Weekly Challenge 13: Structural Induction

CS/MATH 113 Discrete Mathematics

Spring 2024

1. k-ary tree

Definition 5 in Section 5.3 of our textbook defines a *full binary tree*. We extend this definition to a *full k-ary tree* as follows.

Definition 1 (Full k-ary tree)**.**

Basis Step There is a full k-ary tree consisting only of a single vertex r.

Recursive Step If $T_1, T_2, T_3, \ldots, T_k$ are disjoint full k-ary trees, there is a full k-ary tree, denoted by $T_1 \cdot T_2 \cdot T_3 \cdot \ldots \cdot T_k$, consisting of a root r together with edges connecting the root to each of the roots of $T_1, T_2, T_3, \ldots, T_k$.

We also introduce the following definitions of nodes in a tree.

Definition 2 (Leaf node)**.** A leaf node in a tree is a node that has no children.

Definition 3 (Internal node)**.** An internal node in a tree is a node that is not a leaf node.

Use structural induction to prove the following claim.

Claim 1. *The number of internal nodes in a full k-ary tree with n leaves is $\frac{n-1}{k-1}$.*

Solution:

Proof. We will prove the claim by structural induction on the number of leaves n in a full k-ary tree.

Base Case: When $n = 1$, the full k-ary tree consists of a single vertex r which is both the root and the leaf. The number of internal nodes is $0 = \frac{1-1}{k-1}$.

Inductive Hypothesis: Assume that the claim holds for all full k-ary trees with n leaves, where $1 \leq n \leq m$ for some $m \geq 1$.

Inductive Step: We will show that the claim holds for a full k-ary tree with $m+1$ leaves. Let $T_1, T_2, \ldots, T_k$ be disjoint full k-ary trees with $n_1, n_2, \ldots, n_k$ leaves, respectively, such that $n_1 + n_2 + \cdots + n_k = m + 1$. By the inductive hypothesis, the number of internal nodes in T_i is $\frac{n_i-1}{k-1}$ for each $i = 1, 2, \ldots, k$. The total number of internal nodes in the full k-ary tree $T_1 \cdot T_2 \cdot \ldots \cdot T_k$ is:

$$\begin{aligned}
\sum_{i=1}^{k} \frac{n_i - 1}{k-1} &= \frac{1}{k-1} \sum_{i=1}^{k} (n_i - 1) \\
&= \frac{1}{k-1} \left(\sum_{i=1}^{k} n_i - k \right) \\
&= \frac{1}{k-1} (m + 1 - k) \\
&= \frac{m+1-k}{k-1}.
\end{aligned}$$

Since $n_1 + n_2 + \cdots + n_k = m + 1$, we have $k = n_1 + n_2 + \cdots + n_k = m + 1$. Therefore, the number of internal nodes in the full k-ary tree with $m + 1$ leaves is $\frac{m+1-k}{k-1} = \frac{m+1-(m+1)}{k-1} = \frac{m}{k-1} = \frac{m}{k-1}$.

By the principle of structural induction, the claim holds for all full k-ary trees with n leaves, where $n \geq 1$. $\square$