ODE: Qualitative Theory (Spring 2022, Shaobo Gan)

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Contents

1 Basic Concepts

§1.1 Basic notions and results

Assume $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, x) \mapsto f(t, x)$ continuous, consider the **equation** (or **system**)

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x).$$

A differentiable function $\gamma : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ is said to be a **solution** (or **solution curve**), if for every $t \in (a, b)$,

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = f(t, \gamma(t)).$$

The **graph** of γ is

$$\{(t, \gamma(t)) : t \in (a, b)\} \subset \mathbb{R} \times \mathbb{R}^n.$$

For $t_0 \in (a, b)$, let $x_0 = \gamma(t_0)$, then γ is called the solution of the **initial value problem**

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

The initial value problem has a unique solution: Let $\gamma_i : (a_i, b_i) \to \mathbb{R}^n$ be two solutions of the initial value problem. Then there exists $\delta > 0$, $(t_0 - \delta, t_0 + \delta) \subset (a_1, b_1) \cap (a_2, b_2)$, such that $\gamma_1(t) = \gamma_2(t), \forall t \in (t_0 - \delta, t_0 + \delta)$,

Theorem 1.1.1 (Existence and Uniqueness Theorem)
$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, f(t, x)$ continuous, given $t_0 \in \mathbb{R}^n, x_0 \in \mathbb{R}^n, a > 0, b > 0$, consider the region

$$R = R(t_0, x_0, a, b) = \{(t, x) : |t - t_0| \leqslant a, |x - x_0| \leqslant b\}.$$

If f is Lipchitz in x on R, i.e. $\exists L > 0$, $\forall (t, x_1), (t, x_2) \in R$,

$$|f(t, x_1) - f(t, x_2)| \leqslant L|x_1 - x_2|,$$

then the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution on $[t_0 - h, t_0 + h]$, where $h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(t,x)\in R} |f(t, x)|$.

Remark 1.1.2 — The solution is denoted as $\varphi(t; t_0, x_0)$.

Corollary 1.1.3

When $f \in C^1$, the existence and uniqueness theorem holds.

Denotes the **maximal interval** of $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ as $I(t_0, x_0)$, it is an open interval.

Corollary 1.1.4

Assume $f \in C^1$ and $|f(x)| \leqslant A(t)|x| + B(t)$, then the maximal interval of the initial value problem is $(-\infty, +\infty)$.

§1.2 Flows

Now we consider the **autonomous equation**

$$\dot{x} = f(x).$$

$\mathbb{R}^n$ is called the **phase space** and $\mathbb{R} \times \mathbb{R}^n$ is called the **generalized phase space**.

The solution of the initial value problem $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ is denoted as $\varphi(t, x_0)$, the set

$$\operatorname{Orb}(x_0) \coloneqq \{\varphi(t, x_0) : t \in I(x_0)\} \subset \mathbb{R}^n$$

is called the **orbit** pass by x_0.

Corollary 1.2.1 (Continuous Dependence on the Initial Value)

Assume $f \in C^1$, then $U = \{(t,x) : t \in I(x)\}$ is open and $\varphi : U \to \mathbb{R}^n, (t,x) \mapsto \varphi(t,x)$ is continuous.

Theorem 1.2.2

$f(x) \in C^1$, then:

1. $\varphi_0(x) = x$ for every $x \in \mathbb{R}^n$.
2. $\varphi(t, \varphi(s,x)) = \varphi(t+s, x)$ for every $s \in I(x), t \in I(\varphi(s,x))$.

Definition 1.2.3. $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, continuous, is said to be a **(continuous) flow** if:

(i) $\psi(0, x) = x$,

(ii) $\psi(t, \psi(s,x)) = \psi(t+s, x)$.

Remark 1.2.4 — The solution of an autonomous equation is a **local flow.**

Corollary 1.2.5

Let $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a flow, then $\psi_t \coloneqq \psi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are homeomorphisms.

Remark 1.2.6 — Consider the group of self-homeomorphisms of $\mathbb{R}^n$, denotes as $\mathrm{Homeo}(\mathbb{R}^n)$, then $\psi : \mathbb{R} \to \mathbb{R}^n$ is a group homomorphism. More generally, we can consider $G \to \mathrm{Homeo}(\mathbb{R}^n)$ for some group G.

Proposition 1.2.7

Assume f is a C^1 vector field, then the orbits of the flow generated by f are either coincide or disjoint.

$\bigcup_{x\in\mathbb{R}^n} \mathrm{Orb}(x)$ forms a partition of $\mathbb{R}^n$, is called the **orbit space**. For each orbit, orient it to indicate the direction of motion, the family of the oriented orbit $\varphi(t,x)/f(x)$ is called the **phase portrait**.

A point $x_0 \in \mathbb{R}^n$ with $f(x_0) = 0$ is called a **critical point** (or a **singularity**, **equilibrium**). The orbit $\mathrm{Orb}(x_0)$ is a single point $\{x_0\}$.

Example 1.2.8

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t,x_0) = x_0 e^t$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$.

Example 1.2.9

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t,x_0) = \frac{x_0}{1-x_0 t}$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$. But the phase portrait is different from the former examples, because the orientations on $\mathbb{R}_-$ are different.

Theorem 1.2.10

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, $\beta(x) : \mathbb{R}^n \to \mathbb{R} \in C^1$ and $\beta(x) > 0$. Then the equations $\dot{x} = f(x)$ and $\dot{x} = \beta(x)f(x)$ have the same phase portraits.

Proof. $\varphi : I \to \mathbb{R}^n$ a solution of f. Find a C^1 diffeomorphism $h : J \to I$ such that $\varphi \circ h$ is the solution of $\dot{x} = \beta(x)f(x)$. It suffices that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=h(s)} \varphi(t) \cdot \frac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi\circ h(s)) f(\varphi\circ h(s)),$$

i.e. $\frac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi\circ h(s)) > 0$, it is an initial value problem. It shows that the maximal solution curve of f is contain in some solution curve of βf. □

Theorem 1.2.11 (Differentiable Dependence on the Initial Value)
Assume $f \in C^1$, it generates the flow ϕ_t, then $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

Exercise 1.2.12.
$$\frac{\partial}{\partial t}\frac{\partial \phi(t,x)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \phi(t,x)}{\partial t}.$$

Let $\Phi(t,x) = \Phi_t(x) = \frac{\partial \phi(t,x)}{\partial t}$, then Φ is the solution of the equation

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = A(t)y(t), A(t) = Df(\phi_t(x)) \\ y(0) = \mathrm{Id} \end{cases}.$$

The equation is called the **variation equation** of $f(x)$ along $\phi_t(x)$.

Lemma 1.2.13
$f \in C^1$, $\Phi(t,x)$, then
$$\Phi_t(\phi_s(x))\Phi_s(x) = \Phi_{t+s}(x).$$

Remark 1.2.14 — This property is called the **cocycle** condition.

We already know that ϕ_t are self-homeomorphisms of $\mathbb{R}^n$, and lemma 1.2.13 shows that the differential is invertible, hence ϕ_t are diffeomorphisms. Define

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^n \times \mathbb{R}^n \\ (t,x,v) &\mapsto (\phi_t(x), \Phi_t(x)v). \end{aligned}$$

Proposition 1.2.15
$\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a flow.

Remark 1.2.16 — We call Φ_t is a skew product flow of ϕ_t.

Theorem 1.2.17
$$\Phi_t(x)f(x) = f(\phi_t(x)).$$

If ψ is a C^1 flow, let
$$g(x) = \left.\frac{\partial \psi(t,x)}{\partial t}\right|_{t=0},$$
then $\psi(t,x_0)$ solve the initial value problem $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$. Because

$$\frac{\partial \psi(t,x_0)}{\partial t} = \left.\frac{\partial \psi(t+s,x_0)}{\partial s}\right|_{s=0} = \left.\frac{\partial \psi(s,\psi(t,x_0))}{\partial s}\right|_{s=0} = g(\psi(t,x_0)).$$

§1.3 Equations on manifolds

Let M be a closed smooth manifold, X is a C^1 vector field on M. Then X is bounded, hence the maximal intervals are $(-\infty, +\infty)$. Consider the equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = X(x) \\ x(0) = x_0 \end{cases},$$

then the solution $\varphi(t, x)$ generates a flow.

2 Linear Systems

§2.1 Plane linear sigularities

Consider the equation

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}, \quad (x,y) \in \mathbb{R}^2.$$

It is said to be a **plane linear system** if f, g both linear functions of x, y, i.e.

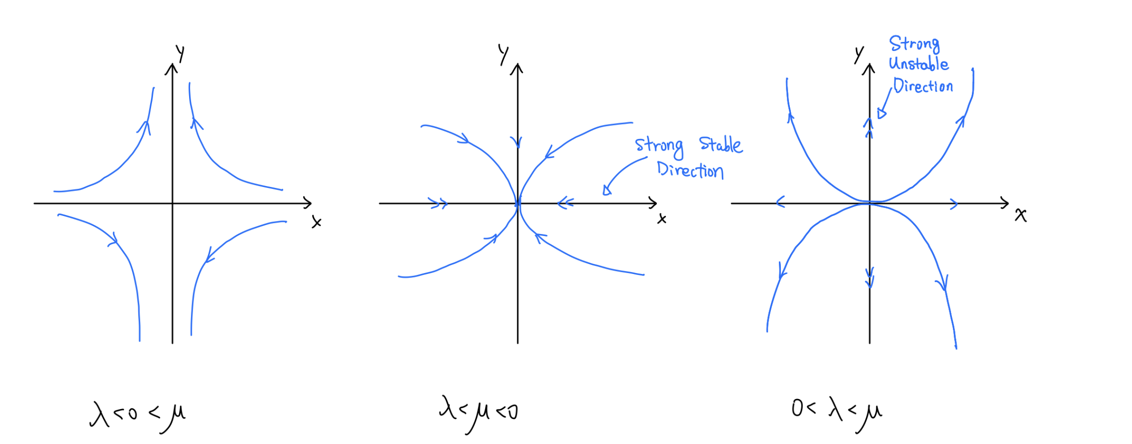
$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad a,b,c,d \in \mathbb{R}.$$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then $(0,0)$ is the only sigularity of the vector field, elementary singularity.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, consider the Jordan form of A. There are four cases:

I. Two different real eigenvalues: $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$.

   i. $\lambda < 0 < \mu$: the origin is called a **saddle point**.
   ii. $\lambda < \mu < 0$: the origin is called a **stable node**.
   iii. $0 < \lambda < \mu$: the origin is called a **unstable node**.

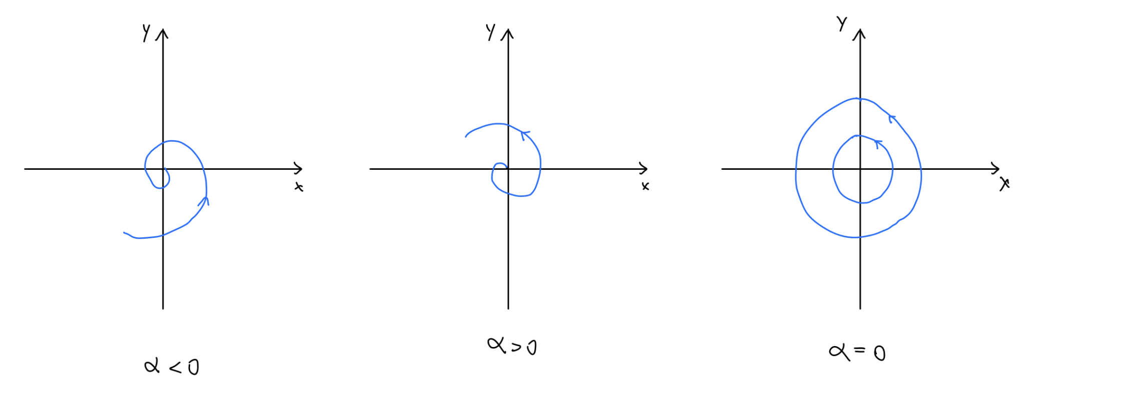


II. Conjugated imaginary eigenvalues: $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \beta > 0$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

   If we consider this equation in the polar coordinates, it turns $\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$.

   i. $\alpha < 0$, the origin is called a **stable focus**.
   ii. $\alpha > 0$, the origin is called a **unstable focus**.
   iii. $\alpha = 0$, the origin is called a **center**.

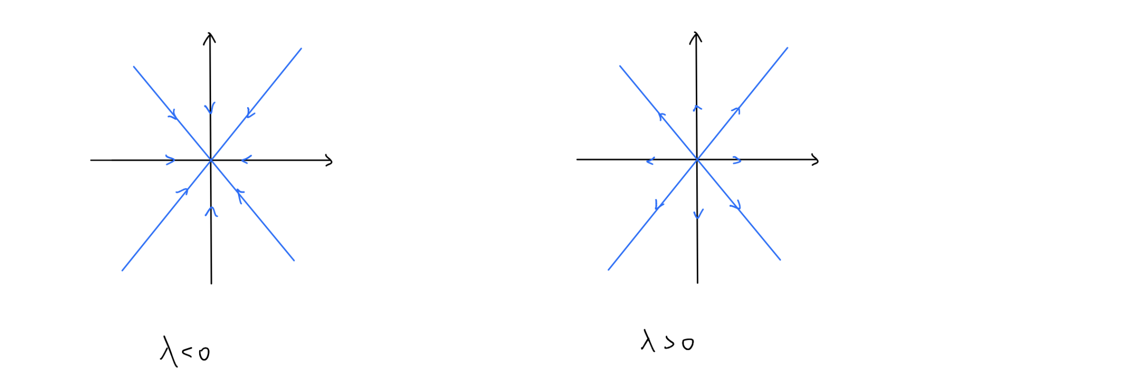
Definition 2.1.1. φ_t a flow. If p is not a singularity and $\exists T > 0$, such that $\varphi_T(p) = p$. Then p is called a **periodic point**, $\text{Orb}(p)$ is called a **periodic orbit**. If p is a periodic point, the smallest $T > 0$ is called the **minimum positive period**.



III. Two same real eigenvalues, diagonalizable: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$.

i. $\lambda < 0$, the origin is called a **stable critical node**.

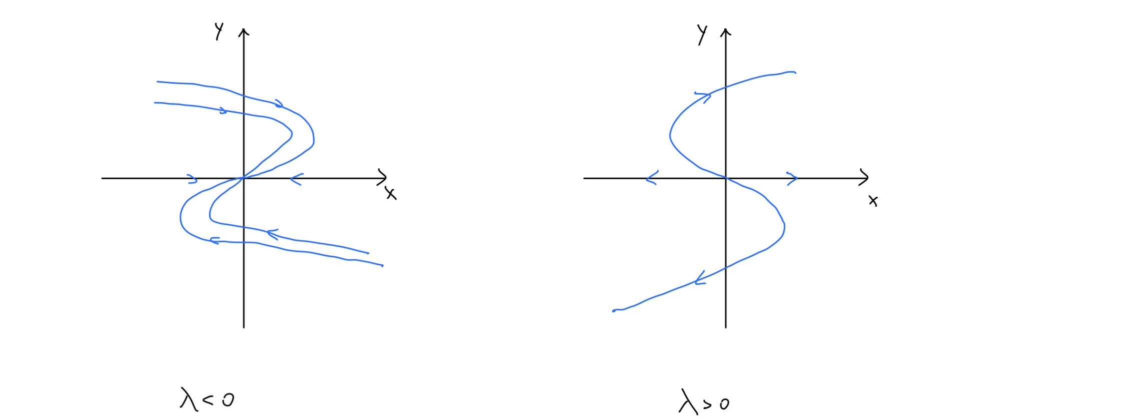
ii. $\lambda > 0$, the origin is called a **unstable critical node**.



IV. Two same real eigenvalues, not diagonalizable: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}(x_0 + t y_0) \\ e^{\lambda t} \end{bmatrix} y_0$, or $x(t) = \frac{x_0}{y_0} y(t) + \frac{y(t)}{\lambda} \ln \frac{y(t)}{y_0}$.

i. $\lambda < 0$, the origin is called a **stable unidirectional node**.

ii. $\lambda > 0$, the origin is called a **unstable unidirectional node**.



Exercise 2.1.2. Draw the phase portraits of non-elementary plane systems (i.e. the determinant is 0).

§2.2 Topological conjugacies between linear systems

Definition 2.2.1. Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ homeomorphisms. f and g are said to be **topologically conjugate** if there exists $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f = g \circ h$.

Remark 2.2.2 — Conjugacy is a equivalence relation.

Definition 2.2.3. Let $\varphi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ be two flows, we call φ_t and ψ_t are conjugate if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ \varphi_t = \psi_t \circ h$. Let X, Y be two C^1 vector fields on $\mathbb{R}^n$, we call X, Y are conjugate if the flows generated by them, respectively, are conjugate.

Example 2.2.4

$A, B \in M(n, \mathbb{R})$ are similar, then $\dot{x} = Ax$ and $\dot{y} = By$ are conjugate.

$f, g : \mathbb{R}^n \to \mathbb{R}^n$ C^1 vector fields, generate flows ϕ_t, ψ_t. Let $x = h(y) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism gives the conjugate, i.e., $h\psi_t(y) = \phi_t h(y)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} h(y) = f(h(y)) \implies D_{h(y)} g(y) = D_{h(y)} \frac{\mathrm{d}y}{\mathrm{d}t} = f(h(y)).$$

If there exists a C^1 diffeomorphism conjugate $e^{Bt}y$ to $e^{At}x$ via $x = h(y)$, i.e. $h(e^{Bt}y) = e^{At}h(y)$. Then $Dh_0 e^{Bt} = e^{At} Dh_0$, hence $Dh_0 B = ADh_0$. It shows that C^1 conjugate generically not hold even if topologically conjugate.

Proposition 2.2.5

Assume f, g C^1 vector fields generate ϕ_t, ψ_t, let h be a conjugate between ϕ_t and ψ_t. Then:

1. $h(\mathrm{Orb}(x, \phi)) = \mathrm{Orb}(hx, \psi)$.
2. h maps the singularities of f to the singularities of g.
3. h maps the periodic orbits of f to the periodic orbits of g. Moreover, it preserves the minimum positive period.

Example 2.2.6

$\dot{x} = -2x$ and $\dot{y} = -4y$ are conjugate.

Let $h : \mathbb{R} \to \mathbb{R}, h(0) = 0$. Take $x_0, y_0 > 0$, let $h(x_0) = y_0$, then $h(e^{-2t}x_0) = e^{-4t}y_0$ or $h(x) = \left(\frac{x}{x_0}\right)^2 y_0$. The construction for the negative part is similar.

Exercise 2.2.7. $\lambda\mu \neq 0$, show that $\dot{x} = \lambda x$ is conjugate to $\dot{y} = \mu y$ if and only if $\lambda\mu > 0$.

Proposition 2.2.8

$\phi_t^i, \psi_t^i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ are topologically conjugate by $h_i, i = 1, 2$. Then $\phi_t^1 \times \phi_t^2$ and $\psi_t^1 \times \psi_t^2$ are topologically conjugate by $h_1 \times h_2$.

Example 2.2.9

$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$ are conjugate.

Proof. $\phi_t(x, y) = e^{-t}(x, y)$ and $\psi_t(x, y) = e^{-t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. For every $(x, y) \neq (0, 0)$, there exists unique $t = t(x, y)$ such that $\phi_t(x, y) \in \mathbb{S}^1$. Let $h(x, y) := \psi_{-t}\phi_t(x, y)$, where $t = t(x, y)$, then h gives the conjugate. $\square$

Exercise 2.2.10. Show that $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -y \end{cases}$ are conjugate.

Classification of elementary plane linear systems:

(I) Stable: node, critical node, unidirectional node, focus.

(II) Unstable: node, critical node, unidirectional node, focus.

(III) Saddle point.

(IV) Center.

Definition 2.2.11. The linear system $\dot{x} = Ax$ in $\mathbb{R}^n$ is called **hyperbolic** if the real parts of eigenvalues of A are nonzero. The **(stable) index** of A is the number of eigenvalues with negative real parts, denoted by $\operatorname{Ind} A$.

Theorem 2.2.12

Two plane hyperbolic linear system $\dot{x} = Ax, \dot{y} = By$ are topologically conjugate if and only if $\operatorname{Ind} A = \operatorname{Ind} B$.

Proof. "$\Longrightarrow$": Let $W_A^s = \left\{x : e^{tA}x \to 0, t \to \infty\right\}$, $W_B^s = \left\{x : e^{tB}x \to 0, t \to \infty\right\}$, then h and h^{-1} preserves the stable manifolds. Then $\operatorname{Ind} A = \dim W_A^s = \dim W_B^s = \operatorname{Ind} B$. $\square$

Example 2.2.13

Consider $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ and $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$ with the same phase portraits are not topologically conjugate. Because the topologically conjugate preserves the minimum positive orbits.

Definition 2.2.14. $\phi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ flows, h is a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ maps the orbit of ϕ to the orbit of ψ preserves the orientation. Then ϕ and ψ is called **topologically equivalent** or **flow equivalent**.

Theorem 2.2.15 (Grobman-Hartman)
If x_0 is a hyperbolic singularity of $f(x)$, then the flows generated by $\dot{x} = f(x)$ and $\dot{y} = Ay$ where $y = Df(x_0)$ are topologically conjugate near 0.

§2.3 Non-autonomous linear systems

$A : \mathbb{R} \to M(n, \mathbb{R})$ continuous, consider the equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n$$

a non-autonomous linear system.

Theorem 2.3.1
The followings hold:

1. The initial problem of the equation exist the unique solution.
2. The maximal interval of any solution is $(-\infty, \infty)$.
3. All solutions of the equation form an n-dimensional linear space S.

Theorem 2.3.2 (Liouville's Formular)
Assume $X(t)$ is a solution of $\dot{x} = A(t)x$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \det X(t) = \operatorname{tr} A(t) \det X(t),$$

hence $\det X(t) = \det X(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(s) \mathrm{d}s$.

Let $X_1(t), X_2(t), \cdots, X_n(t)$ be a basis of S, let

$$X(t) \coloneqq [X_1(t), X_2(t), \cdots, X_n(t)] \in \mathrm{GL}(n, \mathbb{R}),$$

it called a **fundamental solution** of the equation. The fundamental solution of

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = A(t)X \\ X(t_0) = I_n \in \mathrm{GL}(n, \mathbb{R}) \end{cases}$$

is called the **standard fundamental solution**.

If $X(t), Y(t)$ are two fundamental solutions, suppose $Y(0) = X(0)C$, then

$$\frac{\mathrm{d}X(t)C}{\mathrm{d}t} = \frac{\mathrm{d}X(t)}{\mathrm{d}t}C = A(t)X(t)C,$$

is a non-degenerate solution of $\frac{\mathrm{d}X}{\mathrm{d}t} = AX$. By the uniqueness, we get $Y(t) = X(t)C$.

Example 2.3.3

$A(t) \equiv A$, the fundamental solution of $\dot{x} = Ax$ is

$$e^{tA} = \mathrm{Id} + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{k!}t^kA^k + \cdots .$$

Example 2.3.4

$\dot{x} = f(x), x \in \mathbb{R}^n$, where $f \in C^1$, generates the flow $\varphi_t(x)$. Consider $\Phi_t(x) = \frac{\partial}{\partial t}\varphi_t(x)$ and the variation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = Df_{\varphi_t(x)}\Phi_t(x).$$

Given $x \in \mathbb{R}^n$, let $A(t) := Df_{\varphi_t(x)}$, then $\Phi_t(x)$ is the standard fundamental solution $(t_0 = 0)$ of $\dot{x} = A(t)x$. Consider two special types of orbits:

- x is a singularity, denoted by σ. Then $\varphi_t(\sigma) = \sigma$, $\dot{x} = Ax$ where $A = Df(\sigma)$.
- x is a periodic point, denoted by p, the minimum period $T > 0$. Then A is T-periodic.

§2.4 Periodic linear systems

Definition 2.4.1. The equation $\dot{x} = A(t)x$ satisfies $A(t+T) = A(t)$ for some $T > 0$ is called a **periodic linear systems**.

Theorem 2.4.2 (Floquet)

Assume $\dot{x} = A(t)x$ is a T-periodic linear system, if X is a fundamental solution, then $X(t+T)$ is a fundamental solution, i.e. $\exists C \in \mathrm{GL}(n, \mathbb{R})$ such that $X(t+T) = X(t)C$. Moreover, there exists a T-periodic map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ and a constant matrix $B \in M(n, \mathbb{C})$ such that $X(t) = P(t)e^{tB}$.

Lemma 2.4.3

$\forall C \in \mathrm{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{C})$ such that $C = e^B$.

Proof. It suffices to show for Jordan block. This follows by the matrix series

$$\ln(I + N) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} N^k$$

is convergence for nilpotent matrix N. □

Lemma 2.4.4

$\forall C \in \mathrm{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{R})$ such that $C^2 = e^B$.

Proof. Note that the Jordan block of C^2 is either:

(i) $\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & \lambda & 1 \\ 0 & \cdots & & \lambda \end{bmatrix}$, where $\lambda > 0$, or

(ii) $\begin{bmatrix} J & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & J & I_2 \\ 0 & \cdots & & J \end{bmatrix}$, where $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a, b \in \mathbb{R}, b > 0$.

And $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ have a real matrix logarithm because $\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\} \cong \mathbb{C} = \{a + bi\}$. □

Theorem 2.4.5 (Real Form of Floquet Theorem)
Assume $\dot{x} = A(t)x$ is a T-periodic linear system, if X is a fundamental solution. Then there exists a **$2T$-periodic** map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ and a constant matrix $B \in M(n, \mathbb{R})$ such that $X(t) = P(t)e^{tB}$.

Example 2.4.6 ($2T$ is necessary)
Let $\Phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t \right) \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right)$. Let

$$A(t) = \dot{\Phi}(t)\Phi(t)^{-1} = \begin{bmatrix} -\cos t \sin t & -\sin^2 t \\ \cos^2 t & \cos t \sin t \end{bmatrix},$$

then $A(t)$ is π-periodic. Then $\Phi(t)$ is a standard fundamental solution of $\dot{x} = A(t)x$, hence $\exists \pi$-periodic $P(t)$ and B such that $\Phi(t) = P(t)e^{tB}$. Then $e^{\pi B} = \begin{bmatrix} -1 & -\pi \\ 0 & -1 \end{bmatrix}$, there is no real matrix B satisfying this equation.

Definition 2.4.7. In Floquet theorem, $X(t + T) = X(t)C$. We call C is a **monodromy matrix**. The eigenvalues of C are called **Floquet multipliers**. If ρ is a Floquet multiplier with $\rho = e^{\lambda T}$, then λ is called a **Floquet exponent**.

Corollary 2.4.8
Consider a T-periodic linear system $\dot{x} = A(t)x$. Then there exists a linear transformation (non-autonomous) $x = P(t)y$ such that $\dot{y} = By$.

Proof. Let $X(t) = P(t)e^{tB}$ be a fundamental solution, then

$$AX = \dot{X} \implies \dot{P}e^{tB} + PBe^{tB} = APe^{tB},$$

hence $\dot{P} + PB = AP$. Then $APy = \frac{\mathrm{d}}{\mathrm{d}t}(Py) = \dot{P}y + P\dot{y}$, hence $\dot{y} = By$. □

Remark 2.4.9 — This type of equation is called reducible, which means after some reduction, the equation can become independent with time t.

Corollary 2.4.10

Let λ be a Floquet multiplier of $\dot{x} = A(t)x$. Then there exists a T-periodic function $p(t)$ such that $e^{\lambda t}p(t)$ is a solution of the equation $\dot{x} = A(t)x$.

Proof. $e^{\lambda T}$ is an eigenvalue of C, then $\exists x_0$ such that $Cx_0 = e^{\lambda T}x_0$. Then $X(t)x_0$ is a solution. Let $p(t) = e^{-\lambda t}X(t)x_0$ is T-periodic and $e^{\lambda t}p(t)$ is a solution. □

Corollary 2.4.11

The equation admits a nonzero T-periodic solution if and only if 1 is a Floquet multiplier.

Corollary 2.4.12

Assume $\rho_1, \rho_2, \cdots, \rho_n$ are all Floquet multipliers of $\dot{x} = A(t)x$, then

$$\rho_1\rho_2\cdots\rho_n = \det\Phi(T) = \exp\int_0^T \operatorname{tr} A(t)\ \mathrm{d}t.$$

Example 2.4.13

The equation $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^2 t & \frac{1}{2}\sin 2t - 1 \\ \frac{1}{2}\sin 2t + 1 & \sin^2 t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ has an unbounded solution. Because the product of two multipliers is $\exp\int_0^\pi 1\ \mathrm{d}t = e^\pi > 1$.

Consider **Hill equation**

$$\ddot{x} + p(t)x = 0,$$

where $p(t)$ is π-periodic. This is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -p(t)x \end{cases},$$

then $\rho_1\rho_2 = \exp\int_0^\pi \operatorname{tr} A(t)\ \mathrm{d}t = 0$, where $A(t) = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}$.

Lemma 2.4.14

If ρ_1, ρ_2 both are imaginary numbers, then every solution of Hill equation is bounded.

Proof. Because ρ_1, ρ_2 are conjugate imaginary numbers, hence $\Phi(\pi)$ is similar to a rotation. Then $\Phi(\pi)^n$ is bounded independent of n and $\Phi(s)$ is bounded for $s \in [0, \pi]$. □

Definition 2.4.15. A particular Hill equation with $p(t) = a + \varepsilon\cos 2t$ is called **Mathieu equation**.

Exercise 2.4.16. Consider Mathieu equation

$$\ddot{x} + (a + \varepsilon \cos 2t)x = 0.$$

(1) $U = \{(a, \varepsilon) \in [0, 10] \times [-1, 1] :$ every solution is bounded$\}$. Draw the figure of U by some calculation.

(2) Guess some conclusions by the figure of U.

Example 2.4.17

Let $p(t)$ be a π-periodic continuous function satisfying

(i) $p(t) \not\equiv 0$.

(ii) $\int_0^\pi p(t)\mathrm{d}t \geqslant 0$.

(iii) $\pi \int_0^\pi |p(t)|\mathrm{d}t \leqslant 4$.

Then every solution of $\ddot{x} + p(t)x = 0$ is bounded.

Proof. If Floquet multipliers are conjugate imaginary numbers, the statement follows. Otherwise there is a real Floquet multiplier $\rho \neq 0$. There is a solution $x(t) \not\equiv 0$ such that $x(t+T) = \rho x(t)$. If $x(t)$ has no zeros, assume $x(t) > 0$, we have $\frac{\dot{x}}{x}(\pi) = \frac{\dot{x}}{x}(0)$. Note that

$$0 = \frac{\ddot{x}}{x} + p(t) = \left(\frac{\dot{x}}{x}\right)' + \left(\frac{\dot{x}}{x}\right)^2 + p(t) = 0,$$

take the integral and we get a contradiction. Then there must be some zeros, let a, b be two successive zeros, WLOG, $0 < a < b < \pi$. Assume $x(t) > 0$ in (a, b) and $x(c)$ takes the maximum. Then $\exists \alpha \in (a, c), \beta \in (c, b)$ such that $\dot{x}(\alpha) = \frac{x(c)}{c-a}, \dot{x}(\beta) = \frac{-x(c)}{b-c}$. We have

$$\frac{4}{\pi} \geqslant \int_0^\pi |p(t)|\mathrm{d}t > \int_a^b \left|\frac{\ddot{x}}{x}(t)\right| \mathrm{d}t \geqslant \frac{\int_\alpha^\beta |\ddot{x}(t)|\mathrm{d}t}{x(c)} \geqslant \frac{1}{c-a} + \frac{1}{b-c} \geqslant \frac{4}{a-b},$$

the identity holds if and only if $x \equiv 0$, contradiction. □

Back to Mathieu equation, consider

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0, \quad \omega > 0, \varepsilon < \omega^2.$$

We apply the conclusion of the example, for $\omega < \frac{2}{\pi}$,

$$\int_0^\pi (\omega^2 + \varepsilon \cos 2t)\mathrm{d}t = \omega^2 \pi \leqslant \frac{4}{\pi}.$$

Consider $\varepsilon = 0$, then

$$\Phi(t) = \begin{bmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

is a standard fundamental solution. The monodromy matrix for (ω, ε) where $\omega > 0$ is a perturbation of

$$C = \Phi(\pi) = \begin{bmatrix} \cos \omega\pi & \frac{1}{\omega} \sin \omega\pi \\ -\omega \sin \omega\pi & \cos \omega\pi \end{bmatrix}.$$

Note that $|\operatorname{tr} \Phi(\pi)| = |2 \cos \omega\pi| < 2$ for $\omega \notin \mathbb{Z}$. Then there is a small neighborhood U of $(\omega, 0)$ such that every solution is bounded.

Definition 2.4.18. Let $A : \mathbb{R} \to M(n, \mathbb{R})$ continuous, bounded, assume that

$$\sup \{|A(t)| : t \in \mathbb{R}\} < \infty.$$

Let $\Phi(t)$ be a standard fundamental solution of the equation $\dot{x} = A(t)x$. For every $v \neq 0 \in \mathbb{R}^n$, define **Lyapunov exponent** of v

$$\chi(v) := \limsup_{t\to\infty} \frac{\ln \|\Phi(t)v\|}{t}.$$

> **Exercise 2.4.19.** For every $v \neq 0$, show that $\chi(v) \neq \pm\infty$.

Then $\chi : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties

1. $\chi(\alpha v) = \chi(v)$ for every $\alpha \neq 0$.
2. $\chi(v + w) \leqslant \max \{\chi(v), \chi(w)\}$.
3. If $\chi(v) < \chi(w)$, then $\chi(v + w) = \chi(w)$.

Fact 2.4.20. The number of different Lyapunov exponents $\leqslant n$.

Example 2.4.21

$\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and A is a constant matrix. Regard as a T-periodic system, then the eigenvalues λ_1, λ_2 of A are Floquet exponents. Lyapunov exponents are

(1) λ_1, λ_2, if $\lambda_1 \neq \lambda_2$ real.

(2) $\lambda = \lambda_1 = \lambda_2$, if $\lambda_1 = \lambda_2$.

(3) α, if $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$.

For the T-periodic system, assume that λ is a Floquet exponent, then $\chi = \mathrm{Re}(\lambda)$ is a Lyapunov exponent. For $n = 2$, T-periodic system, we always have

$$\chi_1 + \chi_2 = \mathrm{Re}(\lambda_1 + \lambda_2) = \frac{1}{T} \int_0^T \mathrm{tr}\, A(t)\mathrm{d}t.$$

Example 2.4.22

Consider

$$\begin{cases} \dot{x} = (-\mu - (\sin \ln t + \cos \ln t))x \\ \dot{y} = (-\mu + (\sin \ln t + \cos \ln t))y \end{cases},$$

then the solution

$$\begin{cases} x = C_1 e^{-\mu t - t \sin \ln t} \\ y = C_2 e^{-\mu t + t \sin \ln t} \end{cases}.$$

Then $\chi(v) = -\mu+1$ for every $v \neq 0$. But $\chi_1+\chi_2 = -2\mu+2 \neq \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{tr}\, A(t)\mathrm{d}t = -2\mu$. This example is called non-regular.

3 Stability

§3.1 Lyapunov stability

Let $f : \mathbb{R}^n \to \mathbb{R}^n, 0 \in \mathbb{R}^n, f(0) = 0$, generates a (local) flow $\varphi_t(x)$.

Definition 3.1.1. 1. σ is called **(forward Lyapunov) stable**, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that if $|x - \sigma| < \delta$, then $|\varphi_t(x) - \sigma| < \varepsilon$ for $t \geqslant 0$. Otherwise, we call σ is **unstable**.

2. σ is called **asymptotically stable**, if
   (i) σ is stable,
   (ii) there exists $\delta_0 > 0$, such that if $|x - \sigma| < \delta$, then $\lim_{t\to\infty} \varphi_t(x) = \sigma$.

3. σ is called **exponentially stable**, if exists $\delta_0 > 0$, $C \geqslant 1, \lambda > 0$, such that if $|x - \sigma| < \delta$, then $|\varphi_t(x) - \sigma| \leqslant Ce^{-\lambda t}|x - \sigma|$ for $t \geqslant 0$.

Similarly, we can define backward stable, backward asymptotically stable, backward exponentially stable.

Remark 3.1.2 — If we replace the condition of stability by **given** $t \geqslant 0$, then it always holds by the continuous independence of solutions with respect to initial value.

Example 3.1.3

For the equation in polar coordinates

$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2(\theta/2) \end{cases}.$$

Then the fixed point $(1, 0)$ satisfy the second condition of asymptotically stable but it is **not** stable.

In general, we can prove that if $\varphi_t(x) \not\equiv \sigma$ and $\lim_{t\to\pm\infty} \varphi_t(x) = \sigma$, then σ is not stable.

Example 3.1.4

Consider the linear elementary singularities, recall the classification, then

1. Stable type: forward stable.
2. Unstable type: unstable, bet backward stable.
3. Saddle point: unstable.
4. Center: forward and backward stable.

Theorem 3.1.5

Let $A \in M(n, \mathbb{R})$, consider the equation $\dot{X} = AX$, 0 is a singularity, then

1. 0 is stable iff each eigenvalue of A is with non-positive real part and Jordan block are trivial for every eigenvalue with zero real part.

2. 0 is asymptotically stable iff 0 is exponentially stable iff every eigenvalue of A is with negative real part.

Lemma 3.1.6 (Gronwall's Inequality)

Let $u : [0, T] \to \mathbb{R}$ non-negative, continuous. If $C \geqslant 0, K > 0$ such that for every $t \in [0, T]$,

$$u(t) \leqslant C + K \int_0^t u(s)\mathrm{d}s,$$

then $u(t) \leqslant Ce^{Kt}$ for $t \in [0, T]$.

Theorem 3.1.7

$f : \mathbb{R}^n \to \mathbb{R}^n$, C^1, $f(\sigma) = 0$. Assume that every eigenvalue of $A = Df(0)$ is with negative real part, then σ is exponentially stable.

Proof. There $\exists C \geqslant 1, \mu > 0$, such that $|e^{At}| \leqslant Ce^{-\mu t}$ for $t \geqslant 0$. WOLG, $\sigma = 0$. Let $f(x) = Ax + g(x)$ where $g(x) = o(|x|)$, let $\varphi_t(x)$ be a maximal solution of the initial value problem. Then

$$e^{-tA}(\dot{\varphi}_t(x) - A\varphi_t(x)) = e^{-tA}g(\varphi_t(x)),$$

hence

$$\varphi_t(x) = e^{tA}x + \int_0^t e^{(t-s)A}g(\varphi_s(x))\mathrm{d}s.$$

Fix $\varepsilon_0 > 0$ to be determined later, $\exists \delta_0 > 0$ such that $|g(x)| \leqslant \varepsilon_0|x|$ if $|x| \leqslant \delta_0$. Assume the right maximal interval of φ_t is $[0, \beta), \beta > 0$. Let

$$T^* = T^*(x) = \sup\left\{t < \beta : \varphi_{[0,t]}(x) \subseteq \overline{B(\delta_0, \sigma)}\right\}.$$

Then, for every $|x| \leqslant \delta_0, 0 \leqslant t \leqslant T^*$, we have

$$e^{\mu t}|\varphi_t(x)| \leqslant C|x| + C\varepsilon_0 \int_0^t e^{s\mu}|\varphi_s(x)|\mathrm{d}s.$$

By Gronwall's inequality, we have $|\varphi_t(x)| \leqslant C|x|e^{-(\mu - C\varepsilon_0)t}, \forall t < T^*$. Let $C\varepsilon_0 = \frac{\mu}{2}$ is enough. For all $|x| \leqslant \frac{\delta_0}{2C}$, then $|\varphi_t(x)| \leqslant \frac{\delta_0}{2}e^{-\mu t}$ for every $t < T^*$. Then we can show that $T^* = \beta = \infty$ and φ_t is exponentially stable. $\square$

Proposition 3.1.8

f, g, C^1 vector fields. Assume f, g are topologically conjugate, i.e. , $h \circ \varphi_t = \psi_t \circ h$ where φ_t, ψ_t are flows generated by f, g, respectively. Let $\sigma, h\sigma$ be singularities of f, g, respectively, then σ is stable if and only if $h\sigma$ is stable.

Now, we state a celebrated theorem, Hartman-Grobman Theorem. But we will not give a proof here.

Theorem 3.1.9 (Hartman-Grobman)
Let σ be a hyperbolic singularity of f. Then there exists a neighborhood $V \ni \sigma$ and a homeomorphism $h : V \to \mathbb{R}^n$ onto its image, $h(\sigma) = 0$, such that $h \circ \varphi_t(x) = Df(\sigma) \circ h(x)$ for every $x, \varphi_t(x) \in V$.

§3.2 Lyapunov functions

Definition 3.2.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$, be a C^1 vector field, $f(0) = 0$. A C^1 function $V : D \to \mathbb{R}$ where D is a neighborhood of σ is called a **Lyapunov function** of f (for σ) if

(i) $V(\sigma) = 0, V(x) > 0, \forall x \in D \setminus \{\sigma\}$.

(ii) $\forall x \in D \setminus \{\sigma\}, \dot{V}(x) \leqslant 0$, where

$$\dot{V}(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} V(\varphi_t(x)) = DV(x)f(x).$$

V is called a **strict Lyapunov function** if $\dot{V}(x) \leqslant 0$ is replaced by $\dot{V}(x) < 0$.

Theorem 3.2.2
Assume σ is a singularity of f, if there is a Lyapunov function for σ, then σ is stable. If there is a strict Lyapunov function for σ, then σ is asymptotically stable.

Proof. Let V be a Lyapunov function, for every $\varepsilon > 0$, assume $B_\varepsilon(\sigma) = \{x : |x - \sigma| \leqslant \varepsilon\} \subseteq D$. Let $m = \min\{V(x) : x \in \partial B_\varepsilon(\sigma)\} > 0$, take $\delta > 0$ such that $V(x) < m, \forall x \in B_\delta(\sigma)$. By $\dot{V}(x) \leqslant 0$, we have that every solution curve start at $x \in B_\delta(\sigma)$ can not reach $\partial B_\varepsilon(\sigma)$.

If $\dot{V}(x) < 0$ for every $x \in D \setminus \{\sigma\}$, it suffices to show that each convergent subsequence of $\varphi_t(x)$ converges to σ. Otherwise, assume converges to $y \neq \sigma$, but $\dot{V}(y) < 0$, there is some $s > 0$ such that $V(\varphi_s(y)) < V(y)$. Contradiction. ☐

Example 3.2.3
Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = 0$, hence 0 is stable.

Example 3.2.4

Consider the equation

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - y^3 \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2 - 2y^4 < 0$, hence 0 is asymptotically stable.

Example 3.2.5

Consider the equation

$$\begin{cases} \dot{x} = -x - y + x^2 \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2(1 - x) \leqslant 0$, hence 0 is stable. In fact, 0 is asymptotically stable, but we need to consider another Lyapunov function $Q(x, y) = x^2 + y^2 + xy$.

Theorem 3.2.6

If V is a Lyapunov function of f, assume

$$K = \left\{ x \in D \setminus \{\sigma\}, \dot{V}(x) = 0 \right\}$$

does not contain any complete positive orbit $\varphi_{[0,\infty)}(x)$, then σ is asymptotically stable.

Example 3.2.7

Let $f : \mathbb{R} \to \mathbb{R}, C^1, f(0) = 0$, satisfying $xf(x) > 0, \forall x \neq 0$. Consider the stability of $\ddot{x} + f(x) = 0$, or

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}.$$

Let

$$E(x, y) = \frac{1}{2}y^2 + \int_0^x f(z)\mathrm{d}z$$

be an energy function, then $\dot{E}(x, y) \equiv 0$.

Example 3.2.8

Let $V : \mathbb{R}^n \to \mathbb{R}, C^2$, the gradient of V is

$$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}.$$

The system $\dot{x} = -V(x)$ is called the **gradient system** generated by V. Then,

1. $\dot{V}(x) \leqslant 0$.
2. σ is a singularity if and only if $\dot{V}(\sigma) = 0$.
3. If σ is a minimum point of $V(x)$, then σ is stable.

Theorem 3.2.9

Let σ be a singularity of C^1 vector field f, a C^1 function $V : D \to \mathbb{R}$ satisfies

(i) $V(\sigma) = 0$, and V can take positive value on any neighborhood of σ.

(ii) $\dot{V}(x) > 0, \forall x \in D \setminus \{0\}$.

Then σ is unstable.

Example 3.2.10

Consider the equation

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}.$$

Let $V(x, y) = x^2 - y^2$, then $\dot{V}(x, y) = 2x^2 + 2y^2 > 0$, hence 0 is unstable.

Theorem 3.2.11

Let f be a C^1 vector field, $f(\sigma) = 0$. If σ is stable, then every eigenvalue of $Df(\sigma)$ is with non-negative real part.

Proof. Prove for $n = 2$. Assume $\sigma = 0$, the equation is

$$\begin{cases} \dot{x} = \lambda x + \alpha(x, y) \\ \dot{y} = \mu y + \beta(x, y) \end{cases},$$

where $\lambda < \mu, \mu > 0, |\alpha|, |\beta| = o(r)$. Let $V(x, y) = -x^2 + y^2$, then

$$\dot{V}(x, y) = -2\lambda x^2 + 2\mu y^2 - 2x\alpha + 2y\beta.$$

If $\lambda < 0$, then $\dot{V} > 0$ in a neighborhood of 0, then 0 is unstable. If $\lambda \geqslant 0$, consider

$$C = \{(x, y) : V(x, y) \geqslant 0\}.$$

We can show that for some $\varepsilon_0 \geqslant 0$, $\dot{V}(x,y) > 0$ on $C \cap B(0,\varepsilon_0) \setminus \{0\}$. Let $H(x,y) = x^2 + y^2$, then $\dot{H}(x,y) \geqslant \frac{\mu}{2} H(x,y)$ on some neighborhood of 0. Hence

$$H(\varphi_t(x,y)) \geqslant H(x,y) e^{\frac{\mu}{2}t}$$

will be out of $C \cap B_\varepsilon(x,y)$. $\square$

Remark 3.2.12 — In fact, there exists $(x,y) \in B(0,\varepsilon_0) \setminus \{0\}$, such that

$$\lim_{t \to -\infty} \varphi_t(x,y) = 0, \quad \frac{f(\varphi_t(x,y))}{|f(\varphi_t(x,y))|} \to (0,1).$$

$\varphi_t(x,y)$ is called the unstable manifold.

Exercise 3.2.13. Prove the theorem for general dimension n.

Now, we consider a perturbation of a singularity of center type. Consider the system

$$\begin{cases} \dot{x} = -y + \alpha(x,y) \\ \dot{y} = x + \beta(x,y) \end{cases},$$

then

$$\dot{\theta} = 1 + \frac{x\beta - y\alpha}{x^2 + y^2},$$

$$\dot{r} = \frac{x\alpha + y\beta}{r} = \alpha \cos\theta + \beta \sin\theta = R_2(\theta) r^2 + R_3(\theta) r^3 + \cdots .$$

Example 3.2.14

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + (x^2 y + x^3) \end{cases}.$$

Then

$$\dot{r} = \sin\theta (x^2 y + x^3) = r^3 (\cos^2\theta \sin^2\theta + \cos^3\theta \sin\theta),$$

we calculate

$$\overline{R}_3 = \int_0^{2\pi} R_3(\theta) \mathrm{d}\theta = \frac{\pi}{4} > 0.$$

Let $g(\theta) = \int_0^\theta R_3(\theta) \mathrm{d}\theta$, then

$$\varphi_3(\theta) = g(\theta) - \frac{\theta}{2\pi} \int_0^{2\pi} R_3(\theta) \mathrm{d}\theta$$

is 2π-periodic. Let $r = \rho + \varphi_3(\theta) \rho^3$, then

$$\frac{\mathrm{d}\rho}{\mathrm{d}\theta} = \overline{R}_3 \rho^3 + \cdots ,$$

hence ρ is increasing. Therefore, 0 is unstable.

Example 3.2.15

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We want to construct a Lyapunov function of the form $V(x, y) = x^2 + y^2 + F(x, y)$, where $F(x, y)$ is a homogeneous polynomial of deg $= 3$. Then

$$\dot{V}(x, y) = -yF_x + xF_y + 2y^3 + y^2F_y,$$

we want $-yF_x + xF_y + 2y^3 = 0$. Consider $L : H_k \to H_k$, where H_3 is the family of homogeneous polynomials of deg $= k$, $L(F) = -yF_x + xF_y$. After repetition, we can let

$$V(x, y) = \lambda(x^2 + y^2)^k + \cdots .$$

Then 0 is stable if $\lambda < 0$, 0 is unstable if $\lambda > 0$. Or we can find V such that $\dot{V}(x, y) = 0$, then 0 is still a center.

Example 3.2.16

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We can solve this equation,

$$y^2 = -x + \frac{1}{2}(1 - e^{-2x}) + Ce^{-2x},$$

hence $e^{2x}(x^2 + y^2) = C + \cdots$. 0 is still a center.

Example 3.2.17

Consider the equation

$$\begin{cases} \dot{x} = -y & = X(x, y) \\ \dot{y} = x + y^2 & = Y(x, y) \end{cases}.$$

Notice that $X(x, -y) = -X(x, y), Y(x, -y) = Y(x, y)$, hence the solution curve is symmetric with respect to x-axis. We can prove this fact by showing $(x(-t), -y(-t))$ is a solution if $(x(t), y(t))$ is a solution. Then we can show 0 is a center.

§3.3 Stability under perturbations

Definition 3.3.1. Consider an autonomous system $\dot{x} = f(x)$, generating a flow φ_t. For every $x_0 \in \mathbb{R}^n$, the orbit $\varphi_t(x_0)$ is said to be **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi_t(x) - \varphi_t(x_0)| < \varepsilon, \quad \forall t \geqslant 0, x \in B(x_0, \delta).$$

Example 3.3.2

Consider the equation

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = r^2 \end{cases}.$$

Then the orbit of $(r_0, \theta_0) = (1, 0)$ is **not** stable.

Definition 3.3.3. Consider a non-autonomous system $\dot{x} = f(x, t)$, let $\varphi(t; t_0, x_0)$ be the solution of the initial value problem $x(t_0) = x_0$. The orbit $x(t; t_0, x_0)$ is said to be **stable**, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi(t; t_0, x) - \varphi_t(t; t_0, x_0)| < \varepsilon, \quad \forall t \geqslant t_0, x \in B(x_0, \delta).$$

Similarly, we can define the asymptotically stable and the exponentially stable for general orbits of autonomous or non-autonomous systems.

Theorem 3.3.4

$A : \mathbb{R} \to M(n, \mathbb{R})$, consider a non-autonomous system $\dot{x} = A(t)x$. Then

1. Every solution is stable iff 0 is stable.
2. 0 is stable iff $\sup_{t \geqslant 0} |X(t)| < \infty$, where $X(t)$ is a fundamental solution.
3. 0 is asymptotically stable iff $\lim_{t \to \infty} |X(t)| = 0$.

Theorem 3.3.5

Consider a T-periodic system $\dot{x} = A(t)x$. Then

2. 0 is stable iff the Floquet exponents are of non-positive real parts and Jordan block are trivial for every Floquet exponent with zero real part.

2. 0 is asymptotically stable iff Floquet exponents are of negative real parts iff 0 is exponentially stable.

For an autonomous system, let $f(0) = 0, f(x) = Ax + \varphi(x)$, where $\varphi(0) = 0, D\varphi(0) = 0$. Rewrite the system as $\dot{x} = Ax + \varphi(x)$, if every eigenvalue of A is with negative real parts, then 0 is stable.

For a non-autonomous system, assume

$$\dot{x} = Ax + \varphi(t, x), \quad , \varphi(t, 0) = 0, D\varphi(t, 0) = 0,$$

if every eigenvalue of A is with negative real parts, then 0 is stable. In general,

$$\dot{x} = A(t)x + \varphi(t, x),$$

but the negativeness of Lyapunov exponents do **not** imply the stableness. See the following example.

Example 3.3.6

Consider

$$\begin{cases}\dot{x} = (-\mu - (\sin\ln t + \cos\ln t))x \\ \dot{y} = (-\mu + (\sin\ln t + \cos\ln t))y + x^2\end{cases},$$

let $a(t) = t\sin\ln t$, the solutions are

$$\begin{cases}x = C_1 e^{-\mu t - a(t)} \\ y = C_2 e^{-\mu t + a(t)} + C_1^2 e^{-\mu t + a(t)} \displaystyle\int_1^t e^{-\mu s - 3a(s)} \mathrm{d}s\end{cases}.$$

For $\mu = 1 + \sigma, \sigma$ is sufficiently small, then 0 is not stable.

For this case, we need a stronger condition. Let $\Phi(t)$ be a fundamental solution of the linear part, if $\exists \mu > 0$,

$$|\Phi(t)\Phi(-s)| \leqslant Ce^{-\mu(t-s)}, \quad \forall t \geqslant s \geqslant 0,$$

then 0 is also stable under the perturbation .

4 Poincaré-Bendixson Theory

§4.1 Basic notions

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, generating a flow $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 4.1.1. $A \subseteq \mathbb{R}^n$ is said to be $f(\varphi_t)$ **invariant** if for every $t \in \mathbb{R}, \varphi_t(A) = A$.

For every $x \in \mathbb{R}^n$, the orbit $\mathrm{Orb}(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$ is an invariant set. In general, if A is invariant, then

$$A = \mathrm{Orb}_{x \in A}\mathrm{Orb}(x).$$

Definition 4.1.2. Let A be a compact invariant set, A is said to be **Lyapunov orbit stable** if for every neighborhood $U \supseteq A$, there exists a neighborhood $V \supseteq A$ such that

$$\varphi_t(x) \in U, \quad \forall x \in V, t \geqslant 0.$$

Let

$$\mathrm{Orb}^+ := \{\varphi_t(x) : t \geqslant 0\}, \quad \mathrm{Orb}^- := \{\varphi_t(x) : t \leqslant 0\}$$

be the **positive semi-orbit** and the **negative semi-orbit**.

Definition 4.1.3. Given $p \in \mathbb{R}^n$, x is called a **positive limit point** if $\exists t_n \to +\infty$, $\varphi_{t_n} \to x$. The set of all positive limit points is called the **α-limit set** of p, denoted by $\alpha(p)$. Similarly, we can define the **negative limit points**, they form a set is called **ω-limit set**, denoted by $\omega(p)$.

> **Remark 4.1.4** — α is the first Greek letter and ω is the last Greek letter, then the orbit of p ran from α to ω.

Definition 4.1.5. p is said to be **positively recurrent** if $p \in \omega(p)$,

The singularities and periodic points are called trivial recurrent points, other recurrent points are called non-trivial.

Example 4.1.6

Consider the equation

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases}.$$

Then $\omega(0) = \alpha(0) = 0$. For every $p \in \mathbb{S}^1$, we have $\omega(p) = \alpha(p) = \mathbb{S}^1$. Otherwise, let $p = (x, y)$, we have

(1) If $0 < x^2 + y^2 < 1$, then $\omega(p) = \mathbb{S}^1, \alpha(p) = \{0\}$.

(2) If $x^2 + y^2 > 1$, then $\omega(p) = \mathbb{S}^1, \alpha(p) = \varnothing$.

Proposition 4.1.7

$\forall p \in \mathbb{R}^n$, we have

$$\omega(p) = \bigcap_{t \geqslant 0} \overline{\mathrm{Orb}^+(\varphi_t(p))}.$$