# Assignment 4 Solutions

April 15, 2021

## Question 1

The range of $X$ (the set of values that $X$ can take) is $\{1, 2, 3, 4\}$.
Let $y$ be some real number not in the range of $X$. Then $D(y) = Pr(X = y) = 0$.
The sample space can be described as follows:

$$S = \{(b1, b2) \, : \, b1 \neq b2, \, b1, b2 \in \{1, \ldots, 5\}\}.$$

It follows that $|S| = 20$.
Moreover,$Pr(X = 1) = |\{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5), (5, 4)\}|/20 = 8/20$. Thus, $D(1) = Pr(X = 1) = 2/5$. In a similar fashion, we can find that: $D(2) = 6/20$ , $D(3) = 4/20$, and $D(4) = 2/20$.

Remark: We could also have thought of the sample spaces as an unordered set. Namely, sample space $S = \{\text{subset of size 2 of} \{1, \ldots 5\}\}$. It follows that $|S| = \binom{5}{2} = 10$ and $Pr(X = 1) = |\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}|/1 = 4/10$.

## Question 2

$K$ and $H$ are independent random variables if and only if for all pairs of values $k$ and $h$ ($k$ and $h$ are both real numbers) we have:

$$Pr(K = k \wedge H = h) = Pr(K = k)Pr(H = h) \ (1)$$

The range of $K$ is $\{0, 1, 2, 3\}$ and the range of $H$ is $\{0, 1, 2, 3\}$. This makes 16 pairs $(k, h)$. We find one pair for which the equation above is not true.
For instance, we can observe that $Pr(K = 3 \wedge H = 3) = 0$ since we cannot draw more than one King of Hearts. However, $Pr(K = 3) \neq 0$ and $Pr(H = 3) \neq 0$. Therefore, the equation (1) is not satisfied when $k = 3$ and $h = 3$ and so $K$ and $H$ are not independent random variables.

## Question 3

Here's an example of a solution. Denote by $R(X)$ and $R(Y)$ the range of $X$ and $Y$ respectively.

$$
\begin{aligned}
E(X \cdot Y) &= \sum_{x \in R(X), y \in R(Y)} xy \cdot Pr(X \cdot Y = xy) \text{ by def of expected value} \\
&= \sum_{x \in R(X), y \in R(Y)} xy \cdot Pr(X = x \wedge Y = y) \\
&= \sum_{x \in R(X), y \in R(Y)} xy \cdot Pr(X = x) Pr(Y = y) \text{ by independence} \\
&= \sum_{x \in R(X)} \sum_{y \in R(Y)} xy \cdot Pr(X = x) Pr(Y = y) \\
&= \big( \sum_{x \in R(X)} xPr(X = x) \big) \big( \sum_{y \in R(Y)} yPr(Y = y) \big)
\end{aligned}
$$

$= E(X) \cdot E(Y)$by def of expected value

## Question 4

One possible solution: Let $x$ be the (fixed) amount dollars won a roll of the two dice. Let $Y$ be the amount of money earned at the end of the game and let $Z$ be the number of rolls whose sum was greater or equal to 10. We can first notice that:

$$Y = x \cdot Z$$

We want to find $E(Y)$. By the above we have $E(Y) = E(x \cdot Z) = x \cdot E(Z)$, since $x$ is a constant. Let us consider the following indicator variables for $1 \leq k \leq 10$:

$$Z_k = 1 \text{ if the sum of the } k^{th} \text{ roll was} \geq 10. \text{ Otherwise } Z_k = 0.$$

We can observe that (you should convince yourself of this)

$$Z = Z_1 + Z_2 + \cdots + Z_{10}$$

**Note**: Formally this is an equality between functions. However, think of what both sides are counting.

Using linearity of expected value, we get

$$E(Z) = E(\textstyle\sum_{k=1}^{10} Z_k) = \sum_{k=1}^{10} E(Z_k)$$

By def. of expected value, we have $E(Z_K) = 0Pr(Z_k = 0) + 1Pr(Z_k = 1) = Pr(Z_k = 1)$
Note that $Pr(Z_k = 1) = |\{(4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\}|/36 = 1/6$
And so

$$E(Z) = \textstyle\sum_{k=1}^{10} 1/6 = 10/6$$

Thus, $E(Y) = 10x/6$.

Alternatively: We can observe that $Z$ is a binomial random variable for 10 independent and identical trials (the rolls), where one of these trials is called a "success" if the roll yields a sum greater or equal to 10. The probability of "success" is 1/6 (same justification as above).
In class we have seen that the expected value of such a random variable on $n$ trials with probability $p$ of "success" is $n \cdot p$. Therefore, the expected value of $Z$ is 10/6. And so $E(Y) = 10x/6$.

Alternatively: Let $R(Y)$ denote the range of $Y$.
$E(Y) = \sum_{y \in R(Y)} yPr(Y = y)$ by definition of expected value.
Note that $R(Y) = \{0, x, 2x, \ldots, 10x\}$. For $1 \leq k \leq 10$ $Pr(Y = kx) = \binom{10}{k} \cdot (1/6)^k (5/6)^{10-k}$

- $\binom{10}{k}$ is choosing out of the 10 rolls, the $k$ ones that were greater or equal to 10.
- 1/6 is the probability of getting a sum $\geq 10$.
- 5/6 is the probability of not getting a sum $\geq 10$.

Thus, $E(Y) = \sum_{k=0}^{10} x \cdot k \binom{10}{k} (1/6)^k (5/6)^{10-k}$ which simplifies to $10x/6$ (using some online tool).

## Question 5

One possible solution: (Assuming independence of each roll).
Let $X$ be the number of turns for this game to end. In particular, $X$ can be thought as the number of turns until the first "success", where "success" mean rolling 10 or more.
So $X$ is a geometric random variable with probability of "success" $p = 1/6$ (the 1/6 comes from Question 4).
The expected value of a geometric random variable with probability of success $p$ is $1/p$ (seen in class). Thus $E(X) = 6$.

Alternatively: Let $R(X)$ denote the range of $X$.
$E(X) = \sum_{x \in R(X)} xPr(X = x)$ by definition of expected value.
$R(X) = \{1, 2, \ldots, \}$ (as this game could go on forever).
So we can rewrite $E(X)$ as $\sum_{k=1}^{\infty} kPr(X = k)$.
Now we can observe that $Pr(X = k) =$ the probability that the $k^{th}$ roll is a Win and all previous $k - 1$ rolls are failures $= (1/6)(5/6)^{k-1}$.
Thus $E(X) = \sum_{k=1}^{\infty} k(1/6)(5/6)^{k-1} = 6$.

## Question 6

Same solution as Question 4: $Y$ be the number of turns of this game that are Wins.
For $1 \leq k \leq n$, we define the following indicator variables:

$$Y_k = 1 \text{ if the } k^{th} \text{ turn is a Win and } Y_k = 0 \text{ otherwise.}$$

We observe that

$$Y = Y_1 + Y_2 + Y_3 + \cdots + Y_n$$

So

$$E(Y) = \sum_{k=1}^{n} E(Y_k) \text{ by linearity of expected value.}$$

We now observe that (again by def. of expected value)

$$E(Y_K) = 0Pr(Y_k = 0) + 1Pr(Y_k = 1) = Pr(Y_k = 1).$$

$Pr(Y_k = 1) =$ the probability of drawing the same element $= 8/64$.
(Example of justification: Sample space $S = \{(x_1, x_2)\,; x_i \in \{a, b, \ldots, g, h\}\}$. So $|S| = 64$. The desired outcomes are $(a, a), (b, b), \ldots, (g, g), (h, h)$.) Thus, $E(Y) = E(Y) = \sum_{k=1}^{n} 1/8 = n/8$.

Alternatively: We can observe that $X$ is a binomial random variable for $n$ independent and identical trials (the rolls), where one of these trials is called a "success" if both players grabbed the same element. The probability of "success" is $1/8$ (same justification as above). In class we have seen that the expected value of such a random variable on $n$ trials with probability $p$ of "success" is $np$. Therefore, the expected value of $X$ is $n/8$.

## Question 7

Let $X$ be a random variable. We first show the following:
$E(m) = m$ , where $m$ is some real number.
Let $R(m)$ denote the range of $m$. $E(m) = \sum_{m \in R(m)} mPr(m = m)$ (by definition of expected value)
Note $R(m) = \{m\}$. Therefore, $E(m) = mPr(m = m) = m$. as desired.
Finally, observe that, for any random variable $X$, $E(X)$ is some real number.
Thus $E(E(X)) = E(X)$ by what we have just shown.

## Question 8

a) $D(1) =$ probability that the die is $1 = 1/12$.
Similarly $D(2) = D(3) = 1/12$ and $D(4) = D(5) = D(6) = 1/4$. This defines a valid probability distribution since $\sum D(i) = 1$. b) Let $R(X)$ denote the range of $X$.

$E(X) = \sum_{x \in R(X)} xPr(X = x)$ (definition of expected value)
The range of $X$ is $\{1, 2, 3, 4, 5, 6\}$. So we have :

$$E(X) = 1Pr(X = 1) + 2Pr(X = 2) + 3Pr(X = 3) + 4Pr(X = 4) + 5Pr(X = 5) + 6Pr(X = 6)$$

which simplifies to 17/4.

## Question 9

The question is meant to be understood as follows: On a car assembly line of $n$ cars ($n \geq 5$) , the probability of a car being defective is 0.03.

What is the probability that the first defective car is the 5th car?
Let $X$ be the random variable corresponding to the index (in the assembly line) of the first defective car.

$X$ can be thought as a geometric random variable where the probability of "success" (success = defective) $p = 0.03$.

We have seen in class that $Pr(X = k) = (1 - p)^{k-1}p$ for $k \geq 1$. So the probability that the first defective car is the $5^{th}$ in the assembly line is $Pr(X = 5) = (0.97)^4(0.03)$.

Part 2 is asking for $Pr(X \leq 10)$ Using the Sum Rule, we have $Pr(X \leq 10) = \sum_{k=1}^{10} Pr(X = k) = \sum_{k=1}^{10}(0.97)^{k-1}(0.03)$

Part 3 is asking for $Pr(X = 5 \vee 6 \vee \ldots \vee 10) = \sum_{k=5}^{10} Pr(X = k) = \sum_{k=5}^{10}((0.97)^{k-1}(0.03)$