Harmonic Analysis
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Contents

1 Fourier Series and Integrals

§1.1 Fourier series

For $f \in L^1(\mathbb{T})$, define the **Fourier coefficients**

$$\widehat{f}(k) := \int_0^1 f(x)e^{-2\pi ikx}\mathrm{d}x.$$

Let

$$\sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$$

be the **Fourier series** of f. When we discuss the convergence of Fourier series, we consider two types of sum:

$$S_N f = \sum_{k=-N}^{N} \widehat{f}(k)e^{2\pi ikx}, \quad \sigma_N f = \frac{1}{N+1}\sum_{k=0}^{N} S_k f.$$

We concern about the following questions:

> **Question 1.1.1.** The pointwise convergence of $S_N f$.

> **Question 1.1.2.** The L^p convergence of $S_N f$.

> **Question 1.1.3.** The almost everywhere convergence of $S_N f$.

> **Question 1.1.4.** The convergence of $\sigma_N f$.

§1.2 The pointwise convergence

Definition 1.2.1. The **Dirichlet kernel** D_N is given by

$$D_N(t) := \sum_{k=-N}^{N} e^{2\pi ikt} = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}.$$

It satisfies

$$\int_0^1 D_N(t)\mathrm{d}t = 1.$$

Theorem 1.2.2 (Dini's Criterion)

For $x \in \mathbb{T}$, if $\exists\delta > 0$, such that

$$\int_{|t|<\delta} \left|\frac{f(x+t)-f(x)}{t}\right| \mathrm{d}t < \infty,$$

then $S_N f(x) \to f(x)$.

Theorem 1.2.3

If f is bounded variation on a neighborhood of x, then

$$S_N f(x) \to \frac{f(x+) + f(x-)}{2}.$$

Example 1.2.4

$f_1(t) = |t|^{-\alpha} \mathbb{1}_{(0,1/2)}$, $f_2(t) = t^\alpha \sin \frac{1}{t} \mathbb{1}_{(0,1/2)}$, where $\alpha \in (0,1)$.

Theorem 1.2.5 (Riemann Localization Principle)

If f is zero in a neighborhood of x, then $S_N f(x) \to 0$.

Theorem 1.2.6 (Riemann-Lebesgue)

If $f \in L^1(\mathbb{T})$, then $\widehat{f}(k) \to 0 (|k| \to \infty)$.

§1.3 Fourier series of continuous functions

Theorem 1.3.1

There exists $f \in C(\mathbb{T})$ such that $S_N f(0)$ diverges.

Proof. Consider $T_N : C(\mathbb{T}) \to \mathbb{C}, f \mapsto S_N f(0)$, By theorem 1.3.2, it suffices to show $\sup \|T_N\| = \infty$. Suppose $L_N = \|D_N\|_1$, we can prove that$\|T_N\| = L_N$. Consider the functions $f_n(t) = \dfrac{nD_N(t)}{1 + n|D_N(t)|}$ is enough. The statement follows by lemma 1.3.3. □

Theorem 1.3.2 (Uniform Boundedness Principle)

X, Y, Banach Spaces. $\{T_a\}_{a \in A}$ is a family of bounded linear operators from X to Y. Then one of the following holds:

1. $\sup\limits_{a \in A} \|T_a\| < \infty$.
2. $\exists x \in X$, such that $\sup\limits_{a \in A} \|T_a x\| = \infty$.

Lemma 1.3.3

$L_N = \dfrac{4}{\pi^2} \ln N + O(1)$.

§1.4 Convergence in norm

> **Question 1.4.1.** We can ask:
>
> 1. Does $\|S_N f - f\|_p \to 0$ for $f \in L^p(\mathbb{T})$?
> 2. Does $S_N f \to f$ a.e. for $f \in L^p(\mathbb{T})$?

Lemma 1.4.2

$S_N f$ convergence to f in L^p norm, $1 \leqslant p < \infty$, iff exists C_p such that

$$\|S_N f\|_p \leqslant C_p \|f\|_p .$$

§1.5 Summability method

Definition 1.5.1. The **Fejér kernel** is given by

$$F_N(t) := \frac{1}{N+1}\sum_{k=0}^{N} D_k(t) = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)}\right)^2$$

It satisfies

$$\int_0^1 F_N(t)\mathrm{d}t = 1 \text{ and } F_N(t) \geqslant 0.$$

Theorem 1.5.2

If $f \in L^p(\mathbb{T}),\ 1 \leqslant p < \infty$, or $f \in C(\mathbb{T})$ and $p = \infty$, then

$$\|\sigma_N f - f\|_p \to 0.$$

Proof. Applying Minkowski's inequality and it follows by Fejér kernel is a good kernel. □

Corollary 1.5.3

The following holds:

1. The trigonometric polynomials $V = \left\{\sum_{k=-N}^{N} c_k e^{2\pi i k x} : c_k \in \mathbb{C}, N \in \mathbb{Z}_+\right\}$ is dense in $L^p(\mathbb{T})$.
2. If $\in L^1(\mathbb{T})$ and $\widehat{f}(k) = 0$ for every $k \in \mathbb{Z}$, then $f = 0$ a.e. .

Theorem 1.5.4

$$\|f\|_2 = \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 \text{ and } \|S_N f\|_2 \leqslant \|f\|_2 .$$

Define the **Poisson kernel**

$$P_r(t) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi ikt} = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2} = \frac{1-|z|^2}{|1-z|^2},\ z = re^{2\pi it}.$$

Let

$$u(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k + \sum_{k=-\infty}^{-1} \widehat{f}(k) \overline{z}^{|k|}$$

be the Poisson sum, then $u(re^{2\pi i\theta}) = P_r * f(\theta)$.

Theorem 1.5.5

If $f \in L^p(\mathbb{T}),\ 1 \leqslant p < \infty$, or $f \in C(\mathbb{T})$ and $p = \infty$, then

$$\|P_r * f - f\|_p \to 0 (r \to 1^-).$$

Remark 1.5.6 — $\Delta u = 0$ in $D = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{T} \cong \partial D = \mathbb{S}^1$. If $f \in C(\mathbb{T})$, then $u \in C(\overline{D})$ and $u = f$ on ∂D.

Fact 1.5.7. $\sigma_N f \to f$ a.e. and $P_r * f \to f$ a.e. . We will prove these in the next chapter.

§1.6 The Fourier transform of L^1 functions

For $f \in L^1(\mathbb{R}^n)$, let

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i\xi\cdot x} \mathrm{d}x = (\mathcal{F}f)(\xi).$$

Proposition 1.6.1

The following holds:

1. $\widehat{(\alpha f + \beta g)} = \alpha \widehat{f} + \beta \widehat{g}$.
2. $\|\widehat{f}\|_\infty \leqslant \|f\|_1$ and $\widehat{f} \in C(\mathbb{R}^n)$.
3. $\lim\limits_{|\xi|\to\infty} \widehat{f}(\xi) = 0$.
4. $\widehat{f * g} = \widehat{f}\widehat{g}$.
5. $\widehat{\tau_h f} = \widehat{f}(\xi) e^{2\pi ih\cdot\xi}$ where $\tau_h f = f(\,\cdot\, + h)$. $\widehat{fe^{2\pi ih\cdot x}}(\xi) = \widehat{f}(\xi - h)$.
6. $\rho \in \mathrm{O}_n$, then $\widehat{f(\rho\,\cdot\,)}(\xi) = \widehat{f}(\rho\xi)$.
7. If $g(x) = \lambda^{-n} f(\lambda^{-1}x)$, then $\widehat{g}(\xi) = \widehat{f}(\lambda\xi)$ for every $\lambda > 0$.
8. $\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = 2\pi i\xi_j \widehat{f}(\xi)$, if $\frac{\partial f}{\partial x_j} \in L^1$.
9. $\widehat{(-2\pi i x_j f)}(\xi) = \frac{\partial \widehat{f}}{\partial \xi_j}(\xi)$, if $x_j f \in L^1$.

§1.7 The Schwartz class and tempered distributions

Define the **Schwartz class**

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : p_{\alpha\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f|, \forall \alpha, \beta \in \mathbb{N}^n \right\}.$$

Then $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$. Moreover $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and is dense in $L^p(\mathbb{R}^n)(1 \leqslant p < \infty)$. The topology on $\mathcal{S}$ is defined as

$$f_k \to f \text{ in } \mathcal{S} \iff \lim_{k\to\infty} p_{\alpha,\beta}(f_k - f) = 0, \forall \alpha, \beta \in \mathbb{N}^n.$$

We can give a family of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ as

$$\|f\|_{(k)} = \sup\{p_{\alpha,\beta}(f) : \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leqslant k\}$$

and a quasi-norm on $\mathcal{S}(\mathbb{R}^n)$ as

$$\|f\|_{(*)} = \sum_{k=0}^{\infty} \min\left\{\|f\|_{(k)}, 2^{-k}\right\}.$$

Let $d(f,g) := \|f-g\|_{(*)}$, which makes $\mathcal{S}$ a metric space $(\mathcal{S}, d)$ and the topology is identified.

Theorem 1.7.1

The following holds:

1. $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$ and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous.
2. $\displaystyle\int_{\mathbb{R}^n} f\widehat{g} = \int_{\mathbb{R}^n} \widehat{f}g$.

Lemma 1.7.2

If $f(x) = e^{-\pi|x|^2}$, then $\widehat{f}(\xi) = e^{-\pi|\xi|^2}$.

Remark 1.7.3 — $\widehat{e^{-\pi\lambda|x|^2}} = \lambda^{-\frac{n}{2}} e^{-\pi|\xi|^2/\lambda}$ for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.

Theorem 1.7.4

The following holds:

1. If $f \in \mathcal{S}$ (or $f \in L^1$ and $\widehat{f} \in L^1$), then $f(x) = \displaystyle\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi\cdot x} \mathrm{d}\xi$.
2. $\forall f, g \in \mathcal{S}$, $\displaystyle\int_{\mathbb{R}^n} \widehat{f}\overline{\widehat{g}} = \int_{\mathbb{R}^n} f\overline{g}$.

Proof. For $f \in \mathcal{S}$, let $g_\lambda(x) = e^{-\pi\lambda|x|^2}$, by DCT and the identity

$$\int_{\mathbb{R}^n} \widehat{f}(x) g(\lambda x) \mathrm{d}x = \int_{\mathbb{R}^n} f(\lambda x) \widehat{g}(x) \mathrm{d}x.$$

□

Let $\overline{\mathcal{F}}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{2\pi i\xi\cdot x}\mathrm{d}x$, $\sigma f(x) = \widetilde{f}(x) = f(-x)$, $Cf(x) = \overline{f(x)}$. Then $\overline{\mathcal{F}} = C\mathcal{F}C$, $\overline{\mathcal{F}} = \mathcal{F}^{-1}, \mathcal{F}^4 = \mathrm{Id}$.

Corollary 1.7.5 (Plancherel)
$\|f\|_2 = \|\mathcal{F}f\|_2, \forall f \in \mathcal{S}$.

We define the family of **tempered distributions** $\mathcal{S}'$ as the continuous linear function on $\mathcal{S}$. Then $T \in \mathcal{S}'$ if and only if $\exists m \in \mathbb{N}$, such that $|\langle T, f\rangle| \leqslant C\|f\|_{(m)}$ for every $f \in \mathcal{S}$. For every $1 \leqslant p \leqslant \infty$, we have a natural embedding $j_p : L^p \hookrightarrow \mathcal{S}'$.

Definition 1.7.6. $\forall T \in \mathcal{S}'$, define $\widehat{T}(f) = T(\widehat{f}), \forall f \in \mathcal{S}$.

Let $\mathcal{F}_1 : T \mapsto \widehat{T}$. Then $\mathcal{F}_1$ maps $\mathcal{S}'$ to $\mathcal{S}'$ is continuous. Moreover, $\mathcal{F}_1 \circ j_1 = j_\infty \circ \mathcal{F}$.

Proposition 1.7.7
If $T \in \mathcal{S}', \widehat{T} \in L^1$, then $T(x) = \int_{\mathbb{R}^n} \widehat{T}(\xi)e^{2\pi i\xi\cdot x}\mathrm{d}\xi$ a.e. .

§1.8 The Fourier transform on $L^p, 1 < p \leqslant 2$

Theorem 1.8.1
For $\forall f \in L^2(\mathbb{R}^n)$, then $\widehat{f} \in L^2$ and $\|\widehat{f}\|_2 = \|f\|_2$.

Theorem 1.8.2
It holds $\widehat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x)e^{-2\pi i\xi\cdot x}\mathrm{d}x$, $f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \widehat{f}(\xi)e^{-2\pi i\xi\cdot x}\mathrm{d}\xi$, both convergences is in the sense of L^2 norm.

Because $\mathcal{F} : L^1 \to L^\infty, L^2 \to L^2$, then by $L^p \subset L^1 + L^2$ for $1 < p < 2$, we have $\mathcal{F} : L^p \to L^1 + L^\infty$.

Theorem 1.8.3 (Riesz-Thorin Interpolation Theorem)
$p_0, p_1, q_0, q_1 \in [1,\infty], 0 < \theta < 1$, let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. If $T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ such that $\|Tf\|_{q_0} \leqslant M_0\|f\|_{p_0}, \|Tf\|_{q_1} \leqslant M_1\|f\|_{p_1}$, then $\|Tf\|_q \leqslant M_0^{1-\theta}M_1^\theta\|f\|_p$.

Corollary 1.8.4
If $f \in L^p, 1 \leqslant p \leqslant 2$, then $\mathcal{F}f \in L^{p'}$ and $\|\mathcal{F}f\|_{p'} \leqslant \|f\|_p$.

Corollary 1.8.5

$f \in L^p, g \in L^q, p, q, r \in [1, \infty]$ with $1 + \dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$, then $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$.

§1.9 The convergence and summability of Fourier integral

Let $B_R = R \cdot B$ where B is a neighborhood of origin.

> **Question 1.9.1.** $f(x) = \lim\limits_{R\to\infty} \displaystyle\int_{B_R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} \mathrm{d}\xi$?

Let $\widehat{S_R f} = \chi_{B_R} \widehat{f}$, then $\lim\limits_{R\to\infty} \|S_R f - f\|_p = 0$ iff $\|S_R f\|_p \leqslant C_p \|f\|_p$.

Fact 1.9.2. $S_R : L^p \to L^p$ bounded iff $n = 1, 1 < p < \infty$ or $n = 1, p = 2 (B = B(0,1))$ or $n > 1, 1 < p < \infty (B = Q(0,1))$.

$n = 1, B = (-1, 1)$, then $S_R f = D_R * f$, where D_R is the Dirichlet kernel

$$D_R(x) = \int_{-R}^{R} e^{2\pi i \xi \cdot x} \mathrm{d}\xi = \frac{\sin(2\pi R x)}{R x}.$$

Then $D_R \notin L^1$ but $D_R \in L^q (1 < q \leqslant \infty)$.

Almost everywhere convergence Now we consider the almost everywhere convergence, an argument (Carleson-Hunt) shows that

$$\left\| \sup_R |S_R f|_p \right\| \leqslant C_p \|f\|_p \implies \lim_{R\to\infty} S_R f(x) = f(x) \text{ a.e. }, \forall f \in L^p, 1 < p < \infty.$$

Cesàro sum Let $\sigma_R f(x) = \dfrac{1}{R} \displaystyle\int_0^R S_t f(x) \mathrm{d}t = F_R * f(x)$, where F_R is the Fejér kernel

$$F_R(x) = \frac{1}{R} \int_0^R D_t(x) \mathrm{d}t = \frac{\sin^2(\pi R x)}{R(\pi x)^2}.$$

Then $F_R \in L^1$ and $F_R \geqslant 0$. We have $\lim\limits_{R\to\infty} \|\sigma_R f - f\|_p = 0 \forall p \in [1, \infty)$ and $\sigma_R f \to f$ a.e. .

Abel-Poisson sum Let $u(x,t) = \displaystyle\int_{\mathbb{R}^n} e^{-2\pi t|\xi|} \widehat{f}(\xi) e^{-2\pi i \xi \cdot x} \mathrm{d}\xi = P_t * f(x)$, where

$$P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}, \quad \widehat{P_t}(\xi) = e^{-2\pi t|\xi|}.$$

We have $\Delta_{t,x} P_t(x) = 0$, then $\Delta u = 0$ on $\mathbb{R}_+ \times \mathbb{R}^n$. We also have $\lim\limits_{t\to 0+} u(x,t) = f(x)$ a.e. $, \forall f \in L^p(\mathbb{R}^n)$.

Conversely, if $\Delta u = 0$ in $\mathbb{R}_+^{n+1}$, $\sup\limits_{t>0} \displaystyle\int_{\mathbb{R}^n} |u(x,t)|^p \mathrm{d}x < \infty, 1 < p \leqslant \infty$. Then $\exists f \in L^p(\mathbb{R}^n)$ such that $u(x,t) = P_t * f(x)$.

Gauss sum Let $w(x,t) = \displaystyle\int_{\mathbb{R}^n} e^{-2\pi t^2|\xi|^2}\widehat{f}(\xi)e^{-2\pi i\xi\cdot x}\mathrm{d}\xi = W_t * f(x)$, where W_t is the **Gauss kernel** $W_t \coloneqq \overline{\mathcal{F}}(e^{-\pi t^2|\xi|^2}) = E^n e^{-\pi|x^2|/t}$. Let $\widetilde{W}(x,t) = W(x,\sqrt{4\pi t})$, then $\dfrac{\partial\widetilde{W}}{\partial t} - \Delta\widetilde{W} = 0$ in $\mathbb{R}^{n+1}_+$. We have $\lim\limits_{t\to 0+}\widetilde{W}(x,t) = \lim\limits_{t\to 0+} W(x,t) = f(x)$ a.e. $,\forall f \in L^p(\mathbb{R}^n)$.

2 The Hardy-Littlewood Maximal Function

§2.1 Approximations of the identity

$\phi \in L^1(\mathbb{R}^n)$, $\int \phi = 1$. For $t > 0$, let $\phi_t = t^{-n}\phi(t^{-1}x)$. Then $\phi_t \to \delta (t \to 0)$ in $\mathcal{S}'$, hence $\phi_t * g \to g (t \to 0)$.

Example 2.1.1 (Cesàro sum)

$\phi = F_1 = \dfrac{\sin^2 \pi x}{(\pi x)^2}$, then $F_R = \phi_{1/R}$.

Example 2.1.2 (Poisson kernel)

$\phi = P_1 = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \dfrac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$, then $P_t = \phi_t$.

Example 2.1.3 (Gauss kernel)

$\phi = W_1 = e^{-\pi |x|^2}$, then $W_t = \phi_t$.

Theorem 2.1.4

$\int_{\mathbb{R}^n} \phi = A$, $f \in L^p, 1 \leqslant p < \infty$ or $p = \infty$, $f \in C_0(\mathbb{R}^n)$, then $\lim_{t \to 0+} \|\phi_t * f - Af\|_p \to 0$.

Remark 2.1.5 — Then $\exists \{t_k\} \to 0$ such that $\phi_{t_k} * f \to f$ a.e. . Hence,

$$\left| \left\{ x : \lim_{t \to 0} \phi_t * f(x) \text{ exists but not equal to } f(x) \right\} \right| = 0.$$