small wave

LECTURE 06 WAVELETS AND MULTIRESOLUTION PROCESSING

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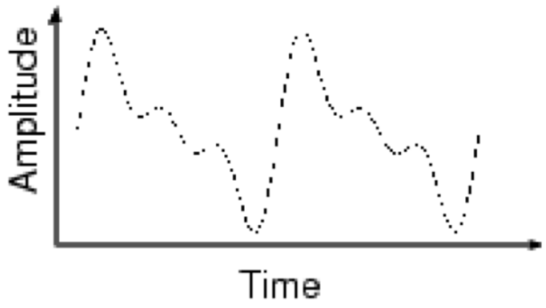
OBJECTIVES

- To be able to understand the basic concept of wavelets (Haar wavelet)
- To be able to apply wavelets transform for multiresolution analysis

CONTENT

- Introduction
- Image Pyramid
- Fast Wavelet Transform (1D & 2D)
- Python Exercises

INTRODUCTION



each pixel have impact to every freq so
FT give a global information

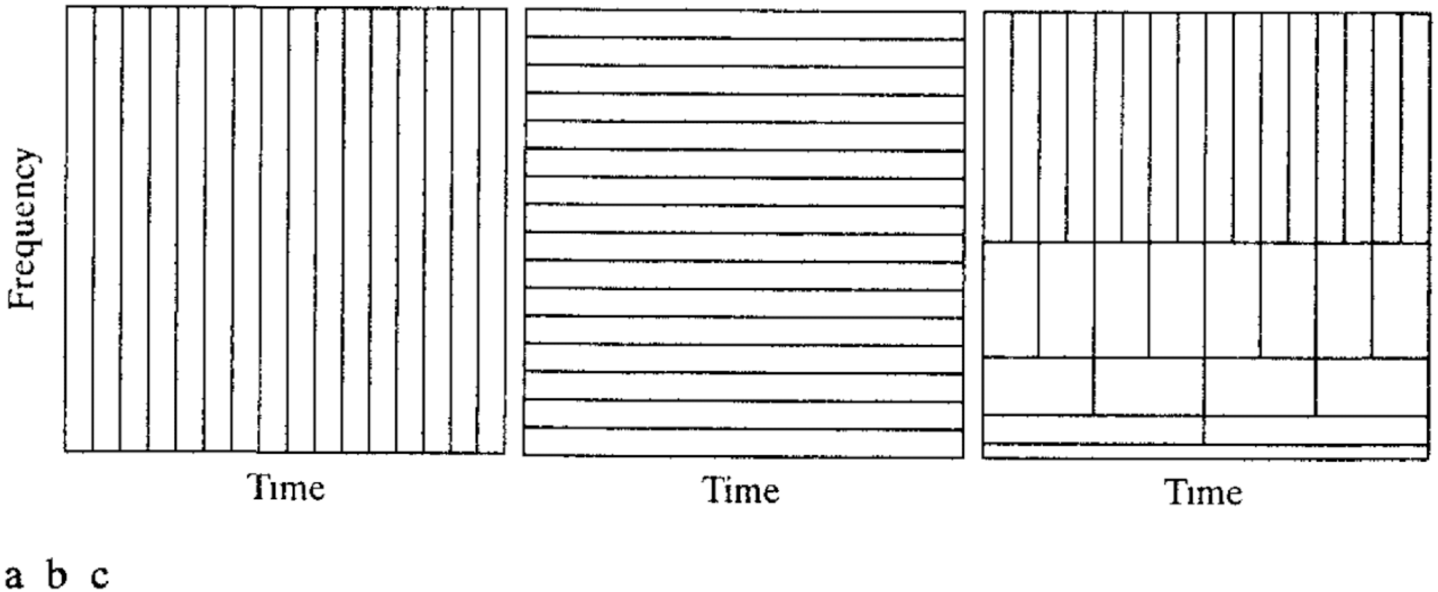
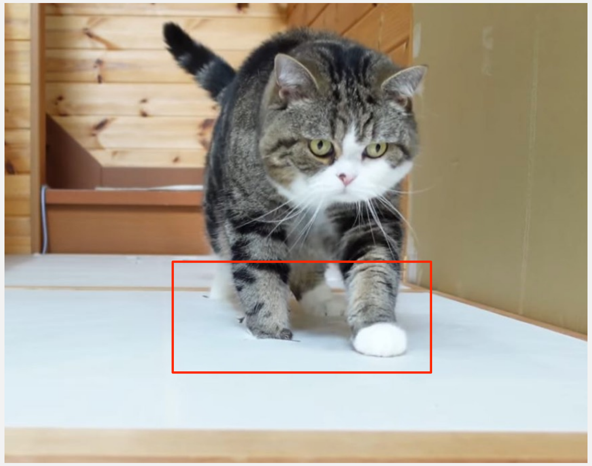


FIGURE 7.21 Time-frequency tilings for (a) sampled data, (b) FFT, and (c) FWT basis functions.

INTRODUCTION

- Small object, low contrast – make it higher resolution
- Large object, high contrast – look at coarse view
- Motivation for multiresolution processing : wavelet
  - All of them together – study them in several resolutions



แฟนคลับแมวกล่อง

INTRODUCTION

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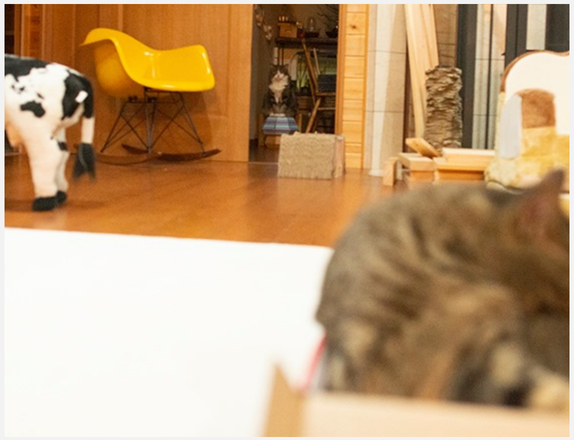
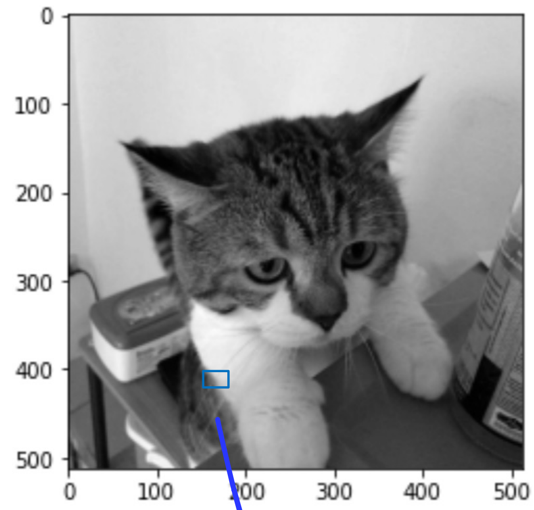
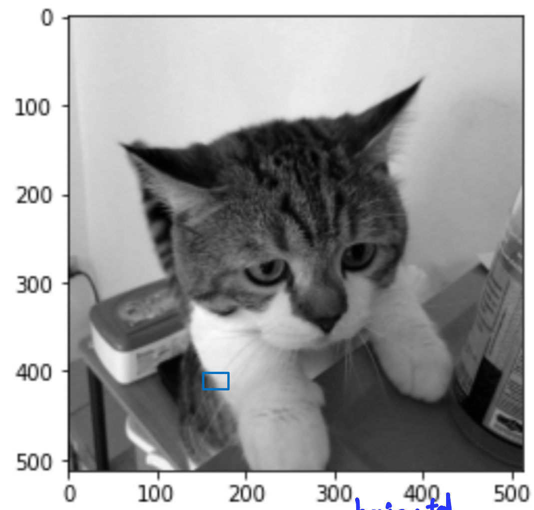


แฟนคลับแมวกล่อง




```
[[ 67  66  70  87  98 114 138 147 151 166 180 183 180 181 186 185]
 [ 56  61  75  87  96 110 127 132 138 155 174 179 178 180 184 184]
 [ 57  68  85  88  89  96 108 115 124 140 156 168 174 177 178 176]
 [ 55  65  77  77  77  84  98 111 122 132 149 163 171 176 179 179]
 [ 48  53  63  63  70  83  97 106 114 122 136 149 157 165 172 175]
 [ 43  50  57  57  65  80  89  90  98 112 118 134 149 162 171 172]
```



horizontal

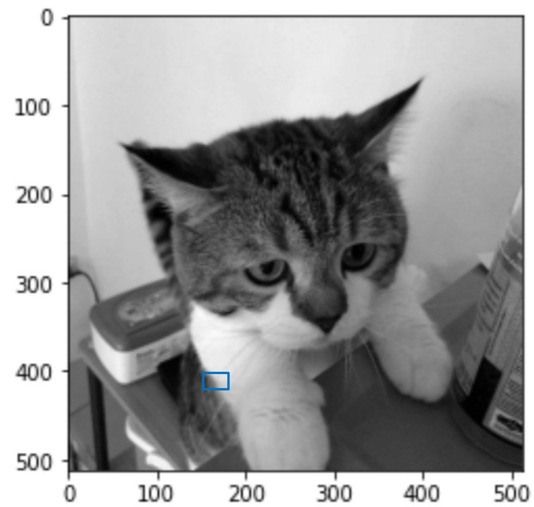
```
[[ 67  66  70  87  98 114 138 147 151 166 180 183 180 181 186 185]
 [ 56  61  75  87  96 110 127 132 138 155 174 179 178 180 184 184]
 [ 57  68  85  88  89  96 108 115 124 140 156 168 174 177 178 176]
 [ 55  65  77  77  77  84  98 111 122 132 149 163 171 176 179 179]
 [ 48  53  63  63  70  83  97 106 114 122 136 149 157 165 172 175]
 [ 43  50  57  57  65  80  89  90  98 112 118 134 149 162 171 172]
```

average

difference

88.5

-1

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 [ 55  65  77  77  77  84  98 111 122 132 149 163 171 176 179 179]
 [ 48  53  63  63  70  83  97 106 114 122 136 149 157 165 172 175]
 [ 43  50  57  57  65  80  89  90  98 112 118 134 149 162 171 172]
```

average

88.5	102
77	91
66.5	90
61	84.5

difference

-1	-12
0	-14
-7	-14
-8	-9

88	89	96	108
77	77	84	98
63	70	83	97
57	65	80	89

vertical

average

88.5	102
77	91
66.5	90
61	84.5

difference

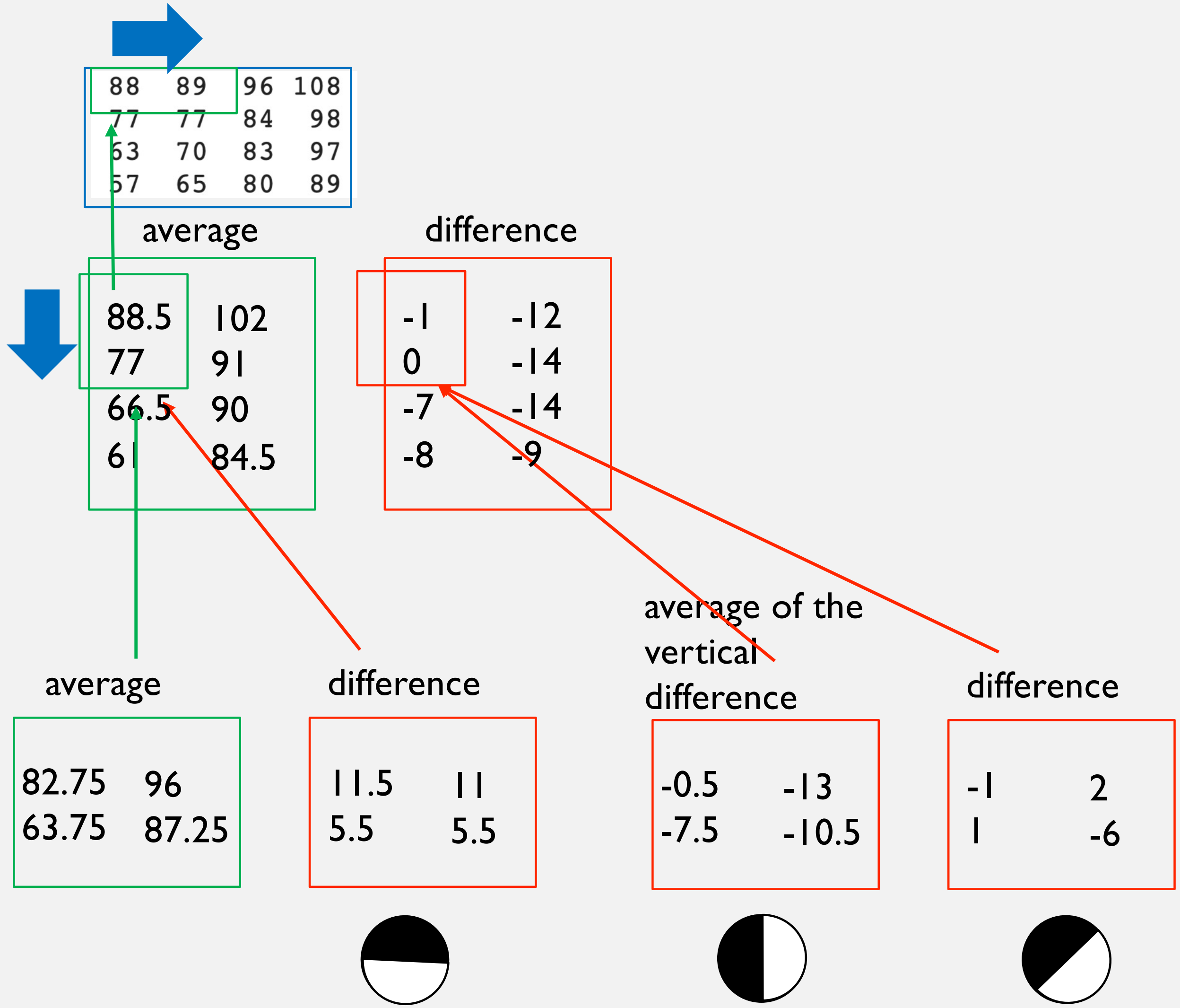
-1	-12
0	-14
-7	-14
-8	-9

average

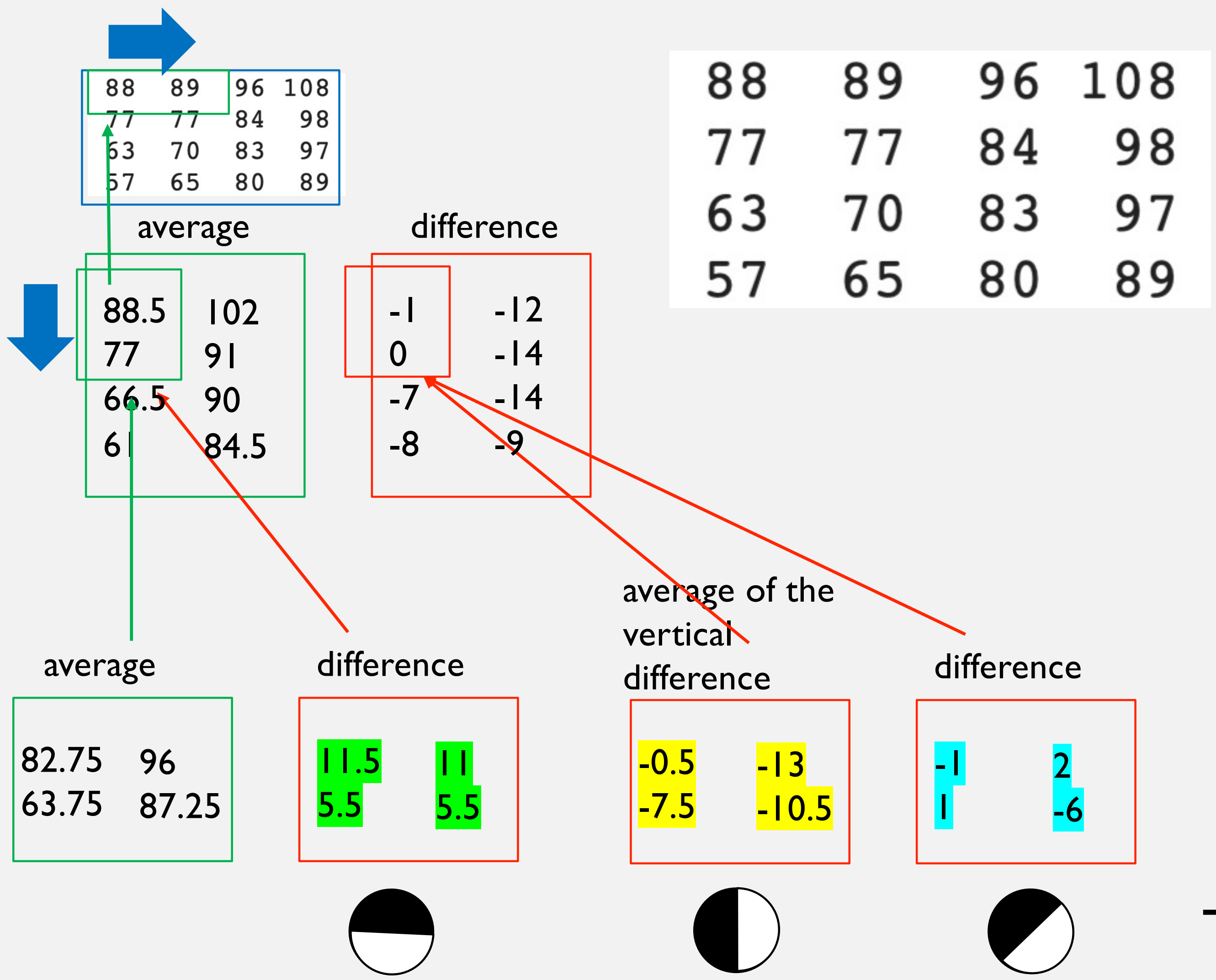
82.75	96
63.75	87.25

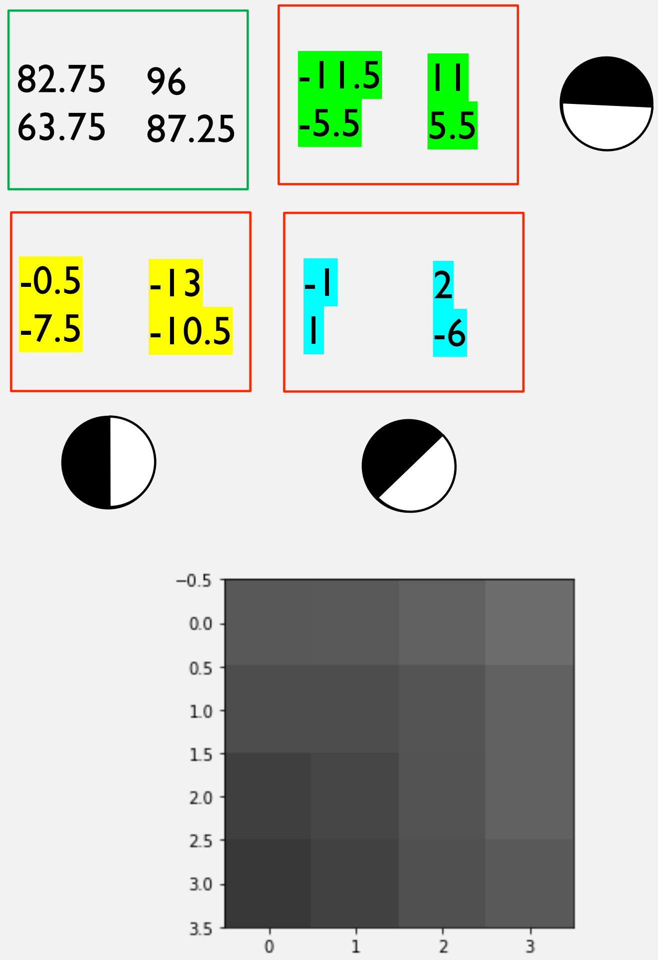
difference

11.5	11
5.5	5.5



Three species of differences





Three species of differences

Three species of differences

88	89	96	108
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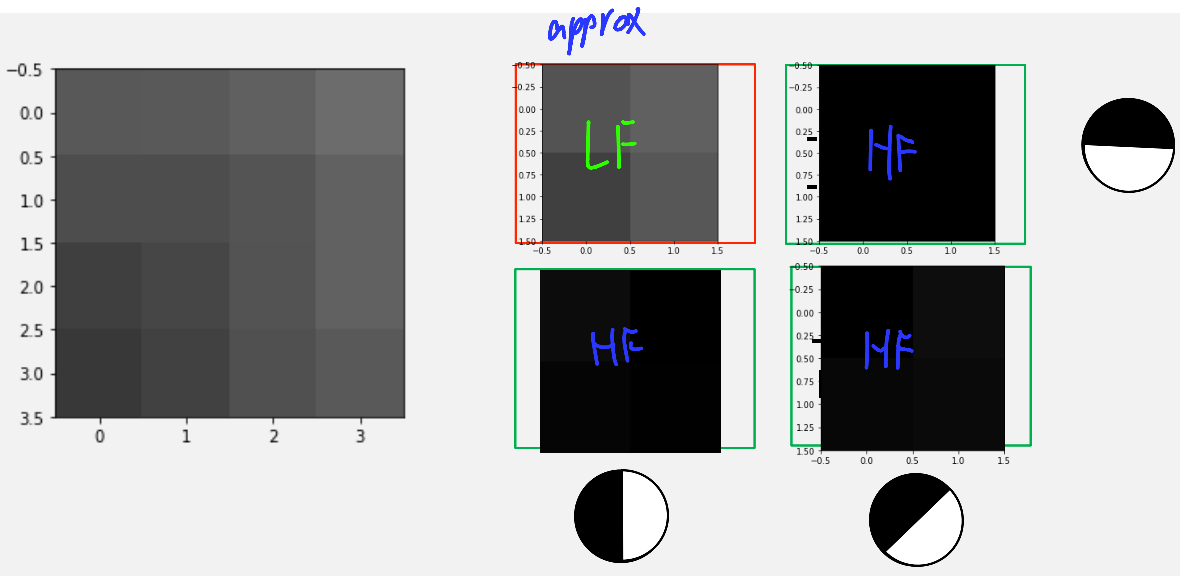


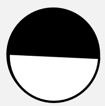
average

82.75	96
63.75	87.25

horizontal detailed

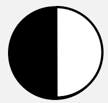
difference

11.5	11
5.5	5.5



average of the vertical difference

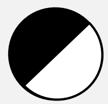
-0.5	-13
-7.5	-10.5



vertical detailed

difference

-1	2
1	-6



diagonal detial

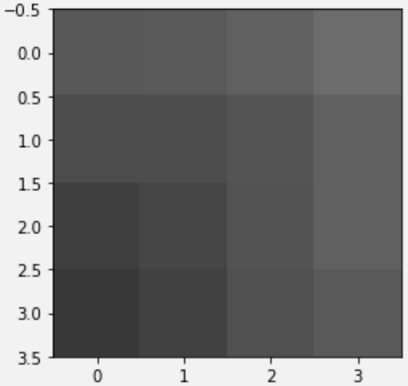
88	89	96	108
77	77	84	98
63	70	83	97
57	65	80	89

average

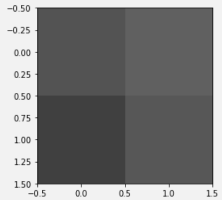
88.5	102
77	91
66.5	90
61	84.5

difference

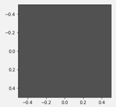
-1	-12
0	-14
-7	-14
-8	-9



88	89	96	108
77	77	84	98
63	70	83	97
57	65	80	89



82.75	96
63.75	87.25



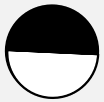
82.44

average

82.75	96
63.75	87.25

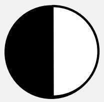
difference

11.5	11
5.5	5.5



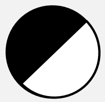
average of the horizontal difference

-0.5	-13
-7.5	-10.5



difference

-1	2
1	-6



Three species of differences

88	89	96	108
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63	70	83	97
57	65	80	89

average

88.5	102
77	91
66.5	90
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difference

-1	-12
0	-14
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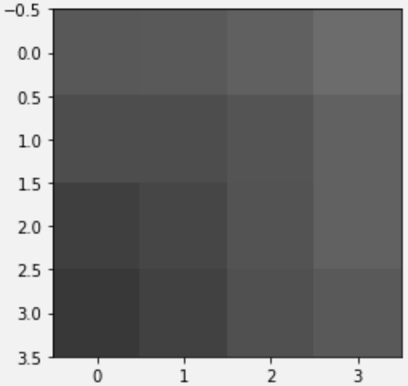
-8	-9



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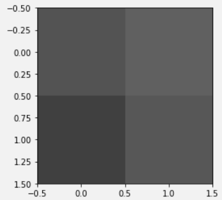
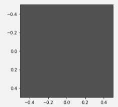
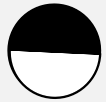




82.75	96
63.75	87.25

82.44

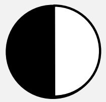
-18.38

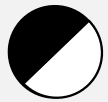


13.875

10.25

89.375	-13.25
75.5	-23.5





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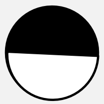
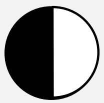


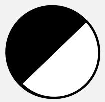
average of the horizontal difference

-0.5	-13
-7.5	-10.5



difference

-1	2
1	-6



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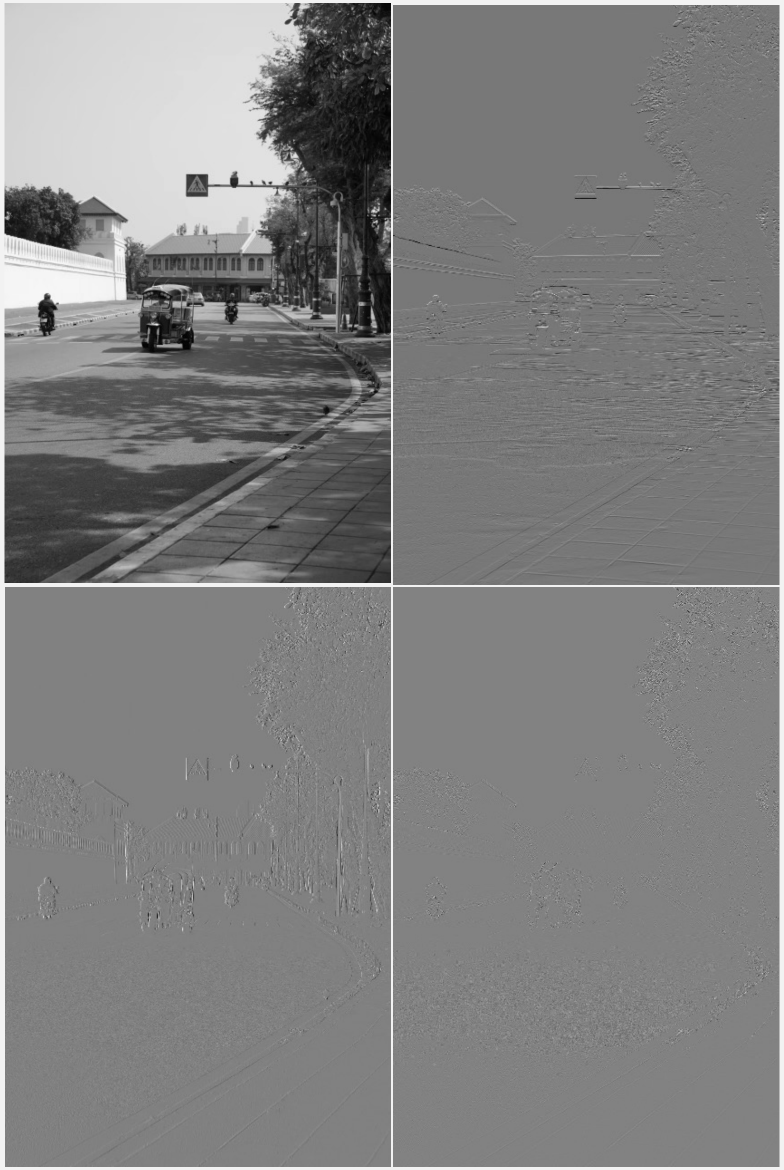
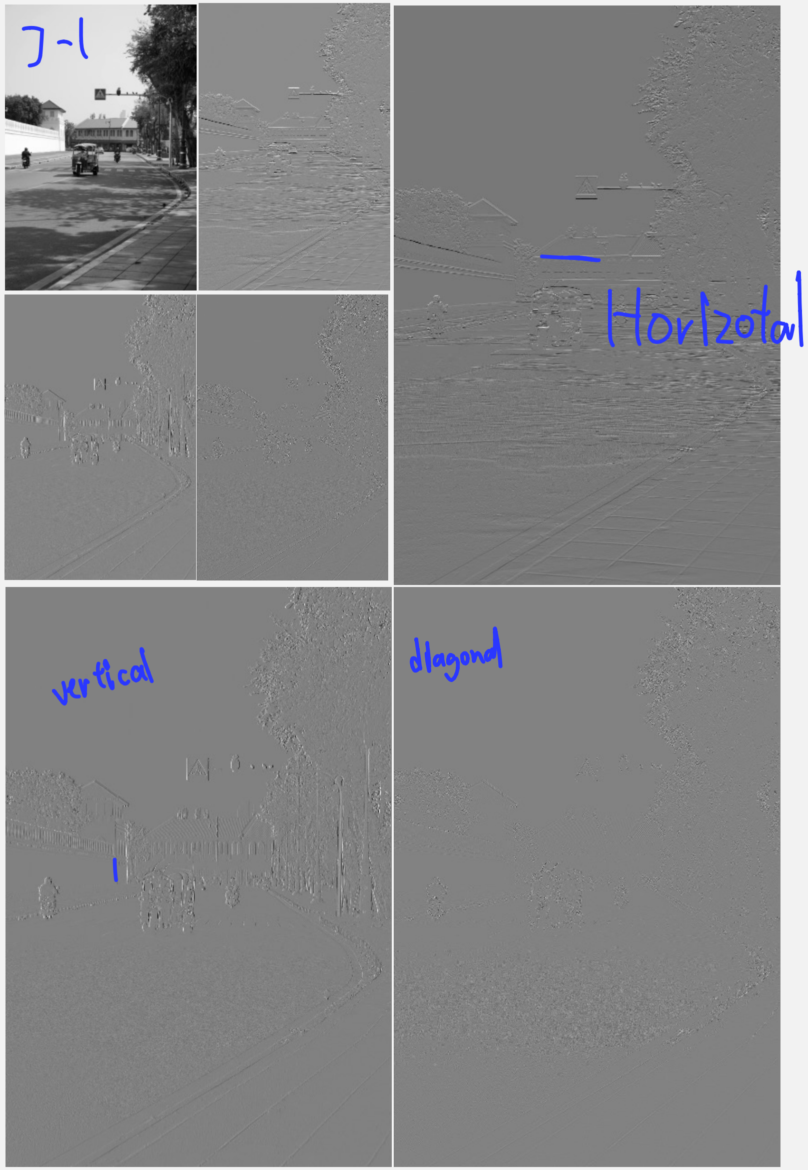
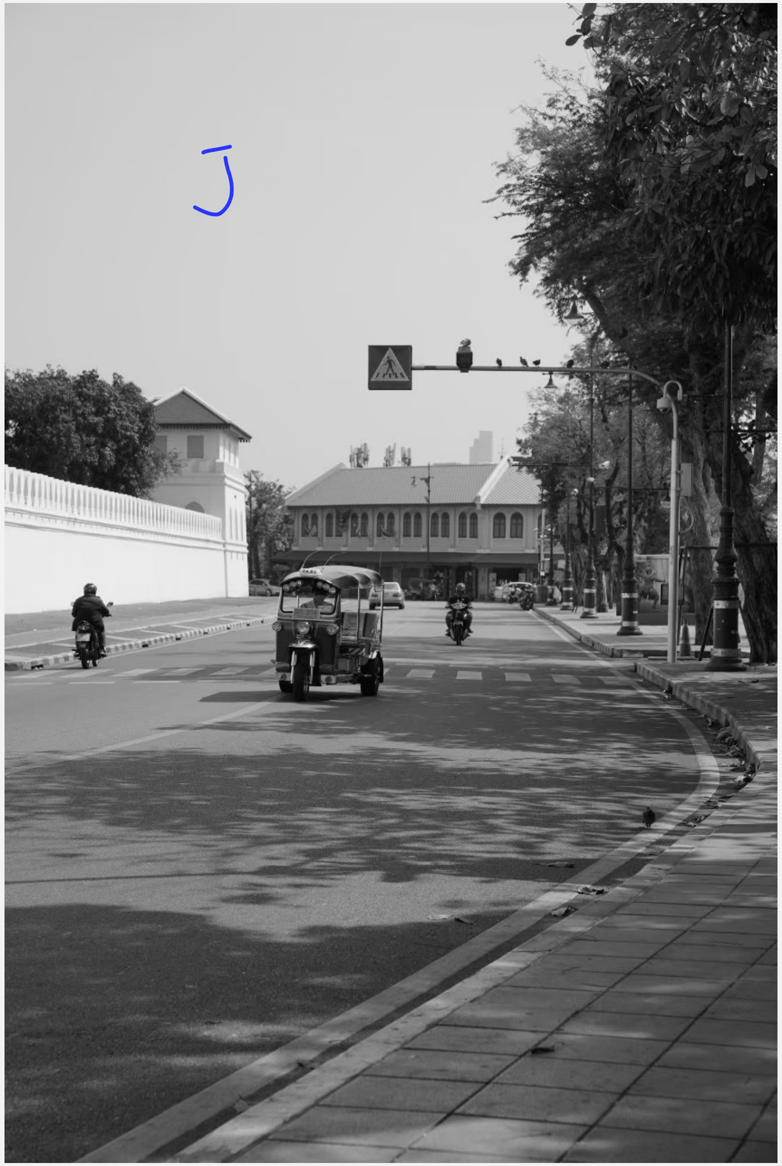


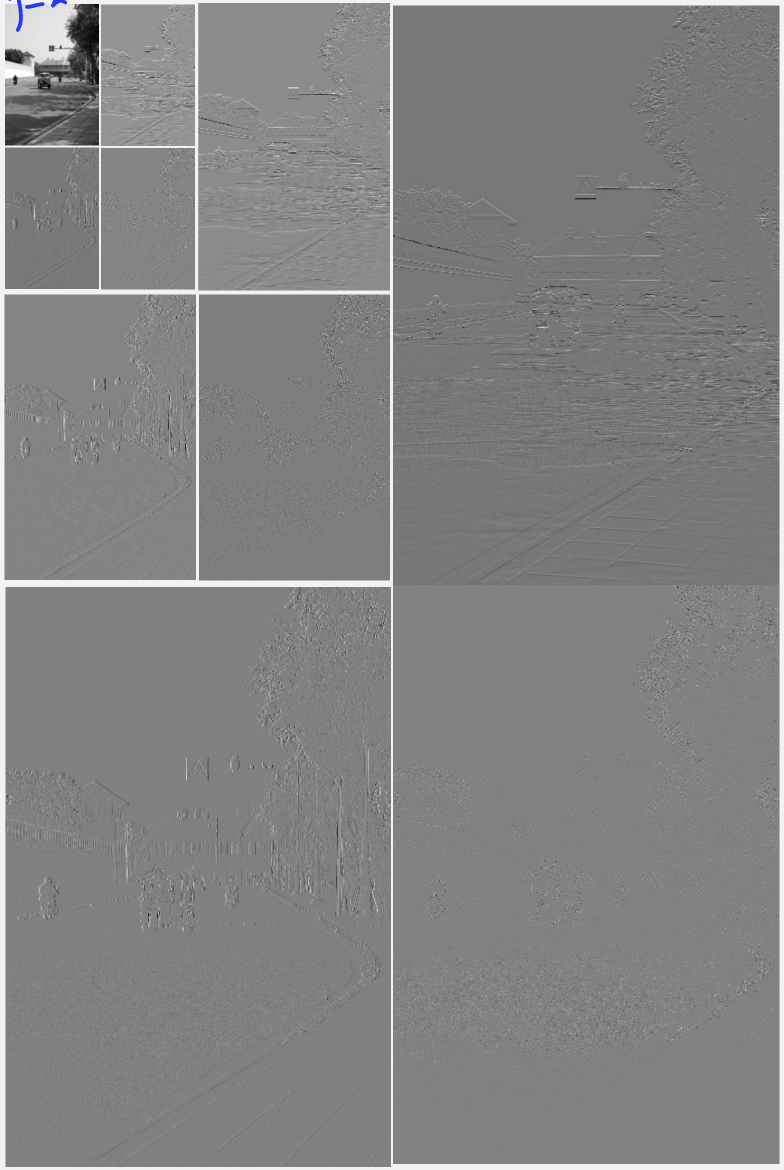

J


J-1
Horizontal
vertical
diagonal

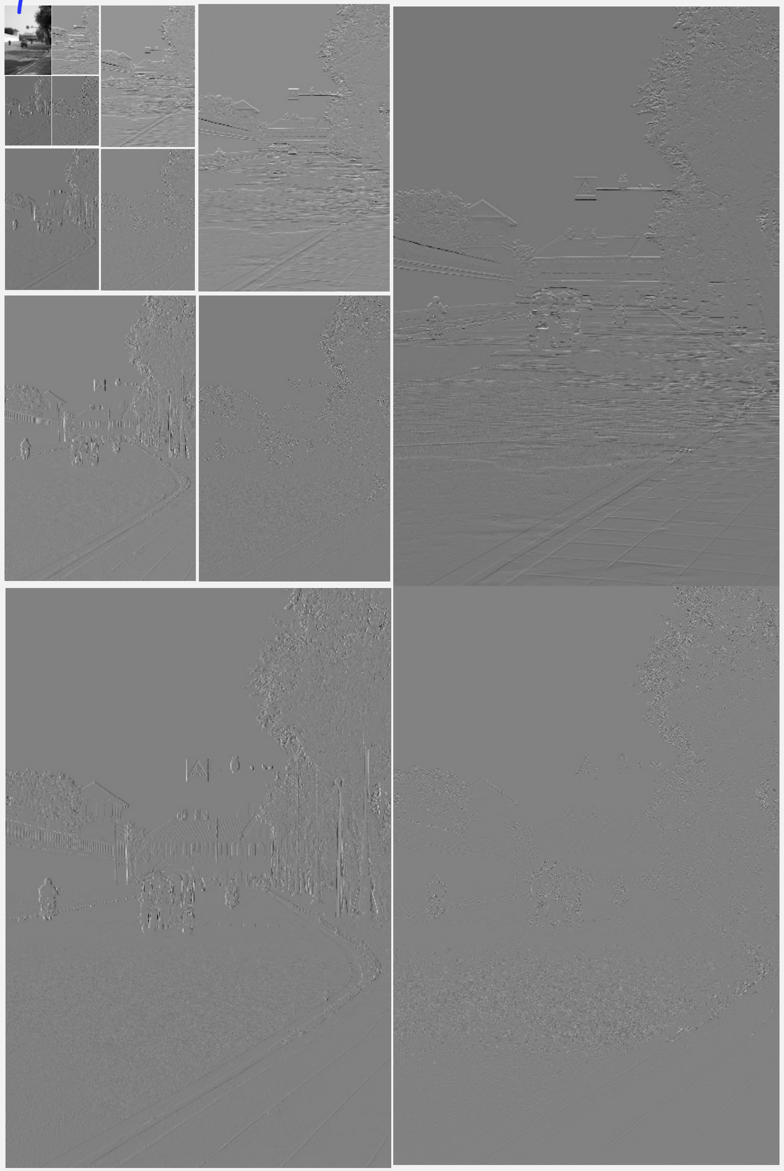
J



J-2

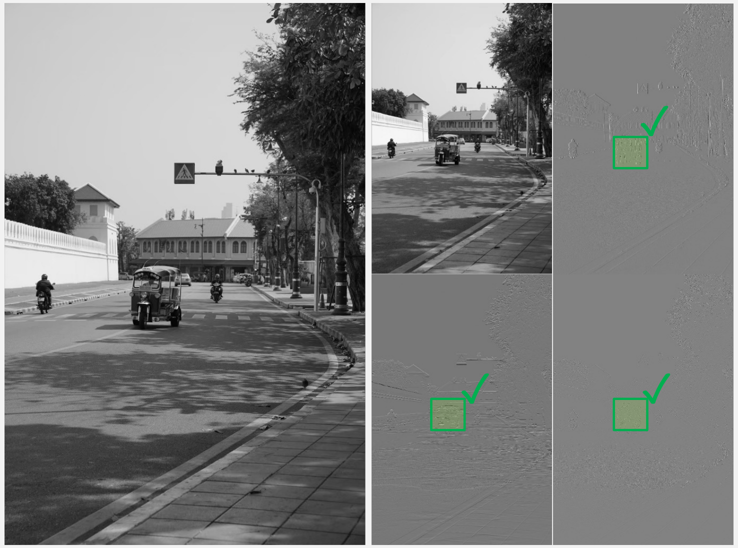


J-3



J-6
Six-scale decomposition

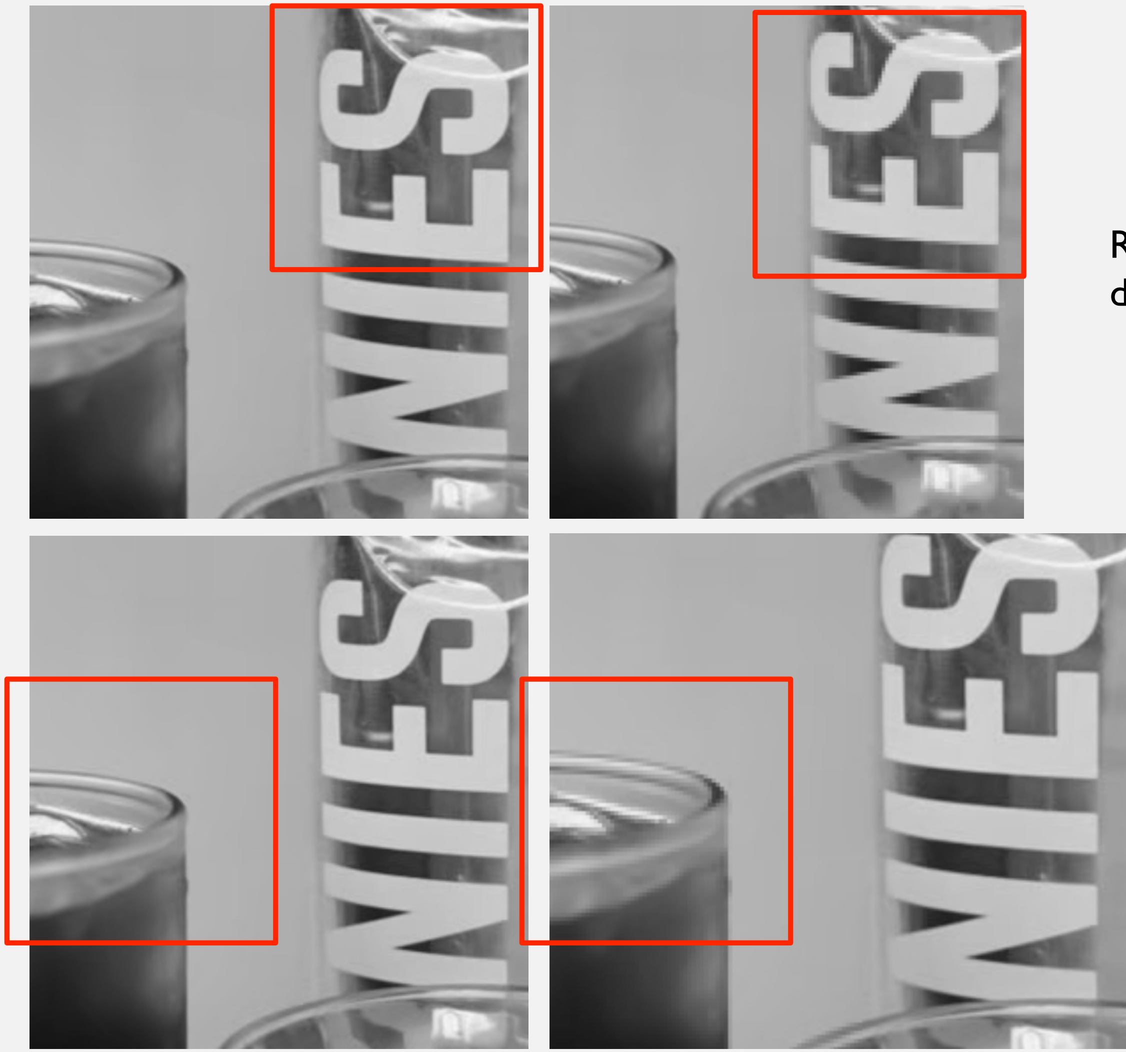
can use to reserve ROI



Original image

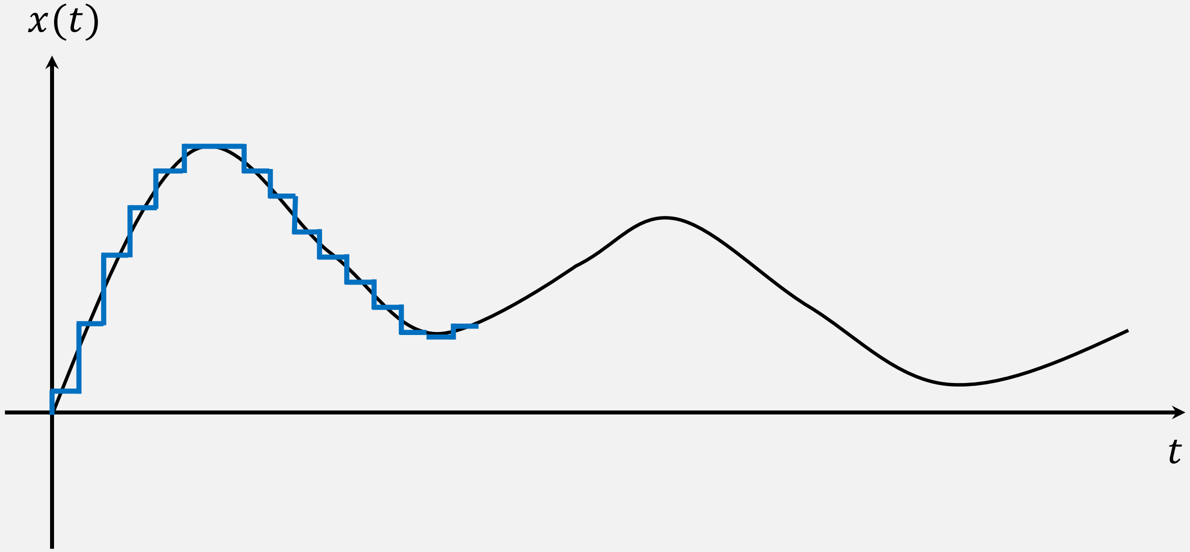
All details are removed except for Tuk-Tuk

All details of Tuk-Tuk are removed

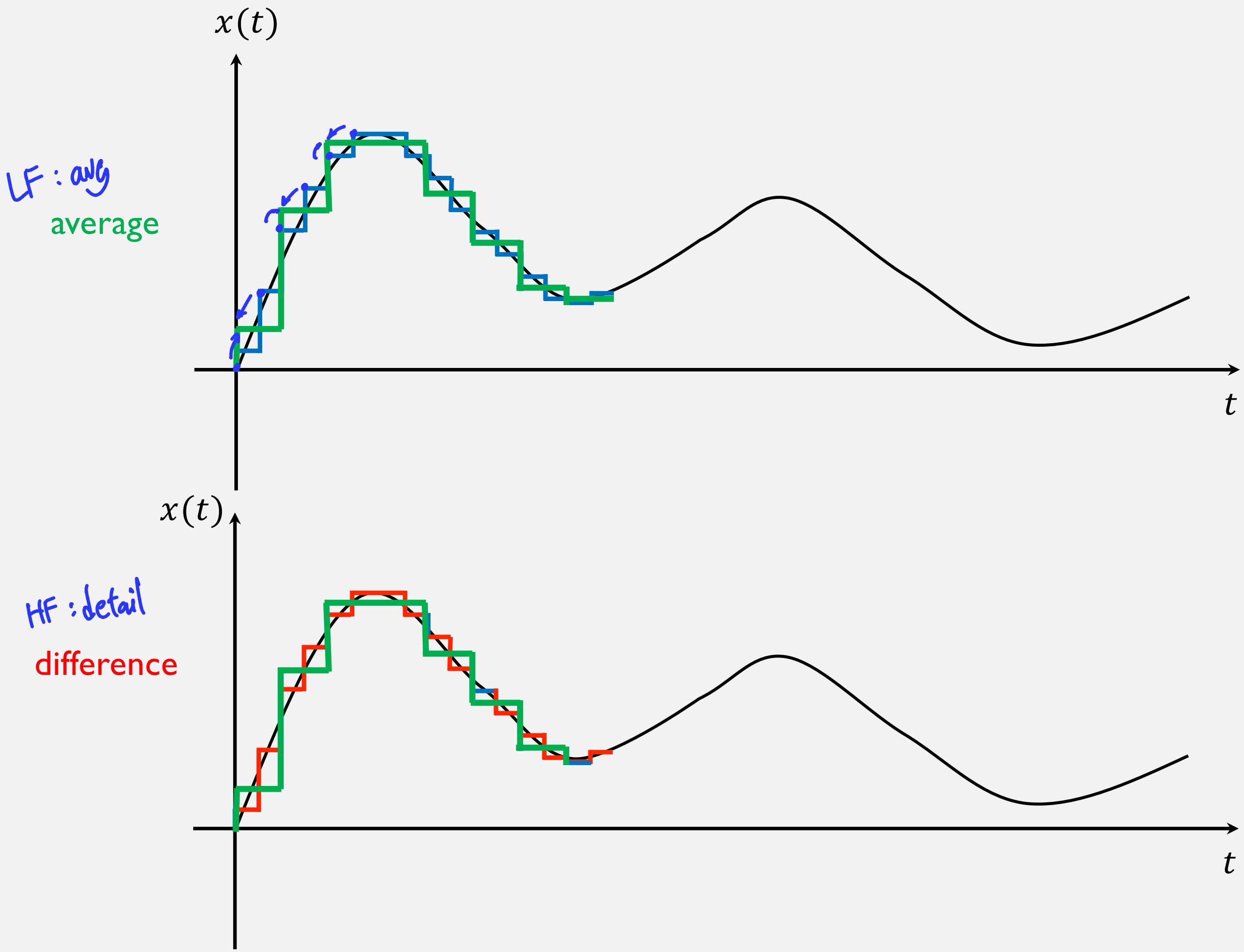


Remove ”ES” details

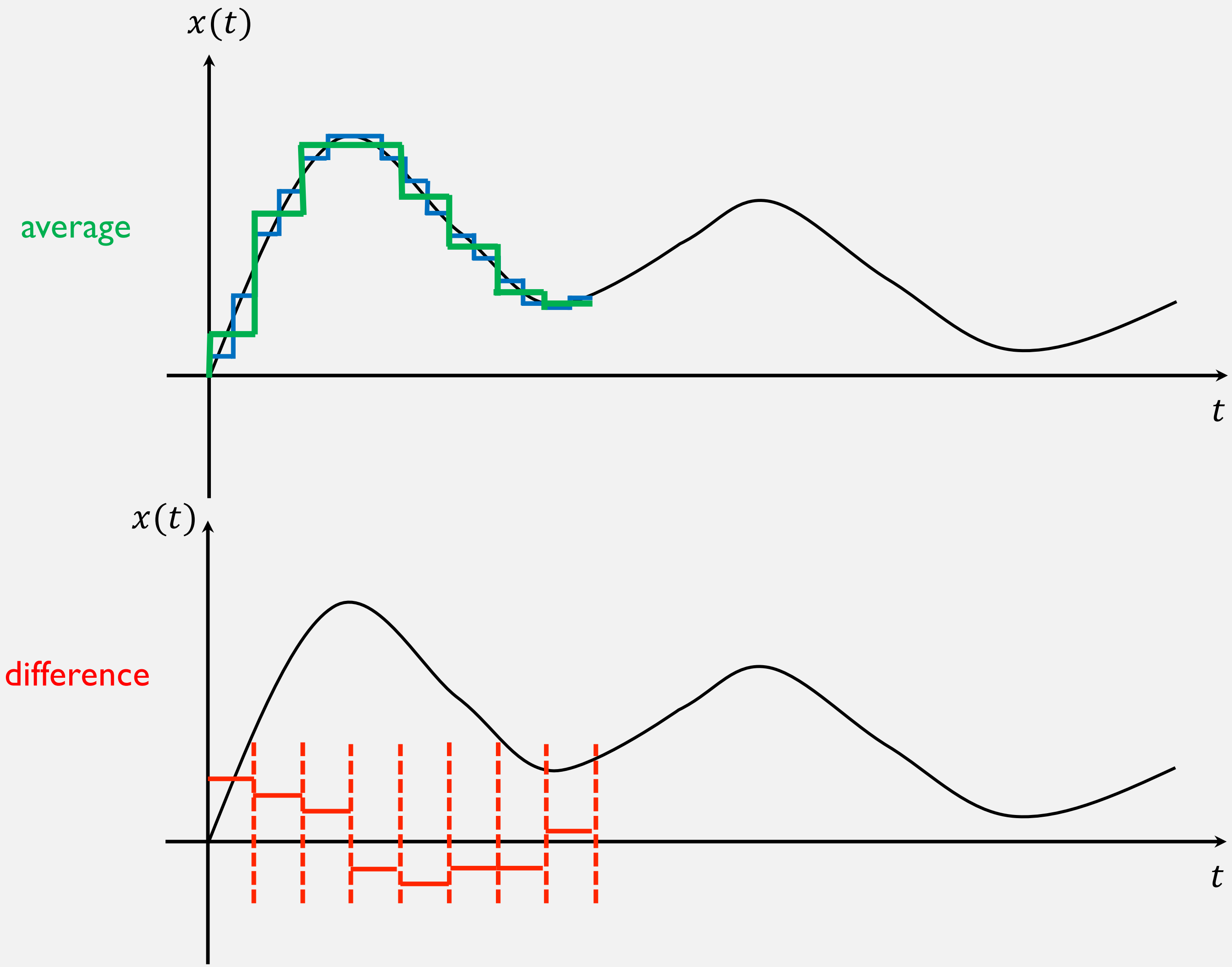
Remove edge of the glass details



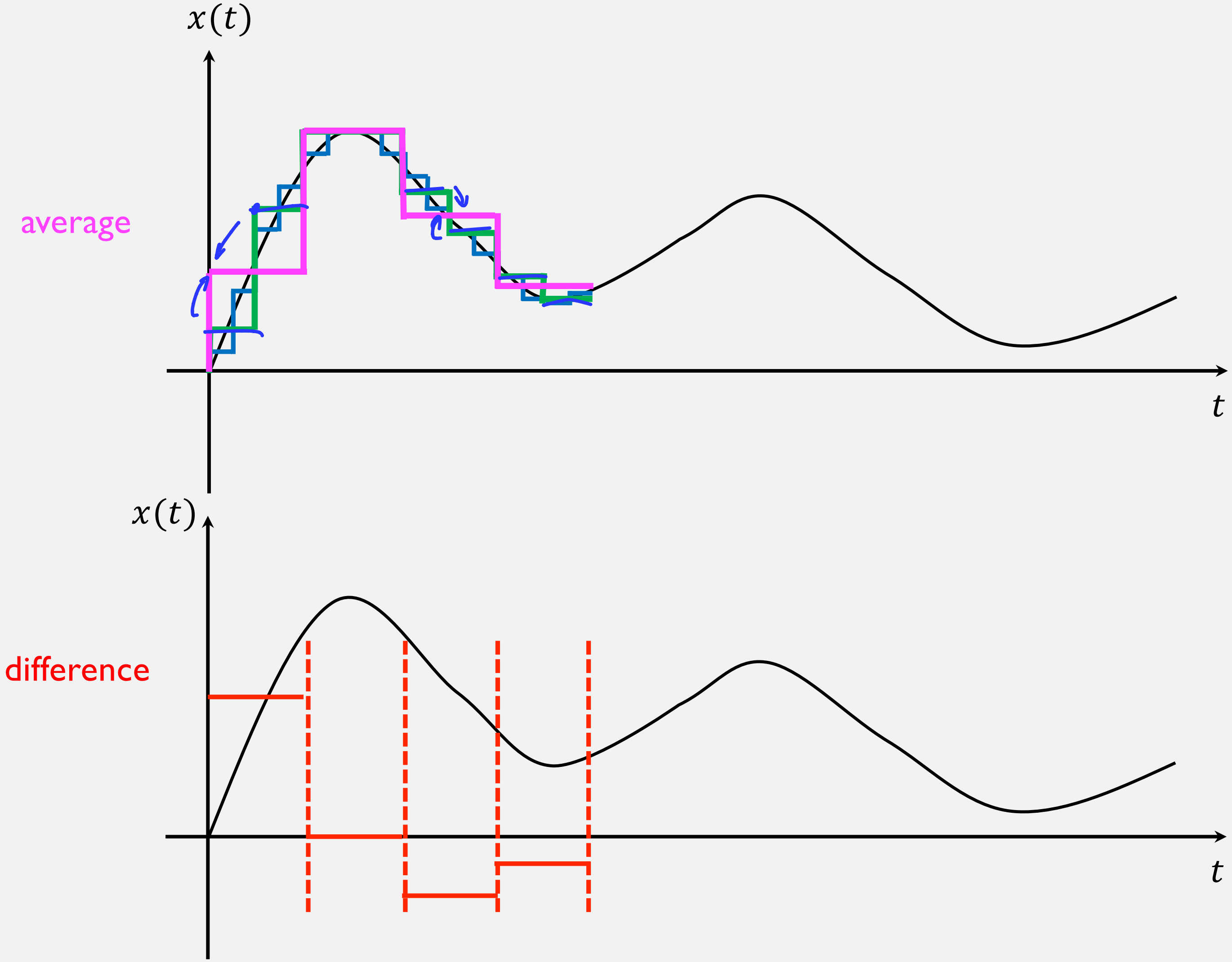
$x(t)$
t



LF : avg
average
x(t)
t
HF : detail
difference
x(t)
t



average
difference
$x(t)$
t
$x(t)$
t



$x(t)$
average
t
$x(t)$
difference
t

IMAGE PYRAMIDS

- Representing an image more than one resolution
  - Base – high resolution
  - Apex – low resolution
  - Base level J is size $2^J \times 2^J$ (or NxN) ($J = \log_2 N$)
  - Level j is size $2^j \times 2^j$, where $0 \leq j \leq J$
  - Fully populated pyramid – from $2^J \times 2^J$ to $2^0 \times 2^0$ (1x1 too small)
  - Most pyramid truncated to P+1 levels ($j = J\text{-}P, J\text{-}P\text{+}1, \ldots J$)

cuz apex is 1x1 = litterally No info

more compact info

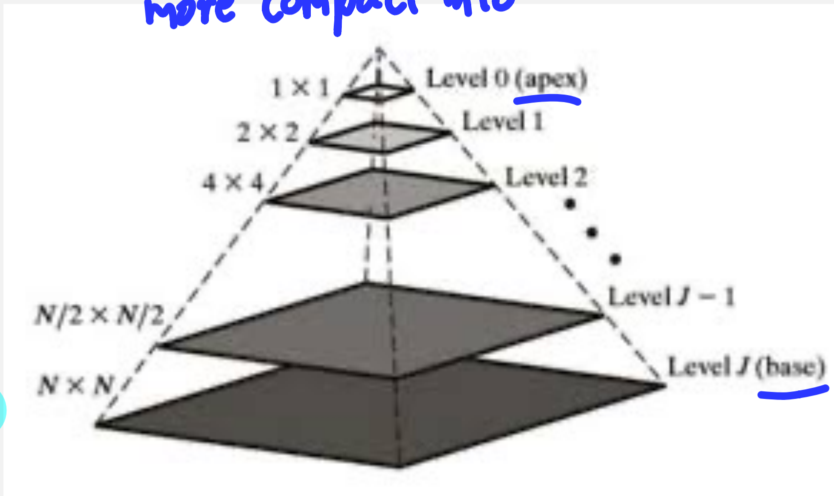


Image from Gonzalez & Woods, Digital Image Processing

IMAGE PYRAMIDS

- System block diagram to create a pyramid

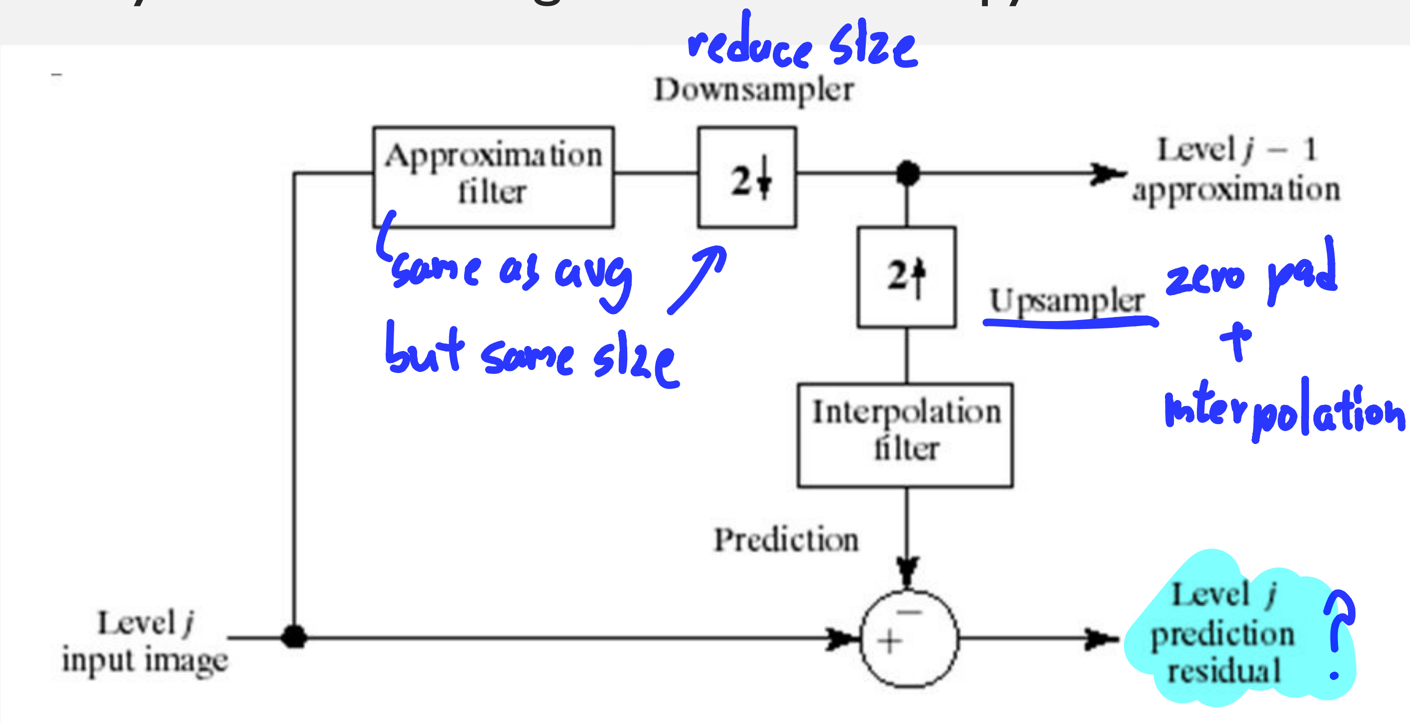


Image from Gonzalez & Woods, Digital Image Processing

Filtering

- averaging filtering creates a mean pyramid

- lowpass Gaussian filtering creates a Gaussian pyramid ป้องกัน $f_0 > 2f_s$

Without filtering, aliasing can be occurred in the upper levels. (subsampling pyramid)

IMAGE PYRAMIDS

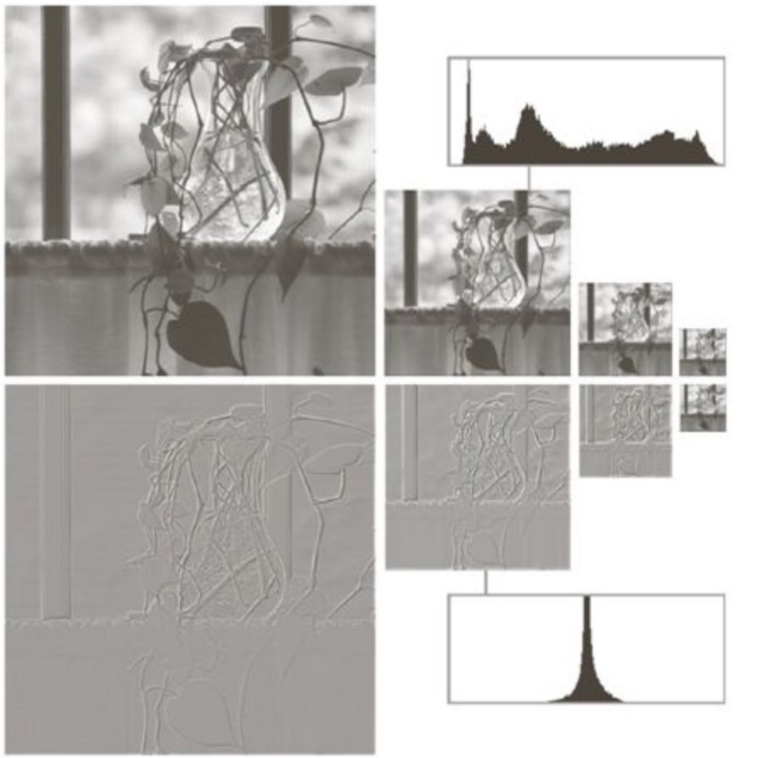


FIGURE 7.3 Two image pyramids and their histograms: (a) an approximation pyramid; (b) a prediction residual pyramid.

Image from Gonzalez & Woods, Digital Image Processing

Peak at around zero. The residuals can be coded, strored, and trasmitted more effectively than approximation

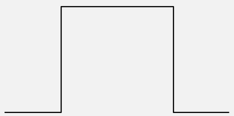
Use 5x5 low pass Gaussian convolution kernel (for filtering)
-Original image 512 x 512 (J = 9)
-P+1 = 4 so j = 6, 7, 8, 9

Laplacian level 6 (64x64) predicts Gaussian pyramid level 7 (128 x 128 approximation) by upsampling and filtering and add the Laplacian level 7.

BASIS FUNCTIONS : unit, and orthogonal

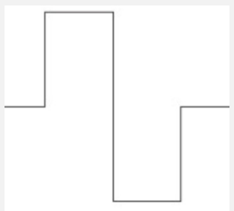
- Basis functions used by the wavelet transform are all generated from one initial function called the mother wavelet. The simplest mother wavelet is the Haar wavelet. The basis functions are orthonormal wavelets.
  - $h_\varphi(n)$ - scaling function coefficients
  - $h_\psi(n)$ - wavelet function coefficients

$$h_\varphi(n) = \begin{cases} 1/\sqrt{2} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$



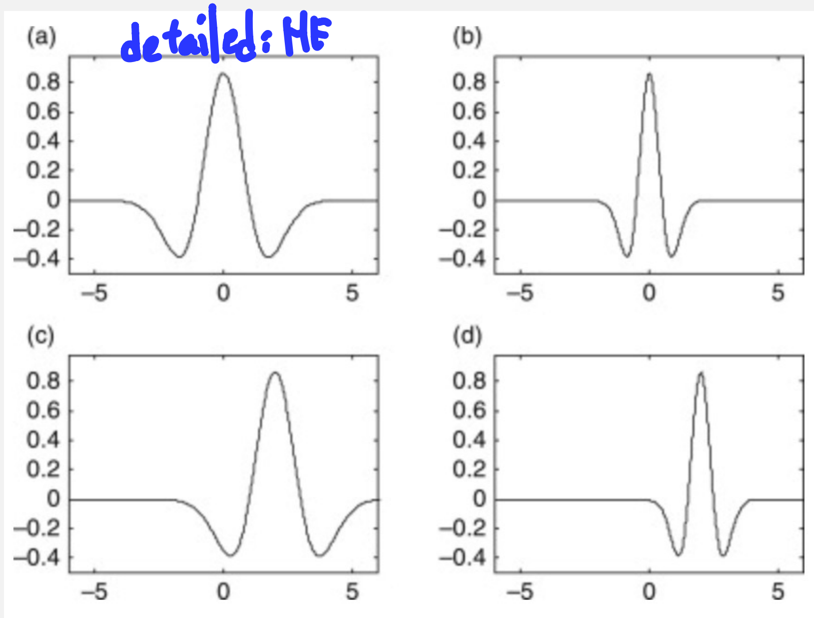
avg LF

$$h_\psi(n) = \begin{cases} 1/\sqrt{2} & n = 0 \\ -1/\sqrt{2} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$



difference HF

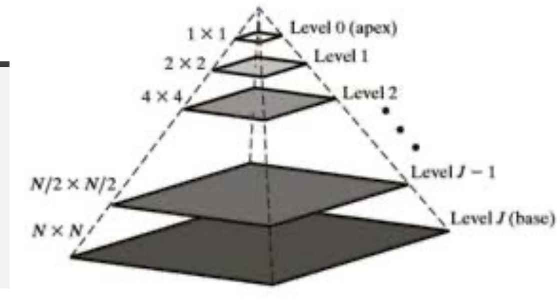
detailed: HF



The Mexican Hat wavelet. (a) Mother wavelet, (b) dilation by a factor of 2, (c) translation of 4 to the right, and (d) combination of (b) and (c).

$$\psi(t) = \frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}} (1 - t^2) e^{-\frac{t^2}{2}}$$

ONE-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

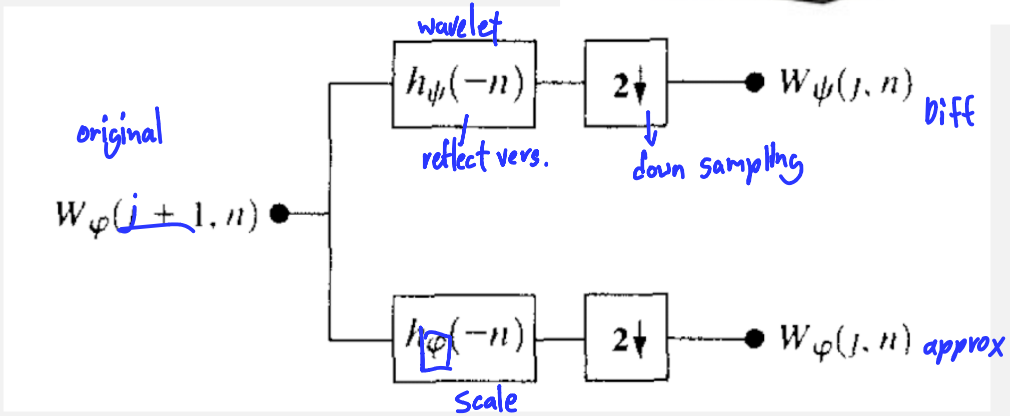


- **FWT Analysis bank**
  - Efficient implementation of discrete wavelet transform

$$W_\psi(j,k) = h_\psi(-n) * W_\varphi(j+1,n)\Big|_{\boxed{n=2k},\ k\geq 0}$$

down sampling by half

$$W_\varphi(j,k) = h_\varphi(-n) * W_\varphi(j+1,n)\Big|_{n=2k,\ k\geq 0}$$

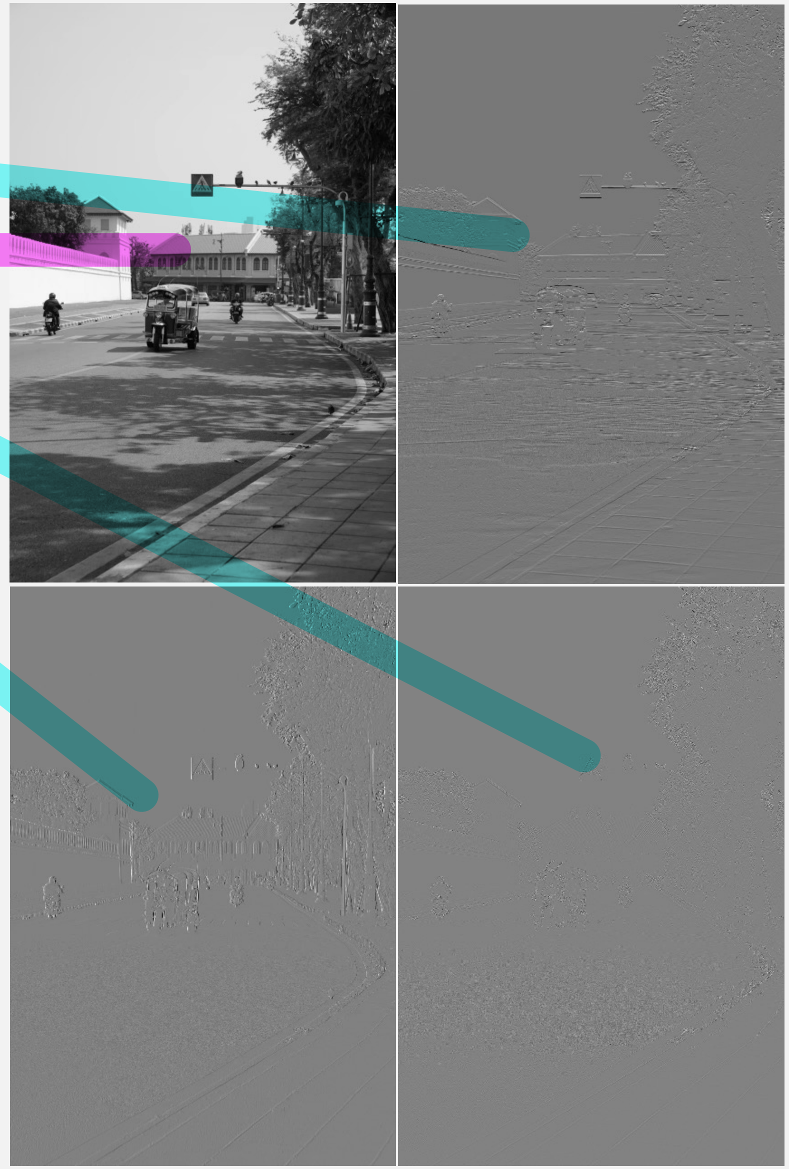


$W_\varphi(j,k)$ - approximation coefficients with scale level j, translating it by k. $W_\psi(j,k)$ - details coefficients. The image in level $j+1$ is subsampled from level j so $2\mathrm{k} = \mathrm{n}$.

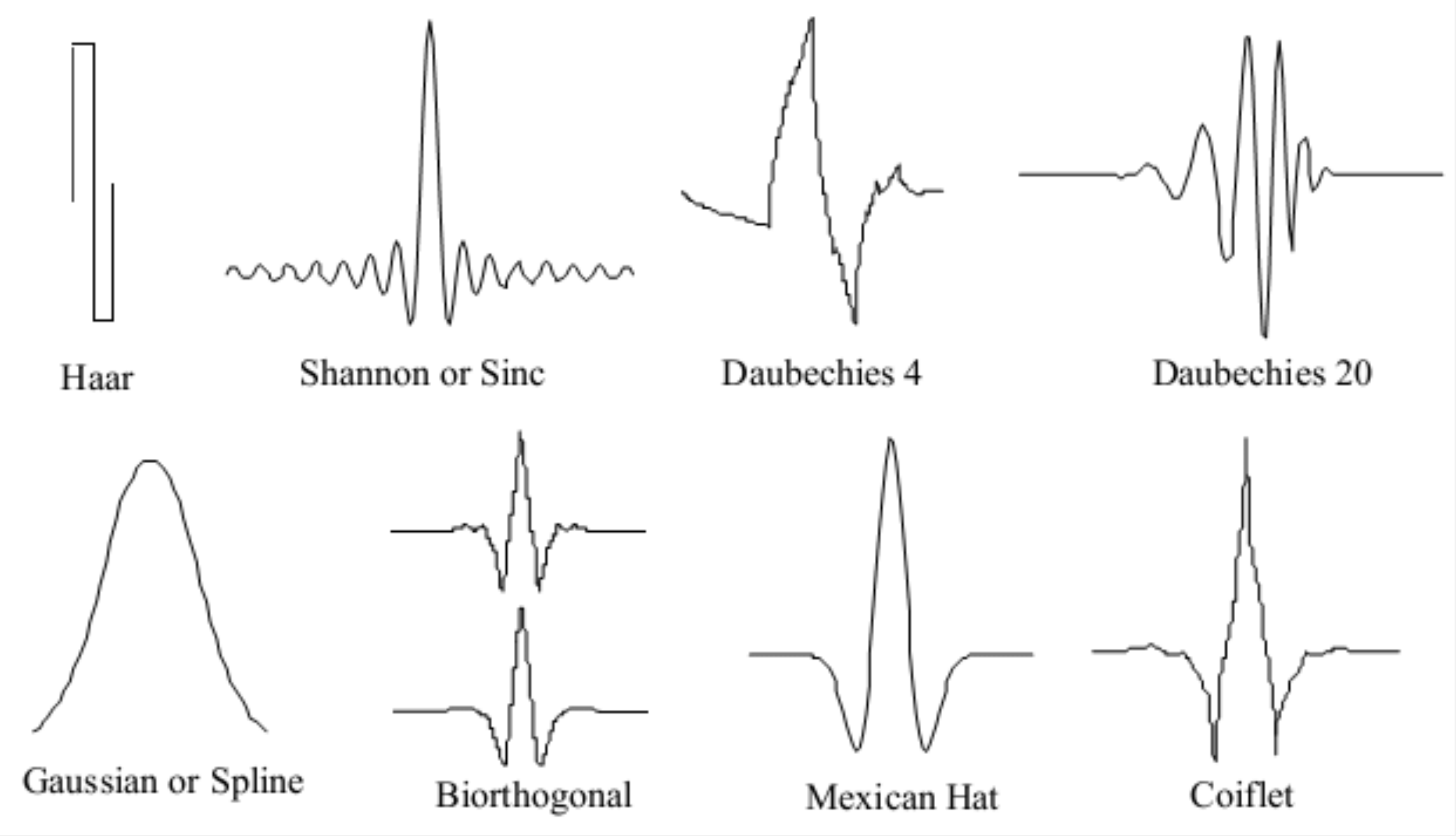
Image from Gonzalez & Woods, Digital Image Processing

$$W_\psi(j,k) = h_\psi(-n) * W_\varphi(j+1,n)\Big|_{n=2k,\,k\geq 0}$$

$$W_\varphi(j,k) = h_\varphi(-n) * W_\varphi(j+1,n)\Big|_{n=2k,\,k\geq 0}$$

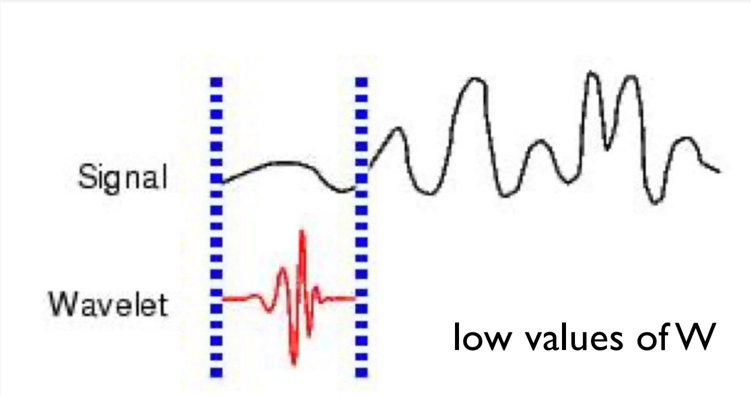


OTHER WAVELETS

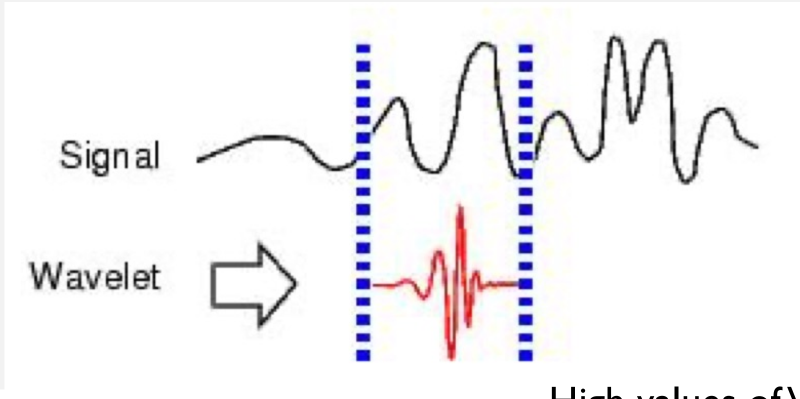


WAVELET TRANSFORM IN ONE DIMENSION

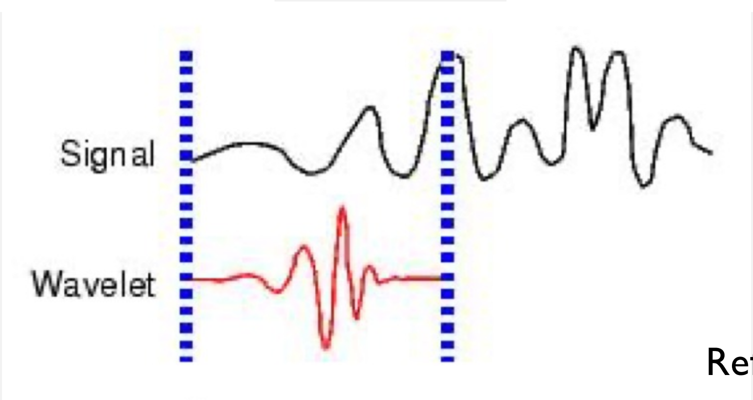
- **Continuous wavelet transform**



low values of W



High values of W



Different scales

Reference: Colorado school of Mines

ONE-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

Fast wavelet Transform

- **FWT Analysis bank**
  - f(n) = {1,4,-3,0} find DWT corresponding scaling and wavelet vectors (Haar)

$$h_\varphi(n) = \begin{cases} 1/\sqrt{2} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_\psi(n) = \begin{cases} 1/\sqrt{2} & n = 0 \\ -1/\sqrt{2} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

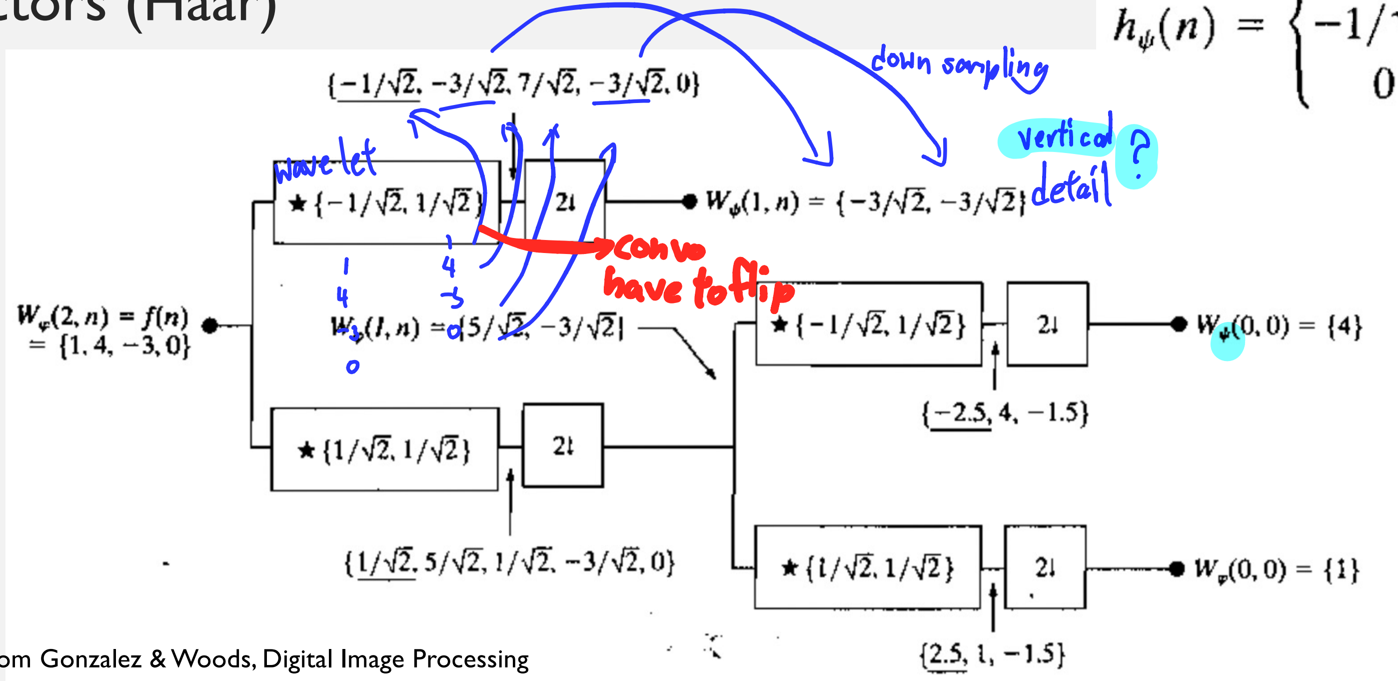
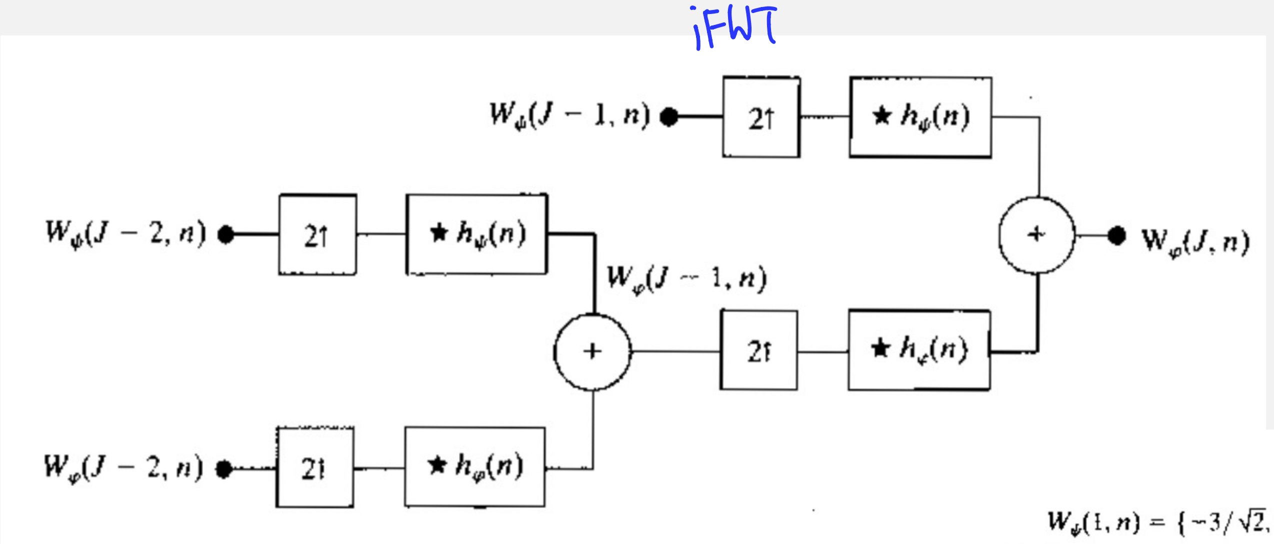


Image from Gonzalez & Woods, Digital Image Processing

ONE-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

- FWT Synthesis bank



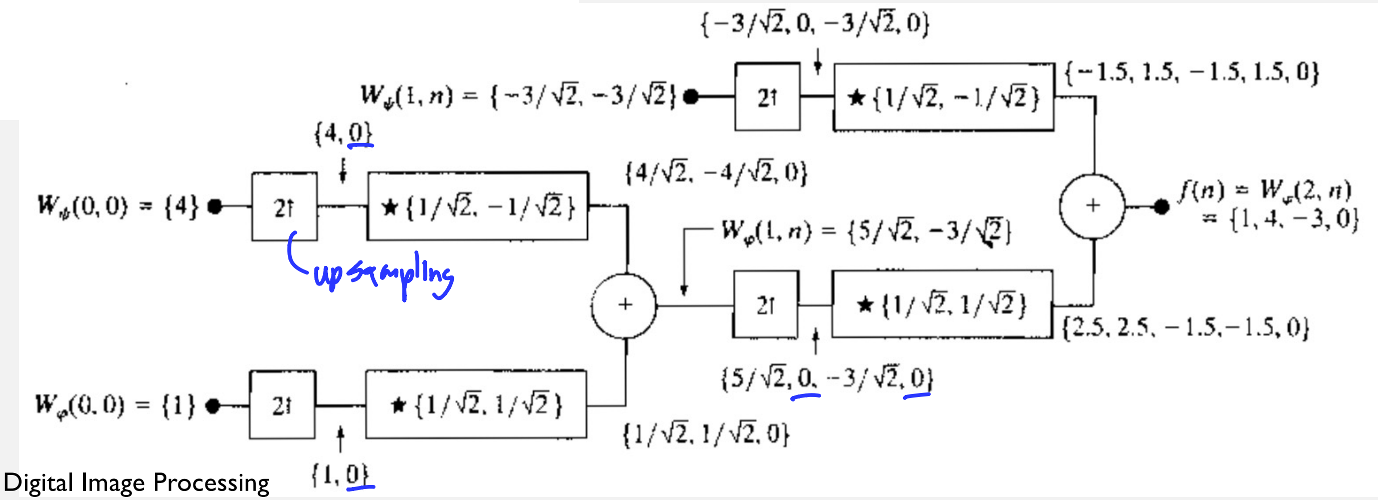
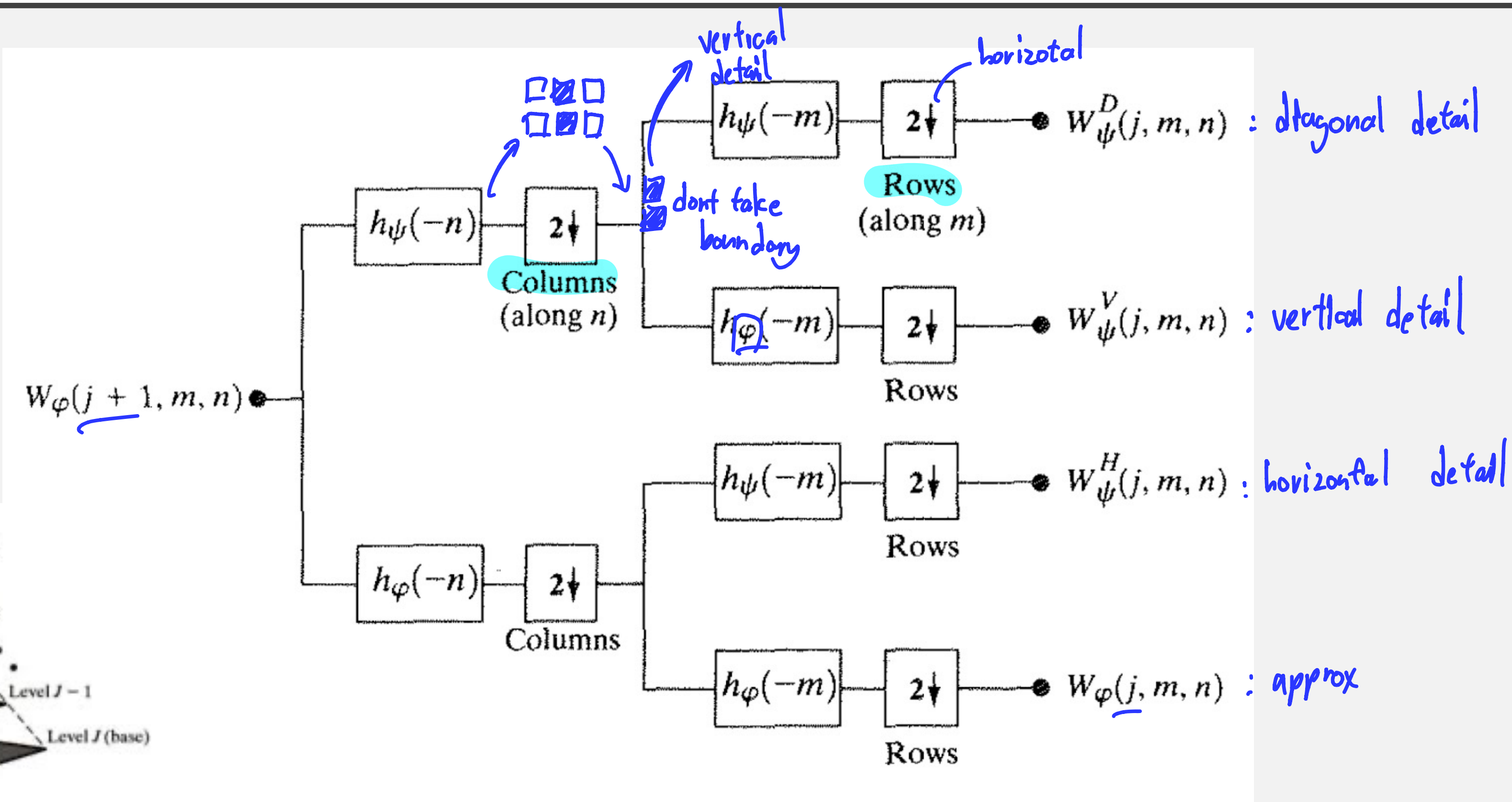


Image from Gonzalez & Woods, Digital Image Processing

TWO-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)



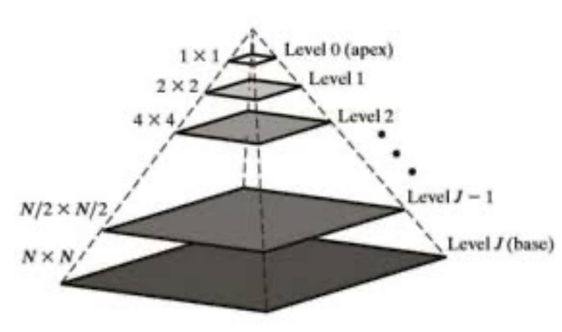


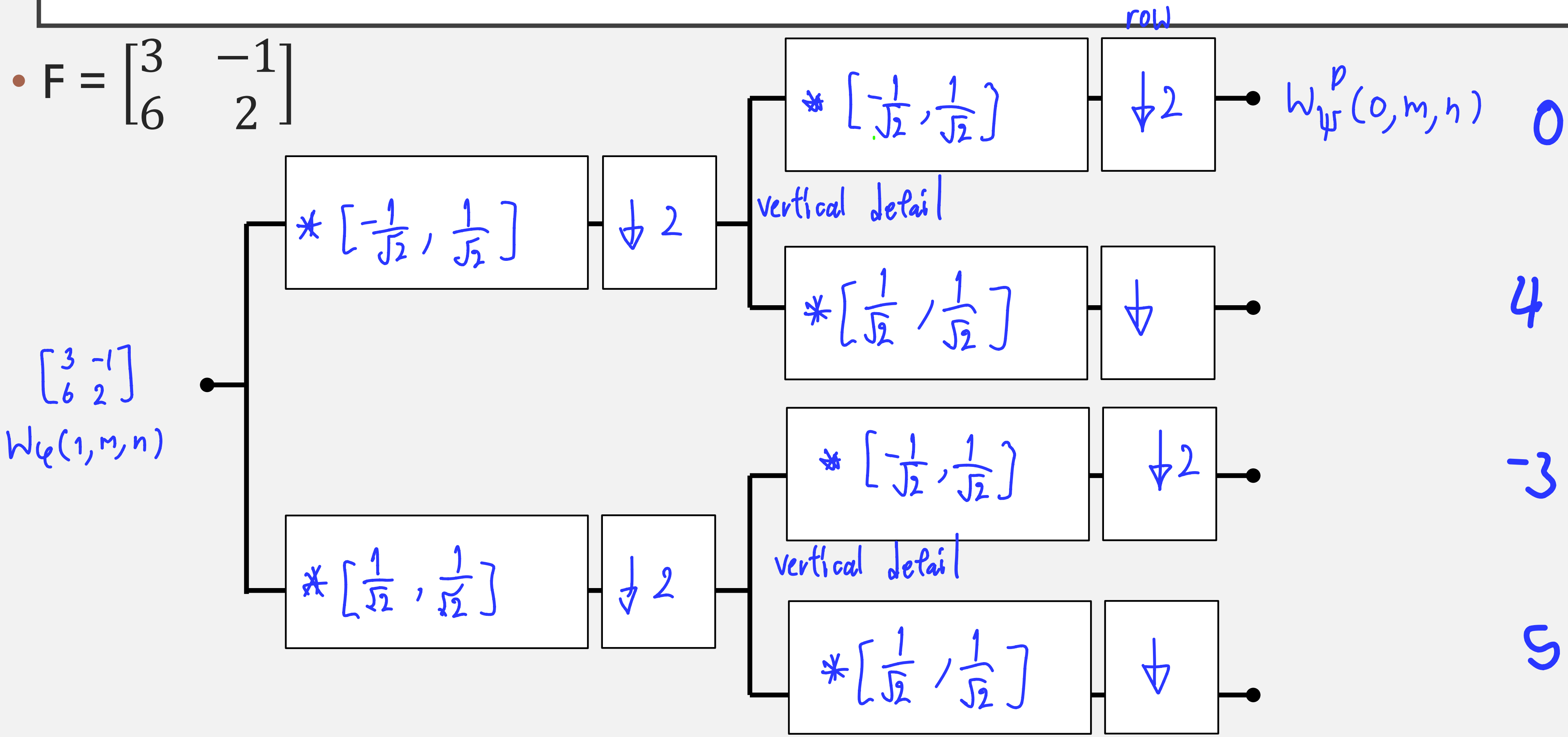
Image from Gonzalez & Woods, Digital Image Processing

TWO-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

- Practice: a small image 2x2 F = $\begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$
- Compute 2D Wavelet transform with respect to Haar wavelets of this image. Draw the required filter bank and label all inputs and outputs with proper arrays.

TWO-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

- $F = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$



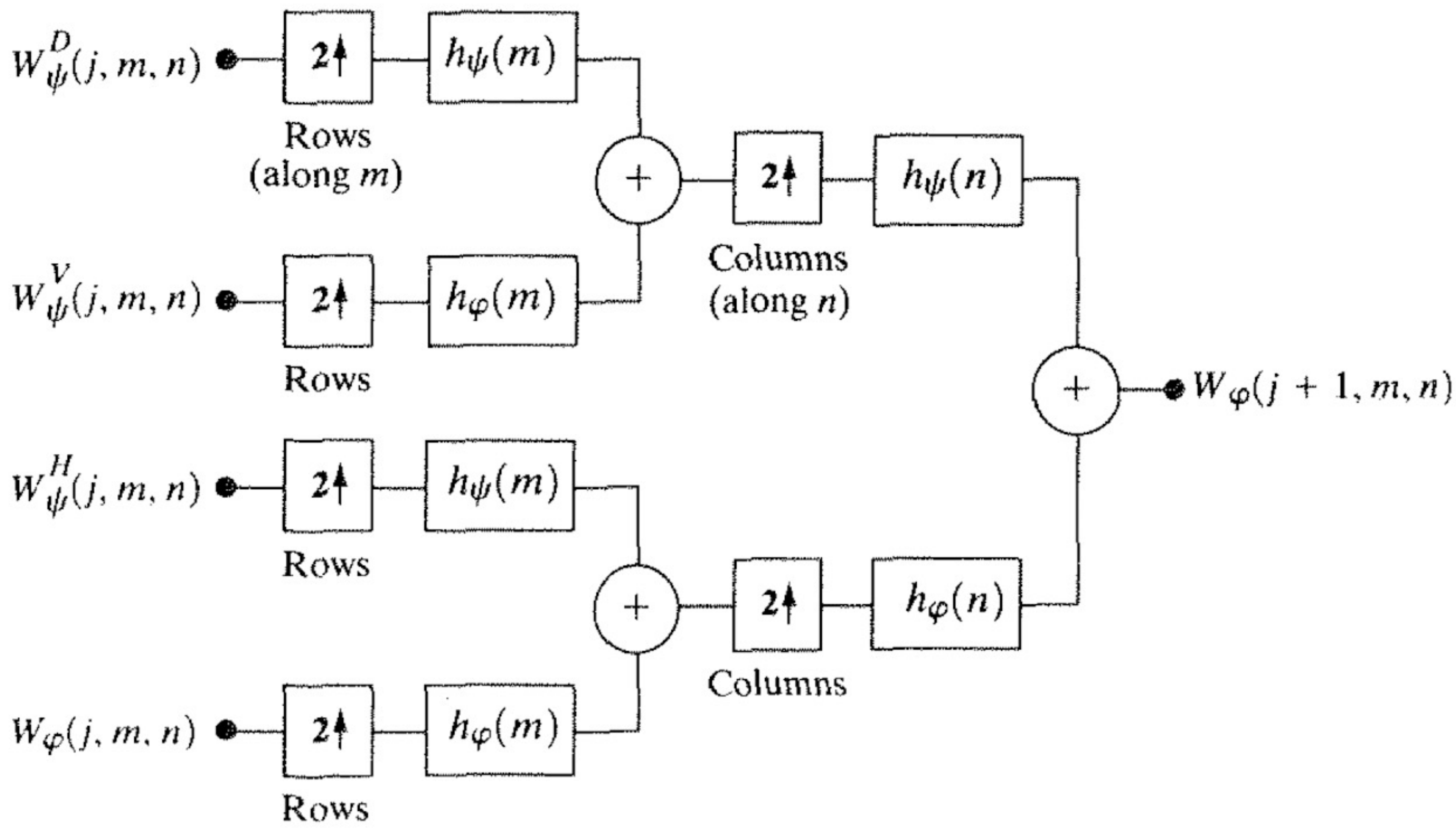
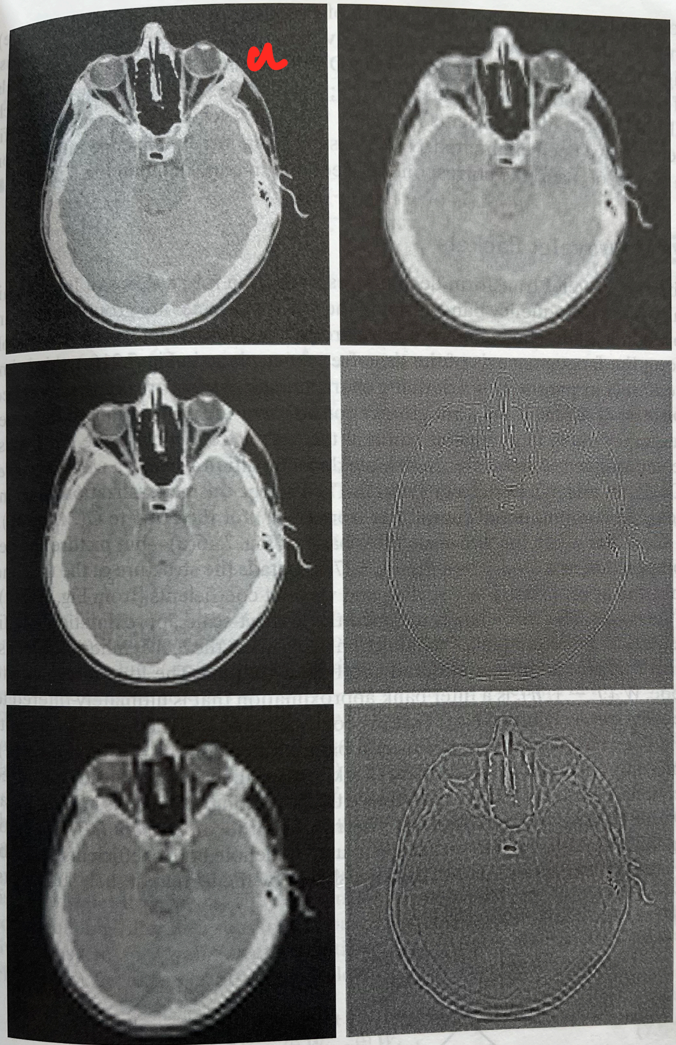


FIGURE 7.22 The two-dimensional fast wavelet transform: (a) the analysis filter bank; (b) the resulting decomposition; and (c) the synthesis filter bank.

Synthesis filter banks

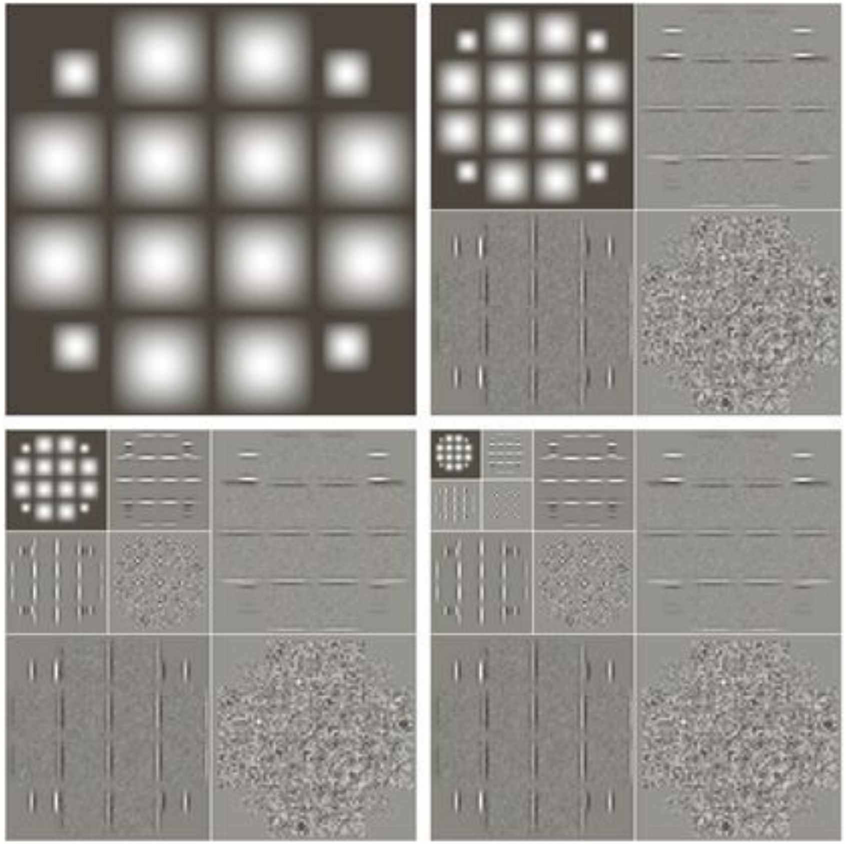
Image from Gonzalez & Woods, Digital Image Processing



a b
c d
e f

FIGURE 7.26 Modifying a DWT for noise removal: (a) a noisy MRI of a human head; (b), (c) and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (c) and (e). (Original image courtesy Vanderbuilt University Medical Center.)

tails) and reconstructing the image. Here, almost all of the background noise has been eliminated and the edges are only slightly disturbed. Figure 7.26(d) shows the information that is lost in the process. This result was generated by computing the inverse FWT of the decomposed image with all but the highest-resolution detail coefficients zeroed. As can be seen, it contains most of the



a b
c d

FIGURE 7.25 Computing a 2-D three-scale FWT: (a) the original image; (b) a one-scale FWT; (c) a two-scale FWT; and (d) a three-scale FWT.

APPLICATIONS OF WAVELETS

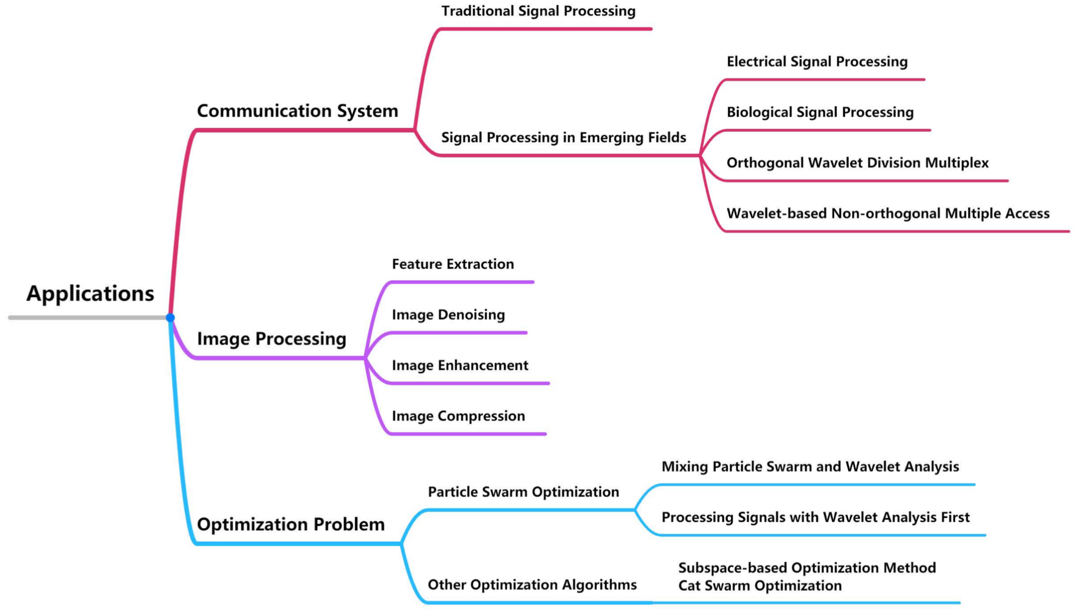


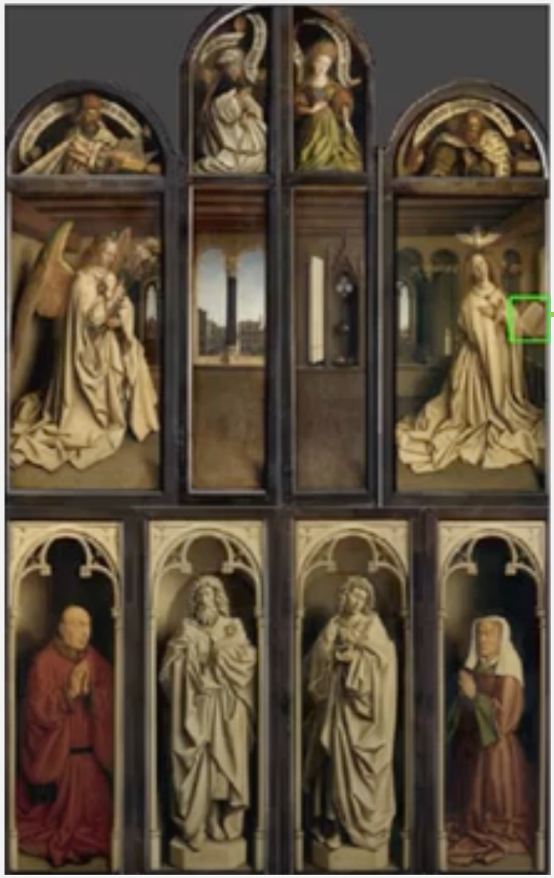
FIGURE 23. The structure of practical applications of wavelets.

Wavelets for image processing

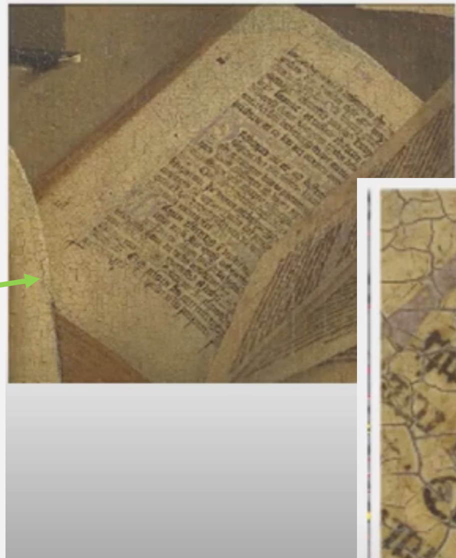
- Optimally extract features in different sizes
  - Palm- print images after two-level wavelet decomposition & maintaining the uniqueness of each palmprint image and can be used for palmprint image classification
- Denoising, enhancement and compression in the image processing area
- Extracting structure and noise information through wavelet transform and generating high-quality images through GAN
- Wavelet kernel-based neural network
  - Initialization
  - Used as the activation function

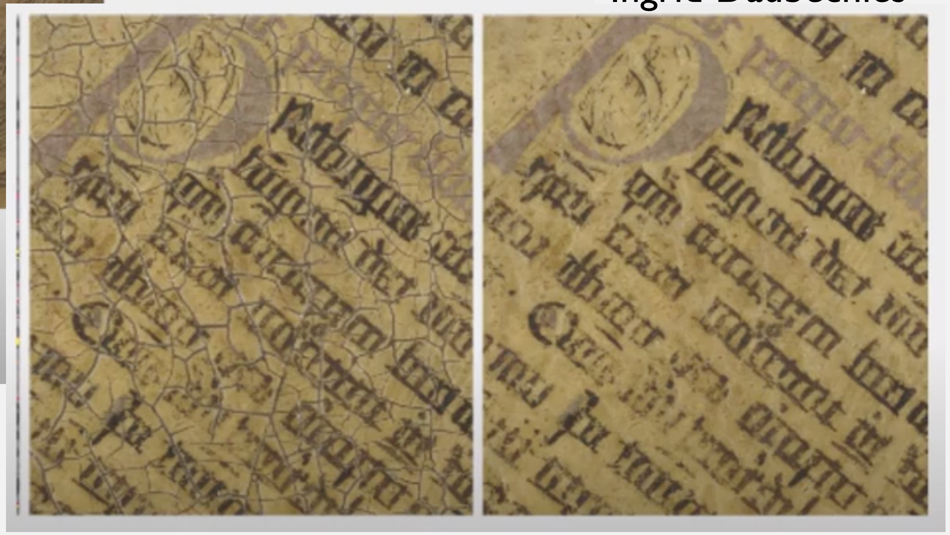
T. Guo, T. Zhang, E. Lim, M. López-Benítez, F. Ma and L. Yu, "A Review of Wavelet Analysis and Its Applications: Challenges and Opportunities," in *IEEE Access*, vol. 10, pp. 58869-58903, 2022, doi: 10.1109/ACCESS.2022.3179517.

WAVELETS FOR ART



Ghent Altarpiece





Ingrid Daubechies

Ref. https://www.youtube.com/watch?v=jrF1SGPyF4g
Using Math to Understand Art | Ingrid Daubechies | TEDxDuke

WAVELETS FOR AI

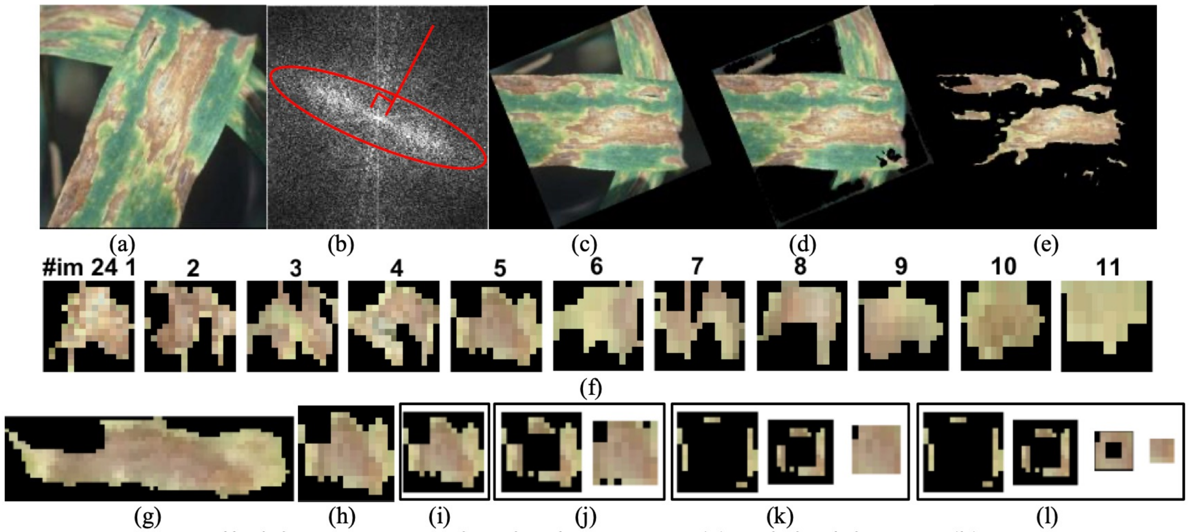


Fig. 4. Uncontrolled image processing in the system (a) original image, (b) DFT on Canny edge, (c) rotated image, (d) partially segmented image, (e) segmented disease, (f) constructed disease patches, (g) original disease blob (#5), (h) normalised disease patch (i) 1-level (j) 2-level (k) 3-level (l) 4-level radial pyramid patches

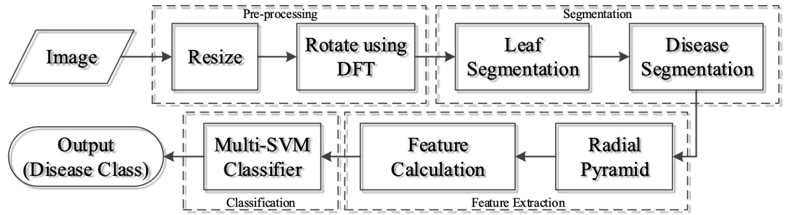


Fig. 1. Our proposed system for wheat disease classification

3.2 Leaf segmentation using multi-resolution discrete Wavelet transform

A single-level two-dimensional Wavelet transform (DWT) is exploited to decompose an image into coefficients of four different components, an approximation component (cA), horizontal, vertical, and diagonal details (cH, cV, and cD). We combined horizontal, vertical and diagonal detail information based on Daubechies wavelets and then thresholded the low coefficient values out to remove part of the background from consideration in the subsequent stages of our process (see Fig. 4 (d)).

Punnarai Siricharoen, Bryan W. Scotney, Philip J. Morrow, Gerard P. Parr:
Automated Wheat Disease Classification Under Controlled and Uncontrolled Image Acquisition. ICIAR 2015: 456-464

EXERCISE #1 USE PYWAVELETS FOR IMAGE SPACE-FREQUENCY ANALYSIS

- Reference: https://pywavelets.readthedocs.io/en/latest/

```python
import pywt
import cv2

original = cv2.imread("kitty.png",0)
LL, (LH, HL, HH) = pywt.dwt2(original, 'haar')
```

```python
# display
fig = plt.figure(figsize=(12, 3))
for i, a in enumerate([LL, LH, HL, HH]):
    ax = fig.add_subplot(1, 4, i + 1)
    ax.imshow(a, interpolation="nearest",
    cmap=plt.cm.gray)
    ax.set_title(titles[i], fontsize=10)
    ax.set_xticks([])
    ax.set_yticks([])
    fig.tight_layout()
plt.show()
```

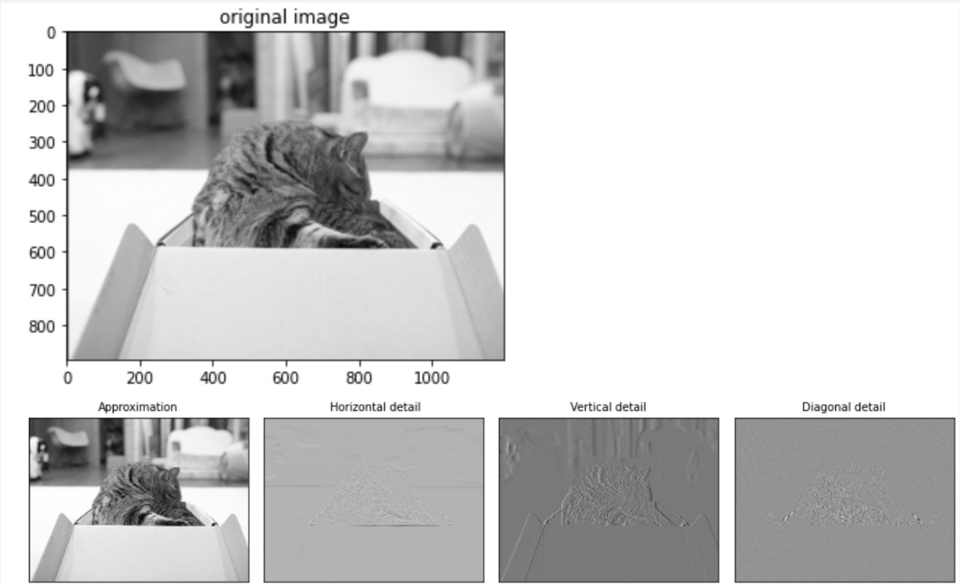
The relation to the other common data layout where all the approximation and details coefficients are stored in one big 2D array is as follows:

```
                        -------------------
                        |        |        |
                        | cA(LL) | cH(LH) |
                        |        |        |
(cA, (cH, cV, cD))  <---> -------------------
                        |        |        |
                        | cV(HL) | cD(HH) |
                        |        |        |
                        -------------------
```

PyWavelets does not follow this pattern because of pure practical reasons of simple access to particular type of the output coeff cients.

EXERCISE #1 USE PYWAVELETS FOR IMAGE SPACE-FREQUENCY ANALYSIS

- Investigate the size of the original image, approximation, horizontal, vertical and diagonal details.
- Investigate the min/max values in the approximation, horizontal, vertical and diagonal details.



Reference:
https://pywavelets.readthedocs.io/en/latest/

EXERCISE #2 CAN YOU SEGMENT THE CAT?

- 1) apply wavelet transform into ”kitty.jpg” image
- 2) Again, investigate the min/max values in the approximation, horizontal, vertical and diagonal details.
- 3) Can you use a threshold to separate the high details from the low and segment the cat out?