作业1：任选第3种方法理解欧拉公式。

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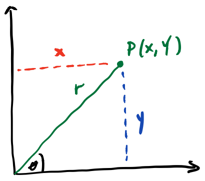

$x = r\cos\theta$
$y = r\sin\theta$ $\quad \frac{y}{x} = \tan\theta$, $\tan^{-1}\left(\frac{y}{x}\right) = \theta$

$$d\theta = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{1+(y/x)^2} \cdot d(y/x)$$

$$= \frac{1}{1+(y/x)^2} \cdot \frac{x\,dy - y\cdot dx}{x^2}$$

$$= \frac{x\,dy - y\cdot dx}{x^2+y^2}$$

因为 $i = \sqrt{-1}$, $i^2 = -1$ $\quad x^2+y^2 = (x+iy)(x-iy)$

所以 $d\theta = \frac{x\cdot dy - y\cdot dx}{(x+iy)(x-iy)}$

$$= \frac{x\cdot dy}{(x+iy)(x-iy)} - \frac{y\cdot dx}{(x+iy)(x-iy)}$$

use partial fraction decomposition to further simplification

$$\frac{x\cdot dy}{(x+iy)(x-iy)} = \frac{A\cdot dy}{x+iy} + \frac{B\cdot dy}{x-iy}$$

$x\cdot dy = A(x-iy)\,dy + B(x+iy)\,dy$
$= (A+B)\,x\,dy + (iB - iA)\cdot y\,dy$

$A+B = 1$ $\quad A = B = \frac{1}{2}$
$B - A = 0$

Similarly for $\frac{y\cdot dx}{(x+iy)(x-iy)} = \frac{C\,dx}{x+iy} + \frac{D\,dx}{x-iy}$ $\quad C = i/2$, $D = -i/2$

所以 $d\theta = \frac{\frac{1}{2}dy}{(x+iy)} + \frac{\frac{1}{2}dy}{(x-iy)} - \frac{\frac{i}{2}dx}{(x+iy)} + \frac{\frac{i}{2}dx}{(x-iy)}$

$$2i\,d\theta = \frac{dx + i\,dy}{(x+iy)} + \frac{-dx + i\,dy}{(x-iy)}$$

$$= \frac{d(x+iy)}{(x+iy)} - \frac{d(x-iy)}{(x-iy)}$$

take integration $\int 2i\,d\theta = \int \frac{d(x+iy)}{x+iy} - \int \frac{d(x-iy)}{x-iy}$

$$2i\theta = \ln(x+iy) - \ln(x-iy) + C$$

$y > 0$, $\theta = 0$, const $= 0$

$$2i\theta = \ln(x+iy) - \ln(x-iy) = \ln\frac{x+iy}{x-iy}$$

$$e^{2i\theta} = \frac{x+iy}{x-iy} \cdot \frac{(x+iy)}{(x+iy)} = \frac{(x+iy)^2}{(x^2+y^2)}$$

$x^2+y^2 = r^2$ $\quad e^{2i\theta} = \frac{(x+iy)^2}{r^2}$

$e^{i\theta} = \frac{x+iy}{r} \Rightarrow e^{i\theta} = \frac{x}{r} + i\frac{y}{r} \Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$ ✓

作业2：证明 $e^{jn\omega_0 t}$ ($n = 0, \pm1, \pm2 \ldots$) 区间 $[-\pi/\omega_0, \pi/\omega_0]$ (ω_0 为实数)上是正交函数集。

$$[\,e^{jn\omega_0 t} \ldots e^{-2j\omega_0 t}, e^{-1j\omega_0 t}, 1, e^{j\omega_0 t}, e^{2j\omega_0 t} \ldots e^{jn\omega_0 t}\,]$$

欧拉公式
$\{ e^{j\omega t} = \cos\omega t + j\sin\omega t$
$\{ e^{-j\omega t} = \cos\omega t - j\sin\omega t$

1) 1 和 $e^{ja\omega_0 t}$ 正交 ($a \neq 0$, $a = \pm1, \pm2, \ldots, \pm n$)

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} e^{ja\omega_0 t} =$$

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} \left(\cos(a\omega_0 t) + j\sin(a\omega_0 t)\right) dt$$

$$\frac{1}{a\omega_0}\left(\sin(a\omega_0 t) + -j\cos(a\omega_0 t)\right)\Big|_{-\pi/\omega_0}^{\pi/\omega_0}$$

$$\frac{1}{a\omega_0}\left[\left(\sin(a\pi) + -j\cos(a\pi)\right) - \left(\sin(-a\pi) + -j\cos(-a\pi)\right)\right]$$

$$\sin(a\pi)\cancel{-j\cos(a\pi)} + \sin(a\pi)\cancel{+j\cos(a\pi)}$$

$$2\sin(a\pi) = 0$$

3) prove that 1 is not 正交

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} 1\cdot 1\,dt = \frac{2\pi}{\omega_0}$$

4) prove that $e^{ja\omega t}$, $a = \pm1, \pm2 \ldots \pm n$

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} e^{ja\omega t} e^{ja\omega t}\,dt = \int_{-\pi/\omega_0}^{\pi/\omega_0} e^{2aj\omega t}\,dt$$

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} [\cos(2a\omega t) + j\sin(2a\omega t)]\,dt$$

$$\frac{1}{2a\omega}\left[\sin(2a\omega t) - j\cos(2a\omega t)\right]\Big|_{-\pi/\omega_0}^{\pi/\omega_0} = 0$$

2) prove for $a \neq b$, $a, b = \pm1, \pm2, \pm3 \ldots \pm n$

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} e^{ja\omega_0 t} e^{jb\omega_0 t}\,dt = \int_{-\pi/\omega_0}^{\pi/\omega_0} e^{(a+b)j\omega_0 t}\,dt$$

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} \cos((a+b)\omega_0 t) + j\sin((a+b)\omega_0 t)$$

$$\frac{1}{(a+b)\omega_0}\left[\sin((a+b)\omega_0 t) - j\cos((a+b)\omega_0 t)\right]\Big|_{-\pi/\omega_0}^{\pi/\omega_0} =$$

$$\frac{1}{(a+b)\omega_0}\left[\sin((a+b)\pi) - j\cancel{\cos((a+b)\pi)}\right] - \left(\sin(-(a+b)\omega_0) - j\cos(-(a+b)\omega_0)\right) =$$

$$\frac{1}{(a+b)\omega_0}\left[\sin((a+b)\pi) - j\cos((a+b)\pi) + \sin((a+b)\omega_0) + j\cancel{\cos((a+b)\omega_0)}\right] =$$

$$= \frac{1}{(a+b)\omega_0} \cdot 2\sin((a+b)\pi) = 0$$