Portifolio optimization using Markowitz

A Project Report submitted by

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Abstract

This paper explores the principles of portfolio optimization based on the Markowitz model, which balances expected returns against risk to construct efficient investment portfolios. The Markowitz framework, also known as mean-variance optimization, quantifies risk through the variance and covariance of asset returns, enabling investors to select portfolios that maximize return for a given level of risk. We discuss the theoretical foundation of expected return, return variance, and covariance in forming the efficient frontier—a set of portfolios offering optimal trade-offs between risk and return. Using the covariance matrix and expected return vector, we derive the mathematical formulations for portfolio return and variance, demonstrating the importance of diversification in reducing unsystematic risk. The Lagrangian method is applied to optimize portfolio weights under specific constraints, providing a structured approach to achieving the highest possible return within an investor's risk tolerance. The findings emphasize the importance of systematic asset selection and diversification strategies, illustrating how the Markowitz model remains a cornerstone in modern portfolio theory and investment decision-making.

Contents

Chapter 1

1.1 Introduction

This project explores Markowitz Portfolio Optimization with 10 assets from the Indian stock market. Using data from October 27, 2023, to January 27, 2024, we calculate daily returns and apply mean-variance optimization to construct the efficient frontier. We analyze two portfolios with risk tolerance levels of 5% and 10% to illustrate the trade-off between risk and return.

1.2 Calculating Returns, Variance, and Covariance

The simple daily return for each asset is calculated as:

$$\text{Daily Return} = \frac{P_t - P_{t-1}}{P_{t-1}}$$

where P_t is the closing price on day t.

1.2.1 Return Expectation Vector

Consider a portfolio consisting of N risky assets, each with its own return. Let $r_1, r_2, \ldots, r_N$ denote the returns of these assets. The **return expectation vector $\boldsymbol{\mu}$** is defined as:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix}, \quad \text{where} \quad \mu_i = \mathbb{E}[r_i], \quad i = 1, 2, \ldots, N$$

Here, $\mathbb{E}[r_i]$ represents the expected return of asset i.

1.2.2 Covariance Matrix

The **covariance matrix** Σ captures the covariances between pairs of asset returns and is defined as:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_{NN} \end{pmatrix}, \quad \text{where} \quad \sigma_{ij} = \text{Cov}(r_i, r_j)$$

In this matrix:

- $\sigma_{ii} = \text{Var}(r_i)$ is the variance of the return of asset i.
- $\sigma_{ij} = \text{Cov}(r_i, r_j)$ is the covariance between the returns of assets i and j.

1.2.3 Portfolio Return and Variance

Let $\boldsymbol{w} = (w_1, w_2, \dots, w_N)^T$ denote the vector of portfolio weights, where w_i is the proportion of the total investment allocated to asset i. The weights satisfy the constraint:

$$\sum_{i=1}^{N} w_i = 1$$

1.2.4 Expected Portfolio Return

The expected return of the portfolio μ_p is given by the weighted sum of the expected returns of the individual assets:

$$\mu_p = \mathbb{E}[r_p] = \boldsymbol{w}^T \boldsymbol{\mu} = \sum_{i=1}^{N} w_i \mu_i$$

1.2.5 Portfolio Variance

The variance of the portfolio return σ_p^2 measures the portfolio's risk and is calculated as:

$$\sigma_p^2 = \text{Var}(r_p) = \boldsymbol{w}^T \Sigma \boldsymbol{w} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

1.2.6 Optimization Framework

In the Markowitz framework, investors aim to optimize their portfolios by maximizing expected returns while minimizing risk. This is typically achieved by solving the following optimization problem:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^T \Sigma \boldsymbol{w} \\ \text{subject to} \quad & \sum_{i=1}^{N} w_i = 1 \end{aligned}$$

- A higher τ implies greater emphasis on minimizing risk.
- A lower τ places more emphasis on maximizing returns.

1.2.7 Lagrangian Formulation

To incorporate the constraint $\sum_{i=1}^{N} w_i = 1$, we construct the Lagrangian:

$$\mathcal{L}(\boldsymbol{w}, \lambda) = \boldsymbol{w}^T \boldsymbol{\mu} - \frac{\tau}{2} \boldsymbol{w}^T \Sigma \boldsymbol{w} + \lambda \left(1 - \sum_{i=1}^{N} w_i \right)$$

Taking the first-order conditions for optimality:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \boldsymbol{\mu} - \tau \Sigma \boldsymbol{w} - \lambda \mathbf{1} = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \sum_{i=1}^{N} w_i = 0$$

Solving these equations yields the optimal portfolio weights.

1.3 Project Implemenatation

1.3.1 Name of Assets

The assets include 3 months data of companies like **TVS Electronics**, **Adani Enterprises**, **Infosys**, **SBI**, **ICICI Bank**, **ITC**, **Infosys**, **Zomato**, **Raymond** and **Vodaphone-Idea** covering diverse sectors to optimize risk and return.
The simple daily return for each asset is calculated as:

$$\text{Daily Return} = \frac{P_t - P_{t-1}}{P_{t-1}}$$

where P_t is the closing price on day t.
Using the daily return, the covariance matrix is given as:

Expected Return, Variance and Standard Deviation of our stocks

	1	2	3	4	5	6	7	8	9	10
Expected Return	0.09063894%	-0.24818322%	-0.24818322%	0.07400191%	0.15391811%	-0.00029276%	-0.20428099%	0.00138562%	-0.19426263%	-0.96883011%
Variance	0.10166072%	0.03304105%	0.03304105%	0.02050736%	0.01621847%	0.01070347%	0.02404433%	0.05538585%	0.05335373%	0.15402382%
Std. Dev	3.18842785%	1.81771965%	1.81771965%	1.43203918%	1.27351762%	1.03457564%	1.55062336%	2.35341989%	2.30984259%	3.92458683%

Covariance Matrix

	1.TVEL	2.ADAN	3.INFY	4.SBI	5.ICBK	6.ITC	7.YESB	8.ZOMT	9.RYMD	10.VODA
1.TVEL	0.001016607218	0.000062295256	0.000069160283	0.000066270388	0.000000515914	0.000028027692	0.000125324835	0.000046802652	0.000074508294	0.000059513434
2.ADAN	0.000062295256	0.000330410473	0.000014136369	0.000185432991	0.000117194691	0.000043697027	0.000061623296	0.000141972379	0.000064187385	0.000110675408
3.INFY	0.000069160283	0.000014136369	0.000330410473	0.000091706375	0.000044972855	0.000028313836	-0.000023171157	0.000075694308	0.000069062632	0.000045975973
4.SBI	0.000066270388	0.000185432991	0.000091706375	0.000205073621	0.000059841627	0.000047287850	0.000064027407	0.000081736846	0.000061964069	0.000070812698
5.ICBK	0.000000515914	0.000117194691	0.000044972855	0.000059841627	0.000162184711	0.000011250530	-0.000018290948	0.000015233347	0.000051561373	0.000002077399
6.ITC	0.000028027692	0.000043697027	0.000028313836	0.000047287850	0.000011250530	0.000107034676	0.000050475169	0.000052077210	0.000063259839	0.000019773516
7.YESB	0.000125324835	0.000061623296	-0.000023171157	0.000064027407	-0.000018290948	0.000050475169	0.000240443279	0.000229490679	0.000078501762	0.000343656994
8.ZOMT	0.000046802652	0.000141972379	0.000075694308	0.000081736846	0.000015233347	0.000052077210	0.000229490679	0.000553858518	0.000080069458	0.000164356854
9.RYMD	0.000074508294	0.000064187385	0.000069062632	0.000061964069	0.000051561373	0.000063259839	0.000078501762	0.000080069458	0.000533537280	0.000226964850
10.VODA	0.000059513434	0.000110675408	0.000045975973	0.000070812698	0.000002077399	0.000019773516	0.000343656994	0.000164356854	0.000226964850	0.001540238179

Calculating return and variance using **1.2.6** , we get minimum variance portfolio as,

Minimum Variance Portfolio											
Weights	-0.002949808828	0.020463253489	0.167338393296	-0.051781985464	0.331615234096	0.305693399178	0.381956316663	-0.102411461235	-0.003674610473	-0.046248730721	1
Portfolio Return	0.000324037688										
Portfolio Variance	0.000054340279										
Portfolio Std dev	0.007371585937										

Using the direct formula for minimum variance portfolio as:

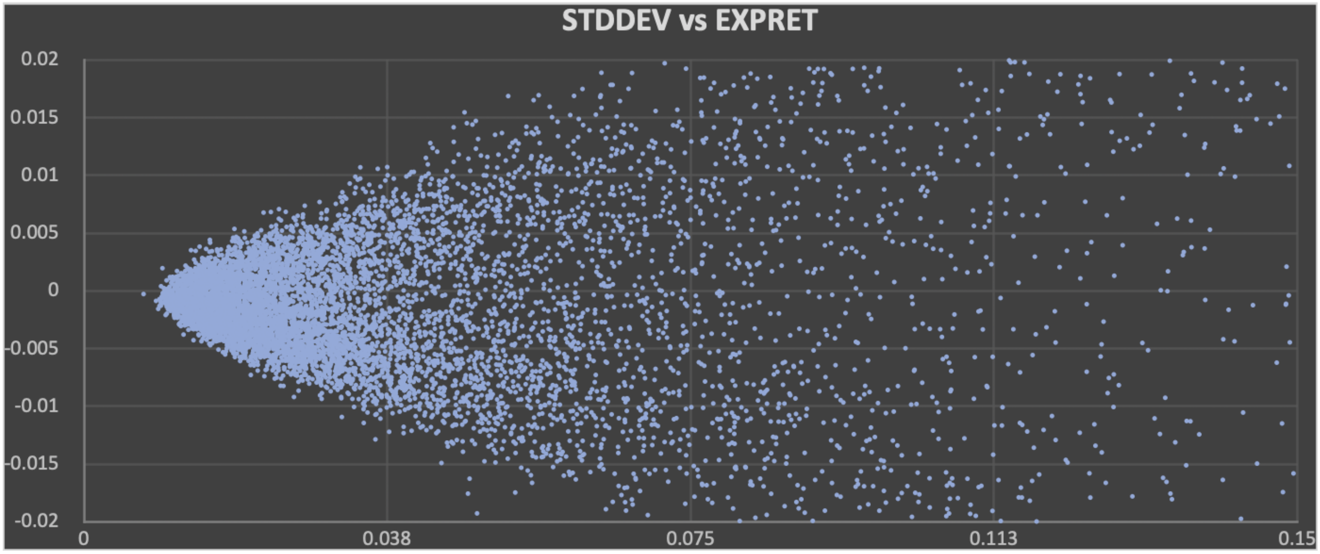
$$w = \frac{uc^{-1}}{uc^{-1}u^T}$$

Covariance Matrix Inverse										
	1259.070805302	-1000.431850637	-984.3247408905	375.0925636956	279.5648503168	501.3882070710	-2309.064977918	330.1723417420	-111.9935420993	433.4184729485
	-1000.431850637	12713.43855868	5597.890739582	-11853.51334042	-4970.437679801	-1918.536206305	7834.972344593	-4635.511557838	260.6055094270	-1757.563577719
	-984.3247408905	5597.890739582	7085.731247349	-7268.567589460	-1897.280681192	-1979.226567291	7791.275756161	-3765.332696606	-120.4144396322	-1532.422376033
	375.0925636956	-11853.51334042	-7268.567589460	18514.40013848	2104.806250708	455.2581939886	-10581.17034977	4928.330013301	-227.2788248847	2039.609398264
	279.5648503168	-4970.437679801	-1897.280681192	2104.806250708	3586.865267975	453.1111266986	-98.51871778877	340.0026030089	-678.1902453139	309.0823536157
	501.3882070710	-1918.536206305	-1979.226567291	455.2581939886	453.1111266986	12569.48708193	-5588.290904413	1566.985345579	-1109.481476755	1237.791531415
	-2309.064977918	7834.972344593	7791.275756161	-10581.17034977	-98.51871778877	-5588.290904413	22693.26579811	-8937.067602792	31.49780897367	-4271.119811249
	330.1723417420	-4635.511557838	-3765.332696606	4928.330013301	340.0026030089	1566.985345579	-8937.067602792	5791.308424059	-151.5993025963	1559.966894823
	-111.9935420993	260.6055094270	-120.4144396322	-227.2788248847	-678.1902453139	-1109.481476755	31.49780897367	-151.5993025963	2243.041389676	-319.9627950701
	433.4184729485	-1757.563577719	-1532.422376033	2039.609398264	309.0823536157	1237.791531415	-4271.119811249	1559.966894823	-319.9627950701	1528.114882798
u vector	1	1	1	1	1	1	1	1	1	1
u*Cinv	-27.10787046967	270.9129395630	2927.328651985	-913.0335460979	6029.005128227	6188.486331926	6625.779343910	-1772.745537319	-123.7759182750	-773.0850262068
u*Cinv*transpose(u)	18431.76449724									
w	-0.001470714888	0.014698155437	0.158819772921	-0.049535873043	0.327098641539	0.335751161146	0.359476128555	-0.096178829627	-0.006715359145	-0.041943082894
Variance	0.000054254165		Which is same as above portfolio Variance which we calculated using 'Solver' Tool							

From both methods, we got the same result for the variance. Which is the minimum variance, and the weights corresponding to this is the minimum variance portfolio.

1.4 Efficient Frontier and Optimization

The efficient frontier represents portfolios with maximum expected returns for a given level of risk. By selecting random weight allocations, we plot the frontier and identify points for specific risk tolerances. The set of optimal portfolios that offer the highest expected return for a defined level of risk forms the **efficient frontier**. Graphically, this is depicted as the upper boundary of the feasible set of portfolios in the mean-variance space.



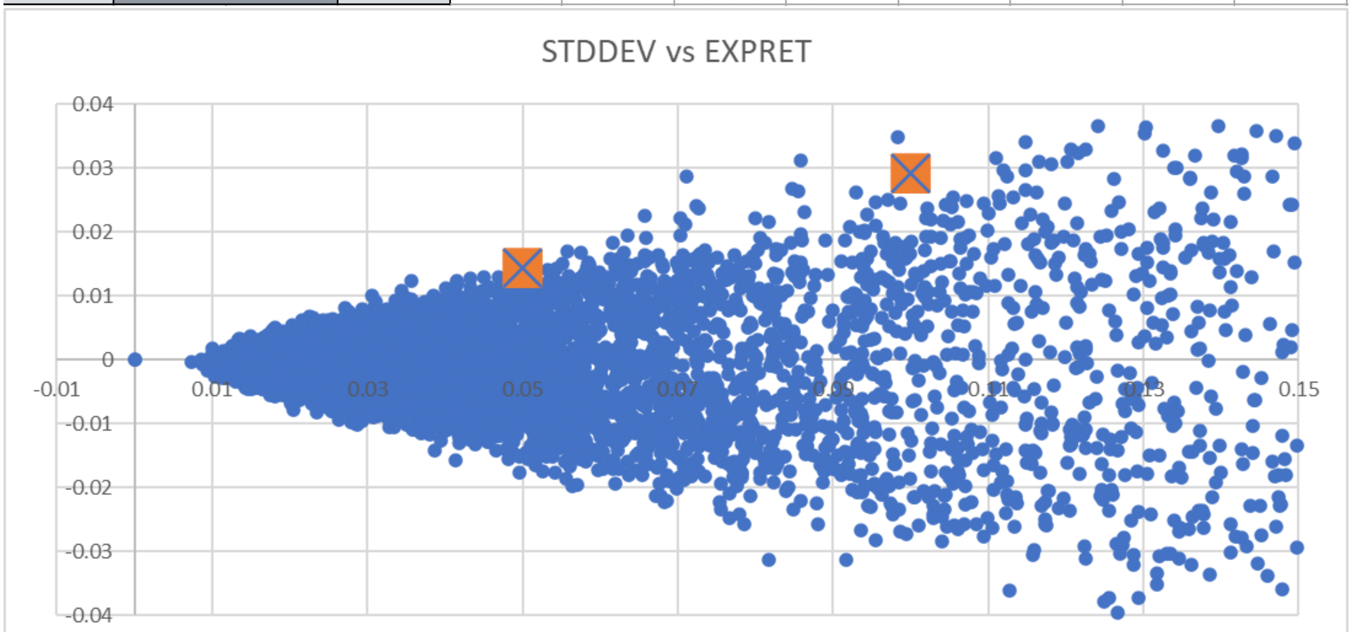
1.5 Portfolio corresponding to varying tolerence

In this study, portfolio optimization was conducted using the Markowitz mean-variance framework for two distinct risk tolerance levels: 0.05 and 0.1. For each level of risk tolerance, the optimal asset weights were determined to maximize the expected return, subject to the constraint of the specified risk. These optimized portfolios correspond to points on the efficient frontier, which represents the set of portfolios offering the highest expected return for a given level of risk. As the risk tolerance increases from 0.05 to 0.1, the portfolio allocation shifts towards higher-risk, higher-return assets, reflecting the investor's willingness to accept more risk for potentially greater returns.

$$
\begin{aligned}
\underset{\boldsymbol{w}}{\text{maximize}} \quad & \boldsymbol{w}^T c \boldsymbol{w} \\
\text{subject to} \quad & \sum_{i=1}^{N} w_i = 1 \\
& \sqrt{wcw^T} = k
\end{aligned}
$$

where k is 0.05 and 0.1.

Consider two risk toleerance level: 5% and 10%											
	w1	w2	w3	w4	w5	w6	w7	w8	w9	w10	sum
	0.6000520	-0.3970875	-0.5124258	-0.1868462	3.2088544	-0.3928394	-0.3478946	-0.1027338	-0.3021492	-0.5669296	1
	Portfolio Return		0.0143918								
5%	Portfolio Std Dev		0.0500000								
	w1	w2	w3	w4	w5	w6	w7	w8	w9	w10	sum
	1.1734038	-0.8780391	-1.1061986	-0.4897717	6.1326967	-0.8901985	-0.7807732	-0.3170480	-0.6943152	-1.1497559	0.99999999
	Portfolio Return		0.0291461								
10%	Portfolio Std Dev		0.1000000								



1.6 Conclusion

This study highlights the foundational principles of the Markowitz model in portfolio optimization, emphasizing the trade-off between expected returns and risk as measured by return variance and covariance. By constructing a portfolio with an optimal balance of assets, investors can aim to maximize returns while managing risk within their tolerance levels. The Markowitz model's use of expected return, covariance matrices, and the concept of efficient frontier provides a robust framework for identifying efficient portfolios that align with various risk preferences.

Through this model, investors are empowered to diversify their investments effectively, reducing exposure to unsystematic risk and achieving a more stable portfolio performance. The efficient frontier represents an essential tool, helping investors visualize the optimal set of portfolios offering the best possible return for a given level of risk. This approach underscores the importance of rigorous mathematical and statistical analysis in financial decision-making and paves the way for more advanced optimization techniques that incorporate additional constraints and real-world complexities.

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