Anosov Diffeomorphisms on Tori (Summer 2022, Yi Shi)

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Basic settings

- M a closed Riemannian manifold.
- $f : M \to M$ a diffeomorphism.
- $\mathrm{Diff}^r(M) := \{C^r\text{-diffeomorphisms on } M\}$, equipped with the C^r-topology

$$d_{C^r}(f,g) := \sup_{x\in M} \{d(f(x),g(x)), d(Df(x),Dg(x)), \cdots, d(D^r f(x), D^r g(x))\}.$$

Definition 1.1. We say $f \in \mathrm{Diff}^r(M)$ is an Anosov, if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^u$ and constants $C \geqslant 1, 0 < \lambda < 1$ such that

(1) $\left\| Df^n|_{E^s(x)} \right\| \leqslant C\lambda^n, \forall x \in M, \forall n \geqslant 0.$

(2) $\left\| Df^{-n}|_{E^u(x)} \right\| \leqslant C\lambda^n, \forall x \in M, \forall n \geqslant 0.$

Remark 1.2 — Df is uniformly contracting on E^s, uniformly expanding on E^u.

Remark 1.3 — E^s, E^u are two continuous plane fields on M.

Example 1.4

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ linearly. Then A is an Anosov diffeomorphism.

Example 1.5

Similarly, $A \in \mathrm{GL}(d,\mathbb{Z})$ has no eigenvalues with modules 1, then $A : \mathbb{T}^d \to \mathbb{T}^d$ is Anosov.

For every C^r-Anosov diffeomorphisms, E^s, E^u are Hölder continuous. But in general, E^s and E^u are not differentiable.

We also concern about if E^s and E^u are integrable. The answer is given by the famous **Stable Manifold Theorem**. It says that both E^s and E^u are locally uniquely integrable, i.e. for every $x \in M$, there exists an immersed C^r-smooth submanifold $\mathscr{F}^s(x)$ such that

(1) $x \in \mathscr{F}^s(x)$ and $\mathscr{F}^s(x)$ is tangent to E^s everywhere, i.e.

$$T_y \mathscr{F}^s(x) = E^s(y), \quad \forall y \in \mathscr{F}^s(x).$$

(2) For every local curve γ containing x and tangent to E^s everywhere, then $\gamma \subset \mathscr{F}^s(x)$.

Theorem 1.6 (Stable Manifold Theorem)
There exists two family of continuous foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ with C^r-leaves tangent to E^s and E^u respectively.

Remark 1.7 — If $\mathscr{F}^s$ has dimension 1, then every leaf $\mathscr{F}^s(x)$ is diffeomorphic to $\mathbb{R}$. (no leaf is $\mathbb{S}^1$) The intuitive is that if a leaf $\Gamma \cong \mathbb{S}^1$, then $\operatorname{diam}(f^k(\Gamma)) \to 0$. Then the orientations of E^s on $f^k(\Gamma)$ will contradict with the uniformly continuous of E^s.

Fact 1.8. $\mathbb{T}^2$ is the only 2-dimensional manifold supports an Anosov diffeomorphism.

Foliations

Let M be an m-dimensional C^∞ closed Riemannian manifold.

Definition 1.9. A **C^r foliation** of dimension n is a C^r-atlas $\mathscr{F}$ on M which is maximal satisfying

(1) For every $(U, \varphi) \in \mathscr{F}$, $U \subset M$ is an open subset such that $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ where U_1, U_2 are open disks in $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$.

(2) For every $(U, \varphi), (V, \psi) \in \mathscr{F}$ such that $U \cap V \neq \varnothing$, then $\psi \circ \phi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ has the form $\psi \circ \phi^{-1}(x, y) = (h_1(x, y), h_2(y))$.

We say M is **foliated** by $\mathscr{F}$.

Remark 1.10 — Stable foliation of an Anosov diffeomorphism is a C^0-foliation with C^r-leaves. That is, for every $(U, \varphi) \in \mathscr{F}$, write

$$\varphi : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}, \quad x \mapsto (\varphi_1(x), \varphi_2(x)),$$

then φ_1 is C^r-smooth but φ_2 is only C^0-continuous.

The space of Anosov diffeomorphisms

Claim 1.11. If $f : M \to M$ is Anosov, then there exists a C^1-neighborhood $\mathcal{U}$ of f in $\operatorname{Diff}^1(M)$ such that $\forall g \in \mathcal{U}$ is Anosov. That is, the space of Anosov diffeomorphisms is open in $\operatorname{Diff}^r(M)$.

Proof. **Cone Arguments.** Take $\gamma \in (0,1)$, denote

$$\mathscr{C}_\gamma^u(x) := \{v \in T_xM : v = v^s + v^u \in E^s(x) \oplus E^u(x), \|v^s\| \leqslant \gamma \|v^u\|\},$$

$\mathscr{C}_\gamma^u = \bigcup_{x\in M} \mathscr{C}_\gamma^u(x)$. Similarly, we can define $\mathscr{C}_\gamma^s(x)$ and $\mathscr{C}_\gamma^s$. For γ small enough, there exists $\lambda \in (0,1)$ such that

$$Df(\mathscr{C}_\gamma^u) \subset \mathscr{C}_{\lambda\gamma}^u, \quad \mathrm{D}f^{-1}(\mathscr{C}_\gamma^s) \subset \mathscr{C}_{\lambda\gamma}^s.$$

Besides, Df expands vectors in $\mathscr{C}_\gamma^u$ and Df^{-1} expands vectors in $\mathscr{C}_\gamma^s$. **This is a robust property.** Then for g which is C^1-closed to f, we have

$$Dg(\mathscr{C}_\gamma^u) \subset \mathscr{C}_{\sqrt{\lambda}\gamma}^u, \quad Dg^{-1}(\mathscr{C}_\gamma^s) \subset \mathscr{C}_{\sqrt{\lambda}\gamma}^s,$$

Dg expands vectors in $\mathscr{C}_\gamma^u$ and Dg^{-1} expands vectors in $\mathscr{C}_\gamma^s$. Take

$$E_g^u(x) = \bigcap_{n\geqslant 0} g^{-n}(\mathscr{C}_\gamma^u(g^n x)), \quad E_g^s(x) = \bigcap_{n\geqslant 0} g^{n}(\mathscr{C}_\gamma^s(g^{-n} x)),$$

which is an Anosov splitting. $\square$

A question is whether our examples on tori are special. But up to now, all known Anosov diffeomorphisms are supported on infra-nilmanifolds.

Open problems

1. Are there Anosov diffeomorphisms not supported on infra-nilmanifolds?
    - All expanding maps are supported on infra-manifold.
2. Are there Anosov diffeomorphisms on simply connected manifolds?
    - Gogolev, etc. Acta Math, 2015. They constructed a partially hyperbolic diffeomorphism on simply connected 6-manifolds.
    - $\mathbb{S}^3 \times \mathbb{S}^3$ has Anosov diffeomorphisms or not?
3. Is every Anosov diffeomorphism transitive?
    - Yes on infra-manifolds, but don't know on other manifolds.

The answer of both problems are unknown on even 4-dimensional manifolds.