Harmonic Analysis
(Spring 2022, Dongyi Wei)

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Contents

1 Fourier Series and Integrals

§1.1 Fourier series

For $f \in L^1(\mathbb{T})$, define the **Fourier coefficients**

$$\widehat{f}(k) := \int_0^1 f(x)e^{-2\pi ikx}\mathrm{d}x.$$

Let

$$\sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$$

be the **Fourier series** of f. When we discuss the convergence of Fourier series, we consider two types of sum:

$$S_N f = \sum_{k=-N}^{N} \widehat{f}(k)e^{2\pi ikx}, \quad \sigma_N f = \frac{1}{N+1}\sum_{k=0}^{N} S_k f.$$

We concern about the following questions:

Question 1.1.1. The pointwise convergence of $S_N f$.

Question 1.1.2. The L^p convergence of $S_N f$.

Question 1.1.3. The almost everywhere convergence of $S_N f$.

Question 1.1.4. The convergence of $\sigma_N f$.

§1.2 The pointwise convergence

Definition 1.2.1. The **Dirichlet kernel** D_N is given by

$$D_N(t) := \sum_{k=-N}^{N} e^{2\pi ikt} = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}.$$

It satisfies

$$\int_0^1 D_N(t)\mathrm{d}t = 1.$$

Theorem 1.2.2 (Dini's Criterion)

For $x \in \mathbb{T}$, if $\exists\delta > 0$, such that

$$\int_{|t|<\delta} \left|\frac{f(x+t)-f(x)}{t}\right| \mathrm{d}t < \infty,$$

then $S_N f(x) \to f(x)$.

Theorem 1.2.3
If f is bounded variation on a neighborhood of x, then

$$S_N f(x) \to \frac{f(x+) + f(x-)}{2}.$$

Example 1.2.4
$f_1(t) = |t|^{-\alpha}\mathbb{1}_{(0,1/2)}$, $f_2(t) = t^\alpha \sin\frac{1}{t}\mathbb{1}_{(0,1/2)}$, where $\alpha \in (0,1)$.

Theorem 1.2.5 (Riemann Localization Principle)
If f is zero in a neighborhood of x, then $S_N f(x) \to 0$.

Theorem 1.2.6 (Riemann-Lebesgue)
If $f \in L^1(\mathbb{T})$, then $\widehat{f}(k) \to 0(|k| \to \infty)$.

§1.3 Fourier series of continuous functions

Theorem 1.3.1
There exists $f \in C(\mathbb{T})$ such that $S_N f(0)$ diverges.

Proof. Consider $T_N : C(\mathbb{T}) \to \mathbb{C}, f \mapsto S_N f(0)$, By theorem 1.3.2, it suffices to show $\sup \|T_N\| = \infty$. Suppose $L_N = \|D_N\|_1$, we can prove that$\|T_N\| = L_N$. Consider the functions $f_n(t) = \frac{nD_N(t)}{1+n|D_N(t)|}$ is enough. The statement follows by lemma 1.3.3. □

Theorem 1.3.2 (Uniform Boundedness Principle)
X, Y, Banach Spaces. $\{T_a\}_{a\in A}$ is a family of bounded linear operators from X to Y. Then one of the following holds:

1. $\sup_{a\in A} \|T_a\| < \infty$.
2. $\exists x \in X$, such that $\sup_{a\in A} \|T_a x\| = \infty$.

Lemma 1.3.3
$L_N = \frac{4}{\pi^2} \ln N + O(1)$.

§1.4 Convergence in norm

Question 1.4.1. We can ask:

1. Does $\|S_N f - f\|_p \to 0$ for $f \in L^p(\mathbb{T})$?
2. Does $S_N f \to f$ a.e. for $f \in L^p(\mathbb{T})$?

Lemma 1.4.2

$S_N f$ convergence to f in L^p norm, $1 \leqslant p < \infty$, iff exists C_p such that

$$\|S_N f\|_p \leqslant C_p \|f\|_p .$$

§1.5 Summability method

Definition 1.5.1. The **Fejér kernel** is given by

$$F_N(t) := \frac{1}{N+1} \sum_{k=0}^{N} D_k(t) = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)} \right)^2$$

It satisfies

$$\int_0^1 F_N(t) \mathrm{d}t = 1 \text{ and } F_N(t) \geqslant 0.$$

Theorem 1.5.2

If $f \in L^p(\mathbb{T}),\ 1 \leqslant p < \infty$, or $f \in C(\mathbb{T})$ and $p = \infty$, then

$$\|\sigma_N f - f\|_p \to 0.$$

Proof. Applying Minkowski's inequality and it follows by Fejér kernel is a good kernel. □

Corollary 1.5.3

The following holds:

1. The trigonometric polynomials $V = \left\{ \sum_{k=-N}^{N} c_k e^{2\pi i k x} : c_k \in \mathbb{C}, N \in \mathbb{Z}_+ \right\}$ is dense in $L^p(\mathbb{T})$.
2. If $\in L^1(\mathbb{T})$ and $\widehat{f}(k) = 0$ for every $k \in \mathbb{Z}$, then $f = 0$ a.e. .

Theorem 1.5.4

$\|f\|_2 = \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2$ and $\|S_N f\|_2 \leqslant \|f\|_2 .$

Define the **Poisson kernel**

$$P_r(t) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k t} = \frac{1 - r^2}{1 - 2r\cos(2\pi t) + r^2} = \frac{1 - |z|^2}{|1 - z|^2},\ z = re^{2\pi i t}.$$

Let

$$u(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k + \sum_{k=-\infty}^{-1} \widehat{f}(k) \overline{z}^{|k|}$$

be the Poisson sum, then $u(re^{2\pi i\theta}) = P_r * f(\theta)$.

Theorem 1.5.5

If $f \in L^p(\mathbb{T}),\ 1 \leqslant p < \infty$, or $f \in C(\mathbb{T})$ and $p = \infty$, then

$$\|P_r * f - f\|_p \to 0 (r \to 1^-).$$

Remark 1.5.6 — $\Delta u = 0$ in $D = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{T} \cong \partial D = \mathbb{S}^1$. If $f \in C(\mathbb{T})$, then $u \in C(\overline{D})$ and $u = f$ on ∂D.

Fact 1.5.7. $\sigma_N f \to f$ a.e. and $P_r * f \to f$ a.e. . We will prove these in the next chapter.

§1.6 The Fourier transform of L^1 functions

For $f \in L^1(\mathbb{R}^n)$, let

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \xi \cdot x} \mathrm{d}x = (\mathcal{F} f)(\xi).$$

Proposition 1.6.1

The following holds:

1. $\widehat{(\alpha f + \beta g)} = \alpha \widehat{f} + \beta \widehat{g}$.
2. $\|\widehat{f}\|_\infty \leqslant \|f\|_1$ and $\widehat{f} \in C(\mathbb{R}^n)$.
3. $\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0$.
4. $\widehat{f * g} = \widehat{f} \widehat{g}$.
5. $\widehat{\tau_h f} = \widehat{f}(\xi) e^{2\pi i h \cdot \xi}$ where $\tau_h f = f(\cdot + h)$. $\widehat{f e^{2\pi i h \cdot x}}(\xi) = \widehat{f}(\xi - h)$.
6. $\rho \in \mathrm{O}_n$, then $\widehat{f(\rho\, \cdot)}(\xi) = \widehat{f}(\rho \xi)$.
7. If $g(x) = \lambda^{-n} f(\lambda^{-1} x)$, then $\widehat{g}(\xi) = \widehat{f}(\lambda \xi)$ for every $\lambda > 0$.
8. $\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$, if $\frac{\partial f}{\partial x_j} \in L^1$.
9. $\widehat{(-2\pi i x_j f)}(\xi) = \frac{\partial \widehat{f}}{\partial \xi_j}(\xi)$, if $x_j f \in L^1$.

§1.7 The Schwartz class and tempered distributions

Define the **Schwartz class**

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : p_{\alpha\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f|, \forall \alpha, \beta \in \mathbb{N}^n \right\}.$$

Then $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$. Moreover $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and is dense in $L^p(\mathbb{R}^n)(1 \leqslant p < \infty)$. The topology on $\mathcal{S}$ is defined as

$$f_k \to f \text{ in } \mathcal{S} \iff \lim_{k\to\infty} p_{\alpha,\beta}(f_k - f) = 0, \forall \alpha, \beta \in \mathbb{N}^n.$$

We can give a family of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ as

$$\|f\|_{(k)} = \sup \{p_{\alpha,\beta}(f) : \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leqslant k\}$$

and a quasi-norm on $\mathcal{S}(\mathbb{R}^n)$ as

$$\|f\|_{(*)} = \sum_{k=0}^{\infty} \min\left\{ \|f\|_{(k)}, 2^{-k} \right\}.$$

Let $d(f, g) := \|f - g\|_{(*)}$, which makes $\mathcal{S}$ a metric space $(\mathcal{S}, d)$ and the topology is identified.

Theorem 1.7.1

The following holds:

1. $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$ and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous.
2. $\int_{\mathbb{R}^n} f\widehat{g} = \int_{\mathbb{R}^n} \widehat{f} g$.

Lemma 1.7.2

If $f(x) = e^{-\pi|x|^2}$, then $\widehat{f}(\xi) = e^{-\pi|\xi|^2}$.

Remark 1.7.3 — $\widehat{e^{-\pi\lambda|x|^2}} = \lambda^{-\frac{n}{2}} e^{-\pi|\xi|^2/\lambda}$ for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.

Theorem 1.7.4

The following holds:

1. If $f \in \mathcal{S}$ (or $f \in L^1$ and $\widehat{f} \in L^1$), then $f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi\cdot x} \mathrm{d}\xi$.
2. $\forall f, g \in \mathcal{S}$, $\int_{\mathbb{R}^n} \widehat{f}\overline{\widehat{g}} = \int_{\mathbb{R}^n} f\overline{g}$.

Proof. For $f \in \mathcal{S}$, let $g_\lambda(x) = e^{-\pi\lambda|x|^2}$, by DCT and the identity

$$\int_{\mathbb{R}^n} \widehat{f}(x) g(\lambda x) \mathrm{d}x = \int_{\mathbb{R}^n} f(\lambda x) \widehat{g}(x) \mathrm{d}x.$$

□

Let $\overline{\mathcal{F}} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \xi\cdot x} \mathrm{d}x$, $\sigma f(x) = \widetilde{f}(x) = f(-x)$, $Cf(x) = \overline{f(x)}$. Then $\overline{\mathcal{F}} = C\mathcal{F}C$, $\overline{\mathcal{F}} = \mathcal{F}^{-1}, \mathcal{F}^4 = \mathrm{Id}$.

Corollary 1.7.5 (Plancherel)
$\|f\|_2 = \|\mathcal{F}f\|_2, \forall f \in \mathcal{S}$.

We define the family of **tempered distributions** $\mathcal{S}'$ as the continuous linear function on $\mathcal{S}$. Then $T \in \mathcal{S}'$ if and only if $\exists m \in \mathbb{N}$, such that $|\langle T, f\rangle| \leqslant C\|f\|_{(m)}$ for every $f \in \mathcal{S}$. For every $1 \leqslant p \leqslant \infty$, we have a natural embedding $j_p : L^p \hookrightarrow \mathcal{S}'$.

Definition 1.7.6. $\forall T \in \mathcal{S}'$, define $\widehat{T}(f) = T(\widehat{f}), \forall f \in \mathcal{S}$.

Let $\mathcal{F}_1 : T \mapsto \widehat{T}$. Then $\mathcal{F}_1$ maps $\mathcal{S}'$ to $\mathcal{S}'$ is continuous. Moreover, $\mathcal{F}_1 \circ j_1 = j_\infty \circ \mathcal{F}$.

Proposition 1.7.7
If $T \in \mathcal{S}', \widehat{T} \in L^1$, then $T(x) = \int_{\mathbb{R}^n} \widehat{T}(\xi) e^{2\pi i \xi \cdot x} \mathrm{d}\xi$ a.e. .

§1.8 The Fourier transform on $L^p, 1 < p \leqslant 2$

Theorem 1.8.1
For $\forall f \in L^2(\mathbb{R}^n)$, then $\widehat{f} \in L^2$ and $\|\widehat{f}\|_2 = \|f\|_2$.

Theorem 1.8.2
It holds $\widehat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x) e^{-2\pi i \xi \cdot x} \mathrm{d}x$, $f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \widehat{f}(\xi) e^{-2\pi i \xi \cdot x} \mathrm{d}\xi$, both convergences is in the sense of L^2 norm.

Because $\mathcal{F} : L^1 \to L^\infty, L^2 \to L^2$, then by $L^p \subset L^1 + L^2$ for $1 < p < 2$, we have $\mathcal{F} : L^p \to L^1 + L^\infty$.

Theorem 1.8.3 (Riesz-Thorin Interpolation Theorem)
$p_0, p_1, q_0, q_1 \in [1, \infty], 0 < \theta < 1$, let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. If $T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ such that $\|Tf\|_{q_0} \leqslant M_0 \|f\|_{p_0}, \|Tf\|_{q_1} \leqslant M_1 \|f\|_{p_1}$, then $\|Tf\|_q \leqslant M_0^{1-\theta} M_1^\theta \|f\|_p$.

Corollary 1.8.4
If $f \in L^p, 1 \leqslant p \leqslant 2$, then $\mathcal{F}f \in L^{p'}$ and $\|\mathcal{F}f\|_{p'} \leqslant \|f\|_p$.

Corollary 1.8.5
$f \in L^p, g \in L^q, p, q, r \in [1, \infty]$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$.

§1.9 The convergence and summability of Fourier integral

Let $B_R = R \cdot B$ where B is a neighborhood of origin.

> **Question 1.9.1.** $f(x) = \lim_{R\to\infty} \int_{B_R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} \mathrm{d}\xi$?

Let $\widehat{S_R f} = \chi_{B_R} \widehat{f}$, then $\lim_{R\to\infty} \|S_R f - f\|_p = 0$ iff $\|S_R f\|_p \leqslant C_p \|f\|_p$.

Fact 1.9.2. $S_R : L^p \to L^p$ bounded iff $n = 1, 1 < p < \infty$ or $n = 1, p = 2 (B = B(0,1))$ or $n > 1, 1 < p < \infty (B = Q(0,1))$.

$n = 1, B = (-1, 1)$, then $S_R f = D_R * f$, where D_R is the Dirichlet kernel

$$D_R(x) = \int_{-R}^{R} e^{2\pi i \xi \cdot x} \mathrm{d}\xi = \frac{\sin(2\pi R x)}{R x}.$$

Then $D_R \notin L^1$ but $D_R \in L^q (1 < q \leqslant \infty)$.

Almost everywhere convergence Now we consider the almost everywhere convergence, an argument (Carleson-Hunt) shows that

$$\left\| \sup_R |S_R f|_p \right\| \leqslant C_p \|f\|_p \implies \lim_{R\to\infty} S_R f(x) = f(x) \text{ a.e. }, \forall f \in L^p, 1 < p < \infty.$$

Cesàro sum Let $\sigma_R f(x) = \frac{1}{R} \int_0^R S_t f(x) \mathrm{d}t = F_R * f(x)$, where F_R is the Fejér kernel

$$F_R(x) = \frac{1}{R} \int_0^R D_t(x) \mathrm{d}t = \frac{\sin^2(\pi R x)}{R(\pi x)^2}.$$

Then $F_R \in L^1$ and $F_R \geqslant 0$. We have $\lim_{R\to\infty} \|\sigma_R f - f\|_p = 0 \forall p \in [1, \infty)$ and $\sigma_R f \to f$ a.e. .

Abel-Poisson sum Let $u(x,t) = \int_{\mathbb{R}^n} e^{-2\pi t |\xi|} \widehat{f}(\xi) e^{-2\pi i \xi \cdot x} \mathrm{d}\xi = P_t * f(x)$, where

$$P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}, \quad \widehat{P_t}(\xi) = e^{-2\pi t |\xi|}.$$

We have $\Delta_{t,x} P_t(x) = 0$, then $\Delta u = 0$ on $\mathbb{R}_+ \times \mathbb{R}^n$. We also have $\lim_{t\to 0+} u(x,t) = f(x)$ a.e. $, \forall f \in L^p(\mathbb{R}^n)$.

Conversely, if $\Delta u = 0$ in $\mathbb{R}_+^{n+1}$, $\sup_{t>0} \int_{\mathbb{R}^n} |u(x,t)|^p \mathrm{d}x < \infty, 1 < p \leqslant \infty$. Then $\exists f \in L^p(\mathbb{R}^n)$ such that $u(x,t) = P_t * f(x)$.

Gauss sum Let $w(x,t) = \int_{\mathbb{R}^n} e^{-2\pi t^2 |\xi|^2} \widehat{f}(\xi) e^{-2\pi i \xi \cdot x} \mathrm{d}\xi = W_t * f(x)$, where W_t is the **Gauss kernel** $W_t := \overline{\mathcal{F}}(e^{-\pi t^2 |\xi|^2}) = E^n e^{-\pi |x|^2 / t}$. Let $\widetilde{W}(x,t) = W(x, \sqrt{4\pi t})$, then $\frac{\partial \widetilde{W}}{\partial t} - \Delta \widetilde{W} = 0$ in $\mathbb{R}_+^{n+1}$. We have $\lim_{t\to 0+} \widetilde{W}(x,t) = \lim_{t\to 0+} W(x,t) = f(x)$ a.e. $, \forall f \in L^p(\mathbb{R}^n)$.

2 The Hardy-Littlewood Maximal Function

§2.1 Approximations of the identity

$\phi \in L^1(\mathbb{R}^n)$, $\int \phi = 1$. For $t > 0$, let $\phi_t = t^{-n}\phi(t^{-1}x)$. Then $\phi_t \to \delta (t \to 0)$ in $\mathcal{S}'$, hence $\phi_t * g \to g (t \to 0)$.

Example 2.1.1 (Cesàro sum)
$\phi = F_1 = \frac{\sin^2 \pi x}{(\pi x)^2}$, then $F_R = \phi_{1/R}$.

Example 2.1.2 (Poisson kernel)
$\phi = P_1 = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$, then $P_t = \phi_t$.

Example 2.1.3 (Gauss kernel)
$\phi = W_1 = e^{-\pi |x|^2}$, then $W_t = \phi_t$.

Theorem 2.1.4
$\int_{\mathbb{R}^n} \phi = A$, $f \in L^p, 1 \leqslant p < \infty$ or $p = \infty$, $f \in C_0(\mathbb{R}^n)$, then $\lim_{t \to 0+} \|\phi_t * f - Af\|_p \to 0$.

Remark 2.1.5 — Then $\exists \{t_k\} \to 0$ such that $\phi_{t_k} * f \to f$ a.e. . Hence,

$$\left| \left\{ x : \lim_{t \to 0} \phi_t * f(x) \text{ exists but not equal to } f(x) \right\} \right| = 0.$$

§2.2 Weak-type inequalities and almost everywhere convergence

$(X, \mu), (Y, \nu)$ measure spaces. $T : L^p(X, \mu) \to m(Y, \nu)$ the measurable functions on Y.

Definition 2.2.1. We say T is **weak** (p, q), $q < \infty$ if $\exists C > 0$, such that

$$\nu(\{y \in Y : |Tf(y)| > \lambda\}) \leqslant \left(\frac{C \|f\|_p}{\lambda} \right)^q, \quad \forall \lambda > 0.$$

Recall strong (p, q), strong (p, ∞), and we define weak (p, ∞) same as strong (p, ∞).

Definition 2.2.2. Define the weak space as

$$L^{p,\infty}(Y, \nu) := \left\{ f \in m(Y, \nu) : \|f\|_{p,\infty} := \sup_{\lambda > 0} \lambda \nu(|Tf| > \lambda)^{\frac{1}{p}} < \infty \right\}.$$

Then weak (p,q) means $\|Tf\|_{q,\infty} \leqslant C\|f\|_p$. Besides, there holds $\|f\|_{p,\infty} \leqslant \|f\|_p$.

Theorem 2.2.3

$\{T_t\}$ are linear operators on $L^p(X,\mu)$, let $T^*f(x) = \sup_t |T_tf(x)|$. If T^* is weak (p,q), then

$$V := \left\{ f \in L^p(X,\mu) : \lim_{t\to 0} T_t(x) = f(x) \text{ a.e. } \right\}$$

is closed in $L^p(X,\mu)$.

Remark 2.2.4 — There are something ambiguous in the theorem. One should notice that the definition of T^* do **not** guarantee the measurablity of T^*f. Besides, if $f = g$ a.e. , there still might be $T^*f \neq T^*g$ on a wet with positive measure.

If $\phi \in L^1$, $\int \phi = 1$, let $T_tf = \phi_t * f$, then $\mathcal{S} \subseteq V$. Then for $1 \leqslant p < \infty$, if we can prove $\sup_{t>0} |T_tf|$ is weak (p,q), then $V = L^p$.

§2.3 The Marcinkiewicz interpolation theorem

$f : X \to \mathbb{C}$ measurable, define $a_f(\lambda) = \mu\{x \in X : f(x)| > \lambda\}, \forall \lambda > 0$.

Proposition 2.3.1

$\phi : [0,\infty) \to [0,\infty)$ C^1 increasing, $\phi(0) = 0$, then

$$\int_X \phi(|f(x)|)\mathrm{d}\mu = \int_0^\infty \phi'(\lambda)a_f(\lambda)\mathrm{d}\lambda.$$

Theorem 2.3.2 (Marcinkiewicz Interpolation Theorem)

$1 \leqslant p_0 < p_1 \leqslant \infty$, $T : L^{p_0}(X,\mu) + L^{p_1}(X,\mu) \to L^{p_0,\infty}(Y,\nu) + L^{p_1,\infty}(Y,\nu)$ sub-linear. If T is weak (p_0,p_0) and weak (p_1,p_1), then T is strong (p,p) for every $p_0 < p < p_1$.

Theorem 2.3.3

$T : L^p(X,\mu)$ weak (p,q), then $\{f : Tf = 0 \text{ a.e. }\}$ is closed in L^p.

§2.4 The Hardy-Littlewood maximal function

Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, for every $f \in L^1_{loc}(\mathbb{R}^n)$, define

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|\mathrm{d}y = \sup_{t>0} \phi_t * |f|(x), \quad \phi = \frac{1}{|B_1|}\mathbb{1}_{B_1}.$$

Besides, for $Q = [-r,r]^n$, $|Q_r| = (2r)^n$, define

$$M'f(x) = \sup_{r>0} \frac{1}{(2r)^n} \int_{Q_r} |f(x-y)|\mathrm{d}y = \sup_{t>0} \phi_t * |f|(x), \quad \phi = \frac{1}{2^n}\mathbb{1}_{Q_1}.$$

Then, if $n = 1$, we have $M = M'$. For $n \geqslant 2$, we have some controlling $c_n M'f \leqslant Mf \leqslant C_n Mf$. We can also define

$$M''f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y, \quad Q \text{ is a box.}$$

Then $M'f \leqslant M''f \leqslant 2^n M'f$.

Lemma 2.4.1

Let $\mathscr{F} = \{B_j = B(x_j, r_j)\}_{j=1}^N$ be open balls in metric space (X, d). Let $mB_j = B(x_j, mr_j)$. Then there exists $\{B_i'\}_{i=1}^l \subseteq \mathscr{F}$ such that $B_i' \cap B_j' = \varnothing$ for every $i \neq j$, and $\bigcup_{j=1}^N B_j \subseteq \bigcup_{i=1}^l 3B_i'$.

Theorem 2.4.2

$\|Mf\|_{1,\infty} \leqslant 3^n \|f\|_1$.

Remark 2.4.3 — This estimate also holds for $M', M'', \widetilde{M}$.

Define the space of non-negative radial decreasing function

$$\nu_0(\mathbb{R}^n) = \left\{\phi(x) = \phi_0(|x|) : \phi_0 : (0, \infty) \to [0, \infty) \text{ decreacing}, \phi \in L^1(\mathbb{R}^n)\right\}.$$

Proposition 2.4.4

If $\phi \in \nu_0(\mathbb{R}^n)$, then $\sup_{t>0} |\phi_t * f(x)| \leqslant \|\phi\|_1 Mf(x), \forall f \in L^1_{loc}(\mathbb{R}^n)$.

Corollary 2.4.5

If $|\phi(x)| \leqslant \psi(x) \in \nu_0(\mathbb{R}^n)$, then $f \mapsto \sup_{t>0} \phi_t * f$ is weak $(1, 1)$.

Define $\nu_1(\mathbb{R}^n) := \left\{\phi \in L^1(\mathbb{R}^n) : \exists \psi \in \nu_0(\mathbb{R}^n), |\phi| \leqslant \psi \text{ a.e. }\right\}$.

Corollary 2.4.6

If $1 \leqslant p \leqslant \infty$, $f \in L^p(\mathbb{R}^n)$, $\phi \in \nu_1(\mathbb{R}^n)$, then $\lim_{t \to 0+} \phi_t * f(x) = \left(\int_{\mathbb{R}^n} \phi\right) f(x)$ a.e. .

Recall Poisson kernel P_1, Gauss kernel W_1, Fejér kernel F_1. Then

$$P_1, W_1 \in \nu_0(\mathbb{R}^n), \quad F_1 \in \nu_1(\mathbb{R}).$$

Corollary 2.4.7

If $f \in L^1_{loc}(\mathbb{R}^n)$, then $\lim_{r \to 0+} \frac{1}{|B_r|} \int_{B_r} f(x - y) \mathrm{d}y = f(x)$ a.e. . Moreover,

$$\lim_{r \to 0+} \frac{1}{|B_r|} \int_{B_r} |f(x - y) - f(x)| \mathrm{d}y = 0 \text{ a.e. }.$$

Lemma 2.4.8

$f \in L^1_{loc}(\mathbb{R}^n)$, $f \neq 0$, then $Mf \notin L^1(\mathbb{R}^n)$.

If $Mf(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, then $f(x) = 0$ a.e. .

Theorem 2.4.9

$B \subset \mathbb{R}^n$ bounded, then $\exists C > 0$ such that

$$\int_B Mf \leqslant 2|B| + C\int_{\mathbb{R}^n} |f| \ln^+ |f| \mathrm{d}x.$$

Proof. Note that

$$\int_B Mf = 2\int_0^\infty |\{x \in B : Mf(x) \geqslant 2\lambda\}| \mathrm{d}\lambda \leqslant 2|B| + \int_1^\infty |\{x \in B : Mf(x) \geqslant 2\lambda\}| \mathrm{d}\lambda.$$

Let $f = f_1 + f_2$, where $f_1 = f\mathbb{1}_{\{x:|f(x)|>\lambda\}}$, $f_2 = f - f_1$, then $|Mf_2| \leqslant \lambda$. Hence

$$|\{x \in B : Mf(x) > 2\lambda\}| \leqslant |\{x \in B : Mf_1(x) > \lambda\}| \leqslant \frac{C}{\lambda}\int_{\{x:|f(x)|>\lambda\}} |f|.$$

Bring back to the integral and apply Fubini theorem. □

Theorem 2.4.10

Let $w \geqslant 0, w \in L^1_{loc}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\mathrm{d}x \leqslant C_p \int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x.$$

$$\int_{\{x:Mf(x)>\lambda\}} w(x)\mathrm{d}x \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|Mw(x)\mathrm{d}x.$$

Proof. WLOG, $w \in L^1(\mathbb{R}^n)$, consider $(X, \mu) = (\mathbb{R}^n, w(x)\mathrm{d}x), (Y, \nu) = (\mathbb{R}^n, Mw(x)\mathrm{d}x)$. For $w \neq 0$, we have estimate $Mw(x) \geqslant \frac{c}{1+|x|^n}$, hence $M : X \to Y$ is (∞, ∞). It suffices to show M is weak $(1, 1)$. Let $E_\lambda = \{x : Mf(x) \geqslant \lambda\}$, for all compact $K \subset E_\lambda$, we have $K \subseteq \bigcup_{j=1}^N 3B_j$ for some B_j. We claim that

$$\int_{3B_j} w(x)\mathrm{d}x \leqslant \frac{4^n}{\lambda}\int_{B_j} |f(x)|Mw(x)\mathrm{d}x.$$

It suffices to show that $4^n|B_j| \inf_{B_j} Mw(x) \geqslant \int_{3B_j} w(x)\mathrm{d}x$. Assume the radius of B_j is r_j, then for every $y \in B_j$, we have $B(y, 4r_j) \supseteq 3B_j$. Hence

$$Mw(y) \geqslant \frac{1}{4^n|B_j|}\int_{B(y,4r_j)} w(z)\mathrm{d}z \geqslant \frac{1}{4^n|B_j|}\int_{3B_j} w(x)\mathrm{d}x.$$

□

Let $\widetilde{F_N} = F_N(t)\mathbb{1}_{\{-\frac{1}{2},\frac{1}{2}\}}$, $\widetilde{P_r}(t) = P_r(t)\mathbb{1}_{\{-\frac{1}{2},\frac{1}{2}\}}$.

Lemma 2.4.11

If $f \in L^p(\mathbb{T}), 1 \leqslant p \leqslant \infty$, then

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \text{ a.e. }, \quad \lim_{r\to 1-} P_r * f(x) = f(x) \text{ a.e. }.$$

Proof. Let $\Omega_1 f = \limsup_{n\to\infty} |\sigma_n f(x) - f(x)|, \Omega_2 f(x) = \limsup_{r\to 1-} |P_r * f(x) - f(x)|$, then $\Omega_1 f = \Omega_2 f = 0$ for every $f \in C(\mathbb{T})$. Note that Ω_1, Ω_2 are both weak $(1,1)$ by the estimate of convolution. Hence $\Omega_1 f, \Omega_2 f = 0$ a.e. . □

§2.5 The dyadic maximal function

Let

$$\mathcal{Q}_k := \left\{ \prod_{i=1}^n \left[\frac{a_i}{2^k}, \frac{a_i+1}{2^k}\right) : a_1, a_2, \cdots, a_n \in \mathbb{Z} \right\}, \quad \mathcal{Q}_* = \bigcup_k \mathcal{Q}_k.$$

Then

1. $\forall x \in \mathbb{R}, k \in \mathbb{Z}$, there exists unique $Q \in \mathcal{Q}_k$ such that $x \in Q$.
2. $\forall A, B \in \mathcal{Q}_*$, then $A \cap B \varnothing$ or $A \subseteq B$ or $B \subseteq A$.
3. $\forall A \in \mathcal{Q}_k, j < k$, there exists unique $B \in \mathcal{Q}_j$ such that $A \subseteq B$.

For all $f \in L^1_{loc}(\mathbb{R}^n)$, define

$$E_k f = \sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \mathbb{1}_Q = \mathbb{E}(f|\sigma(\mathcal{Q}_k)).$$

Definition 2.5.1. Define the **dyadic maximal function** $M_d f(x) = \sup_k |E_k f(x)|$.

Remark 2.5.2 — Note that in the definition of dyadic maximal function, it does not take the absolute value of f directly. Hence, we do **not** have $M_d f = M_d |f|$ in general.

Theorem 2.5.3

The following holds:

1. M_d is weak $(1,1)$. Moreover, $\|M_d f\|_{1,\infty} \leqslant \|f\|_1$.
2. If $f \in L^1_{loc}$, then $E_k f \to f$ a.e. $(k \to +\infty)$.

Proof. For $f \geqslant 0$, for every $\lambda > 0$, let

$$E_\lambda := \{x \in \mathbb{R}^n : M_d f(x) > \lambda\},$$

then $E_\lambda = \bigcup_{k\in\mathbb{Z}} \Omega_k$, where

$$\Omega_k := \{x \in \mathbb{R}^n : E_k f(x) > \lambda, \quad E_j f \leqslant \lambda, \forall j < k\}.$$

Then Ω_k forms a disjoint union of E_λ. The following proof is easy. □

Theorem 2.5.4 (Calderón-Zygmund Decomposition)
$\forall f \in L^1(\mathbb{R}^n)$, $f \geqslant 0$, $\lambda > 0$. $\exists \{A_j\} \subset \mathcal{Q}_*$, $A_i \cap A_j = \varnothing$ for every $i \neq j$ such that

(i) $f(x) \leqslant \lambda$ for almost every $x \notin \bigcup_j A_j$.

(ii) $\left|\bigcup_j A_j\right| \leqslant \frac{1}{\lambda} \|f\|_1$.

(iii) $\lambda < \frac{1}{|A_j|} \int_{A_j} f \leqslant 2^n \lambda$ for every A_j.

Remark 2.5.5 — This theorem also gives somehow "reverse" weak $(1,1)$, that is

$$|E_\lambda| \geqslant \frac{1}{2^n \lambda} \int_{E_\lambda} f.$$

For f with $\operatorname{supp} f \subseteq Q \in \mathcal{Q}_*$, $f \geqslant 0$, we have

$$\int_Q M_d f = \int_0^\infty |E_\lambda \cap Q| \mathrm{d}\lambda \leqslant |Q| + \int_Q f \ln^+ M_d f.$$

Note that $B \ln^+ A \leqslant B \ln^+ B + \frac{A}{e}$. Take $A = M_d f$ and $B = f$ we have

$$(1 - e^{-1}) \int_Q M_d f \leqslant |Q| + \int f \ln^+ f.$$

Conversely, assume $|Q| = 1$ and $\|f\|_1 = 1$, then $E_k f(x) = 0$ for every $x \notin Q$ and $k \geqslant 0$. Moreover, $E_k f(x) \leqslant 2^{nk} \leqslant 2^{-n}$ for $x \notin Q$ and $k \leqslant -1$. Let $\lambda_0 = 2^{-n}$, then $E_\lambda \subseteq Q$ for every $\lambda > \lambda_0$. We have

$$\int_Q M_d f(x) \mathrm{d}x \geqslant \int_{\lambda_0}^\infty |E_\lambda| \mathrm{d}\lambda \geqslant \frac{1}{2^n} \int_Q f(x) \ln^+ \frac{f(x)}{\lambda_0}.$$

Proposition 2.5.6
$M_d f \in L^1(Q)$ if and only if $f \in L \ln L(Q)$.

§2.6 Some other covering lemmas

Theorem 2.6.1 (Vitali)
$\{B_j\}_{j \in \mathscr{J}}$ be a family of open balls in $\mathbb{R}^n$, then there exists countable disjoint $\{B_k\} \subseteq \{B_j\}_{j \in \mathscr{J}}$ such that $\bigcup_{j \in \mathscr{J}} B_j \subseteq_k 5B_k$.

Theorem 2.6.2 (Besicovitch)
$A \subseteq \mathbb{R}^n$ bounded, $\mathscr{F} = \{B_x\}_{x \in A}$ where $B_x = B(x, r_x)$. Then there exists countable $\{B_j\} \subseteq \mathscr{F}$ such that

$$A \subseteq \mathscr{F}, \quad \sum_j \mathbb{1}_{B_j}(x) \leqslant C_n.$$

Let μ be a Radon measure on $\mathbb{R}^n$, define

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \mathrm{d}\mu(y).$$

Corollary 2.6.3

$\|M_\mu f\|_{1,\infty} \leqslant C_n \|f\|_{L^1(\mu)}$.

Remark 2.6.4 — Note that the definition of M_μ is **different** with

$$M'_\mu f(x) := \sup_{r>0} \frac{1}{\mu(B_r)} \int_{B_r} |f(x-y)| \mathrm{d}\mu(y).$$

Because μ may not be invariant under translation. M'_μ is not weak $(1,1)$ in measure μ. Consider $X = \mathbb{R}, \mathrm{d}\mu = e^{|x|}\mathrm{d}x, f = \mathbb{1}_{\{-1,1\}}$. Then $M'_\mu f(x) = \frac{1-e^{-2}}{2} \notin L^{1,\infty}(\mu)$.

3 The Hilbert Transform

§3.1 The conjugate Poisson kernel

$\forall f \in \mathcal{S}(\mathbb{R})$, consider $u(x,t) = P_t * f(x)$, where $P_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}$. Write $z = x + it$, then

$$u(z) = \int_0^\infty \widehat{f}(\xi) e^{2\pi i \xi \cdot z} \mathrm{d}\xi + \int_{-\infty}^0 \widehat{f}(\xi) e^{2\pi i \xi \cdot \overline{z}} \mathrm{d}\xi.$$

Let

$$iv(z) = \int_0^\infty \widehat{f}(\xi) e^{2\pi i z \xi \cdot z} \mathrm{d}\xi - \int_{-\infty}^0 \widehat{f}(\xi) e^{2\pi i \xi \cdot \overline{z}} \mathrm{d}\xi,$$

then v is the conjugate harmonic function of u. We have

$$v(z) = \int_{\mathbb{R}} -i \operatorname{sgn}(\xi) e^{-2\pi t|\xi|} \widehat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}\xi = Q_t * f(x),$$

where $\widehat{Q}(\xi) = -i \operatorname{sgn}(\xi) e^{-2\pi t|\xi|}$, then $Q_t(x) = \frac{1}{\pi}\frac{x}{t^2+x^2}$. Note that

$$P_t(x) + iQ_t(x) = \frac{i}{\pi z},$$

hence Q_t is the conjugate of P_t.

$Q_t \notin L^1(\mathbb{R})$ for every $t > 0$. Even more,

$$\lim_{t \to 0} Q_t(x) = \frac{1}{\pi x} \notin L^1_{loc}(\mathbb{R}).$$

§3.2 The principal value of $1/x$

Definition 3.2.1. We define the **principal value** of $\frac{1}{x}$, which is in $\mathcal{S}'(\mathbb{R})$, by

$$\left\langle \mathrm{p.\,v.}\, \frac{1}{x}, \phi \right\rangle = \lim_{\varepsilon \to 0+} \int_{|x|>\varepsilon} \frac{\phi(x)}{x} \mathrm{d}x, \quad \forall \phi \in \mathcal{S}(\mathbb{R}).$$

To see this is a well-defined tempered distribution, we rewrite it as

$$\left\langle \mathrm{p.\,v.}\, \frac{1}{x}, \phi \right\rangle = \int_{|x|<1} \frac{\phi(x) - \phi(0)}{x} \mathrm{d}x + \int_{|x|>1} \frac{\phi(x)}{x} \mathrm{d}x.$$

Hence $\left|\left\langle \mathrm{p.\,v.}\, \frac{1}{x}, \phi \right\rangle\right| \leqslant C(\|\phi'\|_\infty + \|x\phi\|_\infty)$.