COMP 2804B Assignment1

February 2, 2021

Exercise 1 (10 pts)

We can use the pigeonhole principle to solve this question. Since we have letter grade in the range A+, A, A-, B+, B, B-, C+, C, C-, D+, D, D-, F. A total of 13 grades, we get 13 holes. We have 238 students in COMP 284, consider each student as a pigeon. We can easily see that the number is pigeons is larger than the number of holes, based on the Pigeonhole principle, there must be at least one letter grade that contains at least two students.

Exercise 2 (10 pts)

Treat the string ABC as one letter W, so we have the following set W, D, E, F, G, H, then we use the permutation where $A(n,k) = \frac{n!}{(n-1)!}$. In this case, we get $n = 6$ and $k = 6$. So $A(6,6) = \frac{6!}{(6-6)!} = \frac{720}{1} = 720$. Therefore, there are 720 different words that satisfy the requirements.

Exercise 3 (5 pts)

We can try with $n = 4$, $n! = 24, n = 4, n^n = 256, \log n = \log 4, n^2 = 16, 2n = 8$, we thus have $n^n > n! > n^2 > 2n > n > \log n$.

Exercise 4 (10 pts)

If the bitstring of length n starts with 00 and ends with 11. Then we have four positions fixed, we only need to tackle the remaining $n - 4$ positions. Considering that one position only takes 0 or 1, the result is 2^{n-4}.

Exercise 5 (10 pts)

For a ternary number of length n, it should start with 1 or 2, the remaining (n-1) positions can have three choices (0 or 1 or 2). Then the result is $2 \times 3^{n-1}$.

Exercise 6 (10 pts)

To calculate the number of bitstrings of length 2n whose n positions are filled with '0', we can use the binomial coefficient. We need to choose n positions from a bitstring of length 2n. So we have:

$$\binom{2n}{n} = \frac{2n!}{n!(2n-n)!} = \frac{2n!}{(n!)^2}$$

Exercise 7 (10 pts)

The number of letters: 26;
The number of vowels: 5;
The number of consonants: 21;

Define that the set A contains strings of length n that start with a vowel, so we have $|A| = 5(26)^{n-1}$. Define that the set B contains strings of length n that end with a consonant, so we have $|B| = 21(26)^{n-1}$. $A \cap B$ denotes that the set contains strings that start with a vowel and end with a consonant, we have $|A \cap B| = 5 \times 21(26)^{n-2} = 105(26)^{n-2}$. Since we have $|A \cup B| = |A| + |B| - |A \cap B|$, we hence have $|A \cup B| = 5(26)^{n-1} + 21(26)^{n-1} - 106(26)^{n-2} = 571(26)^{n-2}$.

Exercise 8 (10 pts)

In this question, we have the first position filled with a letter. Then we need to consider two cases: (1) There is no restriction for the other $n-1$ positions; (2) There is no number appearing in the remaining $n-1$ positions. For the first case, we have $26(26+10)^{n-1} = 26 \times 36^{n-1}$. For the second case, we have $26 \times 26^{n-1} = 26^n$. Then the number of alphanumeric strings of length n start with a letter and contain at least one number should be $26 \times 36^{n-1} - 26^n$.

Exercise 9 (10 pts)

For each permutation of A where 'a' appears before 'b', there always exist permutations of where 'b' appears before 'a'. The numbers of permutations for these two cases should be equal. Therefore, the result is $\frac{8!}{2}$.

Exercise 10 (5 pts)

The binomial theorem: $(x+y)^n = \sum_{k=0}^{k} \binom{n}{k} x^{n-k} y^k$.

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k} \times 1^{n-k} = (1+2)^n = 3^n$$

Exercise 11 (10 pts)

For the 6 digit number, 0 occurs 1 time and can't be put in the first digit: $\binom{5}{1}$; 1 occurs 2 times: $\binom{5}{2}$; 2 occurs 2 times: $\binom{3}{2}$; 3 occurs 1 time: 1. By the product rule, we get $\binom{5}{1} \times \binom{5}{2} \times \binom{3}{2} = 150$.

Exercise 12 (10 pts)

Let's assume that there exists a strictly decreasing sequence of n positive integers starting at m, and $n > m$. In this sequence, each element should be a positive integer in the range 1,2,...,m and strictly smaller than the prior one. Thus, we need to have a one-to-one function that maps an element of the sequence to the numbers from 1 to n. Assume we have two sets: A=1,2,...m;B=all elements in the decreasing sequence. $|A| = m, |B| = n$. Based on the Pigeonhole principle and $n > m$, we can induce that there is no such one-to-one function mapping A to B. Thus, we can't have a strictly decreasing sequence of n positive integers starting at m.

Exercise 13 (10 pts)

This is a procedure involving two tasks: (1) assign k balls to first player and (2) assign $n-k$ balls to the all players.

There is one way to assign k balls to one player.

Using the bijection rule, we observe a bijection between the set of assignments of $n-k$ balls to m players and the set of length $n-k+m-1$ bitstrings with $m-1$ ones. In particular, one bijection is:

$f(a) = a_1\, 0s, 1, a_2\, 0s, 1, a_3\, 0s, 1, ..., 1, a_m\, 0s$

where a_i is the number of balls the ith player has in the assignment a.

f is one-to-one because for two different assignments a and b, there must exist some i such that $a_i \neq b_i$, which means $f(a_i)$ and $f(b_i)$ have different number of zeros after the ith one. f is onto because every such bitstring with $n-k$ zeros and $m-1$ ones must be mapped to by some assignment.

The number of ways to assign $n-k$ balls to m players is the number of ways to choose $m-1$ elements to be 1s in an $n-k+m-1$ element set. This is $\binom{n-k+m-1}{m-1}$.

Thus, the number of ways to do this procedure is $1 \times \binom{n-k+m-1}{m-1} = \binom{n-k+m-1}{m-1}$.

Exercise 14 (10 pts)

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k!)} = \frac{n(n-1)!}{(k-1)![(n-1)-(k-1)]!} = n\binom{n-1}{k-1}$$