3.18 How many bitstrings of length 77 are there that start with 010 (i.e., have 010 at positions 1, 2, and 3), or have 101 at positions 2, 3, and 4, or have 010 at positions 3, 4, and 5?

Let

A := the set of bitstrings of length 77 that start with 010

B : = the set of bitstrings of length 77 that have 101 at position 2,3 and 4

C : = the set of bitstrings of length 77 with 010 at position 3, 4 and 5

We want to compute | A u B u C |

By inclusion-exclusion this is equal to

|A | + |B| + |C| - |A n B | - |A n C| - |B n C| + | A n B n C|

|A| : 3 bits are fixed. (in other words: there's only 1 way to start with 010) . So we have 74 remaining bits to fill. There are 2^{74} ways to do this.

|B| : the same as A . So 2^{74} such strings

|C| : the same as A. So again 2^{74} such strings.

|A n B| : 4 bits that are fixed. So we fill 73 other bits (with no constraints). So 2^73 ways to do this.

|A n C| : 5 bits are fixed. So 2^72 such bitstrings.

| B n C| : 4 bits are fixed. So 2^73 such bitstrings

| An B n C| 5 bits are fixed. So 2^72 such bitstrings.

3.21 A string of letters is called a *palindrome*, if reading the string from left to right gives the same result as reading the string from right to left. For example, *madam* and *racecar* are palindromes. Recall that there are five vowels in the English alphabet: a, e, i, o, and u.

In this exercise, we consider strings consisting of 28 characters, with each character being a lowercase letter. Determine the number of such strings that start and end with the same letter, or are palindromes, or contain vowels only.

A : = the set of strings of 28 letters that start and end with the same letter

B : = the set of strings of 28 letters that are palindromes.

C : = the set of strings of 28 letters that contain vowels only

We want to compute | A u B u C | = |A u C|

By inclusion-exclusion this is equal to

|A | + |B| + |C| - |A n B | - |A n C| - |B n C| + | A n B n C|

Note that A contains B (a palindrome have to start and end with the same letter)

Hence A n B = B

So we need to compute

|A | + |C| - |A n C|

|A| =

Task 1 : We pick a starting letter -> 26 options

Task 2: Pick the same letter for the last position -> 1 way to do this

Task 3: We have 26 remaining positions to fill, each with 26 options -> 26^26 ways to do this

By the product rule, there are 26*1*26^26 = 26^27 such strings

| C | =

Each 28 positions of the string have 5 options . BY the product rule, there are 5^28 strings in C.

| A n C| =

Task 1 : We pick a startting vowel -> 5 options

Task 2 : Assign the same vowel for the last postion -> 1 way to do this

Task 3 : Fill the rest (26 positions) with vowels -> 5^26

By the product rule there are 5^27 such strings in A n C

For the sake of it: How would we compute |B|

_ _ _ _ _ _ _ ……. _ _ _ _ _ _

26^14

3.35 Consider strings consisting of 12 characters, each character being a, b, or c. Such a string is called *valid*, if at least one of the characters is missing. For example, *abababababab* is a valid string, whereas *abababacabab* is not a valid string. How many valid strings are there?

One possible approach:

A : = the set of strings of length 12 on a,b,c with no as

B: : = the set of strings of length 12 on a,b,c with no bs

C : = the set of strings of length 12 on a,b,c with no cs

| A u B U C|

|A | + |B| + |C| - |A n B | - |A n C| - |B n C| + | A n B n C|

|A| = |B| = |C| = 2^12 (1 character is missing, so for each 12 positions we have two options)

|A n B | = | A n C| = |B n C| = 1 (cc…cc, bb….bb, aa…aaa respectibely)

|A n B n C| = 0 (cannot make a string of length 12)

3*2^12 – 3

Or try complement rule

We count all invalid strings

A string is invalid if no chracters are missing (meaning the string contains at least 1 a, at leats 1 b and at least 1 c)

Let U be the set of strings of length 12 on a,b,c

Let X be the of set of invalid strings of length 12.

| U \ X | = |U| - |X| = the # of valid strings (by complement rule)

|U| = 3^12 (3 options for 12 positions)

|X| =

Task 1 : Choose 3 of the 12 positions -> (12 C 3) ways to do this

Task 2: Assign 1 a, 1 b and 1 c in the 3 positions -> 3! ways to do this

Task 3 : The remaining 9 positions have 3 options each -> 3^9 ways to fill them

By the product rule , there are (12 C 3) * 3! * 3^9 strings in X

3^12 – ((12 C 3) * 3! * 3^9) is the the number of valid strngs

3.37 A password consists of 100 characters, each character being a digit or a lowercase letter. A password must contain at least two digits. How many passwords are there?

Complement Rule :

Let U be the set of strings of length 100 consisting of lowercase letters or digits.

Let X be the subset of U consisting of strings that are not passwords.

X is the set of strings of U that contain at most 1 digit

|U| = 36^100 (each position has 36 options)

By the complement rule |U \ X | = |U| - |X| = # of strings in U that are passwords.

|X| = |X_0| + |X_1| where X_i is the set of strings in U with exactly i digits

|X_0| = each position has 26 options -> 26^100 such strings

|X_1| =

Task 1: Choose a position for the digit-> 100 ways to do this

Task 2 : Choose the digit to put in that position -> 10 ways

Task 3: Fill the remaining positions with letters -> 26^29 ways to do this

By the product rule 100* 10* 26^99 strings in X

Thus, there are 36^100 – (26^100 + 1000*26^99) passwords.

Or direct approach

Let X_k is the set of strings in U with exactly k digits

We want to compute | X_2 U X_3 U … U X_100|

Notice that the X_k are pairwise disjoint.

So | X_2 U X_3 U … U X_100| = Sum |X_k|

|X_k| =

Task 1 : Choose positions for k digits -> (100 C k)

Task 2 : Fill these k positions with digits -> 10^k

Task 3 : Fill the rest with letters -> 26^{100-k)

So |X_i| = (100 C k)(10^k)(26^{100-k}

And thus the number of passwords is Sum_{k=2 to 100} = (100 C k)(10^k)(26^{100-k} (this looks like something we have seen)

Why not do $(100*10)(99*10)(36^{98})$? is overcounting

1 1 _ _ _ _ _

Or $(100 \text{ C } 2)\ (10)*(10)*36^{98}$

1 1 1 1 a a ….. a

3.38 A password is a string of ten characters, where each character is a lowercase letter, a digit, or one of the eight special characters !, @, #, \$, %, &, (, and).

A password is called *awesome*, if it contains at least one digit or at least one special character. Determine the number of awesome passwords.

Let S = { a, …,z,0, 1, …, 9 , !, @, # , \$,% ,& , (,) }

A := the of strings of length 10 on S with at least 1 digit

B := the set of strings of length 10 on S with at least 1 special character

We want to find $|A \cup B|$

We could use inclusion-exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A|$

$|B|$

$|A \cap B|$

Or we can do the complement rule

We will compute $|\text{not}(A \cup B)| = |\text{not}(A) \cap \text{not}(B)|$

$|\text{not}(A) \cap \text{not}(B)|$ = # of strings of length 10 that contains no digits and no special character

Meaning each position has 26 options. 26^{10} such strings.

Let U be the set of all strings of length 10 on S

$|U| = 44^{10}$

Thus $|A \cup B| = |U \setminus (\text{not}(A \cup B)| = |U| - |\text{not}(A) \cap \text{not}(B)| = 44^{10} - 26^{10}$

3.42 Consider permutations of the 26 lowercase letters a, b, c, $\ldots$, z.

- How many such permutations contain the string *wine*?

- How many such permutations do not contain any of the strings *wine*, *vodka*, or *coke*?

a) We treat "wine" as 1 letter. Which mean we are permuting 26-4 + 1 letters in total. We can call "wine" as !. We are permuting {a,b,c,d,f,g,h,j,k,l,m,o,p,…,v,,x,y,z,!} 23! Permutations.

Let

A:= the set of permutations on the alphabet that contains the string wine

B:= the set of permutations on the alphabet that contains the string vodka

C:= the set of permutations on the alphabet that contains the string coke

We compute |A U B U C|

|A | + |B| + |C| - |A n B | - |A n C| - |B n C| + | A n B n C| (by inclusion-exclusion)

|A| = 23! In similar fashion we find that |B| = 22! and |C| = 23!.

|A n C| = 0 (since we cannot have wine and coke as a substring))

|B n C| = 0

|A n B| = 26-9 + 2 = 19! (vodka and wines are treated as blocks)

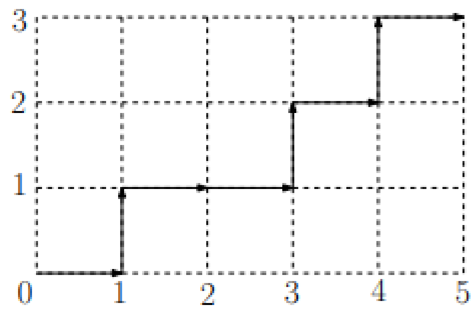
|A n B n C | = 0 (same reasoning)

|A U B U C | = 23! + 22! + 23! – 19!

We remove A U B U C to find the strngs that do not contain wine, vodka, or coke

So 26! – (23! + 22! + 23! – 19!)

3.71 Let $m \geq 1$ and $n \geq 1$ be integers. Consider a rectangle whose horizontal side has length m and whose vertical side has length n. A path from the bottom-left corner to the top-right corner is called *valid*, if in each step, it either goes one unit to the right or one unit upwards. In the example below, you see a valid path for the case when $m = 5$ and $n = 3$.



How many valid paths are there?

In total we need m +n steps (we need to move to the right exactly m times, and we need to move up exactly n times). So for each move we have two options : up or right (we can call them 0, 1 respectively)

So a valid path is string of 1 and 0 with exactly m 1 ones. In class we have seen that the number of such strings is (m+n C m) (or (m+n C n))

3.81 The square in the left figure below is divided into nine cells. In each cell, we write one of the numbers -1, 0, and 1.

0	1	0
1	1	-1
-1	0	-1

Use the Pigeonhole Principle to prove that, among the rows, columns, and main diagonals, there exist two that have the same sum. For example, in the right figure above, both main diagonals have sum 0. (Also, the two topmost rows both have sum 1, whereas the bottom row and the right column both have sum -2.)

Lets say we have p pigeons and n holes

Pigeons: 8 pigeons (the 3 rows, 3 columns and 2 diagonals)

Holes: the possible values of the sum of 3 integers picked from {-1,0,1} . This makes 7

By the P.H.P, there exists at least 1 hole containing at least ceiling (p/h)

Which means that among the diagonals, rows and columns of the square, two have the same sum.

3.82 Let S be a set consisting of 19 two-digit integers. Thus, each element of S belongs to the set $\{10, 11, \ldots, 99\}$.

Use the Pigeonhole Principle to prove that this set S contains two distinct elements x and y, such that the sum of the two digits of x is equal to the sum of the two digits of y.

Let S be such a set 19 two digits numbers

Pigeons: 19 different two digits numbers of S

Holes: the 18 different values of the sum that can be made by a two digit number {1,2,….,18}

By the P.H.P, there exists at least 1 hole containing at least ceiling (p/h) pigeons

Which mean there is at least 1 sun achieved by at least ceil (19/18)= 2 pigeons (two distinct elements of S) .

3.84 Let $n \geq 1$ be an integer. Use the Pigeonhole Principle to prove that in any set of $n+1$ integers from $\{1, 2, \ldots, 2n\}$, there are two elements that are consecutive (i.e., differ by one).

Let Se be a set of n+1 integers from {1,2,3 ,4 ,5 , 6,, .. . 2n}

Pigeons: the n+1 elements of S

Holes: {1,2} , {3,4} , {5,6} , … , {2n-1 ,2n} -> n holes

By the P.H.P, there exists at least 1 hole containing at least ceiling (p/h) pigeons

Which means that there is at least 1 hole containing 2 elements of S. Which means that these two elements are consecutive.

3.86 Let $S_1, S_2, \ldots, S_{50}$ be a sequence consisting of 50 subsets of the set $\{1, 2, \ldots, 55\}$. Assume that each of these 50 subsets consists of at least seven elements.

Use the Pigeonhole Principle to prove that there exist two distinct indices i and j, such that the largest element in S_i is equal to the largest element in S_j.

Pigeons : The 50 subsets S_1, … , S_50

Holes: We classify the subsets with respect to their largest element. This ranges from 7 to 55

55 – 7 +1 = 49

By the P.H.P, there exists at least 1 hole containing at least ceiling (50/49) =2 pigeons

There are two distinct subsets S_i and S_j whose largest element is equal (they were put in the same hole)

4.4 The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 1, \\ f(n) &= \tfrac{1}{2} \cdot 4^n \cdot f(n-1) \quad \text{if } n \geq 1. \end{aligned}$$

Prove that for every integer $n \geq 0$,

$$f(n) = 2^{n^2};$$

this reads as 2 to the power n^2.

Base Case:

n= 0

f(0) = 1 = 2^{0^2}

Induction Step:

Suppose that f(k) = 2^{k^2} for all integers k <= n, for some n > = 0 . (I.H.)

We want to show that f(n+1) = 2^{(n+1)^2} = (2^{n+1})^2

f(n+1) = ½ * 4^{n+1} * f(n) by recursive definition (and n >= 0)

= ½ * 4^{n+1} * 2^{n^2} by I.H.

= ½ * (2*2)^{n+1}) * 2^{n^2}

= ½ * (2^{n+1} * 2^{n+1}(* 2^{n^2}

= 2^{n} * 2^{n+1} * 2^{n^2}

= 2^{n^2 + 2n + 1}

= 2^{(n+1)^2}

4.15 The sequence a_n of numbers, for $n \geq 0$, is recursively defined as follows:

$$\begin{aligned} a_0 &= 5, \\ a_1 &= 3, \\ a_n &= 6 \cdot a_{n-1} - 9 \cdot a_{n-2} \quad \text{if } n \geq 2. \end{aligned}$$

- Determine a_n for $n = 0, 1, 2, 3, 4, 5$.
- Prove that for every integer $n \geq 0$,

$$a_n = (5 - 4n) \cdot 3^n.$$

Part a) a_0 =5 , a_1 = 3 , a_2 = -27, a_3= 6* -27 – 9*3 = -189 , a_4 = -891 , a_5 = -3645

Base Case:

n= 0 : (5-4*0) * 3^0= 5 = a_0

n= 1 : (5-4*1) * 3^1 = 3 = a_1

Induction Step:

Suppose that a_k = (5-4k)*3^k for all integers k <= n, for some n > = 1 . (I.H.)

We want to show that a_{n+1} = (5-4(n+1) * 3^{n+1}

a_(n+1) = 6* a_{n} – 9 * a_{n-1} by recursive definition (and n >= 1)

= 6*(5 – 4n)*3^n - 9 (5-4(n-1)) * 3^{n-1} by I.H.

= 2 (5-4n)* 3^{n+1} – (5-4(n-1)) * 3^{n+1}

= (10 – 8n – (5 -4n-4))* 3^{n+1}

= (5 -4n +4) * 3^{n+1}

= (5 - 4(n+1)) *3^{n+1}

4.39 Ever since he was a child, Nick has been dreaming to be like Spiderman. As you all know, Spiderman can climb up the outside of a building; if he is at a particular floor, then, in one step, he can move up several floors. Nick is not that advanced yet. In one step, Nick can move up either one floor or two floors.

Let $n \geq 1$ be an integer and consider a building with n floors, numbered $1, 2, \ldots, n$. (The first floor has number 1; this is not the ground floor.) Nick is standing in front of this building, at the ground level. There are different ways in which Nick can climb to the n-th floor. For example, here are three different ways for the case when $n = 5$:

1. move up 2 floors, move up 1 floor, move up 2 floors.
2. move up 1 floor, move up 2 floors, move up 2 floors.
3. move up 1 floor, move up 2 floors, move up 1 floor, move up 1 floor.

Let S_n be the number of different ways, in which Nick can climb to the n-th floor.

- Determine, S_1, S_2, S_3, and S_4.
- Determine the value of S_n, i.e., express S_n in terms of numbers that we have seen in this chapter.

Part a)

Part b)

This comes down to counting the number of ways to add to n using 2s and 1s (and order matters)

In other words, we are counting the number of sequences of 2s and 1s which adds up to n.

Such a sequence can start with either a 1 or a 2

If starts with 1: then the rest adds to n-1 and is also a sequence of 1s and 2s. There are S_{n-1} such sequences of 1 sand 2s

If started with a 2: then the rest is a sequence that adds up to n-2. There are S_{n-2} such sequences

S_n = S_{n-1} + S_{n-2} for n>=3

S_0 is not defined

S_1 = 1

S_2 = 2 (1+ 1 or 2)

$S_n = f_n$ for all $n >= 1$

Recall: $f_0 = f_1 = 1$

4.49 The set S of bitstrings is recursively defined in the following way:

- The string 00 is an element of the set S.
- The string 01 is an element of the set S.
- The string 10 is an element of the set S.
- If the string s is an element of the set S, then the string $0s$ (i.e., the string obtained by adding the bit 0 at the front of s) is also an element of the set S.
- If the string s is an element of the set S, then the string $10s$ (i.e., the string obtained by adding the bits 10 at the front of s) is also an element of the set S.

Let s be an arbitrary string in the set S. Prove that s does not contain the substring 11.

Let P(n) be the following statement:

All strings of length n in S do not contain the substring 11.

We want to show that P(n) is true for all n >=2

Base Case

n=2 : the elements of length 2 in S are 00 , 10 , 01. They don't contain 11

n=3 : the elements of length 3 in S are 000, , 010, , 001, They don't contain 11

Induction Step

Suppose that P(k) is true for all k <=n for some n > = 3 .

We want to show that P(n+1) is true. Which means all strings of length n+1 in S do not contain the substring 11.

Let s be a string of length n+1 in S. Since n+1 >=4 then we have either

s=0s' for some s' in S, where s' is a string of S of length n. By induction hypothesis, P(n) is true meaning that s' does not contain the substring 11

Therefore, s does not contain 11 either.

s=10s' for some s' in S where s' is a string of length n-1. By induction hypothesis, P(n-1) is true meaning that s' does not contain 11.

5.17 In Section 5.4.1, we have seen the different cards that are part of a standard deck of cards.

- You choose 2 cards uniformly at random from the 13 spades in a deck of 52 cards. Determine the probability that you choose an Ace and a King.
- You choose 2 cards uniformly at random from a deck of 52 cards. Determine the probability that you choose an Ace and a King.
- You choose 2 cards uniformly at random from a deck of 52 cards. Determine the probability that you choose an Ace and a King of the same suit.

a) Sample space S= { subsets of 2 cards out of 13 spades cards} , |S| = (13 C 2)
   1/(13 C 2). {A,K} = {K,A}

   Or
   Sample space S = ((c1,c2) , where c1 is not = to c2 and c1,c2 are {1,…,13} of spades }

|S| = 13*12

2/13*12

b) Sample space S = { subsets of 2 cards out of 52 cards} . |S| = (52 C 2)
   How many subsets of size 2 contains an A and a King:
   Task 1 : Pick an A -> 4 ways
   Task2 : Pick a King -> 4 ways
   16 such subsets
   So the probability is = 16/(52 C 2)

c) Sample space S = { subsets of 2 cards out of 52 cards} . |S| = (52 C 2)
   How many subsets of size 2 contains an A and a King of the same kind
   Task 1 : Pick an A -> 4 ways
   Task2 : Pick the King matching the suit of the A -> 1 way
   4 such subsets

So the probability is = 4/(52 C 2)

5.20 Let A be an event in some probability space $(S, \Pr)$. You are given that the events A and A are independent[2]. Determine $\Pr(A)$.

Pr(A) = Pr(A n A)= Pr(A)Pr(A) which implies Pr(A) = 1.

5.30 A standard deck of 52 cards contains 13 spades (♠), 13 hearts (♡), 13 clubs (♣), and 13 diamonds (♢). You choose a uniformly random card from this deck. Consider the events

$$\begin{aligned} A &= \text{"the chosen card is a clubs or a diamonds card"}, \\ B &= \text{"the chosen card is a clubs or a hearts card"}, \\ C &= \text{"the chosen card is a clubs or a spades card"}. \end{aligned}$$

- Are the events A, B, and C pairwise independent?
- Are the events A, B, and C mutually independent?

Pr(A) = 26/52 (Pr(clubs) + Pr(diamonds))

Similarly Pr(B = Pr(C)) = 26/52

Pr(A n B) = 13/52 (Pr(clubs))

IN the same manner

Pr(A n C) = Pr(B n C) = 13/52

Pr(A n B) = Pr(A) * Pr(B) is verified therefore A and B are independent

Similarly, A and C are not independent and B and C are independent.

We check whether Pr(A n B n C) = Pr(A)Pr(B)Pr(B)

Pr(A n B n C) = Pr(clubs) = 13/52

Therefore, the equation above is not verified, and A and B and C are not mutually independent.

5.36 You roll a fair die twice. Consider the events

$$A = \text{"the sum of the results is even"},$$
$$B = \text{"the sum of the results is at least 10"}.$$

Determine the conditional probability $\Pr(A \mid B)$.

Pr(A | B) = Pr(A n B) / Pr(B) =

Pr(A n B) = Pr(sum is = 10) + Pr(sum is = 12) = 3/36 + 1/36 = 4/36 = 1/9

Pr(B) = 1/6 in (from the assignment)

So Pr(A | B) = 2/3

5.39 Let A and B be two events in some probability space $(S, \Pr)$ such that $\Pr(A) = 2/5$ and $\Pr\left(\overline{A \cup B}\right) = 3/10$.

- Assume that A and B are disjoint. Determine $\Pr(B)$.
- Assume that A and B are independent. Determine $\Pr(B)$.

If A and B are disjoint Pr(A U B) = Pr(A) + Pr(B)

Pr(B) = Pr(A U B) – Pr(A)

By the complement rule Pr(AU B) = 1- Pr(not(AU B) = 7/10

So Pr(B) = 3/10

If A nd B are independent. Pr(A n B) = Pr(A) Pr(B)

Pr(A U B) = Pr(A) + Pr(B) – Pr(An B)

= Pr(A) + Pr(B) – Pr(A)Pr(B)

7/10 = 2/5 + Pr(B) – 2/5Pr(B) so 3/10 = (1-2/5)Pr(B)

Pr(B) = 15/30 = 1/2

5.41 Elisa and Nick go to Tan Tran's Darts Bar. When Elisa throws a dart, she hits the dartboard with probability p. When Nick throws a dart, he hits the dartboard with probability q. Here, p and q are real numbers with $0 < p < 1$ and $0 < q < 1$. Elisa and Nick throw one dart each, independently of each other. Consider the events

$$\begin{aligned} E &= \text{"Elisa's dart hits the dartboard"}, \\ N &= \text{"Nick's dart hits the dartboard"}. \end{aligned}$$

Determine $\Pr(E \mid E \cup N)$ and $\Pr(E \cap N \mid E \cup N)$.

Pr(E | E U N) = Pr(E n (EU N)) /Pr(E U N) by def. = Pr(E)/ Pr(E U N) since E is contained in EU N = p / (p + q – pq)

Pr(E U N) = p + q – pq

Pr(E) = p

Pr(E n N | E U N) = Pr((E n N) n (E U N)) / Pr(E U N) = Pr(E n N) / Pr(E U N) = pq /(p + q - pq)

5.48 In this exercise, we consider a standard deck of 52 cards.

- We choose, uniformly at random, one card from the deck. Consider the events

$$\begin{aligned} A &= \text{"the rank of the chosen card is Ace"}, \\ B &= \text{"the suit of the chosen card is diamonds"}. \end{aligned}$$

  Are the events A and B independent?

Pr(A n B) = Pr(A) * Pr(B)

Pr(A n B) = 1/52

Pr(A)= 4/52

Pr(B) = 13/ 52

And so the equality holds

5.90 We flip a fair coin repeatedly and independently, and stop as soon as we see one of the two sequences HTT and HHT. Let A be the event that the process stops because HTT is seen.

- Prove that the event A is given by the set

$$\{T^m(HT)^nHTT : m \geq 0, n \geq 0\}.$$

  In other words, event A holds if and only if the sequence of coin flips is equal to $T^m(HT)^nHTT$ for some $m \geq 0$ and $n \geq 0$.

- Prove that $\Pr(A) = 1/3$.

Assume that the sequence of flips = T^m (HT)^n HTT. So (trivially event A happened (there are no HTT in this sequence.

Now suppose that event A happened.

We know that the sequence looks as follows : ……… THTT

The remaining can be either all T (so does not contain any H) -> T^m HTTT for integer m and this sequence is in A

Or contains at least 1 H

Let k be the position of the first H (not included in the last 4 flips). The flip after this H needs to be a T . Otherwise we would have HHT before the first HHT was seen. This entails that starting at this first H (excluding the one at the end) The sequence alternates from H to T.

- ➔ The sequence is = T^k(HT)^jHTT for some k , j >= 0.

Pr(A) = Sum_{m=0 to infinity} Sum_{n= 0 to infinity} (1/2)^m(1/4)^n (1/8)

= 1/8 (Sum_{m=0 to infinity (½)^m Sum_{n=0 to infinity}(¼)^n)

= 1/ 8 (Sum_{m=0 to infinity (½)^m 4/3) (geometric series)

= 4/24 (Sum_{m=0 to infinity} (½)^m)

= 4/24 * 2 (geometric series)

= 1/3

6.4 Lindsay and Simon want to play a game in which the expected amount of money that each of them wins is equal to zero. After having chosen a number x, the game is played as follows: Lindsay rolls a fair die, independently, three times.

- If none of the three rolls results in 6, then Lindsay pays one dollar to Simon.
- If exactly one of the rolls results in 6, then Simon pays one dollar to Lindsay.
- If exactly two rolls result in 6, then Simon pays two dollars to Lindsay.
- If all three rolls result in 6, then Simon pays x dollars to Lindsay.

Determine the value of x.

X is the amount of money that Lindsay wins

We want E(X) = 0

Range of X = {-1, 1,2,x}

Pr(X = -1) = (5/6)^3

Pr(X = 1) = (3 C 1) (1/6)(5/6)^2

Pr(X = 2) = (3 C 2) (1/6)^2(5/6)

Pr(X = x) = (1/6)^3

So E(X) = -1 * (5/6)^3 + 1*(3 C 1) (1/6)(5/6)^2 + 2* (3 C 2) (1/6)^2(5/6) + x* (1/6)^3

We want E(X) = 0

So solve for x= 20

6.13 Consider the sample space $S = \{1, 2, 3 \ldots, 10\}$. We choose a uniformly random element x in S. Consider the following random variables:

$$X = \begin{cases} 0 & \text{if } x \in \{1, 2\}, \\ 1 & \text{if } x \in \{3, 4, 5, 6\}, \\ 2 & \text{if } x \in \{7, 8, 9, 10\} \end{cases}$$

and

$$Y = \begin{cases} 0 & \text{if } x \text{ is even}, \\ 1 & \text{if } x \text{ is odd}. \end{cases}$$

Are X and Y independent random variables?

X and Y are independent random variables iff for all x in the range of X and for all y in the range of Y

Pr(X =x n Y=y) = Pr(X = x) Pr(Y=y)

Range of X = { 0,1,2}

Range of Y = { 0 , 1}]

Pr(X = 0 n Y = 0) = 1/10

Pr(X = 0) = 2/10 and Pr(Y= 0) = 5/10. So the equality is verified

Pr(X= 0 n Y = 1) = 1/10

Pr(X= 0) = 2/10 and Pr(Y = 1) = 5/10,

Pr(X = 1 n Y = 0) = 2/10

Pr(X= 1) = 4/10 and Pr(Y =0) = 5/10

You can check that all equalities are verified

6.17 You are given two independent random variables X and Y, where

$$\Pr(X = 1) = \Pr(X = -1) = \Pr(Y = 1) = \Pr(Y = -1) = 1/2.$$

Consider the random variable

$$Z = X \cdot Y.$$

Are X and Z independent random variables?

The range of Z is {-1,1}

Pr(Z=-1) = Pr(Y=-1 n X=1) or Pr((Y=1 n X= -1) = 2/4 = ½

Pr(Z = 1) = 1/2

Pr (Z = -1 n X = -1) = Pr(Y =1 n X =-1) = ¼

You check the remaining equalities

(They are independent random variables)

6.19 You roll a fair die five times, where all rolls are independent of each other. Consider the random variable

$$X = \text{ the largest value in these five rolls.}$$

Prove that the expected value $\mathbb{E}(X)$ of the random variable X is equal to

$$\mathbb{E}(X) = \frac{14077}{2592}.$$

Hint: What are the possible value for X? What is $\Pr(X = k)$?

E(X) = Sum_ { x in the range of X} x Pr(X =x)

Range of X is {1, …, 6}

Pr(X = k) for 1 <= k <= 6 ?

Sample space = {(d1,…,d5) : each dj is in {1,…,6}}

|S| = 6^5

We break Pr(X=k) with respect to the number of rolls that are equal k. Let m be that number of rolls

Task 1 : Choose m of the 5 rolls to set it as the max - > (5 C m)

Task 2 : Set those chosen rolls as k -> 1 way

Task 3 : For the remaining 5-m rolls , k-1 values -> (k-1)^({5-m}

And so Pr(X= k) = Sum_{m=1 to 5} (5 C m) {k-1}^{5-m} = Sum_{m=1 to 5} (5 C m) {k-1}^{5-m} * 1^m

= Sum_{m=0 to 5} (5 C m) {k-1}^{5-m} * 1^m – (k-1)^5 = k^5 – (k-1)^5

Largest element = k is equivalent that at least 1 roll is equal to k and all rolls are <= k

Let X be the set of rolls where all rolls are <= k (we treat X as the universal set in this case)

|X| = k^5

Let Y be the set of elements of X with at least 1 roll = k

Not(Y) the of elements of X where no rolls = k

|Y| = |X| - |not(Y)|= k^5 – (k-1)^5

So Pr(X= k) = k^5 - (k-1)^5 / 6^5

E(X) = Sum_{k=1 to 6} k Pr(X=k) = Sum_{k=1 to 6} (k^6-k(k-1)^5)/6^5 =

Theorem 3.6.5 (Newton's Binomial Theorem) *For any integer $n \geq 0$, we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

6.41 Let $n \geq 1$ be an integer and consider a uniformly random permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. Define the random variable X to be the number of indices i for which $1 \leq i < n$ and $a_i < a_{i+1}$.

Determine the expected value $\mathbb{E}(X)$ of X.

We use indicator variables

For 1 < = i < n

X_i = 1 if a_i < a_{i+1} and 0 otherwise

Note that X = X_1 + X_2 + … + X_{n-1} (both sides are counting the same thing)

E(X) = Sum_{i=1 to n-1} (E(X_i)) by linearity of expected value

E(X_j) = 0 Pr(X_i = 0) + 1 Pr(X_i = 1)

Pr(X_i =1) = ½ (think about this one)

Thus E(X) = Sum_{i=1 to n-1} ½ = (n-1) /2