Weekly Challenge 13: Structural Induction

CS/MATH 113 Discrete Mathematics

Spring 2024

1. k-ary tree

Definition 5 in Section 5.3 of our textbook defines a *full binary tree.* We extend this definition to a *full k-ary tree* as follows.

Definition 1 (Full k-ary tree)**.**

Basis Step There is a full k-ary tree consisting only of a single vertex r.

Recursive Step If $T_1, T_2, T_3, \ldots, T_k$ are disjoint full k-ary trees, there is a full k-ary tree, denoted by $T_1 \cdot T_2 \cdot T_3 \cdot \ldots \cdot T_k$, consisting of a root r together with edges connecting the root to each of the roots of $T_1, T_2, T_3, \ldots, T_k$.

We also introduce the following definitions of nodes in a tree.

Definition 2 (Leaf node)**.** A leaf node in a tree is a node that has no children.

Definition 3 (Internal node)**.** An internal node in a tree is a node that is not a leaf node.

Use structural induction to prove the following claim.

Claim 1. *The number of internal nodes in a full k-ary tree with n leaves is $\frac{n-1}{k-1}$.*

Solution:

Proof. Let I(n) be the number of internal nodes in a full k-ary tree with n leaves.

Base Case: When $n = 1$, the full k-ary tree consists of a single vertex r which is both the root and the leaf. The number of internal nodes is $\frac{1-1}{k-1} = 0$. This is clearly the case since there are no internal nodes in a single-vertex tree.

Inductive Hypothesis: Assume that the claim holds for all full k-ary trees till n leaves.

Inductive Step: We will show that the claim holds for a full k-ary tree with $n+1$ leaves.
Let T_k be a full k-ary tree and the sum of the number of leaves in T_k be $n+1$.
By the definition of a k-ary three, T_k can be represented as $T_1 \cdot T_2 \cdot T_3 \cdot \ldots \cdot T_{k-1}$
Then the internal nodes in T_k can be written as:
$I(n+1) \equiv \sum_i^{k-1} \frac{n_i - 1}{k-1}$, where $1 \leq i \leq (k-1)$
This is beacuse the number of internal nodes in T_k is the sum of the number of internal nodes in each of the $k-1$ subtrees.

Since $n_{k-1} \leq n$ we can apply the inductive hypothesis to each of the $k-1$ subtrees.

Since, $I(n+1) \equiv \sum_i^{k-1} n_i \equiv n+1$, where $1 \leq i \leq (k-1)$
Finally we get the equation: $\frac{n}{k-1}$

This matches the equation that we get when $n+1$ is plugged into the formula for $I(n)$ i.e. $\frac{n}{k-1}$

Therefore, the claim holds for a full k-ary tree with $m+1$ leaves.
By the principle of structural induction, the claim holds for all full k-ary trees with n leaves, where $n \geq 1$. $\square$