# ODE: Qualitative Theory (Spring 2022, Shaobo Gan)

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# 1 Basic Concepts

## §1.1 Basic notions and results

Assume $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, x) \mapsto f(t, x)$ continuous, consider the **equation** (or **system**)

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x).$$

A differentiable function $\gamma : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ is said to be a **solution** (or **solution curve**), if for every $t \in (a, b)$,

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = f(t, \gamma(t)).$$

The **graph** of $\gamma$ is

$$\{(t, \gamma(t)) : t \in (a, b)\} \subset \mathbb{R} \times \mathbb{R}^n.$$

For $t_0 \in (a, b)$, let $x_0 = \gamma(t_0)$, then $\gamma$ is called the solution of the **initial value problem**

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

**The initial value problem has a unique solution**: Let $\gamma_i : (a_i, b_i) \to \mathbb{R}^n$ be two solutions of the initial value problem. Then there exists $\delta > 0$, $(t_0 - \delta, t_0 + \delta) \subset (a_1, b_1) \cap (a_2, b_2)$, such that $\gamma_1(t) = \gamma_2(t), \forall t \in (t_0 - \delta, t_0 + \delta)$,

**Theorem 1.1.1** (Existence and Uniqueness Theorem)
$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, f(t, x)$ continuous, given $t_0 \in \mathbb{R}^n, x_0 \in \mathbb{R}^n, a > 0, b > 0$, consider the region

$$R = R(t_0, x_0, a, b) = \{(t, x) : |t - t_0| \leqslant a, |x - x_0| \leqslant b\}.$$

If $f$ is Lipchitz in $x$ on $R$, i.e. $\exists L > 0$, $\forall (t, x_1), (t, x_2) \in R$,

$$|f(t, x_1) - f(t, x_2)| \leqslant L|x_1 - x_2|,$$

then the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution on $[t_0 - h, t_0 + h]$, where $h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(t,x)\in R} |f(t, x)|$.

**Remark 1.1.2** — The solution is denoted as $\varphi(t; t_0, x_0)$.

**Corollary 1.1.3**

When $f \in C^1$, the existence and uniqueness theorem holds.

Denotes the **maximal interval** of $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ as $I(t_0, x_0)$, it is an open interval.

**Corollary 1.1.4**

Assume $f \in C^1$ and $|f(x)| \leqslant A(t)|x| + B(t)$, then the maximal interval of the initial value problem is $(-\infty, +\infty)$.

## §1.2 Flows

Now we consider the **autonomous equation**

$$\dot{x} = f(x).$$

$\mathbb{R}^n$ is called the **phase space** and $\mathbb{R} \times \mathbb{R}^n$ is called the **generalized phase space**.

The solution of the initial value problem $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ is denoted as $\varphi(t, x_0)$, the set

$$\mathrm{Orb}(x_0) \coloneqq \{\varphi(t, x_0) : t \in I(x_0)\} \subset \mathbb{R}^n$$

is called the **orbit** pass by $x_0$.

**Corollary 1.2.1** (Continuous Dependence on the Initial Value)

Assume $f \in C^1$, then $U = \{(t, x) : t \in I(x)\}$ is open and $\varphi : U \to \mathbb{R}^n, (t, x) \mapsto \varphi(t, x)$ is continuous.

**Theorem 1.2.2**

$f(x) \in C^1$, then:

1. $\varphi_0(x) = x$ for every $x \in \mathbb{R}^n$.
2. $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for every $s \in I(x), t \in I(\varphi(s, x))$.

**Definition 1.2.3.** $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, continuous, is said to be a **(continuous) flow** if:

(i) $\psi(0, x) = x$,

(ii) $\psi(t, \psi(s, x)) = \psi(t + s, x)$.

**Remark 1.2.4** — The solution of an autonomous equation is a **local flow.**

**Corollary 1.2.5**

Let $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a flow, then $\psi_t \coloneqq \psi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are homeomorphisms.

**Remark 1.2.6 —** Consider the group of self-homeomorphisms of $\mathbb{R}^n$, denotes as $\mathrm{Homeo}(\mathbb{R}^n)$, then $\psi : \mathbb{R} \to \mathbb{R}^n$ is a group homomorphism. More generally, we can consider $G \to \mathrm{Homeo}(\mathbb{R}^n)$ for some group $G$.

**Proposition 1.2.7**

Assume $f$ is a $C^1$ vector field, then the orbits of the flow generated by $f$ are either coincide or disjoint.

$\bigcup_{x\in\mathbb{R}^n} \mathrm{Orb}(x)$ forms a partition of $\mathbb{R}^n$, is called the **orbit space**. For each orbit, orient it to indicate the direction of motion, the family of the oriented orbit $\varphi(t,x)/f(x)$ is called the **phase portrait**.

A point $x_0 \in \mathbb{R}^n$ with $f(x_0) = 0$ is called a **critical point** (or a **singularity**, **equilibrium**). The orbit $\mathrm{Orb}(x_0)$ is a single point $\{x_0\}$.

**Example 1.2.8**

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t,x_0) = x_0 e^t$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$.

**Example 1.2.9**

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t,x_0) = \frac{x_0}{1-x_0 t}$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$. But the phase portrait is different from the former examples, because the orientations on $\mathbb{R}_-$ are different.

**Theorem 1.2.10**

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ vector field, $\beta(x) : \mathbb{R}^n \to \mathbb{R} \in C^1$ and $\beta(x) > 0$. Then the equations $\dot{x} = f(x)$ and $\dot{x} = \beta(x)f(x)$ have the same phase portraits.

*Proof.* $\varphi : I \to \mathbb{R}^n$ a solution of $f$. Find a $C^1$ diffeomorphism $h : J \to I$ such that $\varphi \circ h$ is the solution of $\dot{x} = \beta(x)f(x)$. It suffices that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=h(s)} \varphi(t) \cdot \frac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi \circ h(s)) f(\varphi \circ h(s)),$$

i.e. $\frac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi \circ h(s)) > 0$, it is an initial value problem. It shows that the maximal solution curve of $f$ is contain in some solution curve of $\beta f$. ☐

**Theorem 1.2.11** (Differentiable Dependence on the Initial Value)
Assume $f \in C^1$, it generates the flow $\phi_t$, then $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$.

**Exercise 1.2.12.**

$$\frac{\partial}{\partial t}\frac{\partial \phi(t,x)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \phi(t,x)}{\partial t}.$$

Let $\Phi(t,x) = \Phi_t(x) = \frac{\partial \phi(t,x)}{\partial t}$, then $\Phi$ is the solution of the equation

$$\begin{cases} \frac{\mathrm{d}y(t)}{\mathrm{d}t} = A(t)y(t), A(t) = Df(\phi_t(x)) \\ y(0) = \mathrm{Id} \end{cases}.$$

The equation is called the **variation equation** of $f(x)$ along $\phi_t(x)$.

**Lemma 1.2.13**
$f \in C^1$, $\Phi(t,x)$, then

$$\Phi_t(\phi_s(x))\Phi_s(x) = \Phi_{t+s}(x).$$

**Remark 1.2.14** — This property is called the **cocycle** condition.

We already know that $\phi_t$ are self-homeomorphisms of $\mathbb{R}^n$, and lemma 1.2.13 shows that the differential is invertible, hence $\phi_t$ are diffeomorphisms. Define

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^n \times \mathbb{R}^n \\ (t,x,v) &\mapsto (\phi_t(x), \Phi_t(x)v). \end{aligned}$$

**Proposition 1.2.15**
$\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a flow.

**Remark 1.2.16** — We call $\Phi_t$ is a skew product flow of $\phi_t$.

**Theorem 1.2.17**

$$\Phi_t(x)f(x) = f(\phi_t(x)).$$

If $\psi$ is a $C^1$ flow, let

$$g(x) = \left.\frac{\partial \psi(t,x)}{\partial t}\right|_{t=0},$$

then $\psi(t,x_0)$ solve the initial value problem $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$. Because

$$\frac{\partial \psi(t,x_0)}{\partial t} = \left.\frac{\partial \psi(t+s,x_0)}{\partial s}\right|_{s=0} = \left.\frac{\partial \psi(s,\psi(t,x_0))}{\partial s}\right|_{s=0} = g(\psi(t,x_0)).$$

## §1.3 Equations on manifolds

Let $M$ be a closed smooth manifold, $X$ is a $C^1$ vector field on $M$. Then $X$ is bounded, hence the maximal intervals are $(-\infty, +\infty)$. Consider the equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = X(x) \\ x(0) = x_0 \end{cases},$$

then the solution $\varphi(t, x)$ generates a flow.

# 2 Linear Systems

## §2.1 Plane linear sigularities

Consider the equation

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}, \quad (x,y) \in \mathbb{R}^2.$$

It is said to be a **plane linear system** if $f, g$ both linear functions of $x, y$, i.e.

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad a,b,c,d \in \mathbb{R}.$$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then $(0,0)$ is the only sigularity of the vector field, elementary singularity.

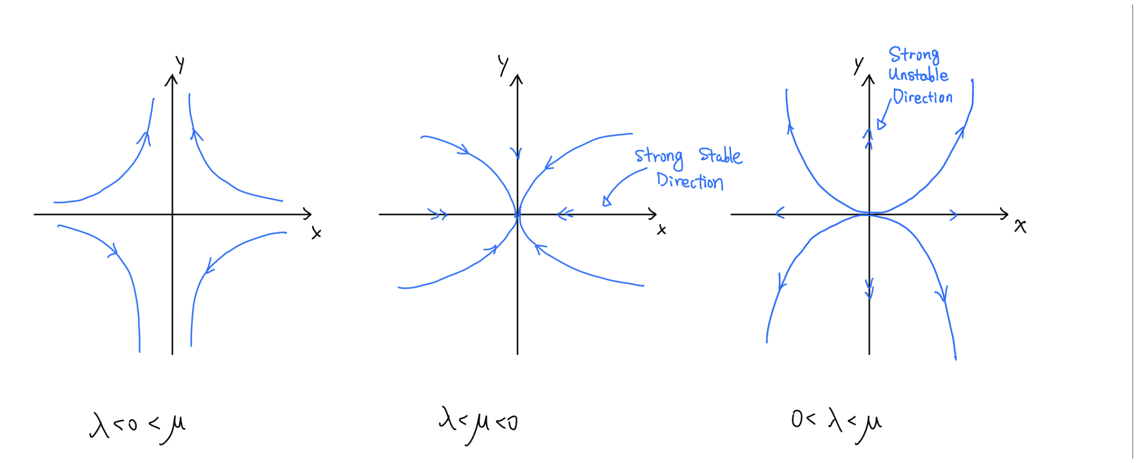
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, consider the Jordan form of $A$. There are four cases:

I. Two different real eigenvalues: $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$.

   i. $\lambda < 0 < \mu$ : the origin is called a **saddle point**.

   ii. $\lambda < \mu < 0$ : the origin is called a **stable node**.

   iii. $0 < \lambda < \mu$ : the origin is called a **unstable node**.




II. Conjugated imaginary eigenvalues: $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \beta > 0$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

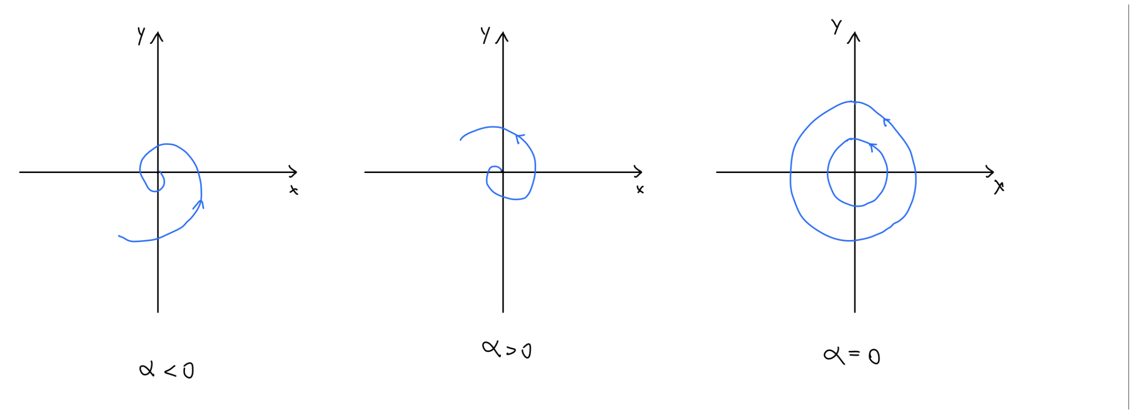
   If we consider this equation in the polar coordinates, it turns $\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$.

   i. $\alpha < 0$, the origin is called a **stable focus**.

   ii. $\alpha > 0$, the origin is called a **unstable focus**.

   iii. $\alpha = 0$, the origin is called a **center**.

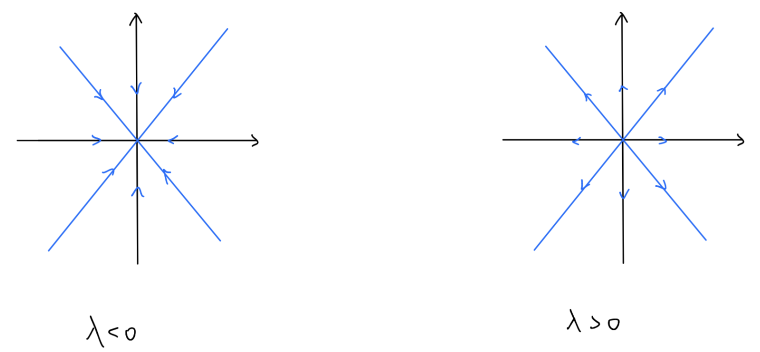
**Definition 2.1.1.** $\varphi_t$ a flow. If $p$ is not a singularity and $\exists T > 0$, such that $\varphi_T(p) = p$. Then $p$ is called a **periodic point**, $\mathrm{Orb}(p)$ is called a **periodic orbit**. If $p$ is a periodic point, the smallest $T > 0$ is called the **minimum positive period**.




III. Two same real eigenvalues, diagonalizable: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$.

  i. $\lambda < 0$, the origin is called a **stable critical node**.

  ii. $\lambda > 0$, the origin is called a **unstable critical node**.

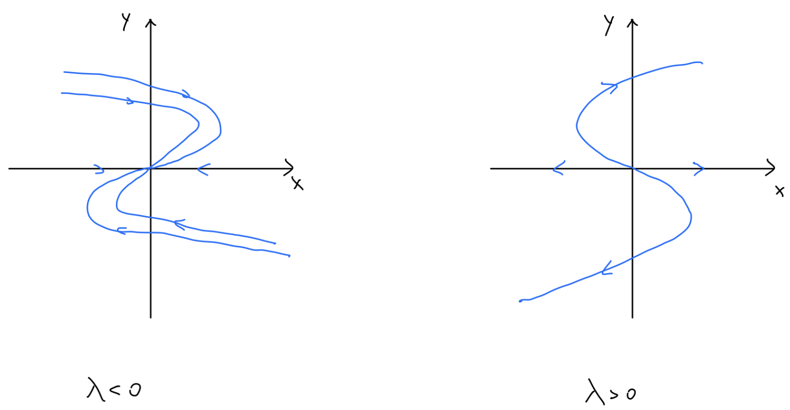



IV. Two same real eigenvalues, not diagonalizable: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}(x_0 + t y_0) \\ e^{\lambda t} \end{bmatrix} y_0$,

or $x(t) = \frac{x_0}{y_0} y(t) + \frac{y(t)}{\lambda} \ln \frac{y(t)}{y_0}$.

  i. $\lambda < 0$, the origin is called a **stable unidirectional node**.

  ii. $\lambda > 0$, the origin is called a **unstable unidirectional node**.



**Exercise 2.1.2.** Draw the phase portraits of non-elementary plane systems (i.e. the determinant is 0).

## §2.2 Topological conjugacies between linear systems

**Definition 2.2.1.** Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ homeomorphisms. $f$ and $g$ are said to be **topologically conjugate** if there exists $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f = g \circ h$.

**Remark 2.2.2** — Conjugacy is a equivalence relation.

**Definition 2.2.3.** Let $\varphi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ be two flows, we call $\varphi_t$ and $\psi_t$ are conjugate if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ \varphi_t = \psi_t \circ h$. Let $X, Y$ be two $C^1$ vector fields on $\mathbb{R}^n$, we call $X, Y$ are conjugate if the flows generated by them, respectively, are conjugate.

**Example 2.2.4**

$A, B \in M(n, \mathbb{R})$ are similar, then $\dot{x} = Ax$ and $\dot{y} = By$ are conjugate.

$f, g : \mathbb{R}^n \to \mathbb{R}^n$ $C^1$ vector fields, generate flows $\phi_t, \psi_t$. Let $x = h(y) : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ diffeomorphism gives the conjugate, i.e., $h\psi_t(y) = \phi_t h(y)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} h(y) = f(h(y)) \implies D_{h(y)} g(y) = D_{h(y)} \frac{\mathrm{d}y}{\mathrm{d}t} = f(h(y)).$$

If there exists a $C^1$ diffeomorphism conjugate $e^{Bt}y$ to $e^{At}x$ via $x = h(y)$, i.e. $h(e^{Bt}y) = e^{At}h(y)$. Then $Dh_0 e^{Bt} = e^{At} Dh_0$, hence $Dh_0 B = ADh_0$. It shows that $C^1$ conjugate generically not hold even if topologically conjugate.

**Proposition 2.2.5**

Assume $f, g$ $C^1$ vector fields generate $\phi_t, \psi_t$, let $h$ be a conjugate between $\phi_t$ and $\psi_t$. Then:

1. $h(\mathrm{Orb}(x, \phi)) = \mathrm{Orb}(hx, \psi)$.
2. $h$ maps the singularities of $f$ to the singularities of $g$.
3. $h$ maps the periodic orbits of $f$ to the periodic orbits of $g$. Moreover, it preserves the minimum positive period.

**Example 2.2.6**

$\dot{x} = -2x$ and $\dot{y} = -4y$ are conjugate.

Let $h : \mathbb{R} \to \mathbb{R}, h(0) = 0$. Take $x_0, y_0 > 0$, let $h(x_0) = y_0$, then $h(e^{-2t}x_0) = e^{-4t}y_0$ or $h(x) = \left(\frac{x}{x_0}\right)^2 y_0$. The construction for the negative part is similar.

**Exercise 2.2.7.** $\lambda\mu \neq 0$, show that $\dot{x} = \lambda x$ is conjugate to $\dot{y} = \mu y$ if and only if $\lambda\mu > 0$.

**Proposition 2.2.8**

$\phi_t^i, \psi_t^i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ are topologically conjugate by $h_i, i = 1, 2$. Then $\phi_t^1 \times \phi_t^2$ and $\psi_t^1 \times \psi_t^2$ are topologically conjugate by $h_1 \times h_2$.

**Example 2.2.9**

$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$ are conjugate.

*Proof.* $\phi_t(x, y) = e^{-t}(x, y)$ and $\psi_t(x, y) = e^{-t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. For every $(x, y) \neq (0, 0)$, there exists unique $t = t(x, y)$ such that $\phi_t(x, y) \in \mathbb{S}^1$. Let $h(x, y) := \psi_{-t}\phi_t(x, y)$, where $t = t(x, y)$, then $h$ gives the conjugate. $\square$

**Exercise 2.2.10.** Show that $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -y \end{cases}$ are conjugate.

Classification of elementary plane linear systems:

(I) Stable: node, critical node, unidirectional node, focus.

(II) Unstable: node, critical node, unidirectional node, focus.

(III) Saddle point.

(IV) Center.

**Definition 2.2.11.** The linear system $\dot{x} = Ax$ in $\mathbb{R}^n$ is called **hyperbolic** if the real parts of eigenvalues of $A$ are nonzero. The **(stable) index** of $A$ is the number of eigenvalues with negative real parts, denoted by $\operatorname{Ind} A$.

**Theorem 2.2.12**

Two plane hyperbolic linear system $\dot{x} = Ax, \dot{y} = By$ are topologically conjugate if and only if $\operatorname{Ind} A = \operatorname{Ind} B$.

*Proof.* "$\Longrightarrow$": Let $W_A^s = \left\{x : e^{tA}x \to 0, t \to \infty\right\}$, $W_B^s = \left\{x : e^{tB}x \to 0, t \to \infty\right\}$, then $h$ and $h^{-1}$ preserves the stable manifolds. Then $\operatorname{Ind} A = \dim W_A^s = \dim W_B^s = \operatorname{Ind} B$. $\square$

**Example 2.2.13**

Consider $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ and $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$ with the same phase portraits are not topologically conjugate. Because the topologically conjugate preserves the minimum positive orbits.

**Definition 2.2.14.** $\phi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ flows, $h$ is a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ maps the orbit of $\phi$ to the orbit of $\psi$ preserves the orientation. Then $\phi$ and $\psi$ is called **topologically equivalent** or **flow equivalent**.

**Theorem 2.2.15** (Grobman-Hartman)
If $x_0$ is a hyperbolic singularity of $f(x)$, then the flows generated by $\dot{x} = f(x)$ and $\dot{y} = Ay$ where $y = Df(x_0)$ are topologically conjugate near 0.

## §2.3 Non-autonomous linear systems

$A : \mathbb{R} \to M(n, \mathbb{R})$ continuous, consider the equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n$$

a non-autonomous linear system.

**Theorem 2.3.1**
The followings hold:

1. The initial problem of the equation exist the unique solution.
2. The maximal interval of any solution is $(-\infty, \infty)$.
3. All solutions of the equation form an $n$-dimensional linear space $S$.

**Theorem 2.3.2** (Liouville's Formular)
Assume $X(t)$ is a solution of $\dot{x} = A(t)x$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \det X(t) = \operatorname{tr} A(t) \det X(t),$$

hence $\det X(t) = \det X(t_0) \exp \int_{t_0}^t \operatorname{tr} A(s) \mathrm{d}s$.

Let $X_1(t), X_2(t), \cdots, X_n(t)$ be a basis of $S$, let

$$X(t) \coloneqq [X_1(t), X_2(t), \cdots, X_n(t)] \in \mathrm{GL}(n, \mathbb{R}),$$

it called a **fundamental solution** of the equation. The fundamental solution of

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = A(t)X \\ X(t_0) = I_n \in \mathrm{GL}(n, \mathbb{R}) \end{cases}$$

is called the **standard fundamental solution**.

If $X(t), Y(t)$ are two fundamental solutions, suppose $Y(0) = X(0)C$, then

$$\frac{\mathrm{d}X(t)C}{\mathrm{d}t} = \frac{\mathrm{d}X(t)}{\mathrm{d}t}C = A(t)X(t)C,$$

is a non-degenerate solution of $\frac{\mathrm{d}X}{\mathrm{d}t} = AX$. By the uniqueness, we get $Y(t) = X(t)C$.

**Example 2.3.3**

$A(t) \equiv A$, the fundamental solution of $\dot{x} = Ax$ is

$$e^{tA} = \mathrm{Id} + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{k!}t^kA^k + \cdots.$$

**Example 2.3.4**

$\dot{x} = f(x), x \in \mathbb{R}^n$, where $f \in C^1$, generates the flow $\varphi_t(x)$. Consider $\Phi_t(x) = \frac{\partial}{\partial t}\varphi_t(x)$ and the variation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = Df_{\varphi_t(x)}\Phi_t(x).$$

Given $x \in \mathbb{R}^n$, let $A(t) := Df_{\varphi_t(x)}$, then $\Phi_t(x)$ is the standard fundamental solution $(t_0 = 0)$ of $\dot{x} = A(t)x$. Consider two special types of orbits:

- $x$ is a singularity, denoted by $\sigma$. Then $\varphi_t(\sigma) = \sigma$, $\dot{x} = Ax$ where $A = Df(\sigma)$.
- $x$ is a periodic point, denoted by $p$, the minimum period $T > 0$. Then $A$ is $T$-periodic.

## §2.4 Periodic linear systems

**Definition 2.4.1.** The equation $\dot{x} = A(t)x$ satisfies $A(t+T) = A(t)$ for some $T > 0$ is called a **periodic linear systems**.

**Theorem 2.4.2** (Floquet)

Assume $\dot{x} = A(t)x$ is a $T$-periodic linear system, if $X$ is a fundamental solution, then $X(t+T)$ is a fundamental solution, i.e. $\exists C \in \mathrm{GL}(n, \mathbb{R})$ such that $X(t+T) = X(t)C$. Moreover, there exists a $T$-periodic map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ and a constant matrix $B \in M(n, \mathbb{C})$ such that $X(t) = P(t)e^{tB}$.

**Lemma 2.4.3**

$\forall C \in \mathrm{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{C})$ such that $C = e^B$.

*Proof.* It suffices to show for Jordan block. This follows by the matrix series

$$\ln(I + N) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}N^k$$

is convergence for nilpotent matrix $N$. □

**Lemma 2.4.4**

$\forall C \in \mathrm{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{R})$ such that $C^2 = e^B$.

*Proof.* Note that the Jordan block of $C^2$ is either:

(i) $\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & \lambda & 1 \\ 0 & \cdots & & \lambda \end{bmatrix}$, where $\lambda > 0$, or

(ii) $\begin{bmatrix} J & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & J & I_2 \\ 0 & \cdots & & J \end{bmatrix}$, where $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a, b \in \mathbb{R}, b > 0$.

And $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ have a real matrix logarithm because $\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\} \cong \mathbb{C} = \{a + bi\}$. □

**Theorem 2.4.5** (Real Form of Floquet Theorem)

Assume $\dot{x} = A(t)x$ is a $T$-periodic linear system, if $X$ is a fundamental solution. Then there exists a **$2T$-periodic** map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ and a constant matrix $B \in M(n, \mathbb{R})$ such that $X(t) = P(t)e^{tB}$.

**Example 2.4.6** ($2T$ is necessary)

Let $\Phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \exp\left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t \right) \exp\left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right)$. Let

$$A(t) = \dot{\Phi}(t)\Phi(t)^{-1} = \begin{bmatrix} -\cos t \sin t & -\sin^2 t \\ \cos^2 t & \cos t \sin t \end{bmatrix},$$

then $A(t)$ is $\pi$-periodic. Then $\Phi(t)$ is a standard fundamental solution of $\dot{x} = A(t)x$, hence $\exists\pi$-periodic $P(t)$ and $B$ such that $\Phi(t) = P(t)e^{tB}$. Then $e^{\pi B} = \begin{bmatrix} -1 & -\pi \\ 0 & -1 \end{bmatrix}$, there is no real matrix $B$ satisfying this equation.

**Definition 2.4.7.** In Floquet theorem, $X(t + T) = X(t)C$. We call $C$ is a **monodromy matrix**. The eigenvalues of $C$ are called **Floquet multipliers**. If $\rho$ is a Floquet multiplier with $\rho = e^{\lambda T}$, then $\lambda$ is called a **Floquet exponent**.

**Corollary 2.4.8**

Consider a $T$-periodic linear system $\dot{x} = A(t)x$. Then there exists a linear transformation (non-autonomous) $x = P(t)y$ such that $\dot{y} = By$.

*Proof.* Let $X(t) = P(t)e^{tB}$ be a fundamental solution, then

$$AX = \dot{X} \implies \dot{P}e^{tB} + PBe^{tB} = APe^{tB},$$

hence $\dot{P} + PB = AP$. Then $APy = \frac{\mathrm{d}}{\mathrm{d}t}(Py) = \dot{P}y + P\dot{y}$, hence $\dot{y} = By$. □

**Remark 2.4.9** — This type of equation is called reducible, which means after some reduction, the equation can become independent with time $t$.

**Corollary 2.4.10**

Let $\lambda$ be a Floquet multiplier of $\dot{x} = A(t)x$. Then there exists a $T$-periodic function $p(t)$ such that $e^{\lambda t}p(t)$ is a solution of the equation $\dot{x} = A(t)x$.

*Proof.* $e^{\lambda T}$ is an eigenvalue of $C$, then $\exists x_0$ such that $Cx_0 = e^{\lambda T}x_0$. Then $X(t)x_0$ is a solution. Let $p(t) = e^{-\lambda t}X(t)x_0$ is $T$-periodic and $e^{\lambda t}p(t)$ is a solution. □

**Corollary 2.4.11**

The equation admits a nonzero $T$-periodic solution if and only if 1 is a Floquet multiplier.

**Corollary 2.4.12**

Assume $\rho_1, \rho_2, \cdots, \rho_n$ are all Floquet multipliers of $\dot{x} = A(t)x$, then

$$\rho_1\rho_2\cdots\rho_n = \det\Phi(T) = \exp\int_0^T \operatorname{tr} A(t)\ \mathrm{d}t.$$

**Example 2.4.13**

The equation $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^2 t & \frac{1}{2}\sin 2t - 1 \\ \frac{1}{2}\sin 2t + 1 & \sin^2 t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ has an unbounded solution. Because the product of two multipliers is $\exp\int_0^\pi 1\ \mathrm{d}t = e^\pi > 1$.

Consider **Hill equation**

$$\ddot{x} + p(t)x = 0,$$

where $p(t)$ is $\pi$-periodic. This is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -p(t)x \end{cases},$$

then $\rho_1\rho_2 = \exp\int_0^\pi \operatorname{tr} A(t)\ \mathrm{d}t = 0$, where $A(t) = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}$.

**Lemma 2.4.14**

If $\rho_1, \rho_2$ both are imaginary numbers, then every solution of Hill equation is bounded.

*Proof.* Because $\rho_1, \rho_2$ are conjugate imaginary numbers, hence $\Phi(\pi)$ is similar to a rotation. Then $\Phi(\pi)^n$ is bounded independent of $n$ and $\Phi(s)$ is bounded for $s \in [0, \pi]$. □

**Definition 2.4.15.** A particular Hill equation with $p(t) = a + \varepsilon\cos 2t$ is called **Mathieu equation**.

**Exercise 2.4.16.** Consider Mathieu equation

$$\ddot{x} + (a + \varepsilon \cos 2t)x = 0.$$

(1) $U = \{(a, \varepsilon) \in [0, 10] \times [-1, 1] :$ every solution is bounded$\}$. Draw the figure of $U$ by some calculation.

(2) Guess some conclusions by the figure of $U$.

**Example 2.4.17**

Let $p(t)$ be a $\pi$-periodic continuous function satisfying

(i) $p(t) \not\equiv 0$.

(ii) $\int_0^\pi p(t)\mathrm{d}t \geqslant 0$.

(iii) $\pi \int_0^\pi |p(t)|\mathrm{d}t \leqslant 4$.

Then every solution of $\ddot{x} + p(t)x = 0$ is bounded.

*Proof.* If Floquet multipliers are conjugate imaginary numbers, the statement follows. Otherwise there is a real Floquet multiplier $\rho \neq 0$. There is a solution $x(t) \not\equiv 0$ such that $x(t+T) = \rho x(t)$. If $x(t)$ has no zeros, assume $x(t) > 0$, we have $\frac{\dot{x}}{x}(\pi) = \frac{\dot{x}}{x}(0)$. Note that

$$0 = \frac{\ddot{x}}{x} + p(t) = \left(\frac{\dot{x}}{x}\right)' + \left(\frac{\dot{x}}{x}\right)^2 + p(t) = 0,$$

take the integral and we get a contradiction. Then there must be some zeros, let $a, b$ be two successive zeros, WLOG, $0 < a < b < \pi$. Assume $x(t) > 0$ in $(a, b)$ and $x(c)$ takes the maximum. Then $\exists \alpha \in (a, c), \beta \in (c, b)$ such that $\dot{x}(\alpha) = \frac{x(c)}{c-a}, \dot{x}(\beta) = \frac{-x(c)}{b-c}$. We have

$$\frac{4}{\pi} \geqslant \int_0^\pi |p(t)|\mathrm{d}t > \int_a^b \left|\frac{\ddot{x}}{x}(t)\right| \mathrm{d}t \geqslant \frac{\int_\alpha^\beta |\ddot{x}(t)|\mathrm{d}t}{x(c)} \geqslant \frac{1}{c-a} + \frac{1}{b-c} \geqslant \frac{4}{a-b},$$

the identity holds if and only if $x \equiv 0$, contradiction. □

Back to Mathieu equation, consider

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0, \quad \omega > 0, \varepsilon < \omega^2.$$

We apply the conclusion of the example, for $\omega < \frac{2}{\pi}$,

$$\int_0^\pi (\omega^2 + \varepsilon \cos 2t)\mathrm{d}t = \omega^2 \pi \leqslant \frac{4}{\pi}.$$

Consider $\varepsilon = 0$, then

$$\Phi(t) = \begin{bmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

is a standard fundamental solution. The monodromy matrix for $(\omega, \varepsilon)$ where $\omega > 0$ is a perturbation of

$$C = \Phi(\pi) = \begin{bmatrix} \cos \omega\pi & \frac{1}{\omega} \sin \omega\pi \\ -\omega \sin \omega\pi & \cos \omega\pi \end{bmatrix}.$$

Note that $|\mathrm{tr}\,\Phi(\pi)| = |2 \cos \omega\pi| < 2$ for $\omega \notin \mathbb{Z}$. Then there is a small neighborhood $U$ of $(\omega, 0)$ such that every solution is bounded.

**Definition 2.4.18.** Let $A : \mathbb{R} \to M(n, \mathbb{R})$ continuous, bounded, assume that

$$\sup \{|A(t)| : t \in \mathbb{R}\} \leqslant \infty.$$

Let $\Phi(t)$ be a standard fundamental solution of the equation $\dot{x} = A(t)x$. For every $v \neq 0 \in \mathbb{R}^n$, define **Lyapunov exponent** of $v$

$$\chi(v) := \limsup_{t\to\infty} \frac{\ln \|\Phi(t)v\|}{t}.$$

> **Exercise 2.4.19.** For every $v \neq 0$, show that $\chi(v) \neq \pm\infty$.

Then $\chi : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties

1. $\chi(\alpha v) = \chi(v)$ for every $\alpha \neq 0$.
2. $\chi(v + w) \leqslant \max \{\chi(v), \chi(w)\}$.
3. If $\chi(v) < \chi(w)$, then $\chi(v + w) = \chi(w)$.

**Fact 2.4.20.** The number of different Lyapunov exponents $\leqslant n$.

**Example 2.4.21**

$\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A$ is a constant matrix. Regard as a $T$-periodic system, then the eigenvalues $\lambda_1, \lambda_2$ of $A$ are Floquet exponents. Lyapunov exponents are

(1) $\lambda_1, \lambda_2$, if $\lambda_1 \neq \lambda_2$ real.

(2) $\lambda = \lambda_1 = \lambda_2$, if $\lambda_1 = \lambda_2$.

(3) $\alpha$, if $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$.

For the $T$-periodic system, assume that $\lambda$ is a Floquet exponent, then $\chi = \mathrm{Re}(\lambda)$ is a Lyapunov exponent. For $n = 2$, $T$-periodic system, we always have

$$\chi_1 + \chi_2 = \mathrm{Re}(\lambda_1 + \lambda_2) = \frac{1}{T} \int_0^T \mathrm{tr}\, A(t)\mathrm{d}t.$$

**Example 2.4.22**

Consider

$$\begin{cases} \dot{x} = (-\mu - (\sin \ln t + \cos \ln t))x \\ \dot{y} = (-\mu + (\sin \ln t + \cos \ln t))y \end{cases},$$

then the solution

$$\begin{cases} x = C_1 e^{-\mu t - t \sin \ln t} \\ y = C_2 e^{-\mu t - t \sin \ln t} \end{cases}.$$

Then $\chi(v) = -\mu+1$ for every $v \neq 0$. But $\chi_1+\chi_2 = -2\mu+2 \neq \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{tr}\, A(t)\mathrm{d}t = -2\mu$. This example is called non-regular.

# 3 Stability and Hyperbolicity

## §3.1 Lyapunov stability

Let $f : \mathbb{R}^n \to \mathbb{R}^n, 0 \in \mathbb{R}^n, f(0) = 0$, generates a (local) flow $\varphi_t(x)$.

**Definition 3.1.1.** 1. $\sigma$ is called **(forward Lyapunov) stable**, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that if $|x - \sigma| < \delta$, then $|\varphi_t(x) - \sigma| < \varepsilon$ for $t \geqslant 0$. Otherwise, we call $\sigma$ is **unstable**.

2. $\sigma$ is called **asymptotically stable**, if
   (i) $\sigma$ is stable,
   (ii) there exists $\delta_0 > 0$, such that if $|x - \sigma| < \delta$, then $\lim_{t\to\infty} \varphi_t(x) = \sigma$.

3. $\sigma$ is called **exponentially stable**, if exists $\delta_0 > 0$, $C \geqslant 1, \lambda > 0$, such that if $|x - \sigma| < \delta$, then $|\varphi_t(x) - \sigma| \leqslant Ce^{-\lambda t}|x - \sigma|$ for $t \geqslant 0$.

Similarly, we can define backward stable, backward asymptotically stable, backward exponentially stable.

**Remark 3.1.2** — If we replace the condition of stability by **given** $t \geqslant 0$, then it always holds by the continuous independence of solutions with respect to initial value.

**Example 3.1.3**

For the equation in polar coordinates

$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2(\theta/2) \end{cases}.$$

Then the fixed point $(1, 0)$ satisfy the second condition of asymptotically stable but it is **not** stable.

In general, we can prove that if $\varphi_t(x) \not\equiv \sigma$ and $\lim_{t\to\pm\infty} \varphi_t(x) = \sigma$, then $\sigma$ is not stable.

**Example 3.1.4**

Consider the linear elementary singularities, recall the classification, then

1. Stable type: forward stable.
2. Unstable type: unstable, bet backward stable.
3. Saddle point: unstable.
4. Center: forward and backward stable.

**Theorem 3.1.5**

Let $A \in M(n, \mathbb{R})$, consider the equation $\dot{X} = AX$, $O$ is a singularity, then

1. $O$ is stable iff each eigenvalue of $A$ is with non-positive real part and Jordan block are trivial for every eigenvalue with zero real part.

2. $O$ is asymptotically stable iff $O$ is exponentially stable iff every eigenvalue of $A$ is with negative real part.

**Lemma 3.1.6** (Gronwall's Inequality)

Let $u : [0, T] \to \mathbb{R}$ non-negative, continuous. If $C \geqslant 0, K > 0$ such that for every $t \in [0, T]$,

$$u(t) \leqslant C + K \int_0^t u(s)\mathrm{d}s,$$

then $u(t) \leqslant Ce^{Kt}$ for $t \in [0, T]$.

**Theorem 3.1.7**

$f : \mathbb{R}^n \to \mathbb{R}^n$, $C^1$, $f(\sigma) = 0$. Assume that every eigenvalue of $A = Df(0)$ is with negative real part, then $\sigma$ is exponentially stable.

*Proof.* There $\exists C \geqslant 1, \mu > 0$, such that $|e^{At}| \leqslant Ce^{-\mu t}$ for $t \geqslant 0$. WOLG, $\sigma = 0$. Let $f(x) = Ax + g(x)$ where $g(x) = o(|x|)$, let $\varphi_t(x)$ be a maximal solution of the initial value problem. Then

$$e^{-tA}(\dot{\varphi}_t(x) - A\varphi_t(x)) = e^{-tA}g(\varphi_t(x)),$$

hence

$$\varphi_t(x) = e^{tA}x + \int_0^t e^{(t-s)A}g(\varphi_s(x))\mathrm{d}s.$$

Fix $\varepsilon_0 > 0$ to be determined later, $\exists\delta_0 > 0$ such that $|g(x)| \leqslant \varepsilon_0|x|$ if $|x| \leqslant \delta_0$. Assume the right maximal interval of $\varphi_t$ is $[0, \beta), \beta > 0$. Let

$$T^* = T^*(x) = \sup\left\{t < \beta : \varphi_{[0,t]}(x) \subseteq \overline{B(\delta_0, \sigma)}\right\}.$$

Then, for every $|x| \leqslant \delta_0, 0 \leqslant t \leqslant T^*$, we have

$$e^{\mu t}|\varphi_t(x)| \leqslant C|x| + C\varepsilon_0 \int_0^t e^{s\mu}|\varphi_s(x)|\mathrm{d}s.$$

By Gronwall's inequality, we have $|\varphi_t(x)| \leqslant C|x|e^{-(\mu - C\varepsilon_0)t}, \forall t < T^*$. Let $C\varepsilon_0 = \frac{\mu}{2}$ is enough. For all $|x| \leqslant \frac{\delta_0}{2C}$, then $|\varphi_t(x)| \leqslant \frac{\delta_0}{2}e^{-\mu t}$ for every $t < T^*$. Then we can show that $T^* = \beta = \infty$ and $\varphi_t$ is exponentially stable. $\square$

**Proposition 3.1.8**

$f, g$, $C^1$ vector fields. Assume $f, g$ are topologically conjugate, i.e. , $h \circ \varphi_t = \psi_t \circ h$ where $\varphi_t, \psi_t$ are flows generated by $f, g$, respectively. Let $\sigma, h\sigma$ be singularities of $f, g$, respectively, then $\sigma$ is stable if and only if $h\sigma$ is stable.

Now, we state a celebrated theorem, Hartman-Grobman Theorem. But we will not give a proof here.

**Theorem 3.1.9** (Hartman-Grobman)

Let $\sigma$ be a hyperbolic singularity of $f$. Then there exists a neighborhood $V \ni \sigma$ and a homeomorphism $h : V \to \mathbb{R}^n$ onto its image, $h(\sigma) = 0$, such that $h \circ \varphi_t(x) = Df(\sigma) \circ h(x)$ for every $x, \varphi_t(x) \in V$.

## §3.2 Lyapunov functions

**Definition 3.2.1.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$, be a $C^1$ vector field, $f(0) = 0$. A $C^1$ function $V : D \to \mathbb{R}$ where $D$ is a neighborhood of $\sigma$ is called a **Lyapunov function** of $f$ (for $\sigma$) if

(i) $V(\sigma) = 0, V(x) > 0, \forall x \in D \setminus \{\sigma\}$.

(ii) $\forall x \in D \setminus \{\sigma\}, \dot{V}(x) \leqslant 0$, where

$$\dot{V}(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} V(\varphi_t(x)) = DV(x)f(x).$$

$V$ is called a **strict Lyapunov function** if $\dot{V}(x) \leqslant 0$ is replaced by $\dot{V}(x) < 0$.

**Theorem 3.2.2**

Assume $\sigma$ is a singularity of $f$, if there is a Lyapunov function for $\sigma$, then $\sigma$ is stable. If there is a strict Lyapunov function for $\sigma$, then $\sigma$ is asymptotically stable.

*Proof.* Let $V$ be a Lyapunov function, for every $\varepsilon > 0$, assume $B_\varepsilon(\sigma) = \{x : |x - \sigma| \leqslant \varepsilon\} \subseteq D$. Let $m = \min\{V(x) : x \in \partial B_\varepsilon(\sigma)\} > 0$, take $\delta > 0$ such that $V(x) < m, \forall x \in B_\delta(\sigma)$. By $\dot{V}(x) \leqslant 0$, we have that every solution curve start at $x \in B_\delta(\sigma)$ can not reach $\partial B_\varepsilon(\sigma)$.

If $\dot{V}(x) < 0$ for every $x \in D \setminus \{\sigma\}$, it suffices to show that each convergent subsequence of $\varphi_t(x)$ converges to $\sigma$. Otherwise, assume converges to $y \neq \sigma$, but $\dot{V}(y) < 0$, there is some $s > 0$ such that $V(\varphi_s(y)) < V(y)$. Contradiction. □

**Example 3.2.3**

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = 0$, hence 0 is stable.

**Example 3.2.4**

Consider the equation

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - y^3 \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2 - 2y^4 < 0$, hence 0 is asymptotically stable.

**Example 3.2.5**

Consider the equation

$$\begin{cases} \dot{x} = -x - y + x^2 \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2(1 - x) \leqslant 0$, hence 0 is stable. In fact, 0 is asymptotically stable, but we need to consider another Lyapunov function $Q(x, y) = x^2 + y^2 + xy$.

**Theorem 3.2.6**

If $V$ is a Lyapunov function of $f$, assume

$$K = \left\{x \in D \setminus \{\sigma\}, \dot{V}(x) = 0\right\}$$

does not contain any complete positive orbit $\varphi_{[0,\infty)}(x)$, then $\sigma$ is asymptotically stable.

**Example 3.2.7**

Let $f : \mathbb{R} \to \mathbb{R}, C^1, f(0) = 0$, satisfying $xf(x) > 0, \forall x \neq 0$. Consider the stability of $\ddot{x} + f(x) = 0$, or

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}.$$

Let

$$E(x, y) = \frac{1}{2}y^2 + \int_0^x f(z)\mathrm{d}z$$

be an energy function, then $\dot{E}(x, y) \equiv 0$.

**Example 3.2.8**

Let $V : \mathbb{R}^n \to \mathbb{R}, C^2$, the gradient of $V$ is

$$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}.$$

The system $\dot{x} = -V(x)$ is called the **gradient system** generated by $V$. Then,

1. $\dot{V}(x) \leqslant 0$.
2. $\sigma$ is a singularity if and only if $\dot{V}(\sigma) = 0$.
3. If $\sigma$ is a minimum point of $V(x)$, then $\sigma$ is stable.

**Theorem 3.2.9**

Let $\sigma$ be a singularity of $C^1$ vector field $f$, a $C^1$ function $V : D \to \mathbb{R}$ satisfies

(i) $V(\sigma) = 0$, and $V$ can take positive value on any neighborhood of $\sigma$.

(ii) $\dot{V}(x) > 0, \forall x \in D \setminus \{0\}$.

Then $\sigma$ is unstable.

**Example 3.2.10**

Consider the equation

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}.$$

Let $V(x, y) = x^2 - y^2$, then $\dot{V}(x, y) = 2x^2 + 2y^2 > 0$, hence 0 is unstable.