Entanglement entropy in non-Hermitian Spin chain system

https://github.com/Ak-ash22/non_hermitian_xxzspin_chain

A Project submitted to the School of Physics, University of Hyderabad

in partial fulfillment for the award of degree of

Master of Science in Physics

By

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Declaration

I hereby declare that, this project entitled **'Entanglement entropy in non-Hermitian Spin chain system'** is carried out by me at the School of Physics, University of Hyderabad under the supervision of Prof. V. Subrahmanyam.

No part of this project has been previously submitted for a degree or diploma or any other qualification at this University or any other.

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Certificate

This is to certify that, this project entitled **'Entanglement entropy in non-Hermitian Spin chain system'** is carried out by Akash M (21PHMP10) under my supervision at the School of Physics, University of Hyderabad, in partial fulfillment of the requirements for the award of the degree of **Master of Science in Physics**. No part of this project has been previously submitted for a degree or diploma or any other qualification at this university or any other.

Prof. V. Subrahmanyam
Project Supervisor

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Abstract

Non-Hermitian (NH) quantum systems have attracted research interest in recent years, among which the $\mathcal{PT}$-symmetric systems are focused more. Motivated by recent advances in this field, this paper analyzes the behaviour of such systems by taking an example. I have used the recently developed theory of 'biorthogonal quantum mechanics' in case for finite dimensional Hilbert space to study the properties of a NH system. In this paper, an interacting NH system is investigated for the presence of exceptional points and the region of $\mathcal{PT}$-symmetry unbroken and $\mathcal{PT}$-symmetry broken phases. The system is also studied for the properties of entanglement entropy and entropy dynamics in $\mathcal{PT}$ symmetry unbroken phase.

Keywords: Non-Hermitian system, exceptional points, $\mathcal{PT}$-symmetry unbroken phase, entanglement entropy, entropy dynamics.

1 Introduction

The Hermiticity condition in standard quantum mechanics is required for the characterisation of physical observables. The key mathematical ingredients required to represent physical observables are that the eigenvalues are real, and that eigenstates are complete. In study of non-hermitian systems, the notion of orthogonality of eigenstates of the observable can be relaxed and substituted by a weaker requirement of biorthogonality. There is a substantial amount of literature on relaxing the notion of Hermiticity requirment. For example, Scholtz *et al.* [7] established a general criterion for a set of non-Hermitian operators, which allows for the normal quantum-mechanical interpretation by the construction of a nontrivial **metric** (if it exists). Bender and others[1] have developed PT-symmetric theory where Hermiticity condition is replaced by the invariance under simultaneous parity and time reversal condition. In this paper, I have adopted the account of Dorje C. Brody approach[2] i.e., the formalism of biorthogonal quantum mechanics.
We shall review basic properties of generic complex Hamiltonians in finite dimensions. Consider $\hat{K}$ to be complex hamiltonian with eigenstates $\{|\phi_n\rangle\,;|\chi_n\rangle\,\}$ and non-degenerate (assuming) eigenvalues $\{\kappa_n;v_n\}$:

$$\hat{K}\,|\phi_n\rangle = \kappa_n\,|\phi_n\rangle\,and\,\langle\phi_n|\,\hat{K}^\dagger = \bar{\kappa_n}\,\langle\phi_n| \tag{1}$$

$$\hat{K}^\dagger\,|\chi_n\rangle = v_n\,|\chi_n\rangle\,and\,\langle\chi_n|\,\hat{K} = \bar{v_n}\,\langle\chi_n| \tag{2}$$

where $\hat{K}^\dagger$ is hermitian adjoint. Here, the eigenstates $|\chi_n\rangle$ are introduced as the eigenstates $|\phi_n\rangle$ are in general not orthogonal. With the help of this conjugate basis $|\chi_n\rangle$, normalization condition is deduced as,

$$\langle\chi_n|\phi_m\rangle = \delta_{nm}\langle\chi_n|\phi_n\rangle \tag{3}$$

Now if we consider an arbitrary state $|\psi\rangle$ and associated state $|\tilde{\psi}\rangle$ i.e.,

$$|\psi\rangle = \sum_n c_n\,|\phi_n\rangle \iff \langle\tilde{\psi}| = \sum_n \tilde{c_n}\,\langle\chi_n| \Rightarrow |\tilde{\psi}\rangle = \sum_n c_n\,|\chi_n\rangle \tag{4}$$

where $|\psi\rangle \in \mathcal{H} \iff |\tilde{\psi}\rangle \in \mathcal{H}^*$. In other words, the state dual to $|\psi\rangle$ is given by $\langle\tilde{\psi}|$. The inner product for such biorthogonal system is thus defined as: If $|\psi\rangle = \sum_n c_n\,|\phi_n\rangle$ and $|\Phi\rangle = \sum_n d_n\,|\phi_n\rangle$, then

$$\langle\Phi,\psi\rangle \equiv \langle\tilde{\Phi}|\psi\rangle = \sum_{n,m} \tilde{d_n} c_m \langle\chi_n|\phi_m\rangle = \sum_n \tilde{d_n} c_n \tag{5}$$

According to the norm defined before, the transition probability between $|\psi\rangle$ and $|\phi_n\rangle$ is defined as,

$$p_n = \frac{\langle\chi_n|\psi\rangle\langle\tilde{\psi}|\phi_n\rangle}{\langle\tilde{\psi}|\psi\rangle\langle\chi_n|\phi_n\rangle}. \tag{6}$$

Further, for the set of fixed biorthogonal basis $\{|\phi_n\rangle\,;|\chi_n\rangle\,\}$, any operator is expressed as,

$$\hat{F} = \sum_{n,m} f_{nm} |\phi_n\rangle \langle\chi_m| . \tag{7}$$

The expectation value of the observable $\hat{F}$ in a pure state $|\psi\rangle$ is defined as,

$$\langle\hat{F}\rangle = \frac{\langle\tilde{\psi}| \hat{F} |\psi\rangle}{\langle\tilde{\psi}| \psi\rangle} \tag{8}$$

The state of physical system which is more commonly characterised by density matrix as,

$$\hat{\rho} = \sum_{n,m} \rho_{nm} |\phi_n\rangle \langle\chi_m| . \tag{9}$$

Here, density matrix $\hat{\rho}$ is not hermitian in orthogonal basis but it is hermitian with respect to biorthogonal basis $\{|\phi_n\rangle ; |\chi_n\rangle\}$. The expectation value of an observable in state $\hat{\rho}$ is defined as,

$$\langle\hat{F}\rangle = tr(\hat{\rho}\hat{F}) = \sum_{n} \langle\chi_n| \hat{\rho}\hat{F} |\phi_n\rangle = \sum_{n,m} \rho_{nm} f_{mn}. \tag{10}$$

2 The non-Hermitian XXZ spin chain

I now report the study of the XXZ spin chain model in a linear chain with periodic boundary condition and an additional non-Hermitian character added at the centre of the chain. The total hamiltonian for a simple three spin system is given as follows:
Consider a spin chain system of length three labelled as A—C—B, then

$$H_C = ih_z\sigma_c^z + h_x\sigma_c^x$$

$$\begin{aligned} H_{AC} &= \sigma_A^x\sigma_C^x + \sigma_A^y\sigma_C^y + \Delta\sigma_A^z\sigma_C^z \\ H_{CB} &= \sigma_C^x\sigma_B^x + \sigma_C^y\sigma_B^y + \Delta\sigma_C^z\sigma_B^z \\ H_{BA} &= \sigma_B^x\sigma_A^x + \sigma_B^y\sigma_A^y + \Delta\sigma_B^z\sigma_A^z \\ H &= H_C + H_{AC} + H_{CB} + H_{BA} \end{aligned}$$

This system is indeed $\mathcal{PT}$-symmetric i.e., $[H,\mathcal{PT}] = 0$, where parity operator $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\mathcal{T}$ is time conjugation operator given by complex conjugation. Few of the observations one can infer from the above hamiltonian are, the parameter h_z can be considered as the degree of non-Hermiticity since H is hermitian iff $h_z = 0$, and real eigenvalues exist within the range where $h_x > h_z$.

2.1 Exceptional points and PT symmetry unbroken phase

The above hamiltonian H is investigated for the region of $\mathcal{PT}$ unbroken phase[6]. Keeping $h_x = 1$ fixed, the spectrum of H as a function of h_z/h_x is studied for various values of Δ (see fig 1). It is observed that in the case of Δ = 1 (isotropic antiferromagnet), $\mathcal{PT}$ unbroken phase is maintained by the system in the region of $h_z/h_x \in$ (-1,+1) (see fig 1). This region remains unperturbed even when the system size is increased (see fig 2a). In other words the first order exceptional points (EP)[4][3] of the system can be observed at $h_z = h_x$ when it is in a isotropic antiferromagnetic phase irrespective of length of the system. Another observation to be noted is that, for spin chain systems with even number of spins, the case of Δ = -1 (isotropic ferromagnet), also shows $\mathcal{PT}$ unbroken phase in the same region of $h_z/h_x \in$ (-1,+1) (see fig 2) which is not observed in spin chain systems with odd number of spins. The fig 2b shows that as h_x is increased, the region of $\mathcal{PT}$ unbroken phase widens for a given value of Δ.

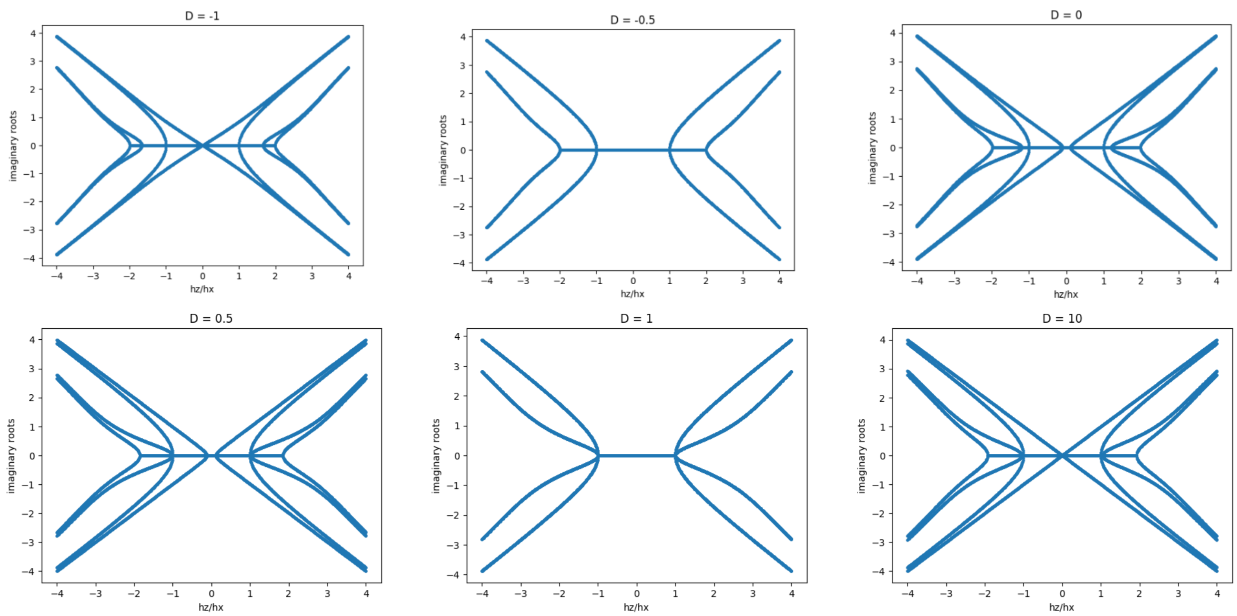

Figure 1: Spectrum of H as a function of hz/hx for system size n = 3.

2.2 Entanglement entropy

Let $|\psi_i\rangle$'s ($|\phi_i\rangle$'s) be eigenstates of hamiltonian H ($H^\dagger$) and satisfy the condition of biorthogonal norm (eqn 3). The biorthogonal density matrix for the ground state of the system is constructed with dual eigenvectors corresponding to the least energy eigenvalue according to eqn 9. Furthermore, the reduced density matrix of the subsystem with non-hermitian character (subsystem C) is found by partial tracing rest of the system with respect to a set of biorthogonal basis. Thus, a set of biorthogonal basis $\left\{ |A_l\rangle \, ; |C_j\rangle \, ; |B_k\rangle \right\}$ and associated dual space vectors $\left\{ |\tilde{A_i}\rangle ; |\tilde{C_j}\rangle ; |\tilde{B_k}\rangle \right\}$ are constructed such that any eigenvector of H ($H^\dagger$) can be represented as,

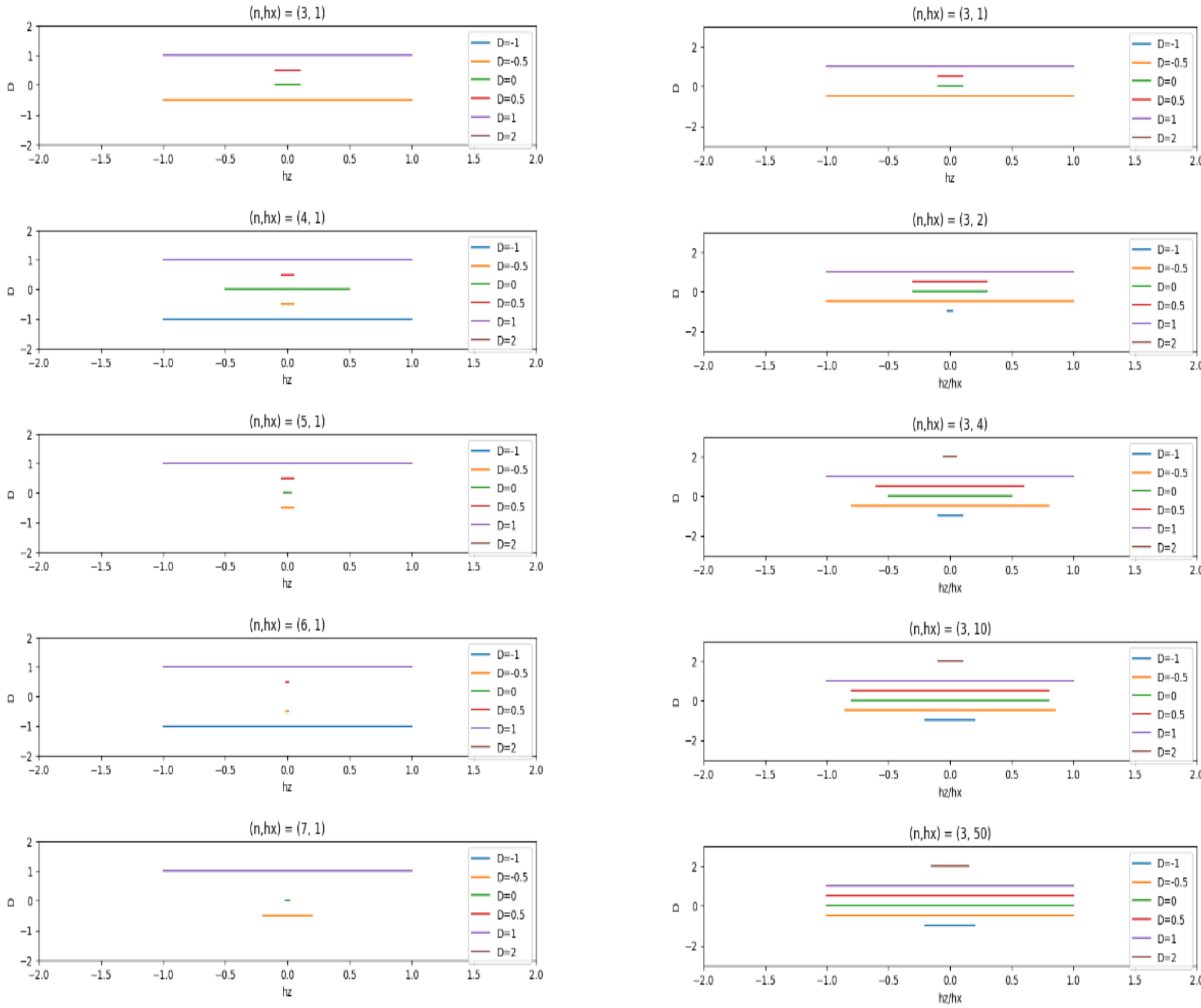

Figure 2: PT symmetry unbroken region (**a**) Keeping $hx = 1$, and increasing the system size (**b**) Keeping system size fixed n = 3 and increasing hx

$$|\psi_{GS}\rangle = \sum_{i,j,k} \alpha_{ijk} |A_i\rangle \otimes |C_j\rangle \otimes |B_k\rangle \tag{11}$$

$$|\phi_{GS}\rangle = \sum_{i,j,k} \alpha_{ijk} |\tilde{A}_i\rangle \otimes |\tilde{C}_j\rangle \otimes |\tilde{B}_k\rangle. \tag{12}$$

The choice of $|A_i\rangle$ and $|B_k\rangle$ can be taken to be set of computational basis $\{|0\rangle\, ; |1\rangle\,\}$, while $|C_j\rangle$ needs to be a biorthogonal basis set $\{|C_a\rangle\, ; |\tilde{C}_b\rangle\}$. Now, the reduced density matrix of subsystem C is given by,

$$\rho_C = Tr_{AB}\rho = \sum_{i,k} \langle \tilde{A}_i I \tilde{B}_k| \, \rho \, |A_i I B_k\rangle \tag{13}$$

where I is identity. The reduced density matrix elements can be written as,

$$(\rho_C)_{ab} = \sum_{i,k} \langle \tilde{A}_i \tilde{C}_a \tilde{B}_k | \, \rho \, | A_i C_b B_k \rangle \tag{14}$$

The von-Neumann entanglement entropy of the subsystem C can be calculated by,

$$S_c = -\sum_k \lambda_k \log_2 \lambda_k \tag{15}$$

where λ_k are eigenvalues of ρ_C. This entanglement entropy is studied as a function of h_z/h_x in $\mathcal{PT}$ unbroken phase for a fixed Δ (see fig 3). Observations to be noted are, in case of systems with 2n number of spins in Δ = -1 phase, the entanglement entropy S_c is fixed at 0.5 and shows no change as h_z/h_x is varied irrespective of system size (see fig 3b). Also, for systems with both 2n and 2n+1 (greater than 3) number of spins S_c varies with h_z/h_x and its behaviour is studied (see fig 3a). And the least value of S_c the system can have is 0.5. The entanglement entropy is also investigated as a function of length of the system in fig 4 and it is believed to saturate at higher lengths.

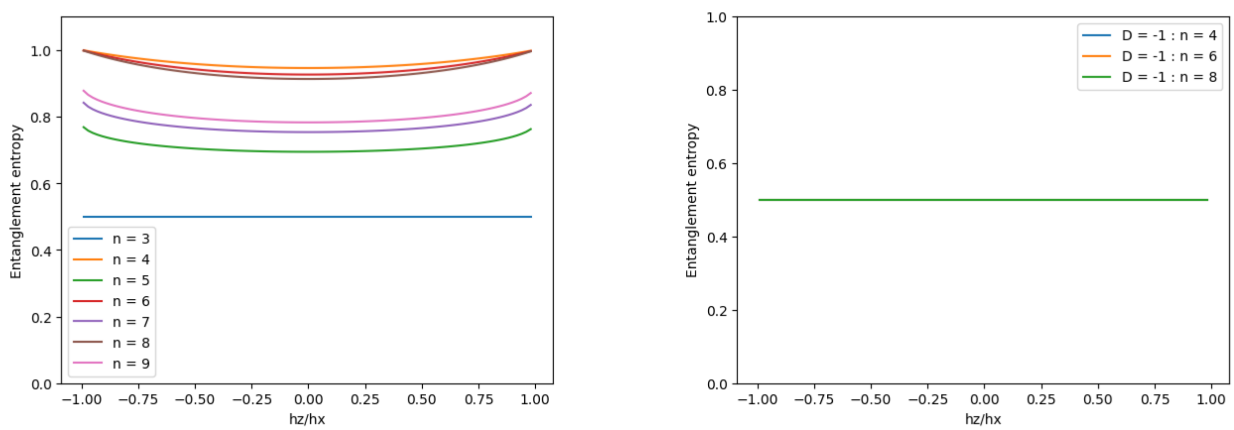

Figure 3: Entanglement entropy as a function of hz/hx (**a**) plot is for Δ = 1 (Isotropic antiferromagnetic phase) for system of size 2n, (**b**) plot is for Δ = -1 (Isotropic ferromagnetic phase) where each line represents non-Hermitian system of corresponding size.

2.3 Time evolution and entropy dynamics

The dynamic evolution of a non-Hermitian system initialized in a state $|\psi\rangle$ is governed by,

$$|\psi(t)\rangle = e^{(-iHt/\hbar)} |\psi\rangle \tag{16}$$

$$\langle\psi(t)| = \langle\psi| e^{(+iH^\dagger t/\hbar)} \tag{17}$$

The corresponding density matrix evolution on setting the norm $\langle\psi(t)|\psi(t)\rangle = 1$, is given by

$$\rho(t) = |\psi(t)\rangle \langle\psi(t)| = e^{-i(H-H^\dagger)t/\hbar} |\psi\rangle \langle\psi| = e^{-i(H-H^\dagger)t/\hbar} \rho \tag{18}$$

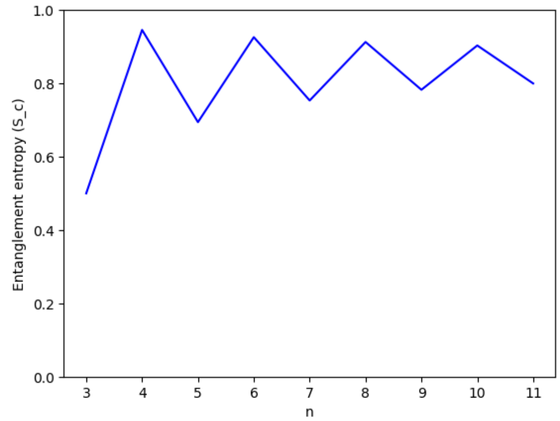

Figure 4: Entanglement entropy of non-Hermitian system for fixed value of parameter $\Delta = 1$ as a function of system size n in $\mathcal{PT}$ symmetry unbroken phase.

Further, the reduced density matrix corresponding to subsystem C is obtained as discussed before and corresponding von-Neumann entanglement entropy dynamics for the system initialized in state $|111\rangle$ is studied in $\mathcal{PT}$ unbroken phase for different values of hz for $\Delta = 1$. The entropy dynamics at $hz = 0$ is plotted so as to get an idea of how non-Hermiticity affects the entropy dynamics. The same is also studied for extended system size of up to 8 spins. For the particular value of hz the entropy dynamics can also be compared to the ground state entropy from the previous section. The observation to be noted is that the entropy shows a kind of oscillating behaviour with respect to time.

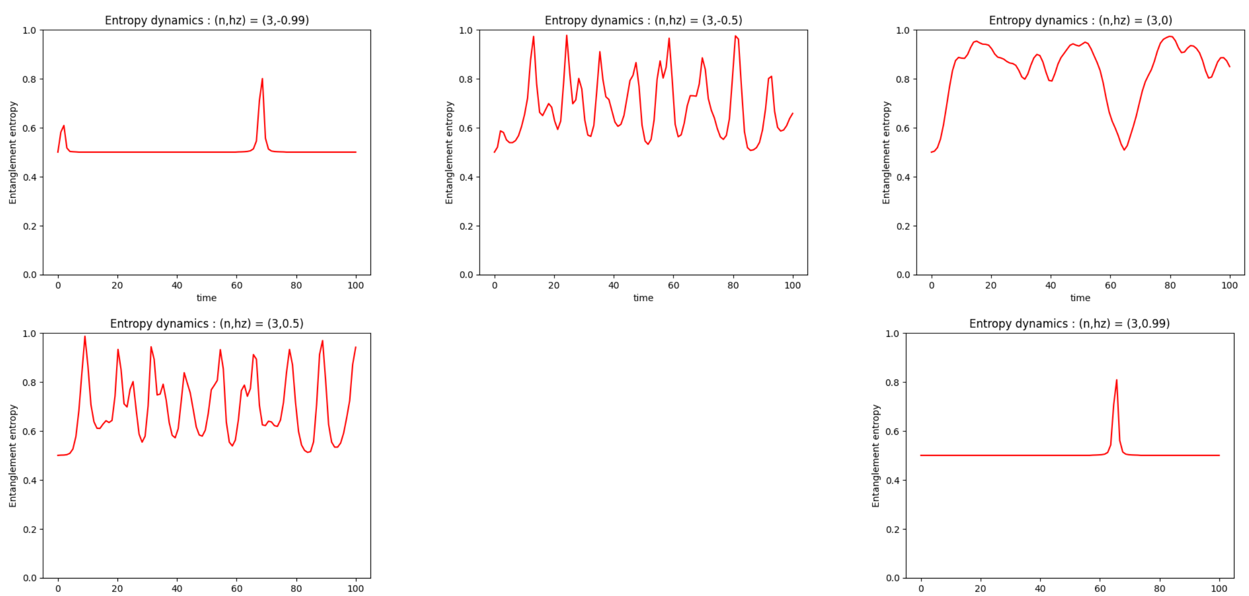

Figure 5: Entropy dynamics of the system of size n = 3 for different values of degree of non-hermiticity (hz) in $\mathcal{PT}$-symmetry unbroken phase keeping $hx = 1$ and $\Delta = 1$.

3 Conclusion

In this paper, I have worked on an example of non-Hermitian systems, which is a hot area of interest since recent years. Unlike the $\mathcal{PT}$ symmetry approach, on which a substantial amount of work has been done before[1][5], I have used the biorthogonal quantum mechanics approach and presented the results here. This approach is elucidated for the reader in section 1. The system I have chosen is the XXZ spin chain model in a linear lattice chain with a non-hermitian character added at its center spin. The system is studied for the presence of PT-symmetry unbroken and PT-symmetry broken phases by finding the presence of exceptional points which is shown in section 2.1. Further, I have investigated entanglement entropy and its dynamics in PT - unbroken phase. The observations are shown in sections 2.2 and 2.3. All the above computational work has been done in Python programming language and with the help of basic packages like Numpy and Scipy. Various numerical techniques like DMRG, variational matrix product state approach, and QMC method are widely used to study such systems. I have used the exact diagonalization technique to access the system's information. The algorithms used here might be less efficient since they have high computational cost. These algorithms though I was able to extend for up to the system of size eight, could be made more efficient by using high performance numerical libraries like JAX and other built-in python quantum toolboxes like QuTip, QuSpin for studying systems of higher length.

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