Final Exam

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Question 1

(1) This statement is true. This is because QuickSort picks an element to be the pivot, and places all the elements less than the pivot into one sequence and recursively calls QuickSort, and all the elements greater than the pivot in another sequence and recursively calls QuickSort. The worst case is if QuickSort picks the largest element every time it is ran. This means that it will have to run on n elements first, and then n-1 elements, and then n-2 elements, and so on until 1. This is equal to n! which we know is equal to $\frac{n(n+1}{2}$. Therefore the worst-case run-time is $O(n^2)$

(2) This is false, given that the worst-case time complexity of insertion sort is also $O(n^2)$ (when the input is in reverse order), it is not always better than QuickSort.

(3) True as we can prove this by treating subsets as bitstrings which relate to a unique subset of the set, and the number of bitstrings of length n is 2^n

(4) No

(5) No

(6) of $A \to B$

(7) linear

Question 2

Consider strings that are made up of characters from the set S = a, b, c, d, e. We call a string "acceptable" if there is at least one character from S that does not appear in that string. For example, abcd is acceptable, bbb is acceptable; edcba is not acceptable. How many acceptable strings of length n exist? Justify your answer.

Instead of counting the valid strings of size n, we can simply count the set of invalid strings of size n and subtract them from all possible strings of size n over the set S. Let:

- U = the set of all strings of length n
- A = the set of all invalid strings of length n. A string is invalid if all characters appear in the string.
- $|U \setminus A|$ = the set of valid strings (by the compliment rule.

We now need to compute the size of U and A to calculate the number of valid strings:

$$|U| = 5^n (BytheProductrule)$$

To compute the size of A we can use the compliment rule:

- The procedure is "write a string of length n with no characters missing"
- Task 1: Choose 5 out of the n positions, there are $\binom{n}{5}$ ways to do this.
- Task 2: Assign each letter to one of these 5 positions, there are 5!
- Task 3: Place letters in the remaining n-5 positions, there are 5^{n-5} ways to do this.

Therefore by the product rule $|A| = \binom{n}{5} \cdot 5! \cdot 5^{n-5}$ strings in A.

$$\begin{aligned} |U \setminus A| &= |U| - |A| \\ &= 5^n - (\binom{n}{5} \cdot 5! \cdot 5^{n-5}) \end{aligned}$$

Question 3

Consider an arbitrary set S of 10 integers on the range [0, 100]. Prove that there exist two distinct subsets of S, call them X and Y, that have equal sums. Hint: Use the pigeonhole principle.

Question 4

Consider the function defined recursively below for $n \geq 1$. Write a non-recursive definition of the function and prove that the two definitions are equivalent.

$$\begin{aligned} f(n) &= f(n-1) + 2n - 1 \\ f(1) &= 0 \end{aligned}$$

After plugging in a few values into the recursive function we can guess the non-recursive solution to be $n^2 - 1$. To prove this we will use induction.

Base Case: n = 1

$$\begin{aligned} f(1) &= 1^2 - 1 \\ &= 0 \end{aligned}$$

Thus it holds for our base case.
Assume $f(k-1) = (k-1)^2 - 1$ (Our Inductive Hypothesis)
Prove $f(k) = k^2 - 1$:

$$\begin{aligned} f(k) &= f(k-1) + 2n - 1 \\ &= (k-1)^2 - 1 + 2n - 1 \\ &= k^2 - 2k + 1 - 1 + 2k - 1 \\ &= k^2 - 1 \end{aligned}$$

Therefore the two definitions are equivalent

Question 5

Consider a coin that has 0 on one side and 1 on the other side. We flip this coin once and roll a die twice, and are interested in the product of the three numbers.

- What is the sample space?
- How many possible events are there?
- If both the coin and the die are fair, how would you define the probability function Pr for this sample space?

The sample space is the 2 possible positions and the $6 \cdot 6$ possible outcomes for the 2 die rolled. Therefore $|S| = 2 \cdot 6 \cdot 6 = 72$.

For each event, you will be choosing whether the coin flip is heads or tails, therefore there are $\binom{2}{1}$ possible outcomes, and choosing 2 numbers out of the 6 possible numbers, therefore there are $\binom{6}{2}$ possible outcomes. This means there are $\binom{2}{1} \cdot \binom{6}{2} = 30$ events.

Let Pr(X = x) be the probability of flipping the coin either heads or tails.
Let Pr(Y = y) be the probability of rolling two six sided die and getting the sum y (y in the range of 2-12).
Therefore the probability function for this sample space is $Pr(X \cap Y)$

Question 6

There are 2 outcomes when flipping a coin, we flip the coin 6 times, therefore the sample space is $2^6 = 64$

Because these are independent events, we can use the formula $Pr(X) = \sum_{i=0}^{6} \cdot Pr(X_i)$. For A, the coin coming up at least 4 times means the number of permutations where it could come up 4, 5, and 6 times. For B, this means we can calculate the number of permutations where heads can come up exactly 3 times. Therefore:

$$\begin{aligned}
Pr(A) &= \frac{\binom{6}{4} \cdot \binom{6}{5} \cdot \binom{6}{6}}{64} \\
&= \frac{22}{64} \\
&= \frac{11}{32} \\
Pr(B) &= \binom{6}{364} \\
&= \frac{20}{64} \\
&= \frac{5}{16}
\end{aligned}$$

Because the coin flips are independent:

$$\begin{aligned}
Pr(C) &= Pr(H) \cdot Pr(H) \cdot Pr(H) \cdot Pr(H) \cdot \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \\
&= \frac{1}{16}
\end{aligned}$$

$$\begin{aligned}
Pr(A \mid B) &= Pr(A \cap B) \setminus Pr(B) \\
Pr(A \cap B) &= 0
\end{aligned}$$

$$\begin{aligned}
Pr(B) &= \frac{5}{16} \\
Pr(A \mid B) &= 0 \\
Pr(C \mid A) &= \frac{1}{16}
\end{aligned}$$

Note that $Pr(C \mid A) = Pr(C)$

Question 7

Given A being the event at least 3 people have the same birthday, $\bar{A}$ is the event that no one shares the same birthday. Using this fact, we can use the compliment rule to give us Pr(A) as $Pr(A) = 1 - Pr(\bar{A})$. To figure out the probability of $\bar{A}$, we can observe that this equals to choosing 5 days out of the 365 available birthdays, and assigning them to 5 people, over the sample space of 365^5 as each person has 365 days to choose a birthday from. Thus the probability equals:

$$Pr(\bar{A}) = \frac{\binom{365}{5} \cdot 5!}{365^5}$$
$$Pr(A) = 1 - (\frac{\binom{365}{5} \cdot 5!}{365^5})$$

Question 8

The sample space is all the different possible hand combinations of cards. This is subsets of 5 cards out of the 52 deck, therefore, let S be the sample space: $|S| = 5^{52}$.
We can solve this using the product rule:

- Procedure: "Draw a straight"
- Task 1: Pick any of the cards out of the deck, there are $\binom{52}{1} = 52$ ways to do this.
- Task 2(repeated 3 times): Pick the next consecutive card in any suit but the one previously chosen (in order to not have all the same suit), there are $\binom{3}{1} waystodothis$
- Therefore this event $= 52 \cdot \binom{3}{1}^3$

Therefore the probability that the hand is a straight is $\frac{1401}{5^{52}}$

Question 9

Let A be the event heads comes up i times before tails or tails comes up

Question 10

We can look at these experiments as coin tosses. Therefore the question would equate to, "How many times do we flip the coin before it comes up head". This resembles a geometric distribution. Using the definition of a geometric distribution, and the definition above (in the question), we get the equation:

$$
\begin{aligned}
Pr(X = k) &= p(1-p)^{k-1} \\
E(X) &= \sum_{k=1}^{\infty} k \cdot Pr(X = k) \\
&= \sum_{k=1}^{\infty} p(1-p)^{k-1} \\
&= p \cdot \frac{1}{(1-(1-p))^2} \\
&= 0.8 \cdot \frac{1}{(1-(1-0.8))^2} \\
&= 0.8 \cdot \frac{1}{(1-0.2)^2} \\
&= 0.8 \cdot \frac{1}{0.8^2} \\
&= \frac{0.8}{0.8^2} \\
&= \frac{1}{0.8} \\
&= \frac{5}{4}
\end{aligned}
$$

Given that we have to pay \$10 per experiment and we can not do a fraction of an experiment, we will have to pay $E(X) = \lceil \frac{5}{4} \rceil \cdot 10 = \$20$

Question 11

Scenario: A TV company has a game show running over several days. Each participant sings one song per day. At the end of each day, they do the following: for each of the candidates active during the day, randomly with probability 0.5 and independently, determine if the candidate is permanently off the show or will participate at least for the next day. They start, on day 1, with n participants and stop when only one participant remains. Please help the TV company to answer the following questions without providing a proof, just state answers:

- (1) What is the expected number of days a participant appears?
- (2) How many participants can we expect to be on the second of the show?
- (3) How many participants can we expect to be on the ith day?
- (4) For how many days can we expect the TV show to run?
- (5) What is the expected total number of songs that will be heard in total?

(1) $= 2$

(2) $= \frac{n}{2}$

(3) $= \frac{n}{(\frac{1}{2})^i}$

(4) $= \sum_{k=1}^{n} k \cdot p(1-p)^{k-1}$

(5) $= n \cdot \sum_{k=1}^{n} k \cdot p(1-p)^{k-1}$

End of The Exam