Discrete Mathematics Notes

Contributor: Tanya Garg
KMV CS(H)

Computer Science Notes

→ Operations on sets.

1. $A \cup B$.

$A_1 \cup A_2 \cup \ldots\ldots A_n$

$= \bigcup_{i=1}^{n} A_i$

set $A = \{1, 2, 3\}$

set $B = \{2, 3, 4\}$

$\Rightarrow$ $A \cup B = \{1, 2, 3, 4\}$.

2. $A \cap B$

A And B both.

$A_1 \cap A_2 \ldots\ldots \cap A_n$

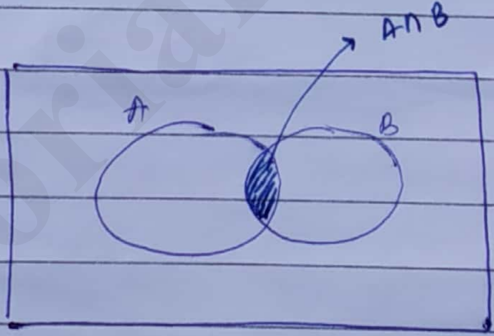
$= \bigcap_{i=1}^{n} A_i$

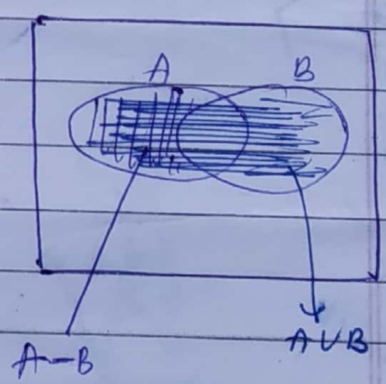
$\Rightarrow$ $A \cap B = \{2, 3\}$

3. Different of Set.

$A - B$, $B - A$.

$= \{1\}$, $= \{4\}$





- $A \oplus B = (A-B) \cup (B-A)$
  $= (A \cup B) - (A \cap B)$

- $A' = U - A$
- $A \cup \phi = A$
- $A \cup B = B \cup A$ (Commutative law)
- $A \cap B = B \cap A$ (,,)

- $A \cup A = A$
- A is subset of B, if & only if $A \cup B = B$

→ • $A \cap \phi = \phi$
- $A \cap A = A$
- $A \subseteq B$, if and only if $A \cap B = A$.

⇒ Commutative $A \cap B = B \cap A$

Associative

Distributive.

$A \cap (B \cup C) = \cancel{(A \cap B) \cup C}$
$= (A \cap B) \cup (A \cap C)$

$(A')' = A$

$(A \cup A') = U$

Demorgan's law

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

→ $A \cup U = U$

→ $A \cap U = A$

⇒ absorption law.

1. $A \cap (A \cup B) = A$.
2. $A \cup (A \cap B) = A$
3. $A \cap (A' \cup B) = A \cap B$
4. $A \cup (A' \cap B) = A \cup B$.

→ if ~~A =~~ we have to Prove $A = B$,
first we have to prove
$A \subseteq B$ & $B \subseteq A$.

→ Proving Distributive law.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof → Let $x \in A \cap (B \cup C)$
⇒ $x \in A$ & $x \in (B \cup C)$
⇒ $x \in A$ & $x \in B$ or $x \in C$
⇒ either $x \in A$ & B or $x \in A$ & C.
⇒ $x \in A \cap B$ or $x \in A \cap C$
⇒ ~~$x \in (A \cup B) \cap (A \cup C)$~~
⇒ $x \in (A \cap B) \cup (A \cap C)$
⇒ L.H.S $\subseteq$ R.H.S

Q→ Prove that A is a proper subset of B if and only if B' is a proper subset of A'.

To Prove → $A \subset B$.

Given → $B' \subset A'$

Proof →

Let

If $A \subset B$ iff $x \in A \Rightarrow x \in B$

iff $x \notin B \Rightarrow x \notin A$

iff $x \in B' \Rightarrow x \in A'$

$\Rightarrow$ iff $B' \subset A'$

Q→ Prove that $A \subseteq B$, where A & B are defined as ~~{x is a prime no.~~

$A = \{ x \mid x \text{ is a prime no } \& \ 42 \le x \le 51 \}$

$B = \{ x \mid x = 4k + 3 \text{ and } k \in N \}$

Solⁿ

Let $x \in N$

x can either be 43 or 47.

$43 = (4 \times 10) + 3$

$47 = 4(11) + 3$

Prove that $A \not\subseteq B$ & $B \not\subseteq A$ where

$A = \{x \mid x = 3k + 1\}$

$B = \{ x \mid x = 4k + 1 \}$ $\Big\} \rightarrow k \in N$

$\Rightarrow A \not\subseteq B, B \not\subseteq A.$

Prove by contradiction,

Suppose

$A \subseteq B$ & $B \subseteq A$.

$A = \{4, 7, 10, 13 \ldots\}$

$B = \{5, 9, 13, \ldots\}$

Hence our assumption is wrong

&

$A \not\subseteq B$ & $B \not\subseteq A$.

Q → Prove that $A = B$, where $A = \{x \mid x$ is a prime no & $12 \le x \le 18\}$

$B = \{x \mid x = 4k+1$ & $k \in \{3, 4\}\}$.

Let $x \in A$.

$\Rightarrow$ x can take form.

13, 17

$\Rightarrow 13 = 4(3) + 1$

$17 = 4 \times (4) + 1$

$\Rightarrow$

$x \in B$. as well

$\Rightarrow A \subseteq B \rightarrow (1)$

From (1) & (2)

$\Rightarrow A = B$

Let $x \in B$

$\Rightarrow x$ can take $\rightarrow 13, 17$

$\Rightarrow x \in A$ as well

$\Rightarrow B \subseteq A \rightarrow (2)$

$A \subseteq B$

$B = \{1,2,3\}$

⇒ Cartesian product of Set.

$A \times B \neq B \times A$ {∵ ordered does matter}

⇒ ~~no~~, commutative law does not hold & associative as well.

$A = \{1, 2\}$
$B = \{3, 4\}$

$(A \times B) = \{(1,3), (1,4), (2,3), (2,4)\}$

$(B \times A) = \{(3,1), (3,2), (4,1), (4,2)\}$

Set

⇒ finite set
~~no of~~ set have a finite cardinality
numerable set
↓
1:1 N cardinality

Infinite Sets

1) Countable
denumerable set
→ approaching to infinity
1:1 N
Ex Set of Integers

2) Uncountable
nondenumerable
→ ∞, list is non exhaustive
Ex → Real nos b/w 0 & 1.

5 Marks

→ Cantor's Diagonalization Method.

To prove : Real nos. b/w 0 & 1 are uncountable

Proof →

Assume that Real nos. b/w 0 & 1 are countable.

$d_1 = 0.a_1 a_2 a_3 \ldots\ldots$

$d_2 = 0.b_1 b_2 b_3 \ldots\ldots$

$d_3 = 0.c_1 c_2 c_3 c_4 \ldots\ldots$

⋮

$d_n = 0.\ldots\ldots\ldots$

Assume that the set of all decimals is countable then we could form the following list containing all such decimals.

Since each of our infinite decimals must appeared somewhere on this list. now contradiction can be constructed as follows.

Construct a no. x

$x = 0.x_1 x_2 x_3 \ldots$

such that,

$x_1 = 1$ if $a_1 = 2$ else $x_1 = 2$.

$x_2 = 1$ if $b_2 = 2$, else $x_2 = 2$

& so on.

$\frac{0.1}{0.2}$ 0.1

0.11
0.10
0.12

this process can be continued indefinitely, hence we can generate any no. by varying the digits on the diagonals of the list.

The resulting no. in that infinite decimal consisting of 1's & 2's ~~by~~ but by its construction x differs from each no. in the list at some position.

Thus, x is not on the list. which is a contradiction to our assumption.

5 marks

⇒ The principle of Inclusion / Exclusion.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

- If we have two disjoint sets, then

$$n(A \cup B) = n(A) + n(B)$$

$$\because n(A \cap B) = \cancel{\phi}\ 0$$

$$\Rightarrow n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Q→ A company wants to hire ~~d~~ 25 for task 1 & 40 for Task 2, out of which 10 are supposed to perform both the task. How many programers are there.

Q→ A survey of 500 T.V watchers produced the following information.

285 watch football.

195 hockey

115 basketball

45 football & basket

70 foot & hockey

50 hockey & basket

& 50 do not ~~not~~ watch any of three kind of games

① How many people watch all 3 kind of games

$n(F \cap H) \cap b)$

② How many people watch exactly one of the sport

$500 - 50$
$= 450$

$$\begin{array}{r} 285 \\ 195 \\ 115 \\ \hline 595 \end{array}$$

⇒ ① $285 + 195 + 115 - 45 - 70 - 50$

→ 20

②

Cardinality of power set → 2^n

Date : / /

Page No. 10

Only football $285 - 70 - 45 + 20 = 190$

Only hockey = 95

Only basketball = 40

⇒ 325

⇒ $\{a, b\} \oplus \{a, b\} \rightarrow \phi$

⇒ $\{a, b\} \oplus \{a, c\} = \{b, c\}$

⇒ $\{a, b\} \oplus \phi = \{a, b\}$

⇒ $\phi \oplus \{\phi\} = \{\phi\}$

⇒ Prove demorgan's law. $(A \cup B)' = A' \cap B'$

$x \in (A \cup B)^c$ ⇒ $x \in A^c \cap B^c$

$(A \cup B)^c \subset A^c \cap B^c$ ⇒ $A^c \cap B^c \subset (A \cup B)^c$

⇒ fundamental Product

$P_1\ P_2 \ldots P_n$

n sets → 2^n fundamental Product

S_1 = my S.P. are

⇒ 21 — cricket Players.
6 — One day
7 — 20-20 match
5 → both

How many players are not taking any part in any game

$21 - (8) = 13$ $(F \cup G \cup S)'$ 20

Suppose 100 out of 120 C.S. students study French, German, Sanskrit.
(65) (45) (42)

20 $(F \cap G)$, 25 $(F \cap S)$, 15 $(G \cap S)$

① Only F & G but not Sanskrit
$F \cap G \cap S'$ → 12

② $F \cap S \cap G'$ → 17

③ $G \cap S \cap F'$ → 7

④ Only French → 28

⑤ Only German → 18

⑥ Only Sanskrit → 10

⑦ none of the three languages → 20

Q→ Determine the no. of integers b/w 1 & 250 which are divisible by any of the integers 2, 3, 5 & 7.

2 →

$$a_n = a + (n-1)d$$

$$250 = 2 + (n-1)2$$

$$\frac{248}{2} = (n-1)$$

$$124 = n-1$$

$$\boxed{125 = n}$$

3 →

$$249 = 3 + (n-1)3$$

$$246 = (n-1)3$$

$$82 = n-1$$

$$\boxed{83 = n}$$

250
24 5
1

5 →

$$250 = 5 + (n-1)5$$

$$\frac{245}{5} = (n-1)$$

$$49 + 1 = n$$

$$50 = n$$

7 →

1 - ~~(244)~~ -

I -

5 marks

Principles of Mathematical Induction

Let P be any ~~any~~ ~~P(n)~~ proposition defined by P(n) is either true / false then As per the P.M.I

1) Bases of Induction.
The statement is true for $n = n_0$ and
2) Induction. Step
The statement is true for $n = k+1$ assuming that the statement is true for $n = k$, such that $k \geq n_0$.

For PMI $\rightarrow$ $n = k$
For strong P.M.I $\rightarrow$ $n_0 \leq n \leq k$

Q→ Show by Mathematical Induction
$$1+2+3+ \cdots + n = \frac{n(n+1)}{2} \quad \forall n \geq 1$$

Sol^n 1) Basis step.
$n_0 = 1$
for $n = n_0$ i.e. $n = 1$
L.H.S = 1 , R.H.S = $\frac{1(1+1)}{2} = 1$
hence,
L.H.S = R.H.S.
$P(n_0 = 1)$ holds true.

2) Inductive step.
Assume that given P is true for some $n = k$, then.
$$1+2+3 \cdots + k = \frac{k(k+1)}{2}$$
$\hookrightarrow$ (1)

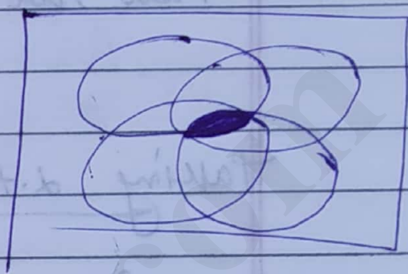
Now, we have to prove, for $n = K+1$.

Taking L.H.S,

$1+2+3+ \ldots\ldots + K + (K+1)$

↓ from (1)

$$\frac{K(K+1)}{2} + (K+1)$$



$$= \frac{K(K+1) + 2(K+1)}{2}$$

$$= \frac{(K+1)(K+2)}{2}$$

which is equal to the value for, $n = K+1$

Hence Proved

Q→ Show that $1^2 + 2^2 + 3^2 \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall n \geq 1$

(1) Basis step.

$n_0 = 1$

for $n = n_0$, i.e. $n = 1$

L.H.S $= 1$, R.H.S $= \frac{1(1+1)(2+1)}{6}$

$$= \frac{2 \times 3}{6} = 1$$

Hence L.H.S = R.H.S

$P(n_0 = 1)$ holds true

(2) Inductive step.

Assume that given P is true for some $n = K$, then

$$1^2 + 2^2 + 3^2 - \cdots + k^2 = \frac{k(2k+1)(k+1)}{6}$$

Now, we have to prove for $n = k+1$
ie $P(n = k+1) =$

Taking L.H.S,

$$\underbrace{1^2 + 2^2 + 3^2 + \cdots + k^2}\ + (k+1)^2$$

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$\frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6}$$

$$\frac{(k+1)\left((2k+1)k + 6(k+1)\right)}{6}$$

$$\frac{(k+1)(2k^2 + k + 6k + 1)}{6}$$

$$\Rightarrow (k+1)\left[\frac{(2k+3)(k+2)}{6}\right]$$

$(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

Q. Let $a_1, a_2, a_3 \dots a_n$ be any n sets, Show by P.M.I that,

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}$$

$i=1$

⇒ Base case :

for $i=1$

$\overline{A_1} = \overline{A_1}$ True.

→ assume

$$\left(\bigcup_{k=1}^{k} A_i\right)^c = \bigcap_{i=1}^{k} A_i^c$$

Now we have to prove,

P(K+1)

L.H.S

$$\left(\bigcup_{i=1}^{k+1} A_i\right)^c = \left(A_1 \cup A_2 \cup A_3 \cup A_4 \dots \cup A_k \cup A_{k+1}\right)^c$$

= using associative

$$= \left(\left(A_1 \cup A_2 \dots \cup A_k\right) \cup A_{k+1}\right)^c$$

$$= \left(A_1 \cup A_2 \dots \cup A_k\right)^c \cap \left(A_{k+1}\right)^c$$

$$= \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}}$$

$$= \bigcap_{i=1}^{k+1} \overline{A_i}$$

Q → Prove that,

$\forall n \geq 1$, $n! \geq 2^{n-1}$

Sol^n $i = 1$

for $i = 1$

L.H.S $= 1! = 1$ R.H.S $= 2^{k-1}$

$= 2^0$

$= 1$

L.H.S = R.H.S

$\boxed{P(k+1)\ (k+1)! \geq 2^k}$

$2^{k+\cancel{1}-\cancel{1}}$

For $n = k$,

$k! = 2^{k-1} \rightarrow (1)$

for $n = k+1$

2^k

$(k+1)! = (k+1)(k)!$

$\geq (k+1)\, 2^{(k-1)}$ from (1) 4!

3! < 4.

$= (k+1)\, 2^{k-1}$ $=$ $\cancel{2^k (k+1)}$

$= \frac{2^k}{2}(k+1)$ neglect $= \cancel{2^{(k-1)}}$ $\cancel{\geq 2^{k-1}}$

$\Rightarrow 2^k = 2 \cdot 2^{k-1}$

$\cancel{\leq 2 \cdot k!}$ $= \cancel{2}\frac{2^k}{\cancel{2}}$

$\boxed{(k+1)! \geq 2^k}$

Hence proved

Homework Wednesday

1. $1+2^n < 3^n \quad \forall\ n \geq 2.$

2. $n < 2^n \quad \forall\ n > 1$

5. $n! > 2^n \quad \forall\ n \geq 4$

3. $1 + a + a^2 + \cdots + a^{n-1} = \frac{a^{n-1}}{a-1}$

4. $1 + 3 + 5 + \cdots + (2n-1) = n^2 \quad \forall\ n \geq 1$

Strong P.M.I

⇒ Q→ Prove that every natural no. $n \geq 2$ is either a prime no. or product of primes.

⇒ $n_0 \quad p \quad q \quad n \quad n+1$

nos in this domain are either prime or product of prime number.

$n+1$

prime ✓ | product of primes

$(p, q) \rightarrow$ always lie b/w nos n.

Q Prove that $3^{2n} - 1$ is divisible by 8. $\forall n \geq 1$

→ For $n = 1$

L.H.S

$3^{2\times1} - 1 = 9 - 1$

$= 8 = 8 = R.H.S$

assume

→ $3^{2k} - 1 = 8 \times m$

$\Rightarrow 3^{2k} = 8m + 1$

$n = k + 1$

$3^{2k+2} - 1$

$= 3^{2k} \cdot 3^2 - 1$

$\cancel{9(3^{2k} - 1)}$

$= (8m + 1) 9 - 1$

$= 72m + 9 - 1$

$= 72m + 8$

$= 8(9m + 1) = 8q$ (q any int)

Q→ Show that $2^n > n^3 \ \forall \ n \geq 10$

$2^{10} > 10^3$ (True)

$1024 > 1000$ ✓

assume,
for $n = K$
$$2^K > K^3$$

$$2^{K+1} > (K+1)^3$$

for, $n = K+1$,

imp in Exam

Well Ordering Principle

Any set of natural nos. will have a least element.

statement
→ Every non empty subset of the natural nos. has a least element.

Let A be a non empty subset of n
we wish to show that A has a least element that is there is an elemement, $a \in A$
$$a \in A \text{ s.t. } a \leq A \ \forall\ n \in A.$$

We will do this by strong Induction, on the following predicate.

$P(n)$: "If $n \in A$ then A has a least element".

Basic step : $P(0)$ is clearly true, since $0 \leq n \ \forall n \in N$

Inductive step: We want to show that
$$[P(0) \wedge P(1) \wedge P(2) \ldots\ldots \wedge P(n) \wedge P(n+1)]$$

To end this, Suppose that, $P(0), P(1) \ldots P(n)$ are all true and that, ~~A≠t~~ $n+1 \in A$,

we have two cases.

case I: There does not exist m s.t. ($m \in A$ s.t. $m < n+1$)

In this case, $n+1$ is the least element of A.

case II: There exist some m. s.t. ($m \in A$ s.t. $m < n+1$)

& either way $P(n+1)$ is true

So, by strong P.M.I we obtain that $P(n)$ is true $\forall\, n \in N$.

Since, A is not empty, we can pick an $n \in A$ moreover, Since $P(n)$ is true

$\Rightarrow$ A has a least element.

→ RELATION

In result of relations, ~~ist~~ we get a~~n~~ set of ordered pairs.

Set A & B.

(a,b) where $a \in A$ & $b \in B$

ordered pair

and $a < b$ → relation of less than

Elements in relation $\subseteq$ ordered pairs in $A \times B$.

⇒ Identity relation

$A \times A$

$\Delta_D = R = \{(a,a) : a \in A\}$

also called equality and diagonal relation

eg → $A = \{1, 2, 3, 4\}$

$B = \{a, b, c, d\}$

$R = \{(1,a), (2,b), (3,a), (4,d), (4,a), (4,b)\}$

Domain = 1, 2, 3, 4

Range = a, b, d.

$R^{-1} = \{(a,1), (b,2), (a,3), (d,4), (a,4), (b,4)\}$

~~Types to represent~~ Relations

(1) Tabular form

	a	b	d
1	✓		
2		✓	
3	✓		
4	✓	✓	✓

a b d → Range

Domain { 1, 2, 3, 4

→ holds a relation

② Relation matrix →

	a	b	d
1	1	0	0
2	0	1	0
3	1	0	0
4	1	1	1

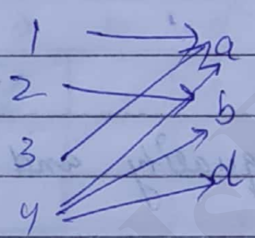
→ No relation (1, d)

→ has a relation

③ Graph →

Directed
(dir^n matters)

Undirected
(dir^n doesn't matter)



Q → $A = \{a, b, c, d\}$, $B = \{\alpha, \beta, \gamma\}$

$R_1 = \{(a, \alpha), (d, \gamma)\}$

$R_2 = \{(a, \alpha), (b, \gamma), (d, \beta), (c, \alpha)\}$

$R_1 \cup R_2 = \{(a, \alpha), (d, \gamma), (b, \gamma), (d, \beta), (c, \alpha)\}$

$R_1 \cap R_2 = \{(a, \alpha)\}$

$R_1 \oplus R_2 = \{(A \cup B) - A \cap B\}$
XOR

$R_1 - R_2 = \{(d, \gamma)\}$

- Properties of binary relations →

① Reflexive

a. Relation is ~~set~~ said to be reflexive for every $a \in A$, there is pair (a, a)

In, Relation matrix, diagonal positive have 1.

② Irreflexive

If for every a, (a, a) doesn't exist, it is Irreflexive

$R = \{(1,2), (1,3)\}$

→ Relations with $\leq$ are reflexive relation but not identity.

- Reflexive closure

$R_1 = \{(1,1), (1,2), (1,3)\}$

$R = \{(2,2), (3,3)\}$

s.t. $R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$

(all identity rel. involved)

- **TRANSITIVE**

If (a,b) and (b,c) → it is necessary to have (a,c) to be transitive relation

If (b,c) of not there, no pt. of checking (a,c), thus relation is contrapositive transitive.

eg → { (1,3) , (2,2) , (1,2) } ✓
{ (1,3) } ✓ transitive.

- **Symmetricity** (~~[illegible]~~ over diagonal → upper half lower half)

If (a,b), then it is imp for (b,a) to exist.

eg → married to relation.

- **Anti-symmetricity** ≠ non sym

eg → <, >, <=

If (a,b) and (b,a), then a=b

R = { (1,2) , (1,1) , (2,2) , (2,3) , (2,3) }

(a,b) & (b,a) exist.

since (2,1) ~~does~~ doesn't exist, it is contrapositive anti symmetric.

→ Whenever (a,b) [a is related to b] and b is related to a, then a = b. Thus, R is not anti-symmetric if there exists (a,b) ∈ A s.t (a,b) & (b,a) ∈ R but a ≠ b

→ if (a,b) is in R ⇒ (b,a) is not in R unless a = b.

* All reflexive relations are anti symmetric but vice-versa is not true.

→ Asymmetric [diagonal elements = 0]
If (a,b) ∈ R, then (b,a) doesn't belong to R.

eg → is father of
It means that R is not asymmetric if for some a & b from A both (a,b) ∈ R and (b,a) ∈ R

$A = \{a, b, c\}$
$B = \{(a,a), (b,b)\}$

→ Symmetric ✓
→ Antisymmetric ✓
→ Asymmetric ✗

$N = \{(a,b), (a,c)\}$ → anti symm.

$N = \{(a,b), (c,c)\}$ → anti symm.

→ Every assymmetric relation is antisymmetric but vice-versa is not true.

• $U = \{ \}$
* Symmetric ✓
Antisymmetric ✓
Asymmetric ✓

$U = \{(a,b), (b,a)\}$
S ✓, Anti ✓, Asy ✗

$X = A \times B$.

⇒ $<$ R → asymmetric ✓
antisymmetric ✓

* Division over integer → symmetric ✓
$(-2, 2)$

* = relation → symmetric & antisymmetric

Q → $R = \{(x,y) : x \in Z, y \in Z, (x-y)$ is div. by 6$\}$

Reflexive :- Let $x \in Z$
$x - x = 0$ div. by 6.
∴ Reflexive

Symmetric :- $(x-y)$ div by 6
$-(x-y)$ is div. by 6
⇒ $(y-x)$ is div by 6

Transitive

$(x-y)$ div. by 6 $\rightarrow$ ①

$(y-z)$ divisible by 6 $\rightarrow$ ②

$x-y+y-z$ = div by 6 [(①+②)]

→ Partition of Sets

divide set into subsets, s.t. $\cap = \phi$ and union all this gives set.

→ Partition of a set A is a set of non empty subsets of A denoted by $\{A_1, A_2 \ldots A_k\}$ s.t. union of A_i's = A and Intersection of any A_i & A_j is empty.

* Partial Ordered Relation

Relation must be reflexive, anti-symm & transitive.

→ A set together with its relation is called a Partial ordered if it is reflexive, anti-symm & transitive.

→ A set S with a partial ordered Relation R is partial ordered set / poset and is denoted by (S, R).

Q→ Prove that $\geq$ is partial order relation R is on set of integers.

antisymmetric → if $(2 \geq 1)$
$\neq (1 \geq 2)$ anti-symmetric ✓
Reflexive → $(1 \geq 1)$, $(2 \geq 2)$ ✓
Transitive → if $a \geq b$, and $b \geq c$
$\Rightarrow a \geq c$ (Transitive) ✓

Q→ A = { set of +ve int }
R = { binary relation on A }
s.t. $(a, b) \in R$ if (a / b)

Every no. divides itself → Reflexive ✓
$a/b \neq b/a$ → antisymmetric ✓
a/b, b/c ∴ a/c ⇒ transitive ✓

② → Totally Ordered (chain) & Partially Ordered

All Elements of sets are comparable over a set

eg → ~~< relation~~

↓
eg → {1, 2, 3, 4, 5}
if a / b
$(1,1)$, ...
$(1, 2)$, $(1, 3)$ . . .
$(2, 4)$ . . . . .
Here 2 cannot be linked with 3 and 5 for division.

Theorem

Q→ If $(A, \leq)$ and $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset with partial order $\leq$ defined by $(a, b) \leq (a', b')$ if $a \leq a'$ (in A) & $b \leq b'$ in B

This partial order is called product partial order over same relation.

Proof → To prove that, partial ordered Relation is reflexive, anti symmetric & transitive

Reflexivity → if $(a, b) \in (A \times B)$
$\therefore (a, b) \leq (a, b)$ [$\because a \leq a \in A$ & $b \leq b \in B$]

Anti-Symmetricity → Suppose that $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$ where a and $a' \in A$ and $(b, b') \in B$. Then, $a \leq a'$ and $a' \leq a$ in A and $b \leq b'$ and $b' \leq b$ in B.
But, we have been given that A & B are posets.
$\Rightarrow a = a'$, $b = b'$ ($\because$ If they are posets $a \leq b$ & $b \leq a$ $\Rightarrow \therefore a = b$)

Transitivity →
Suppose (a, b) is related to (a', b')
$(a, b) \leq (a', b')$
$(a', b') \leq (a'', b'')$
then,

$(a, b) \leq (a'', b'')$

where, a, a', $a'' \in A$
& b, b', $b'' \in B$

Since A & B are posets
$a \leq a'$, and $a' \leq a''$
$\Rightarrow a \leq a''$

Similarly, $b \leq b'$ and $b' \leq b''$
$\Rightarrow b \leq b''$.

Thus, transitivity holds true for $A \times B$.

Q→ $A = \{1, 2\}$
$B = \{2, 3\}$
then prove $A \times B$ is partial ordered or not ??

$A \times B = \{(1,2), (1,3), (2,2), (2,3)\}$

Reflexive → Since $(2,2) \in A \times B$.
Hence relation is ~~reflex~~ reflexive.

Anti-Symmetric → for anti-symm. $(a,b) \in R$ and $(b,a) \notin R$.
$(1,2) \in A \times B$ but $(2,1) \notin A \times B$
Similarly, $(2,3) \notin A \times B$ but $(3,2) \notin A \times B$
and $(2,2) \in A \times B$ when $a = b$
∴ Relation is anti-symmetric

Transitive :-

If $(1,2) \in A \times B$ and $(2,3) \in A \times B$
then $(1,3)$ also $\in A \times B$
similarly, $(1,2) \in A \times B$ and $(2,2) \in A \times B$.
$\therefore (1,2) \in A \times B$
Hence relation is transitive
Thus, it is partial ordered relation.

* Smallest Equivalence relation = Identity relation

→ NULL RELATION

① Reflexivity → $(a,a) \in R \ \forall \ a \in A$
② Symmetricity → $(a,b) \in R$, then $(b,a) \in R$

Only Reflexive if set is null otherwise it is always ~~irref~~ irreflexive.

→ Universal Relation.
(Equivalence relation)

eg → $A = \{1,2,3\}$
$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
$R \leq = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

Since, Ordered pairs of relation are contained in A, thus A is universal relation.

- Reflexive
- Transitive
- Symmetric
- Not antisymmetric
- Not assymmetric.

	Symmetric	Asymmetric
Null relation	✓	✓
$<$	X	✓
$\leq$	X	X

∴ Symmetric can be = assymmetric
" $\neq$ asymmetric

Q→ $A = \{1, 2\}$, $B = \{2, 3\}$ $(\leq)$

Posets of A & B.

$(A, \leq) = \{(1,1), (2,2), (1,2)\}$

$(B, \leq) = \{(2,2), (3,3), (2,3)\}$

$(A \times B, \leq) = \{((1,2),(1,2)), ((1,3),(1,3)), ((1,2),(1,3)), ((2,2),(2,2)), ((2,3),(2,3)), ((2,2),(2,3)), ((1,2),(2,2)), ((1,3),(2,3)), ((1,2),(2,3))\}$

↓ made using theorem of partial order $[a \in A \;\&\; b \in B]$

$A \leq (1,1)$
$B \leq (2,2)$ → $((1,2), (1,2))$

(1,1) (2,2) ✗ ~~(1,2)~~ (1,2)

Reflexive

For $(a,b) \in A \times B$, there must be (a,b)
(1,2), (1,2) ;
(1,3), (1,3) ; [check for all]

Ily, it is anti-symmetric and transitive.

⇒ Posets can only be made of partial ordered relations.
eg → ≤ and ≥

⇒ Hasse Diagram

Every partial order relation can be represented using digraph, where in Hasse diagram of a relation gives a simpler graphical representation of that relation. When given binary relation is partial ordered relation. ~~the~~ ~~binary~~ the graphical representation can be made a simpler by →

1. Since relation is reflexive, we can omit arrows from pointing back to themselves.

2. Since relation is transitive, we can omit arrows b/w pts. that are connected by sequence of arrows

3. We can omit arrow headers assuming all the arrows heads are pointing in 1 dirn (upward, downward, left, right)

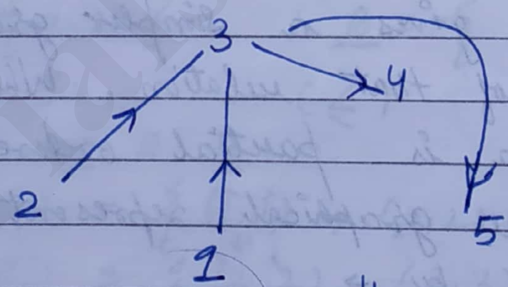
$A = \{1, 2, 3, 4, 5\}$

(1,3)

$$M_R = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 1 & * & * \\ 0 & 1 & 1 & * & * \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

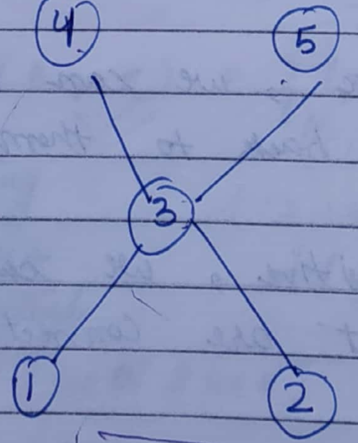
⇒ Remove Reflexivity
→ Since, it is partial ordered, no need to show arrows of (1,1)

⇒ Remove transivity. 1 (X)
→ Since, it is transitive, then (1,3) & (3,4)
∴ (1,4) ∈ M_R and line of (1,4) must be removed.



⇓

Modified version



[arrows are not need to be included].

⇒ In a partial relation, diagraph can't have any cycle of more than length one.

X 

cycle of 3

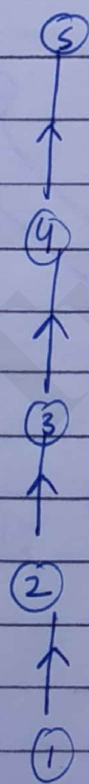
i.e. it has only cycles of length 1 which shows reflexivity.

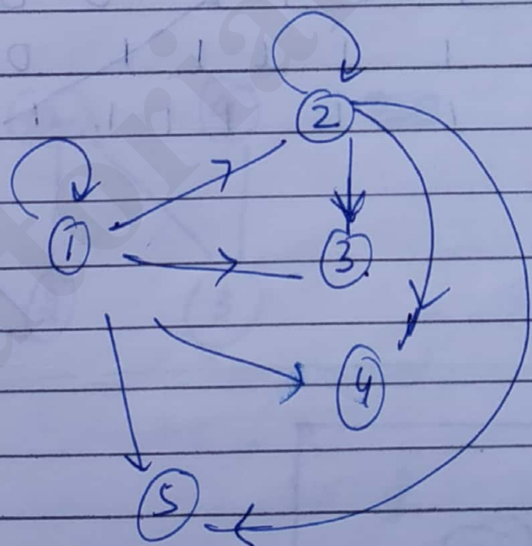
Q→

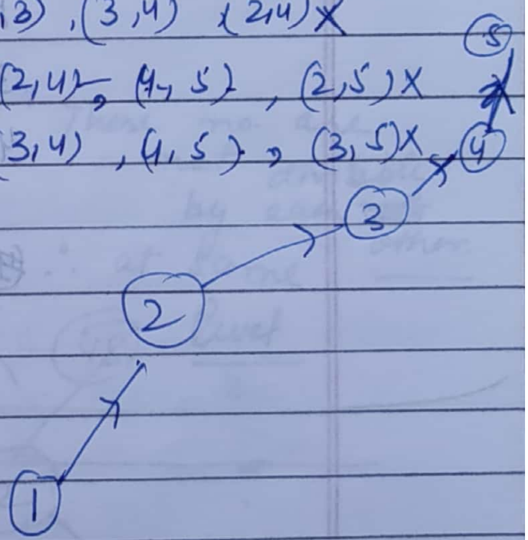
$$M_R = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & \cancel{1} & \cancel{1} & \cancel{1} \\ 2 & 0 & 1 & 1 & \cancel{1} & \cancel{1} \\ 3 & 0 & 0 & 1 & 1 & \cancel{1} \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(1, 2) (2, 3)
(1, 3) X
(1, 3), (3, 4), (1, 4) X
(1, 4), (4, 5) (1, 5) X

(2, 3), (3, 4) (2, 4) X
(2, 4), (4, 5), (2, 5) X
(3, 4), (4, 5), (3, 5) X







linearly ordered

chain

⇒ Closure of Relation

1. Reflexive Closure (R^r) (R_r), R''

If R is given relation, then R' is reflexive closure if only if, $R \subseteq R'$ & R' is reflexive.

$A = \{1, 2, 3\}$

$R = \{(1,1)\}$ $R \subset R'$

$R' = \{(1,1), (2,2), (3,3)\}$

$R \subseteq R'$
&
R' has to the minimal

⇒ If $R \subseteq R'$, $R' \subseteq R''$
then R' has to be a minimal set.

2. Symmetric closure

$R^S = R' = R \cup R^{-1}$ (x,y) (y,x)

$R^S \to$ is sym. if and only if R' is sym
$R \subseteq R'$
& R' has to be minimal

(3) Transitive closure

If R is transitive

$R = \{(1,2), (1,1)\} \cup \{(2,1)\}$

R'

⇒ S & R are two Sets

SOR = $M_R * M_S$

(composition of S & R)

$S \, O \, R$ → composition of sets S & R.

$R^t = R \cup R_1 \cup R_2 \ldots \ldots \cup R_n$

$R_1 = R \cdot O R = M_R * R_R.$

$R_2 = R' O R = M_R * M_{R_1}$

$R_3 = R_2 \, O \, R = M_R * M_{R_2}$

⇒ Q → $A = \{1,2\}$

$R = \{(1,2)\}$ — transitive

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$M_R \times M_R$

$$M_R * M_R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ ⇒ ~~these Matrix start Repeating~~

⇒

Since,

we get a null Matrix ⇒

R is a transitive Relation itself

$\Rightarrow$ S o R
R $\rightarrow$ a to b
S $\rightarrow$ b to c
S o R $\rightarrow$ a to c
$M_R * M_R$

Q $\rightarrow$ $M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 = M_R * M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 = M_R * M_{R_1}$

$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 = M_R * M_{R_2}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cancel{0}1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since,

$R_3 = R_1$

[These Matrix starts repeating ∴ we stop our procedure here] and then find

∴ We stop there

Now $\boxed{R^t = R \cup R_1 \cup R_2}$

OR

$$M_{R^t} = \begin{bmatrix} 0 & \cancel{0}1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q →

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = M_R * M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_2 = M_R * M_{R_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

⇒ This means that (b, a) is occuring twice in Relation.

$R_6 = M_R * R_5$

$$R_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

If → $M_R * M_R = \phi$ Matric → then Relation is Transitive

$$R_3 = M_R * R_2$$

$$\begin{bmatrix} \cancel{0} & 0 & 1 & 0 & 0 \\ \cancel{0} & 1 & 0 & 1 & 0 \\ \cancel{0} & 0 & 0 & 0 & 1 \\ \cancel{0} & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$R_4 = M_R * M_{R_3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2_{,1} & 0 & 2_{,1} & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

⇓

R_4

$$R_5 = M_R * M_{R_4}$$

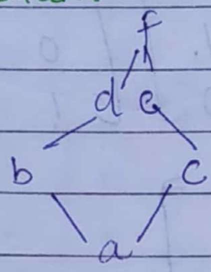
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2_{,1} & 0 & 2_{,1} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Q → Is this anti chain

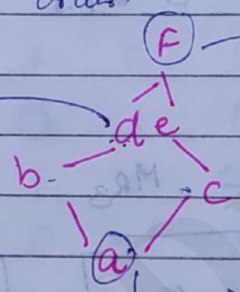
⇒ Hasse Diagram

- Anti chain



{b, c} (~~not relation~~ to each other)
↓
(non comparable)

→ Tropological Sorting
Dictionary Order



F → upper bound for {b, c}

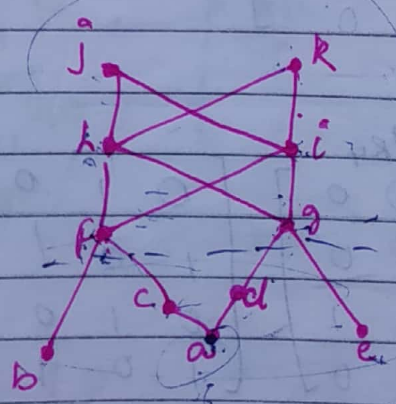
a → lower bound for {b, c}

Here, d, e can't be upper bound as d doesn't bind with c & e doesn't bind with b but f binds with both b & c

{b, c}

(0, 1) → max & min. does not exist.
[0; 1] → max = 1
min = 0

Q →



→ maxima → j and k
minima → a, b, e
as all 3 aren't comparable with each other.

→ {f, g} = ∅

- upper bound = h, i, j, k
- Least upper bound = h ✗ doesn't exist.

least upper bound
greatest lower bound

- Lower bound = a

since lower bound is unique
∴ greatest lower bound exist.

Q → [0, 1) →

min → 0 (exist only in set), max → doesn't exist

L.B → 0 (outside the set), U.P → 1 → (can exist)

L.UB → 1, GLB → 0

⇒ * Covers.

an element that strictly succeeds other An element can have more than 1 cover.

In Ques. above, (a, b, ...) cover of b = f

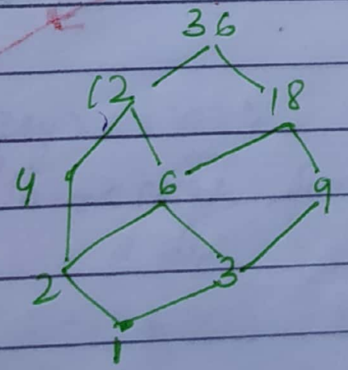
and cover of f = h, i.

Q → (D - 36) Divisors

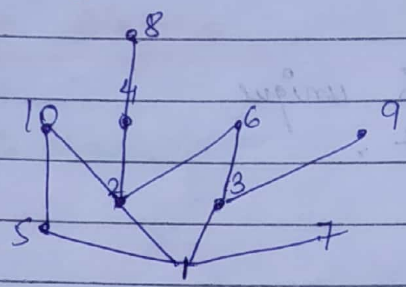
elements which divides 36 are in set.

(D-36) = {1, 2, 3, 4, 6, 9, 12, 18, 36}

Relation → divisibility



Q→ $A = \{ (1, 2, 3 \ldots 10), | \}$
↳ divide



* Lattices

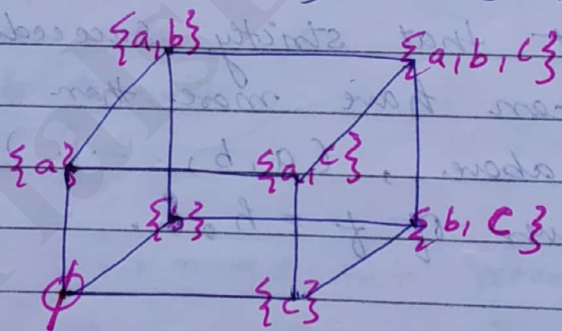
A partial order relation is a lattice if every 2 elements has its own least upper bound & greatest lower bound

eg →

Poset $[P(a, b, c), \subseteq]$

elements of power set

$= \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}$



These are 6 chains of length 3

$\phi \rightarrow \{a\} \rightarrow \{a,b\} \rightarrow \{a,b,c\}$ ①

$\{a,c\} \rightarrow \{a,b,c\}$ ②

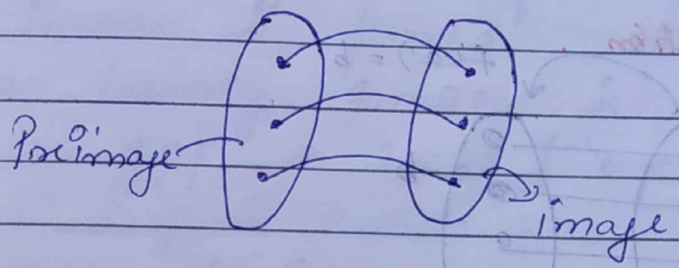
② chains this way

Floor

FUNCTIONS

every element of domain must ~~not~~ map to codomain.

$f(a) = b$



$f : A \rightarrow B$

→ 2 function are said to be equal if they have same domain, codomain & func. for mapping is same.

f_1 & f_2 from A to R

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$.

→ if $S \subseteq A$, then images of $S \subseteq$ Range.

(1) One-one / Injective

(2) Onto / surjective (co-domain = Range)

(3) Tiny function.

if $x \le y$

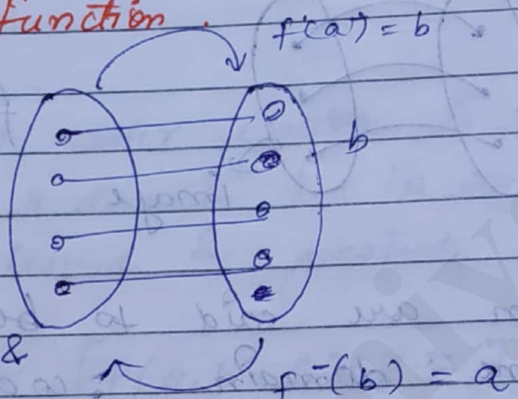
then $f(x) \le f(y)$

if $x < y$ → (strictly Increasing function)

then $f(x) < f(y)$

4. Identity function.
(denoted by I_n)
$f(x) = x$
both one one & onto (Bijective function)

5. Inverse function.



(function should be both one one & onto to define its inverse)

6. composition of functions
$fog \neq gof$
$f[g(x)] \neq g[f(x)]$

Q→ Let g = {a, b, c} to itself
g(a) = b, g(b) = c, g(c) = a
f{abc} to {1, 2, 3}
f{a} = 3, f(b) = 2, f(c) = 1

fog = ?

f(g(x)) = f(g(a)) = 2
f(g(b)) = 1
f(g(c)) = 3

$g(f(a)) = g(3)$ does not exist

$\therefore$ gof doesn't exist.

Theorem

When composition of a function & its inverse is formed in either order, an identity function is observed.

$$fof^{-1} = I_A, \quad f^{-1}of = I_B$$

Suppose f is a one-one correspondence from set A to B, then inverse function exists & has one to one correspondance from B to A. The inverse function reverses the correspondance of the original function. so, if $f^{-1}(b) = a$, when $f(a) = b$ and $f(a) = b$ when $f^{-1}(b) = a$

$$(f^{-1} o f)(a) = f^{-1}(f(a))$$
$$= f^{-1}(b)$$
$$= a$$

$$(f^{-1})^{-1} = f$$

Let f be a mapping from A to B, a mapping g from B to A is called inverse of f when $gof = I_A$ and $fog = I_B$.

⇒ Some other function

1. Floor (greatest int)

$$\lfloor 2.3 \rfloor = 2$$

2. ceil (smallest int)

$\lceil 2.3 \rceil = 3$

⇒ Graph of functions

⇒ PIGEON HOLE PRINCIPLE

If there are more pigeons than pigeon hole, then there must be atleast 1 pigeonhol with atleast 2 pigeons in it.

Theorem

If k is a positive integer and k+1 or more obj are placed in k boxes, then there is atleast one box containing 2 or more of the objects.

Ques 5 nos → (pigeons)

Out of 1 - 8 where to add to 9

{1, 8}, {2, 7}, {3, 6}, {4, 5}

(4 pigeonholes)

since 1, 2, 3, 4 can't add to 9, thus there is one no. added to 1, 2, 3, 4 so asto make sum 9.

Not in Exam

Ques → Show that if any 11 nos are chosen from 1 to 20, then one of them will be a multiple of another.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Ans →

Pigeons – 11 {1, ... , 20}

This problem can be solved using pigeon hole principle which states that if there are more no. of pigeons than pigeons holes, then there will be atleast 1 pigeonhole with more than 1 pigeon in it.

For the given problem, we have to identify no. of pigeons & pigeonholes. here, no. of pigeons = 11

Now we can construct pigeon holes in such a way that each no. chosen can be assigned to only 1 pg. We have to create 10 or fewer pigeon holes. The numbers x & y ∈ same pigeon hole if and only if either x/y or y/x. Lets try & construct pigeon holes on the basis of prime nos.

Case I

Between 1 to 20, ∃ 8 prime nos. Hence, no. of pigeon holes = 8. But we know x & y are multiples of same prime no. will not guarantee x/y or y/x. Hence our case fails.

9, 6, 3 ← 3 divides 6 but 6 doesn't divide 9.

2, 3, 5, 7, 11, 13, 17, 19

Hence, case failed.

Case II

Lets construct pigeon holes on basis of odd no. There are 10 odd nos. b/w 1 to 20 and every positive integer can be written as $n = 2^k \cdot m$, where m is odd and $k \geq 0$. So, if 11 nos. are chosen from the set, then 2 of them must have the same odd part as there are 10 pigeon holes. Let n_1 & n_2 be 2 such nos. with same odd part.

→ $n_1 = 2^{k_1} m$ and $n_2 = 2^{k_2} \cdot m$

if $k_1 > k_2$

n_1 is multiple of n_2

else,

n_2 is multiple of n_1

$\begin{bmatrix} 12 \\ 6 \\ 3 \end{bmatrix}$

Hence proved.

⇒ PERMUTATIONS

→ $P(n, r) = \frac{n!}{(n-r)!}$ (ordering collection)

→ $C(n, r) = \frac{n!}{(n-r)!\, r!}$ (Unordered collection)

→ k doesn't occur = $P(n-k, r)$; $C(n-k, r)$

→ k do occur = $P(n-r, r-k) * P(r, k)$; $C(n-k, r-k)$

→ Permutation with repetition → n^r

Combn with repetition → $\frac{(n+r-1)!}{r!\,(n-1)!}$

→ Circular = $(n-1)!$

→ Ring = $\frac{(n-1)!}{2!}$

Permutn

Ques Each user has a password of 6-8 chars. Each chr. is uppercase or a digit. Atleast 1 digit must be there. How many possible passwords.

→ atleast = All case − none

$P = P_6 + P_7 + P_8$

$P_6 = 36^6 - 26^6$

$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^8$

Ques We have to make a password with 4 distinct letters and 3 distinct nos.

Ans → $^{26}P_4 \times {}^{10}P_3$

Ques How many ways are there to place 2 Red, 1 Blue & 1 white in 10 boxes.

Ans 1W → 10 ways

1B → 9 ways

1R → 8 ways

1R → 7 ways

$\therefore \frac{10 \times 9 \times 8 \times 7}{2!}$ $\quad \frac{^{10}P_4}{2!}$

$\frac{10 \times 9 \times 8 \times 7}{\cancel{4}}$

$^{10}C_4$

$\frac{(10!)}{(6!)\ \cancel{4!}}$

Generalised formula If we have to place r balls in n boxes where Q_1 of these are identical, Q_2 & Q_3 are identical

$\therefore$ formula $= \frac{^nP_r}{Q_1! \times Q_2! \times Q_3! \cdots Q_n!}$

Graphs

$G = (V, E)$

Edges define size of graph (m)

no. of vertices defines order (n) of graph

$G = (n, m)$

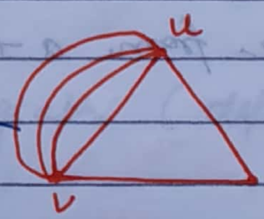
* Every edge in a graph will either ~~to~~ have 1 or 2 end point.

• → Order is 1 & size is 0.

→ Order 1 and size is 1

* Graphs which have infinite order are Infinite graphs.

• Simple graphs → no multiple edges, no ~~sef~~ self loops, and b/w every ~~point~~ pair of vertices, there is single edge.



Multiple edge graph

B/w u & v multiplicity is 4

u and v are adjacent to each other as there is an edge b/w them and thus they are called neighbours

Multiplicity

no. of edges b/w two Points.

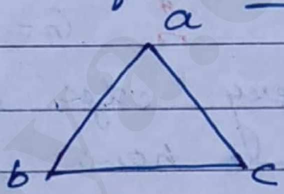
★ **Pseudo ~~graph~~ graphs**

have multiple edges & self loops.

★ **undirected graphs**

edges b/w undirected of pair of vertices.

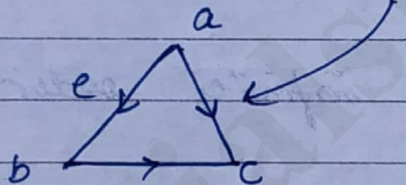
(u to v) = (v to u)



★ **Directed graphs**

edges b/w ordered pair of vertices.

(u to v ≠ v to u)



If there is an edge b/w a and b, then we say that a is adjacent to b and b is adjacent from a.

a → initial point

b → terminal pt.

e → incident edge from a to b.

★ **Mixed graphs**

has both directed and undirected edges.

eg → road map.

⇒ Different graph models

(1) Niche overlap graph in Ecosystem.
→ use of no direction, thus use of simple undirected graph

(2) Acquaintanceship graph.
edge b/w 2 people if they know each other
use of undirected simple graph.

(3) Round robin graph.
simple direct.

(4) Hollywood or Bollywood graph
simple undirected graph.

(5) influence graph
simple direct graph

(6) Collaboration graph
simple undirected.

(7) call graphs
directed graph (depends on the case)

(8) Intersection graph
undirected graph.

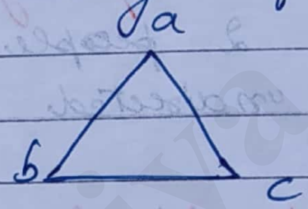
* Graph Terminology

2 nodes are adjacent if there is an edge b/w them.

* Degree of Vertex →

for an undirected graph, no. of edges from a node is degree of vertex.

a has degree 2



Note

Self loops contribute 2 degree to a vertex

* Isolated vertex

no edge related to a node

eg → • has 0 degree.

→ Pendant Vertex

graph with degree 1

⇒ for directed graphs, 2 degrees

(deg⁻) Indegree (no. of edges coming in)

outdegree (deg⁺) (no. of edges ~~to~~ going out)

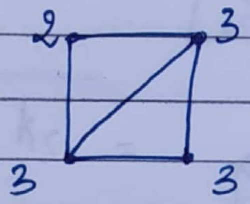
⇒ Self loop in directed graph contribute 1 to indegree & 1 to outdegree

→ Graphs on basis of degree.

ⓐ Handshaking Theorem

use of undirect graphs.

Twice no. of edges = $\sum_{v \in V} \deg(\ )$ → (sum of all degrees of vertices)



$$\boxed{2e = \sum_{v \in V} \deg(\ )}$$

Q→ Calculate the no. of edges in graph with 10 vertices each of degree 6.

$2e = 60 \quad \therefore \quad e = 30$

Theorem 2

There will be even no. of vertices with odd degree.

Let $G = (V, E)$ be an undirected graph with e edges then

Handshaking theorem $\boxed{2e = \sum_{v \in V} \deg(\ )}$ i.e. sum of degree of vertices of an undirected graph is even.

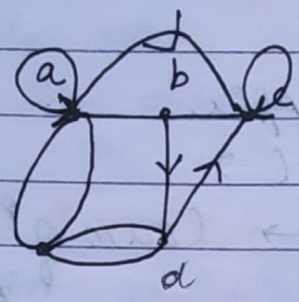
An undirected graph has even no. of vertices of odd degree.

$$2e = \sum_{v \in V} \deg(\ )$$

$$2e = \underbrace{\sum_{v_1 \in V} \deg(\ )}_{\text{(even degree)}} + \underbrace{\sum_{v_2 \in V} \deg(\ )}_{\text{odd degree}}$$

Since ⇒ $\varepsilon + \varepsilon = \varepsilon$ ✓ $0 + \varepsilon = 0$ ✗
$0 + 0 = \varepsilon$ ✗

Q →



Theorem 3

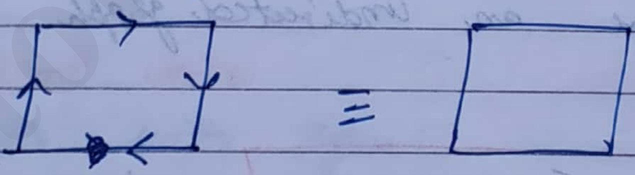
based on indegree & outdegree.

Let $g = (V, E)$ be a graph with directed edges, then no. of edges =

$$2|E| = \sum_{v \in V} deg^{-}() + \sum_{v \in V} deg^{+}().$$

* Underline Undirected Graph

directed graph without directions.



(has same no. of edges & vertices)

V.V. IMP

⇒ Some Special Graphs

1. Complete Graphs (K_n, $n \geq 1$)

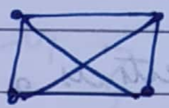
B/w every pair of vertices, there are n edges.

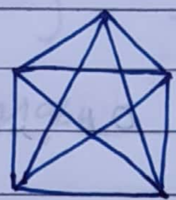
K_n → no. of vertices of graph.

eg → K_1 = •

K_2 = 

K_3 = 

K_4 = 

K_5 = 

No. of edges of K graph = $^nC_2 = \frac{n(n-1)}{2}$

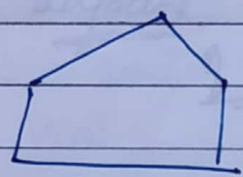
2. Cycle graph

It must be a closed graph defined for 3 or more than 3 vertices.
(C_n for $n \geq 3$)

eg →


C_3


C_4


C_5

C_n

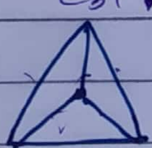
no. of edges = n

~~eg~~ edges b/w adjacent vertices only

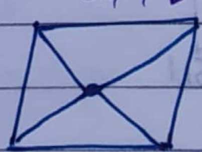
3. Wheel graph

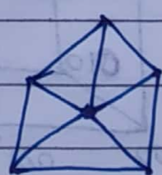
$(n \geq 3)$ one additional vertex is ~~int~~ introduced & is connected to all vertices.

$C_3 + V$


W_3

$C_4 + V$


W_4

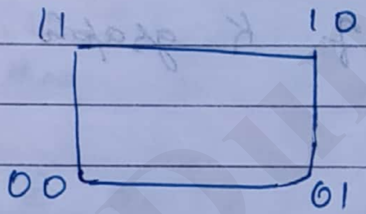


no. of edges = 2n

④ N cubes / n dimensional hypercubes
used to represent 2^n bit strings
If $n = 1$, $2^1 = 2 \rightarrow$ 2 vertices
bit string
↳ 0 – 1

2^n gives no. of vertices, n gives dimensions

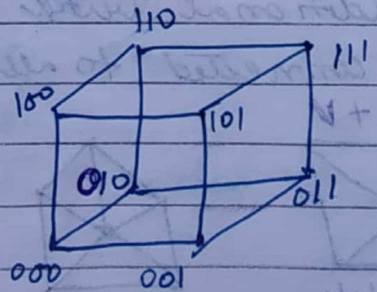
⇒ $2^2 = 4 \{00, 01, 10, 11\}$



(edge b/w 2 vertices only if they differ exactly by 1 bit)

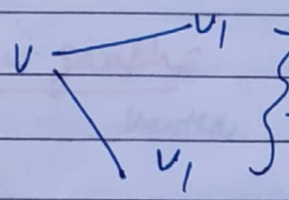
⇒ $2^3 = 8$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



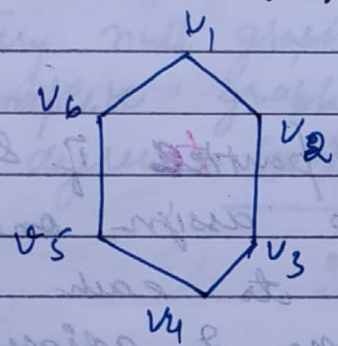
(5) Bi-partite Graphs

If we can partition graph into 2 partition set V_1 & V_2 so that there is no edge b/w vertices of V_1 & V_2 but V_1 and V_2 have an edge.



— disjoint sets

⇒ Draw a C_6 graph

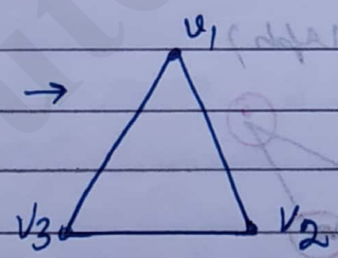


Check (bipartite)

$\{V_1, V_3, V_3\}$
$\{V_2, V_6, V_4\}$

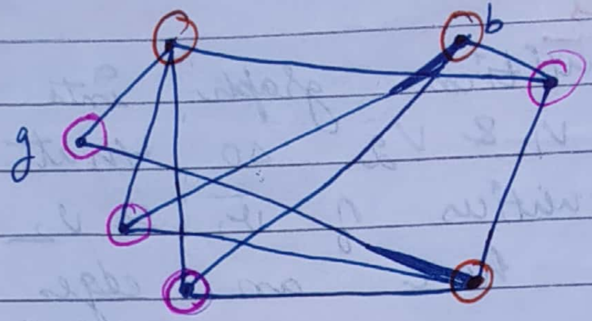
[first check for adjacent vertices of v_4.]

Q→
⇒ K_3 →



(No it is not bi parted)

Q→



(its a bipartite graph)

(no condition on bipartite sets) {a, b, d} 4 (colour)

{g, c, f, e} diff. colour.

Theorem 4

A simple graph is bi-partite if & only if it is possible to assign one of the 2 diff. colours to each vertex of the graph so that no 2 adjacent vertices are assigned the same colour.

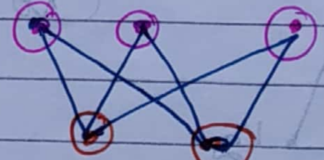
⇒ complete Bipartite graph ÷ ($K_{m,n}$)

Total vertices $= m+n$

Total edges $= mn$

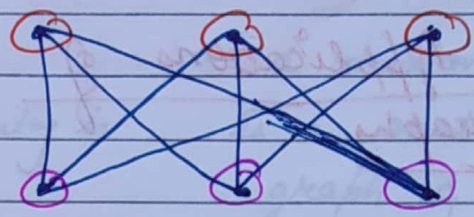
(Since it is a comp. graph)

eg → ($K_{2,3}$) =



complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into 2 subsets m & n. There is an edge b/w 2 vertices if & only if 1 vertex is in the first set & other in 2nd set.

$(K_{3,3}) \Rightarrow$



⇒ Regular graphs

Every vertex has same degree (Rg)

NOTE

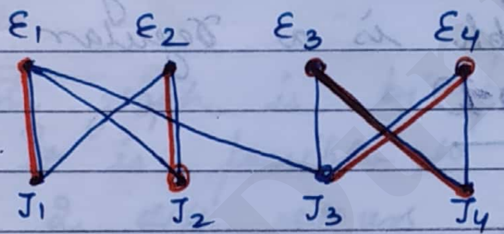
Every null graph is a regular graph.
complete graph is a regular graph with degree = (n-1)

⇒ Some Applications of Special Types of Graphs

1. Job Assignment

If there are m employees in a group & j different jobs that need to be done where $m \le j$. Each Employee is trained to do one or more of these j jobs.

(Using Bipartite graph)



(4 Employees as vertex)

(4 Jobs as vertices)

Matching concept

In case, where Every job must has an employee assigned to it & no employee is assigned more than 1 job.
(coloured lines represent)

- **Matching**

Given a graph $G = (V, E)$, matching M in G is a set of pairwise non adjacent edges i.e. no two edges share a common vertex.

→ Maximal matching

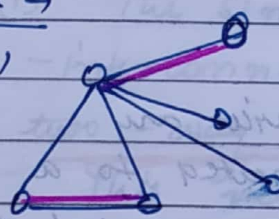
→ Maximum matching

Maximum

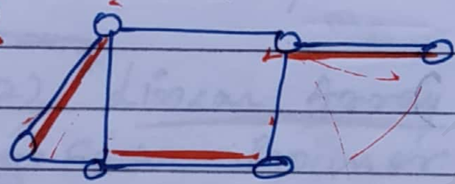
→ a maximum matching is a matching that contains the largest possible no. of edges.

Ex →

(1)



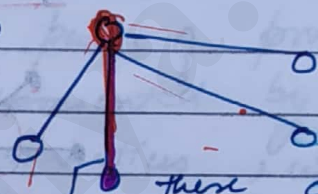
(2)



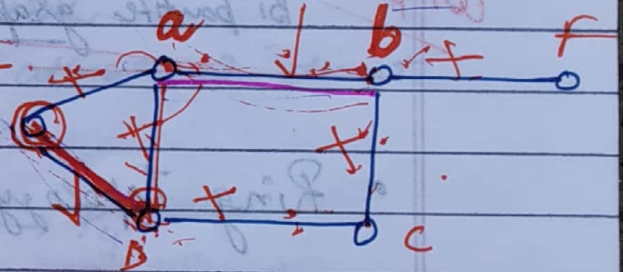
Maximal

→ a maximal matching is a matching M of a graph G with a property that if any edge not in M is added to M, then there is no longer a matching.

Ex → (1)



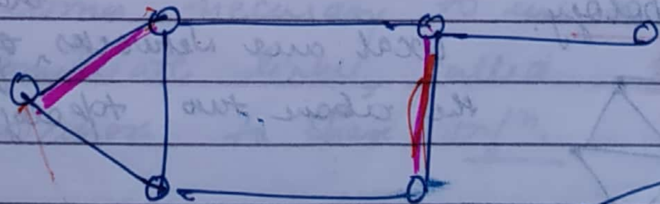
(2)

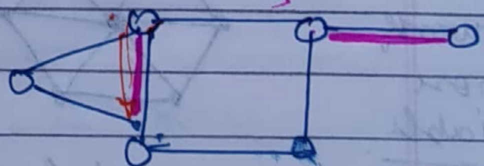


Only 2 edges can be matched with no common vertex

⇒ A graph can have different no. of matching

Ex →





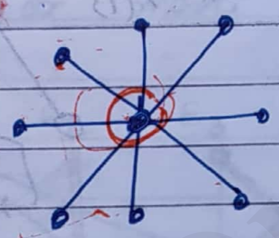
different matching

NOTE Every maximum matching is always maximal matching but not vice-versa

2.

- **Local Area Networks**

  - Star topology

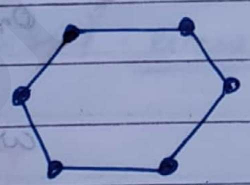


all devices are connected to a central control device.

complete Bi-partite graph ($K_{1,n}$)

  - Ring Topology

⇒ where each device is connected to exactly two others

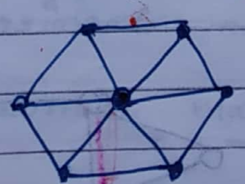


n-cycle graph
(C_n)

  - Hybrid Topology.

Local area Networks that uses a hybrid of the above two topologies.

- makes network most reliable



Wheel Graph (W_n)

3.

Interconnection Networks for Parallel Computation

Parallel Processing

(computers made up of many separate processors) each with its own memory)

(1) ⇒ (1) The simplest, but most expensive, network - interconnecting processors include a two-way link b/w each pair of processors. This network can be represented by K_n, the complete graph on n vertices, when there are n processors 64 → 2016

(2) (2) Linear Array in which
advantage → Each Processor has atmost 2 direct connections at a time.

$P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow P_4$

→ Each Processor P_i, other than P_1 & P_n, is connected to its neighbours P_{i-1} & P_{i+1} via two way link.

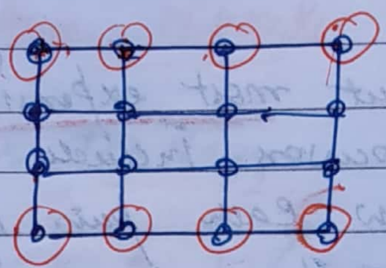
disadvantage

Sometimes necessary to use a large no. of intermediate links, called hops, for processors to share infn.

(3) MESH NETWORK [2-D array]

no. of processors ~~then~~ is a perfect square

cond^n → $\sqrt{n} = m$



(4) HYPERCUBE

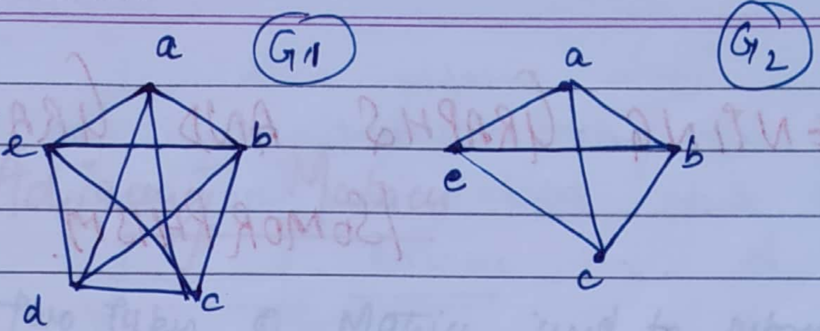
no. of Processors ⇒ $n = 2^m$

(graph → n dimensions Hypercubes)

The ~~hypercup~~ hypercube network balances the number of direct connections for each processor and the number of intermediate connections required so that processors can communicate.

→ Subgraph

when edges and vertices are removed from a graph, without removing endpoints of any remaining edges, a smaller graph is obtained. Such graph is called a Subgraph.

⇒ 

Here G_2 is subgraph of G_1.

⇒ Proper subgraph

$G = (V, E)$, $H = (W, F)$

$W \subseteq V$, $F \subseteq E$

⇒ H is proper subgraph of G if $H \neq G$.

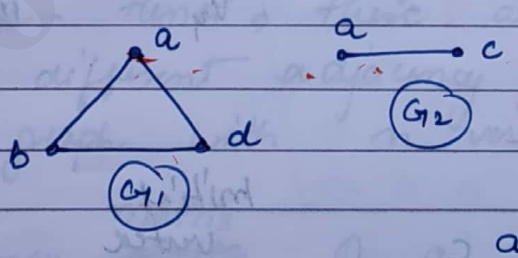
⇒ Union of two Graphs

The new graph that contains all vertices & edges of the two graphs, is union

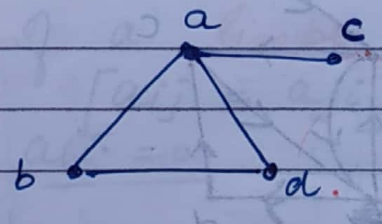
$V \rightarrow V_1 \cup V_2$ ⇒ $G_1 \cup G_2$

$E \rightarrow E_1 \cup E_2$.

Ex →



$G_1 \cup G_2$ =



⇒ REPRESENTING GRAPHS AND GRAPH ISOMORPHISM

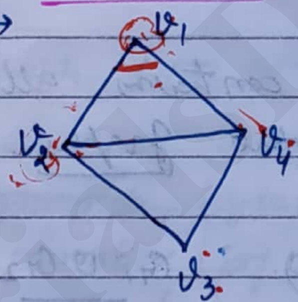
(i) Simple Graphs (no multiple edges)

1. First way to represent is to list all the edges of the graph.

2. **Adjacency list**
   specify,
   the vertices that are adjacent to each vertex of the graph.

undirected

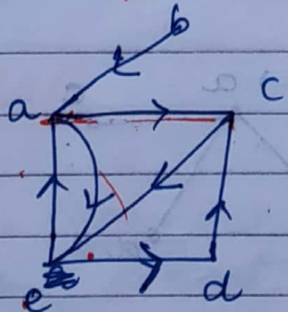
Ex →



$v_1 \to v_2, v_4$
$v_2 \to v_1, v_4, v_3$
$v_3 \to v_2, v_4$
$v_4 \to v_1, v_3$

Vertex	Adjacent vertex
v_1	v_2, v_4
v_2	v_4, v_1, v_3
v_3	v_1, v_4
v_4	v_2, v_3, v_1

Directed



a → c, e
b → a
c → e
d → c
e → a, d

Initial vertex	Terminal vertex
a	c
b	a
c	e
d	c
e	a, d.

⇒ Adjacency Matrices

Two types of Matrices used to represent graphs

① Adjacency of Vertices

$G = (V, E)$

$A = [a_{ij}]$

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

- square Matrix
  size → $n \times n$ (vertices)

⇒ NOTE

→ an adjacency matrix ~~is~~ of a graph is based on the ordering chosen for the vertices. Hence, there are as many as $n!$ different adjacency matrices for a graph with n vertices.

- Adjacency matrix ~~graphs~~ of a Simple graph is symmetric $[a_{ij} = a_{ji}]$
  ⇒ no loops ⇒ $a_{ii} = 0$

- A loop at vertex a_i is represented by a 1 at (i, i) position.

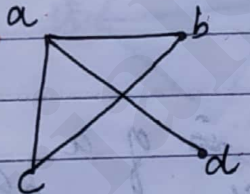
→ we can represent graphs with multiple edges as well. but then Matrix will no-longer be 0 − 1 matrix.

Ex → a to b ⟶ 4 edges.
⇒ then we mark 4 in matrix at that position.

⇒ All undirected graphs, including multigraphs & pseudographs are symmetric

⇒ Questions

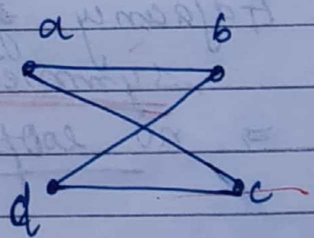
1.

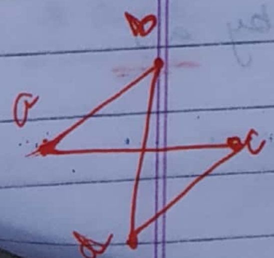


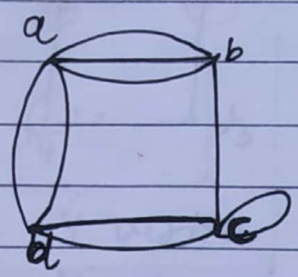
$$\Rightarrow \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

2.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

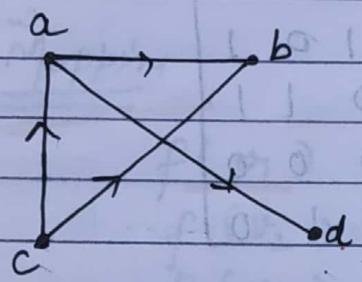




3. 

Pseudo graph

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 3 & 0 & 2 \\ b & 3 & 0 & 1 & 0 \\ c & 0 & 1 & 1 & 2 \\ d & 2 & 0 & 2 & 0 \end{array}$$

4. 

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 0 & 0 & 0 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 0 \end{array}$$

⇒ Trade-offs b/w Adjacency list & Adjacency Matrix

- If simple graph (relatively few edges, (sparse)
  ⇒ ∴ adjacency list
- If simple graph (dense (many edges))
  ⇒ ∴ adjacency Matrix

⇒ ② Incidence Matrices

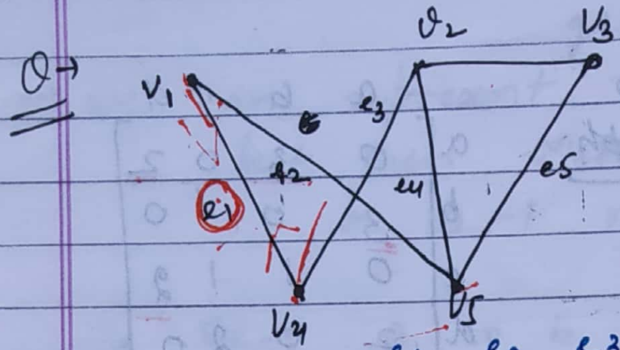
- multiple edges & loops can also be represented.

$G = (V, E)$
↳ is a $n \times m$ matrix
(vertices) (edges)

$M = [m_{ij}]$, where,

$$m_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with } v_i. \\ 0 & \text{otherwise} \end{cases}$$

Q→



$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 1 & 0 & 1 \\ v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$$

⇒ ISOMORPHOSISM

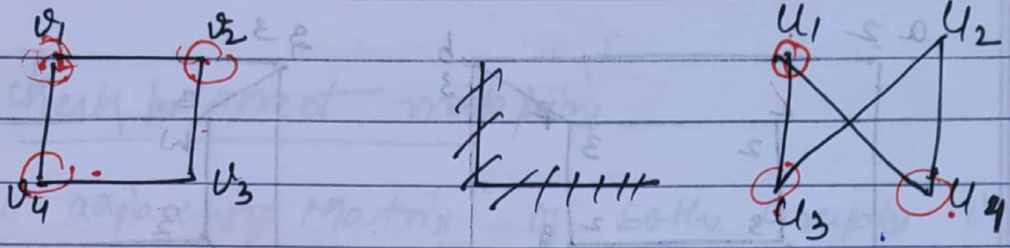
The simple Graph G_1 (V_1, E_1) & G_2 (V_2, E_2) are isomorphic if there is a onetoone & onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) & f(b) are adjacent in G_2 for all a & b in V_1. Such a function is called an Isomorphic function

GRAPH INVARIANT

A property preserved by isomorphosism of graphs is called a graph invariant

- same no. of vertices
- Same no. of edges.
- the degrees of the vertices must be same

Q →



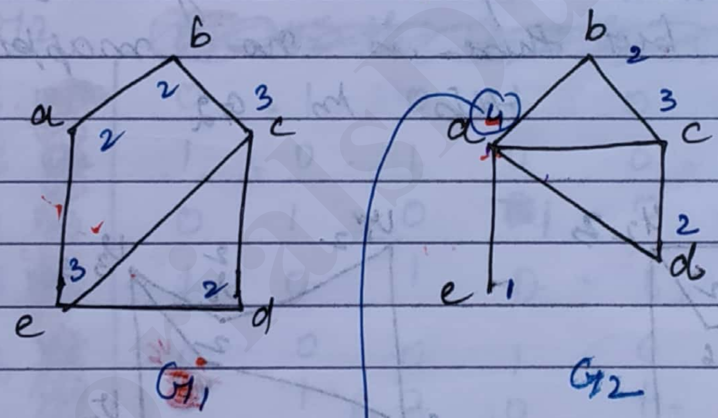
→ 4 vertices
→ 4 edges
→ 4 vertices with 2 degree

⇒ now Mapping

$f(v_1) = u_2$
$f(v_2) = u_4$
$f(v_3) = u_3$
$f(v_4) = u_1$

∴ Isomorphic

Q →

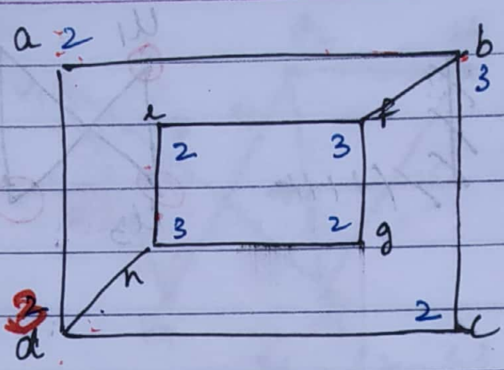


→ there is no such vertex with degree 4 in G_1

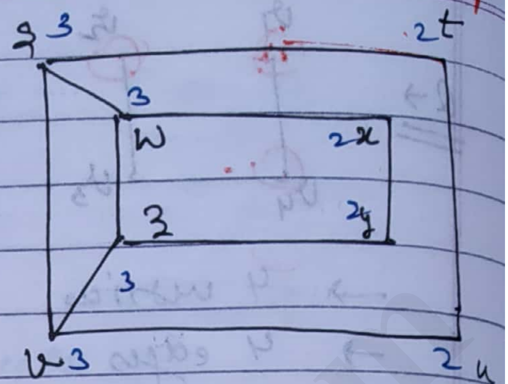
⇒ ∴ not isomorphic

Q →

G_1



G_2



⇒ 8 vertices
⇒ 8 edges
⇒ same no. of vertices with same degree

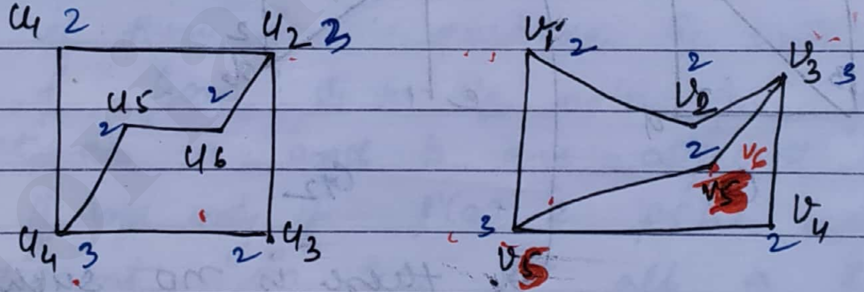
Mapping → $f(a)$ = does not exist.

Since

$f(a)$ is a vertex of degree 2 and adjacent to degree 2 & 3 but there is no mapping of this in G_2

∴ not isomorphic

Q → 5 Marks



Graph Invariant ✓

Mapping

G_1		G_2
$f(u_1)$	=	v_6
$f(u_2)$	=	v_3
$f(u_3)$	=	v_4
$f(u_4)$	=	v_5
$f(u_5)$	=	v_2
$f(u_6)$	=	v_1

∴ Isomorphic.

To check for correct mapping

Use adjacency Matrix of both Graphs in same order of mapping

$$G_1 \quad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 0 & 1 & 0 & 1 & 0 & 0 \\ u_2 & 1 & 0 & 1 & 0 & 0 & 1 \\ u_3 & 0 & 1 & 0 & 1 & 0 & 0 \\ u_4 & 1 & 0 & 1 & 0 & 1 & 0 \\ u_5 & 0 & 0 & 0 & 1 & 0 & 1 \\ u_6 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}$$

$$G_2 \quad \begin{array}{c|cccccc} & v_6 & v_3 & v_4 & v_5 & v_2 & v_1 \\ \hline v_6 & 0 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 1 & 0 & 0 & 1 \\ v_4 & 0 & 1 & 0 & 1 & 0 & 0 \\ v_5 & 1 & 0 & 1 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 0 & 1 & 0 & 1 \\ v_1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}$$

$\Rightarrow$ These two matrices are same. $\therefore$ Our mapping is correct.

Before Mapping

$\Rightarrow$ Length of the Simple Circuit

Graph Invariant.

GRAPHS.... (Continued)

⟹ CONNECTIVITY

• **Path**

A finite or infinite sequence of edges which connects a sequence of vertices distinct from one another is called a path

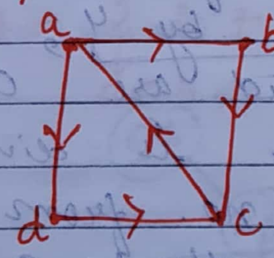
• **Simple path**

A path that does not contain the same edge more than one. i.e. no repeated edge or vertex

• **Circuit**

A path is a circuit, if it begins and ends at the same vertex, & has length greater than zero.

• A path in a directed graph is known as "dipath" [crucial role of arrows]



Here, a, c is not a direct path. First we have to go from $a \to b$ & then from $b \to c$

• **Trail**

A path is a trail in which all vertices except the first & last are distinct.

Hence, trail will not have any repeated edge too., no repetition of edges and vertices. [similar to circuits]

- **Walk**

A Trail is a walk in which all edges are distinct. A walk of length k in a graph is an alternating sequence of vertices of edges.

[edges aren't repeated but vertices can repeat]

Walk can be closed or open.

- **cycle / closed walk**

vertices & edges can't repeat.

Circuit is closed

- **Trivial path**

Path of length 0

→ single vertex

⇒ **Represent path using vertices..**

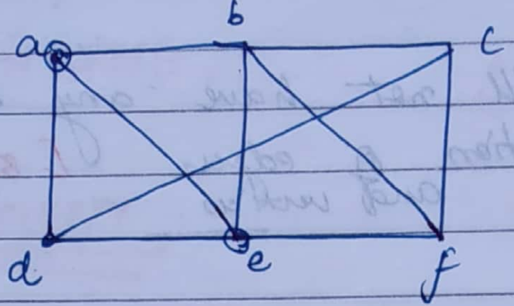
eg → a, c → a to c

$e_1 \rightarrow \{x_0 - x_1\}$

this means edge 1 with $x_0 \rightarrow$ initial vertex &

$x_1 \rightarrow$ is adjacent vertex.

Ex→



- a, b, c, f, e → simple path
- d, e, c, a → not a path (bcoz repeated edge)
- a, b, e, d, a, b → length 5, normal circuit (circuit bcoz it is closed)
- b, c, f, e, b → simple circuit.

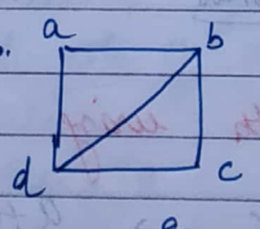
⇒ CONNECTIVITY OF GRAPHS

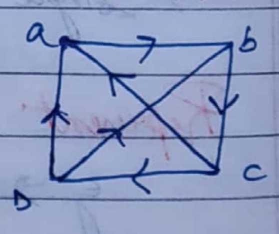
Connectedness

Undirected — Directed

Directed: Weakly connected, Strongly connected

- an undirected graph is called connected, if there is a path b/w every pair of distinct vertices of the graphs.

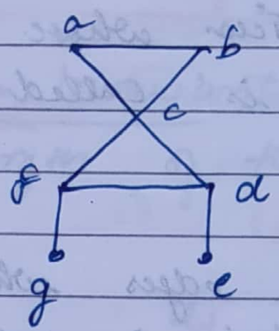




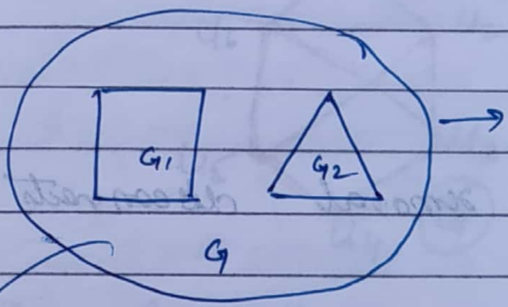
- If underline undirected graph of a directed graph is connected it is weakly connected graph.
- as we can go from each vertex to everywhere.

undirected connected graph

- Ex →



G_1

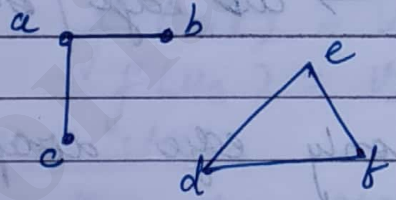


→ G is disconnected graph as there is no common vertex b/w G_1 & G_2.

no. of connected components = no. of disconnected components

→ there 2 connected components.

Ex →



→ 2 conneted components.

⇒ A connected graph always have 1 connected component

THEOREM

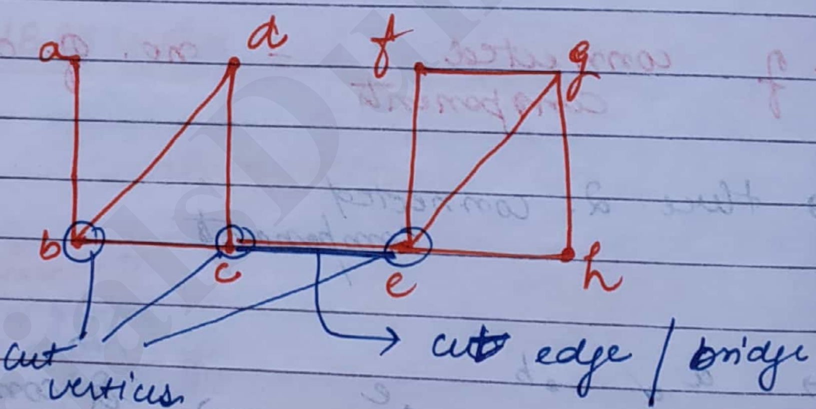
There is a simple path b/w every pair of distinct vertices of a connected undirected graph

- **Cut Vertex Set**
  Minimum set of vertices whose removal renders G disconnected is called cut vertex set.

- **Cut edge** → Min. no. of edges whose removal renders G disconnected is called cut edge. ~~vertex~~ ~~S~~

- **Bridge**
  single edge whose removal disconnects a graph.

Ex



- In cut edge, only edge disappears but not vertex.
- In cut vertex, vertex gets cut.

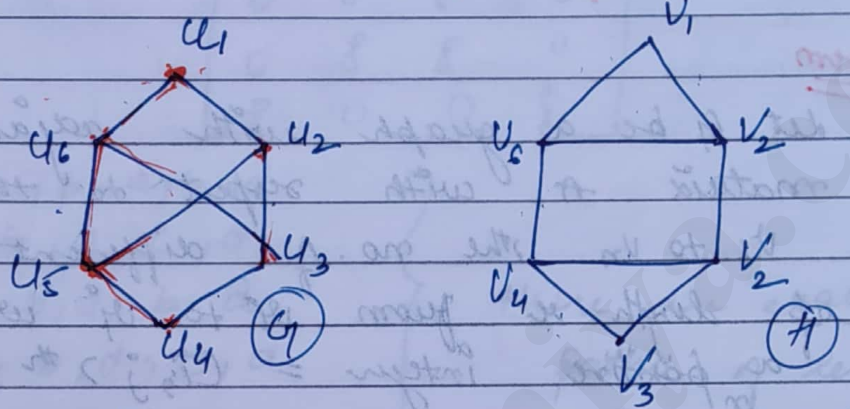
⟹ **Cut Set** → ~~this~~ cut edge set

Path & Isomorphism

Graph Invariants

→ length of the simple circuit

ex →



- 6 vertices
- 8 edges.
- same no. of vertices with a particular degree

Mapping,

$f(u_1) = v_1$ $f(u_4) = v_4$
$f(u_2) = v_2$ $f(u_5) = v_5$
$f(u_3) = v_3$ $f(u_6) = v_6$

But, still,
these two graphs are not isomorphic. as

By this Graph invariant,
H has a simple ~~to~~ circuit of length 3, $\{v_1, v_2, v_6, v_1\}$ whereas G has no simple circuit of length three

Hence, not isomorphic.

⇒ Counting Path b/w Vertices

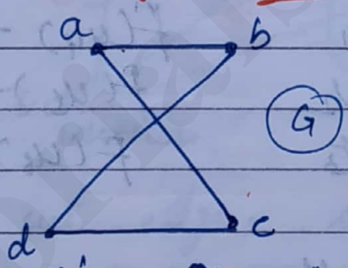
No. of paths b/w 2 vertices can be determined using its adjacency Matrix

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering v_1 to v_n the no. of different paths of length r from v_i to v_j where r is a positive integer = $(i, j)^{th}$ entry of A^r.

(See from book)

Ex How many paths of length 4 are there from a to d.



The adjacency Matrix of G is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Q-46 / 1, 92

Date : / /

Hence, the number of paths of length four from a to d is the (1,4) ~~th~~ entry of A^4. Bcoz

$$A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

$$\left.\begin{array}{l} \rightarrow a, b, a, b, d \\ \rightarrow a, b, a, c, d \\ \rightarrow a, b, d, b, d \\ \rightarrow a, b, d, c, d \\ \rightarrow a, c, a, b, d \\ \rightarrow a, c, a, c, d \\ \rightarrow a, c, d, b, d \\ \rightarrow a, c, d, c, d \end{array}\right\}$$

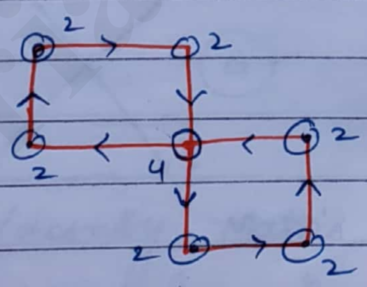
there are exactly 8 paths of length four from a to d

⇒ EULER PATHS / CIRCUITS

An Euler Circuit in a graph G is a simple circuit containing every edge of G, whereas

An Euler Path in G is a simple path containing every edge of G exactly once. It is also called Euler Trail.

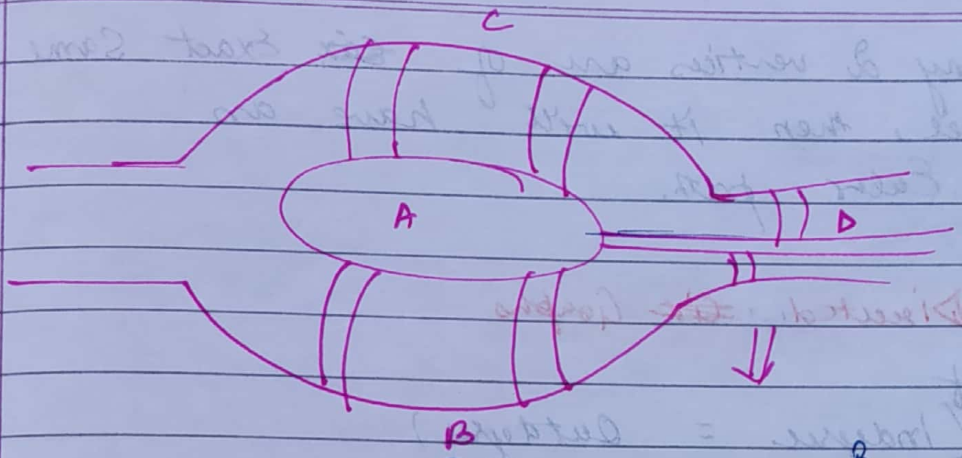
- Edges can't be repeated but vertices can repeat.
- If every vertex is of even degree, then only Euler circuit exists
- If there are exactly 2 vertices of odd degree then Euler paths ~~can~~ exists
-



Euler circuit

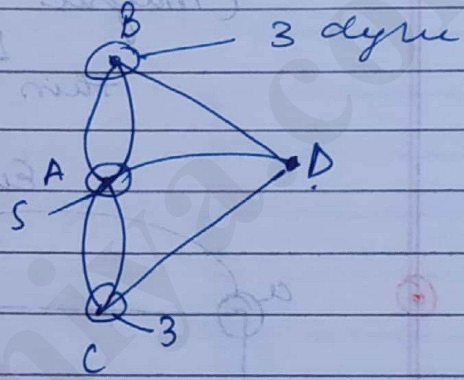
One degree for entrance, One for exit
∴ (even)

Q→

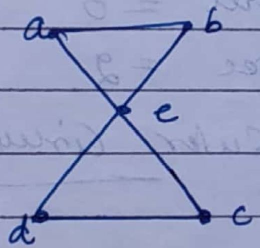


path

∴ Euler circuit/doesn't exist.

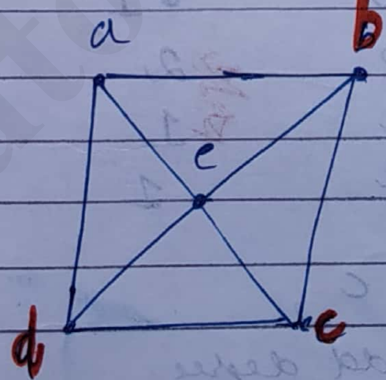


⇒ Q → (1)



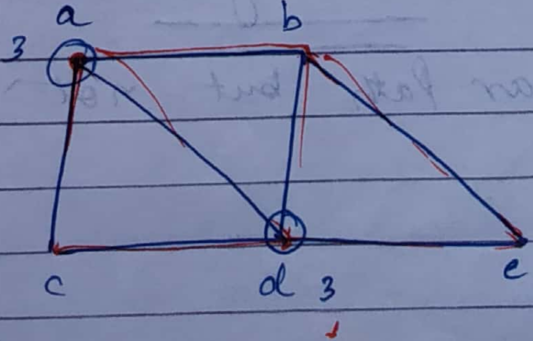
Euler circuit
(even degree)

(2)



not Euler circuit / path
(odd degree)

(3)



not Euler circuit ✗
but
Euler path ✓
(2 odd degree)

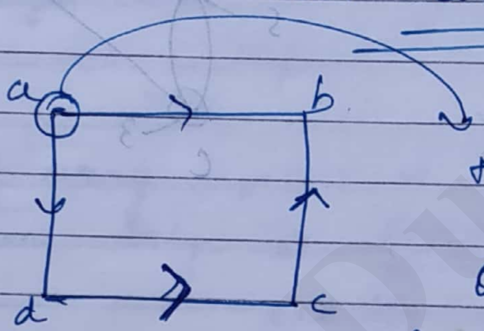
⇒ If any 2 vertices are of Exact Same odd degree, then it will have an Euler path.

⇒ For Directed Graphs

If (Indegree = Outdegree)

then, we have Euler circuits

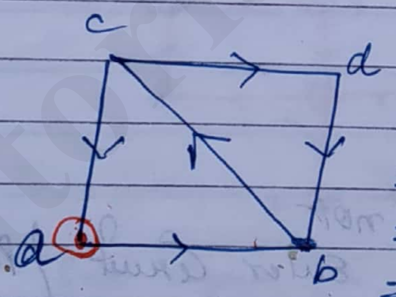
⊙



then
Indegree = 0
Outdegree = 2
∴ No Euler Circuit

Euler path → X

⊙



	Indegree	Outdegree
⇒ a	1	1
⇒ b	2	1
⇒ c	1	2
⇒ d	1	1

Here b and c have odd degree

∴ Euler Path but not Euler circuit.

Date : / /
Page No.

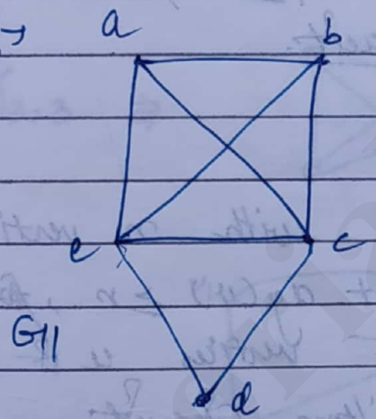
⇒ Hamilton Paths And Circuits

- A simple path in a graph G that passes through every vertex exactly once is called a Hamilton Path
  and
  a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

⇒ Icosian puzzle → dodecahedron

Ex →

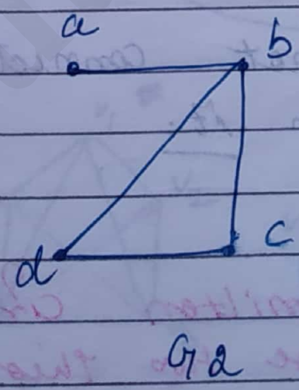
(1)



Hamilton Circuit

a, b, c, d, e, a

(2)



✗ Hamilton Circuit

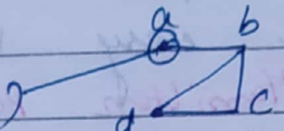
a, b, c, d — not to (a)

Date : / /
Page No.

⇒ a graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the graph circuit.

Ex →

(degree)



∴ ✗ Hamilton circuit

⇒ Complete graph, K_n, $n \geq 3$ is a Hamilton circuit.

DIRAC'S THEOREM

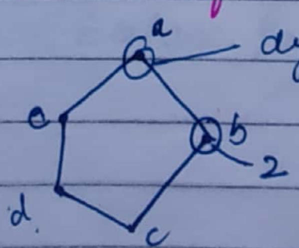
If G is a simple graph with n vertices with $n \geq 3$, such that degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

ORE'S THEOREM

If G is a simple graph with n vertices with $n \geq 3$, such that $deg(u) + deg(v) \geq n$, for every pair of non adjacent vertices u & v in G, then G has a Hamilton circuit.

⇒ A Hamilton circuit cannot connect a smaller circuit within it.

Graph of C_5 has an Hamilton circuit but does not satisfies, these two theorems.



acc. of Dirac's theorem

$2 \neq 5/2 = 2.5$

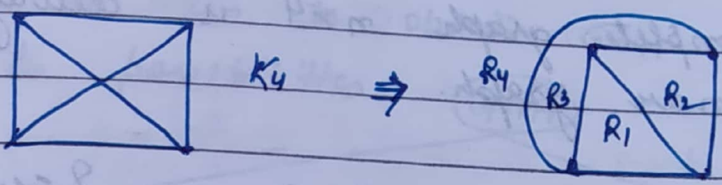
$2+2 \neq 5$ ∴ Ore's theorem ✗

Date : / /
Page No.

PLANAR GRAPHS AND BASIC TERMINOLOGY.

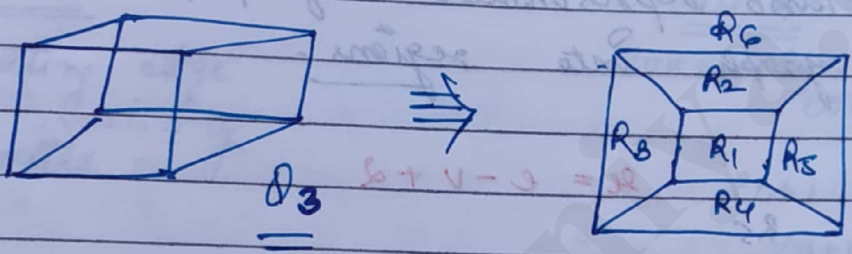
Any graph that can be drawn or redrawn without any edge crossing is called 'planar'.

(1)



Thus it is planar

(2)

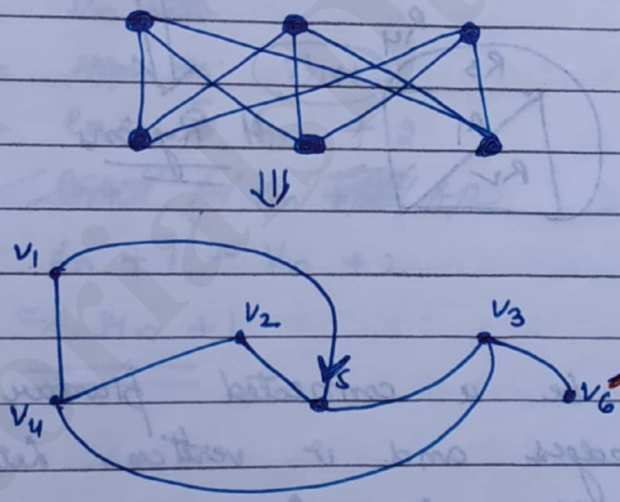


→ 6 Regions

Thus, it is planar.

(3) Not planar

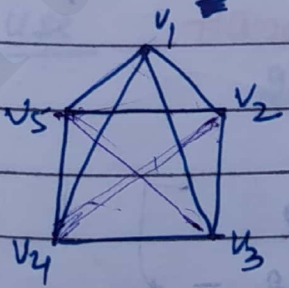
$K_{3,3}$ ⇒



→ can't connect to v_2

∴ It is not planar

(4)



not a planar graph

• Complete bipartite are non-planar graphs.

Exception → $K_{2,2}$ is a planar graph.

Date : / /
Page No.

EULER'S FORMULA

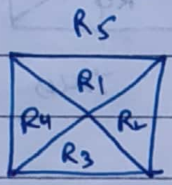
Corollary

Any graph which has $K_{3,3}$ and K_5 as subgraph will not be a planar graph.

→ Any complete graph $n > 4$ is always a non planar graph.

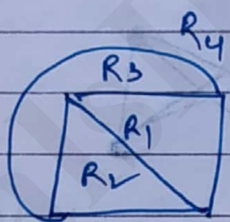
Imp Plano representation of Graph divides graph into regions.

$$r = e - v + 2$$



$= 6 - 4 + 2$

⇓



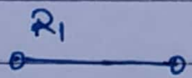
4 Regions

Theorem →

Let G be a connected planar Simple Graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then

Proof: By Induction, $r = e - v + 2$.

~~Basis~~ for $n = 1$, i.e. G_1



$e = 1$, $v = 2$

$\therefore r = 1 - 2 + 2 = 1$

Date : / /
Page No.

For G_n,

We assume that $r_n = e_n - v_n + 2$ is True ↳ ①

Now, let $\{a_{n+1}, b_{n+1}\}$ be the edge that is added to G_n to obtain G_{n+1}. There will be two possibilities

G_{n+1}

Case I

connecting edge to existing vertex

$$e_{n+1} = e_n + 1$$
$$V_{n+1} = V_n$$
$$\therefore r_{n+1} = r_{n+1}$$
$$= e_{n+1} - V_{n+1} + 2$$
$$= e_n + 1 - V_n + 2$$
$$= r_n + 1$$

Case 2

Take one vertex & connect to it

$$e_{n+1} = e_n + 1$$
$$V_{n+1} = V_n + 1$$
$$r_{n+1} = r_n$$
$$= e_n + \cancel{1} - V_n - \cancel{1} + 2$$
$$= r_n$$

⇒ Note

Use Handshaking Theorem
$$2e = \sum_{u \in V} (\deg)$$ for numerical based question.

Date : / /
Page No.

EULER'S FORMULA

sufficient condⁿ ⇒ not necessary

Corollary 1 :

If G is a connected simple planar Graph with ~~e~~ e edges & v vertices where $v \geq 3$, then ~~e~~ $e \leq 3v - 6$.

✓ **Corollary 2 :**

⇒ If G is a connected planar simple Graph, then G has a vertex of degree not exceeding 5.

Proof → of Corollary 2.

By corollar 1, ie. if G has atleast 3 vertices, then we have

$$e \leq 3v - 6$$

$$\overset{\times 2}{\Rightarrow} \quad 2e \leq 6v - 12 \rightarrow (1)$$

and if the degree of every vertex were at least six, then

by Handshaking theorem, we have

$$2e \geq 6v \rightarrow (2)$$

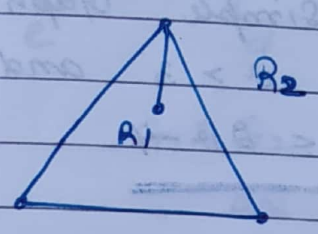
eqⁿ (1) & (2) contradict each other.

∴ corollary 2 is true

Date : / /
Page No.

Proof of corollary 1

Ex →



$R_1 = 5$ degree

r. 5 edges are present (since, we have to come at same pt. We traverse line twice)

$R_2 = 3$ degree

⇒ A connected planar simple graph drawn in the plane divides the plane into regions say r of them.

⇒ The degree of each region is at least 3

~~By Handshaking Theorem~~

$$2e = \sum_{\substack{\text{all} \\ \text{regions} \\ R}} \deg(R) \geq 3r$$

Hence, $(2/3)e \geq r$

using $r = e - v + 2$ (Euler's Formula)

$$e - v + 2 \leq (2/3)e$$

$$\Rightarrow \quad e/3 \leq v - 2$$

$$\Rightarrow \quad e \leq 3v - 6$$

Hence proved

⇒ K_5 is nonplanar by this corollary.

but not

$K_{3,3}$.

Date : / /
Page No.

Corollary 3

If a connected planar simple Graph has e edges and v vertices with $v \geq 3$ and no circuit of length three, then $e \leq 2v - 4$.

∵ There are no circuits of length three implies that the degree of a region must be atleast four.

i.e. $2e \geq 4r$

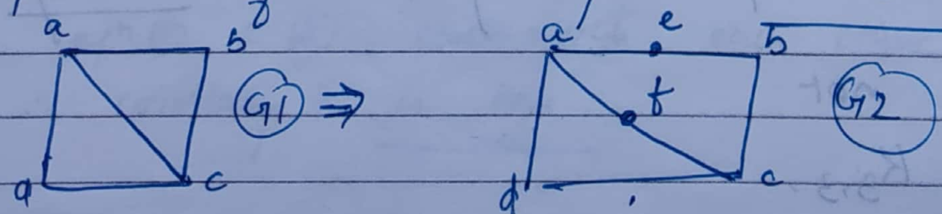
$\Rightarrow \boxed{e \leq 2v - 4}$

$K_{3,3}$ is nonplanar by this corollary.

⇒ Kuratowski's Theorem

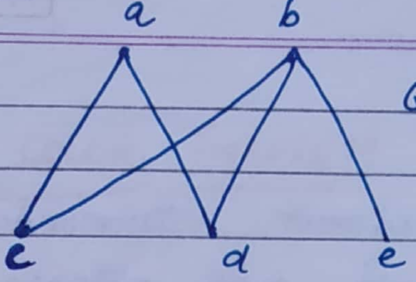
- If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ & $\{w, v\}$. Such an operation is called an Elementary Subdivision.

The Graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are called [homeomorphic] if they can be obtained from the same graph by a sequence of elementary subdivisions.

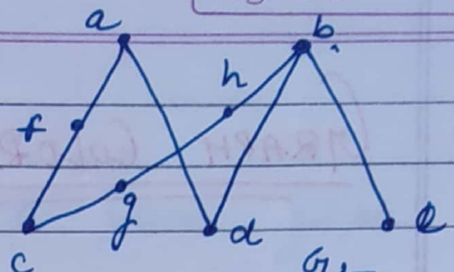


G2 is homeomorphic graph of G1

Ex →



G_1 ⇒



G_2

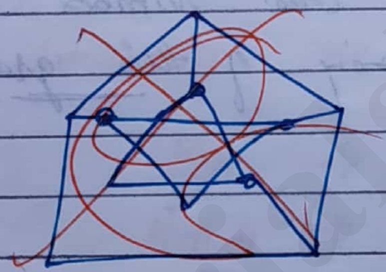
G_1 & G_2 ~~is~~ ~~homeographs~~ are homeomorphic Graphs

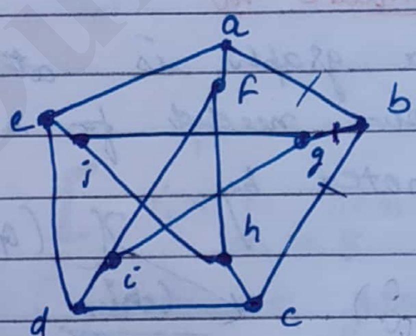
Theorem

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Ex →

Petersen Graph

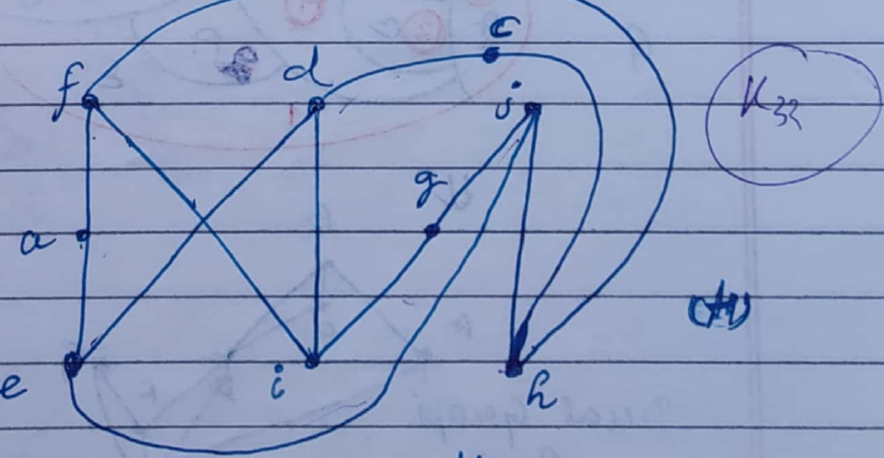




(G)

H is subgraph of G

by removing three edges ~~at~~ from (b)



∴ Non planar

H is Homeomorphic to $K_{3,3}$

{f, d, j} and {e, i, h}

Date : / /
Page No.

⇒ GRAPH COLORING

→ Dual Graphs

Representing a ~~graph~~ map by a graph

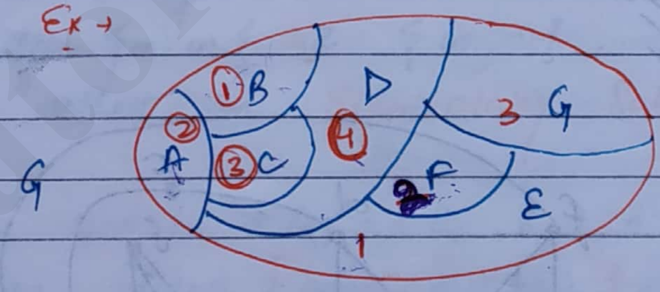
- A colouring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Chromatic no

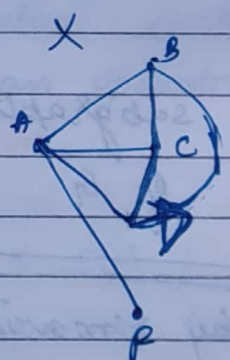
of a graph is ~~at~~ the least number of colours needed for a coloring of this graph.
Denoted by

$\chi(G)$ (chi) (Kai)

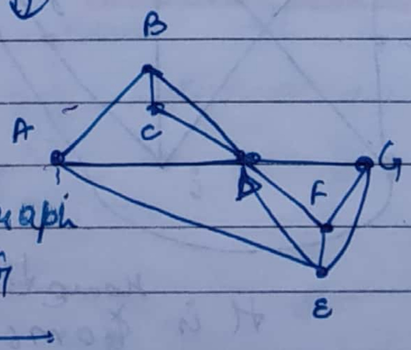
Ex →



→



⇓



Dual Graph of G

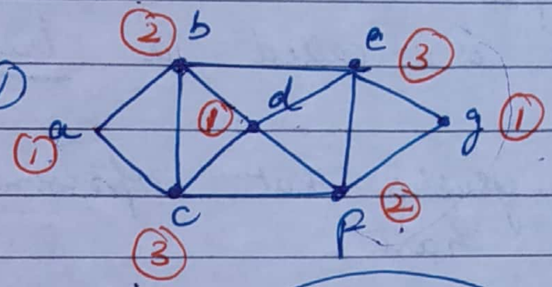
Date : / /
Page No.

Theorem

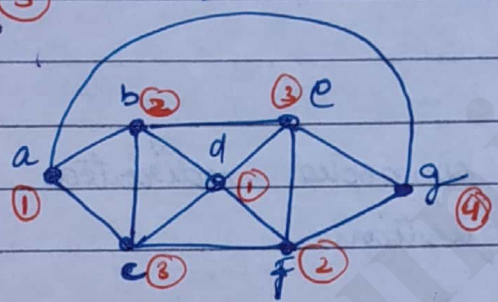
FOUR COLOR THEOREM

The chromatic number of a planar graph is no greater than four.

Ex → (1)



(2)



⇒ chromatic no. of

$K_n \to n$ [K_n is non planar, $n \geq 5$]

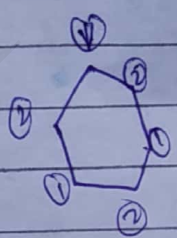
$K_{m,n} \to 2$ (Bipartite graph)

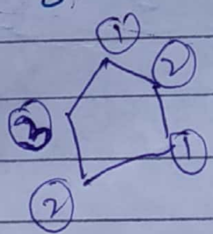
$n \geq 3$, $C_n \to n$ (vertices)

even ⇓ 2

odd ⇓ 3

[$K_{3,3}$ non planar]





Date : / /
Page No.

TREES

{ data structures
used in encryption and decryption
of Huffman coding.

★ A connected graph that contains no simple circuits is called a tree.

- family trees are graphs that represent genealogical charts.

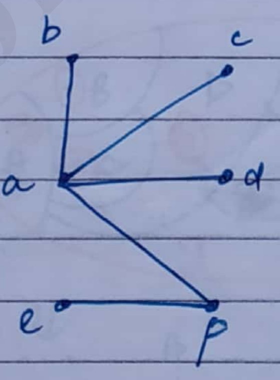
Definition

- A tree is a connected undirected graph with no simple questions.

Trees are often defined as undirected graphs with the property that there is a unique simple path between every pair of vertices.

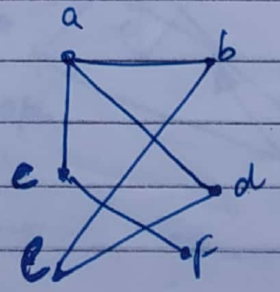
Ex →

(1)



✓ Tree

(2)



X Tree

{a, b, e d, a}
is a simple circuit.

Logics

$\boxed{EQ_{16}}$ $p \to q \equiv \sim p \vee q$

Conditional $p \to q \equiv \bar{q} \to \bar{p}$ contrapositive
converse $q \to p \equiv \bar{p} \to \bar{q}$ inverse

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

→ Modus Ponens
$(p \to q,\ p) \to p$

~~Modus~~

→ Modus Tollens
$(p \to q,\ \bar{q}) \to \bar{p}$

→ (conjuction)
$(p, q) \to p \wedge q$

→ disjunctive syllogism
$(\bar{p},\ (p \vee q)) \to q$

→ Hypothetical syllogism
$(p \to q,\ q \to r) \to (p \to r)$

→ Demorgan's law :
$\overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$

→ Absorption law
$p \vee (p \wedge q) \equiv p$

(1.7 - 1.17)

Logic

* **Propositions** Statements or declarative sentences that are either true or false.

Propositional values: (T), (F) — (P) : Ram is a boy (atomic Proposition)

Proposition variable that denotes the preposition.

* **Logical connectives**
Atomic propositions through logical connectors are combined together to form compound propositions.
4 logical connectors →
→ And ($\wedge$) → Not ($-$)
→ Or ($\vee$) → Exclusive OR $\oplus$

p	q	$p \vee q$	$p \wedge q$	$\bar{p}$	$p \oplus q$
T	T	T	T	F	F
T	F	T	F	F	T
F	T	T	F	T	T
F	F	F	F	T	F

⇒ **Equivalent Propositions.**
When 2 propositions have same truth value.

Ques Construct truth table for $(p \wedge q) \wedge \bar{p}$

$(p \wedge q) \wedge \bar{p}$ = all false

↳ proposition which is all false is called a contradiction.

$(p \wedge \bar{p})$

* all true – tautology $(p \vee \bar{p})$
* combination of both → contigency

⇒ Conditionals & Biconditional

- Conditional

$p \rightarrow q$

(i.e. (p is necessary for q, q is sufficient for p)

Hypothesis/ antecedent — consequence/ conclusion — (not compulsory)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

doesn't matter if it is true or not.

eg → p : temp. exceeds 70°c
q : alarm will be sounded.

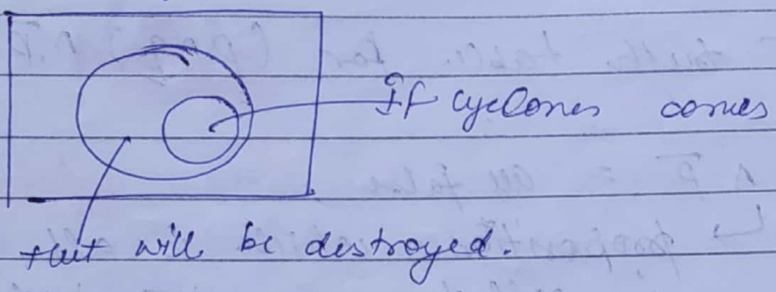
∴ $p \rightarrow q$ ⇒ if temp. exceeds 70°, then alarm will be sounded.

Ques The hut will be destroyed if there is a cyclone

$p \rightarrow q$

p :→ There is a cyclone
q :→ hut will be destroyed.

Venn diagram



Imp

$$\boxed{p \to q \equiv \underset{\text{not}}{\sim} p \vee q}$$

Conditional $(p \to q)$

- Converse $(q \to p)$
- Inverse $(\sim p \to \sim q)$
- Contrapositive $(\sim q \to \sim p)$

p	q	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

$\Rightarrow$ $p \to q$

contrapositive : $\sim q \to \sim p$

$$p \to q \equiv \sim q \to \sim p$$

$\cancel{q \vee \sim p}$

$\Rightarrow$ $q \to p$

Contrapositive :- $\sim p \to \sim q$

$$q \to p \equiv \bar{p} \to \bar{q}$$

Ques $p \to q$: If it rains today, I will go to college tomorrow.

p : rain today
q : college tomorrow

$q \to p$: If I go to college tomorrow, It would have rained today.

contrapositive →
$\sim q \to \sim p$: If I don't go to college tom, It wouldn't have rained today

BICONDITIONAL

$p \leftrightarrow q$
(p if and only if q)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Imp $\boxed{p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)}$

eg → He will swim if & only if water is warm.

Ques p : a new computer will be acquired
q : additional funding is available

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

⇒ Well formed formula

use of proper braces & logical connectors form well formed formula.

eg →

$$p, \bar{p}, p \to q, p \vee \bar{p}, ((p \vee \bar{p}) \wedge r)$$

$p \wedge q \vee r$ ✗ → not a well formed formula.

Note →

- conjuction of 2 tautologies is a tautology.
- If we negate variables of a tautology, then it is a tautology.

eg → $\overline{p \vee \bar{p}} = \bar{p} \wedge p$ (contradiction.)

$\bar{p} \vee p = \bar{p} \wedge p$ (tautology)

(Negation of individual variables)

⇒ Substitution Instance

eg →

$x + 3 + y = 4x$

Now $x = y + 3$

we can replace x by $y+3$

x is not sub^n instance

eg → $((p \wedge q) \vee r) \to p$

$p \equiv p \to r$ → Substitution example

$(((p \to r) \wedge q) \vee r) \to (p \to r)$

Note

Always replace atomic variables for subⁿ instance rather than replacing compound variables.

$p \wedge q \equiv p \to r$ ✗ (wrong)

A substitution is a syntatic transformation on common expression. To apply a substitution to an expression, we have to consistently replace its variable by other expressions. The resulting expression is called the substitution instance.

eg → $p \vee \bar{p}$ (tautology)

$p \equiv p \to q$

$(p \to q) \vee (\overline{p \to q})$

∴ Subⁿ instance formed is also a tautology

→ LOGICAL EQUIVALENCES

Ques Prove $((p \wedge q) \vee (p \wedge r)) \to s$ is equivalent to $(\bar{p} \vee (\bar{q} \wedge \bar{r})) \vee s$.

We know,

$$p \to q \equiv \sim p \vee q$$

$$= \overline{((p \wedge q) \vee (p \wedge r))} \vee s$$

$$= ((\overline{p \wedge q} \wedge (\overline{p \wedge r})) \vee s$$

$$(\text{wrong}) = ((\bar{p} \vee \bar{q}) \wedge (\bar{p} \vee \bar{r})) \vee s$$

$$= (\bar{p} \vee (\bar{q} \wedge \bar{r})) \vee s$$

Ques There are 2 hotels ~~near~~ opp. to each other.
1st says good food is not cheap.
2nd say cheap food is not good.

g = good food

c = food is cheap.

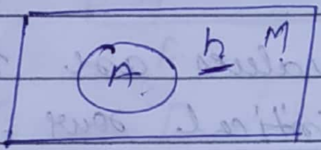
$1^{st} \to g \to \bar{c}$, $2^{nd} : c \to \bar{g}$

Both are logically equivalent

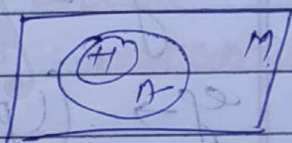
⇒ Concept of Validity & Soundness of argument

→ An arguement is valid if all the premises are true & the conclusion that are desired is also true, i.e. it is impossible for the conclusion to be false if premises are true.

eg → A is human
All humans are mortal
↓
A is mortal (conclusion



eg → A is mortal
All humans are mortal
→ A is human
(invalid)



→ When premises are wrong, conclusion is valid, it is unsounded but valid.

eg → A is a square
all sq. are dogs → (Premise is wrong)
↓
A is a dog (valid but unsounded)

⇒ Theory of Inference

deals with derivation of conclusion from a set of premises by using the standard rule of (interference) is called formal proof or deduction. (inference)

Note Using truth table, we can't draw conclusions but can only validate them.
Using theory of inference, we can do both.

If $(A_1, A_2, \ldots A_n) \rightarrow C$

i.e. these must be tautology as all premises must be true so that conclusion is valid & sounded.

$$(A_1 \wedge A_2 \ldots \wedge A_n) \rightarrow C$$

⇒ When premises ~~are~~ values are true, they are called critical rows.

⇒ Using Truth table Method

eg → $((p \rightarrow q), (q \rightarrow r)) \rightarrow (p \rightarrow r)$

① Identify no. of variables and draw table.

(consistent)

p	q	r	p → q	q → r	p → r
T	T	T	T	T	T ✓
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T ✓
F	T	F	T	F	T
F	F	T	T	T	T ✓
F	F	F	T	T	T ✓

{Inconsistent (contradiction)}

When critical rows are all T., & concl^n is F.

2. Identify critical rows for premises.
3. Check whether conclusions is true for all critical method

Ans Hence conclusion is valid through T.T method.

eg 1.30 (book)

$(p \rightarrow q, p \vee q, \bar{q}) \rightarrow q$

p	q	$p \rightarrow q$	$p \vee q$	$\bar{q}$
T	T	T	T	F
T	F	F	T	T
F	T	T	T	F
F	F	T	F	T

Since, there is no critical ~~point~~ row, then we don't need to check anything and hence it is automatically valid.

⇒ Rules of Inference

1. Rule P → In derivation process, we can pick any premise.

2. Rule T →

eg → $((p \rightarrow q), (q \rightarrow r)) \rightarrow (p, r)$

If implication of $p \rightarrow q$ and $q \rightarrow r$ is $p \rightarrow r$, then conclusion can be inserted in place of $p \rightarrow q$ ~~&~~ or $q \rightarrow r$.

3. Rule CP →

eg → $(A_1 \ldots \ldots A_n) \wedge p \rightarrow q$

(A₁ … An): given set of premise; p: assumed premises

(conditional proof)

used in cases when conclusion is of form $p \rightarrow q$.

- premise 1 : $p \to q$
  2 : p
  ---
  q
  law of modus ponens

- $p \to q$
  $\sim q$

  $\sim p$
  (modus tollens)

- $((p \to q), (q \to r)) \to (p, r)$

- $p \vee q$
  p
  q

- $p \vee q$
  $p \to r$
  $q \to r$
  ---
  r

- $p \vee q$
  $\bar{p}$
  ---
  q

Ans → Show that q is valid from
$(p \to q, p \vee q, \bar{q}) \to q$

{1} (1) $p \to q$ Rule p

{2} (2) $\bar{q}$ Rule p

premise {3} {1, 2} (3) $\bar{p}$ Rule t, (1), (2) modus tollens

premise no. {4} (4) $p \vee q$ Rule p

{1, 2, 4} (5) q Rule t, (3), (4), disjunctive syllogism

(since premise 3 is used which is composed of {1, 2})

Ques $[(\overline{p \wedge q}), \bar{q} \vee r, \bar{r}] \longrightarrow \bar{p}$

{1} (1) $\overline{p \wedge q}$ Rule p

$\bar{p} \vee q \to p \to q$ (rule of equivalence)

{2} (2) $\bar{q} \vee r \longrightarrow q \to r$

$p \to q,\ q \to r$

$p \to r$

$\bar{r}$

$\bar{p}$

Ques $\begin{pmatrix} \sim A \to (C \wedge D) \\ A \to B \\ \sim B \end{pmatrix} \longrightarrow C$

E.Q.16 →

$\bar{A} \vee B \longrightarrow A \to B$ $\quad p \to q \equiv \sim p \vee q$

$\bar{B}$ $\Rightarrow$ Modus tollens

$\bar{A}$

$\bar{A} \to C \wedge D$

$\bar{A}$

$C \wedge D$ — C, D

Derivation

a → $(\bar{p} \vee q,\ \bar{q} \vee r,\ r \to s) \to (p \to s)$

1st method → $\bar{p} \vee q \equiv p \to q$

$\bar{q} \vee r \equiv q \to r$

$r \to s$

$p \to s$

2nd method → (using rule CP)

$\bar{p} \vee q \equiv p \to q$

p ← (additional premise)

q

$\bar{q} \vee r \equiv q \to r$

q

r

$r \to s$

r

s

$\therefore p \to s$ (Rule CP)

⇒ Consistency of premises

⇒ eg → $(p \to q,\ p \to r,\ q \to \bar{r},\ p)$

Show that given system is inconsistent.

(i) $p \to q$
p
$\overline{q}$

(ii) $p \to r$
p
$\overline{r}$

$q \wedge r$ (using rule of conjuction)

(iii) $q \wedge r$
$q \to \bar{r}$
$\overline{\bar{q} \vee \bar{r}} \equiv \overline{q \wedge r}$

Now $(q \wedge r) \wedge \overline{(q \wedge r)}$ = false (inconsistent)

Rule of conjuction

⇒ eg → $(r \to \bar{q},\ r \vee s,\ s \to \bar{q},\ p \to q) \to \bar{p}$

$((r \to \bar{q}),\ r \vee s,\ s \to \bar{q},\ p \to q) \wedge p \to p \wedge \bar{p}$

consider an additional premise.

(i) $p \to q$
p
$\overline{q}$

(ii) $s \to \bar{q}$
q
$\overline{\bar{s}}$ Modus tollens
$r \vee s$
$\overline{r}$
$(r \to \bar{q})$

We get contradiction here

Modus tollens
$p \to q$
$\bar{q}$
$\overline{\bar{p}}$

$p \vee q$
$\bar{p}$
$\overline{q}$ disjunctive

q is imp.

(3) $\dfrac{\bar{r} \vee \bar{q}}{r \wedge q}$

$r \wedge q$ is $\bar{r} \wedge q \wedge (\overline{\bar{r} \wedge \bar{q}})$

p	q	r	s	$r \to \bar{q}$	$r \vee s$	$p \to q$	$s \to \bar{q}$
T	T	T	T	F	T	T	F
T	T	T	F	F	T	T	T
T	T	F	T	T	T	T	F
T	T	F	F	T	F	T	T
T	F	T	T	T	T	F	T
T	F	T	F	T	T	F	T
T	F	F	T	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	F	T	T	F
F	T	T	F	F	T	T	T
F	T	F	T	T	T	T	F
F	T	F	F	T	F	T	T
F	F	T	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T
F	F	F	F	T	F	T	T

- Since value of p for critical rows becomes false, thus we obtain a contradiction.

or system is invalid / inconsistent.

→ **Predicate Calculus** →

based on propositions

(Propositional func.)

stmt. function.

A(x) : x is an animal (predicate)

Predicate

(Predicate variable)

Note

If x gets substituted with any value, like cow, dog, then it will become a proposition.

2 place predicate G(x, y) x is greater than y

Since, use of 2 variables, thus it is called predicate of 2 var.

- Predicates can be combined using logical connectors.

compound stmt. functions $\boxed{V(x) \vee M(y)}$ → combination of 1 place predicates.

Note →

predicates give generalisation of propositions.

- $P(n_1, n_2, n_3 \ldots n_3)$ → n place predicates

⇒ Stmt. obtained by stmt. function is Substitution instance of stmt. func.

eg → x is an animal

Dog is an animal (substitution instance)

* Quantifiers

(Universal) — $\forall$

Existential [$\exists$]

eg → All dogs are animals.
(Universal fact)
for all x, if x is a dog, then x is an animal
$D(x) \rightarrow$ Dogs
$A(x) \rightarrow$ Animals.

$$\boxed{\forall x, (D(x) \rightarrow A(x))}$$

use of conditional connector while using universal quantifier.

eg → Some roses are red.
$\exists(x)\ (R(x) \wedge C(x))$
(use of intersection connector with existential)

Ques Some nos are irrational.
$\exists(x)\ (N(x) \wedge I(x))$

- **Well formed formula**

- **Free & bound Variables**

find scope of quantifier.

eg → $(x)\ \boxed{M(x)}$ — Scope of (x)

$\exists(x)\ (\underbrace{M(x) \wedge N(x)})$ → scope of $\exists(x)$

eg → $m(x) \wedge \exists(x)\ \underline{N(x)}$ → scope of $\exists(x)$ (bound variable)

where no quantifier is used, it is free variable

V.imp

Bound variable can be replaced by Variable but not by a constant.

(x) → for all

$(x)\ M(x,y)$ — scope of (x)

↓

$(x)\ M(z,y)$ (Replacing x by z)

⇒

1. Someone in ur school has visited Agra.
   $\exists(x)\ (S(x) \wedge A(x))$

2. Mary sees everyone.

   Since there is a const. term Mary :-

   $\forall y\ S(M,y)$ — M: Mary, S: sees, y: everyone

   M → It is a const. not a variable.

- If mary sees someone
  $\exists(x)\ S(M,x)$

⇒ **Universalf discourse** →

restriction on domain, talking of a specific domain.

$(x)\ D(x) \rightarrow A(x)$

here Domain is applicabell for all dogs only.

eg →

$P(x)$: x is greater than 2

$(x)\ P(x)$, $\exists(x)\ P(x)$

⇒ **NOTE**

since, universe of discourse has been given in question, do write with universe of discourse.

Q →

$p \rightarrow q$
$q \rightarrow r$
$\underline{p}$
r

I	II	III	IV
{1}	(1)	$p \rightarrow q$	Rule P
{2}	(2)	p	Rule P
{1, 2}	(3)	q	Rule T, (1), (2), modus ponens.
{4}	(4)	$q \rightarrow r$	Rule P
{1, 2, 4}	(5)	r	Rule T, (3), (4), modus ponens

Ans

$(C \vee D)$, $(C \vee D) \rightarrow \bar{H}$

$\bar{H} \rightarrow (A \wedge \bar{B})$

$(A \wedge \bar{B}) \rightarrow (R \vee S)$

Tell that $R \vee S$ is concl^n

$C \vee D \rightarrow \bar{H}$

$C \vee D$

$\bar{H}$

$\bar{H} \rightarrow A \wedge \bar{B}$

$\bar{H}$

$A \wedge \bar{B}$

$A \wedge \bar{B} \rightarrow R \vee S$

$\Rightarrow R \vee S$

Ques $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$
Prove $S \vee R$ is conclusion.

$\boxed{P \vee Q \equiv \sim P \rightarrow Q}$

$P \vee Q$

$\bar{P} \rightarrow Q$

$Q \rightarrow S$

$\bar{P} \rightarrow S$

$\downarrow$

$\sim S \rightarrow P$

$P \rightarrow R$

$\Rightarrow -S \rightarrow R$

$\hookrightarrow$ $S \vee R$

Logically equivalent

$\boxed{\bar{P} \rightarrow S \equiv \sim S \rightarrow P}$

Ques $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$ and $\bar{M}$
conc. $R \wedge (P \vee Q)$

$P \rightarrow M$

$\bar{M}$

$\bar{P}$

[Modus Tollens]

$$P \vee Q \equiv \bar{P} \rightarrow Q$$

$$\frac{\bar{P}}{Q}$$

$Q \rightarrow R$

$\frac{Q}{R}$

2 premises can be combined by using n

$\therefore (R \wedge P \vee Q)$

Note →

If conclusion is of the form $p \rightarrow q$
∴ according to rule cp. p is assumed premise

Ques → Derive using C.P rule

$(\bar{P} \vee q,\ \bar{q} \vee r,\ r \rightarrow s) \rightarrow (p \rightarrow s)$

Refer from back

Ques → Conc. → $R \rightarrow S$

Premise → $P \rightarrow (Q \rightarrow S)$

$\bar{R} \vee P$

Q

assumed premise R

$$\overline{R} \vee P \equiv R \to P$$

$$\frac{R}{P}$$

$$P \to (Q \to S)$$

$$\frac{P}{\Rightarrow Q \to S}$$

$$Q$$

$$\Rightarrow S \to \text{Result}$$

Applicative Example

(1) to identify atomic premises.
(2) Create arguements.
(3) Validate

eg →(1) If there was a ball game, then travelling was difficult
(2) If they arrived on time, then travelling wasn't difficult.
(3) They arrived on time.
concl → ~~They~~ There was no ball game

Soln

$$P \to Q$$

$$R \to Q'$$

$$\frac{R}{P}$$

Now derive

eg →

P1: If A works hard, then either B and C enjoy themselves.

P2: If B enjoys, then A will not work hard.

P3: If D enjoys, then C will not.

Concl^n : If A works hard, D will not enjoy.

$a \to (b \vee c)$

$b \to a'$

$d \to c'$

$a \to d'$ (cond^n)

(atomic premise)

* Indirect method to prove system is inconsistent :-

(to validate)

$(A_1, A_2 \ldots\ldots, A_n) \wedge \bar{B} \longrightarrow B \wedge \bar{B}$

add^n premise is taken and system reaches to a contradiction.

Q ⇒ $(\bar{p} \wedge \bar{q}) \to (\overline{p \wedge q})$

assumed $\to P \wedge Q$

$(\bar{P} \wedge \bar{Q}) \wedge (P \wedge Q) \longrightarrow (\bar{P} \wedge \bar{Q}) \wedge (P \wedge Q)$

$P \wedge Q$

$\hookrightarrow P$ (simplification)

$\bar{P} \wedge \bar{Q}$

$\downarrow$

$\bar{P}$

$\therefore$ from P & $\bar{P} \longrightarrow P \wedge \bar{P}$

contradiction.

NOTE →
additional premise is negation of the conclusion.

Q → Prove it is inconsistent

(i) If Jack misses many classes through illness, then he fails high school

(ii) If Jack fails H.S, then he is uneducated

(iii) If Jack reads a lot of books, then he is not uneducated.

(iv) Jack misses many classes through illness & reads a lot of books.

⇒ $p \to q$
$q \to r$
$s \to r'$
$p \wedge s$

1st Method

$p \wedge s$
↓
p
$p \to q$
q

$s \to r'$
s (from $p \wedge s$)
r'

$q \to r$
q
r

⇒ $r \wedge r'$ (contradiction)

2nd method

$p \to q$
$q \to r$
$p \to r$

$s \to r' \equiv r \to s'$

$p \to s'$
$\overline{p} \vee \overline{s} = (\overline{p \wedge s})$

⇒ $(\overline{p \wedge s}) \wedge (p \wedge s)$
⇓
contradiction

(a) $\{-5, -3, 0, 1, 2\}$ (c) $\{-1, 0, 2, 6\}$

(b) $\{3, 5, 7, 10\}$ ✓

given universe discourse → (a int greater than 2)

Satisfies both $\forall(x) P(x)$ & $\exists(x) P(x)$

These there is atleast 1 value that verifies $\exists(x) P(x)$

⇒ NESTED QUANTIFIERS

$\forall x \ \exists y \ S(x, y)$ → (Everyone sees someone)

$\forall y \ \exists x \ S(x, y)$ → (For every value of y go through loop of x)

Everyone has been seen by someone

Q → Let $Q(x, y)$ be the statement x has send an email message to y, where domain for both x & y consists of all students in your class. Express each of the following quantifiers in English :-

(a) $\exists x \ \exists y \ Q(x, y)$
Someone has send email to someone
(There are some students)

(b) $\exists y \ \forall x$
Someone has received email from everyone