4) a) $\underline{X} = \underline{W}\vec{x} \Rightarrow \underline{W}\,\underline{X} = \underline{W}^2\vec{x}$

b) $x[n] = \cos\left(\frac{2\pi L}{N}n + \phi\right)$, $L \in \mathbb{Z}$, $L \neq N/2$

$$x[n] = \frac{1}{2}\left[e^{j\phi} e^{j\frac{2\pi}{N}Ln} + e^{-j\phi} e^{-j\frac{2\pi}{N}Ln}\right]$$

$$= \frac{1}{2}\left[e^{j\phi} w_L[n] + e^{-j\phi} w_{N-L}[n]\right]$$

$$X[k] = \langle w_k[n], x[n]\rangle$$

$$= \frac{1}{2}\langle w_k[n], \left(e^{j\phi} w_L[n] + e^{-j\phi} w_{N-L}[n]\right)\rangle$$

$$= \frac{1}{2}\left\{e^{j\phi}\langle w_k[n], w_L[n]\rangle + e^{-j\phi}\langle w_k[n], w_{N-L}[n]\rangle\right\}$$

If $k = L$, $X[k] = \frac{1}{2}\left\{e^{j\phi}N + e^{-j\phi}\langle w_L[n], \cancel{w_{N-L}[n]}\rangle\right\}$ 0

$$= \frac{1}{2}e^{j\phi}N$$

If $k = N-L$, $X[k] = \frac{1}{2}\left\{e^{j\phi}\langle w_{N-L}[n], \cancel{w_L[n]}\rangle + e^{-j\phi}N\right\}$ 0

$$= \frac{1}{2}e^{-j\phi}N$$

Otherwise, $X[k] = 0$

c) $x[n] = (-1)^n = e^{-j2\pi n}$, N odd

$$X[k] = \sum_{n=0}^{N-1} e^{-j2\pi n} e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}[nN+k]}$$

$$= \frac{1 - e^{-j2\pi[nN+k]}}{1 - e^{-j\frac{2\pi}{N}[nN+k]}} = \frac{e^{-j\pi[nN+k]}\left[e^{-j\pi[nN+k]} - e^{j\pi[nN+k]}\right]}{e^{-j\frac{\pi}{N}[nN+k]}\left[e^{-j\frac{\pi}{N}[nN+k]} - e^{j\frac{\pi}{N}[nN+k]}\right]}$$

$$= \frac{\sin\left[\pi(nN+k)\right]}{\sin\left[\frac{\pi}{N}(nN+k)\right]} e^{-j\pi\left[nN+k-n+\frac{k}{N}\right]}$$

$$= \frac{\sin\left[\pi(nN+k)\right]}{\sin\left[\frac{\pi}{N}(nN+k)\right]} e^{-j\pi\left[n(N-1)+k(1-\frac{1}{N})\right]}$$

4c cont'd)

$$\sin[\pi(xN+k)] = 0 \iff xN+k \in \mathbb{Z}$$

If $k=\frac{N}{2}$, $xN+k = N(x+\frac{1}{2}) \notin \mathbb{Z} \Rightarrow \sin(\pi(xN+k)) \neq 0$

5) $\vec{W}^{(k)}[n] = e^{-j\frac{2\pi}{N}nk}$, $0 \leq n \leq N-1$

$$\langle \vec{w}^{(i)}, \vec{w}^{(\ell)} \rangle = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}ni} e^{-j\frac{2\pi}{N}n\ell} = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n(i-\ell)}$$

If $i=\ell$, $\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n(0)} = N$

If $i\neq\ell$, $\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n(i-\ell)} = \frac{1-e^{-j2\pi n(i-\ell)}}{1-e^{j\frac{2\pi}{N}n(i-\ell)}} = 0$ since $i-\ell \in \mathbb{N}$ $\Rightarrow e^{-j2\pi N(i-\ell)} = 1$

6) $y_1[n] = 2\cos\frac{2\pi}{64}4n = e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n} = W_{64}^{(4)}[n] + W_{64}^{(60)}[n]$

$y_2[n] = \frac{1}{2}\sin\frac{2\pi}{64}8n = \frac{1}{4j}\left[e^{j\frac{2\pi}{64}8n} - e^{-j\frac{2\pi}{64}8n}\right] = \frac{1}{4}e^{-\frac{\pi}{2}j}\left[W_{64}^{(8)}[n] - W_{64}^{(56)}[n]\right]$

$y_3[n] = 1$

$$Y_3[k] = \sum_{n=0}^{63} e^{-j\frac{2\pi}{64}nk} = 64\delta[k]$$

$$Y_2[k] = \langle W_k[n], y_2[n] \rangle = \frac{1}{4}\langle W_k[n], e^{-\frac{\pi}{2}j}\left(W_{64}^{(8)}[n] - W_{64}^{(56)}[n]\right)\rangle$$

$$= \begin{cases} -\frac{1}{4}j[64] = -16j, & k=8 \\ \frac{1}{4}j[64] = 16j, & k=56 \end{cases}$$

$$Y_1[k] = \langle W_k[n], (W_{64}^{(4)}[n] + W_{64}^{(60)}[n]) \rangle = \begin{cases} 64, & k=4 \\ 64, & k=60 \end{cases}$$

$$\|X\|_2^2 = 2\cdot 64^2 + 2\cdot 16^2 + 64^2 = 3\cdot 4096 + 512 =$$

7) $x[n] = \cos\left(2\pi \frac{L}{M} n\right), \quad 0 < L \le N-1, \quad 0 < M \le N$

$$x[n] = \frac{1}{2}\left(e^{j\frac{2\pi}{M}Ln} + e^{-j\frac{2\pi}{M}Ln}\right)$$

$$X[k] = \frac{1}{2}\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi}{M}Ln} + e^{-j\frac{2\pi}{M}Ln}\right)e^{-\frac{2\pi}{N}nk}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{j2\pi n\left[\frac{L}{M}-\frac{k}{N}\right]} + e^{-j2\pi n\left[\frac{L}{M}+\frac{k}{N}\right]}\right]$$

If $N=M$

$$X[k] = \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{j\frac{2\pi n}{N}[L-k]} + e^{-j\frac{2\pi}{N}n[L+k]}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{j2\pi n[L-k]}}{1-e^{j\frac{2\pi}{N}n[L-k]}} + \frac{1-e^{-j2\pi n[L+k]}}{1-e^{-j\frac{2\pi}{N}n[L+k]}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j2\pi n[L+k]}}{1-e^{-j\frac{2\pi}{N}n[L+k]}}\right] \qquad \text{Since } L-k \in \mathbb{Z}$$

9) $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} x[N-n]^* e^{-j\frac{2\pi}{N}nk}$

let $m := N-n$

$$\Rightarrow X[k] = \sum_{m=N}^{1} x[m]^* e^{-j\frac{2\pi}{N}(N-m)k}$$

$$= \sum_{m=N}^{1} x[m]^* e^{-j2\pi k} e^{j\frac{2\pi}{N}mk}$$

$$= \sum_{m=N}^{1} x[m]^* e^{-j\frac{2\pi}{N}mk^*}$$

$$= \left(\sum_{m=1}^{N} x[m] e^{-j\frac{2\pi}{N}mk}\right)^* = \left(\sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi}{N}mk}\right)^*$$

$$= X[k]^* \iff X[k] \text{ is real}$$

9 cont'd) $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} -x[N-n]^* e^{-j\frac{2\pi}{N}nk}$

$$= \sum_{m=N}^{1} -x[m]^* e^{-j\frac{2\pi}{N}(N-m)k}$$

$$= \left(-\sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}mk}\right)^* = -X[k]^*$$

$\Rightarrow$ $X[k]$ is pure imaginary