Introduction to Queues

September 17, 2020

A queue in "every day" life is a line of people waiting for service.

Outline

1. Motivation
2. Collisions
3. Queues

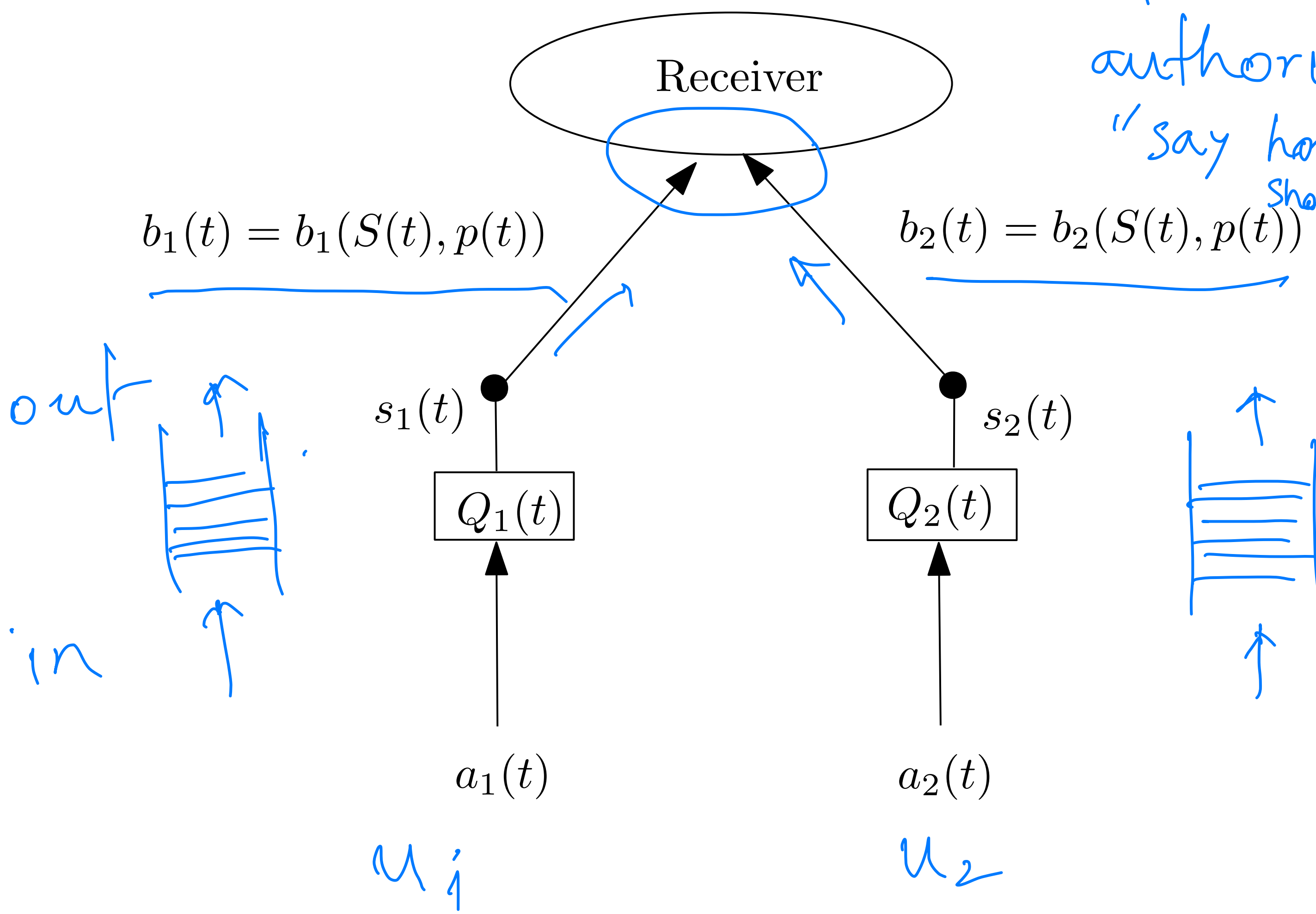
important to understand
Frequency of collision
If we understand that,
then we can design
algorithms, protocols, etc

Motivation

Scheduling in a Simple Queueing System (1/2)

- Consider a 2-user wireless uplink that operates in slotted time $t \in \{0, 1, 2, \ldots\}$.

No central authority to "say how transmiss. should done"



Scheduling in a Simple Queueing System (2/2)

- The unit of transmission of a network is the packet: basically an array of bits transmitted as a "unit".
- Every time slot new data randomly arrives to each user for transmission to a common receiver.
- Let $(a_1(t), a_2(t))$ be the vector of new arrivals on time slot t , in units of bits.
- The data is stored in queues $Q_1(t)$ and $Q_2(t)$ to await transmission.
- The receiver coordinates network decisions every slot.

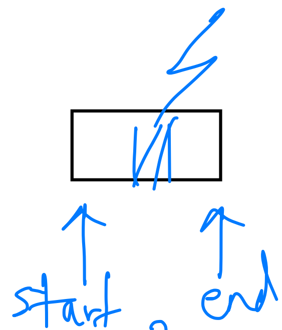
Modelling the Behavior of the System

- Although we have not yet defined what a packet is we can still try to understand what difficulties arise in network transmission.
- In particular, we would like to answer:
  - How can we model a system of packets?
  - What kind of difficulties do we encounter?
  - What is adequate mathematization?
  - Does the model reflect reality?
- Can't do all of them at once!
- Lets just look at packet collisions.

The more realistic the model the more complicated the math

What is a Network Packet (1/2)

- Think of a network packet as a rectangle (in fact a rectangular array) full of bits:



$\frac{1}{10}$

- or better yet an array

1 2 3

$\boxed{0010110011110010001}$

whose length depends on the network technology being used.

- How long" It can be thousands of bits long!
- The rectangle ""occupies" the medium for its transmission from its start to its end.
- To arrive correctly it cannot be interrupted!

Length of packet depends on technology used. Rule of thumb

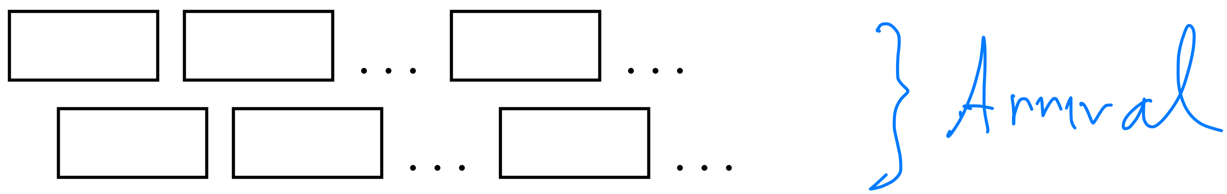
- Wireline Network
  packet are "longer"

- Wireless Medium
  packet are "shorter"

What is a Network Packet (2/2)

- What are some causes for interruption?
- Multiple packets may be transmitted by 2 different hosts, e.g.,

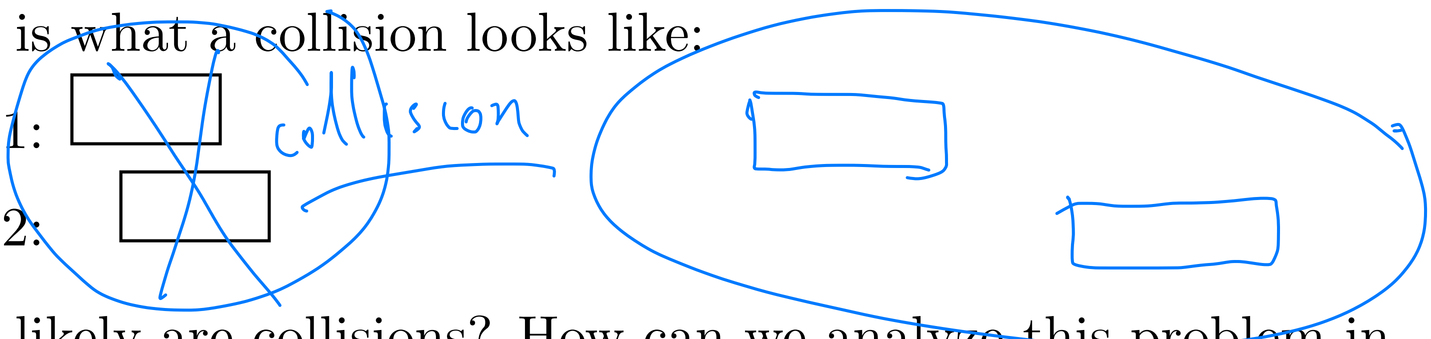
Host1:
Host2:



Arrival

- Notice that the hosts are a bit off sync! Unless they can be differentiated somehow by the receiver the packets may collide!
- This is what a collision looks like:

Host1:
Host2:

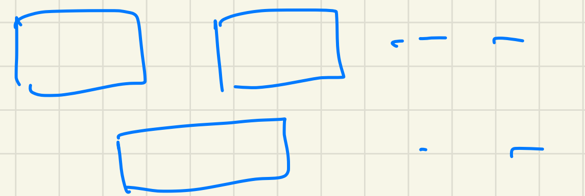


- How likely are collisions? How can we analyze this problem in a transmission system with multiple users sharing the medium?

What is the receiver?

Physical { wire
medium

Protocol
Session }



Collisions

Arrival Times in a Meeting

- There are many instances in every day life where collisions may occur.
  - Waiting for the bus or taxi to arrive
  - Waiting in a bank queue to make a transaction
  - Going to a doctor appointment
  - Waiting for an office hour
  - Shopping at a supermarket
- In the sequel, we use the cafeteria paradigm!

Meeting a Friend at the Cafeteria

- The cafeteria is open for dinner between the hours of 5:30 and 7:30 pm.
- You tend to arrive at the cafeteria at a random time in that interval, and you stay for half an hour eating.

1/2 hour

- The same is true of a friend of yours.

5:30 7:30

- What is the probability that you'll see each other on a given day?
- Why is this the same as the packet collision problem?
- Note: the times you and your friend arrive at the cafeteria for dinner vary in a continuus space since they can be any moments between 5:30 and 7:30 pm.

start finish

Arrival Times

- What does it mean you arrive at the cafeteria at a random time between 5:30 and 7:30 pm?

  the probability that you'll arrive in a given interval is equal to the probability that you'll arrive in any other interval of the same duration.

- Since the whole interval has length 2 hours, if the choice is arriving either between 5:30 and 6:30 or between 6:30 and 7:30 then the probability of each must be exactly 1/2.

  Uniform or Poisson

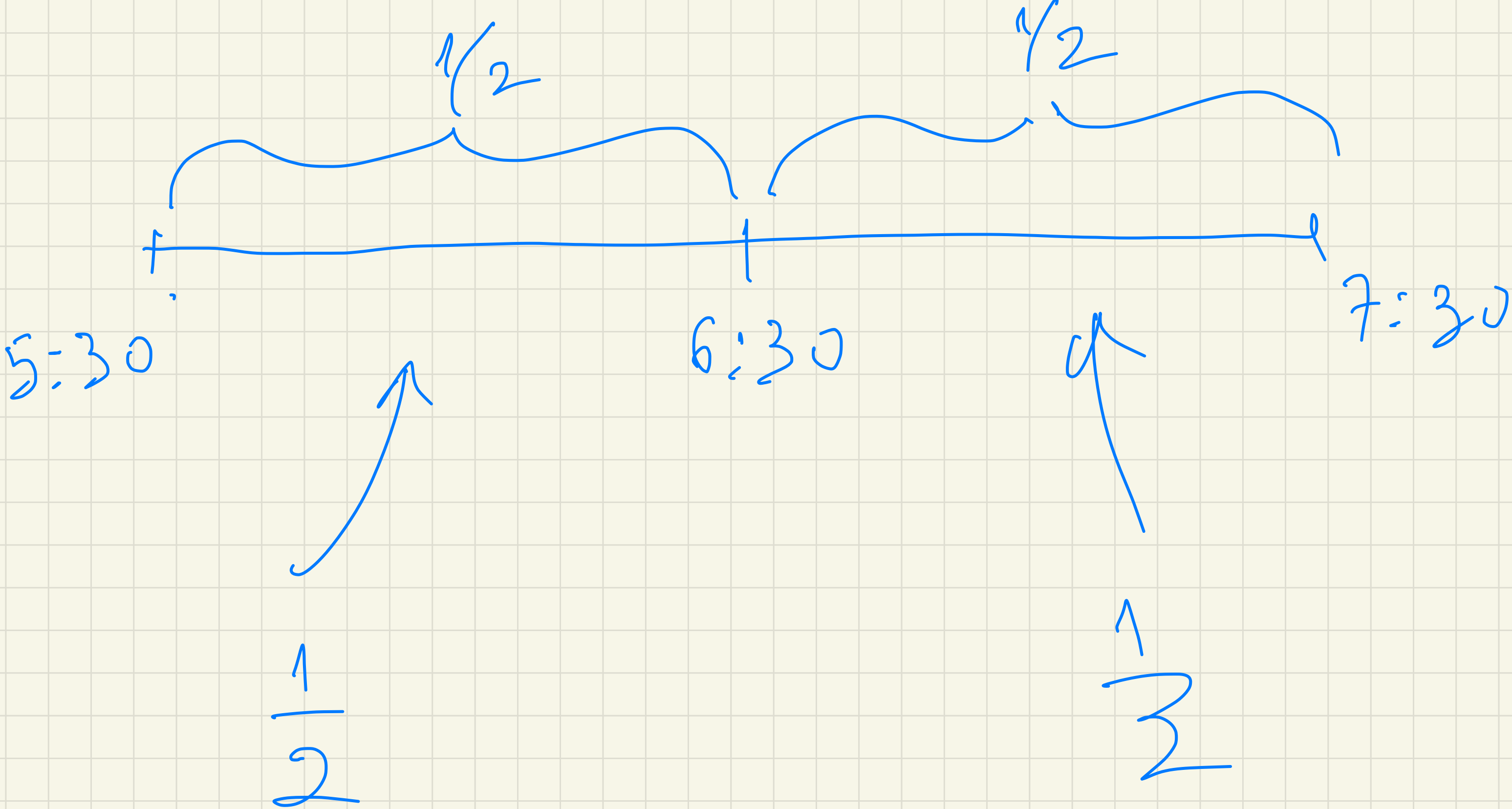
- Similarly, the probability of our arriving in any given hour-long interval, say, between 5:37 and 6:37 must be 1/2 as well.

- Similarly, the probability of arriving in any half-hour interval must be 1/4; and, in general, the probability of our arrival in any interval is one-half the length, in hours, of that interval.

Probability of arrival at an given time is "independent" of the specific time.

Equally likely to arrive

$\left\{ 5:31,\ 6:32,\ 5:17,\ 7:19 \right\}$

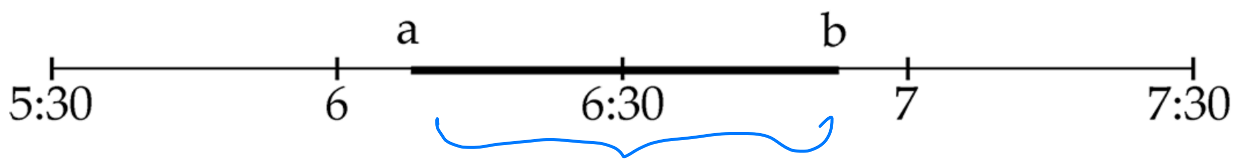

1/2
1/2
5:30
6:30
7:30
1/2
1/2

An Interval Representation

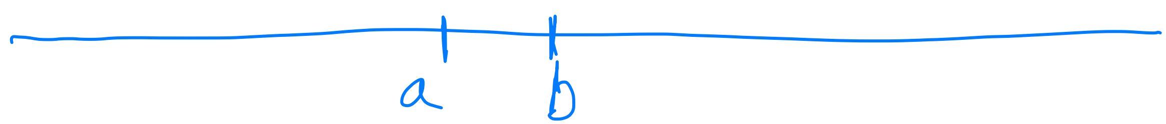
- If we "normalize" the parameters, i.e., think of the entire line (time segment between 5:30 and 7:30 pm) as having length 1 then then the probability that your time of arrival t is between two given times a and b is given by the formula

$$\Pr[a \leq t \leq b] = b - a.$$

- If we represent your arrival time as a point on the line segment $[0, 1]$ (a number t between 0 and 1) then we can represent



your friend's time of arrival as a number s between 0 and 1.



$b = 6:30$ $\quad$ $6:30 - 5:22$

$a = 5:22$ $\quad$ $= 68$ min.

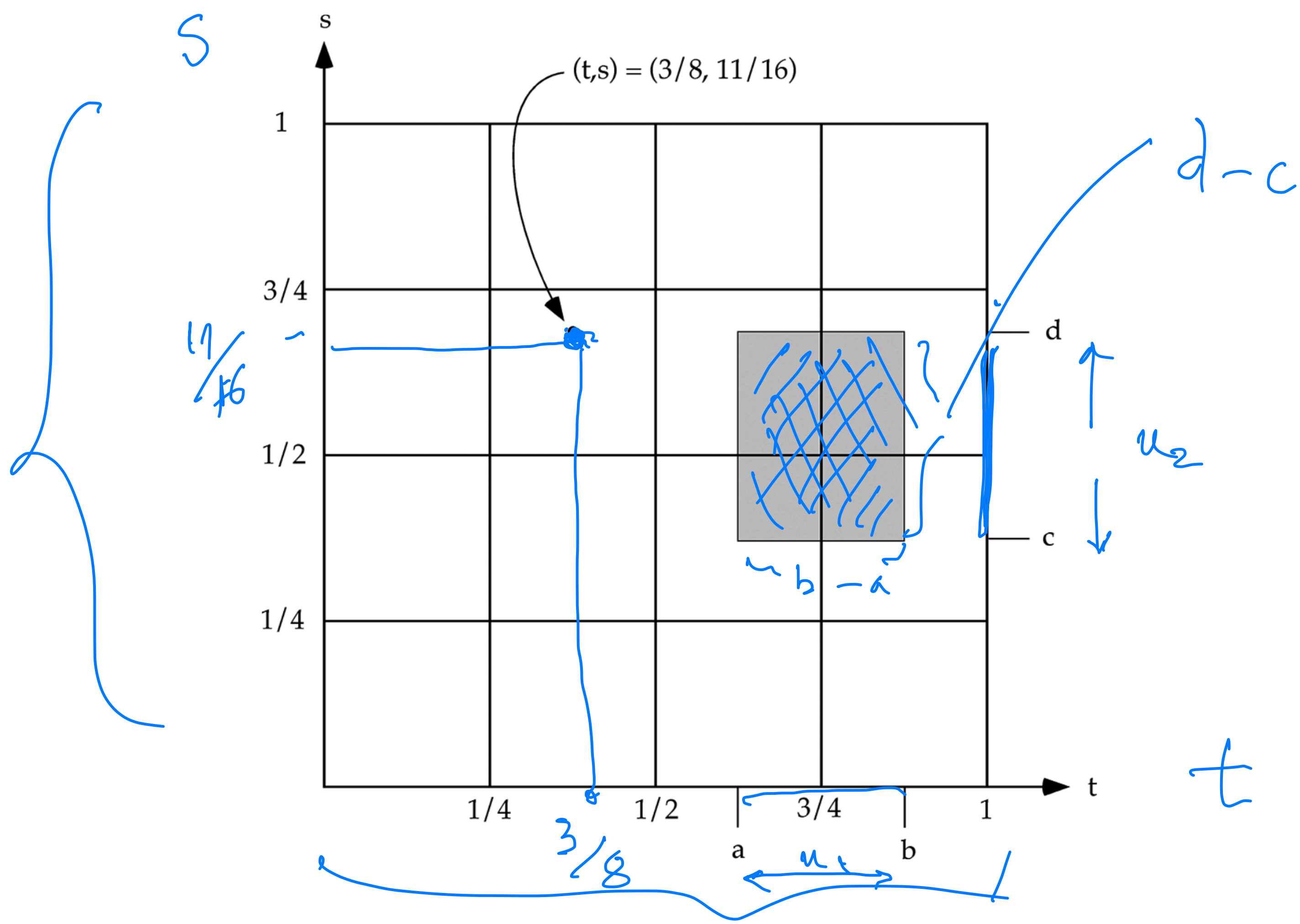
$$\Pr[a < t < b] = \frac{68}{120} = \frac{17}{30}$$

$\approx .56$

$2\text{hrs} = 120$ min

A Unit Square Representation

- We can then view the pair of times s and t as a point in a square of unit side length 1.



Independence of Arrivals

- The next crucial assumption is that your time of arrival and your friend's are independent events

John [a,b] Mary [c,d]

- In that case, the probability that your arrival time is between a and b and your friend's arrival time is between c and d is the product of the probabilities of the two individual events:

$$(b-a)(d-c)$$

  which is the area of a rectangle with side lengths $b-a$ and $d-c$.

- More generally, if your time of arrival and your friends are independent, then the probability that (t, s) falls in any region in the square is equal to the area of that region.

Overlap (1/2)

- Observe that you and your friend's dinners overlap is the same as your arrival times differ by a half hour or less

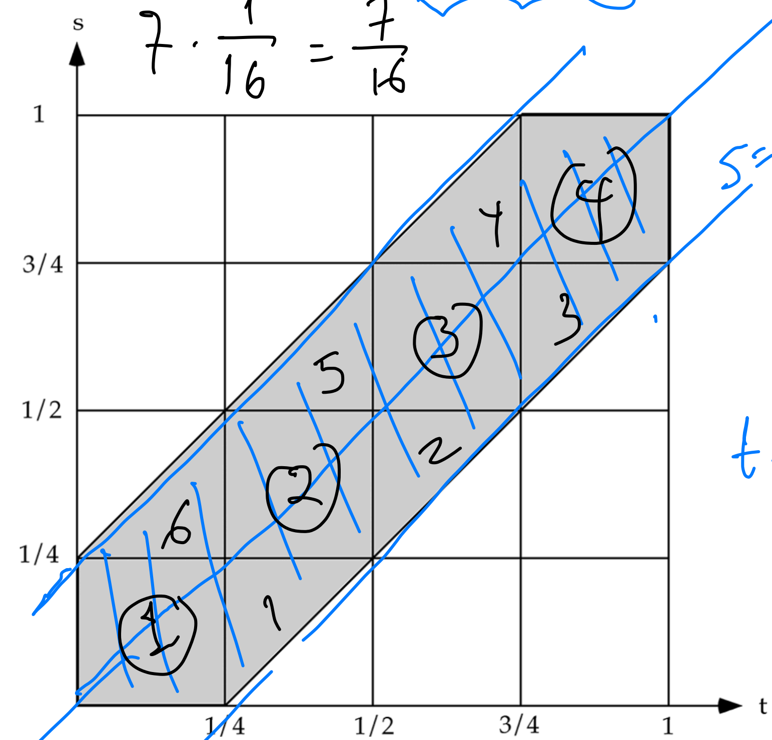
total = 2 hours

$\frac{1}{4}$ of total

$7 \cdot \frac{1}{16} = \frac{7}{16}$

$s - \frac{1}{4} \qquad s \qquad s + \frac{1}{4}$

t



$s = t + \frac{1}{4}$

$s = t$

$s = t - \frac{1}{4}$

$|s - t| \leq \frac{1}{4}$

$t - \frac{1}{4} \leq s \leq t + \frac{1}{4}$

$s = t + \frac{1}{4}$

$s = t - \frac{1}{4}$

- Since a half hour is one-quarter of the time between 5:30 and 7:30, this is represented by the inequality $|s - t| \leq \frac{1}{4}$.

Overlap (2/2)

- What is the area of this shaded region?
- The grid drawn breaks the region up into 4 squares, each 1/4 by 1/4, plus 6 isosceles right triangles with side length 1/4.
- The squares have area 1/16, and the triangles half that, or 1/32; the total area is

$$4 \cdot \frac{1}{16} + 6 \cdot \frac{1}{32} = \frac{7}{16}.$$

- So, the probability their dinners overlap is $\frac{7}{16}$!

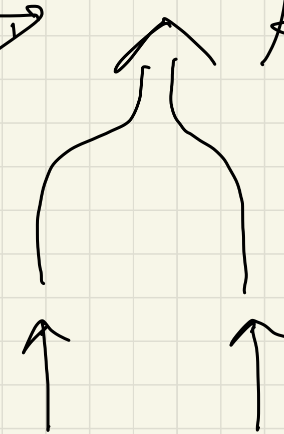
μs

ms

In case of a network: You must know the following things.

1) Length of packet
2) Sending distribution
3) Arrival $\longrightarrow$

Arrival



If prob. of collision is 10^{-3}; 10^{-6} $\longrightarrow$

Are you Being Stalked? (1/2)

- Assume you arrive at the cafeteria at random times between 5:30 and 7:30 pm.
- You notice at one point that you're seeing a lot of this one other person.
- You decide to keep track, and over the course of the next 10 days you see them 8 times at dinner.
- Can this be a random occurrence, or are you being stalked?
- We are asking is the following question:

  what is the likelihood of seeing this person at dinner 8 or more times in 10 days, assuming that their arrival times are in fact random.

Are you Being Stalked? (2/2)

- We've shown, the probability their dinners overlap is $\frac{7}{16}$.
- Treat the 10 days as a series of Bernoulli trials, in which case as we have seen the chances of seeing them exactly 8 times out of 10 or exactly 9 times out of 10 or exactly 10 times would be

Bernoulli trials

$$\begin{cases} \binom{10}{8} \cdot \left(\frac{7}{16}\right)^{8} \cdot \left(\frac{9}{16}\right)^{2} \approx 0.019 \\ \binom{10}{9} \cdot \left(\frac{7}{16}\right)^{9} \cdot \left(\frac{9}{16}\right)^{1} \approx 0.0033 \\ \binom{10}{10} \cdot \left(\frac{7}{16}\right)^{10} \cdot \left(\frac{9}{16}\right)^{0} \approx 0.00025 \end{cases}$$

$\binom{10}{8} p^8 (1-p)^2$

$\binom{10}{9} p^9 (1-p)^1$

$\binom{10}{10} p^{10}$

0.02

- Adding these up, we see that assuming that the other person's arrival times are random, there is only a 1-in-50 chance that you would see them at dinner 8 or more times in 10 days.

Interpretations

- What we have established so far is that the probability of at least 8 sightings in 10 days, assuming random arrival times, is about 0.02.

- To put it more succinctly, let A be the event that you see this person 8 or more times in 10 days, and let B be the event that their arrival times at the cafeteria are random.

- What we know then is that $\Pr[A \text{ assuming } B] \approx 0.02$ $\Pr[A|B]$

- So: it is unlikely that you'd see this person that often, if their arrival times were indeed random.

- What we are actually asking, though, is something different: we are asking, what is the probability that their arrival times are random, given that you have seen then 8 or more times in 10 days? In other words, what is $\Pr[B \text{ assuming } A]$?

know $\Pr[A|B] = \dfrac{\Pr[A \& B]}{\Pr[B]}$

$$\Pr[B|A] = \frac{\Pr[B \& A]}{\Pr[A]}$$

can compute

Bayes' Rule:

Bayes Rule

- Note that Pr[A assuming B] and Pr[B assuming A] are not the same thing!
- According to Bayes' theorem, in order to relate the two we would need to know Pr[B], or, equivalently, $P(notB)$, that is what were the odds that you were being stalked independently of this observation.
- In other words, are stalkers a commonplace part of your life?
- In sum: the fact that you see this person as often as you do may be significant: as a rule of thumb, statisticians view occurrences with less than a 5% probability as not significant, but it certainly doesn't mean that the probability you?re being stalked is 98%.

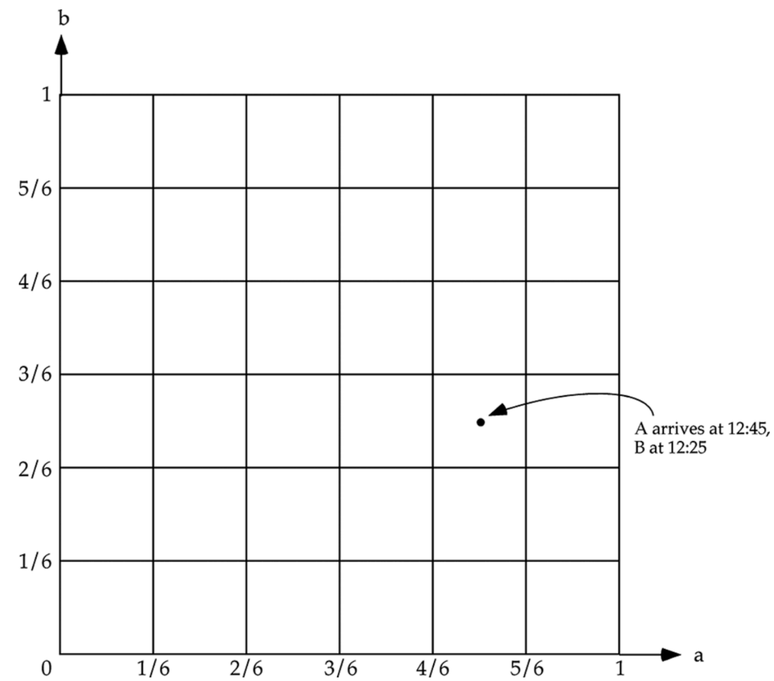
Queues

Customers

- You are the manager of a remote post office branch which employs just one clerk.
- You are concerned that with only one clerk, some of your customers may have to wait for service.
- Observe the traffic over a period of time, and you see
  1. 4 customers come in at midday:
  2. 1 (called Early Bird) comes in every day at noon:
  3. 1 (called Late Bird) comes in every day at 1, and
  4. 2 others (called A, B) come in (in any order) at independent random times between 12 and 1.
- Transactions with each of these customers take 10 minutes.
- Since no one at all shows up between 11 and 12, the teller is always available right at 12:00.

Representation

- Let a (resp. b) be the time in hours after 12 that A (resp. B) shows up.
- Draw a square representing possible arrival times (a, b).



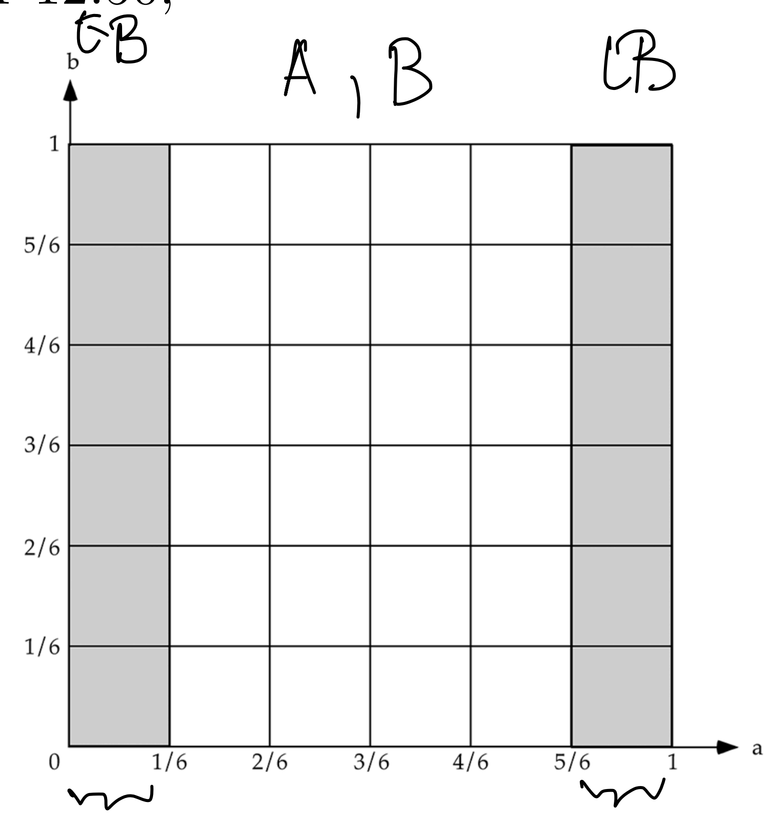
- What is the probability that one or more of the customers will have to wait?

Representation

- The sides of the square represent the hour between 12 and 1, marked off into six increments of 1/6 of an hour, or 10 minutes.
- As before, we make the sides of the square of length 1, so that as before the probability that the pair (a, b) of arrival times lies in a given region in the square is equal to the area of that region.
- Mark off the possible arrival times (a, b) corresponding to scenarios where one or more of the customers will have to wait.

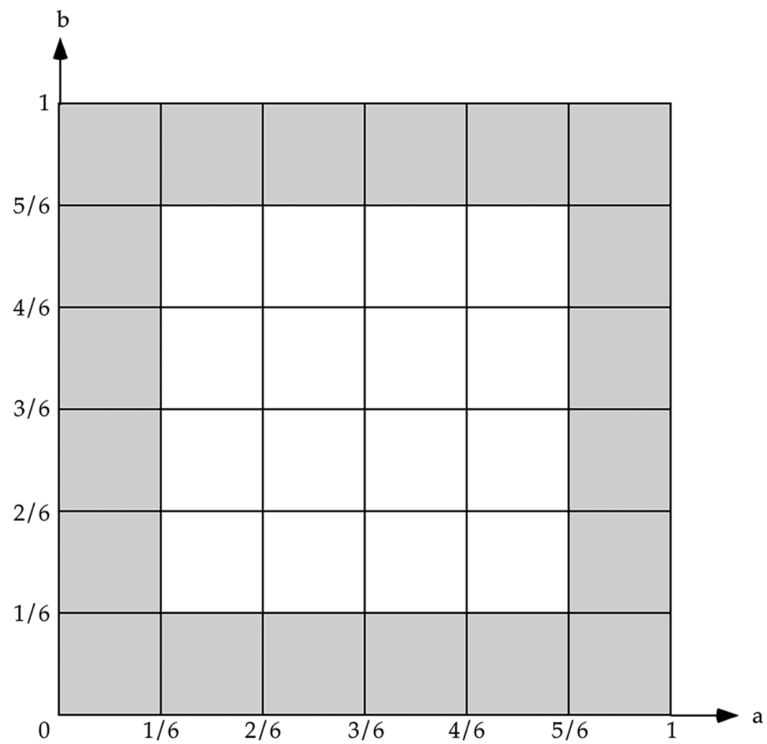
Representation

- Early Bird never has to wait, since the teller is always available when he shows up at 12:00.
- If A arrives before 12:10, she will have to wait; and likewise if she arrives after 12:50,



Representation

- The same is true for B as well *if she arrives before 12:10 or after 12:50, someone is going to have to wait) and we mark off those outcomes as well:

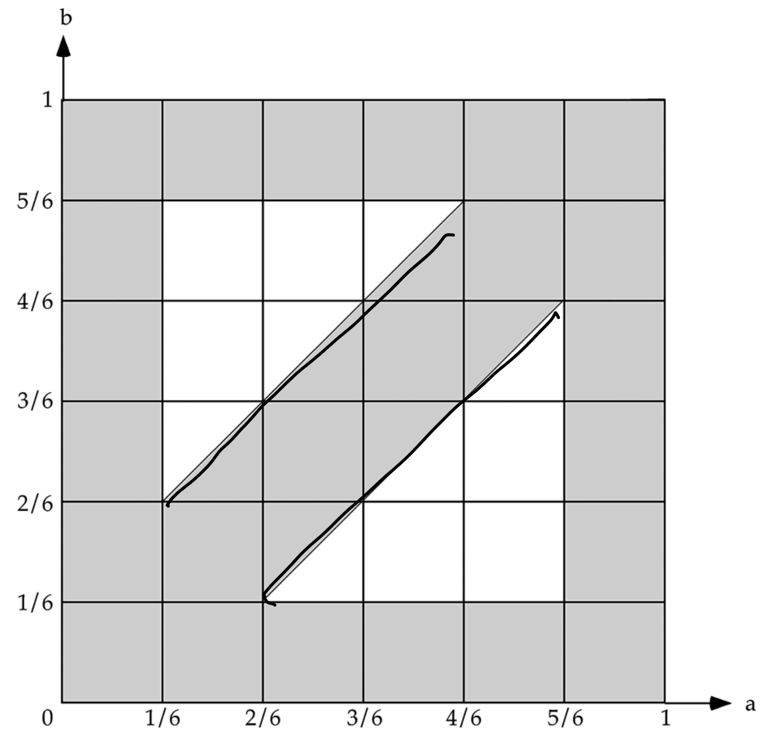


A: vertical

B: horizontal

Representation

- Finally, if A and B arrive within 10 minutes of each other, one of them will have to wait, so we mark off points (a, b) with a and b within $1/6$ of each other, that is, within $1/6$ of the diagonal either horizontally or vertically.



white squares

$\parallel$

non-waiting

- And this is what we are left with!

Outcome

- Finally, we are ready to answer the question of how likely it is that someone will have to wait: this is just the area of the shaded portion of our square.
- To calculate it, it might be easier just to find the area of the rest (the locus of (a, b) corresponding to outcomes where no one has to wait(.
- This region consists of two triangles, that fit together to form a $1/2 \times 1/2$ square; so the probability that no one will have to wait is $\frac{1}{2} \cdot \frac{1}{2} = 1/4$, and the likelihood that someone will have to wait is $1 - 1/4 = 3/4$, or three in four

Conclusion

- That was an extremely simple example of a more general problem, belonging to an area called queuing theory.
- It is concerned with what happens when a number of people show up at random times, but where the number of people may be large, and the total number of people is unknown.
- It's relevant not only to service businesses, as in the example ~~we just did~~, but also to things like comp~~o~~uter, data packet, and phone networks: if people make calls at random times, how large does the bandwidth of the network have to be to have, say, only a 5% chance of an overload occurring during a given call?

Enlarg

Exercises[a]

1. What is the probability that you arrive before 6, assuming that you and your friend see each other?
2. What is the probability that at least one of you and your friend will arrive before 6?
3. What is the probability that at least one of you and your friend arrives before 6, assuming that you see each other?
4. Say the cafeteria serves lobster, but they run out around 6.You and your friend agree that if either of you arrives before 6, the first one to arrive will get two so you'll each have one (assuming that the second one arrives while the first is still there). What is the probability that at least one of you will arrive before 6 and that you'll see each other?
5. What is the probability that you and your friend will overlap,

[a] Not to submit

assuming that you arrive between 5:30 and 6? In symbols, what is Pr[overlap assuming $0 \le t \le 1/4$]?

6. Assume the model is as before, however you stay for 30 minutes upon arrival but your friend is a fast eater and finishes dinner in 15 minutes. What are the chances your dinners will overlap?

7. If bank transactions took only 5 minutes instead of 10, what would be the probability that someone would have to wait?

8. Suppose now that banking transactions take only 5 minutes, but B is old friends with the post office clerk and, what with all the chatting, spends 15 minutes at the window. What is the probability that someone will have to wait?

References

- Benedict Gross, Joe Harris, Emily Riehl, Fat Chance: Probability from 0 to 1, Cambridge University Press, Jun. 13, 2019
- Neely, Stochastic Network Optimization with Application to Communication and Queueing Systems, Morgan & Claypool Publishers series, Synthesis Lectures on Communication Networks, 2010.