COMP 2804 Assignment 4

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March 2021

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1 Question 1 [10 Marks]

Suppose we have a bag with 5 balls in it, numbered 1 to 5. Consider the drawing of two balls from the bag without replacement. Define the random variable X to be the absolute difference between the values of the two balls. Compute the distribution function for X.

- The distribution function can be calculated using the absolute values from the table below:

Table 1: Distribution Matrix

	1	2	3	4	5
1		1	2	3	4
2	1		1	2	3
3	2	1		1	2
4	3	2	1		1
5	4	3	2	1	

Therefore the distribution functions D(X) = Pr("X=x") can be seen as:

$$D(1) = Pr(1) = \frac{8}{20}$$
$$D(2) = Pr(2) = \frac{6}{20}$$
$$D(3) = Pr(3) = \frac{4}{20}$$
$$D(4) = Pr(4) = \frac{2}{20}$$

2 Question 2 [10 Marks]

Suppose we draw three cards from a shuffled standard 52 card deck (without replacement). Define the random variable H to be the number of hearts in the three card hand. Define the random variable K to be the number of kings in the three card hand. Show whether or not K and H are independent random variables.

- In order to prove whether or not K and H are independent random variables, it would be easier to assume (based on intuition) they are not and prove that.
- There are 4 types of each card in a deck each with a different suit.
- We can look at the specific case when H = 3 and K = 3.
- When K = 3, as stated before all 3 kings will have different suits.
- therefore by Definition 6.2.1 two events are independent when:

$$Pr(X = x^Y = y) = Pr(X = x) \cdot Pr(Y = y)$$

This definition will not hold when K = 3 and H = 3 because you can not have a hand of 3 kings that all have the suit hearts. Due to this fact, the definition fails and the random variables K and H are not independent

3 Question 3 [10 Marks]

Suppose X and Y are independent random variables on a probability space (S, P). Show that $E(X \cdot Y) = E(X) \cdot E(Y)$

- We can use the same approach as Linearity of expectation in Theorem 6.5.1, although this is used for the addition of expected values this approach can be used for the product of two expected values
- As the two expected values are independent, we can take the product of them without adding any dependency
- We can write $E(X \cdot Y)$ as $Z = aX \cdot bY$
- This can be proved by using the definition of expected value from Definition 6.4.1

$$E(X) = \sum_{\omega \in W}^{n} X(\omega) \cdot Pr(\omega)$$

- Now applying this to our proof:

$$\begin{aligned} E(Z) &= \sum_{\omega \in W}^{n} (a \cdot X(\omega) \cdot Y(\omega)) \cdot Pr(\omega) \\ E(Z) &= a \sum_{\omega \in W}^{n} (X(\omega) \cdot Pr(\omega)) \cdot b \sum_{\omega \in W}^{n} (Y(\omega) \cdot Pr(\omega)) \\ E(Z) &= E(X) \cdot E(Y) \\ \therefore E(X \cdot Y) &= E(X) \cdot E(Y) \end{aligned}$$

4 Question 4 [10 Marks]

Consider a game of chance where you roll two fair six-sided dice 10 times. Suppose you win x dollars every time the sum of the two dice is greater than or equal to 10 (assume you win zero dollars otherwise). What is the expected amount of money you win at the end of this game?

- A win is when $X \geq 10$ so we must first find the probability that the player will roll a 10 or above in a single roll.
- There are $6 \cdot 6 = 36$ possible combinations of dice rolls when using two fair six-sided die. Looking at the combinations of dice rolls that give us the products 10, 11 or 12 we can see that:

  Three combinations of dice rolls that add up to 10 therefore probability of rolling a 10 $= \frac{3}{36}$
  Two combinations of dice rolls that add up to 11 therefore probability of rolling an 11 $= \frac{2}{36}$
  One combinations of dice rolls that add up to 12 therefore probability of rolling a 12 $= \frac{1}{36}$

- This gives us the Probabilities:

$$Pr(X = 10) = \frac{3}{36}$$
$$Pr(X = 11) = \frac{2}{36}$$
$$Pr(X = 12) = \frac{1}{36}$$

$Finally,\ Pr(X \geq 10) = Pr(X = 10 \vee X = 11 \vee X = 12$

$= \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$

$= \frac{1}{6}$

Using the Binomial Distribution from Theorem 6.3, tossing two die n times with probability p, where p is the probability $X \geq 10$ we get the formula:

Let n be the number of times the dice is thrown
Let B be the number of wins

$$E(B) = \sum_{\omega \in S}^{n} k \cdot \binom{n}{k} \cdot (1-p)^{n-k} \cdot p^k = n \cdot p$$

$$n \cdot p = 10 \cdot \frac{1}{6} = \frac{5}{3}$$

This tells us that the player can expect to win the game $\frac{5}{3} = 1.\overline{6666}$ times if rolling the dice 10 times. If you win x dollars every time you win the game the player can expect to win $x \cdot \frac{5}{3}$ dollars.

5 Question 5 [10 Marks]

Consider a game of chance where you roll two fair six-sided dice. Suppose you win the game once you roll such that the sum of the two dice is greater than or equal to 10. What is the expected number of turns for this game (consider rolling the two dice once to be a turn)?

- We are looking at how many times we have to roll to get the event "X = 10", this resembles a geometric distribution.
- This can be viewed as a similar problem as repeatedly flipping a fair coin a number of times until heads comes up.
- If we look at a given dice roll as tails (T) if it is <10 and heads (H) if it is ≥ 10
- We can take the same approach to solve said coin toss with our problem, from Theorem 6.2:

$$Pr(X = k) = Pr(T^{k-1}H) = p(1-p)^{k-1}$$

- With that, by lemma 6.4.3:

$$E(X) = p\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p}$$

- There are 6 out of 36 ways to get a roll with a sum of 10 or above, 3/36 for 10, 2/36 for 11, and 1/36 for 12. Therefore 6/36 or 1/6 = p
- Using the lemma above: $E(X) = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6$
- Therefore you would need to an expected number of 6 turns for this game

6 Question 6 [10 Marks]

Consider a game where two players select elements from the set X = a, b, c, d, e, f, g, h. We consider it a "win" if the two Players select the same element. Suppose Player A and Player B use the following strategy: each player independently selects an element from X with uniform probability. What is the expected number of "wins" if Player A and Player B play this game n times?

- We consider it a win when two players select the same element from the set X = a, b, c, d, e, f, g, h therefore we need to find the probability of this happening.
- there are 8 x 8 = 64 possible combination of pairs. If both players select the same element that means they either both select a, both select b, c, and so on. There are 8 ways to do this out of the 64 possible pairs.
- Therefore p = 8/64 = 1/8 as the elements are chosen from X with uniform probability
- Using the Binomial Distribution from Theorem 6.3, two players selecting elements from the given set n times with probability p, where p is the probability both players select the same element we get the formula:

  Let n be the total number of times elements are selected
  Let k denote the number of games played currently
  Let B be the number of wins

$$E(B) = \sum_{\omega \in S}^{n} k \cdot \binom{n}{k} \cdot (1-p)^{n-k} \cdot p^k = n \cdot p$$

$$n \cdot p = n \cdot \frac{1}{8}$$

∴ the expected number of wins if Player A and B play this game n times is $n \cdot \frac{1}{8}$

7 Question 7 [10 Marks]

Suppose X is a random variable on a probability space (S, P). Show that $E(E(X)) = E(X)$

- We can view X as random variable of a constant, and x to be a constant. The probability $Pr(x) = 1$ as the probability of a constant is 1. $\therefore$

$$\begin{aligned} E(X) &= \sum_{\omega \in S} X(\omega) \cdot Pr(\omega) \\ &= X(\omega) \cdot 1 = X(\omega) \end{aligned}$$

- It then follows that because the expected value of some random variable will be a constant, taking the expected value of that constant will return that constant itself. $\therefore E(E(X)) = E(X)$

8 Question 8 [10 Marks]

Consider a die that is not fair. When rolled you obtain: 1 or 2 or 3 with probability $\frac{1}{12}$ (each) or 4, 5, or 6 with probability $\frac{1}{4}$ (each)

- Is this a valid probability distribution?
- What is the expected value of a roll?
- For this to be a valid probability space all of the probabilities must equal up to 1.
- There are three outcomes where the probability equals $\frac{1}{12}$ and three outcomes where the probability equals $\frac{1}{4}$
- Therefore the check for this is as follows:

$$= \frac{1}{12} \cdot 3 + \frac{1}{4} \cdot 3 = 1 \tag{1}$$

- Now that we know this is a valid probability space, we can calculate the expected value as follows:

$$E(X) = 1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{1}{4}$$
$$E(X) = \frac{17}{4} \, or \, 4.25$$

9 Question 9 [10 Marks]

On a car assembly line, out of 100 cars 3 are defective and need additional work.

1. What is the probability that the first defective car is the 5^{th} inspected?

- We can again look at this problem as coin flips, so in this case it would be what is the probability we get a heads on the 5^{th} flip? We can use the formula given for this problem

$$\begin{aligned} Pr(T^{k-1}H) &= p(1-p)^{k-1} \\ &= \frac{3}{100} \cdot (\frac{97}{100})^{5-1} \\ &= 0.026558 \end{aligned}$$

2. What is the probability that the first defective car is among the first 10 cars?

- This can be found by looking at the probabilities over the range S = 1, 2, 3, ..., 10
- We will use the above a formula again but within a summation.
- i will represent the current car we are calculating the probability for.

$$\sum_{i \in S}^{|S|} p(1-p)^{k-1}$$

3. What is the probability that the first defective car is the 5^{th}, 6^{th}, 7^{th}, ..., or 10^{th} car?

- We will take the same approach as above (a summation) but this time with a shorter range of S = 5, 6, 7, 8, 9, 10

$$\sum_{i \in S}^{|S|} p(1-p)^{k-1}$$

For items 2 and 3 do NOT compute each of the 10 (6, respectively) probabilities explicitly.