Correlated Equilibrium

Roger Myerson (2007, Nobel Memorial Prize in Economics): “ If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium”

Correlated Equilibrium

(Solution Concept for games with communication but no binding agreement)

Example: Crossing Game

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

Player I and Player II are using their own randomization device to play the crossing game.

	Cross	Yield
Cross	$(-10, -10)$	$(5, 0)$
Yield	$(0, 5)$	$(-1, -1)$

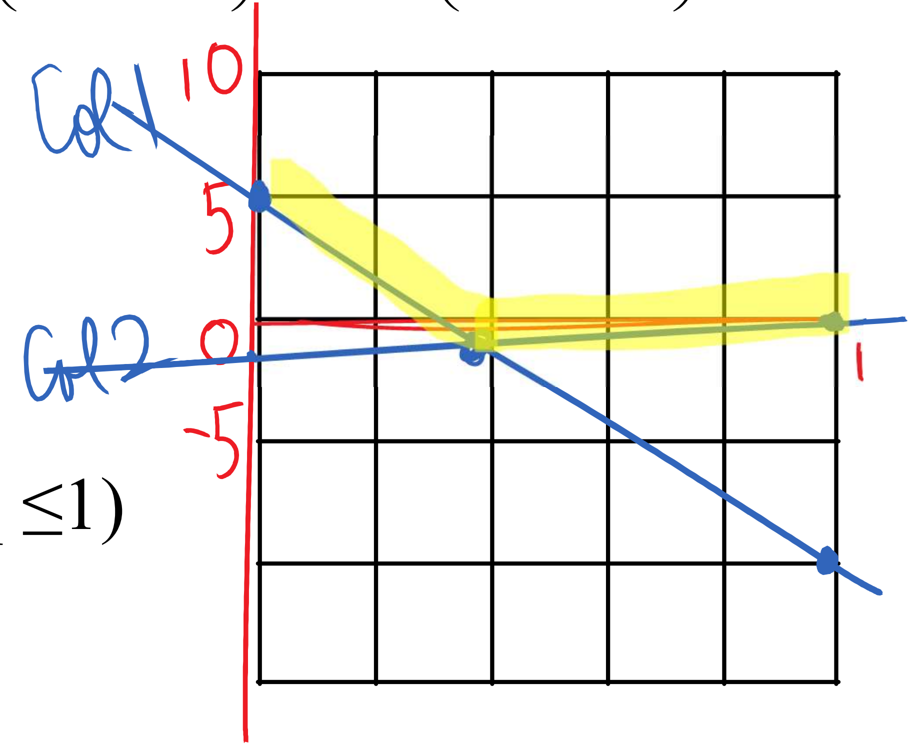
By the Tetraskelion method we can find SE's for this game.

$$\begin{array}{c|cc|} & q & 1-q \\ \hline p & (-10,-10) & (5,0) \\ 1-p & (0,5) & (-1,-1) \\ \hline \end{array} \qquad A=\begin{pmatrix} -10 & 5 \\ 0 & -1 \end{pmatrix}, \quad B=\begin{pmatrix} -10 & 0 \\ 5 & -1 \end{pmatrix}$$

BR of II if I uses (p, 1-p):

By the Tetraskelion method we can find SE's for this game.

$$
\begin{array}{c|cc|}
 & q & 1-q \\
\hline
p & (-10,-10) & (5,0) \\
1-p & (0,5) & (-1,-1) \\
\hline
\end{array}
\qquad
A = \begin{pmatrix} -10 & 5 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -10 & 0 \\ 5 & -1 \end{pmatrix}
$$



BR of II if I uses (p, 1-p):

0≤p<6/16: BR is Col 1 (q=1)

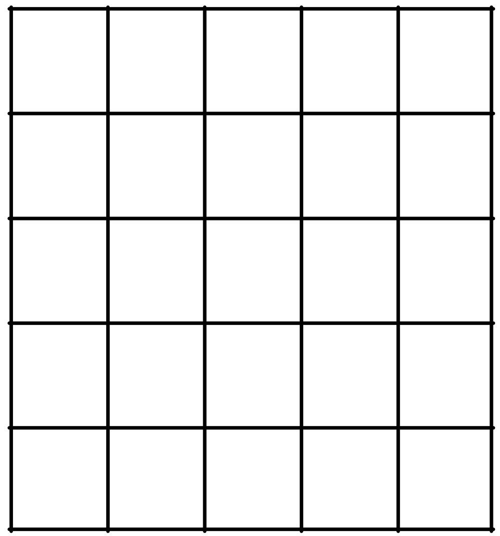
p=6/16: BR is everything (0 ≤q ≤1)

6/16<p ≤1: BR is Col2 (q=0)

By the Tetraskelion method we can find SE's for this game.

$$\begin{array}{c|cc|} & q & 1-q \\ \hline p & (-10,-10) & (5,0) \\ 1-p & (0,5) & (-1,-1) \\ \hline \end{array} \qquad A = \begin{pmatrix} -10 & 5 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -10 & 0 \\ 5 & -1 \end{pmatrix}$$

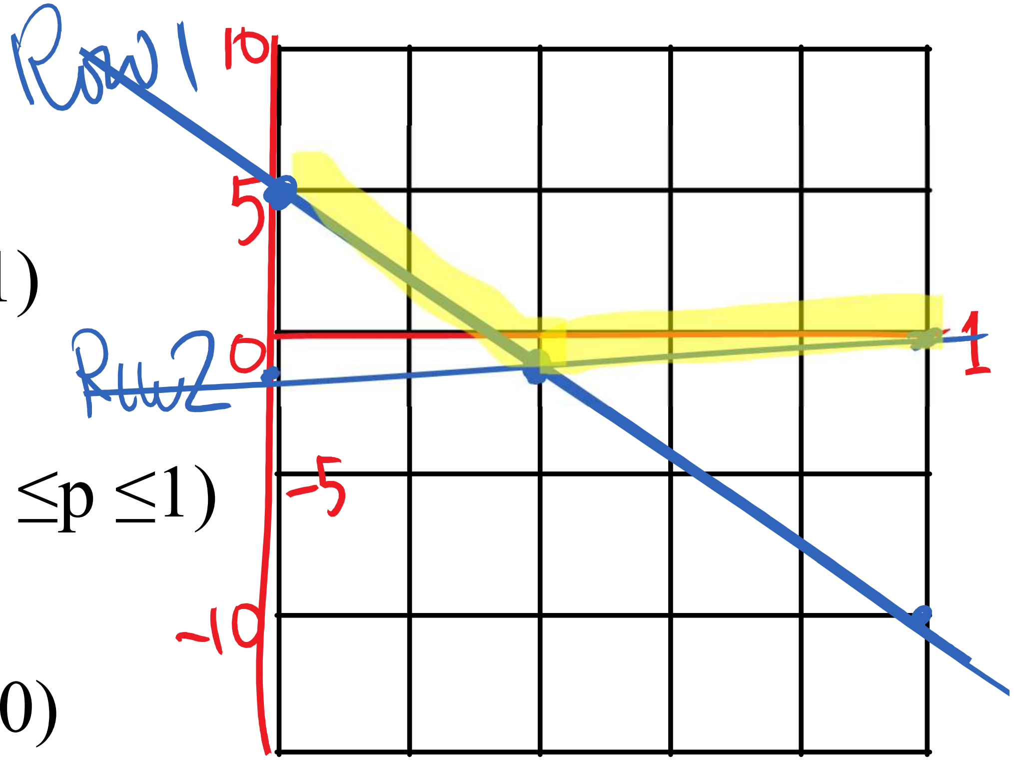
BR of I if II uses (q, 1-q):



By the Tetraskelion method we can find SE's for this game.

$$\begin{array}{c|cc|} & q & 1-q \\ \hline p & (-10,-10) & (5,0) \\ 1-p & (0,5) & (-1,-1) \\ \hline \end{array}$$

$$A=\begin{pmatrix} -10 & 5 \\ 0 & -1 \end{pmatrix}, \quad B=\begin{pmatrix} -10 & 0 \\ 5 & -1 \end{pmatrix}$$



BR of I if II uses (q, 1-q):

0≤q<6/16: BR is Row1 (p=1)

q=6/16: BR is everything (0 ≤p ≤1)

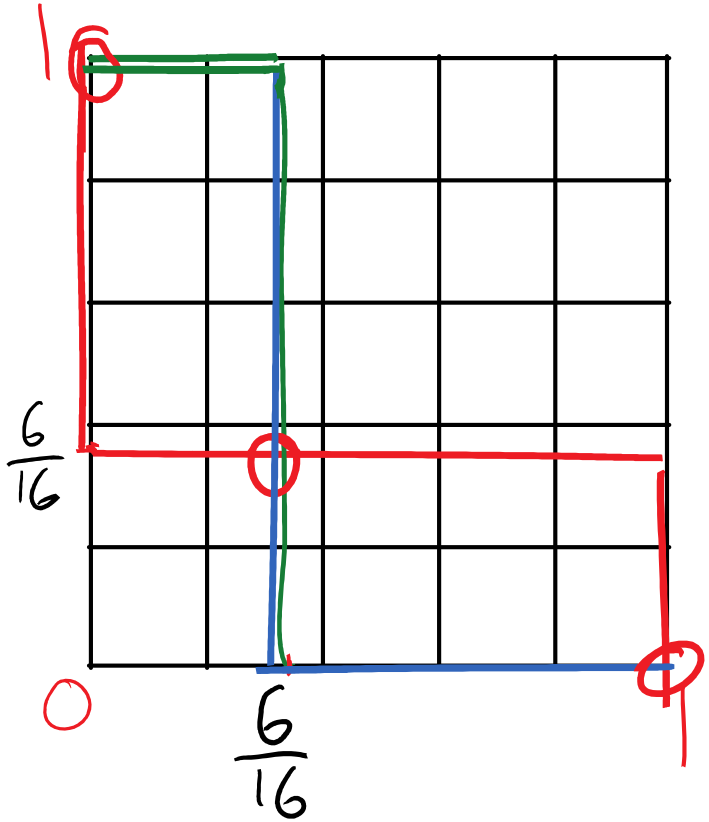
6/16<q ≤1: BR is Row2 (p=0)

SE are

<Row 1, Col 2>

<(6/16, 10/16), (6/16, 10/16)>

<Row 2, Col 1>



By the Tetraskelion method we get three SE's. They are <Cross, Yield>, <Yield, Cross> and <(6/16, 10/16), (6/16, 10/16)>.

The payoff vectors are respectively, (5,0), (0,5) and

$$(6/16)^2(-10,-10) + 60/16(5,0) + 60/16(0,5) + (10/16)^2(-1,-1)$$
$$= (-160/256, -160/256) = (-10/16, -10/16)$$

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

For the SE <Cross, Yield>, we will see (Row1, Colum2) with prob=1.
For the SE <Yield, Cross>, we will see (Row2, Colum1) with prob=1.

For the SE <(6/16,10/16), (6/16, 10/16)>, we will see
(Row1, Column1) with prob=36/256,
(Row1, Column2) with prob=60/256,
(Row2, Column1) with prob=60/256,
(Row2, Column2) with prob=100/256.

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

Which SE we should use?

We can interpret the mixed strategy (6/16, 10/16) for Player I as the instruction of a randomization device to Player I to play row 1 with probability 6/16.

Similarly, we can interpret the mixed strategy (6/16, 10/16) for Player II as the instruction of a randomization device to Player II to play column 1 with probability 6/16.

Note that for the SE, <(6/16,10/16), (6/16,10/16)> the two randomization devices are independent.

What if the randomization devices are correlated, i.e. a single randomization device giving instructions to the players?

What if the randomization devices are correlated?

For example, both players base their randomization on the observation on the outcome of a certain event such as the result of a football match, the weather etc. For example, Player I may get instruction from the performance of Shanghai stock market and Player II will get instruction from the performance of Shenzhen stock market.

Specifically, we may suppose a single randomization device giving instructions to the players.

Road Crossing Game

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

In the interest of fairness and efficiency, and to avoid misunderstandings, a traffic light might be installed.

The traffic light is merely a randomization device which, with a certain probability, tells each of the two drivers whether to Cross or to Yield. This is not much different from a pair of mixed strategies. Except that now the two players' probability distribution on strategies can be correlated. For example, the traffic light can, with prob=0.55 instruct (Cross, Yield), prob=0.4 instruct (Yield, Cross), prob=0.05 instruct (Yield, Yield). If the players comply, then the payoff vector is (2.7, 1.95).

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

Will the players comply or will they deviate from the traffic light's instruction? If the traffic light is operated in a way that both players see that it is in their best interest to comply then we have an EQUILIBRIUM! We will call this a Correlated Equilibrium.

Suppose both players know the mechanism of the traffic light such that it gives instructions (Cross, Yield) with prob=0.55, (Yield, Cross) with prob=0.4, (Yield, Yield) with prob=0.05, (Cross, Cross) with prob=0.

Suppose Player I knows that Player II will comply with the traffic light instructions. When Player I sees the signal to Cross, he will not deviate because he knows that (Cross, Cross) occurs with prob=0.

When Player I sees the signal to Yield, he can compute the conditional probability of instructions to Player II given that his instruction is Yield. Then,

Prob(II to Cross| I to Yield)=

Prob(II to Cross, I to Yield)/Prob(I to Yield)=0.4/0.45.

Similarly, Prob(II to Yield|I to Yield)=

Prob(II to Yield, I to Yield)/Prob(I to Yield)=0.05/0.45.

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

Then, the expected payoff to Player I if he complies with the instruction to Yield is 0(8/9) + (-1)(1/9)= -1/9.

The expected payoff to Player I if he deviates from the instruction to Yield is -10(8/9) + 5(1/9)= -75/9. This is far inferior to his payoff if he complies with the instruction.

For Player II, if he sees Cross, he will not deviate because the instruction of (Cross, Cross) has prob=0.

Suppose II knows that Player I will comply with the traffic light instructions. When II sees the instruction of Yield. He can compute the conditional probability of Player I's instruction.

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

Prob(I to Cross| II to Yield)=

Prob(I to Cross, II to Yield)/Prob(II to Yield)=0.55/0.6.

Prob(I to Yield| II to Yield)=

Prob(I to Yield, II to Yield)/Prob(II to Yield)=0.05/0.6.

Then, the expected payoff to Player II to comply with the instruction is 0(11/12) + (-1)(1/12)= -1/12.

The expected payoff to Player II to deviate from the instruction is -10(11/12) + 5(1/12)= -105/12. Again, this is inferior to the payoff if II complies.

Therefore, it is for the best interest of both players to comply with the instruction of the traffic light that instructs

(Cross, Cross) with prob=0, (Cross, Yield) with prob=0.55, (Yield, Cross) with prob=0.4, (Yield, Yield) with prob=0.05.

This probability distribution over the outcomes is then called a Correlated Equilibrium.

	Cross	Yield
Cross	$(-10,-10)$	$(5,0)$
Yield	$(0,5)$	$(-1,-1)$

In general, let [A, B] be a an mxn bimatrix game where $A=(a_{ij})$, $B=(b_{ij})$. A probability distribution of the outcomes of this game is given by p_{ij}, $p_{ij}\geq 0$, $\sum_{i,j}p_{ij}=1$. This means that the instruction of playing

(Row i, Column j) to the players with probability p_{ij}.

This probability distribution is called a Correlated Equilibrium (CE) if it is for the best interest for players to comply with the instruction.

We will write down all the inequalities characterizing CE later.

Observation: If $<(p_1,\ldots p_m), (q_1,\ldots,q_n)>$ is a SE then $p_{ij}=p_iq_j$ is a CE. Therefore, all SE' s are CE' s.

Inequalities characterizing CE

Prob(Column j for II| Row i for I)=
Prob(Column j for II, Row i for I)/Prob(Row i for I) $=p_{ij}/\sum_k p_{ik}$.

For Player I to comply with the instruction to play Row i, the payoff must be better than deviating the instruction in playing Row r instead. This means

$$\sum_j (p_{ij} / \sum_k p_{ik}) a_{ij} \geq \sum_j (p_{ij} / \sum_k p_{ik}) a_{rj}$$

for r =1,…,m.

Equivalently,

$$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0 \text{ for } r=1,\ldots m.$$

Inequalities characterizing CE

Prob(Row i for I| Column j for II)=
Prob(Row i for I, Column j for II)/Prob(Column j for II) $=p_{ij}/\sum_k p_{kj}$.

For Player II to comply with the instruction to play Column j, the payoff must be better than deviating the instruction in playing Column s instead. This means

$$\sum_i (p_{ij} /\sum_k p_{kj})b_{ij} \geq \sum_i (p_{ij} /\sum_k p_{kj})b_{is}$$

for $s = 1,\ldots,n$.

Equivalently,

$$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0 \text{ for } s=1,\ldots n.$$

Definition of CE

A probability distribution p_{ij} over the outcomes of the mxn bimatrix game [A, B] is called a Correlated Equilibrium whenever

$$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0 \text{ for } i, r=1,\ldots m.$$

and

$$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0 \text{ for } j, s=1,\ldots n.$$

Remark: For an mxn game, there are m(m-1) +n(n-1) linear inequalities to characterize a CE.

Computing CE's for the Crossing Game

For 2x2 Game, the inequalities are

$p_{11}(a_{11} - a_{21}) + p_{12}(a_{12} - a_{22}) \geq 0$

$p_{21}(a_{21} - a_{11}) + p_{22}(a_{22} - a_{12}) \geq 0$

$p_{11}(b_{11} - b_{12}) + p_{21}(b_{21} - b_{22}) \geq 0$

$p_{12}(b_{12} - b_{11}) + p_{22}(b_{22} - b_{21}) \geq 0$

$p_{11}+p_{12}+p_{21}+p_{22}=1$, $p_{ij} \geq 0$

$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0$ for $i, r=1,\ldots m$.

and

$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0$ for $j, s=1,\ldots n$.

Computing CE's for the Crossing Game

For the Crossing Game, the inequalities are

$p_{11}(-10) + p_{12}(6) \geq 0$

$p_{21}(10) + p_{22}(-6) \geq 0$

$p_{11}(-10) + p_{21}(6) \geq 0$

$p_{12}(10) + p_{22}(-6) \geq 0$

$p_{11} + p_{12} + p_{21} + p_{22} = 1,\ p_{ij} \geq 0$

	Cross	Yield
Cross	$(-10, -10)$	$(5, 0)$
Yield	$(0, 5)$	$(-1, -1)$

Computing CE's for the Crossing Game

For the Crossing Game, the inequalities are

$3p_{12} \geq 5\ p_{11}$

$5p_{21} \geq 3p_{22}$

$3p_{21} \geq 5\ p_{11}$

$5p_{12} \geq 3p_{22}$

$p_{11}+p_{12}+p_{21}+p_{22}=1,\ p_{ij} \geq 0$

Computing CE's for the Crossing Game

For the Crossing Game, the inequalities are

$p_{12} \geq \text{Max}(5/3\ p_{11},\ 3/5\ p_{22})$

$p_{21} \geq \text{Max}(5/3\ p_{11},\ 3/5\ p_{22})$

$p_{11}+p_{12}+p_{21}+p_{22}=1,\ p_{ij} \geq 0$

There are many CE's.

Example: Find the CE that maximizes the expected sum of the two players' payoffs.

Solution: the expected sum of the two players' payoff is

$-20p_{11}+ 5p_{12} + 5p_{21} -2p_{22}$

We then want to maximize

$-20p_{11}+ 5p_{12} + 5p_{21} -2p_{22}$

subject to

$p_{12} \geq Max(5/3\ p_{11}, 3/5\ p_{22})$

$p_{21} \geq Max(5/3\ p_{11}, 3/5\ p_{22})$

$p_{11}+p_{12}+p_{21}+p_{22}=1,\ p_{ij} \geq 0$

Note that

$p_{12}+p_{21}=1-p_{11}-p_{22}$.

Hence, $-20p_{11}+ 5p_{12} + 5p_{21} -2p_{22}$

$$=5 - 25p_{11} -7p_{22}.$$

To maximize it, we set $p_{11}=p_{22}=0$. Then, $p_{12}\geq 0$, $p_{21}\geq 0$, $p_{12}+p_{21}=1$. Therefore, the CE are (p_{ij}) such that $p_{11}=p_{22}=0$, $p_{12}\geq 0$, $p_{21}\geq 0$, $p_{12}+p_{21}=1$. The maximum of the sum of expected payoffs is 5.

Remark: Recall that if $p=(p_i)$, $q=(q_j)$ are mixed strategies for Player I and Player II respectively such that $< (p_i), (q_j) >$ a SE, then (p_iq_j) is a CE.

It is then natural to ask the following question.

Question: Given $(p_{ij})= (p_iq_j)$ is a CE, is $< (p_i), (q_j) >$ a SE?

Theorem: Let $p=(p_i)$, $q=(q_j)$ be mixed strategies for Player I and Player II respectively. Then, (p_iq_j) is a CE if and only if <p, q> is a SE.

$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0$ for i, r=1,…m.

and

$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0$ for j, s=1,…n.

Proof: (p_iq_j) is a CE whenever

(*)$\sum_j p_iq_j (a_{ij} - a_{rj}) \geq 0$ for i, r=1,…m,

(**)$\sum_i p_iq_j (b_{ij} - b_{is}) \geq 0$ for j, s=1,…n, are satisfied.

(*) means $p_i>0$ implies Row i is a BR to q

(**) means $q_j>0$ implies Col j us a BR to p.

Therefore, (p_iq_j) is a CE iff <p, q> is a SE.

Corollary: Suppose (p_{ij}) is a CE such that $p_{ij}=1$ for $i=k$ and $j=l$, and $p_{ij}=0$ otherwise. Then, <Row k, Col l> is a PSE.

Proof: Let s be the pure strategy representing Row k.

Let t be the pure strategy representing Col l.

Then, $(s_i t_j) = (p_{ij})$. The Theorem then says that <s, t> is a SE i.e. <Row k, Col l> is a PSE.

Theorem: The convex combination of two CE's is a CE.

Proof: The result follows from observing that CE is defined by a set of linear inequalities in the following.

$$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0 \text{ for } i, r=1,\ldots m.$$

and

$$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0 \text{ for } j, s=1,\ldots n.$$

Theorem: Let (u_1, u_2), (u'_1, u'_2) be payoffs to two CE's. Then, any convex combination of these two payoffs is the payoff of a certain CE.

Proof: Let (p_{ij}), (q_{ij}) be the corresponding CEs' for (u_1, u_2), (u'_1, u'_2) respectively. Any convex combination of them is again a CE and its payoff is a convex combination of (u_1, u_2), (u'_1, u'_2) .

Remark: The set of payoff of CE's is a convex set. The following example shows that this set is not generated by payoffs of SE's.

Example: Given the following bimatrix [A, B] game

$$\boxed{\begin{matrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{matrix}}, \quad A = \begin{pmatrix} 6 & 2 \\ 7 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 7 \\ 2 & 0 \end{pmatrix}$$

We can use the Tetraskelion method to find the SE's and solve the following inequalities for CE's.

$p_{11}(-1) + p_{12}(2) \geq 0$

$p_{21}(1) + p_{22}(-2) \geq 0$

$p_{11}(-1) + p_{21}(2) \geq 0$

$p_{12}(1) + p_{22}(-2) \geq 0.$

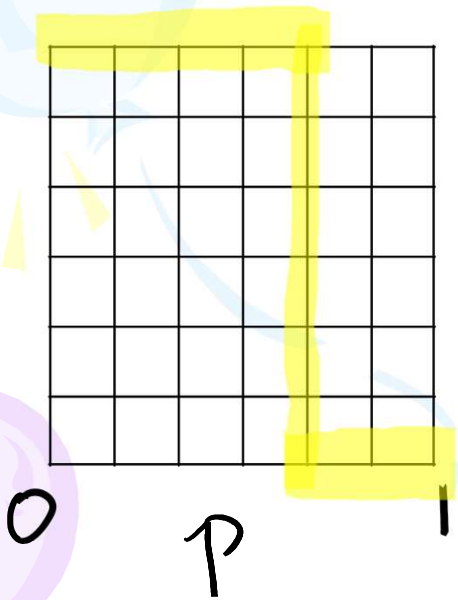
Tetraskelion method to find the SE's:

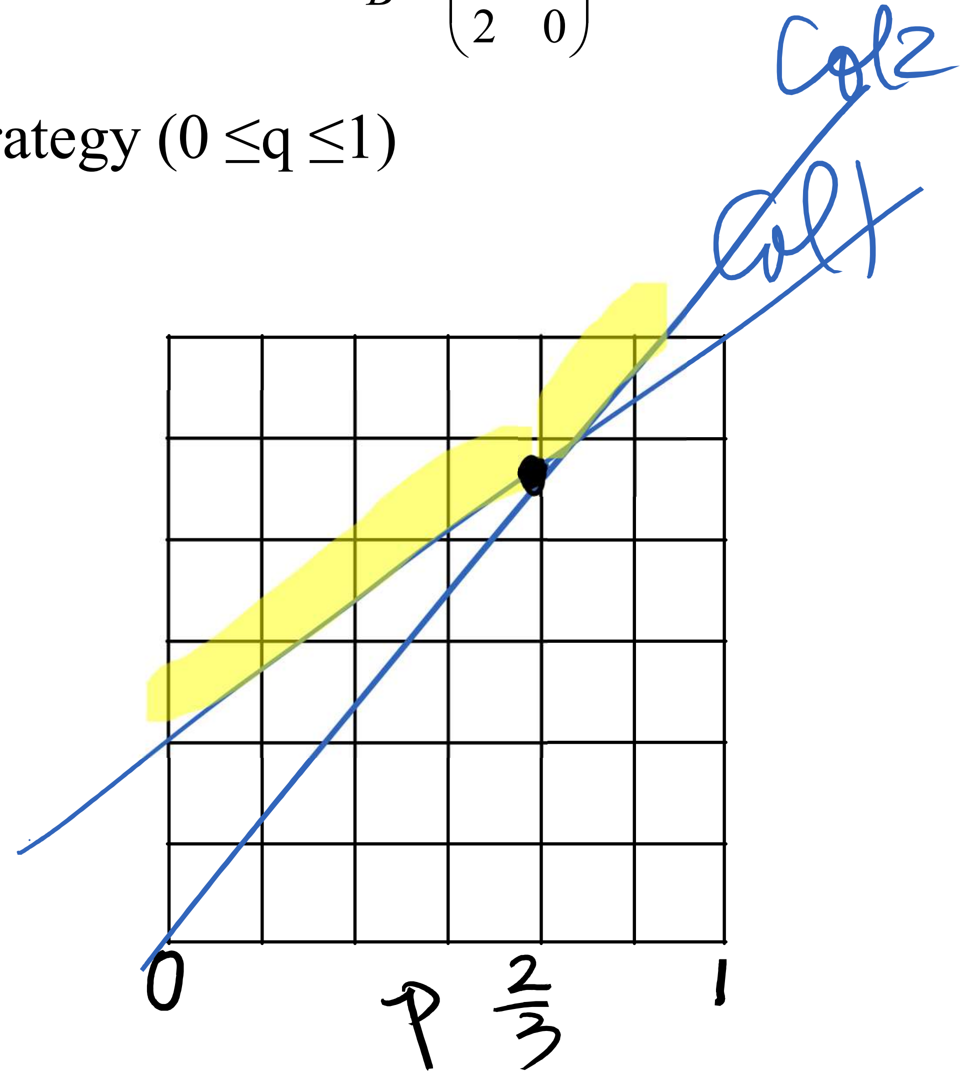
BR to Player I's mixed strategies (p,1-p):

$$B = \begin{pmatrix} 6 & 7 \\ 2 & 0 \end{pmatrix}$$

*$0 \le p \le 2/3$, BR is Col 1 (q=1)

- $p=2/3$, BR is every mixed strategy ($0 \le q \le 1$)
- $2/3 \le p \le 1$, BR is Col 2 (q=0)



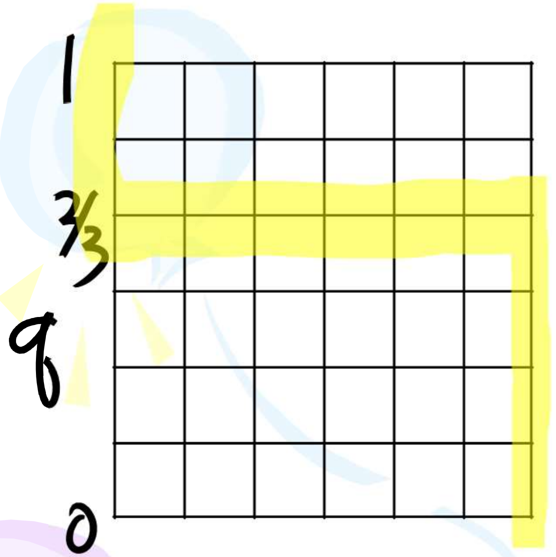


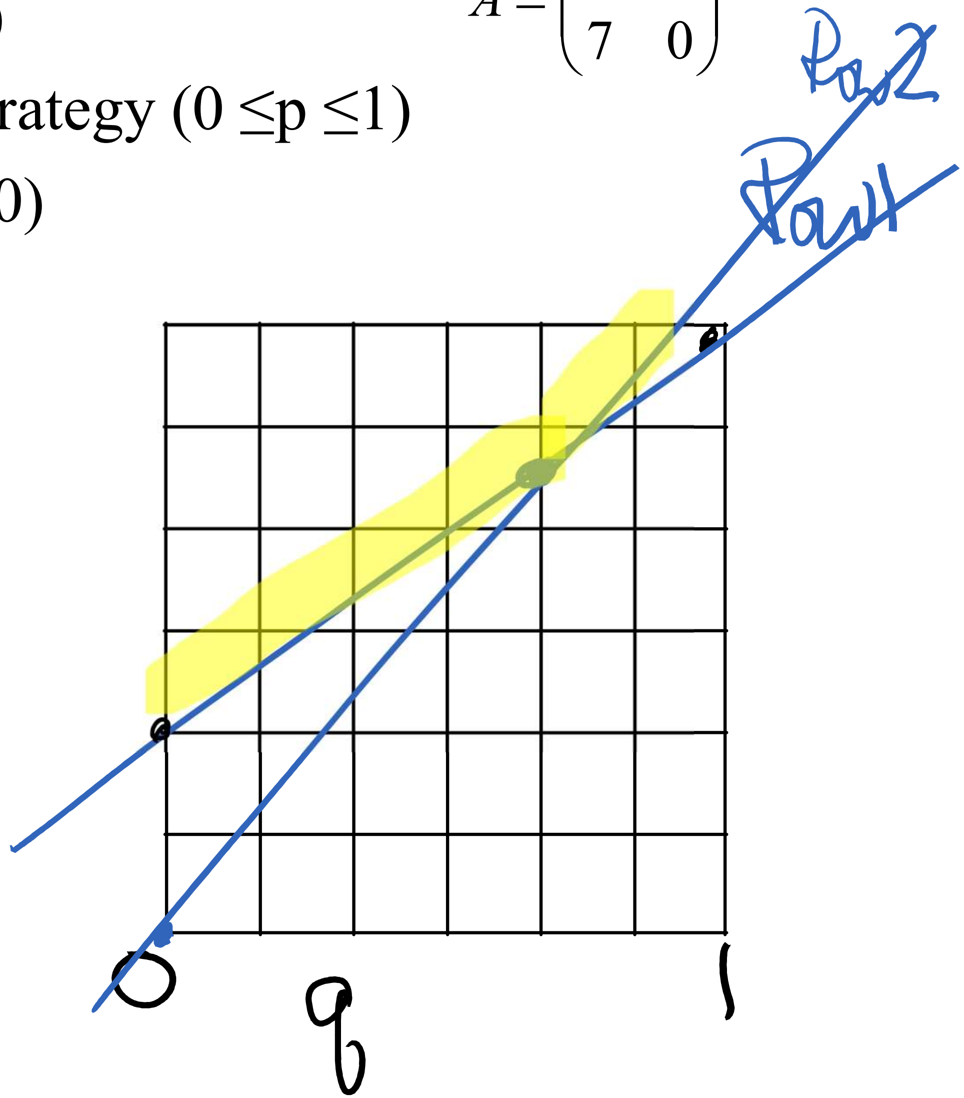
BR to Player II's strategy of (q, 1-q):

*$0 \le q \le 2/3$, BR is Row 1 (p=1)

- $q=2/3$, BR is every mixed strategy ($0 \le p \le 1$)
- $2/3 \le q \le 1$, BR is Row 2 (p=0)

$$A = \begin{pmatrix} 6 & 2 \\ 7 & 0 \end{pmatrix}$$





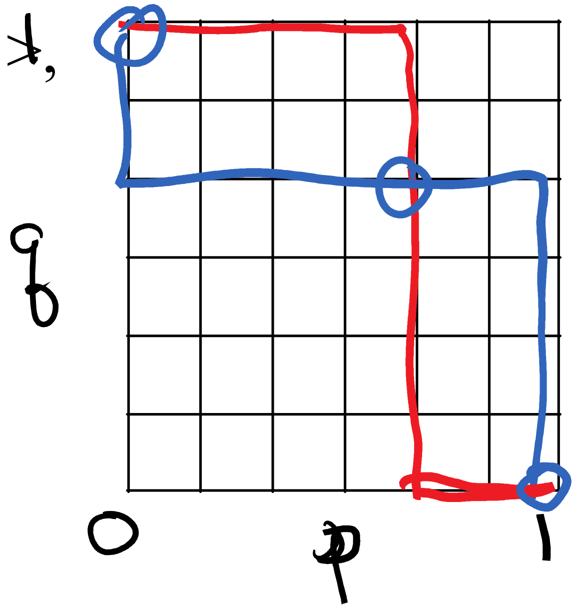
Putting two BR curves together, we get the SEs’ as their intersections. The solutions are then

<Row 1, Column 2>, <Row 2, Column 1>,

<(2/3, 1/3), (2/3, 1/3)>.

The corresponding payoffs are

(2, 7), (7, 2), (14/3, 14/3).



Note that $p_{11} = 1/3$, $p_{12} = 1/3$, $p_{21} = 1/3$, $p_{22} = 0$, is a CE as it satisfies

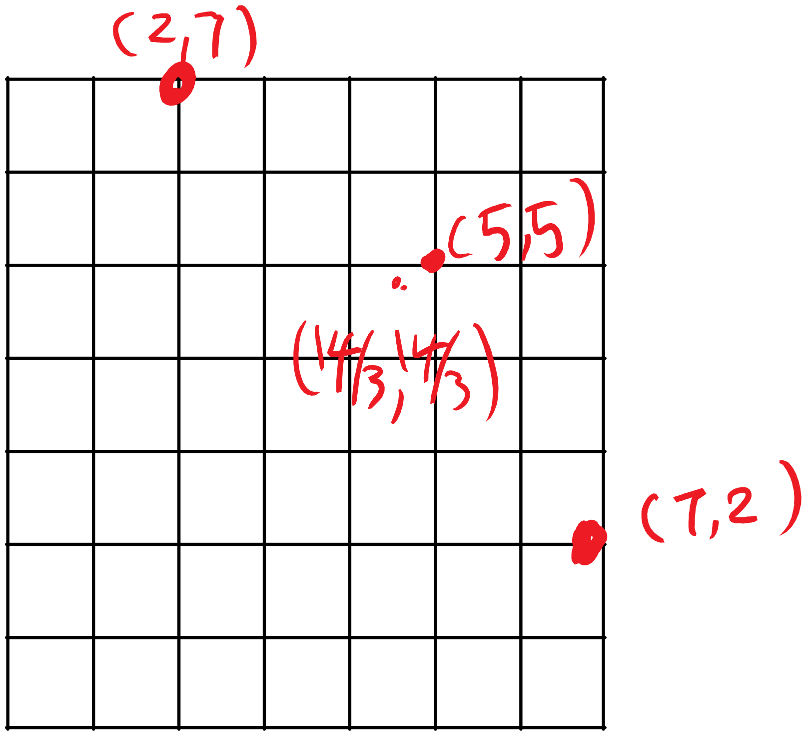
$$p_{11}(-1) + p_{12}(2) \geq 0$$

$$p_{21}(1) + p_{22}(-2) \geq 0$$

$$p_{11}(-1) + p_{21}(2) \geq 0$$

$$p_{12}(1) + p_{22}(-2) \geq 0.$$

The corresponding payoff is (5, 5) which is outside the convex hull of (2, 7), (7, 2), (14/3, 14/3).



Remark: For a given bimatrix game [A, B], we have shown that the set of payoffs of CEs of [A, B] is a convex set containing the set of payoffs of SEs of [A, B]. In other words, we may achieve better payoff by using CE. We may then ask the question that how good are the payoffs from CE. Can they be worse than their safety levels?

Theorem: The payoff of each player from a CE is better than their own safety level.

Proof:

Now let (p_{ij}) be a CE of $[A, B]$.

For each $i=1,\ldots,m$, let $p_{i\bullet}=\sum_j p_{ij}$.

When $p_{i\bullet}>0$, define $k_i = (p_{i1}/p_{i\bullet}, \ldots , p_{in}/ p_{i\bullet})^T$. k_i is a mixed strategy of Player II.

By definition of CE, the maximum of entries of Ak_i is attained at the i^{th} entry.

$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0$ for $i, r=1,\ldots m$.

and

$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0$ for $j, s=1,\ldots n$.

Let h denotes a mixed strategy of Player I.

Then, $\max_h h^T A\, k_i = i^{th}$ entry of $Ak_i = (\sum_j a_{ij} p_{ij}) / p_i$.

Let Val(A) = Safety Level of Player I.

Then, $Val(A) \leq \max_h h^T A\, k_i = (\sum_j a_{ij} p_{ij}) / p_i$.

Therefore, when $p_{i\bullet} > 0$ we have

(*) $\mathrm{Val}(A)\, p_{i\bullet} \le \sum_j a_{ij} p_{ij}$.

Note that (*) is also true when $p_{i\bullet} = 0$. This means that (*) is true for all i=1, … , m.

Summing on i on both sides of (*), we get

(**) $\mathrm{Val}(A) \le \sum_i \sum_j a_{ij} p_{ij}$.

The RHS of (**) is the payoff for Player I w.r.t. (p_{ij}).

Similarly, we can prove $\mathrm{Val}(B) \le \sum_i \sum_j b_{ij} p_{ij}$.

For 2-person 0-sum games, we have a very satisfactory solution concept of Value for the game.

Question: Do we get any other payoffs in applying the concept of CE to 2-person 0-sum games?

Proof: Given a 2-person 0-sum game $[(a_{ij}), (-a_{ij})]$, let (p_{ij}) be a CE.

Then,

$\sum p_{ij} a_{ij} \geqq \mathrm{Val}\,(a_{ij})$, $\sum p_{ij}(-a_{ij}) \geqq \mathrm{Val}\,((-a_{ij})^T) = -\,\mathrm{Val}\,(a_{ij})$.

Hence, $\sum p_{ij} a_{ij} = \mathrm{Val}\,(a_{ij})$.

For the concept of SE, the Principle of Elimination of Dominated Strategies is effective in reducing a game to a smaller one.

Question: Does the Principle of Elimination of Dominated Strategies work for solving for CE?

We discuss this question in the following.

Let $[(a_{ij}), (b_{ij})]$ be an mxn bimatrix game. Without loss of generality, we will discuss the special case when the first row (strictly) is dominated by the last row.

Let (p_{ij}) be a CE. Since $(a_{1j} - a_{nj})<0$, we must have $p_{1j}=0$ for all j. In other words, row 1 is not used in a CE because it is dominated by row n.

Theorem: The Principle of Elimination of Dominated Strategies is valid for computing CEs.

Question: What is the computational complexity of finding CE's?

Computation of CE:

Find (p_{ij}) such that

$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0$ for i, r=1,…m.

$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0$ for j, s=1,…n.

$p_{ij} \geq 0$, $\sum_{i.j} p_{ij} = 1$

We can represent the above inequalities as finding an optimal strategy, (p_{ij}), of the row chooser of a certain 2-person 0-sum game whose value is nonnegative. The payoff matrix $\mathcal{A}$ of this game is an mn by (m(m-1) + n(n-1)) matrix.

For 2x2 Game, the inequalities are

$p_{11}(a_{11} - a_{21}) + p_{12}(a_{12} - a_{22}) \geq 0$

$p_{21}(a_{21} - a_{11}) + p_{22}(a_{22} - a_{12}) \geq 0$

$p_{11}(b_{11} - b_{12}) + p_{21}(b_{21} - b_{22}) \geq 0$

$p_{12}(b_{12} - b_{11}) + p_{22}(b_{22} - b_{21}) \geq 0$

$p_{11}+p_{12}+p_{21}+p_{22}=1,\ p_{ij} \geq 0$

p_{11}	$a_{11} - a_{21}$	0	$b_{11} - b_{12}$	0
p_{12}	$a_{12} - a_{22}$	0	0	$b_{12} - b_{11}$
p_{21}	0	$a_{21} - a_{11}$	$b_{21} - b_{22}$	0
p_{22}	0	$a_{22} - a_{12}$	0	$b_{22} - b_{21}$

General mxn bimatrix games

Find (p_{ij}), $p_{ij}\geq 0$, $\sum_{i,j} p_{ij} = 1$, such that

$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0$ for i, r=1,…m.

$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0$ for j, s=1,…n.

Define $\mathcal{A}$, a payoff matrix for 2-person 0-sum game, as in the following.

(i) The rows of $\mathcal{A}$ is labeled by ij, i=1,…m, j=1,…n.

(ii)The columns of $\mathcal{A}$ is labeled by (i, r), $i\neq r$, i, r=1,…,m and by (j, s) $j\neq s$, j, s=1,…,n.

(iii) The entry at row ij and column (i, r) is $(a_{ij} - a_{rj})$

(iv) The entry at row ij and column (j, s) is $(b_{ij} - b_{is})$

(v) All the other entries are 0.

The existence of SE implies the existence of CE. Hence, the following must have a solution

Find (p_{ij}) , $p_{ij} \geq 0$, $\sum_{i.j} p_{ij} = 1$, such that

$\sum_j p_{ij} (a_{ij} - a_{rj}) \geq 0$ for i, r=1,…m.

$\sum_i p_{ij} (b_{ij} - b_{is}) \geq 0$ for j, s=1,…n.

It implies that the value of the matrix game $\mathcal{A}$ is nonnegative. The problem of finding CE is then reduced to finding the optimal strategy of the matrix game $\mathcal{A}$. Therefore, there is a polynomial time algorithm to find CE's.