Sum Product Theorems and Applications (Spring 2022, Weikun He)

Ajorda Jiao

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Theorem 0.1 (Erdos-Szemeredi Theorem)
The exists an absolute constant $c > 0$, such that

$$\max \{\#(A+A), \#AA\} \geqslant c(\#A)^{1+c}.$$

§1 Basic additive combinatorics

$(E, +)$ abelian group. $A, B \subseteq E$.

Notation 1.1. $A + B \coloneqq \{a + b : a \in A, b \in B\}$.

> **Question 1.2** (Freiman)**.** If $\#(A+A) \leqslant K\#A$, for some parameter K, what can we say about A?

Observation 1.3. If A is a **arithmetic progression**, then $\#(A+A) \leqslant 2\#A$. If A is a **generalized A.P.** of **rank** r, i.e.

$$A = \{a_0 + t_1 d_1 + \cdots + t_r d_r : \forall i, 1 \leqslant t_i \leqslant N_i\},$$

then $\#(A+A) \leqslant 2^r \#A$.

Freiman Type Theorem: If $\#(A+A) \leqslant K\#A$, then exists

(i) $P \subseteq E$ is a generalized arithmetic progression of rank $O_K(1)$, $\#P = O_K(\#A)$.

(ii) $X \subseteq E$ finite, $\#X = O_K(1)$.

Such that $A \subseteq P + X$.

Theorem 1.4 (Szemerédi)
$A \subseteq \mathbb{N}$ with positive upper density, then A contains arbitrarily long A.P.

Lemma 1.5 (Ruzsa Triangle Inequality)
$A, B, C \subseteq (E, +)$ finite, then

$$\#(A - C)\#B \leqslant \#(A - B)\#(B - C).$$

Proof. Construct a map $(A - C) \times B \to (A - B) \times (B - C), (x, b) \mapsto (a_x - b, b - c_x)$, where $x = a_x - b_x$ is a typical decomposition, this map is an injective. $\square$

Definition 1.6. Define the **Ruzsa distance** between A, B by

$$d(A, B) = \log \frac{\#(A - B)}{(\#A)^{\frac{1}{2}}(\#B)^{\frac{1}{2}}}.$$

Lemma 1.7 (Ruzsa Covering Lemma)
$A, B \subseteq (E, +)$ finite, $K \geqslant 1$. If $\#(A+B) \leqslant K\#A$, then $\exists X \subseteq E, \#X \leqslant K$, such that $B \subset A - A + X$.

Proof. Let $X \subseteq B$ be the maximal set such that $(x+A)_{x\in X}$ is pointwise disjoint. □

Notation 1.8. $\mathbb{O}(K)$ denotes some subset of cardinality $\leqslant K$.

Remark 1.9 — Ruzsa Covering Lemma $\iff B \subseteq A - A + \mathbb{O}\left(\frac{\#(A+B)}{\#A}\right)$.

Proposition 1.10 (Plünnecke-Ruzsa Inequality)
$A, B \subseteq E$ finite, $K \geqslant 1$. If $\#(A+B) \leqslant K\#A$, then $\forall k, l \geqslant 0$, we have

$$\#\left(\sum_k B - \sum_l B\right) \leqslant K^{k+l}\#A,$$

where $\sum_k B := \underbrace{B + B + \cdots + B}_{k\ Bs}$.

Lemma 1.11 (Petridis)
If $\#(A+B) \leqslant K\#A$, then $\exists A_0 \subseteq A$, such that for every $C \subset E$ finite,

$$\#(C + A_0 + B) \leqslant K\#(C + A_0).$$

Proof. Let $K_0 := \inf_{A' \subseteq A} \frac{\#(A'+B)}{\#A'} \leqslant K$ and $A_0 \subseteq A$ such that $K_0 = \frac{\#(A_0+B)}{\#A_0}$. Applying induction to $\#C$, consider $C' = C \cup \{c\}$, where $c \notin C$. WLOG, assume $c = 0$. Then

$$\#(C' + A_0 + B) = \#(C + A_0 + B) + \#(A_0 + B) - \#((C + A_0 + B) \cap (A_0 + B)).$$

Observe that $((C + A_0) \cap A_0) + B \subseteq (C + A_0 + B) \cap (A_0 + B)$. By assumption,

$$(C + A_0) \cap A_0 \subseteq A \implies \#((C + A_0) \cap A_0) + B \geqslant K_0\#((C + A_0) \cap A_0).$$

Hence by inductive assumption,

$$\#(C' + A_0 + B) \leqslant K_0(\#(C + A_0) + \#A_0 - \#((C + A_0) \cap A_0)) = K_0\#(C' + A_0).$$

□

Proof of Plünnecke-Ruzsa Inequality 1.10. Applying lemma, we have

$$\#(B + A_0) \leqslant K\#A_0, \quad \#(B + B + A_0) \leqslant K\#(B + A_0) \leqslant K^2\#A_0, \quad \cdots.$$

Hence, $\#\left(\sum_k B + A_0\right) \leqslant K^k\#A_0$. Finally, applying Ruzsa Triangle Inequality, we have

$$\#\left(\sum_k B - \sum_l B\right) \leqslant \frac{\#\left(\sum_k B + A_0\right)\#\left(\sum_l B + A_0\right)}{\#A_0} \leqslant K^{k+l}\#A_0 \leqslant K^{k+l}\#A.$$

□

Exercise 1.12. If E is not an abelian group, does the argument still hold?