5.70 You flip three fair coins independently of each other. Consider the events

$$A = \text{"there is at most one tails"}$$

and

$$B = \text{"not all flips are identical".}$$

Are A and B independent?

We want to check if Pr(A n B) = Pr(A)Pr(B)

Pr(A n B) = Pr(exactly 1 Tails)= (3 C 1) (1/2)^2 (1/2) = 3/8

Pr(A) = Pr(exactly 0 Tails) + Pr(exactly 1 Tails) =1 /8 + 3/8 = ½

Pr(B) = 1- Pr(all flips are identical) = 1 – 2/8 = 6/8

Note 3/8 = ½ * 6/8

Therefore, A and B are independent

5.87 Two players P_1 and P_2 take turns rolling two fair and independent dice, where P_1 starts the game. The first player who gets a sum of seven wins the game. Determine the probability that player P_1 wins the game.

(A comment: Let X be the random variable corresponding to the number of rolls that Player 1 makes.

What kind of variable is X?)

Pr(sum = 7 on one roll)= |{ (1,6) ,(2,5),(3,4),(4,3),(5,2),(6,1)}| / 36 = 1/6

Pr(Player 1 wins) = Pr(Wins on his first roll) + Pr(Wins on his 2nd) + Pr(Wins on his 3rd) + …..

= 1/6 + (5/6)^2(1/6) + (5/6)^4(1/6)

Pr(Player 1 wins on the kth round) = (5/6)^{2(k-1}}(1/6) , in the previous k-1 rounds both player fails. ON the kth round Player 1 succeeds.

So Pr(Player 1 wins) = Sum_{k=1 to infinity} Pr(Player 1 wins on the kth round)

= Sum_{k=1 to infinity} (5/6)^{2(k-1}}(1/6)

= (1/6) Sum_{k=0 to infinity} (25/36)^(k}

= (1/6) (1/(1-25/36)) = 6/11

(Note: Pr(Player 2 wins) = 5/11 .

Pr(player 2 wins) = Pr(that he wins in game where he goes first | that player 1 fails his first toss) Pr(that player 1 fails his first toss) + Pr(Player 2 wins | Player 1 succeeds on his first toss)Pr(Player 1 succeeds on his first toss) (Total Law probability)

= (6/11)(5/6) = 5/11

5.88 By flipping a fair coin repeatedly and independently, we obtain a sequence of H's and T's. We stop flipping the coin as soon as the sequence contains either HH or TH.

Two players P_1 and P_2 play a game, in which P_1 wins if the last two symbols in the sequence are HH. Otherwise, the last two symbols in the sequence are TH, in which case P_2 wins. Determine the probability that player P_1 wins the game.

Pr(Player 1 Wins) = Pr(getting HH) = Pr(getting HH | first is H)Pr(first toss is H) + Pr(getting HH | first is T) Pr(first is T) (Total Law probability)

Pr(getting HH | first is T) = 0 (why?)

= Pr(getting HH | first is H)Pr(first toss is H) = ¼

Pr(getting HH n first is H) = Pr(getting HH) /Pr(first toss is H) * Pr(first toss is H)

6.5 You are given a fair coin.

- You flip this coin twice; the two flips are independent. For each heads, you win 3 dollars, whereas for each tails, you lose 2 dollars. Consider the random variable

$$X = \text{ the amount of money that you win.}$$

  - Use the definition of expected value to determine $\mathbb{E}(X)$.
  - Use the linearity of expectation to determine $\mathbb{E}(X)$.

- You flip this coin 99 times; these flips are mutually independent. For each heads, you win 3 dollars, whereas for each tails, you lose 2 dollars. Consider the random variable

$$Y = \text{ the amount of money that you win.}$$

  Determine the expected value $\mathbb{E}(Y)$ of Y.

Part a)

Let R(X) denote the range of X.

E(X) = Sum_{x in R(X)} x Pr(X=x)

R(X) = {6, 1, -4}

Pr(X = 6) = Pr(HH) = ¼

Pr(X=1) = Pr(HT or TH) = 2/4

Pr(X=-4) = Pr(TT) = ¼

E(X) = -4 *1/4 + 1*2/4 + 6*1/4 = 4/4 = 1

Let V = the # of Heads and W= the # of T

X = 3V– 2W

E(W) = 0 * Pr(0 T) + 1 Pr(1 T) + 2 Pr(2 T) = ½ + ½ = 1

E(W)= E(V) (since the coin is fair)

E(X) = E(3Y- 2T) = 3E(Y)-2E(T) = E(T) = 1

Part b)

For 1 <= k <=99

Let Y_k be the amount of money one on toss k

$E(Y) = E(\text{Sum}_{\{k=1 \text{ to } 99\}}\ Y_k) = \text{Sum}_{\{k=1 \text{ to } 99\}}\ E(Y_k)$

$E(Y_K) = -2\ \Pr(Y_k = -2) + 3\ \Pr(Y_k = 3) = -2 * ½ + 3 * ½ = ½$

Thus, $E(Y) = \text{Sum}_{\{k=1 \text{ to } 99\}}\ ½ = 99/2$

Let V be the number of H

And W be the number of T

$Y = 3V – 2T$

Note that V is a binomial random variable with probability of success (getting H) is = ½

$E(V) = 99/2$ (from theorem seen in class0

$E(W) = E(V)$

6.8 Assume we flip a fair coin twice, independently of each other. Consider the following random variables:

$$\begin{aligned} X &= \text{the number of heads,} \\ Y &= \text{the number of tails,} \\ Z &= \text{the number of heads times the number of tails.} \end{aligned}$$

- Determine the expected values of these three random variables.
- Are X and Y independent random variables?
- Are X and Z independent random variables?
- Are Y and Z independent random variables?

$E(X) = E(Y) = 1$ (from question 6.5)

$E(Z) = E(X .Y)$

Range of $Z = \{ 0, 1 , 2 ,4\}$

$\Pr(Z = 1) = \Pr(X =1 \text{ and } Y=1) = \Pr(\text{ HT or TH}) = 2/4 = ½$

$\Pr(Z =2) = \Pr((X=2 \text{ and } Y =1) \text{ or } (X=1 \text{ and } Y=2)) = 0 + 0$ (there are only 2 flips)

$\Pr(Z=4) = \Pr(X=2 \text{ an } Y =2) = 0$ (same reason)

$E(Z) = 0 * Pr(Z=0) + 1*1/2 + 2*0 + 4*0 = ½$

Note $Pr(Z = 0) = Pr(X=0 \text{ or } Y = 0) = Pr(HH \text{ or } TT) = 1/2$

X and Y are independent random variables iff for all pairs of values x and y in the range of X and the range of Y respectively we

$Pr(X = x \cap Y = y) = Pr(X = x) Pr(Y = y)$

Note $Pr(X=0 \cap Y = 0) = 0$ (we have to have some T or some H)

Also $Pr(X= 0) = Pr(TT) = ¼$ and $Pr(Y=0) = Pr(HH) = ¼$. So X and Y are not independent.

Similarly

$Pr(X = 2 \text{ and } Z = 4) = Pr(Z=4) = 0$

$Pr(X = 1 \cap Z = 1) = Pr(Z = 1) = ½$

$Pr(X = 1) = Pr(HT \text{ or } TH) = ½$. Therefore, X and Z are not independent.

Same argument for Y (try Y= 1 and Z=1)

6.11 Assume we roll two fair and independent dice, where one die is red and the other die is blue. Let (i, j) be the outcome, where i is the result of the red die and j is the result of the blue die. Consider the random variables

$$X = i + j$$

and

$$Y = i - j.$$

Are X and Y independent random variables?

X and Y are independent random variables iff for all pairs of values x and y in the range of X and the range of Y respectively we

$Pr(X = x \cap Y = y) = Pr(X = x) Pr(Y = y)$

(Note : Range of X : { 2, …, 12} , Range of Y : { -5, … , 5} , that would make 121 equalities to check)

$Pr(X = 3 \cap Y = 0) = 0$ since if Y= 0 implies that i = j. And X = 2i which is even.

Pr(Y = 0) = 1/6 and Pr(X =3) = 2/36

Therefore, X and Y are not independent random variables.

6.20 Consider the following algorithm, which takes as input a large integer n and returns a random subset A of the set $\{1, 2, \ldots, n\}$:

```
Algorithm RandomSubset(n):

    // all coin flips are mutually independent
    A = ∅;
    for i = 1 to n
    do flip a fair coin;
        if the result of the coin flip is heads
        then A = A ∪ {i}
        endif
    endfor;
    return A
```

Define

$$\max(A) = \begin{cases} \text{the largest element in } A & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

$$\min(A) = \begin{cases} \text{the smallest element in } A & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

and the random variable

$$X = \max(A) - \min(A).$$

- Prove that the expected value $\mathbb{E}(X)$ of the random variable X satisfies

$$\mathbb{E}(X) = n - 3 + f(n),$$

where $f(n)$ is some function that converges to 0 when $n \to \infty$.

E(X) = E(max(A) – min(A)) = E(max(A)) – E(min(A) (linearity of expected value)

Let M(A) denote the range of max(A)

E(max(A)) = sum_{ k in M(A)} k Pr(max(A) = k) (definition of expected value)

What is M(A) ? {0, …, n}

(For 1<= k <=n) Pr(max(A) = k) = Pr(last the H on kth flip) = 2^{k-1}/2^n = 2^{(k-1)-n} = (1/2)^{n-k+1)

So we need

_ _ _ _ _ _ _ _ _ H T T T T T T T T T

Two ways to argue

Counting

Sample Space = {sequences of H and T of length n}

$|S| = 2^n$

So we are counting elements of S such that the at the kth flip we have H and after all T. So we only have two options for the first k-1 flips

And so Pr(last the H on kth flip) = $2^{k-1}/2^n$ $k>=1$

Or

Independence (disregard the firt k-1 flips) On flip k you get H -> ½. On the n-k flips after you get T -> $(1/2)^{n-k}$. So Pr(last the H on kth flip) = $(1/2)^{n-k+1}$ $k>=1$

It follows that $E(\max(A)) = \sum_{k=0}^{n} k (1/2)^{n-k+1}$

$E(\min(A))$

The range of $\min(A) = \{0, \ldots, n\}$

For $1 <= k <= n$, $\Pr(\min(A) = k)$ = Pr(the first H is at position k) = $(1/2)^{k}$

T T T T T T T T H _ _ _ _ _ _ _ _

$E(\min(A)) = \sum_{k=0}^{n} k (1/2)^{k}$

So

$E(X) = \sum_{k=0}^{n} k \left((1/2)^{n-k+1} - (1/2)^{k} \right)$

$E(X) = n/2 - n/2^n + \sum_{k=0}^{n-1} k \left((1/2)^{n-k+1} - (1/2)^{k} \right)$

STOP HERE (or try the last part of the question if you want to)

6.29 You repeatedly flip a fair coin and stop as soon as you get tails followed by heads. (All coin flips are mutually independent.) Consider the random variable

$$X = \text{ the total number of coin flips.}$$

For example, if the sequence of coin flips is $HHHTTTTH$, then $X = 8$.

- Determine the expected value $\mathbb{E}(X)$ of X.

Npte that Pr(X =1) = 0

E(X) = Sum_{k in range of X} k Pr(X=k)

Range of X is { 2. …, infinity

Pr(X = k) = ______T_______H where the yellow part does not contain TH

We will partition X= k in terms of the following events

For 1 < = j <= k-1

X_j : X=k and the first T is at position j

Pr(X= k) = Pr(X_1) + Pr(X_2) + …+ Pr(X_{k-1})

Pr(X_j) = 1/2^k = (1/2)^k

Pr(X = k) = Sum_{j=1 to k-1} (1/2)^k = (k-1)(1/2)^k

So E(X) = Sum_{k=2 to infinity} k (k-1)(1/2)^k = 4

(Try to do this question with the theorem from exercise 6.31)

6.35 When Lindsay and Simon have a child, this child is a boy with probability $1/2$ and a girl with probability $1/2$, independently of the gender of previous children. Lindsay and Simon stop having children as soon as they have a girl. Consider the random variables

$$B = \text{the number of boys that Lindsay and Simon have}$$

and

$$G = \text{the number of girls that Lindsay and Simon have.}$$

Determine the expected values $\mathbb{E}(B)$ and $\mathbb{E}(G)$.

E(G) = 0 * Pr(G=0) + 1 * Pr(G =1) = 1 (they have to have exactly 1 girl)

E(B) = Sum_{k=0 to infinity) k Pr(X= k)

Pr(X = k) = (1/2)^{k+1}

X=k if and only if   BBBBBBBBBBB

E(B) = sum_{k=0 to infinity} k (1/2)^{k+1} = 1

Or

Consider X to be the number of children until the first girl. This follows a geometric distribution with probability of success p= ½

E(X) = 1/p = 2 (seen in class)

Note that X = G + B -> E(X) = E(G) + E(B)

6.40 A *block* in a bitstring is a maximal consecutive substring of 1's. For example, the bitstring 1000111110100111 has four blocks: 1, 11111, 1, and 111.

Let $n \geq 1$ be an integer and consider a random bitstring of length n that is obtained by flipping a fair coin, independently, n times. Define the random variable X to be the number of blocks in this bitstring.

- Use indicator random variables to determine the expected value $\mathbb{E}(X)$ of X.

X_1 = 1 if position 1 is equal to 1 and 0 otherwise

For 2 <= k <= n

X_k = 1 if kth bit is equal to 1 and the (k-1)th bit is equal to 0. Otherwise X_k = 0

X = X_1 + X_2 + …+ X_n (they are counting the same thing: on the R.H.S we counted the number of 1s which introduced a block)

E(X) = Sum_{k=1 to n} {E(X_K)) = E(X_1) + Sum_{k=2 to n} {E(X_K))

E(X_1) = ½

For n >=k>=2 E(X_k) = 0* Pr(X_k=0) + 1*Pr(X_k =1) = Pr(X_k =1)

Pr(X_k =1) = ¼

Thus, E(X) = 1/2 + Sum_{k=2 to n} ¼ = ½ + (n-1)/4

6.46 Consider the following recursive algorithm TwoTails, which takes as input a positive integer n:

Algorithm TwoTails(n):

> // all coin flips are mutually independent
> flip a fair coin twice;
> **if** the coin came up tails exactly twice
> **then** return 2^n
> **else** TwoTails($n+1$)
> **endif**

- You run algorithm TwoTails(1), i.e., with $n = 1$. Define the random variable X to be the value of the output of this algorithm. Let $k \geq 1$ be an integer. Determine $\Pr\left(X = 2^k\right)$.
- Is the expected value $\mathbb{E}(X)$ of the random variable X finite or infinite?

Pr(X =2^k) = Pr(that there was exactly k rounds for this algorithm to terminate) (= Pr(flipping the coin twice, exactly k times) = Pr(the first k-1 rounds to not give TT, followed by TT on the kth round) = (3/4)^{k-1}(1/4) since Pr(TT) = ¼ and and so Pr(not TT) = ¾ .

Part b) Note that X corresponds to a geometric random variable with probability p of success (getting TT) ¼

We have seen in class that E(X) = 1/p for such random variables. Thus E(X) = 1/(1/4) = 4

(Now compare this solution to exercise 6.29, this is much nicer)

7.8 Elisa Kazan is having a party at her home. Elisa has a round table that has 52 seats numbered $0, 1, 2, \ldots, 51$ in clockwise order. Elisa invites 51 friends, so that the total number of people at the party is 52. Of these 52 people, 15 drink cider, whereas the other 37 drink beer.

In this exercise, you will prove the following claim: No matter how the 52 people sit at the table, there is always a consecutive group of 7 people such that at least 3 of them drink cider.

From now on, we consider an arbitrary (which is not random) arrangement of the 52 people sitting at the table.

- Let k be a uniformly random element of the set $\{0, 1, 2, \ldots, 51\}$. Consider the consecutive group of 7 people that sit in seats $k, k+1, k+2, \ldots, k+6$; these seat numbers are to be read modulo 52. Define the random variable X to be the number of people in this group that drink cider. Prove that $\mathbb{E}(X) > 2$.

  Hint: Number the 15 cider drinkers arbitrarily as $P_1, P_2, \ldots, P_{15}$. For each i with $1 \leq i \leq 15$, consider the indicator random variable

$$X_i = \begin{cases} 1 & \text{if } P_i \text{ sits in one of the seats } k, k+1, k+2, \ldots, k+6, \\ 0 & \text{otherwise.} \end{cases}$$

- For the given arrangement of the 52 people sitting at the table, prove that there is a consecutive group of 7 people such that at least 3 of them drink cider.

  Hint: Assume the claim is false. What is an upper bound on $\mathbb{E}(X)$?

X = X_1 + X_2 + … + X_15 (they are counting the same thing

E(X) = Sum_{k=1 to 15} E(X_k)   (by linearity of expectation)

E(X_K) = 0*Pr(X_k=0) + 1 *Pr(X_k =1) = Pr(X_k=1)

Pr(X_k=1) = 7/52

Therefore, E(X) = Sum_{k=1 to 15} 7/52 = 15*7 /52 =105/52 > 2

Part b) Suppose that for all groups of 7 consecutive chairs you have at most 2 cider drinker. This would entail that E(X) <= 2, which contradicts part a). Therefore, there must be such a group of 7 chairs containing at least 3 cider drinkers.

6.31 Let X be a random variable that takes values in $\{0, 1, 2, 3, \ldots\}$. By Lemma 6.4.3, we have

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot \Pr(X = k).$$

As in Exercise 6.30, define an infinite matrix and use it to prove that

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \Pr(X \geq k).$$

Pr(X >= 1) = Pr(X =1) +Pr(X =2) + Pr(X=3) + ….

Pr(X >= 2) = Pr(X =2) + Pr(X=3) + Pr(X =4)+….

Each Pr(X >= k) contributes increments the number of times Pr(X = j)is accounted in the the summation Sum_{k=1 to infinity) Pr(X >=k) for j > = k.

Therefore, the number of time s Pr(X =k) occurs in this summation is exactly

For notation purposes ,We abbreviate Pr(X >= k) x_k

x_1 0 0 0 0 0 0 0 0 0 0 ……

x_2 x_2 0000000000……

x_3 x_3 x_3 0000000000

x_4 x_4 x_4 x_4 000000000

Looking the sum of each rows : the kth row adds up to k* x_k (k * Pr(X=k))

So sum of all the elements of this matrix = Sum_{k=1 to inf} k Pr(X=k)

But using the column sum we get: the kth column is Sum _{j=k to inf} Pr(X =j) = Pr(X >= k)

So sum of all the elements of this matrix = Sum_{k=1 to inf} Pr(X >= K)