ODE: Qualitative Theory (Spring 2022, Shaobo Gan)

Ajorda Jiao

Contents

1 Basic Concepts

§1.1 Basic notions and results

Assume $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, x) \mapsto f(t, x)$ continuous, consider the **equation** (or **system**)

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x).$$

A differentiable function $\gamma : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ is said to be a **solution** (or **solution curve**), if for every $t \in (a, b)$,

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = f(t, \gamma(t)).$$

The **graph** of γ is

$$\{(t, \gamma(t)) : t \in (a, b)\} \subset \mathbb{R} \times \mathbb{R}^n.$$

For $t_0 \in (a, b)$, let $x_0 = \gamma(t_0)$, then γ is called the solution of the **initial value problem**

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

The initial value problem has a unique solution: Let $\gamma_i : (a_i, b_i) \to \mathbb{R}^n$ be two solutions of the initial value problem. Then there exists $\delta > 0$, $(t_0 - \delta, t_0 + \delta) \subset (a_1, b_1) \cap (a_2, b_2)$, such that $\gamma_1(t) = \gamma_2(t), \forall t \in (t_0 - \delta, t_0 + \delta)$,

Theorem 1.1.1 (Existence and Uniqueness Theorem)
$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, f(t, x)$ continuous, given $t_0 \in \mathbb{R}^n, x_0 \in \mathbb{R}^n, a > 0, b > 0$, consider the region

$$R = R(t_0, x_0, a, b) = \{(t, x) : |t - t_0| \leqslant a, |x - x_0| \leqslant b\}.$$

If f is Lipchitz in x on R, i.e. $\exists L > 0$, $\forall (t, x_1), (t, x_2) \in R$,

$$|f(t, x_1) - f(t, x_2)| \leqslant L|x_1 - x_2|,$$

then the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution on $[t_0 - h, t_0 + h]$, where $h = \min\{a, b/M\}, M = \max\limits_{(t,x) \in R} |f(t, x)|$.

Remark 1.1.2 — The solution is denoted as $\varphi(t; t_0, x_0)$.

Corollary 1.1.3

When $f \in C^1$, the existence and uniqueness theorem holds.

Denotes the **maximal interval** of $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ as $I(t_0, x_0)$, it is an open interval.

Corollary 1.1.4

Assume $f \in C^1$ and $|f(x)| \leqslant A(t)|x| + B(t)$, then the maximal interval of the initial value problem is $(-\infty, +\infty)$.

§1.2 Flows

Now we consider the **autonomous equation**

$$\dot{x} = f(x).$$

$\mathbb{R}^n$ is called the **phase space** and $\mathbb{R} \times \mathbb{R}^n$ is called the **generalized phase space**.

The solution of the initial value problem $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ is denoted as $\varphi(t, x_0)$, the set

$$\operatorname{Orb}(x_0) \coloneqq \{\varphi(t, x_0) : t \in I(x_0)\} \subset \mathbb{R}^n$$

is called the **orbit** pass by x_0.

Corollary 1.2.1 (Continuous Dependence on the Initial Value)

Assume $f \in C^1$, then $U = \{(t,x) : t \in I(x)\}$ is open and $\varphi : U \to \mathbb{R}^n, (t,x) \mapsto \varphi(t,x)$ is continuous.

Theorem 1.2.2

$f(x) \in C^1$, then:

1. $\varphi_0(x) = x$ for every $x \in \mathbb{R}^n$.
2. $\varphi(t, \varphi(s,x)) = \varphi(t+s, x)$ for every $s \in I(x), t \in I(\varphi(s,x))$.

Definition 1.2.3. $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, continuous, is said to be a **(continuous) flow** if:

(i) $\psi(0, x) = x$,

(ii) $\psi(t, \psi(s,x)) = \psi(t+s, x)$.

Remark 1.2.4 — The solution of an autonomous equation is a **local flow.**

Corollary 1.2.5

Let $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a flow, then $\psi_t \coloneqq \psi(t, \cdot\,) : \mathbb{R}^n \to \mathbb{R}^n$ are homeomorphisms.

Remark 1.2.6 — Consider the group of self-homeomorphisms of $\mathbb{R}^n$, denotes as $\mathrm{Homeo}(\mathbb{R}^n)$, then $\psi : \mathbb{R} \to \mathbb{R}^n$ is a group homomorphism. More generally, we can consider $G \to \mathrm{Homeo}(\mathbb{R}^n)$ for some group G.

Proposition 1.2.7

Assume f is a C^1 vector field, then the orbits of the flow generated by f are either coincide or disjoint.

$\bigcup_{x\in\mathbb{R}^n} \mathrm{Orb}(x)$ forms a partition of $\mathbb{R}^n$, is called the **orbit space**. For each orbit, orient it to indicate the direction of motion, the family of the oriented orbit $\varphi(t,x)/f(x)$ is called the **phase portrait**.

A point $x_0 \in \mathbb{R}^n$ with $f(x_0) = 0$ is called a **critical point** (or a **singularity**, **equilibrium**). The orbit $\mathrm{Orb}(x_0)$ is a single point $\{x_0\}$.

Example 1.2.8

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t,x_0) = x_0 e^t$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$.

Example 1.2.9

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t,x_0) = \dfrac{x_0}{1 - x_0 t}$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$. But the phase portrait is different from the former examples, because the orientations on $\mathbb{R}_-$ are different.

Theorem 1.2.10

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, $\beta(x) : \mathbb{R}^n \to \mathbb{R} \in C^1$ and $\beta(x) > 0$. Then the equations $\dot{x} = f(x)$ and $\dot{x} = \beta(x)f(x)$ have the same phase portraits.

Proof. $\varphi : I \to \mathbb{R}^n$ a solution of f. Find a C^1 diffeomorphism $h : J \to I$ such that $\varphi \circ h$ is the solution of $\dot{x} = \beta(x)f(x)$. It suffices that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=h(s)} \varphi(t) \cdot \frac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi \circ h(s)) f(\varphi \circ h(s)),$$

i.e. $\dfrac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi \circ h(s)) > 0$, it is an initial value problem. It shows that the maximal solution curve of f is contain in some solution curve of βf. $\square$

Theorem 1.2.11 (Differentiable Dependence on the Initial Value)
Assume $f \in C^1$, it generates the flow ϕ_t, then $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

Exercise 1.2.12.

$$\frac{\partial}{\partial t}\frac{\partial \phi(t,x)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \phi(t,x)}{\partial t}.$$

Let $\Phi(t,x) = \Phi_t(x) = \dfrac{\partial \phi(t,x)}{\partial t}$, then Φ is the solution of the equation

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = A(t)y(t), A(t) = Df(\phi_t(x)) \\ y(0) = \mathrm{Id} \end{cases}.$$

The equation is called the **variation equation** of $f(x)$ along $\phi_t(x)$.

Lemma 1.2.13
$f \in C^1$, $\Phi(t,x)$, then

$$\Phi_t(\phi_s(x))\Phi_s(x) = \Phi_{t+s}(x).$$

Remark 1.2.14 — This property is called the **cocycle** condition.

We already know that ϕ_t are self-homeomorphisms of $\mathbb{R}^n$, and lemma 1.2.13 shows that the differential is invertible, hence ϕ_t are diffeomorphisms. Define

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^n \times \mathbb{R}^n \\ (t,x,v) &\mapsto (\phi_t(x), \Phi_t(x)v). \end{aligned}$$

Proposition 1.2.15
$\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a flow.

Remark 1.2.16 — We call Φ_t is a skew product flow of ϕ_t.

Theorem 1.2.17

$$\Phi_t(x)f(x) = f(\phi_t(x)).$$

If ψ is a C^1 flow, let

$$g(x) = \left.\frac{\partial \psi(t,x)}{\partial t}\right|_{t=0},$$

then $\psi(t,x_0)$ solve the initial value problem $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$. Because

$$\frac{\partial \psi(t,x_0)}{\partial t} = \left.\frac{\partial \psi(t+s,x_0)}{\partial s}\right|_{s=0} = \left.\frac{\partial \psi(s,\psi(t,x_0))}{\partial s}\right|_{s=0} = g(\psi(t,x_0)).$$

§1.3 Equations on manifolds

Let M be a closed smooth manifold, X is a C^1 vector field on M. Then X is bounded, hence the maximal intervals are $(-\infty, +\infty)$. Consider the equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = X(x) \\ x(0) = x_0 \end{cases},$$

then the solution $\varphi(t, x)$ generates a flow.

2 Linear Systems

§2.1 Plane linear sigularities

Consider the equation

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}, \quad (x,y) \in \mathbb{R}^2.$$

It is said to be a **plane linear system** if f, g both linear functions of x, y, i.e.

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad a,b,c,d \in \mathbb{R}.$$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then $(0,0)$ is the only sigularity of the vector field, elementary singularity.

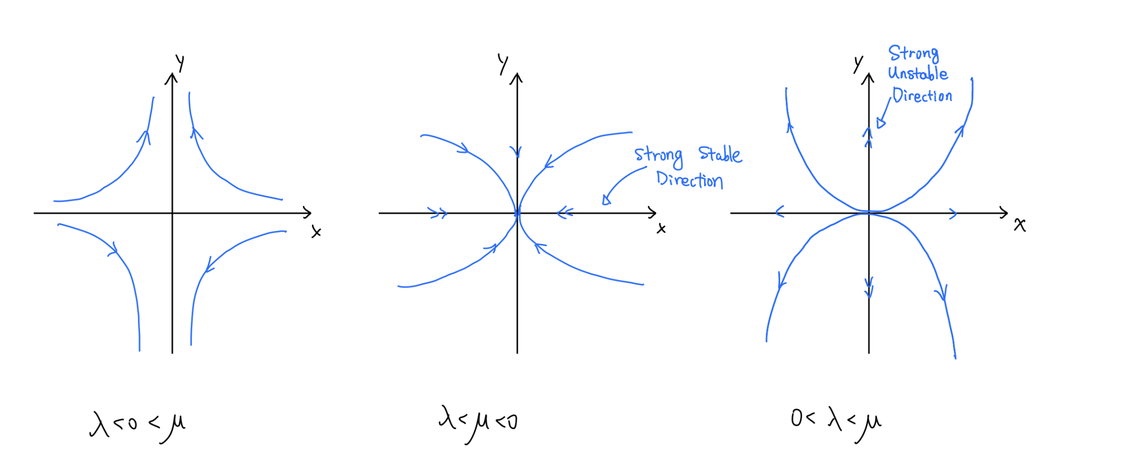
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, consider the Jordan form of A. There are four cases:

I. Two different real eigenvalues: $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$.

  i. $\lambda < 0 < \mu$: the origin is called a **saddle point**.

  ii. $\lambda < \mu < 0$: the origin is called a **stable node**.

  iii. $0 < \lambda < \mu$: the origin is called a **unstable node**.



II. Conjugated imaginary eigenvalues: $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \beta > 0$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

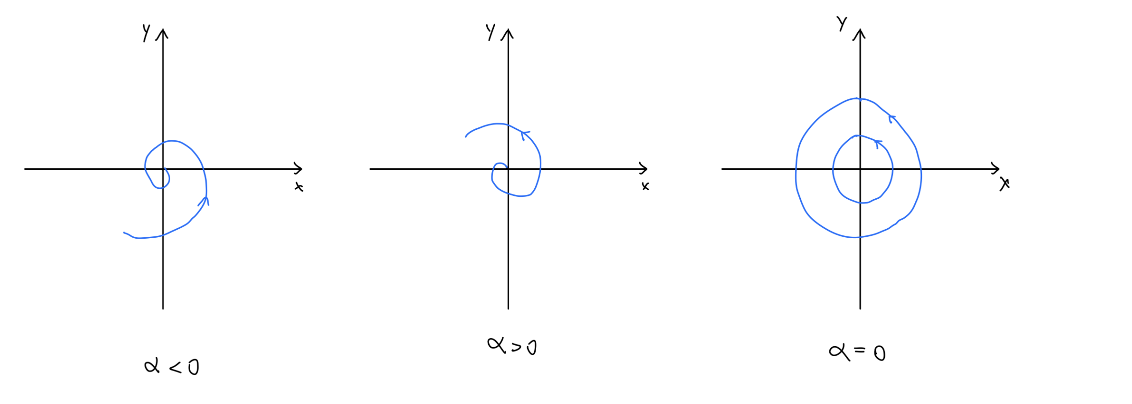
  If we consider this equation in the polar coordinates, it turns $\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$.

  i. $\alpha < 0$, the origin is called a **stable focus**.

  ii. $\alpha > 0$, the origin is called a **unstable focus**.

  iii. $\alpha = 0$, the origin is called a **center**.

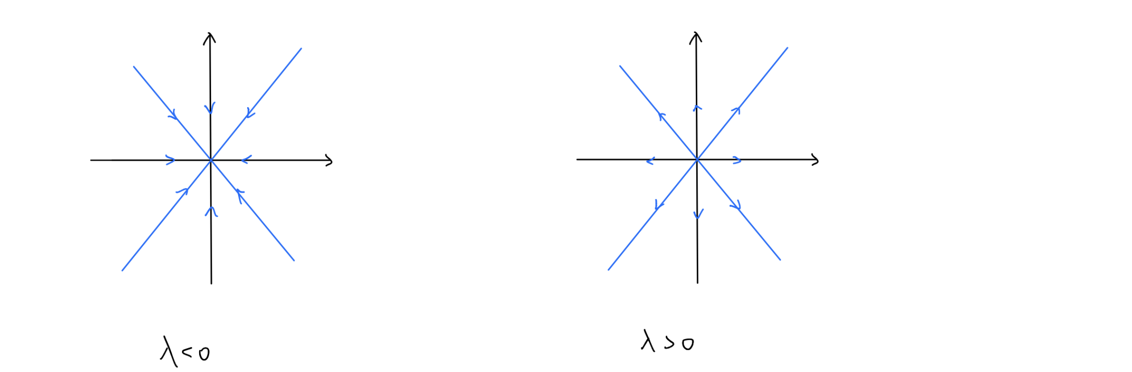
Definition 2.1.1. φ_t a flow. If p is not a singularity and $\exists T > 0$, such that $\varphi_T(p) = p$. Then p is called a **periodic point**, $\mathrm{Orb}(p)$ is called a **periodic orbit**. If p is a periodic point, the smallest $T > 0$ is called the **minimum positive period**.



III. Two same real eigenvalues, diagonalizable: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$.

i. $\lambda < 0$, the origin is called a **stable critical node**.

ii. $\lambda > 0$, the origin is called a **unstable critical node**.

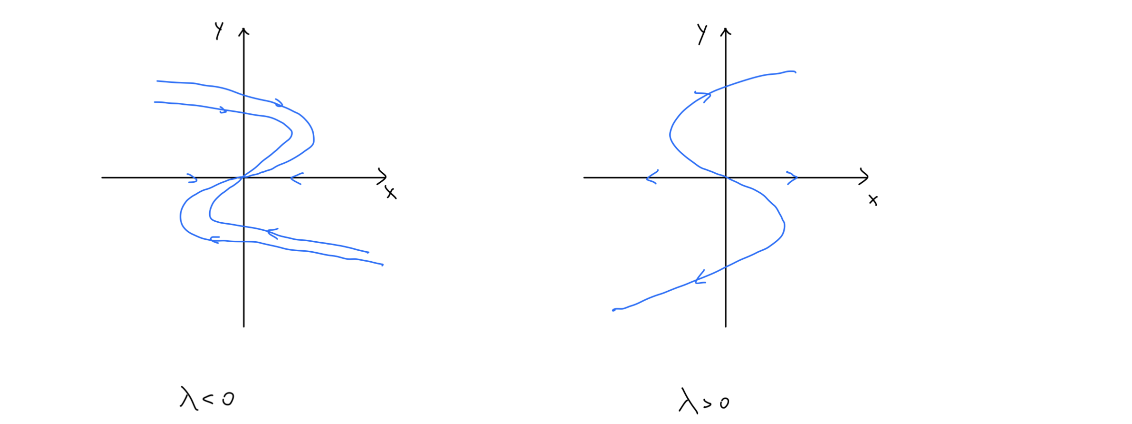


IV. Two same real eigenvalues, not diagonalizable: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}(x_0 + t y_0) \\ e^{\lambda t} \end{bmatrix} y_0$,

or $x(t) = \dfrac{x_0}{y_0} y(t) + \dfrac{y(t)}{\lambda} \ln \dfrac{y(t)}{y_0}$.

i. $\lambda < 0$, the origin is called a **stable unidirectional node**.

ii. $\lambda > 0$, the origin is called a **unstable unidirectional node**.



Exercise 2.1.2. Draw the phase portraits of non-elementary plane systems (i.e. the determinant is 0).

§2.2 Topological conjugacies between linear systems

Definition 2.2.1. Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ homeomorphisms. f and g are said to be **topologically conjugate** if there exists $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f = g \circ h$.

Remark 2.2.2 — Conjugacy is a equivalence relation.

Definition 2.2.3. Let $\varphi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ be two flows, we call φ_t and ψ_t are conjugate if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ \varphi_t = \psi_t \circ h$. Let X, Y be two C^1 vector fields on $\mathbb{R}^n$, we call X, Y are conjugate if the flows generated by them, respectively, are conjugate.

Example 2.2.4

$A, B \in M(n, \mathbb{R})$ are similar, then $\dot{x} = Ax$ and $\dot{y} = By$ are conjugate.

$f, g : \mathbb{R}^n \to \mathbb{R}^n$ C^1 vector fields, generate flows ϕ_t, ψ_t. Let $x = h(y) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism gives the conjugate, i.e., $h\psi_t(y) = \phi_t h(y)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} h(y) = f(h(y)) \implies D_{h(y)} g(y) = D_{h(y)} \frac{\mathrm{d}y}{\mathrm{d}t} = f(h(y)).$$

If there exists a C^1 diffeomorphism conjugate $e^{Bt}y$ to $e^{At}x$ via $x = h(y)$, i.e. $h(e^{Bt}y) = e^{At}h(y)$. Then $Dh_0 e^{Bt} = e^{At} Dh_0$, hence $Dh_0 B = ADh_0$. It shows that C^1 conjugate generically not hold even if topologically conjugate.

Proposition 2.2.5

Assume f, g C^1 vector fields generate ϕ_t, ψ_t, let h be a conjugate between ϕ_t and ψ_t. Then:

1. $h(\mathrm{Orb}(x, \phi)) = \mathrm{Orb}(hx, \psi)$.
2. h maps the singularities of f to the singularities of g.
3. h maps the periodic orbits of f to the periodic orbits of g. Moreover, it preserves the minimum positive period.

Example 2.2.6

$\dot{x} = -2x$ and $\dot{y} = -4y$ are conjugate.

Let $h : \mathbb{R} \to \mathbb{R}, h(0) = 0$. Take $x_0, y_0 > 0$, let $h(x_0) = y_0$, then $h(e^{-2t}x_0) = e^{-4t}y_0$ or $h(x) = \left(\frac{x}{x_0}\right)^2 y_0$. The construction for the negative part is similar.

Exercise 2.2.7. $\lambda\mu \neq 0$, show that $\dot{x} = \lambda x$ is conjugate to $\dot{y} = \mu y$ if and only if $\lambda\mu > 0$.

Proposition 2.2.8

$\phi_t^i, \psi_t^i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ are topologically conjugate by $h_i, i = 1, 2$. Then $\phi_t^1 \times \phi_t^2$ and $\psi_t^1 \times \psi_t^2$ are topologically conjugate by $h_1 \times h_2$.

Example 2.2.9

$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$ are conjugate.

Proof. $\phi_t(x, y) = e^{-t}(x, y)$ and $\psi_t(x, y) = e^{-t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. For every $(x, y) \neq (0, 0)$, there exists unique $t = t(x, y)$ such that $\phi_t(x, y) \in \mathbb{S}^1$. Let $h(x, y) := \psi_{-t}\phi_t(x, y)$, where $t = t(x, y)$, then h gives the conjugate. $\square$

Exercise 2.2.10. Show that $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -y \end{cases}$ are conjugate.

Classification of elementary plane linear systems:

(I) Stable: node, critical node, unidirectional node, focus.

(II) Unstable: node, critical node, unidirectional node, focus.

(III) Saddle point.

(IV) Center.

Definition 2.2.11. The linear system $\dot{x} = Ax$ in $\mathbb{R}^n$ is called **hyperbolic** if the real parts of eigenvalues of A are nonzero. The **(stable) index** of A is the number of eigenvalues with negative real parts, denoted by $\operatorname{Ind} A$.

Theorem 2.2.12

Two plane hyperbolic linear system $\dot{x} = Ax, \dot{y} = By$ are topologically conjugate if and only if $\operatorname{Ind} A = \operatorname{Ind} B$.

Proof. "$\Longrightarrow$": Let $W_A^s = \left\{x : e^{tA}x \to 0, t \to \infty\right\}$, $W_B^s = \left\{x : e^{tB}x \to 0, t \to \infty\right\}$, then h and h^{-1} preserves the stable manifolds. Then $\operatorname{Ind} A = \dim W_A^s = \dim W_B^s = \operatorname{Ind} B$. $\square$

Example 2.2.13

Consider $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ and $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$ with the same phase portraits are not topologically conjugate. Because the topologically conjugate preserves the minimum positive orbits.

Definition 2.2.14. $\phi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ flows, h is a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ maps the orbit of ϕ to the orbit of ψ preserves the orientation. Then ϕ and ψ is called **topologically equivalent** or **flow equivalent**.

Theorem 2.2.15 (Grobman-Hartman)
If x_0 is a hyperbolic singularity of $f(x)$, then the flows generated by $\dot{x} = f(x)$ and $\dot{y} = Ay$ where $y = Df(x_0)$ are topologically conjugate near 0.

§2.3 Non-autonomous linear system

$A : \mathbb{R} \to M(n, \mathbb{R})$ continuous, consider the equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n$$

a non-autonomous linear system.

Theorem 2.3.1
The followings hold:

1. The initial problem of the equation exist the unique solution.
2. The maximal interval of any solution is $(-\infty, \infty)$.
3. All solutions of the equation form an n-dimensional linear space S.

Theorem 2.3.2 (Liouville's Formular)
Assume $X(t)$ is a solution of $\dot{x} = A(t)x$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \det X(t) = \operatorname{tr} A(t) \det X(t),$$

hence $\det X(t) = \det X(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(s) \mathrm{d}s$.

Let $X_1(t), X_2(t), \cdots, X_n(t)$ be a basis of S, let

$$X(t) := [X_1(t), X_2(t), \cdots, X_n(t)] \in \mathrm{GL}(n, \mathbb{R}),$$

it called a **fundamental solution** of the equation. The fundamental solution of

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = A(t)X \\ X(t_0) = I_n \in \mathrm{GL}(n, \mathbb{R}) \end{cases}$$

is called the **standard fundamental solution**.

If $X(t), Y(t)$ are two fundamental solutions, suppose $Y(0) = X(0)C$, then

$$\frac{\mathrm{d}X(t)C}{\mathrm{d}t} = \frac{\mathrm{d}X(t)}{\mathrm{d}t} C = A(t)X(t)C,$$

is a non-degenerate solution of $\dfrac{\mathrm{d}X}{\mathrm{d}t} = AX$. By the uniqueness, we get $Y(t) = X(t)C$.

Example 2.3.3

$A(t) \equiv A$, the fundamental solution of $\dot{x} = Ax$ is

$$e^{tA} = \mathrm{Id} + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{k!}t^kA^k + \cdots.$$

Example 2.3.4

$\dot{x} = f(x), x \in \mathbb{R}^n$, where $f \in C^1$, generates the flow $\varphi_t(x)$. Consider $\Phi_t(x) = \dfrac{\partial}{\partial t}\varphi_t(x)$ and the variation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = Df_{\varphi_t(x)}\Phi_t(x).$$

Given $x \in \mathbb{R}^n$, let $A(t) \coloneqq Df_{\varphi_t(x)}$, then $\Phi_t(x)$ is the standard fundamental solution ($t_0 = 0$) of $\dot{x} = A(t)x$. Consider two special types of orbits:

- x is a singularity, denoted by σ. Then $\varphi_t(\sigma) = \sigma$, $\dot{x} = Ax$ where $A = Df(\sigma)$.
- x is a periodic point, denoted by p, the minimum period $T > 0$. Then A is T-periodic.

§2.4 Periodic linear system

Definition 2.4.1. The equation $\dot{x} = A(t)x$ satisfies $A(t+T) = A(t)$ for some $T > 0$ is called a **periodic linear system**.

Theorem 2.4.2 (Floquet)

Assume $\dot{x} = A(t)x$ is a T-periodic linear system, if X is a fundamental solution, then $X(t+T)$ is a fundamental solution, i.e. $\exists C \in \mathrm{GL}(n, \mathbb{R})$ such that $X(t+T) = X(t)C$. Moreover, there exists a T-periodic map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ and a constant matrix $B \in M(n, \mathbb{C})$ such that $X(t) = P(t)e^{tB}$.

Lemma 2.4.3

$\forall C \in \mathrm{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{C})$ such that $C = e^B$.

Proof. It suffices to show for Jordan block. This follows by the matrix series

$$\ln(I + N) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} N^k$$

is convergence for nilpotent matrix N. □

Lemma 2.4.4

$\forall C \in \mathrm{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{R})$ such that $C^2 = e^B$.

Proof. Note that the Jordan block of C^2 is either:

(i) $\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & \lambda & 1 \\ 0 & \cdots & & \lambda \end{bmatrix}$, where $\lambda > 0$, or

(ii) $\begin{bmatrix} J & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & J & I_2 \\ 0 & \cdots & & J \end{bmatrix}$, where $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a, b \in \mathbb{R}, b > 0$.

And $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ have a real matrix logarithm because $\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\} \cong \mathbb{C} = \{a + bi\}$. □

Theorem 2.4.5 (Real Form of Floquet Theorem)
Assume $\dot{x} = A(t)x$ is a T-periodic linear system, if X is a fundamental solution. Then there exists a **$2T$-periodic** map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ and a constant matrix $B \in M(n, \mathbb{R})$ such that $X(t) = P(t)e^{tB}$.

Example 2.4.6 (2T is necessary)
Let $\Phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t \right) \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right)$. Let

$$A(t) = \dot{\Phi}(t)\Phi(t)^{-1} = \begin{bmatrix} -\cos t \sin t & -\sin^2 t \\ \cos^2 t & \cos t \sin t \end{bmatrix},$$

then $A(t)$ is π-periodic. Then $\Phi(t)$ is a standard fundamental solution of $\dot{x} = A(t)x$, hence $\exists \pi$-periodic $P(t)$ and B such that $\Phi(t) = P(t)e^{tB}$. Then $e^{\pi B} = \begin{bmatrix} -1 & -\pi \\ 0 & -1 \end{bmatrix}$, there is no real matrix B satisfying this equation.

Definition 2.4.7. In Floquet theorem, $X(t+T) = X(t)C$. We call C is a **monodromy matrix**. The eigenvalues of C are called **Floquet multipliers**. If ρ is a Floquet multiplier with $\rho = e^{\lambda T}$, then λ is called a **Floquet exponent**.

Corollary 2.4.8
Consider a T-periodic linear system $\dot{x} = A(t)x$. Then there exists a linear transformation (non-autonomous) $x = P(t)y$ such that $\dot{y} = By$.

Proof. Let $X(t) = P(t)e^{tB}$ be a fundamental solution, then

$$AX = \dot{X} \implies \dot{P}e^{tB} + PBe^{tB} = APe^{tB},$$

hence $\dot{P} + PB = AP$. Then $APy = \dfrac{\mathrm{d}}{\mathrm{d}t}(Py) = \dot{P}y + P\dot{y}$, hence $\dot{y} = By$. □

Remark 2.4.9 — This type of equation is called reducible, which means after some reduction, the equation can become independent with time t.

Corollary 2.4.10

Let λ be a Floquet multiplier of $\dot{x} = A(t)x$. Then there exists a T-periodic function $p(t)$ such that $e^{\lambda t}p(t)$ is a solution of the equation $\dot{x} = A(t)x$.

Proof. $e^{\lambda T}$ is an eigenvalue of C, then $\exists x_0$ such that $Cx_0 = e^{\lambda T}x_0$. Then $X(t)x_0$ is a solution. Let $p(t) = e^{-\lambda t}X(t)x_0$ is T-periodic and $e^{\lambda t}p(t)$ is a solution. □

Corollary 2.4.11

The equation admits a nonzero T-periodic solution if and only if 1 is a Floquet multiplier.

Corollary 2.4.12

Assume $\rho_1, \rho_2, \cdots, \rho_n$ are all Floquet multipliers of $\dot{x} = A(t)x$, then

$$\rho_1\rho_2\cdots\rho_n = \det\Phi(T) = \exp\int_0^T \operatorname{tr} A(t)\ \mathrm{d}t.$$

Example 2.4.13

The equation $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^2 t & \frac{1}{2}\sin 2t - 1 \\ \frac{1}{2}\sin 2t + 1 & \sin^2 t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ has an unbounded solution. Because the product of two multipliers is $\exp\displaystyle\int_0^\pi 1\ \mathrm{d}t = e^\pi > 1$.

Consider **Hill equation**

$$\ddot{x} + p(t)x = 0,$$

where $p(t)$ is π-periodic. This is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -p(t)x \end{cases},$$

then $\rho_1\rho_2 = \exp\displaystyle\int_0^\pi \operatorname{tr} A(t)\ \mathrm{d}t = 0$, where $A(t) = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}$.

Lemma 2.4.14

If ρ_1, ρ_2 both are imaginary numbers, then every solution of Hill equation is bounded.

Proof. Because ρ_1, ρ_2 are conjugate imaginary numbers, hence $\Phi(\pi)$ is similar to a rotation. Then $\Phi(\pi)^n$ is bounded independent of n and $\Phi(s)$ is bounded for $s \in [0, \pi]$. □

Definition 2.4.15. A particular Hill equation with $p(t) = a + \varepsilon \cos 2t$ is called **Mathieu equation**.

Exercise 2.4.16. Consider Mathieu equation

$$\ddot{x} + (a + \varepsilon \cos 2t)x = 0.$$

(1) $U = \{(a, \varepsilon) \in [0, 10] \times [-1, 1] : \text{ every solution is bounded}\}$. Draw the figure of U by some calculation.

(2) Guess some conclusions by the figure of U.