COMP 2804: Assignment 3 Solution

Winter 2021

Question 1 [10 marks]

Consider a set of $n \geq 3$ people sitting in a circle around a campfire. What is the probability that three particular people, call them A, B, and C, sit together contiguously? Justify your answer.

Solution: The sample space is choosing 3 positions from n for A, B and C and permutate them, i.e. $|S| = \binom{n}{3} \cdot 3!$. The event is choosing one triplet from n for A, B and C and permutate them, i.e. $|A| = n \cdot 3!$. Thus, the probability is $Pr = \frac{|A|}{|S|} = \frac{6}{(n-1)(n-2)}$.

Question 2 [10 marks]

Consider a set of $n \leq 12$ people. What is the probability that two or more people in the set share the same birth month? Assume there is an equal probability of being born in each month. Justify your answer.

Solution: Denote A as the event that two or more people in the set share the same birth month, $\bar{A}$ as the even that no one shares the same birth month. According to the complementary rule, $Pr(A) = 1 - Pr(\bar{A})$. As for $Pr(\bar{A})$, the sample space is that each person has 12 choices as the birth month, i.e. 12^n. The event space is choosing n months from 12 months and assign them to the n people, i.e. $\binom{12}{n} \cdot n!$. Thus, the probability is

$$Pr(A) = 1 - Pr(\bar{A}) = 1 - \frac{\binom{12}{n} \cdot n!}{12^n} = 1 - \frac{12!}{12^n \cdot (12-n)!}.$$

Question 3 [10 marks]

Provide an example of a probability space and events where the events are pairwise independent, but not mutually independent. Try to make your sample space as small as possible. Show that your example is correct.

Solution: A existing example is shown in section 5.11.3 of the textbook. Pairwise independence means for any i and j, $Pr(A_i \cap A_j) = Pr(A_i) \cdot Pr(A_j)$. Mutually independence means for all k, $Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = Pr(A_{i_1}) \cdot Pr(A_{i_2}) \cdots Pr(A_{i_k})$.
Here is the example. Consider flipping a coin three times and assume that the result is a uniformly random element from the sample space

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},$$

where H means heads and T means tails. For $i = 1, 2, 3$, let f_i denote the result of the i-th flip, and consider the events

$$A = "f_1 = f_2",$$
$$B = "f_2 = f_3",$$
$$C = "f_1 = f_3".$$

We will see that $Pr(A) = 4/8$, $Pr(B) = 4/8$, $Pr(C) = 4$, $Pr(A \cap B) = 2/8$, $Pr(A \cap C) = 2/8$, $Pr(B \cap C) = 2/8$, $Pr(A \cap B \cap C) = 2/8$. Thus, the sequence A, B, C is pairwise independent. Since $Pr(A \cap B \cap C) \neq Pr(A) \cdot Pr(B) \cdot Pr(C)$, the sequence A, B, C is not mutually independent.

Question 4 [10 marks]

Suppose you draw cards with replacement from a standard 52 card deck. First, you draw one card. If it is a face card (i.e. Jack, Queen, or King), you draw one more card. If it is not a face card, you draw two more cards. Compute the probability that you draw exactly one face card and exactly one ace. Assume that the card is replaced after each draw. Justify your answer.

Solution: Drawing cards with replacement means each draw is independent. Denote cards as face, ace and other. There are five cases satisfying the event "exactly one face card and exactly one ace".

First draw	Second draw	Third draw
face	ace	None
ace	face	other
ace	other	face
other	face	ace
other	ace	face

Then, the probability will be

$$Pr = \frac{12}{52} \times \frac{4}{52} + \frac{4}{52} \times \frac{12}{52} \times \frac{36}{52} \times 2 + \frac{36}{52} \times \frac{12}{52} \times \frac{4}{52} \times 2 = \frac{147}{2197} = 0.0669$$

Question 5 [10 marks]

Consider a three player game where players take turns rolling a standard six-sided die. The winner is the first player who rolls a six. What is the probability the player who rolls first wins the game? What is the probability the player who rolls second wins the game? What is the probability the player who rolls third wins the game? Justify your answer.

Solution1: Suppose the three players can play this game only one round. The events each player rolls the die are independent. Denote the three players as A, B, C. The second player wins means that the first player lost. $Pr(A \text{ wins}) = \frac{1}{6} = 0.17, Pr(B \text{ wins}) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36} = 0.14, Pr(C \text{ wins}) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216} = 0.12.$

Solution2: Suppose the three players can play this game more than one round. The events each player rolls the die are independent. Denote the three players as A, B, C. Player A wins in the k-th round means that player A, B, C all lost in the first $k - 1$ rounds. Player B wins in the k-th round means that player A, B, C all lost in the first $k - 1$ rounds and player A lost in the k-th round.

We know that

$$Pr(A \text{ wins in the } k\text{-th round}) = \left(\frac{5}{6}\right)^{3(k-1)} \times \frac{1}{6}$$

$$Pr(B \text{ wins in the } k\text{-th round}) = \left(\frac{5}{6}\right)^{3(k-1)} \times \frac{5}{6} \times \frac{1}{6}$$

$$Pr(C \text{ wins in the } k\text{-th round}) = \left(\frac{5}{6}\right)^{3(k-1)} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

Thus,

$$Pr(A \text{ wins}) = \sum_{k=1}^{\infty} Pr(A \text{ wins in the } k\text{-th round}) = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{3(k-1)} \times \frac{1}{6}$$
$$= \frac{1}{6} \times \sum_{k=0}^{\infty} \left(\frac{125}{216}\right)^{k} = \frac{1}{6} \times \frac{1}{1-\frac{125}{216}} = \frac{36}{91} = 0.396$$

$$Pr(B \text{ wins}) = \sum_{k=1}^{\infty} Pr(B \text{ wins in the } k\text{-th round}) = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{3(k-1)} \times \frac{5}{6} \times \frac{1}{6} = \frac{5}{6} \times Pr(A \text{ wins}) = \frac{30}{91} = 0.330$$

$$Pr(C \text{ wins}) = \sum_{k=1}^{\infty} Pr(C \text{ wins in the } k\text{-th round}) = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{3(k-1)} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{36} \times Pr(A \text{ wins}) = \frac{25}{91} = 0.274$$

Question 6 [10 marks]

Consider the set X = {a, b, c, d, e, f, g, h}. Suppose we choose a random 5-element subset of X, call it Y. Define the events:

- A is the event a and b are in Y
- B is the event g or h is in Y

Compute $P(A|B)$ and $P(B|A)$. Justify your answer.

Solution: The sample space is always choosing 5 from 8, i.e. $\binom{8}{5}$. The event space for A is choosing 3 more elements from the left 6 elements, i.e. $|A| = \binom{6}{3}$. Thus,

$$P(A) = \frac{\binom{6}{3}}{\binom{8}{5}} = \frac{20}{56}.$$

The event space for B includes 3 cases, {g in, h not}, {g not, h in} and {g in, h in}. Thus,

$$P(B) = \frac{\binom{6}{4}\binom{2}{1} + \binom{6}{3}\binom{2}{2}}{\binom{8}{5}} = \frac{50}{56}.$$

The event space for AB includes 3 cases, {ab in, g in, h not}, {ab in, g not, h in} and {ab in, g in, h in}. Thus,

$$P(AB) = \frac{\binom{4}{2}\binom{2}{1} + \binom{4}{1}\binom{2}{2}}{\binom{8}{5}} = \frac{16}{56}.$$

According to the definition of conditional probability,

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{16}{50} = 0.32.$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{16}{20} = 0.8.$$

Question 7 [10 marks]

Consider a scenario where we flip a coin 8 times. Define the events:

- A is the event the first flip is Heads
- B is the event exactly four flips are Heads and exactly four flips are Tails

Are the events A and B are independent? Justify your answer.

Solution: The samples space is flipping a coin 8 times, i.e. $|S| = 2^8$. Thus,

$$P(A) = \frac{2^7}{2^8} = \frac{1}{2}, \quad P(B) = \frac{\binom{8}{4}}{2^8}, \quad P(AB) = \frac{\binom{7}{3}}{2^8} = \frac{35}{2^8}, \quad P(A) \cdot P(B) = \frac{35}{2^8}.$$

Since $P(AB) = P(A) \cdot P(B)$ and according to the definition of independent events, events A and B are independent.

Question 8 [5 marks]

Prove or disprove the following: Event A does not depend on whether event B occurs or not if and only if A does not depend on whether event $\bar{B}$ occurs or not.

Solution: The question asks to prove or disprove: A and B are independent if and only if A and $\bar{B}$ are independent. Note that $0 \leq P(B) \leq 1$. According to the definition of independent events, the statement equals to this,

$$P(AB) = P(A) \cdot P(B) \Leftrightarrow P(A\bar{B}) = P(A) \cdot P(\bar{B}).$$

What is known includes that $P(\bar{B}) = 1 - P(B), P(A) = P(AB) + P(A\bar{B})$. Combining the two rules, we have

$$P(A) = P(A) \cdot [P(B) + P(\bar{B})] = P(A) \cdot P(B) + P(A) \cdot P(\bar{B}) = P(AB) + P(A\bar{B})$$

From the left to the right, if $P(AB) = P(A) \cdot P(B)$ we will have $P(A\bar{B}) = P(A) \cdot P(\bar{B})$. From the right to the left, if $P(A\bar{B}) = P(A) \cdot P(\bar{B})$ we will have $P(AB) = P(A) \cdot P(B)$. Thus, the statement is true.

Question 9 [10 marks]

We have two events A and B. A: D.T. is an idiot with probability 0.6. B: D.T. is a crook with probability 0.7. We also know that neither is true with probability 0.25.

- What is the probability that D.T. is an idiot or a crook, but not both?
- What is the conditional probability that D.T is a crook, given that he is not an idiot.

Solution: Given that $P(A) = 0.6$, $P(B) = 0.7$ and $P(\bar{A}\bar{B}) = 0.25$.
(1) What is $P(A\bar{B} + \bar{A}B)$?
We know that

$$P(A\bar{B} + \bar{A}B) = P(A\bar{B}) + P(\bar{A}B) = P(\bar{B}) \cdot P(A|\bar{B}) + P(\bar{A}) \cdot P(B|\bar{A}) = 0.3P(A|\bar{B}) + 0.4P(B|\bar{A}).$$

We also know that

$$P(\bar{A}\bar{B}) = P(\bar{A}) \cdot P(\bar{B}|\bar{A}) = 0.4(1 - P(B|\bar{A})) = 0.25$$

and

$$P(\bar{A}\bar{B}) = P(\bar{B}) \cdot P(\bar{A}|\bar{B}) = 0.3(1 - P(A|\bar{B})) = 0.25$$

Thus, we have $P(B|\bar{A}) = \frac{3}{8}$ and $P(A|\bar{B}) = \frac{1}{6}$. So we have $P(A\bar{B} + \bar{A}B) = 0.3 \times \frac{1}{6} + 0.4 \times \frac{3}{8} = 0.2$.

(2) What is $P(B|\bar{A})$?
According to (1), we already have $P(B|\bar{A}) = \frac{3}{8}$.