Weekly Challenge 15: Matching in Bipartite Graphs

CS/MATH 113 Discrete Mathematics

Spring 2024

1. n-doku (10 points)

Let us define an n-doku as an $n \times n$ grid which contains all the numbers from 1 to n inclusive in the following manner.

- Each number appears exactly n times in the grid.
- Each number appears exactly once in each row of the grid.
- Each number appears exactly once in each column of the grid.

For example here is a 4-doku.

4	3	1	2
3	4	2	1
2	1	4	3
1	2	3	4

(a) $\boxed{\text{2 points}}$ Below is a partially completed 5-doku.

1	2	5	3	4
3	5	2	4	1
5	1	4	2	3

Copy and complete the 5-doku.

Solution:

1	2	5	3	4
3	5	2	4	1
5	1	4	2	3
4	3	1	5	2
2	4	3	1	5

(b) $\boxed{\text{4 points}}$ Show that filling in the next row of an n-doku is equivalent to finding a matching in some 2n-vertex bipartite graph.

Solution:

Proof. Consider a bipartite graph $G = (V, E)$ where there are two disjoint sets of vertices in V, $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, a_n\}$ such that $|A| = |B|$.

For each $i \in \{1, 2, \ldots, n\}$, there is an edge between a_i and b_j if and only if the ith column of the n-doku is missing the number j.
Then, finding a matching in this bipartite graph is equivalent to filling in the next row of the n-doku.
Therefore, filling in the next row of an n-doku is equivalent to finding a matching in some 2n-vertex bipartite graph. □

(c) $\boxed{\text{4 points}}$ Prove that a matching must exist in this bipartite graph and, consequently, that an incomplete n-doku can always be completed.

Solution:

Proof. Applying Hall's Marriage Theorem to the bipartite graph defined in part(b):
Let $S \subseteq A$. We need to show that $|N(S)| \geq |S|$ for every subset S of A.
Consider any subset S of A. Each mumber in S needs to be placed in a different position in B to satisfy the rules.
Since teach position in B can only be occupied by one number, the neighbourhood $N(S)$ contains atleast $|S|$ distict positions i.e. $|N(S)| \geq |S|$.
This satisfies the theorem implying that a perfect matching is guaranteed. This perfect matching corresponds to a valid completion of the n-doku.
Therefore, an incomplete n-doku can always be completed in this manner. □