Homogeneous Dynamics
(2022, Spring, Runlin Zhang)

Ajorda Jiao

Contents

1 Introduction of Homogeneous Dynamics

§1.1 Motivations and applications

§1.1.i Horocycles on constant negative curvature surfaces

Equip $\mathbb{H}^2 := \{x + iy \in \mathbb{C}, y > 0\}$ with the metric $\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$. Let $\Gamma \leqslant \mathrm{Isom}(\mathbb{H}^2)$ be a discrete (torsion free) subgroup such that $\Gamma \setminus \mathbb{H}^2$ is compact (such a subgroup is called a uniform lattice). Then $\Gamma \setminus \mathbb{H}^2$ is a compact surface of constant negative curvature.

Let $\pi : \mathbb{H}^2 \to \Gamma \setminus \mathbb{H}^2 = M$ be the quotient map. Consider a horocycle $\mathcal{H} \subset \mathbb{H}^2$.

Theorem 1.1.1

For every $\mathcal{H}$, $\pi(\mathcal{H})$ is dense in M.

Theorem 1.1.2

If $M = \Gamma \setminus \mathbb{H}^2$ ($\Gamma \leqslant \mathrm{Isom}(\mathbb{H}^2)$ still discrete) is just of finite volume, then:

1. $\pi(\mathcal{H})$ is either closed or dense in M.
2. Consider a sequence of closed horocycles $\pi(\mathcal{H}_i)$ with length $\to \infty$, then $\pi(\mathcal{H}_i)$ becomes dense in $\Gamma \setminus \mathbb{H}^2$.

§1.1.ii Isometric immersion of hyperbolic spaces

Let $\mathbb{H}^3$ be the three dimensional hyperbolic space $\{(x + iy, z) \in \mathbb{C} \times \mathbb{R}, z > 0\}$ equipped with the metric $\frac{1}{z^2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$. Let $\Gamma \leqslant \mathbb{H}^3$ be a discrete (torsion free) subgroup, such that $\mathbb{H}^3$ is compact (finite volume suffices). Consider an isometric embedding $\iota : \mathbb{H}^2 \to \mathbb{H}^3$. The image of ι can be explicitly described.

Theorem 1.1.3

The following holds:

1. $\pi(\iota(\mathbb{H}^2))$ is either closed or dense in M;
2. Given an infinite sequence of distinct closed $\pi(\iota_i(\mathbb{H}^2))$, then $\lim_i \pi(\iota_i(\mathbb{H}^2))$ is dense in M.

§1.1.iii Oppenheim conjecture/Margulis theorem

Let Q be a real quadratic form in 3 variables, indefinite and non-degenerated. Consider Q as a function $\mathbb{R}^3 \to \mathbb{R}$.

Theorem 1.1.4

Assume Q is NOT proportional to a quadratic form with $\mathbb{Q}$-coefficients. Then $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.

Remark 1.1.5 — It is true for $k \geqslant 3$ variables. But it is false for Q only has two variables.

Theorem 1.1.6 (Eskin-Margulis-Mozes)

Further assume Q has at least signature $(3,1)$, then for every $a < b \in \mathbb{R}$,

$$\begin{aligned} &\# \left\{ v \in \mathbb{Z}^4 : \|v\| \leqslant T, Q(v) \in (a,b) \right\} \\ \sim\ &\mathrm{Vol} \left\{ v \in \mathbb{R}^4 : \|v\| \leqslant T, Q(v) \in (a,b) \right\} \\ \sim\ &C_Q(b-a)T^2. \end{aligned}$$

§1.1.iv Littlewood conjecture

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we have $\inf \{n\langle n\alpha\rangle : n \in \mathbb{Z}_+\} < 1$.

Fact 1.1.7. There exists α such that $\inf \{n\langle n\alpha\rangle : n \in \mathbb{Z}_+\} > 0$.

Conjecture 1.1.8

For all $(\alpha,\beta) \in \mathbb{R}^2,\ \alpha,\beta \notin \mathbb{Q}$,

$$\inf \{n\langle n\alpha\rangle\langle n\beta\rangle : n \in \mathbb{Z}_+\} = 0.$$

Remark 1.1.9 — The conjecture is reasonable in some sense:

1. $\forall \delta > 0$, $\inf \left\{ n^{1-\delta}\langle n\alpha\rangle\langle n\beta\rangle : n \in \mathbb{Z}_+ \right\} = 0$.
2. $\forall \delta > 0, \exists (\alpha,\beta)$, such that $\inf \left\{ n^{1+\delta}\langle n\alpha\rangle\langle n\beta\rangle : n \in \mathbb{Z}_+ \right\} > 0$.

§1.1.v Quantum unique ergodicity

Consider $M^2 = \Gamma \setminus \mathbb{H}^2$ is a closed hyperbolic surface. Consider $\Delta \coloneqq -y^2 \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \right)$ acts on $C^\infty(M)$. Then:

1. $\exists \lambda_0 = 0 < \lambda_1 < \cdots,\ \lambda_i \to \infty$,
2. Let $E_{\lambda_i} \coloneqq \{f \in C^\infty(M) : \Delta f = \lambda_i f\}$, then $E_{\lambda_i} \neq \varnothing$ and $\dim E_{\lambda_i} < \infty$.

For each i, choose $f_i \in E_{\lambda_i}$. Consider $(|f_i|^2 \mathrm{Vol})_i$ a sequence of measure on M, normalized to be probability measure.

Conjecture 1.1.10

$|f_i|^2\mathrm{Vol}$ tends to $\dfrac{1}{\mathrm{Vol}(M)}\mathrm{Vol}$ in the weak* topology.

Further assume Γ is a "congruence subgroup". In this situation, there is an additional supply of operators, called Hecke operators, that commute with the Laplacian. Let $f_i \in E_{\lambda_i}$ which is also an eigenfunction of Hecke operator.

Theorem 1.1.11 (Lindenstrauss-Bourgain)

In such settings, the conjecture holds.

§1.2 Measure rigidity

§1.2.i Unipotent rigidity

Let $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma \leqslant G$ a discrete subgroup. G has a right G-invariant Riemannian metric. It induces a volume measure Vol on G/Γ.

Fact 1.2.1. Vol is left G-invariant.

Let $U = \left\{u_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R}\right\}$.

Theorem 1.2.2

If G/Γ is compact, then Vol is the unique U-invariant finite measure(up to a scalar).

Theorem 1.2.3

If Vol is finite (normalized to be probability measure). Then every U-invariant probability measure is a "convex combination" of:

(i) the U-invariant measure supported on a closed(and compact) orbit.

(ii) Vol.

Theorem 1.2.4 (Measure Rigidity Theorem)

Let G be a (conneted) Lie group, let $U = \{u_s : s \in \mathbb{R}\}$ be an Ad-unipotent one-parameter subgroup of G. Let $\Gamma \leqslant G$ be a closed subgroup. Then every U-invariant ergodic probability measure on G/Γ is "homogeneous".

Theorem 1.2.5 (Equidistribution and Topological Rigidity)
Assume Γ is a lattice in G, then for any $x \in G/\Gamma$:

1. There exists a probability "homogeneous" measure μ such that
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_s x)\mathrm{d}s = \int f(x)\mathrm{d}\mu(x), \quad \forall f \in C_c(G/\Gamma).$$
2. The closure of the orbit Ux is "homogeneous", which means $\exists H \leqslant G$ closed such that $\overline{Ux} = Hx$.

§1.2.ii Higher rank diagonalizable flow

Let $G = \mathrm{SL}(2,\mathbb{R}),\ \Gamma \leqslant G$ lattice. Consider $A = \left\{\begin{bmatrix} e^t & 0 \\ 0 & e^{-t}\end{bmatrix}\right\}$ acts on G/Γ.

Conjecture 1.2.6
$G = \mathrm{SL}(3,\mathbb{R}),\ \Gamma = \mathrm{SL}(3,\mathbb{Z})$. Consider
$$\mathbb{R}^2 \cong A := \left\{\begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3}\end{bmatrix} : t_1 + t_2 + t_3 = 0\right\}$$
acts on G/Γ.

1. Every A-ergodic probability measure is homogeneous.
2. Every bounded A-orbit is closed.

Theorem 1.2.7
A, G, Γ as in the conjecture, then:

1. Every A-invariant ergodic probability measure with "positive entropy" is homogeneous.
2. The Hausdorff dimension of $\{x \in G/\Gamma : Ax \text{ is bounded}\}$ is equal to 2.

2 Oppenheim Conjecture

§2.1 22.2.25: The unipotent flow is minimal on compact space

- Let $G = \mathrm{SL}(2,\mathbb{R})$, let $\Gamma \leqslant G$ be a discrete subgroup.
- Assume for today $X = G/\Gamma$: is compact.
- $U^+ = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \geqslant 0 \right\}$.

Theorem 2.1.1
For all $x \in X$, U^+x is dense in X.

Definition 2.1.2. Let A be a semigroup acting on a topological space Z :

1. We say the action is **minimal** if every A-orbit is dense in Z.
2. We say the subset $W \subset Z$ is **A-minimal** if W is A-stable, closed and $A \curvearrowright W$ is minimal.

Theorem 2.1.3
Let Y be a U^+-minimal subset of X. Then $Y = \varnothing$ or $Y = X$.

Claim 2.1.4. Theorem 2.1.3 implies Theorem 2.1.1

Proof. Zorn's lemma + compactness of X. We can always find a nonempty U^+-minimal subset of X, which must be X. $\square$

Fact 2.1.5. $\mathrm{SL}(2,\mathbb{R})$ admits a right-invariant metric compatible with its topology.

Now we fix such a metric $d : G \times G \to \mathbb{R}$. It induces a "quotient" metric $d_X : X \times X \to \mathbb{R}$ by

$$d_X(g\Gamma, h\Gamma) := \inf_{\gamma_1,\gamma_2 \in \Gamma} d(g\gamma_1, h\gamma_2) = \inf_{\gamma \in \Gamma} d(g\gamma, h).$$

For $x \in X = G/\Gamma$, define the **injective radius** of x as

$$\mathrm{InjRad}(x) := \sup \{\delta > 0 : \text{such that } g \mapsto g.x \text{ is injective on } g \in B(\mathrm{Id}, \delta) \subseteq G\}.$$

Exercise 2.1.6. For all $x \in X$, $\mathrm{InjRad}(x) > 0$.

Proof. By Γ is discrete. $\square$

Exercise 2.1.7. $\exists r_X > 0$, such that $\forall x \in X$, $\mathrm{InjRad}(x) > r_X$.

Proof. By the compactness of X. Because Γ is cocompact, there exists $C \subseteq G$ compact, then $\forall x \in X, \exists g_x \in C$, $x = g_x\Gamma$. $\square$

Lemma 2.1.8

$U^+ \curvearrowright X = G/\Gamma$ has no closed(compact) orbit.

Proof. Say: we have a compact orbit $\{u_s.x : s \geqslant 0\}$. Define $s_0 = \inf\{s > 0 : u_s.x = x\}$, then

$$\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} u_{s_0} \begin{bmatrix} e^{t} & \\ & e^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x = \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x.$$

This shows that $\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} .x$ is invariant under $\begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} u_{s_0} \begin{bmatrix} e^{t} & \\ & e^{-t} \end{bmatrix} = u_{e^{-2t} s_0}$. □

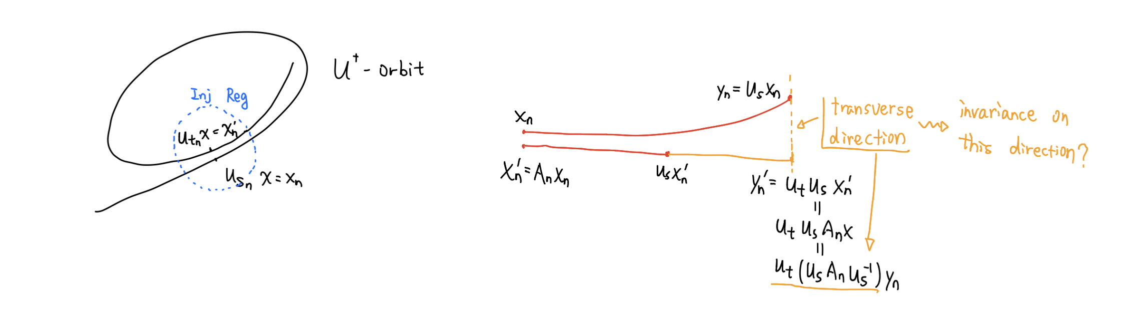
Corollary 2.1.9

Γ contains no nontrivial unipotent matrix.

Corollary 2.1.10

The following holds:

1. $\forall x \in X$, the map $s \mapsto u_s.x$ is injective.
2. $\forall x, \exists s_n, t_n \to \infty$ with $|s_n - t_n| \to \infty$, such that $d_X(u_{s_n}.x, u_{t_n}.x) \to 0$.



Proof of Theorem 2.1.3. By corollary 2.1.10, we can find $A_n \in G \backslash U$ and $x_n, x'_n \in U^+ x \subseteq X$ with $d_X(x_n, x'_n) \to 0$ and $x'_n = A_n.x_n$. Moreover, we can choose $A_n \to \mathrm{Id}$ (use the fact that injective radius is larger than r_X).

Say $A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$, where $a_n, d_n \to 1, b_n, c_n \to 0$. We have

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} A_n \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_n + sc_n & b_n - sa_n + sd_n - s^2 c_n \\ c_n & d_n - sc_n \end{bmatrix}.$$

We want to choose $t = t_s$ such that

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_n + sc_n & b_n - sa_n + sd_n - s^2 c_n \\ c_n & d_n - sc_n \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Take $t = t_s = \dfrac{-(b_n - sa_n + sd_n - s^2 c_n)}{d_n - sc_n}$. Then

$$u_t u_s A_n u_s^{-1} = \begin{bmatrix} \dfrac{1}{d_n - sc_n} & 0 \\ c_n & d_n - sc_n \end{bmatrix}.$$

Fix $\delta > 0$, choose $s = s_{\delta,n} \geqslant 0$ such that $d_n - sc_n = 1-\delta$ or $1+\delta$. Let $y_n = u_s.x_n$, $y_n' = u_t u_s A_n.x_n = \begin{bmatrix} (1+\delta)^{-1} & 0 \\ c_n & (1+\delta) \end{bmatrix}.y_n$. By passing to a subsequence, assume that $y_n \to y_\infty$ and $y_n' \to y_\infty'$ both in Y, where $y_\infty' = \begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix}.y_\infty$. Then

$$Y = \overline{U^+ y_\infty'} = \begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix} \overline{U^+ y_\infty} = \begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix} Y.$$

Let $B^+ = \{a_t u_s : s \in \mathbb{R}_+, t \in \mathbb{R}\}$, where $a_t = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}$, then Y is B^+ invariant. The theorem is immediate by the following lemma. □

Lemma 2.1.11

We have:

1. $B \curvearrowright \mathrm{SL}(2,\mathbb{R})/\Gamma$ is minimal.
2. $B^+ \curvearrowright \mathrm{SL}(2,\mathbb{R})/\Gamma$ is minimal.

§2.2 22.3.4: Weak Oppenheim conjecture I

Theorem 2.2.1 (Weak Version of Oppenheim Conjecture)

Let Q be a nondegenerate, indefinite, quadratic form with real coefficients in 3 variables. Assume Q is not proportional to a quadratic form with $\mathbb{Q}$-coefficients. Then $\overline{Q(\mathbb{Z}^3 \setminus (0))}$ contains 0.

Example 2.2.2

$Q(x,y,z) = xy - \sqrt{2}z^2$, the statement is trivial for Q because $Q(1,0,0) = 0$.

Definition 2.2.3. Define the special orthogonal group of Q as

$$\mathrm{SO}(Q,\mathbb{R}) := \{g \in \mathrm{SL}(3,\mathbb{R}), Q \circ g = Q\}, \quad \mathrm{SO}(Q,\mathbb{Z}) := \{g \in \mathrm{SL}(3,\mathbb{Z}), Q \circ g = Q\}.$$

Definition 2.2.4. A subgroup $\Lambda \leqslant \mathbb{R}^N$ is a **lattice** if Γ is discrete and cocompact.

Definition 2.2.5. $\Lambda \leqslant \mathbb{R}^n$ is a **unimodular lattice** if Λ is a lattice and $\mathrm{Vol}(\mathbb{R}^N/\Lambda) = 1$.

Definition 2.2.6. Let $X_N := \left\{\text{unimodular lattice in } \mathbb{R}^N\right\}$ equipped with the **Chabauty topology**.

Remark 2.2.7 — A sequence $\{\Lambda_N\} \subseteq X_N$ converges to $\Lambda_\infty \in X_N$ iff we can find a basis $\{v_1^n, v_2^n, \cdots, v_N^n\}$ of Λ_n such that for every $i = 1, 2, \cdots, N$, $v_i^n \to v_i^\infty \in \mathbb{R}^N$, and $\Lambda_\infty = \mathbb{Z}v_1^\infty \oplus \mathbb{Z}v_2^\infty \oplus \cdots \oplus \mathbb{Z}v_N^\infty$.

Remark 2.2.8 — $\mathrm{SL}(N, \mathbb{R})$ naturally acts on X_N.

Lemma 2.2.9

The map $g \mapsto g \cdot \mathbb{Z}^N$, induces a homeomorphism $\mathrm{SL}(N, \mathbb{R})/\mathrm{SL}(N, \mathbb{Z}) \cong X_N$.

Definition 2.2.10. For a discrete subgroup $\Lambda \leqslant \mathbb{R}^N$, define $\delta(\Lambda) := \inf\{\|v\| : v \neq 0 \in \Lambda\}$.

Fact 2.2.11. $\delta : X_N \to \mathbb{R}_{>0}$ is continuous.

Lemma 2.2.12 (Mahler's Criterion)

$\delta : X_N \to \mathbb{R}_{>0}$ is proper, i.e. $(x_n) \subseteq X_N$ diverges iff $\delta(x_n) \to 0$.

Remark 2.2.13 — (x_n) diverges iff for every compact $K \subseteq X_N$, (x_n) will eventually out of K. This is equivalent to (x_n) has no convergent subsequence.

Proof. **The "if" part:** If $\delta(x_n) \to 0$, we need to show (x_n) is divergent. This is immediate by (x_n) has a convergence subsequence.

The "only if" part: By passing to a subsequence, $\exists \varepsilon > 0$ such that $\delta(x_n) \geqslant \varepsilon > 0$. The statement follows by the following claim. □

Claim 2.2.14. $\exists C = C(N, \varepsilon) > 0$, such that every Λ with $\delta(\Lambda) > \varepsilon$ has a basis $(v_1, v_2, \cdots, v_N)$ with $\|v_i\| \leqslant C(N, \varepsilon), i = 1, 2, \cdots, N$.

Proof. Consider the projection $p : \mathbb{R}^N \to \mathbb{R}^N/\Lambda$. Then p is not injective restricted to $[-1, 1]^N$. There will be $v \neq w \in [-1, 1]^N$ such that $v - w \in \Lambda$ and $\|v - w\| \leqslant 2\sqrt{N}$. Now we pick $w_1 \in \Lambda$ that minimize $\{\|v\| : v \neq 0 \in \Lambda\}$, then $\|w_1\| \leqslant 2\sqrt{N}$.

Let $\pi_1^{\perp} : \mathbb{R}^N \to w_1^{\perp}$ be the orthogonal projection. Consider $\pi_1^{\perp}(\Lambda) \leqslant w_1^{\perp} \cong \mathbb{R}^{N-1}$. Then:

1. $\pi_1^{\perp}(\Lambda)$ is discrete and is a lattice in $w_1^{\perp}$.
2. $1 = \|\Lambda\| = \|w_1\| \, \|\pi_1^{\perp}(\Lambda)\| \geqslant \varepsilon \|\pi_1^{\perp}(\Lambda)\|$.

Then $\|\pi_1^{\perp}(\Lambda)\| \leqslant \varepsilon^{-1}$ and $\delta(\pi_1^{\perp}(\Lambda))$ is controlled by a function of ε. We can reduce to the situation of dimensional $N - 1$. □

Lemma 2.2.15

Let Q be a nondegenerate quadratic form in N variables with real coefficients, then the followings are equivalent:

(i) $\overline{Q(\mathbb{Z}^N \setminus \{0\})}$ contains 0.

(ii) $\mathrm{SO}(Q, \mathbb{R}) \cdot \mathbb{Z}^N$ is unbounded in X_N.

Proof. **(ii) $\Longrightarrow$ (i):** By assumption, $\exists g_n \in \mathrm{SO}(Q, \mathbb{R})$ such that $(g_n \cdot \mathbb{Z}^N)_n$ diverges in X_N. By Mahler's Criterion 2.2.12, $\delta(g_n \cdot \mathbb{Z}^N) \to 0$, hence $\exists v_n \neq 0 \in \mathbb{Z}^N$ such that $g_n v_n \to 0$. □

Consider $N = 3$, Q indefinite.

Fact 2.2.16. $\exists g_Q \in \mathrm{SL}(3,\mathbb{R})$ such that $Q = \lambda(Q_0 \circ g_Q)$, where $\lambda \in \mathbb{R} \setminus \{0\}$ and $Q_0 = 2xz - y^2$.

Then $\mathrm{SO}(Q,\mathbb{R}) = g_Q^{-1}\mathrm{SO}_{Q_0}(\mathbb{R})g_Q$, hence $\mathrm{SO}(Q,\mathbb{R})\cdot\mathbb{Z}^3$ is unbounded iff $\mathrm{SO}(Q_0,\mathbb{Z})g_Q\cdot\mathbb{Z}^3$ is unbounded.

Theorem 2.2.17

Every orbit of $\mathrm{SO}(Q_0,\mathbb{R})$ on $X_3 \cong \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ either unbounded or is closed.

Proof of Theorem 2.2.1 assuming Theorem 2.2.17. Otherwise, $\mathrm{SO}(Q,\mathbb{R})\cdot\mathbb{Z}^3$ is compact. Then $\mathrm{SO}(Q,\mathbb{Z}) := \mathrm{SO}(Q,\mathbb{R}) \cap \mathrm{SL}(3,\mathbb{Z})$ is cocompact in $\mathrm{SO}(Q,\mathbb{R})$. We want to show that Q is proportional to a $\mathbb{Q}$-coefficient quadratic form. Otherwise, $\exists \alpha, \beta$ coefficients of Q such that $\dfrac{\alpha}{\beta} \notin \mathbb{Q}$. Then $\exists \sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ such that $\sigma(Q)$ is not proportional to Q.

Step 1: $\mathrm{SO}(Q,\mathbb{R})^0 = \mathrm{SO}(\sigma(Q),\mathbb{R})^0 = \sigma(\mathrm{SO}(Q,\mathbb{R}))^0$.
$\mathrm{SO}(Q,\mathbb{R})^0 \supseteq \mathrm{SO}(Q,\mathbb{Z}) \cap \mathrm{SO}(Q,\mathbb{R})^0 = \Gamma \subseteq \sigma(\mathrm{SO}(Q,\mathbb{R}))^0$. Consider

$$\mathrm{SL}(3,\mathbb{R}) \curvearrowright \mathrm{Sym} := \{\mathbb{R} - \text{Symmetric matrices}\}\,, \quad g.M = gMg^t.$$

Let $\psi : \mathrm{SO}(Q,\mathbb{R}) \to \mathrm{Sym}, g \mapsto g.\sigma(Q)$, then ψ factors through $\mathrm{SO}(Q,\mathbb{R})/\mathrm{SO}(Q,\mathbb{Z}) \to \mathrm{Sym}$. Hence, the image of ψ is compact. The following two facts shows that $\mathrm{SO}(Q,\mathbb{R})^0$ fixes $\sigma(Q)$ and the statement follows immediately:

1. $\mathrm{SO}(Q,\mathbb{R})^0$ is generated by one-parameter unipotent flows.
2. For every unipotent flow $\{u_t\}$ and $M \in \mathrm{Sym}$, either $\{u_t.M\}$ is unbounded or M is fixed by $\{u_t\}$.

Step 2: A direct compute shows that $\mathrm{SO}(Q,\mathbb{R})^0 = \mathrm{SO}(\sigma(Q),\mathbb{R})^0$ implies $\sigma(Q)$ is proportional to Q. □

§2.3 22.3.8: Weak Oppenheim conjecture II

Theorem 2.3.1

An orbit of $H = \mathrm{SO}(Q_0,\mathbb{R})$ on X_3 is either:

(i) unbounded.

(ii) compact.

(iii) its closure contains a $\{v_s\}_{s\geqslant 0}$-orbit or a $\{v_s\}_{s\leqslant 0}$-orbit, where $v_s = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Fact 2.3.2. Theorem 2.3.1 $\implies$ Theorem 2.2.17.

Now, we calculate H. Let $\mathfrak{h}$ be the Lie algebra of H, then

$$\mathfrak{h} = \left\{ X \in \mathfrak{gl}(3,\mathbb{R}) : M_0 X = -X^t M_0, M_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \right\}.$$

After some tough work, we get

$$\mathfrak{h}=\left\{\begin{bmatrix}x_1 & x_2 & 0\\ x_4 & 0 & x_2\\ 0 & x_4 & -x_1\end{bmatrix}: x_1,x_2,x_4\in\mathbb{R}\right\}.$$

In particular,

$$u_t:=\exp\left(t\begin{bmatrix} & 1 & \\ & & 1\\ & & \end{bmatrix}\right)=\begin{bmatrix}1 & t & t^2/2\\ & 1 & t\\ & & 1\end{bmatrix}, a_t=\exp\left(t\begin{bmatrix}1 & & \\ & 0 & \\ & & -1\end{bmatrix}\right)=\begin{bmatrix}e^t & & \\ & 1 & \\ & & e^{-t}\end{bmatrix}\in H.$$

Proof of Theorem 2.3.1. Take $x_0\in X_3$ such that $Y_0=\overline{H.x_0}\neq H.x_0$ and $H.x_0$ is bounded. Let $\Omega:=\{y\in Y_0: Hy \text{ is open in } Y_0\}$. We need the following lemma.

Lemma 2.3.3

$\Omega\neq Y_0$.

Proof. Otherwise, every orbit of H in Y_0 is closed, in particular Hx_0 is closed. Contradiction. □

Continued proof of Theorem 2.3.1. Let Y_1 be a nonempty U-minimal nonempty subset of $Y_0\setminus\Omega$, where $U=\{u_t\}$. If $y\in Y_0\setminus\Omega$, then $H.y$ is NOT open in Y_0, hence $\exists y_n\in Y_0$ such that $y_n\notin H.y, y_n\to y$.

Case 1: Y_1 is closed U-orbit. Impossible.

Case 2: Y_1 is NOT a closed U-orbit but Y_1 is A-stable, where $A=\{a_t\}$. We want to find a $\{v_s\}_{s\geqslant 0}$-orbit or a $\{v_s\}_{s\leqslant 0}$-orbit inside Y_0.

Fact 2.3.4. The map $\mathfrak{h}\oplus\mathfrak{h}^\perp\to X_3, (h,w)\mapsto\exp(h)\exp(w).x_1$ is a local diffeomorphism.

$$\mathfrak{h}=\left\{X\in\mathfrak{gl}(3,\mathbb{R}): M_0X=-X^tM_0, M_0=\begin{bmatrix} & & 1\\ & -1 & \\ 1 & & \end{bmatrix}\right\}.$$

$$\mathfrak{h}^\perp=\left\{X\in\mathfrak{gl}(3,\mathbb{R}): \operatorname{tr}X=0, M_0X=XM_0, M_0=\begin{bmatrix} & & 1\\ & -1 & \\ 1 & & \end{bmatrix}\right\}.$$

Fact 2.3.5. $\mathfrak{sl}(3,\mathbb{R})=\mathfrak{h}\oplus\mathfrak{h}^\perp$, moreover $\mathfrak{h}^\perp$ is invariant under $\mathrm{Ad}(H)$.

In this case, there exists $x_1\in Y_1, A_n\to\mathrm{Id}, A_n.x_1\in Y_0$ where $A_n\notin H$. Write $A_n=\exp(h_n)\exp(w_n), h_n\in\mathfrak{h}, w_n\neq 0\in\mathfrak{h}^\perp$. Let $x_n=\exp(w_n)x_1\in Y_0$, $\|w_n\|\to 0$.

Lemma 2.3.6

For δ sufficiently small, n sufficiently large, there exists $t_{n,\delta}\in\mathbb{R}$ such that:

(i) $\|\mathrm{Ad}(u_{t_{n,\delta}})w_n\|\in\left[10^{-10}\delta, 10^{10}\delta\right]$.

(ii) Every limit of $\mathrm{Ad}(u_{t_{n,\delta}})w_n$ is in Lie algebra of $\{v_s\}$.

Let $y_{n,\delta}=u_{t_{n,\delta}}.x_1$, $z_{n,\delta}=u_{t_{n,\delta}}.x_n$. As $x_n=\exp(w_n)x_1$, hence $z_{n,\delta}=\exp(\mathrm{Ad}(u_{t_{n,\delta}})w_n)y_{n,\delta}$. By passing to a subsequence, we assume that

$$z_{n,\delta}\to z_{\infty,\delta},\quad \mathrm{Ad}(u_{t_{n,\delta}})w_n\to w_{\infty,\delta},\quad y_{n,\delta}\to y_{\infty,\delta}.$$

Then $z_{n,\delta}\in Y_0, y_{\infty,\delta}\in Y_1$ and $w_{\infty,\delta}$ is in Lie algebra of $\{v_s\}$. Note that v_s commutes with u_t, we get $\exp(w_{\infty,\delta})Y_1\subseteq Y_0$. By assumption, Y_1 is A-stable, after some calculation, $a_t\exp(w_{n,\delta})a_t^{-1}$ can go through ever v_s for $s\geqslant 0$ or $s\leqslant 0$.

Case 3: Y_1 is NOT A-stable. Because

Lemma 2.3.7

For δ sufficiently small, for n sufficiently large. There exists $s_{n,\delta}, t_{n,\delta} \in \mathbb{R}$, $h_{n,\delta} \oplus w_{n,\delta} \in \mathfrak{h} \oplus \mathfrak{h}^{\perp}$, such that:

(i) $u_{s_{n,\delta}} \exp(\mathrm{Ad}(u_t)h_n) \exp(\mathrm{Ad}(u_t)w_n) = \exp(h_{n,\delta}) \exp(w_{n,\delta})$.

(ii) $\max\{\|h_{n,\delta}\|, \|w_{n,\delta}\|\} \in \left[10^{-100}\delta, 10^{100}\delta\right]$.

(iii) Every limit of $h_{n,\delta}$ is in Lie algebra of $\{a_t\}$, every limit of $w_{n,\delta}$ is in Lie algebra of $\{v_s\}$.