), (t, x_2) \in R$ ,

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|,$$

then the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution on  $[t_0 - h, t_0 + h]$ , where  $h = \min\{a, b/M\}$ ,  $M = \max_{(t,x) \in R} |f(t, x)|$ .

**Remark 1.1.2** — The solution is denoted as  $\varphi(t; t_0, x_0)$ .

**Corollary 1.1.3**

When  $f \in C^1$ , the existence and uniqueness theorem holds.

Denotes the **maximal interval** of  $\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$  as  $I(t_0, x_0)$ , it is an open interval.

**Corollary 1.1.4**

Assume  $f \in C^1$  and  $|f(x)| \leq A(t)|x| + B(t)$ , then the maximal interval of the initial value problem is  $(-\infty, +\infty)$ .

**§1.2 Flows**

Now we consider the **autonomous equation**

$$\dot{x} = f(x).$$

$\mathbb{R}^n$  is called the **phase space** and  $\mathbb{R} \times \mathbb{R}^n$  is called the **generalized phase space**.

The solution of the initial value problem  $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$  is denoted as  $\varphi(t, x_0)$ , the set

$$\text{Orb}(x_0) := \{\varphi(t, x_0) : t \in I(x_0)\} \subset \mathbb{R}^n$$

is called the **orbit** pass by  $x_0$ .

**Corollary 1.2.1** (Continuous Dependence on the Initial Value)

Assume  $f \in C^1$ , then  $U = \{(t, x) : t \in I(x)\}$  is open and  $\varphi : U \rightarrow \mathbb{R}^n, (t, x) \mapsto \varphi(t, x)$  is continuous.

**Theorem 1.2.2**

$f(x) \in C^1$ , then:

1.  $\varphi_0(x) = x$  for every  $x \in \mathbb{R}^n$ .
2.  $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$  for every  $s \in I(x), t \in I(\varphi(s, x))$ .

**Definition 1.2.3.**  $\psi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , continuous, is said to be a **(continuous) flow** if:

- (i)  $\psi(0, x) = x$ ,
- (ii)  $\psi(t, \psi(s, x)) = \psi(t + s, x)$ .

**Remark 1.2.4** — The solution of an autonomous equation is a **local flow**.

**Corollary 1.2.5**

Let  $\psi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a flow, then  $\psi_t := \psi(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are homeomorphisms.

**Remark 1.2.6** — Consider the group of self-homeomorphisms of  $\mathbb{R}^n$ , denoted as  $\text{Homeo}(\mathbb{R}^n)$ , then  $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$  is a group homomorphism. More generally, we can consider  $G \rightarrow \text{Homeo}(\mathbb{R}^n)$  for some group  $G$ .

**Proposition 1.2.7**

Assume  $f$  is a  $C^1$  vector field, then the orbits of the flow generated by  $f$  are either coincide or disjoint.

$\bigcup_{x \in \mathbb{R}^n} \text{Orb}(x)$  forms a partition of  $\mathbb{R}^n$ , is called the **orbit space**. For each orbit, orient it to indicate the direction of motion, the family of the oriented orbit  $\varphi(t, x)/f(x)$  is called the **phase portrait**.

A point  $x_0 \in \mathbb{R}^n$  with  $f(x_0) = 0$  is called a **critical point** (or a **singularity**, **equilibrium**). The orbit  $\text{Orb}(x_0)$  is a single point  $\{x_0\}$ .

**Example 1.2.8**

$$\begin{cases} \frac{dx}{dt} = x \\ x(0) = x_0 \end{cases},$$

the solutions are  $\varphi(t, x_0) = x_0 e^t$ . There are three orbits  $\mathbb{R}_+, \mathbb{R}_-, \{0\}$ .

**Example 1.2.9**

$$\begin{cases} \frac{dx}{dt} = x^2 \\ x(0) = x_0 \end{cases},$$

the solutions are  $\varphi(t, x_0) = \frac{x_0}{1 - x_0 t}$ . There are three orbits  $\mathbb{R}_+, \mathbb{R}_-, \{0\}$ . But the phase portrait is different from the former examples, because the orientations on  $\mathbb{R}_-$  are different.

**Theorem 1.2.10**

Assume  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$  vector field,  $\beta(x) : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1$  and  $\beta(x) > 0$ . Then the equations  $\dot{x} = f(x)$  and  $\dot{x} = \beta(x)f(x)$  have the same phase portraits.

*Proof.*  $\varphi : I \rightarrow \mathbb{R}^n$  a solution of  $f$ . Find a  $C^1$  diffeomorphism  $h : J \rightarrow I$  such that  $\varphi \circ h$  is the solution of  $\dot{x} = \beta(x)f(x)$ . It suffices that

$$\frac{d}{dt} \Big|_{t=h(s)} \varphi(t) \cdot \frac{dh(s)}{ds} = \beta(\varphi \circ h(s))f(\varphi \circ h(s)),$$

i.e.  $\frac{dh(s)}{ds} = \beta(\varphi \circ h(s)) > 0$ , it is an initial value problem. It shows that the maximal solution curve of  $f$  is contain in some solution curve of  $\beta f$ .  $\square$

**Theorem 1.2.11** (Differentiable Dependence on the Initial Value)

Assume  $f \in C^1$ , it generates the flow  $\phi_t$ , then  $\phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^1$ .

**Exercise 1.2.12.**

$$\frac{\partial}{\partial t} \frac{\partial \phi(t, x)}{\partial x} = \frac{\partial}{\partial x} \frac{\partial \phi(t, x)}{\partial t}.$$

Let  $\Phi(t, x) = \Phi_t(x) = \frac{\partial \phi(t, x)}{\partial t}$ , then  $\Phi$  is the solution of the equation

$$\begin{cases} \frac{dy(t)}{dt} = A(t)y(t), A(t) = Df(\phi_t(x)) \\ y(0) = \text{Id} \end{cases}.$$

The equation is called the **variation equation** of  $f(x)$  along  $\phi_t(x)$ .

**Lemma 1.2.13**

$f \in C^1$ ,  $\Phi(t, x)$ , then

$$\Phi_t(\phi_s(x))\Phi_s(x) = \Phi_{t+s}(x).$$

**Remark 1.2.14** — This property is called the **cocycle** condition.

We already know that  $\phi_t$  are self-homeomorphisms of  $\mathbb{R}^n$ , and lemma 1.2.13 shows that the differential is invertible, hence  $\phi_t$  are diffeomorphisms. Define

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (t, x, v) &\mapsto (\phi_t(x), \Phi_t(x)v). \end{aligned}$$

**Proposition 1.2.15**

$\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  is a flow.

**Remark 1.2.16** — We call  $\Phi_t$  is a skew product flow of  $\phi_t$ .

**Theorem 1.2.17**

$$\Phi_t(x)f(x) = f(\phi_t(x)).$$

If  $\psi$  is a  $C^1$  flow, let

$$g(x) = \left. \frac{\partial \psi(t, x)}{\partial t} \right|_{t=0},$$

then  $\psi(t, x_0)$  solve the initial value problem  $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$ . Because

$$\frac{\partial \psi(t, x_0)}{\partial t} = \left. \frac{\partial \psi(t+s, x_0)}{\partial s} \right|_{s=0} = \left. \frac{\partial \psi(s, \psi(t, x_0))}{\partial s} \right|_{s=0} = g(\psi(t, x_0)).$$

### §1.3 Equations on manifolds

Let  $M$  be a closed smooth manifold,  $X$  is a  $C^1$  vector field on  $M$ . Then  $X$  is bounded, hence the maximal intervals are  $(-\infty, +\infty)$ . Consider the equation

$$\begin{cases} \frac{dx}{dt} = X(x) \\ x(0) = x_0 \end{cases},$$

then the solution  $\varphi(t, x)$  generates a flow.

# 2 Linear Systems

## §2.1 Plane linear singularities

Consider the equation

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}, \quad (x, y) \in \mathbb{R}^2.$$

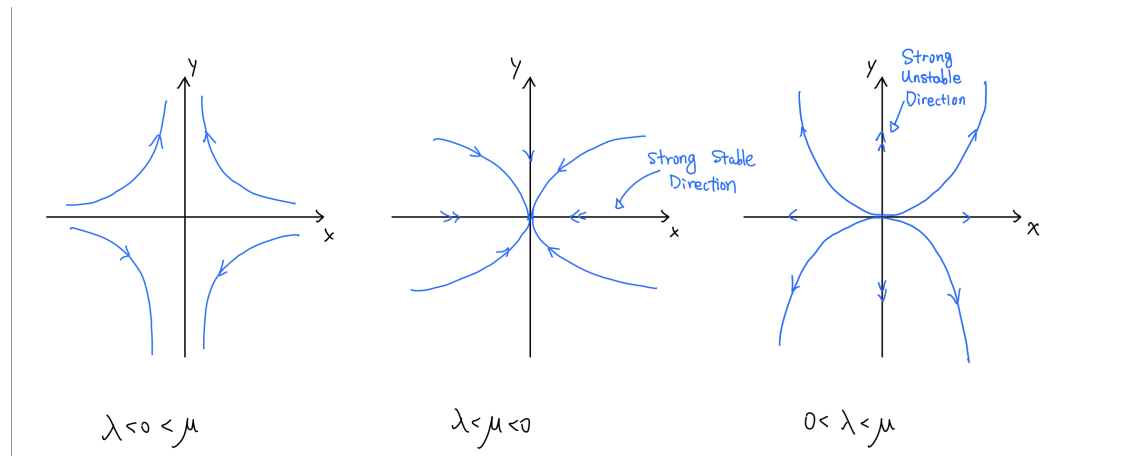
It is said to be a **plane linear system** if  $f, g$  both linear functions of  $x, y$ , i.e.

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad a, b, c, d \in \mathbb{R}.$$

If  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ , then  $(0, 0)$  is the only singularity of the vector field, elementary singularity.

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , consider the Jordan form of  $A$ . There are four cases:

- I. Two different real eigenvalues:  $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$ .
  - i.  $\lambda < 0 < \mu$ : the origin is called a **saddle point**.
  - ii.  $\lambda < \mu < 0$ : the origin is called a **stable node**.
  - iii.  $0 < \lambda < \mu$ : the origin is called a **unstable node**.



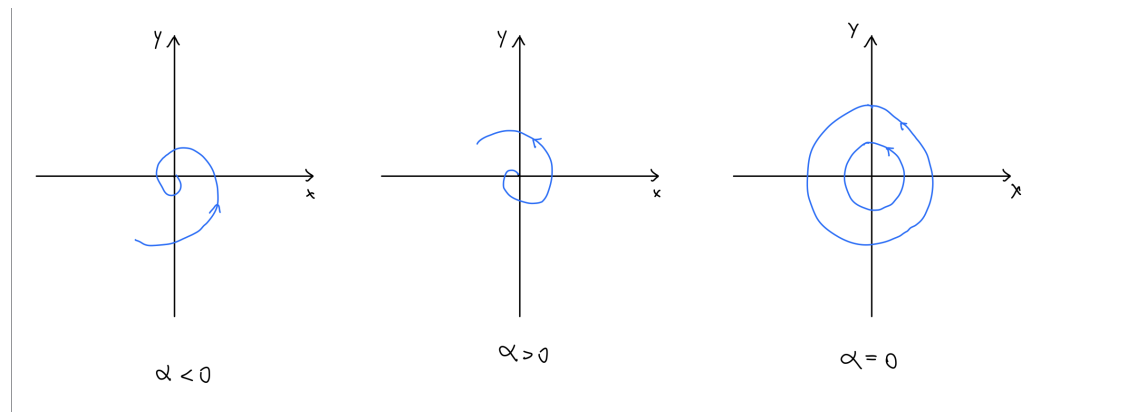
- II. Conjugated imaginary eigenvalues:  $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ ,  $\beta > 0$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ .

If we consider this equation in the polar coordinates, it turns  $\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$ .

- i.  $\alpha < 0$ , the origin is called a **stable focus**.
- ii.  $\alpha > 0$ , the origin is called a **unstable focus**.
- iii.  $\alpha = 0$ , the origin is called a **center**.

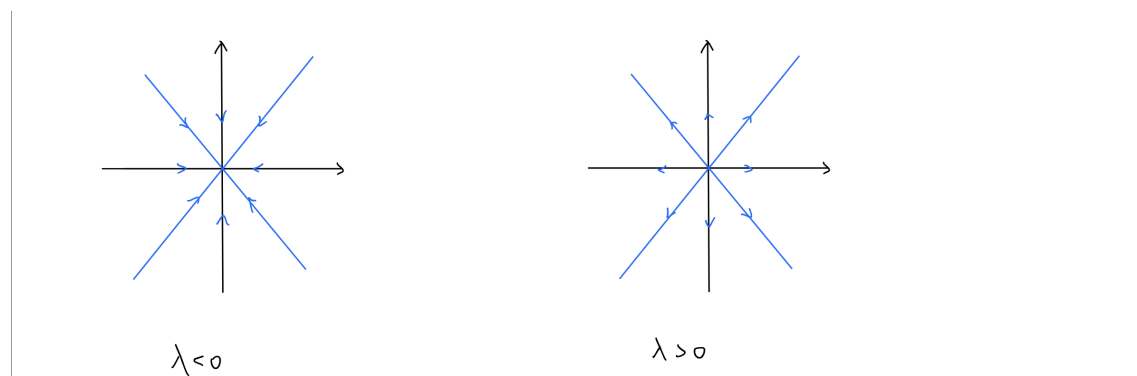
**Definition 2.1.1.**  $\varphi_t$  a flow. If  $p$  is not a singularity and  $\exists T > 0$ , such that  $\varphi_T(p) = p$ . Then  $p$  is called a **periodic point**,  $\text{Orb}(p)$  is called a **periodic orbit**. If  $p$  is a periodic point, the smallest  $T > 0$  is called the **minimum positive period**.





III. Two same real eigenvalues, diagonalizable:  $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$ .

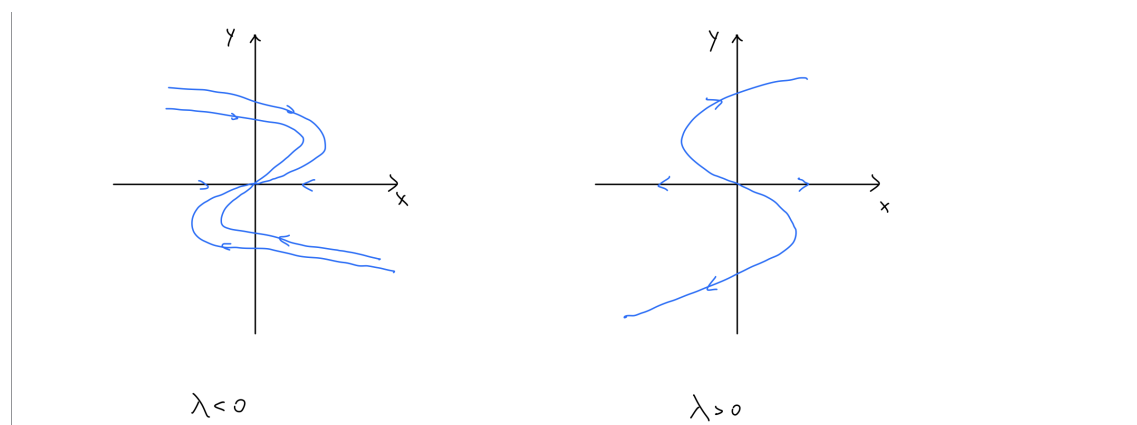
- i.  $\lambda < 0$ , the origin is called a **stable critical node**.
- ii.  $\lambda > 0$ , the origin is called a **unstable critical node**.



IV. Two same real eigenvalues, not diagonalizable:  $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ , then  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}(x_0 + ty_0) \\ e^{\lambda t}y_0 \end{bmatrix}$ ,

or  $x(t) = \frac{x_0}{y_0}y(t) + \frac{y(t)}{\lambda} \ln \frac{y(t)}{y_0}$ .

- i.  $\lambda < 0$ , the origin is called a **stable unidirectional node**.
- ii.  $\lambda > 0$ , the origin is called a **unstable unidirectional node**.



**Exercise 2.1.2.** Draw the phase portraits of non-elementary plane systems (i.e. the determinant is 0).

## §2.2 Topological conjugacies between linear systems

**Definition 2.2.1.** Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  homeomorphisms.  $f$  and  $g$  are said to be **topologically conjugate** if there exists  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $h \circ f = g \circ h$ .

**Remark 2.2.2** — Conjugacy is an equivalence relation.

**Definition 2.2.3.** Let  $\varphi_t, \psi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be two flows, we call  $\varphi_t$  and  $\psi_t$  are conjugate if there is a homeomorphism  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $h \circ \varphi_t = \psi_t \circ h$ . Let  $X, Y$  be two  $C^1$  vector fields on  $\mathbb{R}^n$ , we call  $X, Y$  are conjugate if the flows generated by them, respectively, are conjugate.

### Example 2.2.4

$A, B \in M_n(\mathbb{R})$  are similar, then  $\dot{x} = Ax$  and  $\dot{y} = By$  are conjugate.

$f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$   $C^1$  vector fields, generate flows  $\phi_t, \psi_t$ . Let  $x = h(y) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  diffeomorphism gives the conjugate, i.e.,  $h\psi_t(y) = \phi_t h(y)$ . Then

$$\frac{d}{dt}h(y) = f(h(y)) \implies D_{h(y)}g(y) = D_{h(y)}\frac{dy}{dt} = f(h(y)).$$

If there exists a  $C^1$  diffeomorphism conjugate  $e^{Bt}y$  to  $e^{At}x$  via  $x = h(y)$ , i.e.  $h(e^{Bt}y) = e^{At}h(y)$ . Then  $D_{h(0)}e^{Bt} = e^{At}D_{h(0)}$ , hence  $D_{h(0)}B = AD_{h(0)}$ . It shows that  $C^1$  conjugate generically not hold even if topologically conjugate.

### Proposition 2.2.5

Assume  $f, g$   $C^1$  vector fields generate  $\phi_t, \psi_t$ , let  $h$  be a conjugate between  $\phi_t$  and  $\psi_t$ . Then:

1.  $h(\text{Orb}(x, \phi)) = \text{Orb}(hx, \psi)$ .
2.  $h$  maps the singularities of  $f$  to the singularities of  $g$ .
3.  $h$  maps the periodic orbits of  $f$  to the periodic orbits of  $g$ . Moreover, it preserves the minimum positive period.

### Example 2.2.6

$\dot{x} = -2x$  and  $\dot{y} = -4y$  are conjugate.

Let  $h : \mathbb{R} \rightarrow \mathbb{R}$ ,  $h(0) = 0$ . Take  $x_0, y_0 > 0$ , let  $h(x_0) = y_0$ , then  $h(e^{-2t}x_0) = e^{-4t}y_0$  or  $h(x) = \left(\frac{x}{x_0}\right)^2 y_0$ . The construction for the negative part is similar.

**Exercise 2.2.7.**  $\lambda\mu \neq 0$ , show that  $\dot{x} = \lambda x$  is conjugate to  $\dot{y} = \mu y$  if and only if  $\lambda\mu > 0$ .

**Proposition 2.2.8**

$\phi_t^i, \psi_t^i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$  are topologically conjugate by  $h_i, i = 1, 2$ . Then  $\phi_t^1 \times \phi_t^2$  and  $\psi_t^1 \times \psi_t^2$  are topologically conjugate by  $h_1 \times h_2$ .

**Example 2.2.9**

$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$  and  $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$  are conjugate.

*Proof.*  $\phi_t(x, y) = e^{-t}(x, y)$  and  $\psi_t(x, y) = e^{-t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ . For every  $(x, y) \neq (0, 0)$ , there exists unique  $t = t(x, y)$  such that  $\phi_t(x, y) \in \mathbb{S}^1$ . Let  $h(x, y) := \psi_{-t}\phi_t(x, y)$ , where  $t = t(x, y)$ , then  $h$  gives the conjugate.  $\square$

**Exercise 2.2.10.** Show that  $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$  and  $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -y \end{cases}$  are conjugate.

Classification of elementary plane linear systems:

- (I) Stable: node, critical node, unidirectional node, focus.
- (II) Unstable: node, critical node, unidirectional node, focus.
- (III) Saddle point.
- (IV) Center.

**Definition 2.2.11.** The linear system  $\dot{x} = Ax$  in  $\mathbb{R}^n$  is called **hyperbolic** if the real parts of eigenvalues of  $A$  are non zero. The **(stable) index** of  $A$  is the number of eigenvalues with negative real parts, denoted by  $\text{Ind } A$ .

**Theorem 2.2.12**

Two plane hyperbolic linear system  $\dot{x} = Ax, \dot{y} = By$  are topologically conjugate if and only if  $\text{Ind } A = \text{Ind } B$ .

*Proof.* “ $\implies$ ”: Let  $W_A^s = \{x : e^{tA}x \rightarrow 0, t \rightarrow \infty\}$ ,  $W_B^s = \{x : e^{tB}x \rightarrow 0, t \rightarrow \infty\}$ , then  $h$  and  $h^{-1}$  preserves the stable manifolds. Then  $\text{Ind } A = \dim W_A^s = \dim W_B^s = \text{Ind } B$ .  $\square$

**Example 2.2.13**

Consider  $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$  and  $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$  with the same phase portraits are not topologically conjugate. Because the topologically conjugate preserves the minimum positive orbits.

**Definition 2.2.14.**  $\phi_t, \psi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  flows,  $h$  is a homeomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  maps the orbit of  $\phi$  to the orbit of  $\psi$  preserves the orientation. Then  $\phi$  and  $\psi$  is called **topologically equivalent** or **flow equivalent**.

**Theorem 2.2.15** (Grobman-Hartman)

If  $x_0$  is a hyperbolic singularity of  $f(x)$ , then the flows generated by  $\dot{x} = f(x)$  and  $\dot{y} = Ay$  where  $y = Df(x_0)$  are topologically conjugate near 0.

## §2.3 Nonautonomous linear system

$A : \mathbb{R} \rightarrow M(n, \mathbb{R})$  continuous, consider the equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n$$

a nonautonomous linear system.

**Theorem 2.3.1**

The followings hold:

1. The initial problem of the equation exist the unique solution.
2. The maximal interval of any solution is  $(-\infty, \infty)$ .
3. All solutions of the equation form an  $n$ -dimensional linear space  $S$ .

**Theorem 2.3.2** (Liouville's Formular)

Assume  $X(t)$  is a solution of  $\dot{x} = A(t)x$ , then

$$\frac{d}{dt} \det X(t) = \text{tr } A(t) \det X(t),$$

hence  $\det X(t) = \det X(t_0) \exp \int_{t_0}^t \text{tr } A(s) ds$ .

Let  $X_1(t), X_2(t), \dots, X_n(t)$  be a basis of  $S$ , let

$$X(t) := [X_1(t), X_2(t), \dots, X_n(t)] \in \text{GL}(n, \mathbb{R}),$$

it called a **fundamental solution** of the equation. The fundamental solution of

$$\begin{cases} \frac{dX}{dt} = A(t)X \\ X(t_0) = I_n \in \text{GL}(n, \mathbb{R}) \end{cases}$$

is called the **standard fundamental solution**.

If  $X(t), Y(t)$  are two fundamental solutions, suppose  $Y(0) = X(0)C$ , then

$$\frac{dX(t)C}{dt} = \frac{dX(t)}{dt}C = A(t)X(t)C,$$

is a nondegenerate solution of  $\frac{dX}{dt} = AX$ . By the uniqueness, we get  $Y(t) = X(t)C$ .

**Example 2.3.3**

$A(t) \equiv A$ , the fundamental solution of  $\dot{x} = Ax$  is

$$e^{tA} = \text{Id} + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{k!}t^kA^k + \cdots.$$

**Example 2.3.4**

$\dot{x} = f(x)$ ,  $x \in \mathbb{R}^n$ , where  $f \in C^1$ , generates the flow  $\varphi_t(x)$ . Consider  $\Phi_t(x) = \frac{\partial}{\partial t}\varphi_t(x)$  and the variation equation

$$\frac{d}{dt}\Phi_t(x) = Df(\varphi_t(x))\Phi_t(x).$$

Given  $x \in \mathbb{R}^n$ , let  $A(t) := Df(\varphi_t(x))$ , then  $\Phi_t(x)$  is the standard fundamental solution ( $t_0 = 0$ ) of  $\dot{x} = A(t)x$ . Consider two special types of orbits:

- $x$  is a singularity, denoted by  $\sigma$ . Then  $\varphi_t(\sigma) = \sigma$ ,  $\dot{x} = Ax$  where  $A = Df(\sigma)$ .
- $x$  is a periodic point, denoted by  $p$ , the minimum period  $T > 0$ . Then  $A$  is  $T$ -periodic.

**Definition 2.3.5.** The equation  $\dot{x} = A(t)x$  satisfies  $A(t+T) = A(t)$  for some  $T > 0$  is called a **periodic linear system**.

**Theorem 2.3.6** (Floquet)

Assume  $\dot{x} = A(t)x$  is a  $T$ -periodic linear system, if  $X$  is a fundamental solution, then  $X(t+T)$  is a fundamental solution, i.e.  $\exists C \in \text{GL}(n, \mathbb{R})$  such that  $X(t+T) = X(t)C$ . Moreover, there exists a  $T$ -periodic map  $P : \mathbb{R} \rightarrow \text{GL}(n, \mathbb{C})$  and a constant matrix  $B \in \text{GL}(n, \mathbb{C})$  such that  $X(t) = P(t)e^{tB}$ .

**Lemma 2.3.7**

$\forall C \in \text{GL}(n, \mathbb{R})$ ,  $\exists B \in M(n, \mathbb{C})$  such that  $C = e^B$ .