COMP 2804: Final Exam

Winter 2021

School of Computer Science — Carleton University

Please note:

- There are **11 questions** and you have **3 hours** to complete the exam.
- Please give enough time to process your exam as per instructions given by Carleton University.
- Plagiarism will be heavily sanctioned.

Question 1 [15 marks]

For each of the following statements, indicate whether the statement is **true or false**. You must **give a brief explanation** to justify your answer.

- The worst-case run-time of QuickSort is $O(n^2)$.
- InsertionSort is always better than QuickSort.
- The number of subsets of an n-element set is 2^n.

Only here, provide an answer only: "Yes" or "No", without additional explanation.

- $\log n$ is asymptotically larger than $\sqrt{n}$.
- DT claims he has a plane embedding of a planar graph with 10 vertices, 5 edges, 5 faces and 1 connected component. Is DT lying?
- If the number of crossings of an embedded graph is 10, does that mean that the crossing number of this graph is 10?

Complete each sentence listed below.

- We can prove that a set is countable by providing a bijection .....
- A sorting algorithm divides a problem into 2 equal-sized subproblems and then recursively sorts those subproblems. The time for dividing the problem into the two subproblems and for combining the solutions to the two subproblems into a solution for the original sorting problem is linear. (Sorting a list with 1 element takes constant time.) The run-time of the algorithm is then .....

Question 2 [15 marks]

Consider strings that are made up of characters from the set $S = \{a, b, c, d, e\}$. We call a string "acceptable" if there is at least one character from S that does not appear in that string. For example, *abcd* is acceptable, *bbb* is acceptable; *edcba* is not acceptable. How many acceptable strings of length n exist? Justify your answer.

Question 3 [10 marks]

Consider an arbitrary set S of 10 integers on the range [0, 100]. Prove that there exist two distinct subsets of S, call them X and Y, that have equal sums.

Hint: Use the pigeonhole principle.

Question 4 [15 marks]

Consider the function defined recursively below for $n \geq 1$. Write a non-recursive definition of the function and prove that the two definitions are equivalent.

$$\begin{aligned} f(n) &= f(n-1) + 2n - 1 \\ f(1) &= 0 \end{aligned} \tag{1}$$

Question 5 [15 marks]

Consider a coin that has 0 on one side and 1 on the other side. We flip this coin once and roll a die twice, and are interested in the product of the three numbers.

- What is the sample space?
- How many possible events are there?
- If both the coin and the die are fair, how would you define the probability function Pr for this sample space?

Question 6 [12 marks]

You flip a fair coin, independently, six times.

- What is the sample space?
- Consider the events:
  A = "the coin comes up heads at least four times",
  B = "the number of heads is equal to the number of tails",
  C = "there are at least four consecutive heads".
  Determine $Pr(A), Pr(B), Pr(C), Pr(A|B), and Pr(C|A)$.

Question 7 [12 marks]

Consider five people, each of which has a uniformly random and indepedent birthday. (We ignore leap years.) Consider the event A = “at least three people have the same birthday”.

Determine Pr(A).

Question 8 [12 marks]

In this course, we have seen the different cards that are part of a standard deck of cards. Assume you get a uniformly random hand of cards. A hand of cards is a subset consisting of five cards. A hand of cards is called a straight, if the ranks of these five cards are consecutive and the cards are not all of the same suit. An Ace and a 2 are considered to be consecutive, whereas a King and an Ace are also considered to be consecutive. Each of these three hands is a straight:

$$8\spadesuit, 9\heartsuit, 10\diamondsuit, J\spadesuit, Q\clubsuit$$

$$A\diamondsuit, 2\heartsuit, 3\spadesuit, 4\spadesuit, 5\clubsuit$$
$$10\diamondsuit, J\heartsuit, Q\spadesuit, K\spadesuit, A\clubsuit$$

Figure 1: Relevant Hands

Determine the probability that this hand is a straight.

Question 9 [18 marks]

You repeatedly flip a fair coin and stop as soon as you get tails followed by heads. (All coin flips are mutually independent.) Consider the random variable X = the total number of coin flips. for example, if the sequence of coin flips is HHHTTTTH, then X = 8.

- Determine the expected value of X.

  Hint: Use the linearity of expectation.

Question 10 [8 marks]

Consider an experiment that is successful with probability 0.8. We repeat this experiment (independently) until it is successful for the first time. The first 5 times we do the experiment, we have to pay $10 per experiment. After this, we have to pay $5 per experiment. Define the random variable X to be the total amount of money that we have to pay during all experiments. Determine the expected value of X.

Recall:

$$\sum_{n=1}^{\infty} kx^{k-1} = 1/(1-x)^2$$

Question 11 [10 marks]

Scenario: A TV company has a game show running over several days. Each participant sings one song per day. At the end of each day, they do the following: for each of the candidates active during the day, randomly with probability 0.5 and indepdently, determine if the candidate is permanently off the show or will participate at least for the next day. They start, on day 1, with n participants and stop when only one participant remains.

Please help the TV company to answer the following questions without providing a proof, just state answers:

- What is the expected number of days a participant appears?
- How many participants can we expect to be on the second of the show?
- How many participants can we expect to be on the ith day?
- For how many days can we expect the TV show to run?
- What is the expected total number of songs that will be heard in total?

End of The Exam.