?? A FURTHER GENERALIZATION??

- Let $k \geqq 1$ and $n \geqq 0$ be integers. The number of solutions to the inequality $x_1 + x_2 + \ldots + x_k \geqq n$, where each $x_i \geqq 0, i=1,\ldots,k$ is an integer, is equal to: ????

- Why can't we count that?

PIGEONHOLE PRINCIPLE

Pigeonhole Principle: Let $k \geqq 1$ be an integer. If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more objects.

Corollary: There is no bijection between two sets which have different cardinality. In fact, there is not even a one-to-one function $f: A \to B$ if $|A|>|B|$.

INDIA PALE ALE

President of the Carleton Computer Science Society	Favorite Drink
Simon Pratt (2013–2014)	India Pale Ale
Lindsay Bangs (2014–2015)	Wheat Beer
Connor Hillen (2015–2016)	Black IPA
Elisa Kazan (2016–2019)	Cider
William So (2019–2020)	Amber Lager

Simon drinks each day of April, at least one bottle of India Pale Ale a day. He drinks a total of 45 bottles.
The claim is that there must be a sequence of consecutive days, during which Simon drink exaxcly 14 bottles.

INDIA PALE AL

How to do prove this?

- Let b_i: = # of bottles Simon drank on April i, for i = 1, ..., 30.

  Note: b_i is a positive integer and $b_1 + b_2 + \ldots + b_{30} = 45$
- Let a_i: = # of bottles Simon drank during the first i days of April, for i = 1, ..., 30.
- thus, $a_i = b_1 + b_2 + \ldots + b_i$

  Note that all a_i's must be different. Why?

INDIA PALE ALE

Now consider: $a_1, a_2, ..., a_{30}$, $a_1 + 14, a_2 + 14, ..., a_{30} + 14$.

These are 60 numbers the first 30 as well as the last 30 of which each are pairwise distinct.

Each number in that set is a number within the rage {1, ..., 59} . Why?

By Pigeonhole therefore not all 60 numbers can be distinct and there must be two different indices i, j such that $a_i = a_j + 14$. Clearly $j < i$ therefore

$$14 = a_i - a_j = b_{j+1} + b_{j+2} + ... + b_{i.}$$

Thus, in the period April j+1 until April i Simon drinks exactly 14 bottles.

LONG MONOTONE SUBSEQUENCES

- Let n=3 and consider this sequence: 20, 10, 9, 7, 11, 2, 21, 1, 20, 31 of

  $10 = n^2 + 1$ numbers.

This sequence contains an increasing subsequence of length 4 = n+1 namely 10, 11, 21, 31.

This idea holds in general.

The series 20, 10, 9, 7, 11, 2, 21, 1, 20, 31 also contains a decreasing subsequence, here of length 6.

LONG MONOTONE SUBSEQUENCES

Theorem 3.10.2 Let $n \geqq 1$ be an integer. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either increasing or decreasing.

Proof For each i, $1 \leqq i \leqq n^2+1$, let inc_i, denote the length of the longest increasing subsequence that starts at a_i. Analogously, let dec_i, denote the length of the longest decreasing subsequence that starts at a_i.

Now we will prove the result by contradiction. So, let us assume that both $inc_i \leqq n$ and $dec_i \leqq n$, for all i with $1 \leqq i \leqq n^2+1$.

LONG MONOTONE SUBSEQUENCES

Consider the set

$B = \{(b,c) \mid 1 \leqq b \leqq n,\ 1 \leqq c \leqq n\}$. Think about these elements as being boxes.

For each i with $1 \leqq i \leqq n^2+1$, the pair (inc_i, dec_i) is in B. We have n^2+1 elements placed in the n^2 boxes of B. By Pigeon Hole Principle, one of the boxes must have two elements.

There are two integers $i<j$ such that $(inc_i, dec_i) = (inc_j, dec_j)$

LONG MONOTONE SUBSEQUENCES

The elements in the sequence are distinct, so $a_i \neq a_j$, for $i \neq j$. We now have two cases:

1. $a_i < a_j$ and 2. $a_i < a_j$

1. $a_i < a_j$ Then, the length of the longest increasing subsequence starting at a_i is at least 1 + inc_j. Why? We can make the longest increasing subsequence starting at a_j longer by starting at a_i. Thus, inc_i , $\neq$ inc_j which is a contradiction.
2. $a_j < a_i$ Then, the length of the decreasing longest subsequence starting at a_i is at least 1 + dec_j. Why? We can make the longest decresing subsequence starting at a_j longer by starting at a_i. Thus, dec_i , $\neq$ dec_j which is again a contradiction. q.e.d.

EXERCISES

- It is really important to do some exercises yourself.
- One of TAs has done a set of exercises.
- There are many many many exercises in the book. There are also many in other textbooks, also online.
- Start by following some exercises on the web, then more and more hiding the solution and thinking about it yourself. Once done, you verify the solution.
- Like in anything sports, programming , ..., with practice it gets easier.

HOW MANY PRIME NUMBERS ARE THERE?

Theorem 3.10.3 There are infinitely many prime numbers.

Proof We carry out the proof by contradiction. So, we assume that the number of prime numbers is finite, say us say there are k primes p_i. Let us order them $2= p_1<p_2<...<p_k$, where k is a fixed integer.

Observe now: that $\lim_{n\to\infty}\left(\frac{2^n}{(n+1)^k}\right) = \infty$. We can therefore choose an integer n such that:

$$2^n > (n+1)$$

$$\mathbf{N} := \{0, 1, ..., n\}$$

INFINITELY MANY PRIME NUMBERS

- Next, we define the function f: $\{1, 2, ..., 2^n\} \rightarrow \mathbf{N}^k$ as follows:

  Look at the prime factorization of $x = p_1^{m_1} * p_2^{m_2} * .... * p_k^{m_k}$ and define

  $f(x) = (m_1, m_2, ..., m_k)$. The vector has k elements.

  Can we say something about each m_i?

  $m_i \leqq m_1 + m_2 + ... + m_k \leqq m_1 \log p_1 + m_2 \log p_2 + ... + m_k \log p_k$

  $\leqq \log(p_1^{m_1} * p_2^{m_2} * .... * p_k^{m_k}) = \log x \leqq n.$

  So, each $m_i \leqq n$.

  Therefore, $f(x) \in \{0, 1, 2, ..., n\}^k$

INFINITELY MANY PRIME NUMBERS

Therefore, we obtain that f is a **one-to-one** function with

$$f : \{1, \ldots, 2^n\} \rightarrow \{0, 1, 2, \ldots, n\}^k.$$

$$|\{1, \ldots, 2^n\}| = 2^n \qquad |\{0, 1, 2, \ldots, n\}^k| = (n+1)^k$$

Since f is one-to-one $(n+1)^k \geqq 2^n$. (Otherwise, apply Pigeonhole Principle and get a contradiction to f being one-to-one).

So, $(n+1)^k \geqq 2^n$ and from earlier $2^n > (n+1)^k$ which is a contradiction to our choice of n. q.e.d.

RECURSION

To understand recursion, you must first understand recursion.

RECURSION

A concept in which an object such as a function, a set , an algorithm, is defined as follow:

- There is at least one base case
- There is at least one rule that defines the object in terms of "smaller" objects that have already been defined.

RECURSION

Example: Let $\mathbf{N} := \{0, 1, ..., n\}$ and $f: \mathbf{N} \to \mathbf{N}$

We recursive define f:

<u>Base case</u>: $f(0) = 3$ (so $n=0$)

<u>Recursive step</u>: $f(n) = 2 * f(n-1) + 3$, if $n \geqq 1$.

We claim that f is uniquely defined for all $n \in \mathbf{N}$

RECURSION

f(0) is defined and for any $n \geqq 1$, f(n) is defined in terms of f(n-1). So, if we know f(n-1), we know f(n). Therefore, by induction, we get that f(n) is defined for all natural numbers n.

NOTE: do not forget the base case – otherwise you have nothing

RECURSION

- $f(0) = 3$
- $f(1) = 2*f(0) + 3 = 2*3 + 3 = 9$
- $f(2) = 2*f(1) + 3 = 2*9 + 3 = 21$
- $f(3) = 2*f(2) + 3 = 45$
- ....

Is there a direct (i.e., non-recursive) way to determine f(n)? That means to solve the recurrence.
Let us guess the solution and them prove that it is a solution.

RECURSION

by studying the values for a while you might guess that:

$$f(n) = 3*2^{n+1} - 3$$

Let us try to prove this by induction.

<u>Base case</u>: $n = 0$ then $f(n) = f(0) = 3$ and also $3*2^{n+1} - 3 = 3*2^1 - 3 = 6-3 = 3$.

let $n \geqq 1$.

<u>Induction hypothesis</u>: Assume that $f(n) = 3*2^{n+1} – 3$ is true for $n-1$, so

$$f(n-1) = 3*2^n – 3$$

RECURSION

Induction step: $f(n) = 2 * f(n-1) + 3$ $= 2(3*2^n - 3) + 3$ $= 3*2^{n+1} - 3$ q.e.d.

by definition by induction hypothesis

RECURSIVE DEFINITION OF FACTORIAL

We can recursively define factorial:

$g(0) = 1$ (base case $n=0$)

$g(n) = n*g(n-1)$ if $n \geqq 1$.

It is very easy to prove that $g(n) = n!$

RECURSIVE DEFINITION OF BINOMIAL COEFFICIENTS

We can recursively define binomial coefficients:

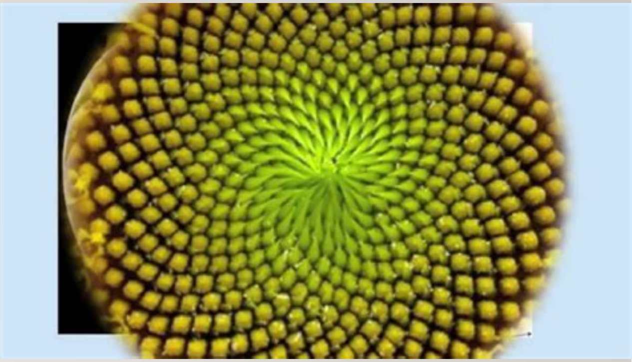
Base Case 1:	$B(n,0) = 1$ if $n \geqq 0$
Base Case 2:	$B(n,n) = 1$ if $n \geqq 0$
Recursive Step:	$B(n,k) = B(n-1,k-1) + B(n-1, k)$ if $n \geqq 2$ and $1 \leqq k \leqq n-1$.

It is very easy to prove by induction that $B(n,k) = \binom{n}{k}$ for $n \geqq 0$ and $1 \leqq k \leqq n$.

For this, recall Theorem 3.7.2 (Pascal) for the recursive step.

FIBONACCI NUMBERS

The number of spirals going in each direction is a Fibonacci Number. Here, there are 13 spirals that turn clockwise and 21 curving counterclockwise.

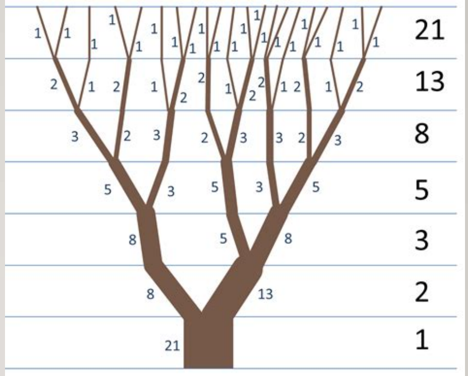


FIBONACCI NUMBERS

- The first 21 numbers are:

  0, 1,1, 2, 3, 5, *8,* 13, 21, 34, 55, 89, 144, 233 377, 610, 987,1597, 2584, 4181, 6765

- What can we say about this sequence?



FINONACCI

Non-negative

Rapidly increasing almost as fast as 2^n, in fact a good approximation is: $20^{0.694n}$

If we call f_n the n^{th} number then

$f_0 = 0$ and $f_1 = 1$

$f_n = f_{n-2} + f_{n-1}$ where $n>1$.

FIBONACCI

Solve the recurrence or find a closed form as it is alternatively called.

<u>Theorem 4.2.1</u>: Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ be the two solutions of the quadratic equation $x^2 = x+1$. Then, for all $n \geqq 0$, we have

$$f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

Note that the pronunciation is as follows: ϕ *: phi* ψ*: psi*

PROOF - FIBONACCI CLOSED FORM

Base cases:

1. $\frac{\varphi 0 - \psi 0}{\sqrt{5}} = 0$

2. $\frac{\varphi 1 - \psi 1}{\sqrt{5}} = 1$

why do we need two base cases?

PROOF - FIBONACCI CLOSED FORM

- Recursive Step: Let n ≧ 2 and assume that the claim is true for n-2 and n-1.

$$f_{n-2} = \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}}$$

$$f_{n-1} = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}.$$

We must establish the claim for n.

PROOF CONT'D

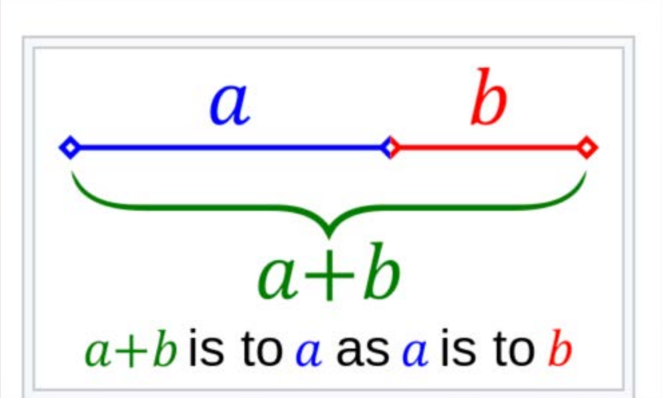
- We will use the definition of f_n the fact that $\varphi^2 = \varphi + 1$ and that $\psi^2 = \psi + 1$ in the following:
- induction
- multiplying and regrouping
- removing the ()'s
- combining

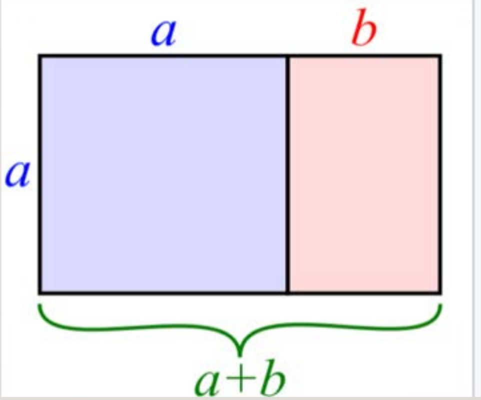
$$
\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \\
&= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\varphi^{n-2}(\varphi + 1)}{\sqrt{5}} - \frac{\psi^{n-2}(\psi + 1)}{\sqrt{5}} \\
&= \frac{\varphi^{n-2} \cdot \varphi^2}{\sqrt{5}} - \frac{\psi^{n-2} \cdot \psi^2}{\sqrt{5}} \\
&= \frac{\varphi^n - \psi^n}{\sqrt{5}}.
\end{aligned}
$$

q.e.d.

GOLDEN RATIO

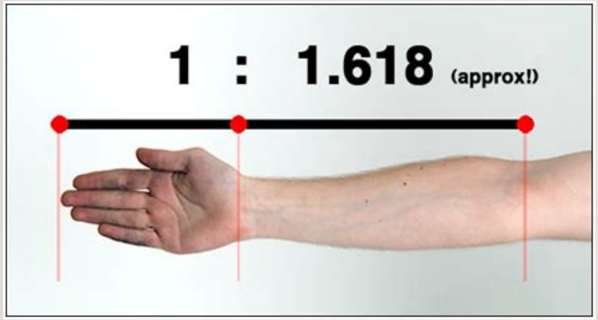
- Where do these strange numbers φ and ψ come from?
- They are solutions to the equation $x^2 = x+1$
- φ is also equal to the golden ration.
- Two numbers a and b are in the golden ration if: $\frac{a+b}{a} = \frac{a}{b} = \varphi$

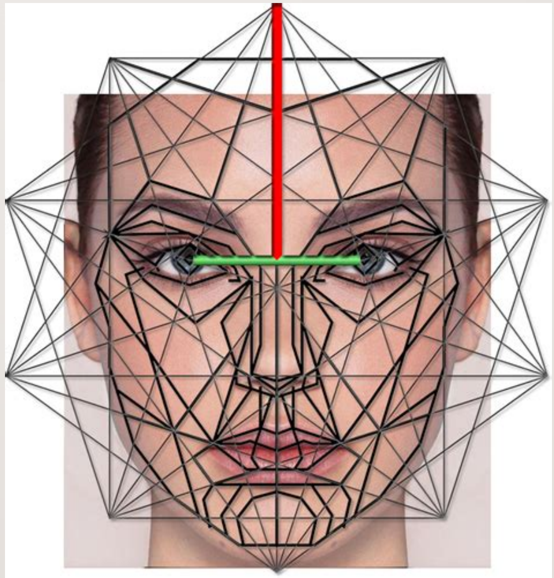




GOLDEN RATIO

- $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339...$





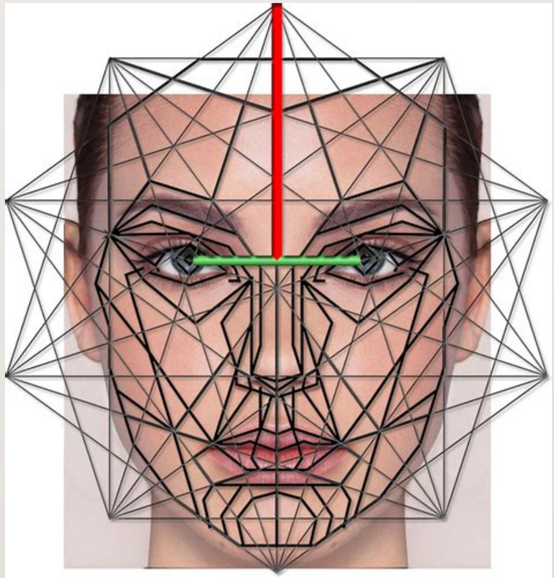
A number of facial features are related via the golden ratio.

What do the Pyramids of Giza and DaVinci's Mona Lisa have in common with Twitter and Pesi?
https://www.canva.com/learn/what-is-the-golden-ratio/
https://www.nationalgeographic.org/media/golden-ratio/

GOLDEN RATIO

- $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339...$

- Center of the pupil ● Bottom of the teeth ● Bottom of the chin
- Outer and inner edge of the eye ● Center of the nose
- Outer edges of the lips ● Upper ridges of the lips
- Width of the center tooth ● Width of the second tooth
- Width of the eye ● Width of the iris



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https://www.canva.com/learn/what-is-the-golden-ratio/
https://www.nationalgeographic.org/media/golden-ratio/

GOLDEN RATIO

- The golden ratio is seen in these flowers in terms of petal arrangement. All the petals exhibit a twisting of about 1.618034°, in order to optimize exposure to sunlight.

https://sciencestruck.com/real-life-examples-of-golden-ratio

GOLDEN RATIO

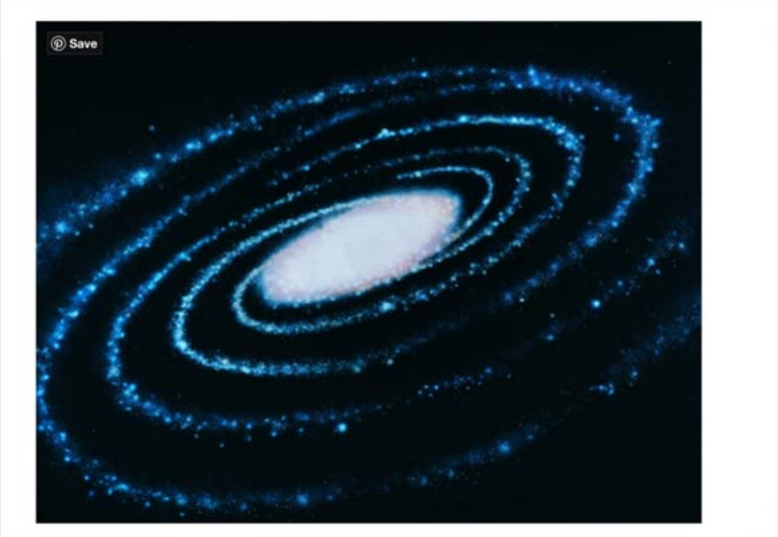
- Each cone consists of pairs of alternating whorls, each oriented in the opposite direction to the other whorl. The ratio of the turn of each pod and the ratio between the number of pods in successive whorls is the golden ratio, i.e., 1.618.

GOLDEN RATIO

A DNA molecule is 34Å in length and 21Å in width. The ratio is approximately equal to the golden ratio. The same is true for the ratio of the two grooves of the helical DNA molecule, i.e., the major (21Å) and the minor (13Å) groove.

GOLDEN RATIO

Each cone consists of pairs of alternating whorls, each oriented in the opposite direction to the other whorl. The ratio of the turn of each pod and the ratio between the number of pods in successive whorls is the golden ratio, i.e., 1.618.



https://sciencestruck.com/real-life-examples-of-golden-ratio

COUNTING 00-FREE BITSTRINGS

Definition: A bitstring is 00-free if it contains no two consecutive "0"s.

Ex.: 010101010, 11011110111, are 00-free but 01010100101010 is not.

How many 00-free bitstrings of length n are there? We call that number B_n.

If you have no idea, then it is always good to get some insight looking at smaller values of n.

$B_1 = 2$ We have only the strings 0 and 1. Both are 00-free.

$B_2 = 3$ We have four length 2 bitstrings 00, 01, 10, and 11 and only three of them are 00-free.

COUNTING 00-FREE BITSTRINGS

$B_3 = 5$ there are eight bitstrings of length 3. 000, 001, 010, 011, 100, 101, 110, 111

There is something recursive going on; we will try to explore this.

Let us put the 00-free bitstrings into a matrix. One bitstring per row.

B_n
here n=3

1	0	1
1	1	0
1	1	1
0	1	0
0	1	1

010

here 3 (length of bistring, n)

COUNTING 00-FREE BITSTRINGS

Note that we arranged the rows so that all bistrings starting with a 1 are listed above all starting with a 0.

1	0	1
1	1	0
1	1	1
0	1	0
0	1	1

010

This submatrix (obtained by removing the first 1) contains all 00-free bitstrings of length 2, in general n-1.

COUNTING 00-FREE BITSTRINGS

Note that we arranged the rows so that all bistrings starting with a 1 are listed above all starting with a 0.

1	0	1
1	1	0
1	1	1
0	1	0
0	1	1

010

This submatrix (obtained by removing the first 0) does not contains all 00-free bitstrings of length 2, in general n-1 because no string is allowed to start with a 0 there. So, they all start with a 1. Let us remove that 1.

COUNTING 00-FREE BITSTRINGS

Note that we arranged the rows so that all bistrings starting with a 1 are listed above all starting with a 0.

1	0	1
1	1	0
1	1	1
0	1	0
0	1	1

010

This submatrix (obtained by removing the first 0 and the next 1) contains all 00-free bitstrings of length 1, i.e.,, n-2.

COUNTING 00-FREE BITSTRINGS

So, we know the matrix has B_n rows. If we look at the two submatrices and count their rows, we get B_{n-1} and B_{n-2}. So, $B_n = B_{n-1} + B_{n-2}$. this gives us a recursion!

$B_1 = 2$ (n=1)

$B_2 = 3$ (n-2)

$B_n = B_{n-1} + B_{n-2}$ for $n \geqq 3$.

COUNTING 00-FREE BITSTRINGS

This is almost! Fibonacci except that the base cases are different,

$B_1 = 2$	$(n=1)$
$B_2 = 3$	$(n-2)$
$B_n = B_{n-1} + B_{n-2}$ for	$n \geqq 3.$

2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Fibonacci: 0, 1,1, **2, 3, 5, 8, 13, 21, 34, 55, 89, 144**, 233 377, 610, 987,1597, 2584, ...

So, it is easily proven that $\boxed{B_n = f_{n+2}.}$

OUTLOOK

We will do a lot more on recursion in 3804.

Recursion is a common strategy for algorithm designs.

It makes the analysis of the time complexity of an algorithm manageable.

On the other hand, recursion needs to be maintained by the OS at a cost

(e..g, recursion stack) and thus the observed run-time may suffer from that.

So techniques have been developed to unroll recursions automatically by the compiler.

RECURSIVELY DEFINED SETS

We can also use recursion to define sets elegantly.

Define a set S as through via two rules:

1. 5 is an element of S
2. If x and y are elements of S, then x – y is also an element of S.

So, what is S?

RECURSIVELY DEFINED SET

By 1.,		$5 \in S$
by 2.,	$5-5$	$= 0 \in S$
	$0-5$	$= -5 \in S$
...	$5-(-5)$	$= 10 \in S$
...	$0-10$	$= -10 \in S$
...	$5-(-10)$	$= 15 \in S$
...	$0-15$	$= -15 \in S$
....		

$\{-15, -10, -5, 0, 5, 10, 15\}$

RECURSIVELY DEFINED SET

- $\{-15, -10, -5, 0, 5, 10, 15\}$
- So, the guess is that $S = \{5n \mid n \in Z\}$
- We have to prove the equality of two sets S and $\{5n \mid n \in Z\}$.
- For this we prove the containment in both directions.
  1. $S \subseteq \{5n \mid n \in Z\}$
  2. $S \supseteq \{5n \mid n \in Z\}$

Then the two sets are equal.

$S \subseteq \{5n \mid n \in Z\}$

- We need to establish that every element of S is a multiple of 5.
  - S is recursively defined. So, we must start with the base case.
  - Clearly, 5 is a multiple of 5.
  - Let x and y be two elements of S and assume that both are multiples of 5. Then, also x-y is a multiple of 5. Easy.

$S \supseteq \{5n \mid n \in Z\}$

We prove that for all $n \in Z$, both $5n \in S$ and $-5n \in S$.

by induction:

base case (of induction proof) $n=0$.

By Base case (of definition of S), 5 is in S. So take $x=y=5$ $x-y=0 \in S$

*induction hypo*thesis: for all $n \geqq 0$, assume that $5n \in S$ and $-5n \in S$.

$S \supseteq \{5n \mid n \in Z\}$

induction hypothesis: for all $n \geqq 0$, assume that $5n \in S$ and $-5n \in S$.

induction step: Show that both $5(n+1) \in S$ and $-5(n+1) \in S$.

Both $x = 5$ and $y = -5n$ are in S. Why?

Then, $x-y = 5(n+1) \in S$.

Analogously, both $y = 5$ and $x = -5n$ are in S.

Then, $x-y = -5(n+1) \in S$.

FINISHING THE ARGUMENT

Since both 1 and 2 are true, i.e.,

1. $S \subseteq \{5n \mid n \in Z\}$
2. $S \supseteq \{5n \mid n \in Z\}$

we get that $S = \{5n \mid n \in Z\}$. q.e.d.