Module 1: Basics of Digital Signal Processing

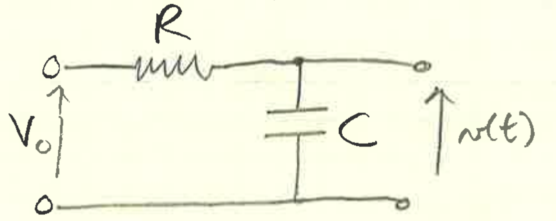
1.1 Introduction to digital signal processing

Signal: Description of evolution of a physical phenomenon

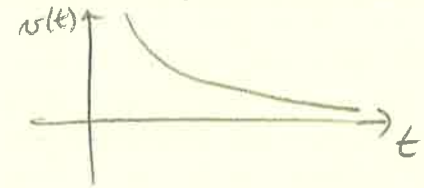
- Weather → temperature
- Sound → pressure
- Sound → magnetic deviation
- Light intensity → gray level on paper

Analysis: understanding the information carried by the signal

Synthesis: creating a signal to contain the given information



$$v(t) = V_0\left(1 - e^{-t/RC}\right)$$

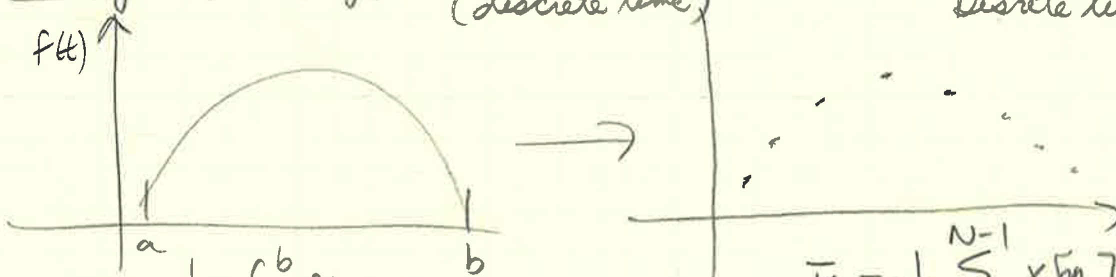


2 dof R, C

Analog signals $f: \mathbb{R} \to \mathbb{V}$

From analog to digital: $f(t) \to$ sample

The digital paradigm (discrete time) Discrete time



$$\bar{x} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

$$\bar{x} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

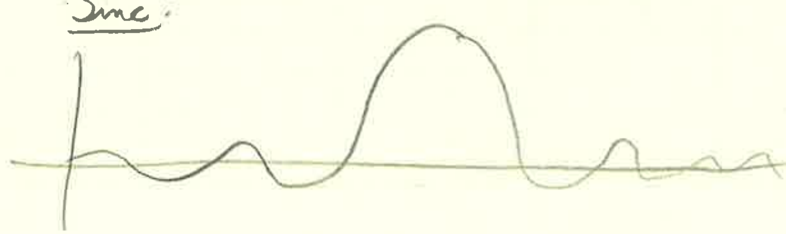
"time"

$$x: \mathbb{Z} \to \mathbb{V} \quad (\mathbb{Z} \text{ or } \mathbb{R})$$

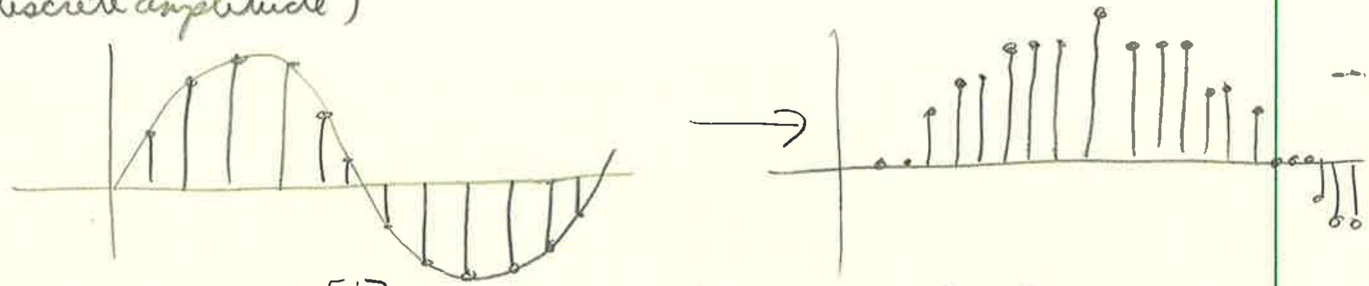
The Sampling Theorem (1920)

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Sinc: Infinite support



(discrete amplitude)

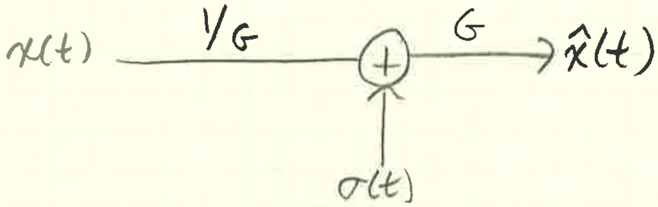


Why is it important?
- storage
- processing
- transmission

<u>Digital Storage</u> : $\{0, 1\}$

<u>Data Transmission</u>

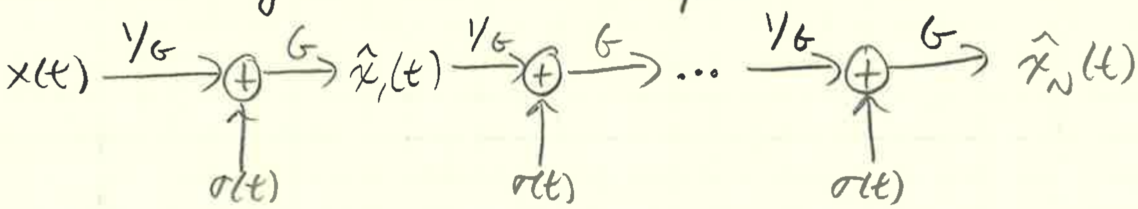
$TX \rightarrow \boxed{\text{channel}} \rightarrow RX$



$$\hat{x}(t) = x(t)/G + G\sigma(t)$$

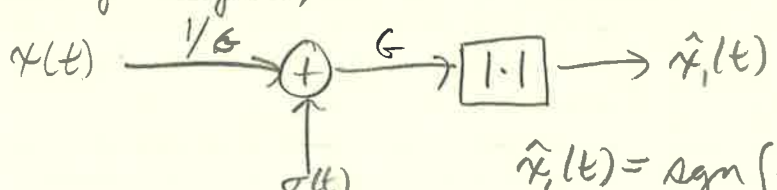
now, it is noisy

For a long channel, we need repeaters

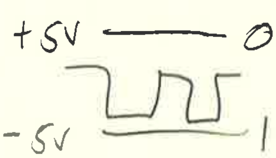


$$\hat{x}_N(t) = x(t) + NG\sigma(t)$$

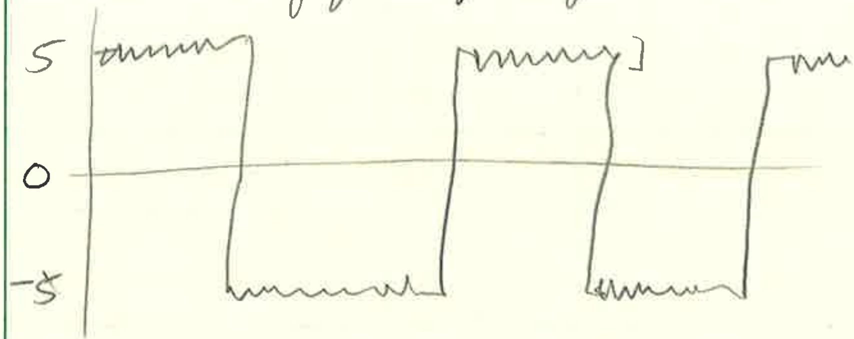
In digital signals, we can threshold



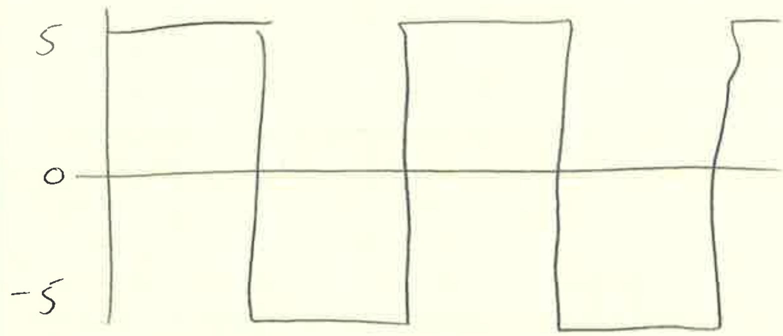
$$\hat{x}_1(t) = \operatorname{sgn}[x(t) + G\sigma(t)]$$



Transmission of quantized signals



$$G\left(x(t)/G + \sigma(t)\right) = x(t) + G(\sigma(t))$$



$$\hat{x}(t) = G \, \mathrm{sgn}\left[x(t) + G\sigma(t)\right]$$

(after thresholding operator)

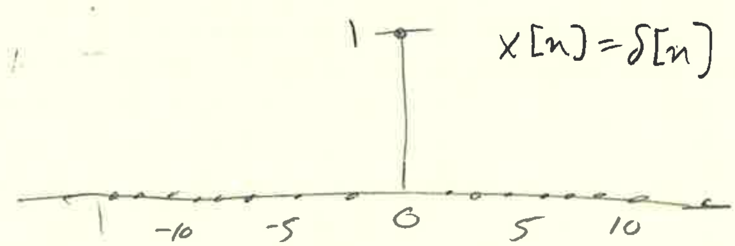
Digital Signal Processing: Key Ideas

- Discretization of time:
  - samples replace idealized models
  - simple math replaces calculus
- Discretization of values:
  - general-purpose of storage
  - general-purpose processing (CPU)
  - noise can be controlled

1.2 Discrete-time signals

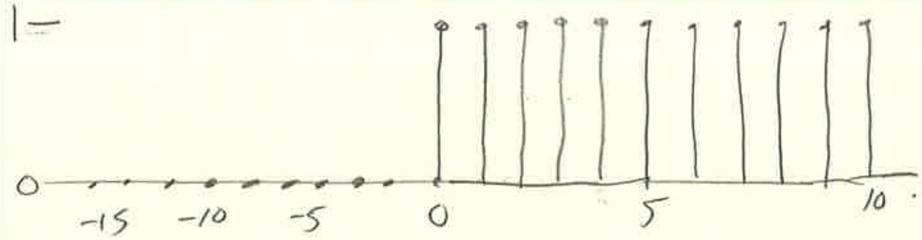
discrete-time signal : a sequence of complex numbers

- One dimension (for now)
- notation : $x[n]$
- two-sided sequences : $x: \mathbb{Z} \to \mathbb{C}$
- n is a-dimensional "time"
- analysis : periodic measurement
- synthesis : stream of generated samples



$x[n] = \delta[n]$

Ex: Used to synchronize audio and video in a movie



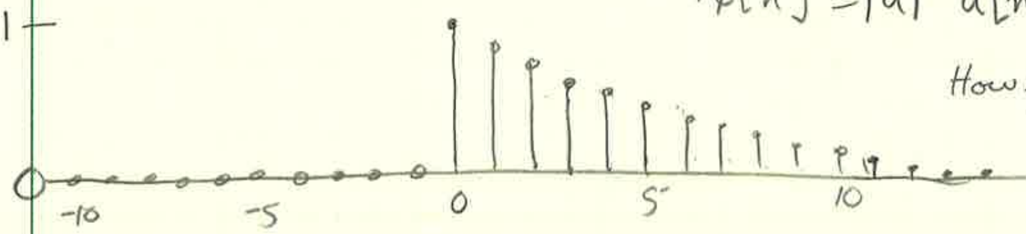
$x[n] = u[n]$

$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

(switch)

Exponential decay

$x[n] = |a|^n u[n]$, $|a| < 1$



How fast does coffee get cold?

Newton's Law of cooling $\frac{dT}{dt} = -c(T - T_{env}) \Rightarrow T(t) = T_{env} + (T_0 - T_{env})e^{-ct}$

Sinusoid $x[n] = \sin(\omega_0 n + \theta)$, ω_0, θ in rad

<u>Four signal classes</u>

- finite-length
- infinite-length
- periodic
- finite-support

<u>Finite-length signals</u>

- sequence notation: $x[n]$, $n = 0, 1, \ldots, N-1$
- vector notation: $\bar{x} = [x_0\ x_1 \ldots x_{N-1}]^T$
- practical entities, good for numerical packages (e.g. numpy)

<u>Infinite-length signals</u>

- sequence notation: $x[n]$, $n \in \mathbb{Z}$
- abstraction, good for theorems

<u>Periodic signals</u>

- N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + kN]$, $n, k, N \in \mathbb{Z}$
- same information as finite-length of length N
- "natural" bridge between finite and infinite lengths

Finite-support signals

- Finite-support sequence:

$$\bar{x}[n] = \begin{cases} x[n], & 0 \le n < N \\ 0, & \text{otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

- same information as finite-length of length N
- another bridge between finite and infinite lengths

Elementary operators

- scaling: $y[n] = \alpha x[n]$, $\alpha \in \mathbb{C}$
- sum: $y[n] = x[n] + z[n]$
- product: $y[n] = x[n] \cdot z[n]$

$\left.\right\} \; 0 \le n \le N-1$

- shift by k (delay): $y[n] = x[n-k]$, $k \in \mathbb{Z}$

Shift of a finite-length: finite-support

$$\ldots 0\ 0\ 0\ \underbrace{x_0\ x_1\ \ldots,\ x_7}_{\bar{x}[n]}\ 0\ 0\ 0 \ldots$$

$$\cdot\ 0\ 0\ 0\ \overbrace{0\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6}^{\bar{x}[n-1]}\ x_7\ 0\ 0 \ldots$$

$$\cdot\ 0\ 0\ 0\ \overbrace{0\ 0\ 0\ 0\ x_0\ x_1\ x_2\ x_3}^{\bar{x}[n-4]}\ x_4\ x_5\ x_6\ x_7\ 0\ 0\ 0 \ldots$$

Shift of a finite length: periodic extension

$$\ldots\ \overbrace{x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7}^{x[n]}\ \ldots$$

$$\ldots\ x_5\ x_6\ x_7\ \overbrace{x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7}^{\tilde{x}[n]}\ x_0\ x_1\ x_2 \ldots$$

$$\ldots\ x_4\ x_5\ x_6\ \overbrace{x_7\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6}^{\tilde{x}[n-1]}\ x_7\ x_0\ x_1 \ldots$$

$$\cdots\ x_1\ x_2\ x_3\ \overbrace{\boxed{x_4\ x_5\ x_6\ x_7\ x_0\ x_1\ x_2\ x_3}}^{\tilde{x}[n-4]}\ x_4\ x_5\ x_6\ \cdots$$

Energy and power

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Energy and power: periodic signals

$$E_{\tilde{x}} = \infty$$

$$P_{\tilde{x}} \equiv \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

1.3 Basic signal processing

1.3.a How your PC plays discrete-time sounds

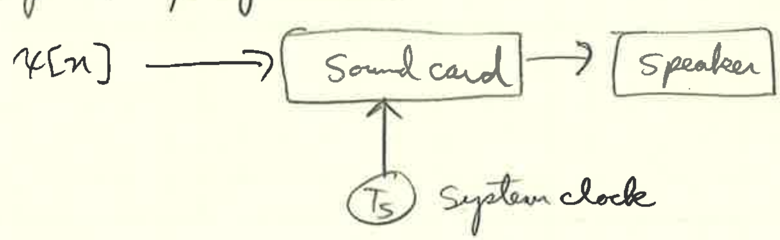
The discrete-time sinusoid

$$x[n] = \sin(\omega_0 n + \theta)$$

Digital vs. physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- Physical world:
  - periodicity: how many seconds before pattern repeats
  - frequency measured in Hz (s^{-1})

How your PC plays sounds



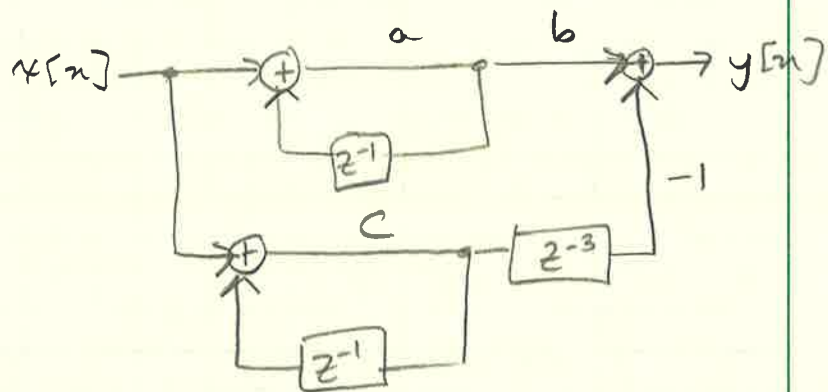
- set T_s, time in seconds between samples
- periodicity of M samples $\rightarrow$ periodicity of MT_s seconds
- real world frequency: $f = \dfrac{1}{MT_s}$ Hz

- usually we choose F_s, the number of samples per second
- $T_s = 1/F_s$

  Eg. for a typical value, $F_s = 48000\ Hz$, $T_s \approx 20.8\ \mu s$.
  If $M = 110$, $f \approx 440\ Hz$

1.3b The Karplus-Strong algorithm

DSP as Meccano



Building blocks:

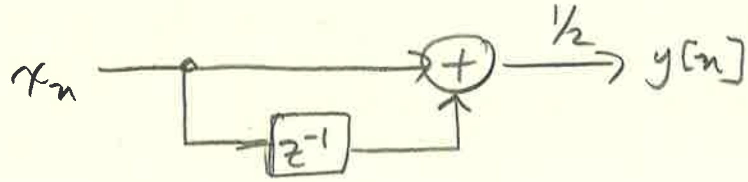
- Adder: $x[n],\ y[n] \longrightarrow \oplus \longrightarrow x[n]+y[n]$
- Multiplier: $x[n] \xrightarrow{\alpha} \alpha x[n]$
- Unit Delay: $x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1]$
- Arbitrary Delay: $x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow x[n-N]$

The 2-point Moving Average

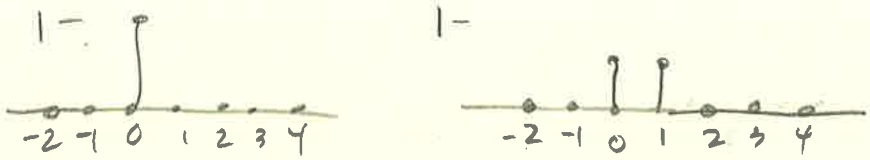
- simple average: $m = \dfrac{a+b}{2}$
- moving average: take a "local" average

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

- DSP Blocks:



Ex: $x[n] = \delta[n]$



$$y[0] = \frac{x[0]+x[-1]}{2} = \frac{1}{2}$$

$$y[1] = \frac{x[1]+x[0]}{2} = \frac{1}{2}$$

- $x[n] = u[n]$

$$y[0] = \frac{x[0]+x[-1]}{2} = \frac{1+0}{2} = \frac{1}{2}$$

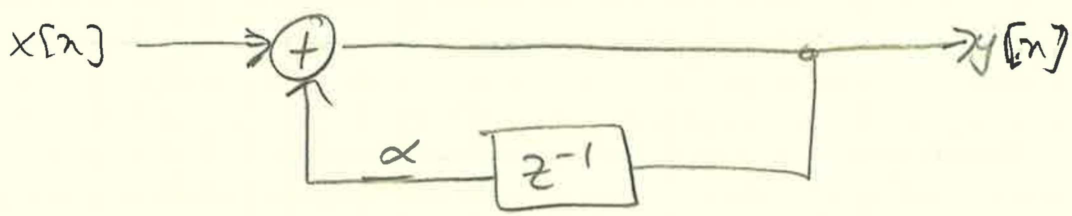
$$y[1] = \frac{x[1]+x[0]}{2} = \frac{1+1}{2} = 1$$

- $x[n] = \cos(\omega n)$, $\omega = \pi/10$

$$y[n] = \frac{\cos \omega n - \cos \omega(n-1)}{2} = \cos(\omega n + \theta)$$

- $x[n] = (-1)^n \Rightarrow y[n] = 0, \forall n$

What if we reverse the loop?



$$y[n] = x[n] + \alpha y[n-1] \quad , \alpha \in \mathbb{R}$$

(recursion)

How we solve the chicken-and-egg problem

Zero Initial conditions

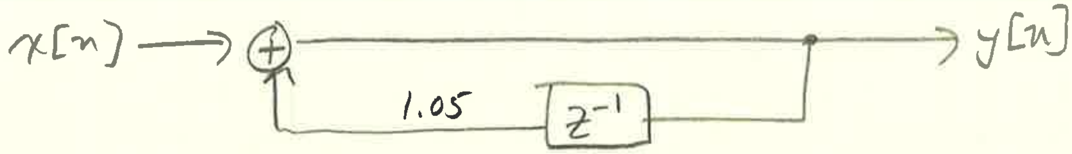
- set a start time (usually $n_0 = 0$)
- assume input and output are zero for all time before n_0

Ex: A simple model for banking

A simple equation to describe compound interest:

- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

$$y[n] = 1.05\, y[n-1] + x[n]$$

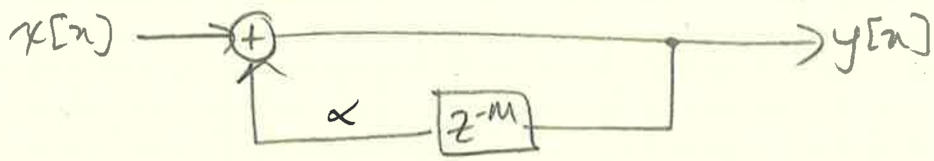


$$y[n] = 1.05\,y[n-1] + x[n]$$

Ex: One-time investment $x[n] = 100\,\delta[n]$

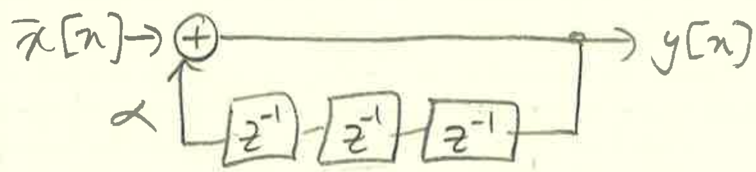
- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25$, $y[3] = 115.7625$, etc.
- In general: $y[n] = (1.05)^n \cdot 100\, u[n]$

An interesting generalization



$$y[n] = \alpha\, y[n-M] + x[n]$$

- Creating loops



$$y[n] = \alpha\, y[n-3] + \bar{x}[n]$$

Ex: $M=3$, $\alpha = 0.7$, $x[n] = \delta[n]$

- $y[0] = 1$, $y[1] = 0$, $y[2] = 0$
- $y[3] = 0.7$, $y[4] = 0$, $y[5] = 0$
- $y[6] = 0.7^2$, $y[7] = 0$, $y[8] = 0$, etc.

Ex: $M=3$, $\alpha = 1$, $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

- $y[0] = 1$, $y[1] = 2$, $y[2] = 3$
- $y[3] = 1$, $y[4] = 2$, $y[5] = 3$
- $y[6] = 1$, $y[7] = 2$, $y[8] = 3$, etc.

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \le n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

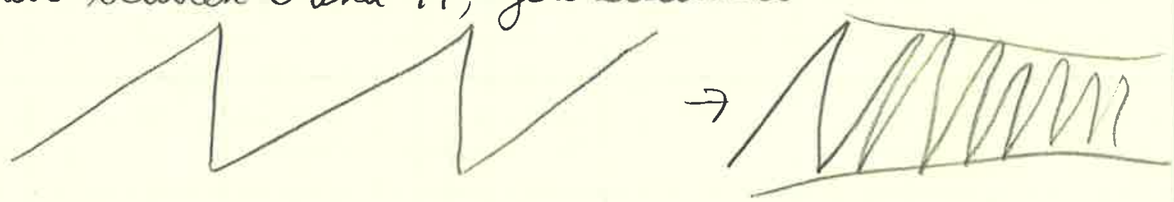
Ex: $M=100$, $\alpha=1$, $\bar{x}[n] = \sin(2\pi n/100)$ for $0 \le n < 100$ and zero elsewhere

$F_S = 48\,kHz \Rightarrow 480\,Hz$

Introducing some realism

- M controls frequency (pitch)
- α controls envelope (decay)
- $\bar{x}[n]$ controls color (timbre)

Proto-violin: $M=100$, $\alpha=0.95$, $\bar{x}[n]$: zero-mean sawtooth wave between 0 and 99, zero elsewhere



The Karplus-Strong Algorithm

$M=100$, $\alpha=0.9$, $\bar{x}[n]$: 100 random values in $[-1,1]$ between 0 and 99, zero elsewhere.

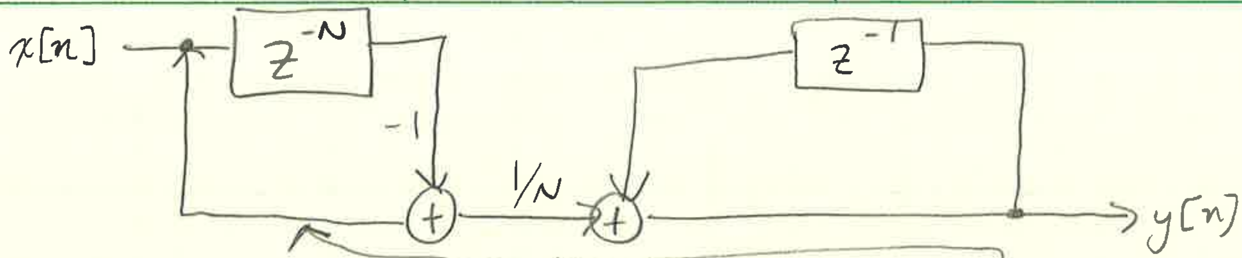
Similar to a harpsichord.

Signal of the Day: Goethe's Temperature Measurement

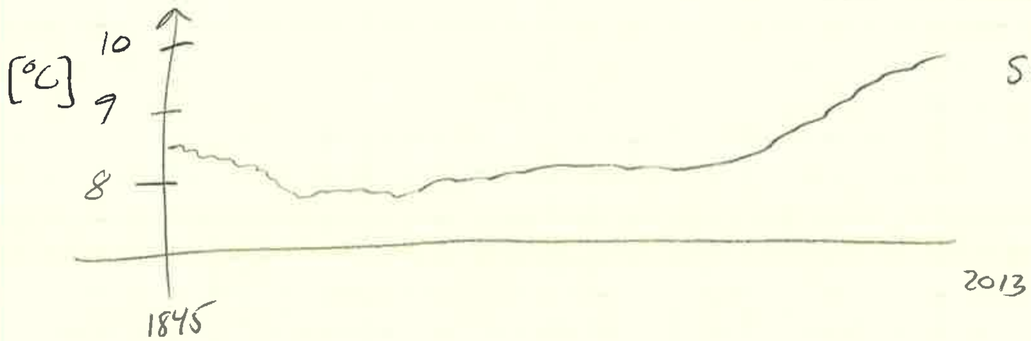
Smoothing
- Moving average: $y[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[n-m]$
- N: window of last observations over which the average is computed

A recursive method

$$y[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[n-m]$$
$$= \frac{1}{N}x[n] + \underbrace{\frac{1}{N}\sum_{m=1}^{N-1} x[n-m] + \frac{1}{N}x[n-N]}_{y[n-1]} - \frac{1}{N}x[n-N]$$
$$= y[n-1] + \frac{1}{N}\left(x[n] - x[n-N]\right)$$



$$y[n] = y[n-1] + \boxed{\tfrac{1}{N}\left(x[n] - x[n-N]\right)}$$



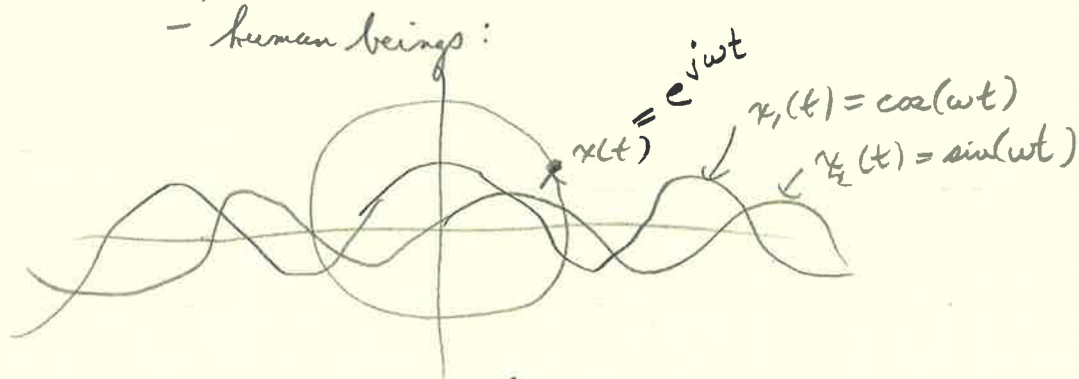
Smoothed by $N=25$

1.4 Complex exponentials

$j = \sqrt{-1}$

Oscillations are everywhere!

- Sustainable **dynamic systems exhibit** oscillatory behavior
- Intuitively: things that don't move in circles can't last:
  - bombs
  - rockets
  - human beings:



- The discrete-time oscillatory heartbeat

  Ingredients:
  - a frequency ω (units: radians)
  - an initial phase ϕ (units: radians)
  - an amplitude A

$$x[n] = Ae^{j(\omega n + \phi)}$$
$$= A\left[\cos(\omega n+\phi) + j\sin(\omega n+\phi)\right]$$

Why complex exponentials?

- we can use complex numbers in digital systems, so why not?
- it makes sense: every sinusoid can always be written as a sum of sine and cosine
- math is simpler: trigonometry becomes algebra

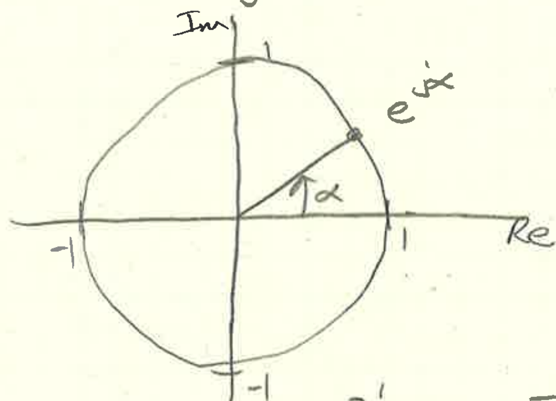
Ex: change the phase of a cosine the "old-school" way

$$\cos(\omega n+\phi) = a\cos(\omega n) - b\sin(\omega n), \quad a=\cos\phi,\ b=\sin\phi$$

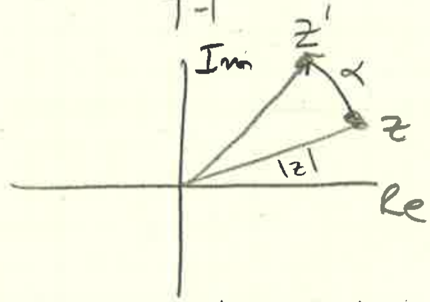
use complex exponentials

$$\cos(\omega n+\phi) = Re\left[e^{j(\omega n+\phi)}\right] = Re\left[e^{j\omega n}e^{j\phi}\right]$$

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$



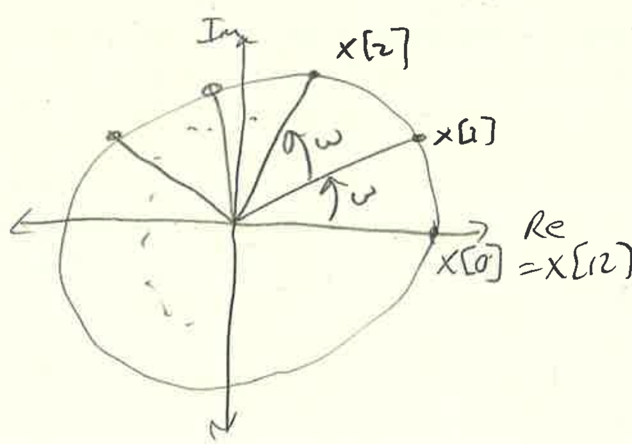
$$|e^{j\alpha}| = 1$$



rotation $z' = z e^{j\alpha}$

The complex exponential generating machine

$$x[n] = e^{j\omega n}; \quad x[n+1] = e^{j\omega}x[n]$$

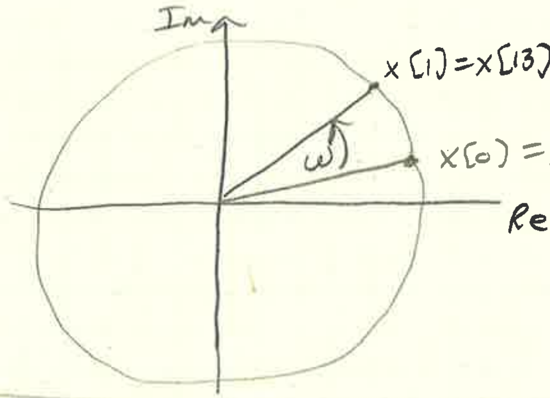


$x[0]=1$

$$\omega = \frac{2\pi}{12}$$

Initial phase

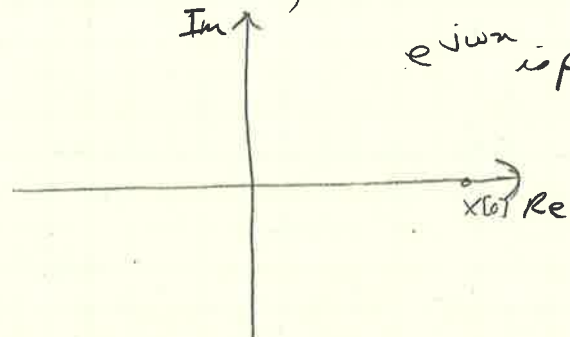
$$x[n] = e^{j(\omega n+\phi)}; \quad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



$$\omega = \frac{2\pi}{12}$$

$$\boxed{\text{Careful: not every sinusoid is periodic in discrete time}}$$

$$x[n] = e^{j\omega n}; \quad x[n+1] = e^{j\omega} x[n]$$



$e^{j\omega n}$ is periodic in $n \Leftrightarrow \omega = \frac{M}{N} 2\pi$, $M, N \in \mathbb{Z}$

$$x[n] = x[n+N]$$

$$e^{j(\omega n+\phi)} = e^{j(\omega(n+N)+\phi)}$$

$$e^{j\omega n} e^{j\phi} = e^{j\omega n} e^{j\omega N} e^{j\phi}$$

$$e^{j\omega N} = 1 \iff \omega N = 2M\pi, \ M \in \mathbb{Z}$$

$$\omega = \frac{M}{N} 2\pi$$

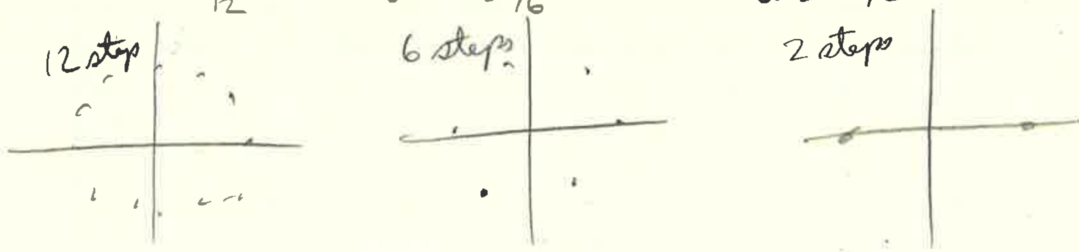
<u>2π-periodicity</u>: one point, many names

$$e^{j\alpha} = e^{j(\alpha + 2\pi k)}, \ \forall k \in \mathbb{Z}$$

One point, many names: Aliasing

How "fast" can we go?

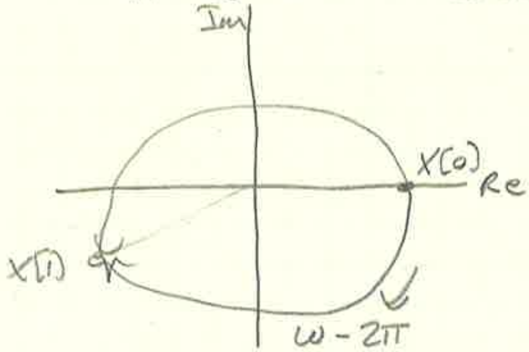
$\omega = \frac{2\pi}{12}$ $\quad$ $\omega = 2\pi/6$ $\quad$ $\omega = 2\pi/2$



What if we go faster?

$\pi < \omega < 2\pi$

corresponds to going slower in opposite direction



$\omega = 2\pi - \alpha$, α small

very slow in opposite direction