Anosov Diffeomorphisms on Tori (Summer 2022, Yi Shi)

Ajorda Jiao

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Basic settings

- M a closed Riemannian manifold.
- $f : M \to M$ a diffeomorphism.
- $\mathrm{Diff}^r(M) := \{C^r\text{-diffeomorphisms on } M\}$, equipped with the C^r-topology

$$d_{C^r}(f,g) := \sup_{x\in M}\left\{d(f(x),g(x)), d(\mathrm{D}f(x),\mathrm{D}g(x)),\cdots,d(\mathrm{D}^r f(x),\mathrm{D}^r g(x))\right\}.$$

Definition 1.1. We say $f \in \mathrm{Diff}^r(M)$ is an **Anosov diffeomorphism**, if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^u$ and constants $C \geqslant 1, 0 < \lambda < 1$ such that

(i) $\left\|\mathrm{D}f^n|_{E^s(x)}\right\| \leqslant C\lambda^n, \forall x \in M, \forall n \geqslant 0.$

(ii) $\left\|\mathrm{D}f^{-n}|_{E^u(x)}\right\| \leqslant C\lambda^n, \forall x \in M, \forall n \geqslant 0.$

Remark 1.2 — $\mathrm{D}f$ is uniformly contracting on E^s, uniformly expanding on E^u.

Remark 1.3 — E^s, E^u are two continuous plane fields on M.

Example 1.4

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ linearly. Then A is an Anosov diffeomorphism.

Example 1.5

Similarly, $A \in \mathrm{GL}(d, \mathbb{Z})$ has no eigenvalues with modules 1, then $A : \mathbb{T}^d \to \mathbb{T}^d$ is Anosov.

For every C^r-Anosov diffeomorphisms, E^s, E^u are Hölder continuous. But in general, E^s and E^u are not differentiable.

We also concern about if E^s and E^u are integrable. The answer is given by the famous **Stable Manifold Theorem**. It says that both E^s and E^u are locally uniquely integrable, i.e. for every $x \in M$, there exists an immersed C^r-smooth submanifold $\mathscr{F}^s(x)$ such that

(i) $x \in \mathscr{F}^s(x)$ and $\mathscr{F}^s(x)$ is tangent to E^s everywhere, i.e.

$$T_y\mathscr{F}^s(x) = E^s(y), \quad \forall y \in \mathscr{F}^s(x).$$

(ii) For every local curve γ containing x and tangent to E^s everywhere, then $\gamma \subset \mathscr{F}^s(x)$.

Theorem 1.6 (Stable Manifold Theorem)

There exists two family of continuous foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ with C^r-leaves tangent to E^s and E^u respectively.

Remark 1.7 — If $\mathscr{F}^s$ has dimension 1, then every leaf $\mathscr{F}^s(x)$ is diffeomorphic to $\mathbb{R}$. (no leaf is $\mathbb{S}^1$) The intuitive is that if a leaf $\Gamma \cong \mathbb{S}^1$, then $\mathrm{diam}(f^k(\Gamma)) \to 0$. Then the orientations of E^s on $f^k(\Gamma)$ will contradict with the uniformly continuous of E^s.

Fact 1.8. $\mathbb{T}^2$ is the only 2-dimensional manifold that supports an Anosov diffeomorphism.

Foliations

Let M be an m-dimensional C^∞ closed Riemannian manifold.

Definition 1.9. A **C^r foliation** of dimension n is a C^r-atlas $\mathscr{F}$ on M which is maximal satisfying

(i) For every $(U, \varphi) \in \mathscr{F}$, $U \subset M$ is an open subset such that $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ where U_1, U_2 are open disks in $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$.

(ii) For every $(U, \varphi), (V, \psi) \in \mathscr{F}$ such that $U \cap V \neq \varnothing$, then $\psi \circ \phi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ has the form $\psi \circ \phi^{-1}(x, y) = (h_1(x, y), h_2(y))$.

We say M is **foliated** by $\mathscr{F}$.

Remark 1.10 — Stable foliation of an Anosov diffeomorphism is a C^0-foliation with C^r-leaves. That is, for every $(U, \varphi) \in \mathscr{F}$, write

$$\varphi : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}, \quad x \mapsto (\varphi_1(x), \varphi_2(x)),$$

then φ_1 is C^r-smooth but φ_2 is only C^0-continuous.

The space of Anosov diffeomorphisms

Claim 1.11. If $f: M \to M$ is Anosov, then there exists a C^1-neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that $\forall g \in \mathcal{U}$ is Anosov. That is, the space of Anosov diffeomorphisms is open in $\mathrm{Diff}^r(M)$.

Proof. **Cone Arguments.** Take $\gamma \in (0,1)$, denote

$$\mathscr{C}^u_\gamma(x) := \{v \in T_x M : v = v^s + v^u \in E^s(x) \oplus E^u(x), \|v^s\| \leqslant \gamma \|v^u\|\},$$

$\mathscr{C}^u_\gamma = \bigcup_{x \in M} \mathscr{C}^u_\gamma(x)$. Similarly, we can define $\mathscr{C}^s_\gamma(x)$ and $\mathscr{C}^s_\gamma$. For γ small enough, there exists $\lambda \in (0,1)$ such that

$$\mathrm{D}f(\mathscr{C}^u_\gamma) \subset \mathscr{C}^u_{\lambda\gamma}, \quad \mathrm{D}f^{-1}(\mathscr{C}^s_\gamma) \subset \mathscr{C}^s_{\lambda\gamma}.$$

Besides, $\mathrm{D}f$ expands vectors in $\mathscr{C}^u_\gamma$ and $\mathrm{D}f^{-1}$ expands vectors in $\mathscr{C}^s_\gamma$. **This is a robust property.** Then for g which is C^1-closed to f, we have

$$\mathrm{D}g(\mathscr{C}^u_\gamma) \subset \mathscr{C}^u_{\sqrt{\lambda}\gamma}, \quad \mathrm{D}g^{-1}(\mathscr{C}^s_\gamma) \subset \mathscr{C}^s_{\sqrt{\lambda}\gamma},$$

$\mathrm{D}g$ expands vectors in $\mathscr{C}^u_\gamma$ and $\mathrm{D}g^{-1}$ expands vectors in $\mathscr{C}^s_\gamma$. Take

$$E^u_g(x) = \bigcap_{n \geqslant 0} g^{-n}(\mathscr{C}^u_\gamma(g^n x)), \quad E^s_g(x) = \bigcap_{n \geqslant 0} g^{n}(\mathscr{C}^s_\gamma(g^{-n} x)),$$

which is an Anosov splitting. $\square$

A question is whether our examples on tori are special. But up to now, all known Anosov diffeomorphisms are supported on infra-nilmanifolds.

Open problems

1. Are there Anosov diffeomorphisms not supported on infra-nilmanifolds?
   - All expanding maps are supported on infra-nilmanifold.
2. Are there Anosov diffeomorphisms on simply connected manifolds?
   - Gogolev, etc. Acta Math, 2015. They constructed a partially hyperbolic diffeomorphism on simply connected 6-manifolds.
   - $\mathbb{S}^3 \times \mathbb{S}^3$ has Anosov diffeomorphisms or not?
3. Is every Anosov diffeomorphism transitive?
   - Yes on infra-nilmanifolds, but don't know on other manifolds.

The answer of these problems are unknown on even 4-dimensional manifolds.

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Let $f : M \to M$ be an Anosov diffeomorphism. TM admits a splitting $E^s \oplus E^u$ with parameters (C_0, λ_0),

- $\left\|\mathrm{D}f^n|_{E^s(x)}\right\| \leqslant C_0\lambda_0^n, \forall x \in M, \forall n \geqslant 0.$
- $\left\|\mathrm{D}f^{-n}|_{E^u(x)}\right\| \leqslant C_0\lambda_0^n, \forall x \in M, \forall n \geqslant 0.$

Theorem 2.1 (Anosov Closing Lemma)

There exists $\delta_0 > 0, C > 0, \lambda \in (\lambda_0, 1)$ such that if $d(x, f^k x) < \delta_0$, then there exists $p \in \mathrm{Per}(f)$ such that

(i) $f^k p = p$.

(ii) $d(f^i(p), f^i(x)) \leqslant C\lambda^{\min\{i,k-i\}} d(x, f^k x), \forall 0 \leqslant i \leqslant k$.

Remark 2.2 — λ can be chose arbitrarily closed to λ_0, but $\delta_0 \to 0$ when $\lambda \to \lambda_0^+$.

Notation 2.3. Denote $\Omega(f)$ be the set of non-wandering points of f.

Corollary 2.4

If f is Anosov, then $\Omega(f) = \overline{\mathrm{Per}(f)}$. In particular, if f is transitive, then $\overline{\mathrm{Per}(f)} = M$.

Definition 2.5. Let $f : M \to M$ be a diffeomorphism. For every continuous function $\varphi : M \to \mathbb{R}$, we say φ is a **coboundary** if $\exists \psi : M \to \mathbb{R}$ continuous such that

$$\varphi(x) = \psi \circ f(x) - \psi(x), \quad \forall x \in M.$$

[Can also read T.Tao's blog *Cohomology for dynamical systems* for reference.]

A Necessary condition for a coboundary: for every k-periodic point p, $\sum_{i=0}^{k-1} \varphi(f^i p) = 0$. Moreover, if μ is an f-invariant ergodic measure, then we take a generic point x, we have

$$\int \varphi \mathrm{d}\mu = \lim_{k\to\infty} \frac{1}{k}[\psi(f^k(x)) - \psi(x)] = 0.$$

Theorem 2.6 (Livsic)

Let $f : M \to M$ be a C^1 transitive Anosov diffeomorphism. Let $\varphi : M \to \mathbb{R}$ be a Hölder continuous function such that for every $p \in \mathrm{Per}(f)$ with $f^k p = p$, it satisfies

$$\sum_{i=1}^{k-1} \varphi(f^i p) = 0.$$

Then $\exists \psi : M \to \mathbb{R}$ a Hölder continuous function such that $\varphi = \psi \circ f - \psi$.

Remark 2.7 — If φ is α-Hölder continuous for some $\alpha \in (0, 1]$, then ψ is also α-Hölder.

Remark 2.8 — ψ is unique up to a constant.

Proof. Because f is transitive, there exists an $x_0 \in M$ such that $\overline{\mathrm{Orb}^+(x_0)} = M$. Fix $\varphi(x_0) \in \mathbb{R}$, define

$$\psi(f^n x_0) = \psi(x_0) + \sum_{i=0}^{n-1} \varphi(f^i x_0).$$

This is unique candidate for solving the cohomologous function.

Claim 2.9. ψ is α-Hölder continuous on $\mathrm{Orb}^+(x_0)$, that is, there exists $K_0 > 0$ such that

$$|\psi(f^n x_0) - \psi(f^m x_0)| \leqslant K_0 \cdot d(f^n x_0, f^m x_0)^\alpha.$$

Proof. φ is α-Hölder, $|\varphi(x_1) - \varphi(x_2)| \leqslant K \cdot d(x_1, x_2)^\alpha$. By Anosov closing lemma, $\exists \delta_0 > 0, C > 1, \lambda \in (0,1)$ such that if $d(f^n x_0, f^m x_0) < \delta_0$, $0 < n < m$, then there exists $p \in \mathrm{Per}(f)$, $f^{m-n}(p) = p$ and

$$d(f^{n+i}x_0, f^{n+i}p) \leqslant C \cdot \lambda^{\min\{i, m-n-i\}} d(f^n x_0, f^m x_0).$$

Then we have an estimate

$$\begin{aligned} |\psi(f^n x_0) - \psi(f^m x_0)| &= \left| \sum_{i=0}^{m-n-1} \varphi(f^{n+i}x_0) \right| \\ &= \left| \sum_{i=0}^{m-n-1} (\varphi(f^{n+i}x_0) - \varphi(f^{n+i}p)) + \sum_{i=0}^{m-n-1} \varphi(f^{n+i}p) \right| \\ &\leqslant \sum_{i=0}^{m-n-1} K \cdot d(f^{n+i}x_0, f^{n+i}p)^\alpha \leqslant 2KC^\alpha \cdot d(f^n x_0, f^m x_0)^\alpha \sum_{i=0}^{m-n-1} \lambda^{\alpha i} \\ &\leqslant \frac{2KC^\alpha}{1 - \lambda^\alpha} \cdot d(f^n x_0, f^m x_0)^\alpha. \end{aligned}$$

Then we can extend ψ uniquely to a α-continuous function on $M = \overline{\mathrm{Orb}^+(x_0)}$. □

Remark 2.10 — If $f : M \to M$ is a C^∞-Anosov diffeomorphism and φ is C^∞, then ψ is C^∞. Moreover, if φ is C^r, then ψ is $C^{r-\varepsilon}$ for every $\varepsilon > 0$. [Applying Journé theorem.]

Remark 2.11 — A more general setting: α-Hölder continuous linear cocycle over Anosov diffeomorphism f. [Non-abelian Livsic Theorem.]

Shadowing lemma

Definition 2.12. Let $f : M \to M$ be a diffeomorphism, $\delta > 0$. We say $\{x_n\}_{n\in\mathbb{Z}} \subset M$ is a **δ-pseudo-orbit** if for every $n \in \mathbb{Z}$, $d(fx_n, x_{n+1}) < \delta$.

Theorem 2.13 (Anosov Shadowing Lemma)

Let $f : M \to M$ be an Anosov diffeomorphism. There exists $\delta_0 > 0$ and $L_0 > 0$ such that for every $\delta \leqslant \delta_0$ and every δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{Z}}$, there exists a unique point $z \in M$ such that $d(f^n z, x_n) < L_0 \cdot \delta$.

Remark 2.14 — For every pair of δ-pseudo-orbit $\{x_n\}, \{y_n\}$, $K = \dim M$, define

$$d^0(\{x_n\}, \{y_n\}) = \sum_{n\in\mathbb{Z}} \frac{d(x_n, y_n)}{(K+1)^n} < \infty.$$

Denote z_x, z_y be the points shadowing $\{x_n\}, \{y_n\}$, respectively. Then $d(z_x, z_y) \to 0$ as

$d^0(\{x_n\}, \{y_n\}) \to 0$. This is the continuity of shadowing.

Remark 2.15 — By the uniqueness of shadowing point, we can deduce the closing lemma by the shadowing lemma.

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Corollary 3.1

Every Anosov diffeomorphism $f : M \to M$ is structurally stable, i.e. there exists a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that for every $g \in \mathcal{U}$, there exists a homeomorphism $h : M \to M$ which is C^0-close to Id such that $h \circ g = f \circ h$.

Proof. Let $TM = E^s_f \oplus E^u_f = E^s_g \oplus E^u_g$ be the Anosov splitting with respect to f, g, respectively, with a same hyperbolic constant (C, λ). Take the neighborhood sufficiently small such that $\{g^n x\}$ is a $\frac{1}{2}\delta_0$-pseudo-orbit of f. There exists a unique $h(x)$ such that $\{f^n(hx)\}$ $\frac{1}{2}L_0\delta_0$-shadows $\{g^n x\}$. Besides, we can consider the sequence $\left\{g^{n+1}x\right\}$, which can be shadows by the orbit of $h(gx)$ and $f(hx)$. Hence $h \circ g = f \circ h$. But we still do not know that h is a homeomorphism. We can shadow $\{f^n x\}$ by some g-orbit, it gives the inverse of h. □

Global stability of diffeomorphisms on $\mathbb{T}^2$

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism which is homotopic to A. Then $f_* : \pi_1(\mathbb{T}^2) = \mathbb{Z}^2 \to \pi_1(\mathbb{T}^2)$ which is equal to $A \in \mathrm{GL}(2, \mathbb{Z})$.

Proposition 3.2

There exists a continuous surjective map $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h \circ f = A \circ h$.

Proof. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f, then $F(x) = Ax + \varphi(x)$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and $\mathbb{Z}^2$-periodic.

Claim 3.3. There exists a unique $H : \mathbb{R}^2 \to \mathbb{R}$ which is continuous and satisfies:

(i) $\|H - \mathrm{Id}\|_{C^0} = \sup_{x \in \mathbb{R}^2} \|H(x) - x\| < K$.

(ii) $H(x + n) = H(x) + n, \forall x \in \mathbb{R}^2, n \in \mathbb{Z}^2$.

(iii) $H \circ F = A \circ H$.

Proof. Denote $K_0 = \sup_{x \in \mathbb{R}^2} \|\varphi(x)\| < \infty$, then $\|F(x) - Ax\| \leqslant K_0$. Hence $\left\{F^k(x)\right\}_{k \in \mathbb{Z}}$ is a K_0-pseudo-orbit of A. Note that A is linear, we can draw rectangles $R_k(x) \ni F^k(x)$ with a large size such that $A(R_k(x))$ is transverse to $R_{k+1}(x)$. This implies a global shadowing. That is, there exists a unique point $H(x) \in \mathbb{R}^2$ such that

$$\sup_{k \in \mathbb{Z}} \left\{ \left\| A^k(H(x)) - F^k(x) \right\| \right\} < \frac{\sqrt{2}}{1 - \lambda} K_0.$$

The uniqueness of H guarantees that $H(x + n) = H(x) + n$ and $H \circ F = A \circ H$. □

Then H induces a continuous map $h : \mathbb{T}^2 \to \mathbb{T}^2$ which is homotopic to Id. It suffices to show $h(\mathbb{T}^2) = \mathbb{T}^2$. Otherwise, there exists an open disc contained in $\mathbb{T}^2 \setminus h(\mathbb{T}^2)$. Then $H(\mathbb{R}^2)$ is not simply connected, a contradiction. □

Another proof of claim 3.3. Denote $H(x) = x + \psi(x)$, where

$$\psi \in C_b^0(\mathbb{R}^2) = \left\{\psi : \mathbb{R}^2 \to \mathbb{R}^2 : \text{continuous}, \mathbb{Z}^2\text{-periodic}\right\}.$$

Define $\|\psi\| = \max_{x\in\mathbb{R}} \{\|\psi_s(x)\|, \|\psi_u(x)\|\}$, then $(C_b^0(\mathbb{R}^2), \|\cdot\|)$ forms a Banach space. Consider the equation

$$A \circ H(x) = H \circ F(x).$$

Or, $A\psi(x) = \varphi(x) + \psi \circ F(x)$. We write this equation in the component form with respect to $L^s \oplus L^u$, which is,

$$\begin{cases} \psi_s = (\lambda\psi_s - \varphi_s) \circ F^{-1}, \\ \psi_u = \lambda(\psi_u \circ F + \varphi_u). \end{cases}$$

We consider an operator $T : (C_b^0(\mathbb{R}^2), \|\cdot\|) \to (C_b^0(\mathbb{R}^2), \|\cdot\|)$ given by last equation. Then T is well-defined. Furthermore, T is contracting. Then there exists a unique $\psi \in C_b^0(\mathbb{R}^2)$ such that $T(\psi) = \psi$. □

Theorem 3.4 (Franks-Manning)
Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ has a hyperbolic linear part $f_* \in \mathrm{GL}(d, \mathbb{Z})$ and f is topological conjugate to f_*.

Theorem 3.5 (Franks-Newhouse)
Let $f : M \to M$ be an Anosov diffeomorphism, if $\dim E^s = 1$ or $\dim E^u = 1$, then $M = \mathbb{T}^d$ and f conjugates to f_*.

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§6 Exercises

Exercise 6.1. Let $A \in \mathrm{GL}(d, \mathbb{Z})$ such that every eigenvalue of A has absolute value not equal to 1. Prove that $A : \mathbb{T}^d \to \mathbb{T}^d$ is an Anosov diffeomorphism.

Exercise 6.2. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism. We say (x, y) is a rational point if $\exists p_1, p_2, q_1, q_2$ such that $(x, y) = (p_1/q_1, p_2/q_2) \bmod \mathbb{Z}^2$. Prove that $(x, y) \in \mathbb{T}^2$ is a periodic point if and only if (x, y) is a rational point.

Exercise 6.3. Prove that $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is transitive.

Exercise 6.4. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ preserves the Lebesgue measure on $\mathbb{T}^2$, prove that Lebesgue measure is an ergodic measure of A.

Remark 6.5 — Anosov have shown that every $C^{1+\alpha}$ conservative Anosov diffeomorphism is ergodic with respect to Lebesgue measure.