Little's Law

Network Performance

- Development of efficient network algorithms is influenced by transmission delays of packets from source to destination.
  - Which network protocol gives the best delay-throughput characteristics under specified conditions?
  - What size buffers must be employed by a network's users in order to keep the probability of buffer overflow below a particular value?
  - What is the maximum number of voice calls that can be accepted by a network in order to keep the voice packet transfer delay to a minimum?
  - How many users can a satellite link support and still maintain a reasonable response time?

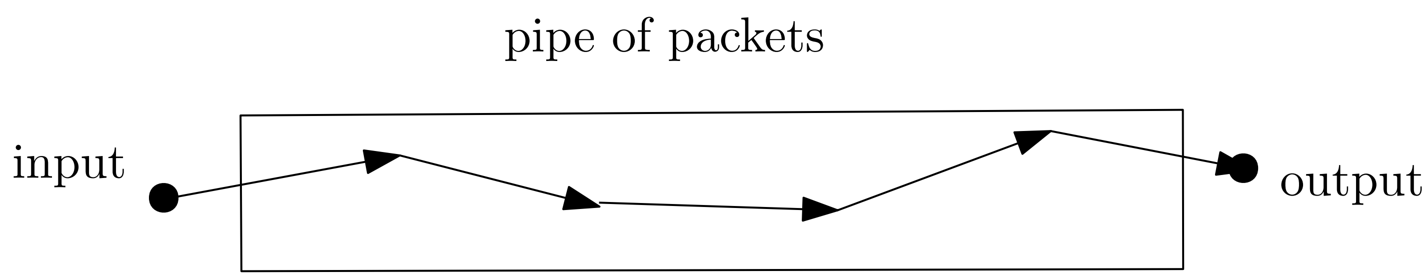
Types of Delay

- They are all measured in time units.
  - **Processing:** Delay between time packet is correctly received and the time it is correctly assigned to an outgoing link.
  - **Queuing:** Delay between time packet is assigned to a queue for transmission and the time it starts being transmitted.
  - **Transmission:** Delay between time that the first and the last bits of the packet are transmitted.
  - **Propagation:** Delay between the time that the last bit is transmitted and the bit is received.
- Is there a general principle underlying the various types of delay?

Packet Pipes

- The transmission of packets from source (input) to destination (output) ...

... resembles a pipe of packets



in that you can observe only the input and the output.

- You don't know precisely what is going on inside the pipe!
- Can observations of the input and output teach us something about the performance of the system?

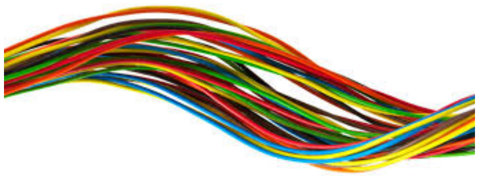
Example

- You are a spy from Burger King trying to figure out how many people are inside MacDonald's.

- You cannot sit inside MacDonald's all day;
- You must derive the answer based only on observing traffic.

Example

- It's like having a wire with packets entering from the left and exiting from the right.



- You can count how many packets enter the wire in a given time interval: of course the count will be on the average!
- You can count how long a packet stays in the wire before exiting: of course for many customers the measurement will be on the average!
- But you cannot see inside the wire!
- This is observed in many network traffic applications.

Example

- Back to the restaurant example:
- You observe that on the average 40 customers per hour go into the restaurant.
- You observe that on the average a customer stays 15 minutes.
- Any given time there are, on the average, 10 customers inside the restaurant, because

$$40 \text{ customers per hr} \times 1/4 \text{ of an hour} = 10$$

- This sounds like a fundamental principle in networing!.

Modeling Delay

- In typical queueing systems customers (i.e. packets) arrive at random times to obtain service.
- If L = packet length in bits, C = link transmission capacity in bits/sec Then:

$$\text{service time } = \frac{L}{C}$$

- We ignore the distinction between frame and packet.
- We will be interested in estimating:
  - Average # of customers in the system (either waiting in queue or undergoing service);
  - Average delay per customer.
- These will be estimated in terms of customer arrival and service rates.

Little's Theorem

- Little's theorem concerns time averages in the limit.
- Suppose we observe a sample history of a system from the starting time $t = 0$,

$$\boxed{\alpha(t)} \Rightarrow \overbrace{- - - - - - - - - - - - - -}^{N(t)} \Rightarrow \boxed{\beta(t)}$$

- Consider the quantities:

$$\begin{aligned} N(t) &= \# \text{ of customers in the system at time } t; \\ \alpha(t) &= \# \text{ of customers who arrived in the interval } [0, t]; \\ \beta(t) &= \# \text{ of customers who departed in the interval } [0, t]; \\ T(i) &= \text{time spent in the system by } i\text{-th customer.} \end{aligned}$$

- How are these quantities related?

Little's Theorem

- **Theorem 1 (Little's theorem)** *Assuming a system with steady state behavior, i.e., the rate of arrival and departure are the same (in the limit), we have that*

$$N = \lambda T,$$

where N, T, λ are the averages of the quantities defined before (and will be defined in the course of the proof).

Proof of Little's Theorem

- We will be interested in finding a relation between these parameters. We define the time average
  - arrival rate over interval $[0, t]$: $\lambda_t = \alpha(t)/t$
  - of the customer delay up to time t:

$$T_t = \frac{1}{\alpha(t)} \sum_{i=1}^{\alpha(t)} T(i)$$

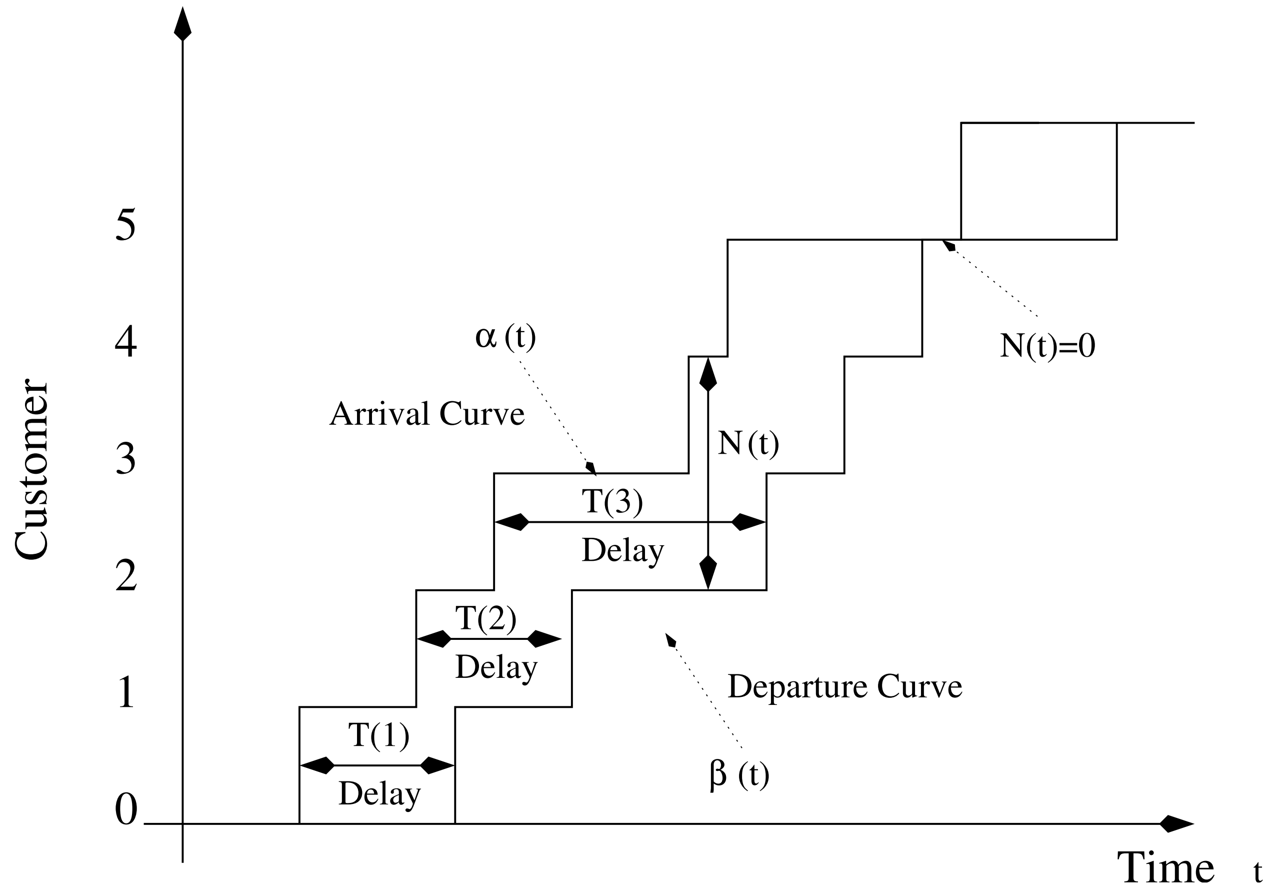
  - of the number of customers up to time t:

$$N_t = \frac{1}{t} \sum_{i=1}^{t} N(i) \approx \frac{1}{t} \int_0^t N(i) di$$

- In many systems of interest, these quantities tend to a steady state:

$$\lambda := \lim_{t \to \infty} \lambda_t, T := \lim_{t \to \infty} T_t, N := \lim_{t \to \infty} N_t$$

Proof of Little's Theorem



Illustrated are the functions $\beta(t), \alpha(t)$ and $\beta(t) \leq \alpha(t)$, $T(i)$, is the delay of customer i.

Proof of Little's Theorem

- $\alpha(t), \beta(t)$ is the number of arrivals and departures up to time t.
- Their difference $\alpha(t) - \beta(t)$ is the number $N(t)$ in the system at time t.
- The area between the arrival and departure curves $\alpha(t), \beta(t)$ is equal to
$$\int_0^t N(\tau) d\tau$$
- If $N(t) = 0$ then the area between the arrival and departure curves is also equal to
$$\sum_{i=1}^{\alpha(t)} T(i).$$

Proof of Little's Theorem

- From the picture $\sum_{i=1}^{\beta(t)} T(i) \leq \int_0^t N(\tau)d\tau \leq \sum_{i=1}^{\alpha(t)} T(i)$. Therefore

$$\begin{aligned}
\lambda_t T_t &= \frac{\alpha(t)}{t}\frac{1}{\alpha(t)}\sum_{i=1}^{\alpha(t)} T(i) \\
&= \frac{1}{t}\sum_{i=1}^{\alpha(t)} T(i) \\
&\geq \frac{1}{t}\int_0^t N(\tau)d\tau \\
&\geq \frac{1}{t}\sum_{i=1}^{\beta(t)} T(i) \\
&= \frac{\beta(t)}{t}\frac{1}{\beta(t)}\sum_{i=1}^{\beta(t)} T(i)
\end{aligned}$$

Proof of Little's Theorem

- Hence:

$$\frac{\beta(t)}{t} \cdot \frac{\sum_{i=1}^{\beta(t)} T(i)}{\beta(t)} \leq N_t \leq \frac{\alpha(t)}{t} \cdot \frac{\sum_{i=1}^{\alpha(t)} T(i)}{\alpha(t)}$$

- Taking the limit we have that

$$\lambda T \leq N \leq \lambda T$$

- Which proves, $N = \lambda T$, i.e., Little's Theorem.

Remarks

- Note in the proof we used the fact that

$$\lambda = \lim_{t\to\infty} \frac{\beta(t)}{t} = \lim_{t\to\infty} \frac{\alpha(t)}{t}$$

$$T = \lim_{t\to\infty} \frac{\sum_{i=1}^{\beta(t)} T(i)}{\beta(t)} = \frac{\sum_{i=1}^{\alpha(t)} T(i)}{\alpha(t)}$$

$$N = \lim_{t\to\infty} N_t = \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} N(i)$$

- In a way, these are stability conditions!
- The significance of Little's result is that it holds for any system that reaches a steady state.
- System need not consist of a single queue provided that the terms N, λ, T are properly interpreted.

Example 1

- Suppose we have a closed full system of K servers and N customers, $N \geq K$ (closed means departing customers are always replaced).
- Say average customer service time is $\bar{X}$; we want to find the average customer time T in the system. Apply Little's Theorem on the whole system: $N = \lambda T$.
- Apply Little's Theorem on the service portion: $K = \lambda \overline{X}$ since all K servers are always busy
- If follows that:

$$\frac{N}{T} = \frac{K}{\overline{X}}$$

- Hence:

$$T = N\frac{\overline{X}}{K}$$

Example 2

- Consider now the system under the assumption that customers arrive at a rate λ and are lost (or blocked) if they find the system full.
- In this case the number of busy servers may be less than K. Let $\bar{K}$ be the average number of busy servers, β the proportion of customers that are blocked from entering the system. From Little's theorem we derive that

$$\bar{K} = (1 - \beta)\lambda\bar{X},$$

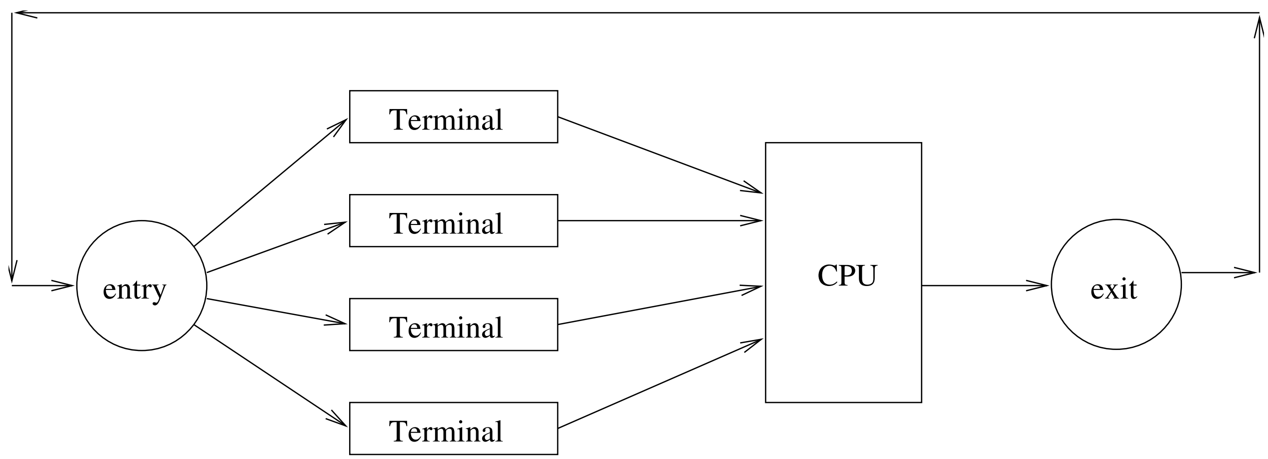
$$\beta = 1 - \frac{\bar{K}}{\lambda\bar{X}}.$$

Since $\bar{K} \leq K$ we obtain the lower bound

$$\beta \geq 1 - \frac{K}{\lambda\bar{X}}.$$

Example 3 (1/3)

- Suppose a system consisting of N terminals connected to a single CPU. Users login through a terminal.



- After reflection R, a user submits a job requiring average processing time P.
- Applying Little's Theorem between entry and exit portion of the system, we have: $N = \lambda T$, where T is average time a user spends in the system, and λ the attainable system throughput.

Example 3 (2/3)

- However, $T = R + D$, where D is the average delay between the time a job is submitted and the time its execution is completed. Clearly D may vary.

$$
\begin{array}{ccccc}
P & \leq & D & \leq & NP \\
\uparrow & & & & \uparrow \\
\text{case of no other job submitted} & & & & \text{waiting for other job to be completed} \\
R + P & \leq & T & \leq & R + NP
\end{array}
$$

- Hence: $\frac{N}{R+NP} \leq \lambda = \frac{N}{T} \leq \frac{N}{R+P}$
- However, λ is also bounded above by the processing capacity of the computer. Since the CPU can not process more than one terminal per P time units.

Example 3 (3/3)

- We have: $\lambda \leq \frac{1}{P}$
- Hence:

$$\begin{array}{ccccc} \frac{N}{R+NP} & \leq & \lambda & \leq \min & \{\frac{1}{P}, \frac{N}{R+P}\} \\ \downarrow & & & & \\ (N \to \infty) & & & & \\ \downarrow & & & & \\ \frac{1}{P} & \leq & \lambda & \leq & \frac{1}{P} \end{array}$$

  which means that in the limit $\lambda = 1/P$.
- By using:

$$T = \frac{N}{\lambda}$$

- We obtain:

$$\max\{R + P, NP\} \leq T \leq R + NP$$

Exercises[a]

1. What happens in Little's theorem if the rate of arrival and departure differ?
   (a) Consider the case: where rate of arrival is less than the rate of departure
   (b) Consider the case: where rate of departure is less than the rate of arrival
2. Packets arrive every k seconds at a regular rate; first packet arrives at time $= 0$. All packets have equal length and require αk seconds for transmission ($\alpha \leq 1$). Suppose delay and propagation time is P seconds. Then:
   (a) What is the arrival rate λ of the packets as a function of k?
   (b) How much time T does a packet spend in the system?
   (c) Now use plug in the formulas for λ and T to Little's

[a] Not to submit

theorem $N = \lambda T$ to give an expression for N.

3. Suppose we have a network of n different nodes $1, 2, \ldots, n$. Suppose the packets arrive at node i at a rate λ_i; let N_i= average number of packets in the system arriving at the node i; and assume T_i = average delay of packets at node i.
   (a) Apply Little's Theorem to each node i. What does the theorem say?
   (b) Now look at it as a whole system, and let N is the average number of packets in the system. Apply Little's theorem to the entire system to derive the average time T a packet is in the system as a function of N and λ_i.
4. Formulate and prove a form of Little's theorem when arriving packets are lost uniformly and independently with probability p.
5. Formulate and prove a form of Little's theorem when departing

packets are lost uniformly and independently with probability p.

6. Formulate and prove a form of Little's theorem when packets are lost inside the wire uniformly and independently with probability p.

7. Formulate and prove a form of Little's theorem when packet faults may occur in any of the arriving and departing packets at the same time.