COMP 2804 Assignment 5

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Contents

Question 1 [10 Marks]

Consider two random variables X and Y . If X and Y are independent, then it can be shown that $E(XY) = E(X) \cdot E(Y)$. In this exercise, you will show that the converse of this statement is, in general, not true. Let X be the random variable that takes each of the values -1, 0, and 1 with probability 1/3. Let Y be the random variable with value $Y = X^2$.

- Prove that X and Y are not independent.
- Prove that $E(XY) = E(X) \cdot E(Y)$.

First to prove X and Y are not independent we can use Definition 6.2.1 for independent random variables:

$$Pr(X = x \cap Y = y) = Pr(X = x) \cdot Pr(Y = y)$$

Using our given numbers and subbing in 0 for X and 1 for Y we see:

$$Pr(X = 0 \cap Y = 1) = 0$$
$$Pr(X = 0) \cdot Pr(Y = 1) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

These two things are clearly not equal so therefore X and Y are not independent. To prove $E(XY) = E(X) \cdot E(Y)$ we can look at the definition of expected value: $E(X) = \sum_i x_i \cdot Pr(X = x_i)$, from this:

$$E(XY) = \sum_i \sum_j x_i y_j Pr(x_i y_j)$$
$$E(XY) = \sum_i \sum_j x_i y_j Pr(x_i) Pr(y_j)$$
$$E(XY) = \sum_i x_i Pr(x_i) \cdot \sum_j y_j Pr(y_j)$$
$$E(XY) = E(X) \cdot E(Y)$$

Specifically in our case:

$$E(XY) = (-1)(-1^2) \cdot \frac{1}{3} + (1)(1^2) \cdot \frac{1}{3} + (0)(0) \cdot \frac{1}{3}$$
$$E(XY) = 0$$

$$E(X) \cdot E(Y) = ((-1) \cdot \frac{1}{3} + (1) \cdot \frac{1}{3} + (0) \cdot \frac{1}{3}) \cdot ((-1)^2 \cdot \frac{1}{3} + (1)^2 \cdot \frac{1}{3} + (0) \cdot \frac{1}{3})$$
$$E(X) \cdot E(Y) = 0 \cdot \frac{1}{9}$$
$$E(X) \cdot E(Y) = 0$$

$\therefore E(XY) = E(X) \cdot E(Y)$

Question 3 [10 Marks]

The Ottawa Senators and the Toronto Maple Leafs play a best-of-seven series: These two hockey teams play games against each other, and the first team to win four games wins the series. Assume that

- each game has a winner (thus, no game ends in a tie),
- in any game, the Sens have a probability of 3/4 of defeating the Leafs,
- the results of the games are mutually independent.

Determine the probability that seven games are played in this series.

In order for there to be 7 games played, there needs to be 6 games played and each team needs to win exactly k = 3 games. Let the random variable X be the number of games won. With p = 3/4 being the probability the Sens won the game and n = 6 being the total number of games played, we can use the Binomial Distribution to determine the probability X = 3 as the results of the games are mutually independent. Using the following equation:

$$Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

we get a probability of:

$$Pr(X = 3) = \binom{6}{3} \frac{3}{4}^3 (1 - \frac{3}{4})^{6-3}$$

$$Pr(X = 3) = 20(\frac{27}{64})(\frac{1}{64})$$

$$Pr(X = 3) = \frac{135}{1024}$$

$$or:$$

$$Pr(X = 3) = 0.131836$$

Question 4 [10 Marks]

Prove that, for any graph G with m edges, the sequence $X_1, X_2, ..., X_m$ of random variables in the proof of Theorem 7.1.1 is pairwise independent. Give an example of a graph for which this sequence is not mutually independent.

Question 5 [10 Marks]

Prove that Theorem 7.5.4 also holds if G is not connected.

We can look at a connected graph G_1 with the vertices V = A, B, C, D, E, F, G, H and the edges E = AB, BC, CD, DA, BE, CH, EF, FG, GH, HE (basically 2 sqaures connected with the edges BE and CH). Using this graph G_1, we can see that there are a total of 8 vertices, 10 edges and 3 faces.

This satisfies both the conditions of a planar graph with $v \geq 3$ vertices, $10 \leq 3(8) - 6 = 18$ edges and $3 \leq 2(8) - 4 = 12$ faces.
Now if we remove the connecting edges BE and CH to make this graph G_1 a graph G_2 that is not connected, we can see if this still holds.

$$8 \leq 3(8) - 6 = 18 \ edges$$
$$and$$
$$2 \leq 2(8) - 4 = 12 \ faces$$

It is clear that G_2 also satisfies Theorem 7.5.4
Because these conditions simply state that G has at most 3v - 6 edges and 2v - 4 faces, it is clear that this will always hold if G is not connected as creating an unconnected graph from a connected graph will always result in less edges and faces. $\therefore$ Theorem 7.5.4 also holds if G is not connected

Question 6 [10 Marks]

Let K_5 be the complete graph on 5 vertices. In this graph, each pair of vertices is connected by an edge. Prove that K_5 is not planar.

To prove K_5 is not planar, we can use a proof by contradiction and attempt to prove that it is planar. By Euler's formula, Theorem 7.5.3, if v = the number of vertices, e = the number of edges, and f = the number of faces, then,

$$v - e + f = 2$$

In addition, by Theorem 7.5.4, any planar graph with ≥ 3 verticies has at most $3v - 6$ edges and $2v - 4$ faces. Looking at K_5, there is 5 vertices by definition, and therefore $\binom{5}{2} = 10$ edges. $\therefore$

$$\begin{aligned} 5 - 10 + f &= 2 \\ f &= 2 + 5 \\ f &= 7 \end{aligned}$$

Given that K_5 fails the condition regarding edges as $10 \leq 3(5) - 6 = 10$ is not true, and fails the condition regarding faces as $7 \leq 2(5) - 4 = 6$ is also not true. We can confidently say K_5 is not a planar graph.

Question 7 [10 Marks]

Suppose we have a graph $G = (V, E) with m$ edges. Prove that there exists a partition of V into three subsets A, B, C such that there are $\frac{2m}{3}$ edges between these subsets (i.e. between A and B, between B and C, or between A and C).

To prove, we can use a similar process as described in Theorem 7.1.1. Given that there are now three subsets, A, B, and C, we cannot use coin flips to decide what subset the vertices from the graph go into. However, we can use the same random process. First initializing subsets to the empty set, i.e.: $A = \emptyset, B = \emptyset, and\ C = \emptyset$. We then place the vertices a and b in these sets at random with uniform probability 1/3. For each edge $e \in E$ define

$$X_e = \begin{cases} 1 & if\ e\ has\ two\ endpoints\ in\ different\ sets \\ 0 & otherwise \end{cases}$$

For there to be an edge in between these subsets, two of the endpoints must be in different sets, the probability of this happening is just the probability that that two endpoints were chosen to be in different sets (chosen at random with uniform probability 1/3). This probability can be determined by creating a table with each of the 9 possibilities for where a and b can be placed. We then ignore the possibilities where a and b are in the same set, this happens 3 out of the 9 times. Therefore a and b are in different sets 6 out of nine times Now, looking at the expected number of edges that cross either (A,B), (B,C), or (A,C) is:

$$E(\sum_{e \in E} X_e) = \sum_{e \in E}^{m} E(X_e)$$

$$= \sum_{e \in E}^{m} Pr(X_e = 1)$$

$$= \sum_{e \in E}^{m} \frac{6}{9}$$

$$= m \cdot \frac{2}{3}$$

$$= \frac{2m}{3}$$

Question 8 [10 Marks]

Suppose we randomly draw two integers from the range [1, n] with uniform probability. Define X to be the value of the first integer drawn; define Y to be the value of the second integer drawn. Define $Z = |X - Y|$. Compute E(Z).

Using definition 6.4.1, we can determine the expected value of Z as it is defined to be

$$E(Z) = \sum_{i=1}^{n} Z \cdot Pr(Z = z_i)$$

Using this equation, $Z = |X - Y|$ and all that is left is to determine $Pr(z_i)$. To do this we can examine cases when z_i is small and attempt to find a pattern. For example, assuming n = 5, we can create a distribution matrix and then determine the distribution functions.

Therefore the distribution functions D(X) = $Pr(Z = z_i$ can be seen as:

$$\begin{aligned}
D(1) = Pr(1) = \frac{8}{20} \\
D(2) = Pr(2) = \frac{6}{20} \\
D(3) = Pr(3) = \frac{4}{20} \\
D(4) = Pr(4) = \frac{2}{20}
\end{aligned}$$

If this was not enough to determine a pattern we can increase the size of n and repeat the process. Using this process we can see that numerator for the probability is always 2(n - z) and the denominator is always n^2 as that is the number of choices X and Y. Therefore, now that we have this probability function we can now determine the expected value of Z:

$$\begin{aligned}
E(Z) &= \sum_{Z=1}^{n-1} Z \cdot Pr(Z = Z) \\
E(Z) &= \sum_{Z=1}^{n-1} Z \cdot \frac{2(n - z)}{n^2} \\
E(Z) &= \frac{2}{n^2} \sum_{Z=1}^{n-1} Z \cdot (n - Z) \\
E(Z) &= \frac{2}{n^2} \sum_{Z=1}^{n-1} (Zn - Z^2) \\
E(Z) &= \frac{2}{n^2} \sum_{Z=1}^{n-1} Z - \frac{2}{n^2} \sum_{Z=1}^{n-1} Z^2
\end{aligned}$$

$$E(Z) = \frac{2}{n^2} \cdot \frac{n(n-1)(2n-1)}{6} - \frac{2}{n^2} \cdot \frac{n^2 - n}{2}$$

$$E(Z) = \frac{2}{n^2} \cdot (\frac{n(n-1)(2n-1)}{6} - \frac{n^2 - n}{2})$$