Module 1: Basics of Digital Signal Processing

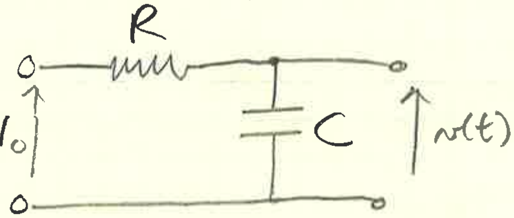
1.1 Introduction to digital signal processing

__Signal__: Description of evolution of a physical phenomenon

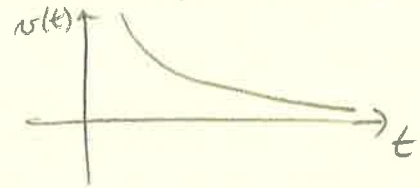
- Weather → temperature
- Sound → pressure
- Sound → magnetic deviation
- Light intensity → gray level on paper

__Analysis__: understanding the information carried by the signal

__Synthesis__: creating a signal to contain the given information



$v(t) = V_0(1-e^{-t/RC})$

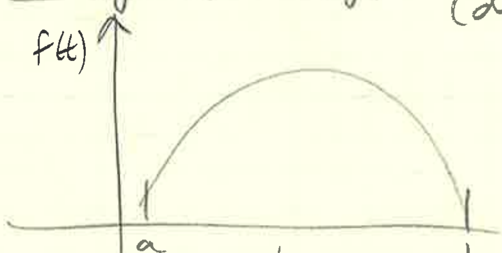


2 dof R, C

__Analog signals__ $f: \mathbb{R} \to \mathbb{V}$

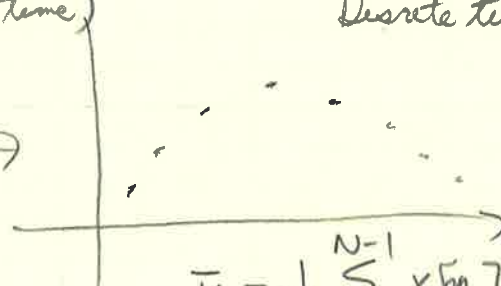
__From analog to digital__: $f(t) \to$ sample

__The digital paradigm__ (discrete time)



$$\bar{x} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

Discrete time



$$\bar{x} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

"time"

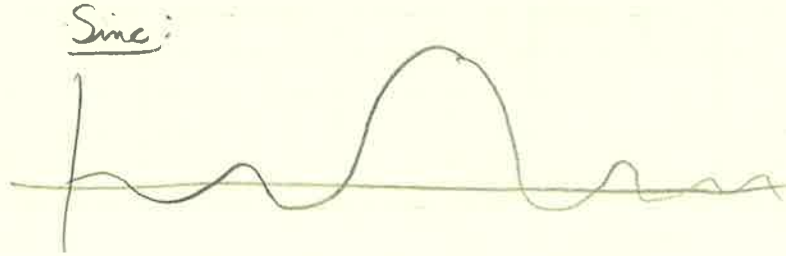
$x: \mathbb{Z} \to \mathbb{V}$ ($\mathbb{Z}$ or $\mathbb{R}$)

__The Sampling Theorem (1920)__

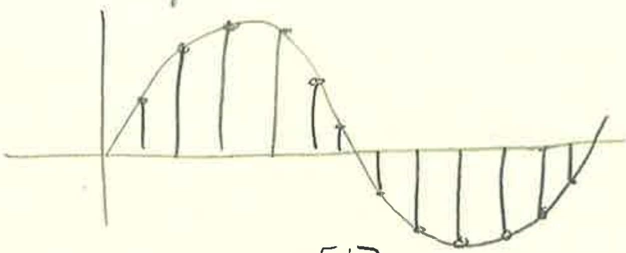
$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right)$$

__Sinc__:

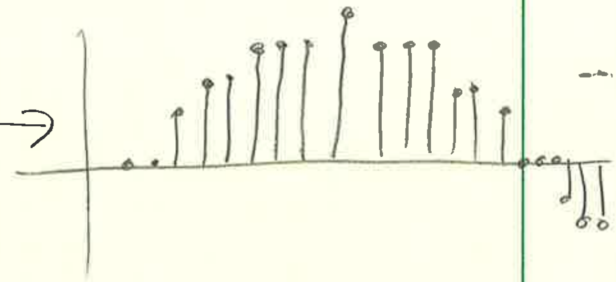
Infinite support



(discrete amplitude)



$x[n]$



$\hat{x}[n]$

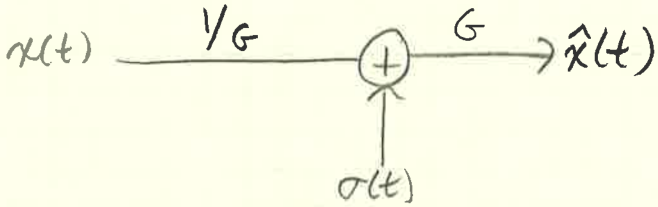
Countable set of levels

Why is it important?
- storage
- processing
- transmission

<u>Digital Storage</u> : $\{0, 1\}$

<u>Data Transmission</u>

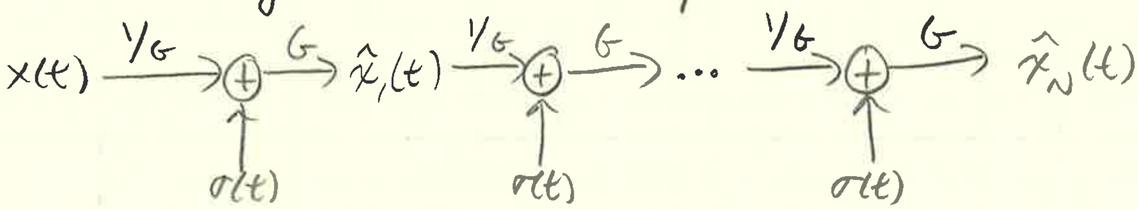
TX → $\boxed{\text{channel}}$ → RX



$\hat{x}(t) = x(t)/G + G\sigma(t)$

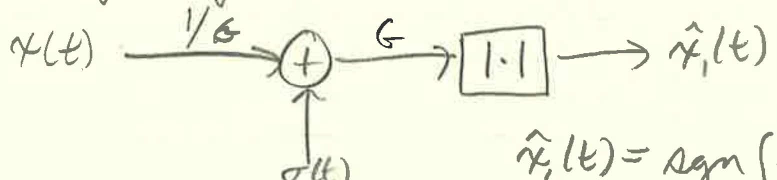
now, it is noisy

For a long channel, we need repeaters

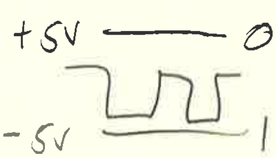


$$\hat{x}_N(t) = x(t) + NG\sigma(t)$$

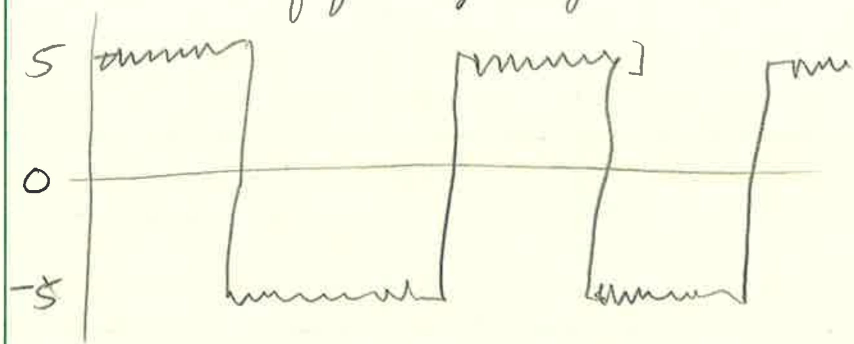
In digital signals, we can threshold



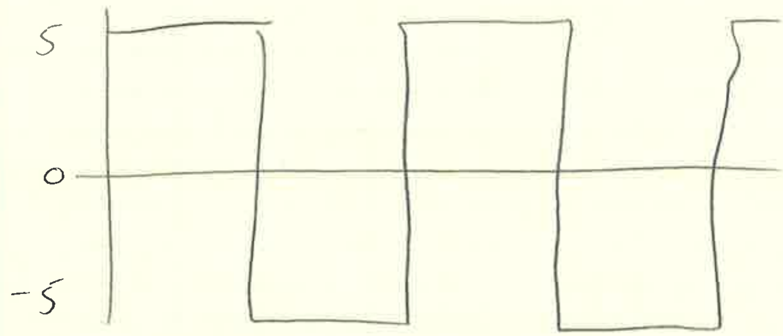
$$\hat{x}_1(t) = \operatorname{sgn}\left[x(t) + G\sigma(t)\right]$$



Transmission of quantized signals



$$G\left(x(t)/G+\sigma(t)\right)=x(t)+G(\sigma(t))$$



$$\hat{x}(t) = G\,\mathrm{sgn}\left[x(t)+G\sigma(t)\right]$$

(after thresholding operator)

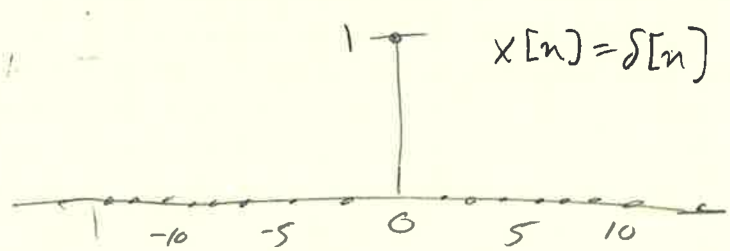
Digital Signal Processing: Key Ideas

- Discretization of time:
  - samples replace idealized models
  - simple math replaces calculus
- Discretization of values:
  - general-purpose of storage
  - general-purpose processing (CPU)
  - noise can be controlled

1.2 Discrete-time signals

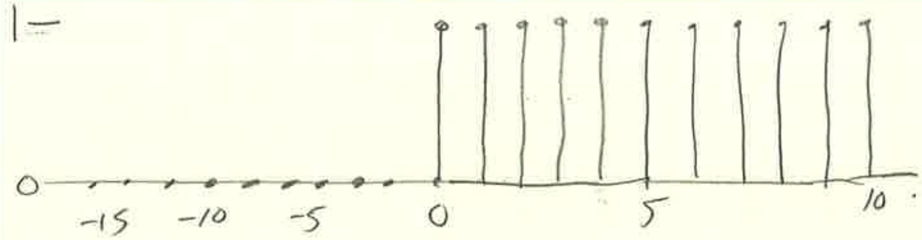
discrete-time signal: a sequence of complex numbers

- One dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x: \mathbb{Z} \to \mathbb{C}$
- n is a-dimensional "time"
- analysis: periodic measurement
- synthesis: stream of generated samples



$x[n] = \delta[n]$

Ex: Used to synchronize audio and video in a movie



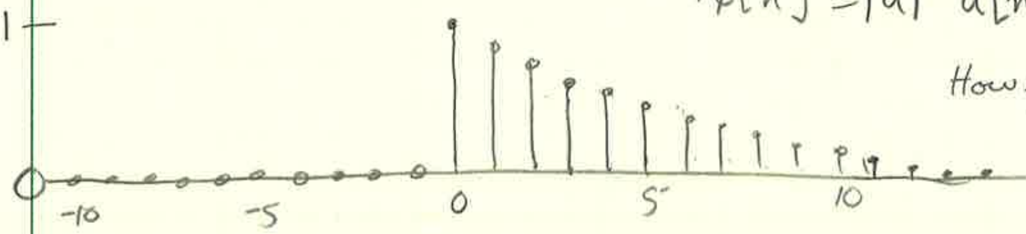
$x[n] = u[n]$

$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

(switch)

Exponential decay

$x[n] = |a|^n u[n]$, $|a| < 1$



How fast does coffee get cold?

Newton's Law of cooling $\frac{dT}{dt} = -c(T - T_{env}) \Rightarrow T(t) = T_{env} + (T_0 - T_{env})e^{-ct}$

Sinusoid $x[n] = \sin(\omega_0 n + \theta)$, ω_0, θ in rad

Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

Finite-length signals

- sequence notation: $x[n]$, $n = 0, 1, \ldots, N-1$
- vector notation: $\bar{x} = [x_0 \; x_1 \ldots \; x_{N-1}]^T$
- practical entities, good for numerical packages (e.g. numpy)

Infinite-length signals

- sequence notation: $x[n]$, $n \in \mathbb{Z}$
- abstraction, good for theorems

Periodic signals

- N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + kN]$, $n, k, N \in \mathbb{Z}$
- same information as finite-length of length N
- "natural" bridge between finite and infinite lengths

Finite-support signals

- Finite-support sequence:

$$\bar{x}[n] = \begin{cases} x[n], & 0 \le n < N \\ 0, & \text{otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

- same information as finite-length of length N
- another bridge between finite and infinite lengths

Elementary operators

- scaling: $y[n] = \alpha x[n]$, $\alpha \in \mathbb{C}$
- sum: $y[n] = x[n] + z[n]$
- product: $y[n] = x[n] \cdot z[n]$

$\Big\}\ 0 \le n \le N-1$

- shift by k (delay): $y[n] = x[n-k]$, $k \in \mathbb{Z}$

Shift of a finite-length: finite-support

$$\ldots 0\ 0\ 0\ \underbrace{x_0\ x_1\ \ldots,\ x_7}_{\bar{x}[n]}\ 0\ 0\ 0 \ldots$$

$$\ldots 0\ 0\ 0\ \overbrace{0\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6}^{\bar{x}[n-1]}\ x_7\ 0\ 0 \ldots$$

$$\ldots 0\ 0\ 0\ \overbrace{0\ 0\ 0\ 0\ x_0\ x_1\ x_2\ x_3}^{\bar{x}[n-4]}\ x_4\ x_5\ x_6\ x_7\ 0\ 0\ 0 \ldots$$

Shift of a finite length: periodic extension

$$\ldots \overbrace{x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7}^{x[n]} \ldots$$

$$\ldots x_5\ x_6\ x_7\ \overbrace{x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7}^{\tilde{x}[n]}\ x_0\ x_1\ x_2 \ldots$$

$$\ldots x_4\ x_5\ x_6\ \overbrace{x_7\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6}^{\tilde{x}[n-1]}\ x_7\ x_0\ x_1 \ldots$$

$$\cdots x_1\, x_2\, x_3 \underbrace{\boxed{x_4\, x_5\, x_6\, x_7\, x_0\, x_1\, x_2\, x_3}}_{} x_4\, x_5\, x_6 \cdots$$

$\tilde{x}[n-4]$

Energy and power

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Energy and power: periodic signals

$$E_{\tilde{x}} = \infty$$

$$P_{\tilde{x}} \equiv \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

1.3 Basic signal processing

1.3.a How your PC plays discrete-time sounds

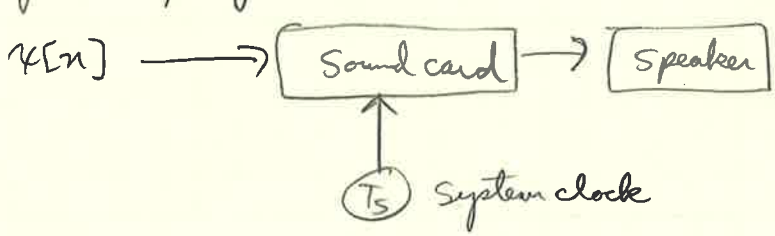
The discrete-time sinusoid

$$x[n] = \sin(\omega_0 n + \theta)$$

Digital vs. physical frequency

- **Discrete** time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- Physical world:
  - periodicity: how many seconds before pattern repeats
  - frequency measured in Hz (s^{-1})

How your PC plays sounds



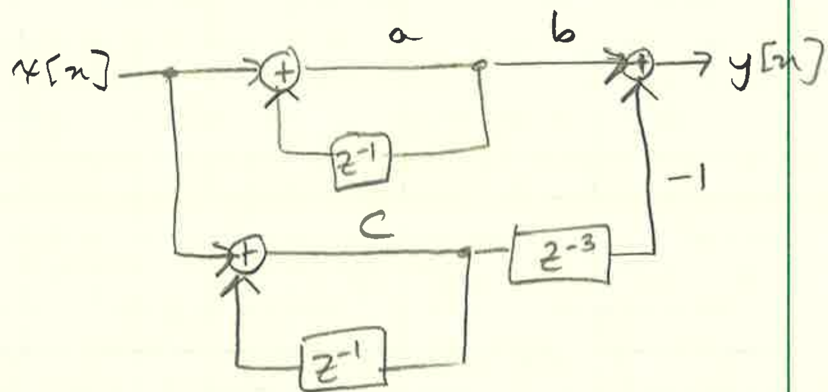
- set T_s, time in seconds between samples
- periodicity of M samples → periodicity of MT_s seconds
- real world frequency: $f = \dfrac{1}{MT_s}$ Hz

- usually we choose F_s, the number of samples per second
- $T_s = 1/F_s$

  Eg. for a typical value, $F_s = 48000$ Hz, $T_s \approx 20.8\mu s$.
  If $M = 110$, $f \approx 440$ Hz

1.3 b The Karplus-Strong algorithm

DSP as Meccano



Building blocks:

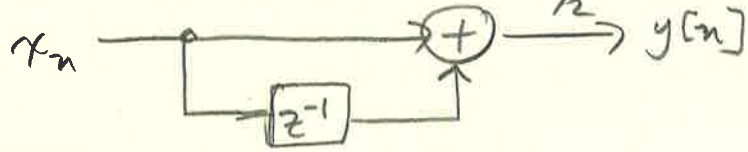
- Adder: $x[n]$, $y[n]$ $\to \oplus \longrightarrow x[n]+y[n]$
- Multiplier: $x[n] \xrightarrow{\alpha} \alpha x[n]$
- Unit Delay: $x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1]$
- Arbitrary Delay: $x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow x[n-N]$

The 2-point Moving Average

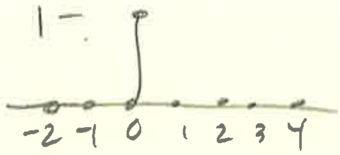
- simple average: $m = \dfrac{a+b}{2}$
- moving average: take a "local" average

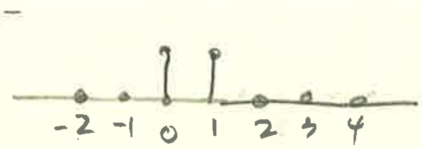
$$y[n] = \frac{x[n] + x[n-1]}{2}$$

- DSP Blocks:



Ex: $x[n] = \delta[n]$





$$y[0] = \frac{x[0] + x[-1]}{2} = \frac{1}{2}$$

$$y[1] = \frac{x[1] + x[0]}{2} = \frac{1}{2}$$

- $x[n] = u[n]$

$$y[0] = \frac{x[0] + x[-1]}{2} = \frac{1+0}{2} = \frac{1}{2}$$

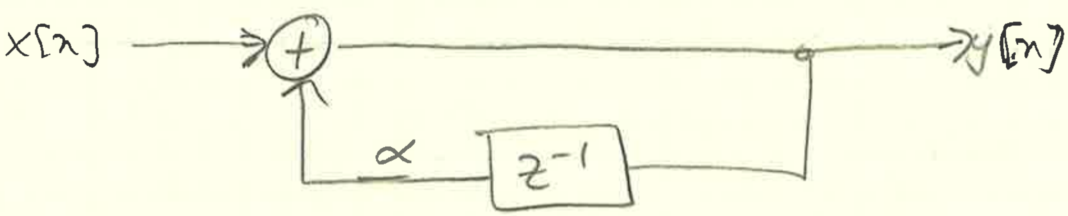
$$y[1] = \frac{x[1] + x[0]}{2} = \frac{1+1}{2} = 1$$

- $x[n] = \cos(\omega n)$, $\omega = \pi/10$

$$y[n] = \frac{\cos \omega n - \cos \omega(n-1)}{2} = \cos(\omega n + \theta)$$

- $x[n] = (-1)^n \Rightarrow y[n] = 0, \forall n$

What if we reverse the loop?



$$y[n] = x[n] + \alpha y[n-1] \quad , \alpha \in \mathbb{R}$$

(recursion)

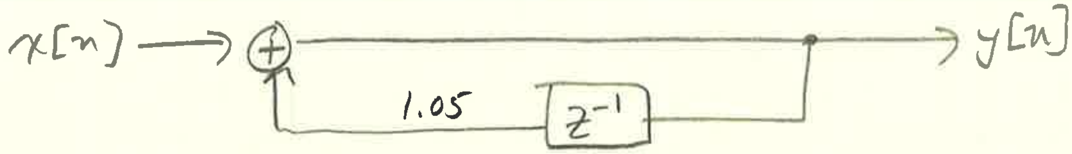
How we solve the chicken-and-egg problem

Zero Initial conditions
- set a start time (usually $n_0 = 0$)
- assume input and output are zero for all time before n_0

Ex: A simple model for banking

A simple equation to describe compound interest:
- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

$$y[n] = 1.05\, y[n-1] + x[n]$$

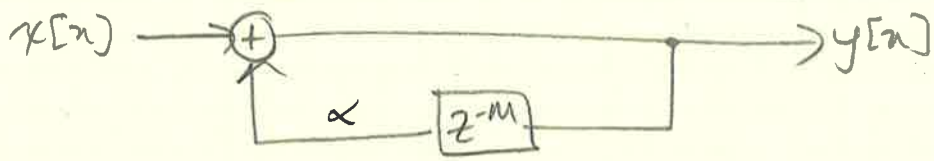


$$y[n] = 1.05\,y[n-1] + x[n]$$

Ex: One-time investment $x[n] = 100\,\delta[n]$

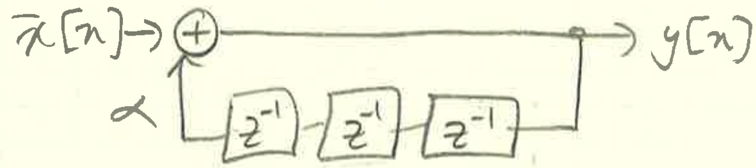
- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25$, $y[3] = 115.7625$, etc.
- In general: $y[n] = (1.05)^n \cdot 100\, u[n]$

An interesting generalization



$$y[n] = \alpha\, y[n-M] + x[n]$$

- Creating loops



$$y[n] = \alpha\, y[n-3] + \bar{x}[n]$$

Ex: $M = 3$, $\alpha = 0.7$, $x[n] = \delta[n]$

- $y[0] = 1$, $y[1] = 0$, $y[2] = 0$
- $y[3] = 0.7$, $y[4] = 0$, $y[5] = 0$
- $y[6] = 0.7^2$, $y[7] = 0$, $y[8] = 0$, etc.

Ex: $M = 3$, $\alpha = 1$, $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

- $y[0] = 1$, $y[1] = 2$, $y[2] = 3$
- $y[3] = 1$, $y[4] = 2$, $y[5] = 3$
- $y[6] = 1$, $y[7] = 2$, $y[8] = 3$, etc.

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \le n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

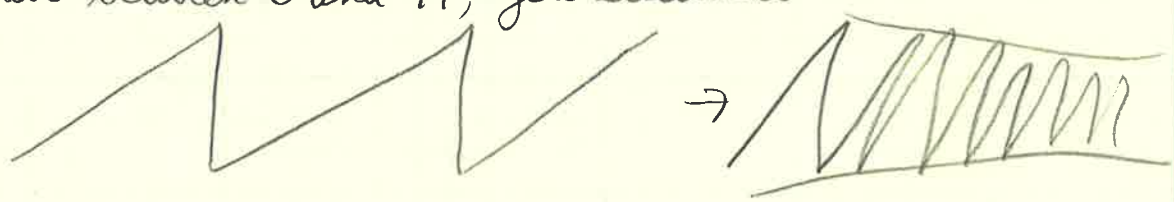
Ex: $M = 100$, $\alpha = 1$, $\bar{x}[n] = \sin(2\pi n/100)$ for $0 \le n < 100$ and zero elsewhere

$$F_S = 48\,\text{kHz} \to 480\,\text{Hz}$$

Introducing some realism

- M controls frequency (pitch)
- α controls envelope (decay)
- $\bar{x}[n]$ controls color (timbre)

Proto-violin: $M = 100$, $\alpha = 0.95$, $\bar{x}[n]$: zero-mean sawtooth wave between 0 and 99, zero elsewhere



The Karplus-Strong Algorithm

$M = 100$, $\alpha = 0.9$, $\bar{x}[n]$: 100 random values in $[-1, 1]$ between 0 and 99, zero elsewhere.

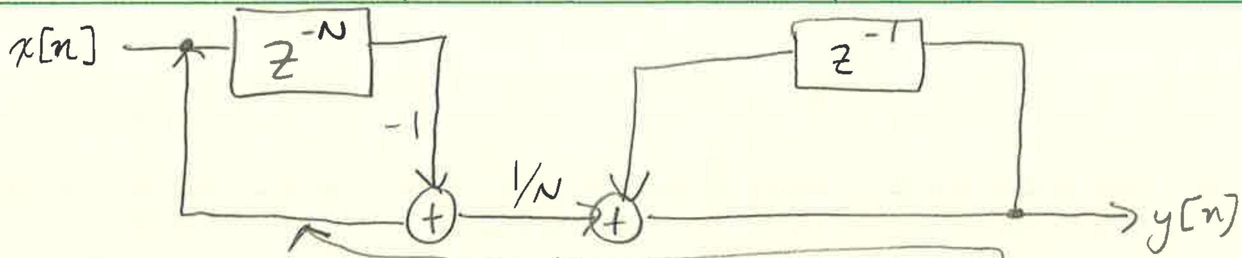
Similar to a harpsichord.

Signal of the Day: Goethe's Temperature Measurement

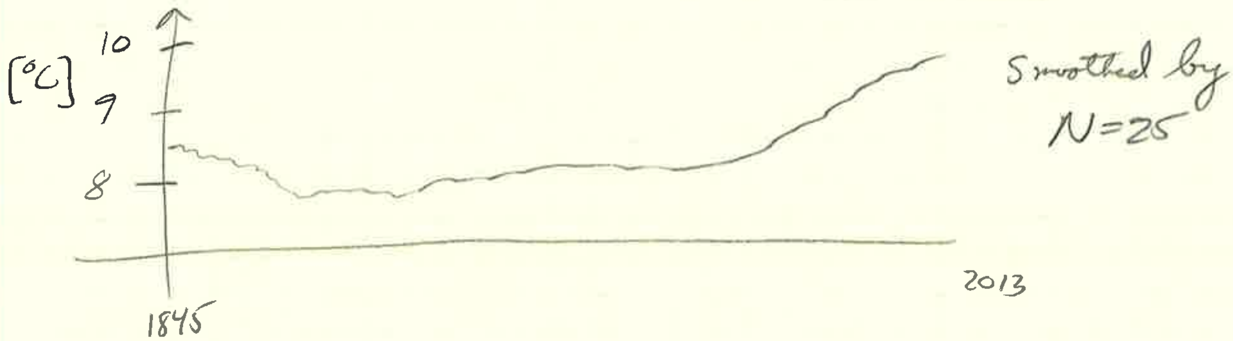
Smoothing
- Moving average: $y[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[n-m]$
- N: window of last observations over which the average is computed

A recursive method

$$
\begin{aligned}
y[n] &= \frac{1}{N}\sum_{m=0}^{N-1} x[n-m] \\
&= \frac{1}{N}x[n] + \underbrace{\frac{1}{N}\sum_{m=1}^{N-1} x[n-m] + \frac{1}{N}x[n-N]}_{y[n-1]} - \frac{1}{N}x[n-N] \\
&= y[n-1] + \frac{1}{N}\left(x[n] - x[n-N]\right)
\end{aligned}
$$



$$y[n] = y[n-1] + \boxed{1/N \left(x[n] - x[n-N] \right)}$$

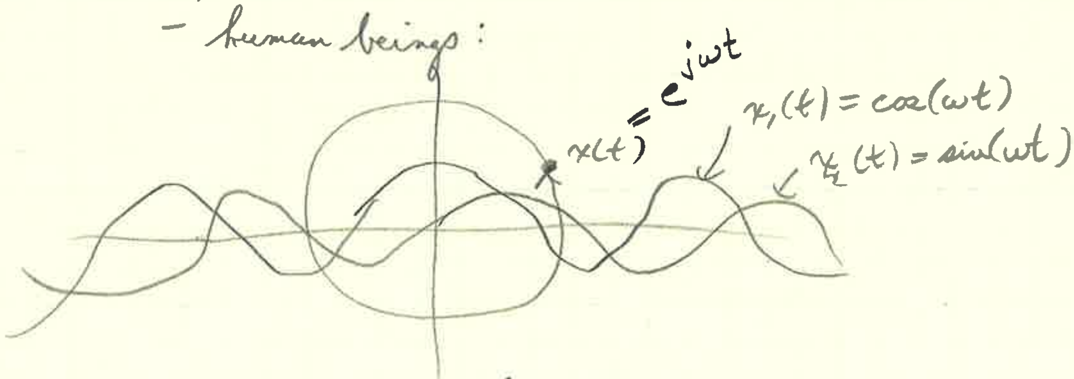


1.4 Complex exponentials

$j = \sqrt{-1}$

Oscillations are everywhere!

- Sustainable dynamic systems exhibit oscillatory behavior
- Intuitively: things that don't move in circles can't last:
  - bombs
  - rockets
  - human beings:



- The discrete-time oscillatory heartbeat
  Ingredients:
  - a frequency ω (units: radians)
  - an initial phase ϕ (units: radians)
  - an amplitude A

$$x[n] = Ae^{j(\omega n + \phi)}$$
$$= A\left[\cos(\omega n + \phi) + j \sin(\omega n + \phi)\right]$$

Why complex exponentials?

- we can use complex numbers in digital systems, so why not?
- it makes sense: every sinusoid can always be written as a sum of sine and cosine
- math is simpler: trigonometry becomes algebra

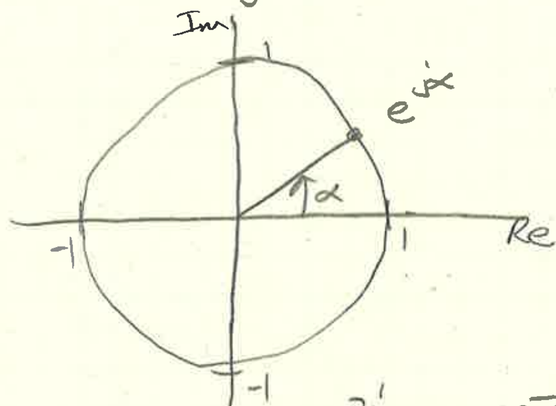
Ex: change the phase of a cosine the "old-school" way

$$\cos(\omega n+\phi) = a\cos(\omega n) - b\sin(\omega n), \quad a=\cos\phi,\ b=\sin\phi$$

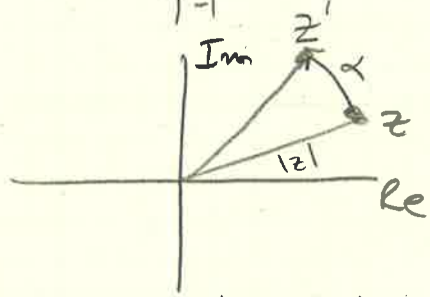
use complex exponentials

$$\cos(\omega n+\phi) = Re\left[e^{j(\omega n+\phi)}\right] = Re\left[e^{j\omega n}e^{j\phi}\right]$$

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$



$|e^{j\alpha}| = 1$

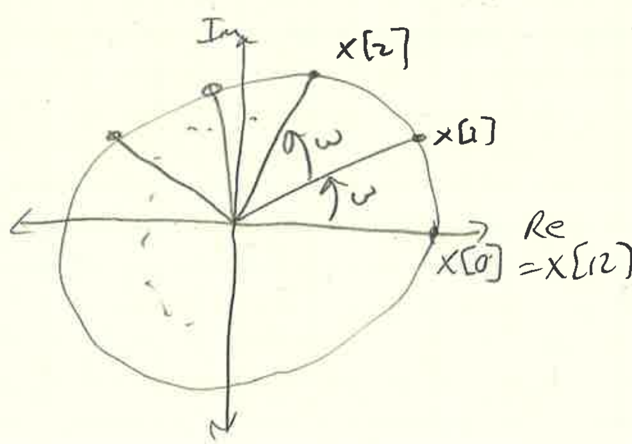


rotation $z' = ze^{j\alpha}$

The complex exponential generating machine

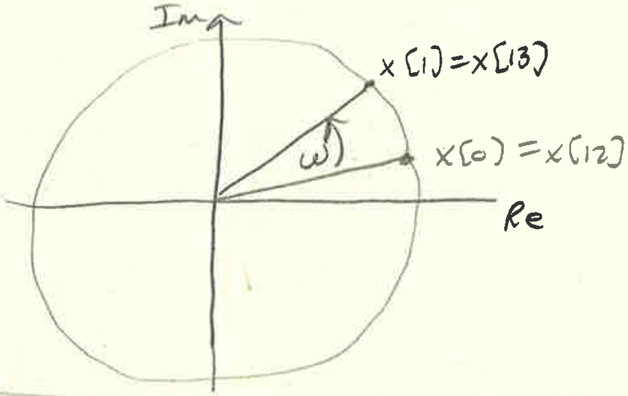
$$x[n] = e^{j\omega n}; \quad x[n+1] = e^{j\omega}x[n]$$

$x[0]=1$



$\omega = \frac{2\pi}{12}$

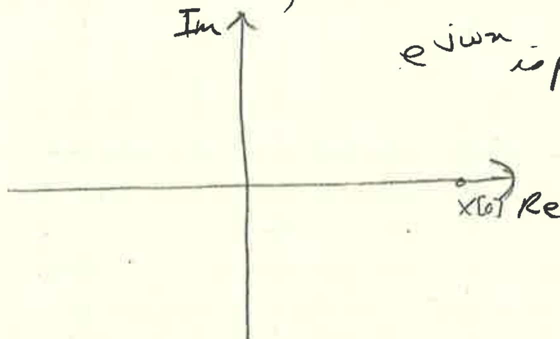
Initial phase
$x[n] = e^{j(\omega n+\phi)}$; $x[n+1] = e^{j\omega} x[n]$, $x[0] = e^{j\phi}$



$\omega = \frac{2\pi}{12}$

$\boxed{\text{Careful: not every sinusoid is periodic in discrete time}}$

$x[n] = e^{j\omega n}$; $x[n+1] = e^{j\omega} x[n]$



$e^{j\omega n}$ is periodic in $n \iff \omega = \frac{M}{N} 2\pi$,
$M, N \in \mathbb{Z}$

$$x[n] = x[n+N]$$
$$e^{j(\omega n+\phi)} = e^{j(\omega(n+N)+\phi)}$$
$$e^{j\omega n} e^{j\phi} = e^{j\omega n} e^{j\omega N} e^{j\phi}$$
$$e^{j\omega N} = 1 \iff \omega N = 2M\pi,\ M \in \mathbb{Z}$$
$$\omega = \frac{M}{N} 2\pi$$

2π-periodicity : one point, many names
$$e^{j\alpha} = e^{j(\alpha + 2\pi k)},\ \forall k \in \mathbb{Z}$$

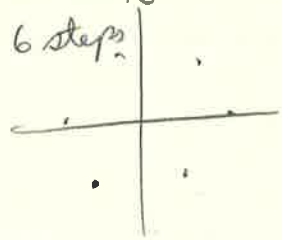
One point, many names : Aliasing

How "fast" can we go?

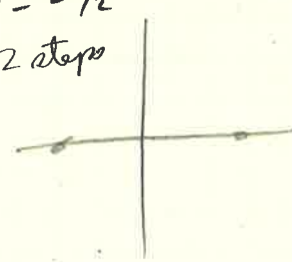
$\omega = \frac{2\pi}{12}$ $\qquad$ $\omega = 2\pi/6$ $\qquad$ $\omega = 2\pi/2$

12 steps

6 steps

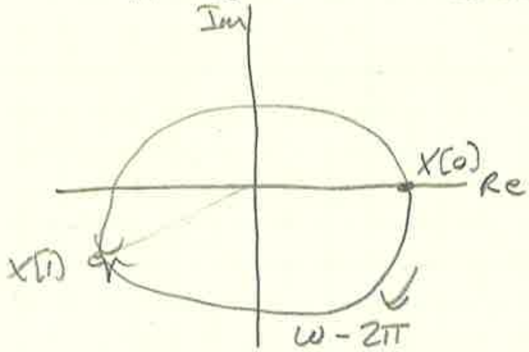


2 steps



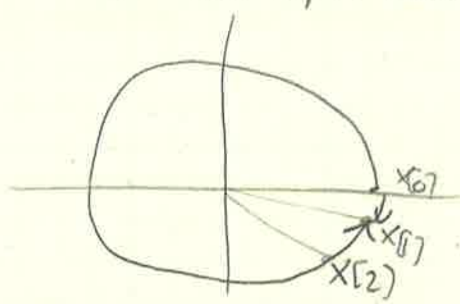
What if we go faster?

$\pi < \omega < 2\pi$ corresponds to going slower in opposite direction



$\omega = 2\pi - \alpha$, α small

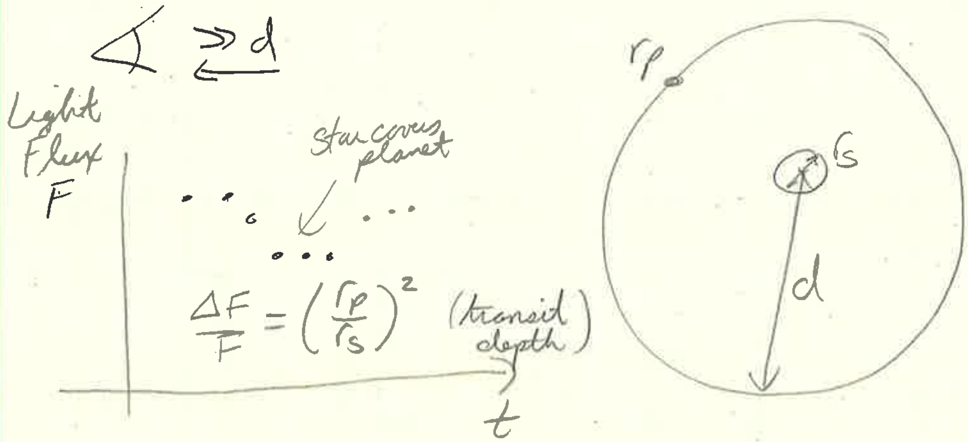
very slow in opposite direction



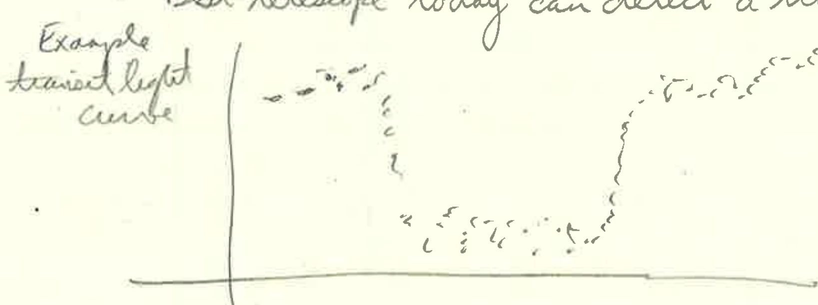
2.1 Signal processing and vector spaces

Common framework: vector space
- vector spaces are very general objects
- vector spaces are defined by their properties
- once you know the properties are satisfied, you can use all the tools for the space

Signal of the day: exoplanet hunting



- Earth: $\frac{\Delta F}{F} = \left(\frac{r_p}{r_s}\right)^2 = \left(\frac{6,371}{696,000}\right)^2 \approx 0.01\%$
- Jupiter: $\frac{\Delta F}{F} = \left(\frac{69,911}{696,000}\right)^2 \approx 1\%$
- Best telescope today can detect a transit depth of 0.1%



2.2 Vector Spaces

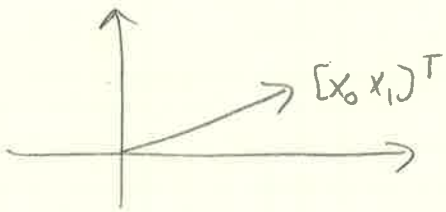
2.2.a Vector space

Some familiar examples
- $\mathbb{R}^2, \mathbb{R}^3$: Euclidean space
- $\mathbb{R}^N, \mathbb{C}^N$: linear algebra

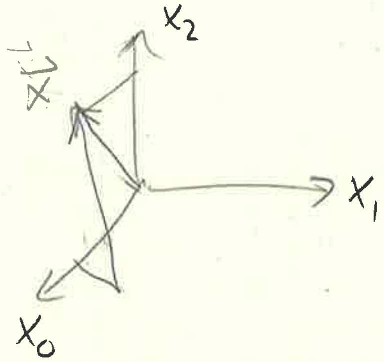
Other examples
- $l_2(\mathbb{Z})$: space of square-summable infinite sequences
- $L_2([a,b])$: space of square-integrable functions over an interval

Some can be represented geometrically

$\mathbb{R}^2$: $\vec{x} = [x_0\ x_1]^T$

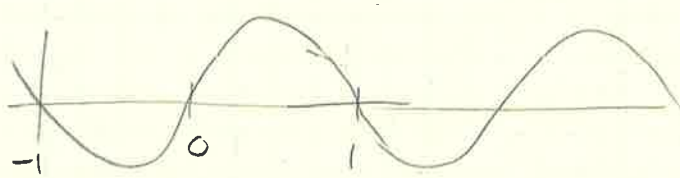


$\mathbb{R}^3$: $\vec{x} = [x_0\ x_1\ x_2]^T$



$L_2([-1,1])$: $\vec{x} = x(t),\ t \in [-1,1]$

$\vec{x} = \sin(\pi t)$



Can't plot $\mathbb{R}^N, N > 3$ or $\mathbb{C}^N, N > 1$

<u>Ingredients</u>

- the set of vectors V
- a set of scalars (say $\mathbb{C}$)

We need at least to be able to:

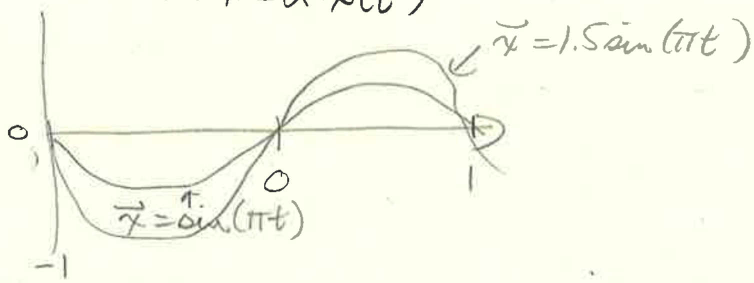
- resize vectors i.e., multiply a vector by a scalar
- combine vectors together, i.e., sum them

<u>Formal properties</u>: For $\vec{x}, \vec{y}, \vec{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- $\alpha(\vec{x} + \vec{y}) = \alpha\vec{y} + \alpha\vec{x}$
- $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$
- $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$
- $\exists\, 0 \in V : \vec{x} + 0 = 0 + \vec{x} = \vec{x}$
- $\forall \vec{x} \in V, \exists(-\vec{x}) : \vec{x} + (-\vec{x}) = 0$

Scalar multiplication in $L_2[-1,1]$

$$\alpha\vec{x} = \alpha x(t)$$



We need something more: inner product (aka dot product)

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$$

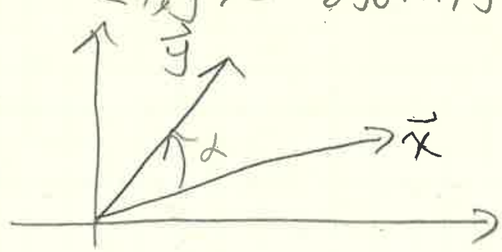
- measure of similarity between vectors
- inner product is zero? vector are orthogonal (maximally different)

Formal properties of the inner product

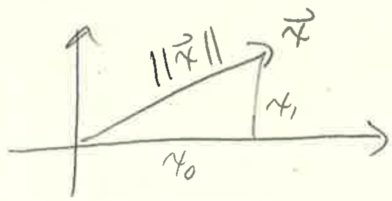
For $\vec{x},\vec{y},\vec{z} \in V$, $\alpha \in \mathbb{C}$:

- $\langle \vec{x}+\vec{y}, \vec{z}\rangle = \langle \vec{x},\vec{z}\rangle + \langle \vec{y},\vec{z}\rangle$
- $\langle \vec{x},\vec{y}\rangle = \langle \vec{y},\vec{x}\rangle^*$
- $\langle \alpha\vec{x},\vec{y}\rangle = \alpha^*\langle \vec{x},\vec{y}\rangle$
- $\langle \vec{x},\alpha\vec{y}\rangle = \alpha\langle \vec{x},\vec{y}\rangle$
- $\langle \vec{x},\vec{x}\rangle \geq 0$
- $\langle \vec{x},\vec{x}\rangle = 0 \Leftrightarrow \vec{x} = 0$
- If $\langle \vec{x},\vec{y}\rangle = 0$ and $\vec{x},\vec{y} \neq 0$, then $\vec{x}$ and $\vec{y}$ are called orthogonal

$$\langle \vec{x},\vec{y}\rangle = x_0 y_0 + x_1 y_1 = \|\vec{x}\|\,\|\vec{y}\|\cos\alpha$$



$$\langle x,x\rangle = x_0^2 + x_1^2 = \|\vec{x}\|^2$$

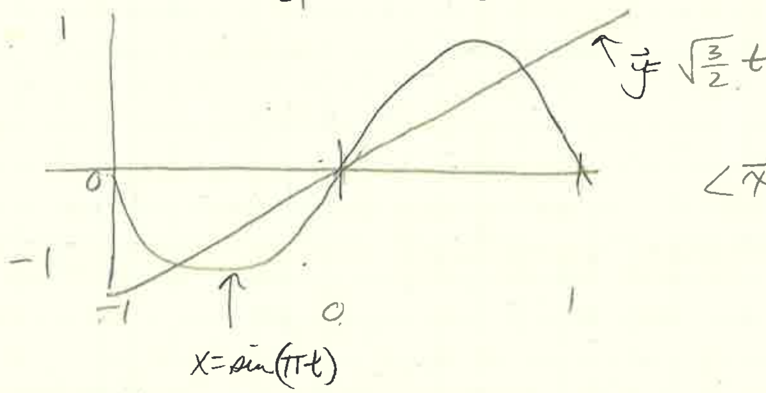


Inner product in $L_2[-1,1]$

$$\langle \vec{x},\vec{y}\rangle = \int_{-1}^{1} x(t)\,y(t)\,dt$$

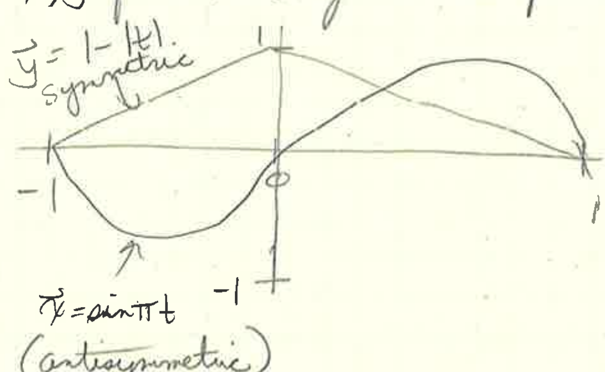
$$\|\sin(\pi t)\|^2 = \int_{-1}^{1} \sin^2\pi t\,dt = 1$$

$\vec{y}=t$: $\|\vec{y}\|^2 = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$



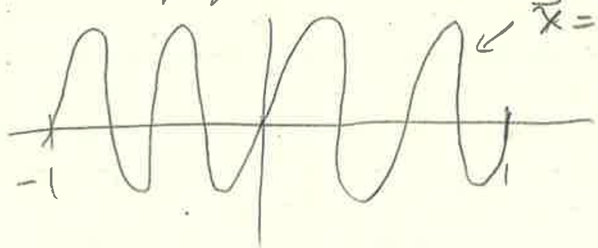
$$\langle \vec{x},\vec{y}\rangle = \int_{-1}^{1} \sqrt{\tfrac{3}{2}}\,t\sin\pi t\,dt = \frac{2}{\pi}\sqrt{\frac{3}{2}} \approx 0.78$$

$\vec{x},\vec{y}$ from orthogonal subspaces:



$$\langle \vec{x},\vec{y}\rangle = 0$$

Sinusoids with frequencies that are integer multiples of a fundamental



$\vec{y} = \sin 5\pi t$

$$\langle \vec{x},\vec{y}\rangle = 0$$

Norm vs Distance

- inner product defines a norm: $\|\vec{x}\| = \sqrt{\langle \vec{x},\vec{x}\rangle}$
- norm defines a distance: $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

Distance in $L_2[-1,1]$: the Mean Square Error

$$\|\vec{x}-\vec{y}\|^2 = \int_{-1}^{1} |x(t)-y(t)|^2\,dt$$

$\vec{x} = \sin 4\pi t$, $\vec{y} = \sin 5\pi t$, , $\|\vec{x}-\vec{y}\|^2 = \int_{-1}^{1} |\sin 4\pi t - \sin 5\pi t|^2 dt = 2$

2.2.b Signal Spaces

Finite-Length Signals

finite-length and periodic signals live in $\mathbb{C}^N$

- vector notation : $\vec{x} = [x_0 \; x_1 \; \dots \; x_{N-1}]^T$
- all operations well-defined and intuitive
- space of N-periodic signals sometimes indicated by $\tilde{\mathbb{C}}^N$

Inner product for signals

$$\langle \vec{x}, \vec{y} \rangle = \sum_{n=0}^{N-1} x^*[n] y[n]$$

well-defined for all finite-length vectors

Infinite Signals?

$$\langle \vec{x}, \vec{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n] y[n]$$

We require sequences to be square-summable : $\sum |x[n]|^2 < \infty$

i.e. in $\ell_2(\mathbb{Z})$ (finite-energy)

many interesting signals are not in $\ell_2(\mathbb{Z})$, such as, $x[n] = 1$, $x[n] = \cos(\omega n)$, etc.

Completeness

Limiting operations must yield vector space elements

An incomplete space : $\mathbb{Q}$ $\quad x_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$,

but $\lim_{n\to\infty} x_n = e \notin \mathbb{Q}$

Hilbert Space

1. a vector space : $H(V, \mathbb{C})$
2. an inner product : $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$
3. Complete

Bases

Linear combination is the basic operation in vector spaces:

$$\vec{y} = \alpha \vec{x} + \beta \vec{y}$$

Can we find a set of vectors $\{\vec{w}^{(k)}\}$ so that we can write any vector as a linear combination of the $\{\vec{w}^{(k)}\}$?

Canonical $\mathbb{R}^2$ basis

$$\vec{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Another $\mathbb{R}^2$ basis

$$\vec{v}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \alpha_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \alpha_0 = x_0 - x_1, \quad \alpha_1 = x_1$$

Not a basis for $\mathbb{R}^2$

$$\vec{g}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{g}^{(1)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$ not linearly independent

What about infinite-dimensional spaces?

$$\vec{x} = \sum_{k=0}^{\infty} \alpha_k \vec{w}^{(k)}$$

a basis for $\ell_2(\mathbb{Z})$

$$\vec{e}^{(k)} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad 1 \text{ in } k^{th} \text{ position}, \ k \in \mathbb{Z}$$

What about function vector spaces?

$$f(t) = \sum_k \alpha_k h^{(k)}(t)$$

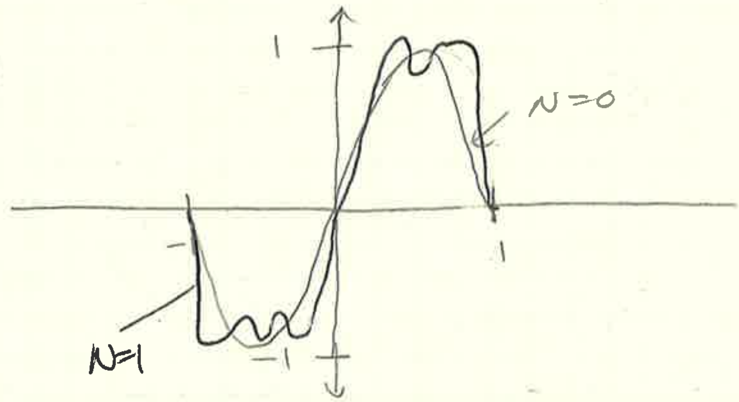
A basis for the functions over an interval?

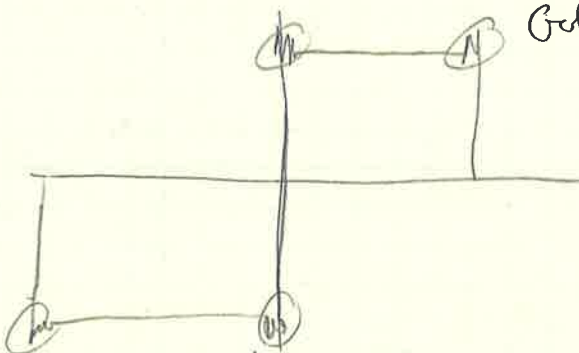
the Fourier basis for $[-1,1]$

$$\left\{ \frac{1}{\sqrt{2}}, \cos \pi t, \sin \pi t, \cos 2\pi t, \sin 2\pi t, \cos 3\pi t, \sin 3\pi t, \ldots \right\}$$

Using the Fourier Basis (approximating a square wave)

$$\sum_{k=0}^{N} \frac{\sin(2k+1)\pi t}{2k+1} = \sum_{k=0}^{N} \frac{w^{(4k+2)}}{2k+1}$$





Bases: formal definition

Given:

- a vector space H
- a set of K vectors from H: $W = \{\vec{w}^{(k)}\}_{k=0,1,\dots,K-1}$

W is a basis for H if:

1. We can write for all $x \in H$:

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

2. the coefficients α_k are unique

Uniqueness implies linear independence

$$\sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} = 0 \iff \alpha_k = 0, \quad k = 0, 1, \dots, K-1$$

Special bases

Orthogonal basis:

$$\langle \vec{w}^{(k)}, \vec{w}^{(n)} \rangle = 0, \quad k \neq n$$

Orthonormal basis: $\langle \vec{w}^{(k)}, \vec{w}^{(n)} \rangle = \delta[n-k]$

We can use Gram-Schmidt to normalize any orthogonal basis

Basis expansion

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)}$$, how do we find the α's?

Orthonormal bases are the best: $\alpha_k = \langle \vec{w}^{(k)}, \vec{x} \rangle$

Change of basis

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \vec{v}^{(k)}$$

If $\{\vec{v}^{(k)}\}$ is orthonormal:

$$\beta_h = \langle \vec{v}^{(h)}, \vec{x} \rangle$$

$$= \langle \vec{v}^{(h)}, \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} \rangle = \sum_{k=0}^{K-1} \alpha_k \langle \vec{v}^{(h)}, \vec{w}^{(k)} \rangle$$

$$= \sum_{k=0}^{K-1} \alpha_k c_{hk}$$

$$= \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0(K-1)} \\ & & \vdots & \\ c_{(K-1)0} & c_{(K-1)1} & \cdots & c_{(K-1)(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{K-1} \end{bmatrix}$$

Change of basis: example

- canonical basis $E = \{\vec{e}^{(0)}, \vec{e}^{(1)}\}$
- $\vec{x} = \alpha_0 \vec{e}^{(0)} + \alpha_1 \vec{e}^{(1)}$
- new basis $V = \{\vec{v}^{(0)}, \vec{v}^{(1)}\}$ with $\vec{v}^{(0)} = [\cos\theta \ \sin\theta]^T$, $\vec{v}^{(1)} = [-\sin\theta \ \cos\theta]^T$

$$\vec{x} = \beta_0 \vec{v}^{(0)} + \beta_1 \vec{v}^{(1)}$$

- new basis is orthonormal: $c_{hk} = \langle \vec{v}^{(h)}, \vec{e}^{(k)} \rangle$
- in compact form:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = R\vec{\alpha}$$

- R: Rotation matrix
- $R^T R = I$

2.4 Subspaces and approximations

Vector Subspace

- A subset of vectors closed under addition and scalar multiplication
- Example: $R^2 \subset R^3$

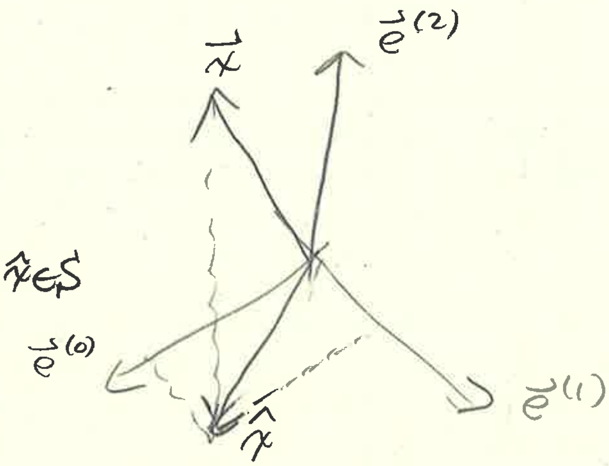
- Subspace of symmetric functions over $L_2[-1,1]$

$$\vec{x} = \cos \pi t$$
$$\vec{y} = \cos 5\pi t$$
to name a couple

- Subspaces have their own bases

$$\left\{ \vec{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ basis for a plane

Approximation



Problem:
- vector $x \in V$
- subspace $S \subseteq V$

- approximate $\vec{x}$ with $\hat{x} \in S$

Least-Squares Approximation

- $\{\vec{s}^{(k)}\}_{k=0,1,\ldots,K-1}$ orthonormal basis for S
- orthogonal projection:

$$\hat{x} = \sum_{k=0}^{K-1} \langle \vec{s}^{(k)}, \hat{x} \rangle \vec{s}^{(k)}$$

- orthogonal projection is the "best" approximation over S
- orthogonal projection has minimum-norm error:

$$\arg\min_{\vec{y} \in S} \|\vec{x} - \vec{y}\| = \hat{x}$$

- error is orthogonal to approximation:

$$\langle \vec{x} - \hat{x}, \hat{x} \rangle = 0$$

draw concentric circles until hitting $\vec{s}$, this radius vector is $\vec{x}-\hat{x}$.

Example: polynomial approximation

- vector space $P_N[-1,1] \subset L_2[-1,1]$
- $\vec{p} = a_0 + a_1 t + \ldots + a_{N-1} t^{N-1}$
- a self-evident, naive basis: $\vec{s}^{(k)} = t^k$, $k = 0, 1, \ldots, N-1$
- naive basis is not orthonormal

goal: approximate $\vec{x} = \sin t \in L_2[-1,1]$ over $P_3[-1,1]$

- build orthonormal basis from naive basis
- project $\vec{x}$ over the orthonormal basis
- compute approximation error
- compare error to Taylor approximation (well known but not optimal over the interval)

Building an orthonormal basis

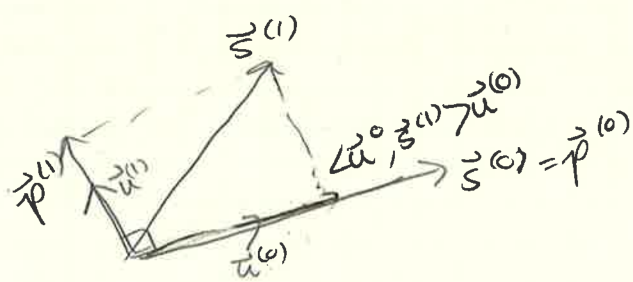
Gram-Schmidt orthonormalization procedure:

$$\{\vec{s}^{(k)}\} \longrightarrow \{\vec{u}^{(k)}\}$$

original set $\qquad$ orthonormal set

Algorithmic procedure: at each step k

1. $\vec{p}^{(k)} = \vec{s}^{(k)} - \sum_{n=0}^{k-1} \langle \vec{u}^{(n)}, \vec{s}^{(k)} \rangle \vec{u}^{(n)}$
2. $\vec{u}^{(k)} = \vec{p}^{(k)} / \| \vec{p}^{(k)} \|$



Apply Gram-Schmidt to $S = \{1, t, t^2, t^3, \dots\}$

$$\langle \vec{x}, \vec{y} \rangle = \int_{-1}^{1} x(t)\, y(t)\, dt$$

- $\vec{s}^{(0)} = 1$
  - $\vec{p}^{(0)} = \vec{s}^{(0)} = 1$
  - $\|\vec{p}^{(0)}\|^2 = 2$
  - $\vec{u}^{(0)} = \vec{p}^{(0)} / \|\vec{p}^{(0)}\| = \frac{1}{\sqrt{2}}$
- $\vec{s}^{(1)} = t$
  - $\langle \vec{u}^{(0)}, \vec{s}^{(1)} \rangle = \int_{-1}^{1} \frac{t}{\sqrt{2}} = 0$
  - $\vec{p}^{(1)} = \vec{s}^{(1)} = t$
  - $\|\vec{p}^{(1)}\|^2 = \frac{2}{3}$
  - $\vec{u}^{(1)} = \sqrt{3/2}\, t$
- $\vec{s}^{(2)} = t^2$
  - $\langle \vec{u}^{(0)}, \vec{s}^{(2)} \rangle = \int_{-1}^{1} \frac{t^2}{\sqrt{2}} dt = \frac{2}{3\sqrt{2}}$
  - $\langle \vec{u}^{(1)}, \vec{s}^{(2)} \rangle = \int_{-1}^{1} t^3/\sqrt{2} = 0$
  - $\vec{p}^{(2)} = \vec{s}^{(2)} - \frac{2}{3\sqrt{2}} \vec{u}^{(0)} = t^2 - \frac{1}{3}$
  - $\|\vec{p}^{(2)}\|^2 = 8/45$
  - $\vec{u}^{(2)} = \sqrt{\frac{5}{8}}\,(3t^2 - 1)$

Legendre Polynomials

The Gram-Schmidt algorithm leads to an orthonormal basis for $P_N([-1,1])$

$$\vec{u}^{(0)} = \sqrt{\frac{1}{2}}, \quad \vec{u}^{(1)} = \sqrt{\frac{3}{2}}\, t, \quad \vec{u}^{(2)} = \sqrt{\frac{5}{8}}(3t^2-1), \quad \vec{u}^{(3)} = \dots$$

Orthogonal projection over $P_3[-1,1]$

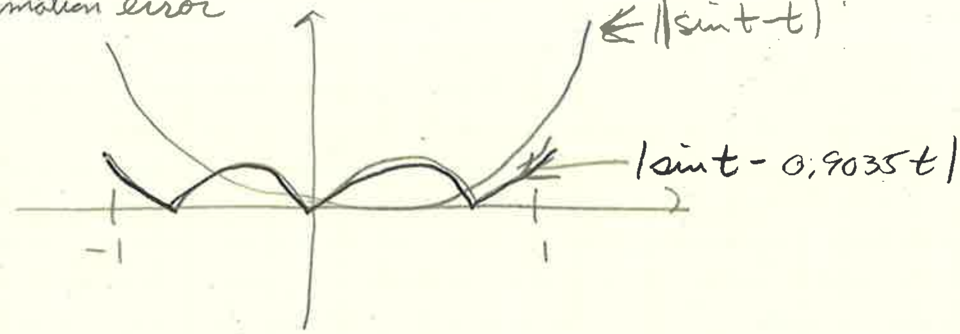
$$\alpha_k = \langle \vec{u}^{(k)}, \vec{x} \rangle = \int_{-1}^{1} u_k(t) \sin t \, dt$$

- $\alpha_0 = \langle \frac{1}{\sqrt{2}}, \sin t \rangle = 0$
- $\alpha_1 = \langle \sqrt{\frac{3}{2}}\, t, \sin t \rangle \approx 0.7377$
- $\alpha_2 = \langle \sqrt{\frac{5}{8}}(3t^2-1), \sin t \rangle = 0$

$\sin t \to \alpha_1 \vec{u}^{(1)} \approx 0.9035\, t$

Taylor Series : $\sin t \approx t$

Approximation error



Error norm:

Orthogonal projection over $P_3[-1,1]$:

$$\|\sin t - \alpha_1 \vec{u}^{(1)}\| \approx 0.0337$$

Taylor series: $\|\sin t - t\| \approx 0.0857$