Equilibrium Principle: BR to each other

Maximin Principle: Safety First

For Player I: Find p so that $\text{Min}_q p^T A q$ is the largest. p is called the Safety Strategy or Optimal Strategy.

$\text{Min}_q pTAq$ is the lower value.

For Player II: Find q so that $\text{Max}_p p^T A q$ is the smallest. q is called the Safety Strategy or Optimal Strategy.

$\text{Max}_p p^T A q$ is the upper value.

Minimax Theorem: Maximin=Minimax

Value = Lower Value (Maximin) = Upper Value (Minimax)

Finding safety strategies for 2-strategy games

A 2-strategy game is a 2-person game such that either player has 2 strategies.

First consider a 2x2 game. $A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$

We will use the Maximin Principle to find the safety strategies and the lower value.

Given a mixed strategy (p,1-p) of Player I, we will find the minimum of the payoff to Player I.

Recall that the minimum is achieved by a pure strategy of Player II.

$$
\begin{array}{c}
p \\
1-p
\end{array}
\begin{array}{|cc|}
\hline
a & b \\
d & c \\
\hline
\end{array}
$$

If Player II uses Column 1, payoff to Player I: pa+(1-p)d

If Player II uses Column 2, payoff to Player I: pb+(1-p)c

$$\begin{array}{r} p \\ 1-p \end{array}\begin{array}{|cc|} \hline a & b \\ d & c \\ \hline \end{array}$$

The minimum is then Min(pa+(1-p)d, pb+(1-p)c), for $0 \le p \le 1$.

The graph of the function

$p \mapsto$ Min(pa+(1-p)d, pb+(1-p)c) is the lowest part of the graphs of the two linear functions

$p \mapsto pa+(1-p)d$, for $0 \le p \le 1$

and

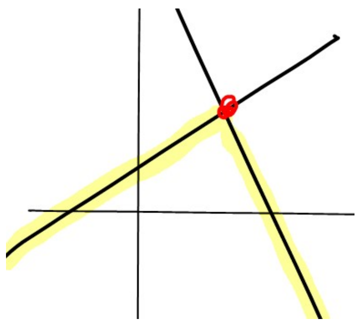
$p \mapsto pb+(1-p)c$, for $0 \le p \le 1$.

We get a piecewise linear function!

The maximum of Min(pa + (1-p)d, pb + (1-p)c) is the highest point of the graph of

$p \mapsto$ Min(pa + (1-p)d, pb + (1-p)c), for $0 \leq p \leq 1$

The highest point or the maximum is the lower value of the game. The corresponding (p, 1-p) is the safety strategy of Player I.

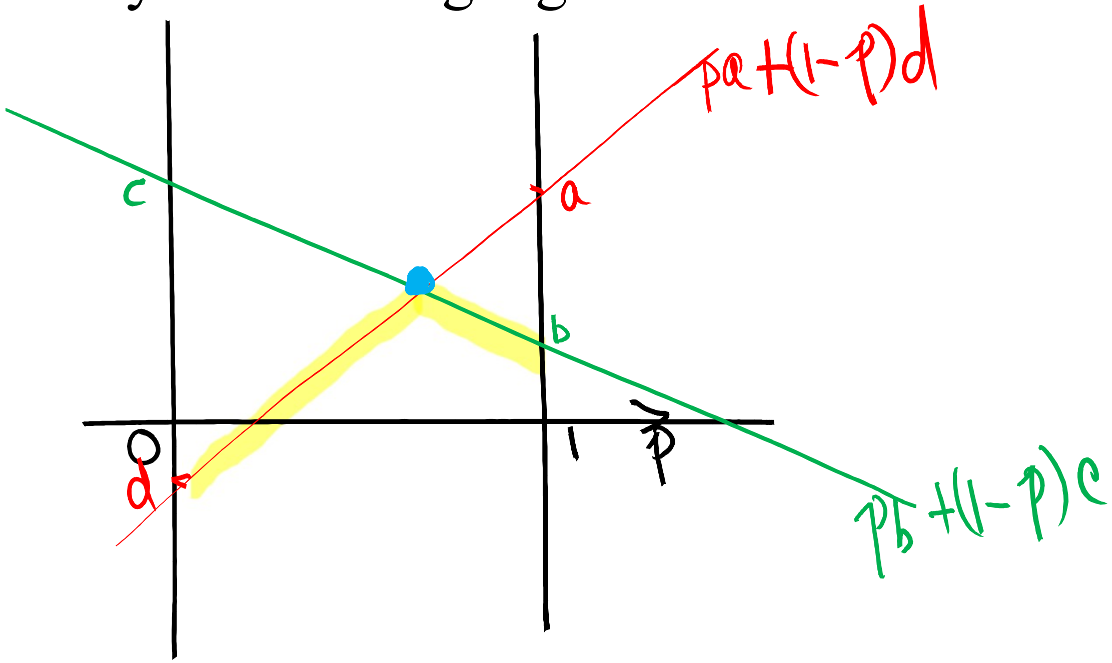


This is easy to figure it out geometrically if we draw the two graphs— straight lines—for the two functions

$p \mapsto pa + (1-p)d$, for $0 \le p \le 1$,

$p \mapsto pb + (1-p)c$, for $0 \le p \le 1$.

If the two lines do not meet in [0,1], we have one row dominates the other row. Then we will have a saddle point that could be detected by the labeling algorithm.



If the two line meet within [0,1], then we can solve for p in the following equation.

$$pa + (1-p)d = pb + (1-p)c$$

We get

$$p=c-d/(a-b +c-d) = |c-d|/(|c-d|+|a-b|)$$

$$1-p =|a-b|/(|c-d|+|a-b|)$$

$|c-d|$

$$\begin{array}{|cc|}\hline a & b \\ d & c \\ \hline\end{array}$$

$|a-b|$

$$\begin{array}{|cc|}\hline a & b \\ d & c \\ \hline\end{array}$$

$$\frac{|c-d|}{|a-b|+|c-d|}$$

$$\frac{|a-b|}{|a-b|+|c-d|}$$

$$\begin{array}{|cc|}\hline a & b \\ d & c \\ \hline\end{array}$$

How do we know that whether the two lines will meet within [0,1]?

Answer:

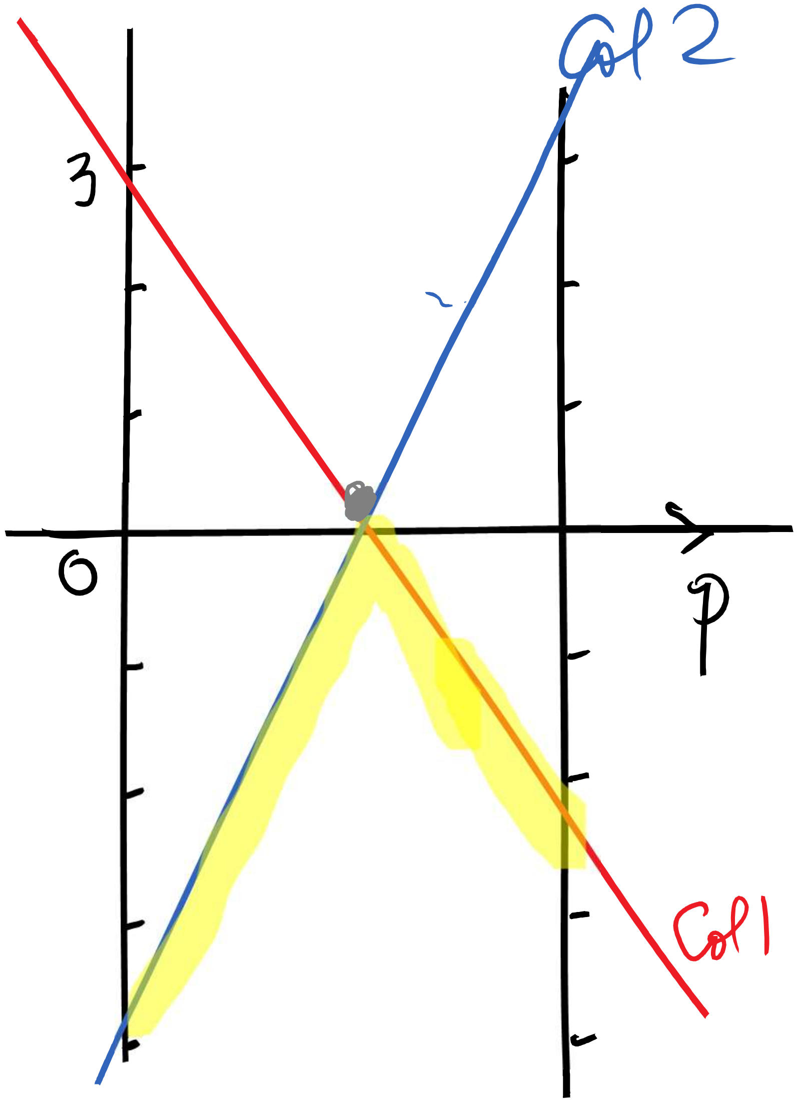
The two lines will meet within (0,1) iff No saddle point. (Problem 2 in Assignment)

Example: Find the safety strategy of Player I and the value of the game.

$$\begin{vmatrix} -2 & 3 \\ 3 & -4 \end{vmatrix}$$

We will draw the response graphs when Player II uses Col 1 and Col 2 in the following.

$$\begin{array}{c} p \\ 1-p \end{array}\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$



Answer: The safety strategy is (7/12, 5/12).

The lower value is 1/12.

Safety strategy of Player II for 2x2 games:

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

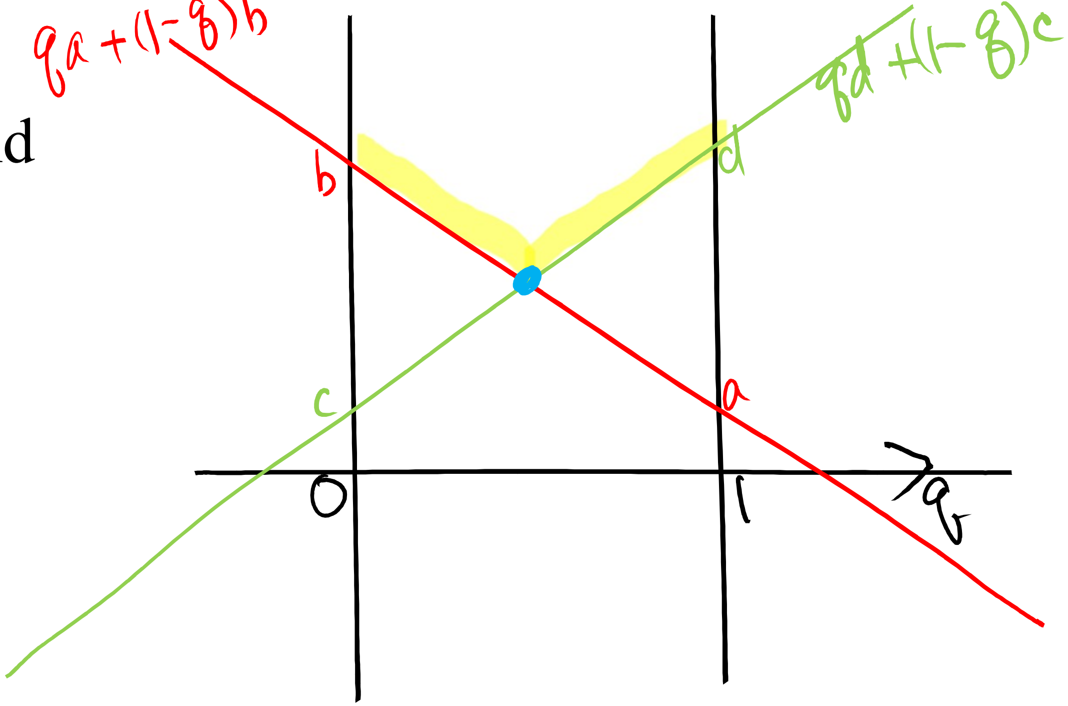
For Player II, he looks for the lowest point of the graph of

$q \mapsto \mathrm{Max}(\ qa+(1-q)b,\ qd+(1-q)c\ )$

The graph is the upper envelope of the two lines

$q \mapsto qa+(1-q)b$ and

$q \mapsto qd+(1-q)c$



If the two lines do not meet in [0,1], then one column dominates the other. Then, we have a saddle point that could be detected by the labeling algorithm.

If the two line meet within [0,1], then we can solve for q in the following equation.

$qa + (1-q)b = qd + (1-q)c$

We get

$q= c-b/(a-d+c-b) = |c-b|/(|c-b|+|a-d|)$

$1-q =a-d /(a-d + c-b) = |a-d|/(|b-c|+|a-d|)$

$|c-b|$

$$\begin{array}{|cc|}\hline a & b \\ d & c \\ \hline\end{array}$$

$|a-d|$

$$\begin{array}{|cc|}\hline a & b \\ d & c \\ \hline\end{array}$$

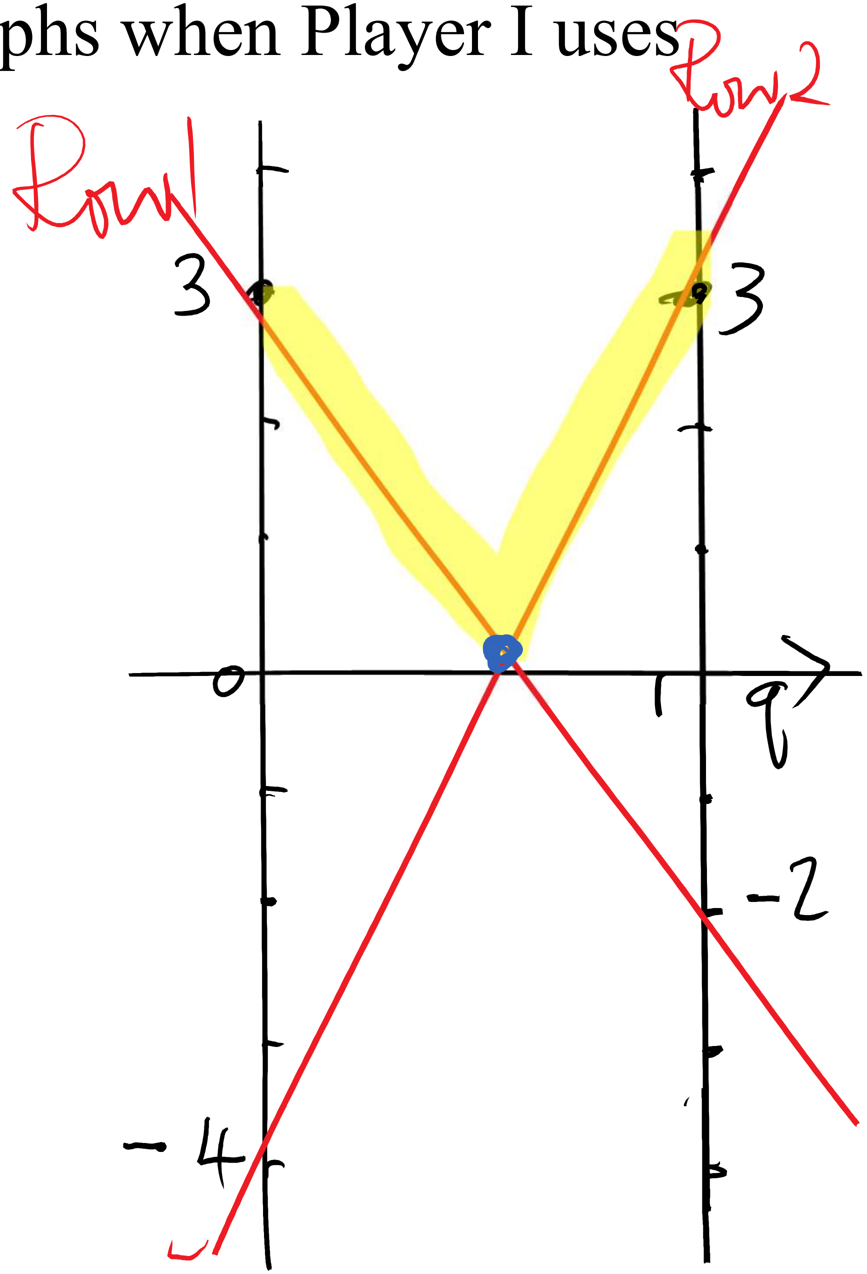
$$\frac{|c-b|}{|a-d|+|c-b|} \qquad \frac{|a-d|}{|a-d|+|c-b|}$$

$$\begin{array}{|cc|}\hline a & b \\ d & c \\ \hline\end{array}$$

Example: Find safety strategies of Player II and the value of the game.

$$\begin{array}{cc} q & 1-q \\ \end{array}$$
$$\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

We will draw the response graphs when Player I uses Row 1 and Row 2.



Answer: The safety strategy is (7/12,5/12).

The upper value is 1/12.

Solving $2 \times n$ and $m \times 2$ games.

$2 \times n$ games:

(i) Check for saddle point first. If there is a saddle point then we solve the game already.

(ii)Suppose there is no saddle point.

Using the graphical method, we can find the safety strategy of Player I for 2xn games. Since the Maximin occurs at the intersection of two lines, Player II, knowing that Player I will play his/her safety strategy, will play the two Columns giving rise to the two lines. The situation then reduces to 2x2 games.

Solving $2 \times n$ **and** $m \times 2$ **games.**

$m \times 2$ **games:**

(i) Check for saddle point first. If there is a saddle point then we solve the game already.

(ii)Suppose there is no saddle point.

Using the graphical method, we can find the safety strategy of Player II for mx2 games. Since the Minimax occurs at the intersection of two lines, Player I, knowing that Player II will play his/her safety strategy, will play the two Rows giving rise to the two lines. The situation then reduces to 2x2 games.

Example: Find the value and the safety strategies of both players.

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$$

Example: Find the value and the safety strategies of both players.

$$\begin{array}{cc} & \begin{array}{cc} \tfrac{5}{10} & \tfrac{5}{10} \end{array} \\ \begin{array}{c} \tfrac{6}{10} \\ \tfrac{4}{10} \end{array} & \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \end{array}$$

$$\text{Value} = -1 \times \frac{6}{10} + 4 \times \frac{4}{10} = 1$$

$$\text{Safety Strategy for I} = \left(\frac{6}{10}, \frac{4}{10}\right)$$

$$\text{Safety Strategy for II} = \left(\frac{5}{10}, \frac{5}{10}\right)$$

Example: Find the value and the safety strategies of both players.

$$\begin{vmatrix} 5 & 0 \\ -1 & 2 \end{vmatrix}$$

Example: Find the value and the safety strategies of both players.

$$\begin{array}{c|cc|} & 2/8 & 6/8 \\ \hline 3/8 & 5 & 0 \\ 5/8 & -1 & 2 \\ \hline \end{array}$$

Value $= \frac{10}{8}$

Safety Strategy for I: $(3/8, 5/8)$

Safety Strategy for II: $(2/8, 6/8)$

Example: Find the value and the safety strategies.

$$\begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix}$$

Example: Find the value and the safety strategies.

$$\begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix}$$

Saddle point at Row1 Column2.

Safety Strategy for I: Row1

Safety Strategy for II: Column2

Value = 1

Example: Find the value and the safety strategies.

$$\begin{array}{|cc|}\hline 0 & -1 \\ 2 & 3 \\ \hline\end{array}$$

Example: Find the value and the safety strategies.

$$\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$$

Safety Strategy for I : Row2

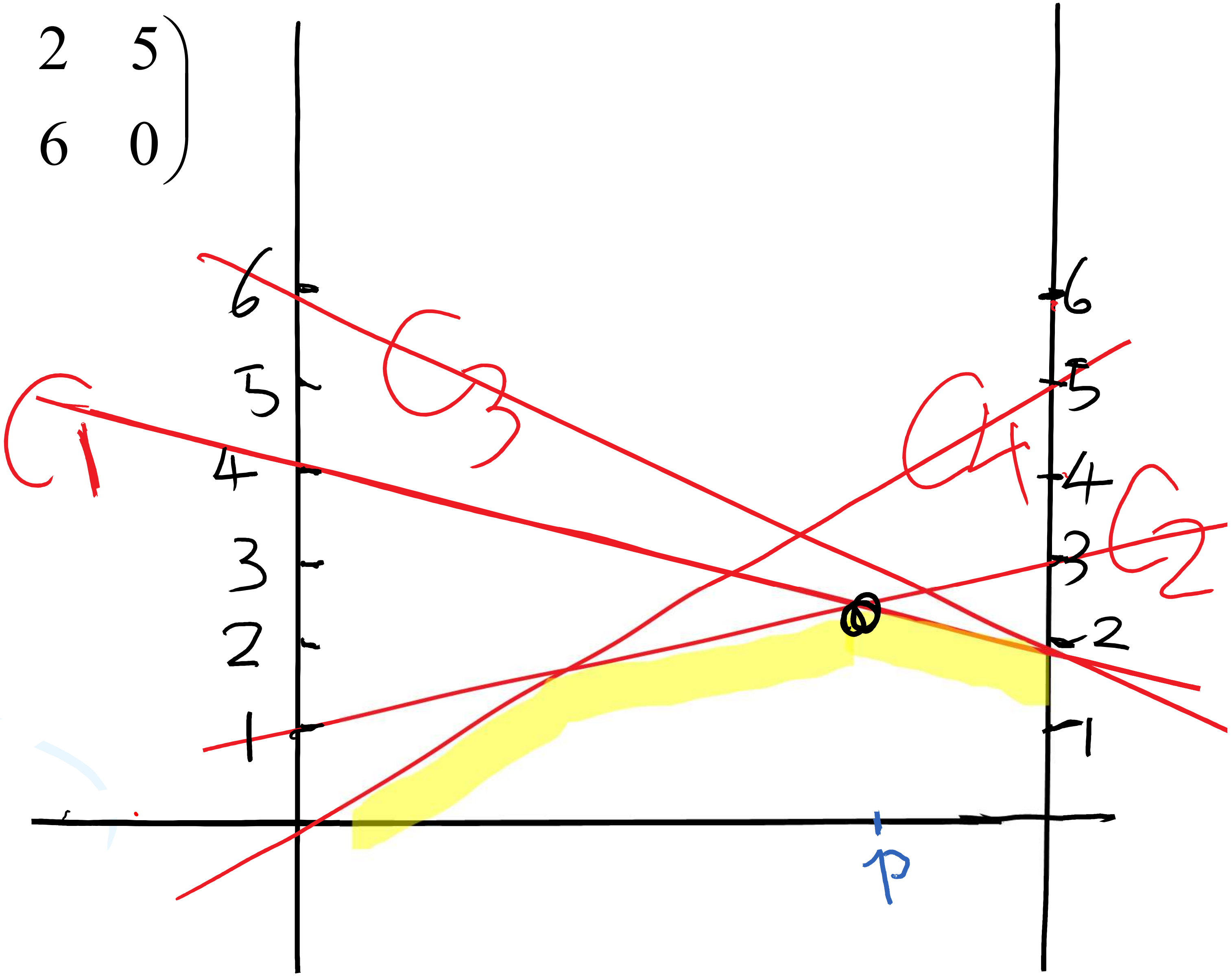
Safety Strategy for II : Column 1

Value = 2

Example of 2xn game: Find the value and the safety strategies.

$$\begin{pmatrix} 2 & 3 & 2 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 2 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$



Answer: To solve for p, we use Col1 and Col2, to get p=3/4. Therefore (3/4,1/4) is a safety (optimal) strategy for Player I. (2/4,2/4, 0,0) is a BR to (3/4, 1/4)

Clearly, (3/4,1/4) is a BR to (2/4,2/4,0,0).

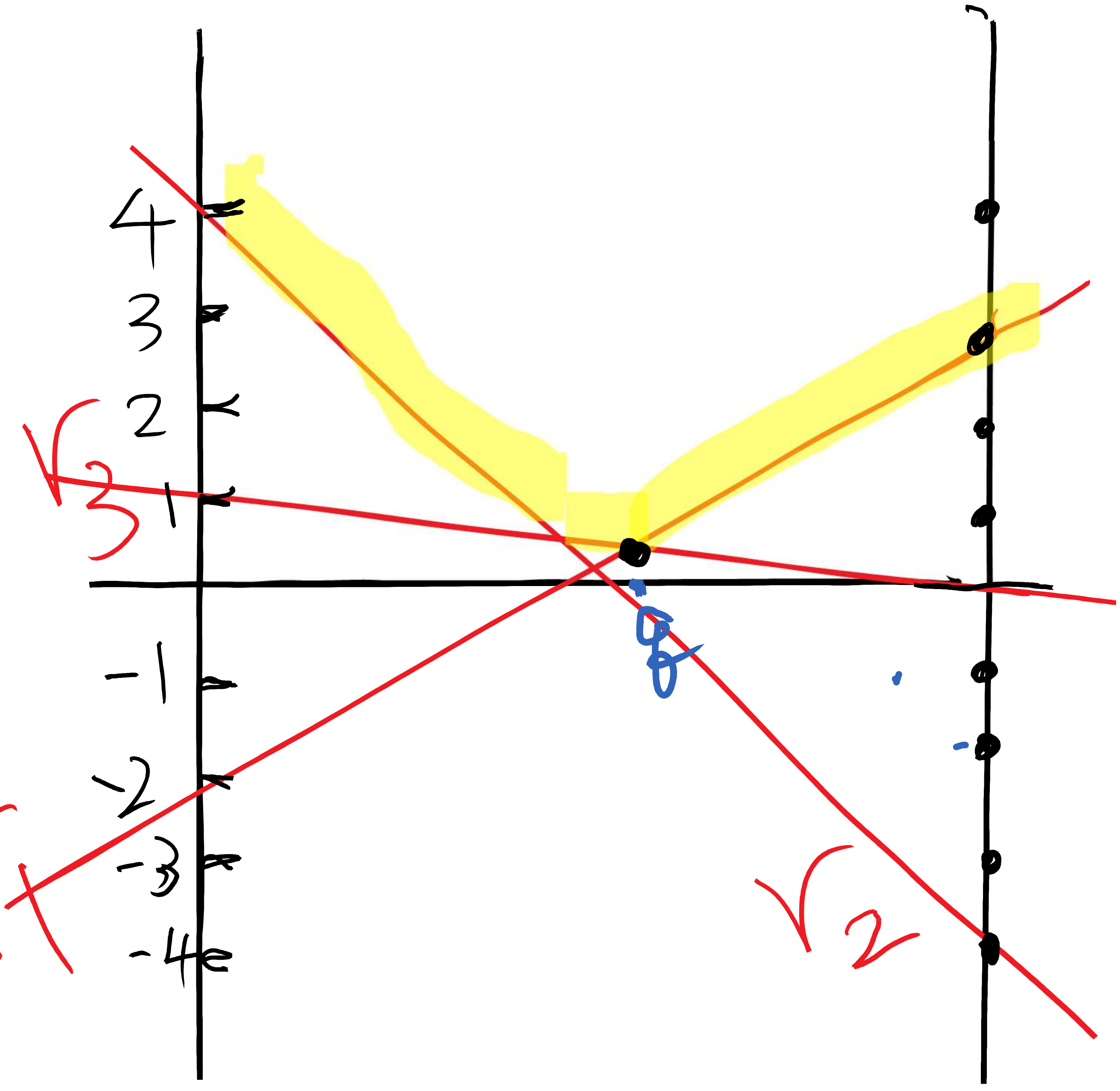
$$\begin{array}{cc} & \begin{array}{cccc} 2/4 & 2/4 & 0 & 0 \end{array} \\ \begin{array}{c} 3/4 \\ 1/4 \end{array} & \begin{pmatrix} 2 & 3 & 2 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix} \end{array}$$

Ans: Value $= \frac{3}{4} \times 2 + \frac{1}{4} \times 4 = \frac{10}{4}$

Example of mx2 game : Find the value and the safety strategies.

$$\begin{bmatrix} 3 & -2 \\ -4 & 4 \\ 0 & 1 \end{bmatrix}$$

q	$1-q$
3	−2
−4	4
0	1



$$
\begin{array}{c|cc|}
 & \frac{3}{6} & \frac{3}{6} \\
\hline
\frac{1}{6} & 3 & -2 \\
0 & -4 & 4 \\
\frac{5}{6} & 0 & 1 \\
\hline
\end{array}
$$

Value $= \frac{3}{6}$

Reduction by removing dominated strategies.

1	2	3
2	0	4
3	1	0
4	1	2

0	4	6
5	7	4
9	6	3

1	2	3
2	0	4
3	1	0
4	1	2

Reduction by removing dominated strategies.

$$\begin{array}{c|ccc}
 & 1/4 & 3/4 & 0 \\
\hline
3/4 & 1 & 2 & 3 \\
0 & 2 & 0 & 4 \\
0 & 3 & 1 & 0 \\
1/4 & 4 & 1 & 2
\end{array}$$

$V = \frac{7}{4}$

Safety strategy for I $= (3/4, 0, 0, \frac{1}{4})$

Safety strategy for II $= (1/4, 3/4, 0)$

Reduction by removing dominated strategies.

0	4	6
5	7	4
9	6	3

0	4	6
5	7	4
9	6	3

We can eliminate column 2 by a mixture of Column 1 & 3.

$$\frac{2}{6}\begin{pmatrix}0\\5\\9\end{pmatrix}+\frac{4}{6}\begin{pmatrix}6\\4\\3\end{pmatrix}=\begin{pmatrix}4\\\frac{26}{6}\\\frac{30}{6}\end{pmatrix}\leq\begin{pmatrix}4\\7\\6\end{pmatrix}$$

0	6
5	4
9	3

We can eliminate row 2 by a mixture of row 1 & 3.

$$\frac{4}{9}(0, 6) + \frac{5}{9}(9, 3) = \left(5, \frac{39}{9}\right) \geq (5, 4)$$

$$\begin{bmatrix} 0 & 6 \\ 9 & 3 \end{bmatrix}$$

Answer:

$$
\begin{array}{c|ccc|}
 & 3/12 & 0 & 9/12 \\
\hline
6/12 & 0 & 4 & 6 \\
0 & 5 & 7 & 4 \\
6/12 & 9 & 6 & 3 \\
\hline
\end{array}
$$

$$\text{Value} = \frac{54}{12}$$

Equivalence of the Maximin Principle and the Equilibrium Principle

We have studied 2-person 0-sum games from the Maximin Principle. We were successful to define the safety strategy for Player I and got $\underline{V}$, the lower value.

Also, we defined the safety strategy for Player II and got $\overline{V}$ the upper value.

Then, the Minimax Theorem of von Neumann says that $\underline{V}=\overline{V}$.

In the following we will prove that

Maximin Principle $\Leftrightarrow$ Equilibrium Principle

The main tool is von Neumann's Minimax Theorem.

MiniMax Theorem (John von Neumann, 1928)

$\underline{V}(A) = \overline{V}(A)$, or equivalently

$$\operatorname*{Min}_{q\in Y^*} \operatorname*{Max}_{p\in X^*} p^T Aq = \operatorname*{Max}_{p\in X^*} \operatorname*{Min}_{q\in Y^*} p^T Aq$$

Then, $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the value of the game.

We first recall the concept of Best Response.

Let (X, Y, A) be a 2-person 0-sum game.

Let X^*, Y^* be the corresponding set of mixed strategies.

Definition: Let p~belong to X^*, q~ is called the best response (BR) to p~ whenever

$$A(p^{\sim},q) \geqslant A(p^{\sim}, q^{\sim}), \text{ for all } q \text{ belong to } Y^*.$$

Let q~ belong to Y^*, p~ is called the best response (BR) to q~ whenever

$$A(p,q^{\sim}) \leqslant A(p^{\sim}, q^{\sim}), \text{ for all } p \text{ belong to } X^*.$$

$$
\begin{array}{c|cccc|}
\cline{2-5}
1/3 & 2 & 3 & 1 & 5 \\
2/3 & 4 & 1 & 6 & 0 \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\frac{10}{3}} & \multicolumn{1}{c}{\frac{5}{3}} & \multicolumn{1}{c}{\frac{13}{3}} & \multicolumn{1}{c}{\frac{5}{3}} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\text{BR}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\text{BR}}
\end{array}
$$

$$
\begin{array}{|cc|ll}
\multicolumn{1}{c}{1/4} & \multicolumn{1}{c}{3/4} & & \\
\hline
0 & 6 & 18/4 & \text{BR} \\
5 & 4 & 17/4 & \\
9 & 3 & 18/4 & \text{BR} \\
\hline
\end{array}
$$

Lemma: Let p belong to X^* and we let

J_p={j: Column j is a BR to p}, the set of best response columns to p.

Then,

$q=(q_1,\ldots q_n)^T$ a BR to p iff $q_j>0$ implies j belongs to J_p.

Let q belong to Y^* and we let I_q={i: Row i is a BR to q}, the set of best response rows to q.

Then,

$p=(p_1,\ldots p_m)^T$ a BR to q iff $p_i>0$ implies i belongs to I_q.

Remark: The Lemma says that a BR to p of Player I uses only the BR columns to p.

Similarly, a BR to q of Player II uses only the BR rows to q.

Remark: The payoffs of BR columns to p are equal.

Similarly, the payoffs of BR rows to q are equal.

We can then prove easily the following Principle of Indifference for equilibrium pairs.

Theorem (Principle of Indifference): Let $p=(p_1,\ldots p_m)^T$ lie in X^* and $q=(q_1,\ldots q_n)^T$ lie in Y^* such that $<p,q>$ is an equilibrium pair. Let $p^TAq=V$. Then,

$a_{i1}q_1+\ldots+a_{in}q_n=V$ for all i for which $p_i>0$

and

$a_{1j}p_1+\ldots+a_{mj}p_m=V$ for all j for which $q_j>0$

Sketch of proof:

$p_i>0$ means the i^{th} Row is a BR row to q.

Hence, $a_{i1}q_1+\ldots+a_{in}q_n= a_{k1}q_1+\ldots+a_{kn}q_n$ whenever $p_i,p_k >0$. Call this value U. Then,

$V = \sum p_i a_{ij} q_j = \sum p_i U=U$. (sum of positive p_i is 1).

A similar argument then completes the proof of $a_{1j}p_1+\ldots+a_{mj}p_m=V$ for all j for which $q_j>0$.

The Principle of Indifference gives the following recipe to find all the best response strategies to a given strategy:

(i) Find all best response pure strategies

(ii) Use only those BR pure strategy to compose your mixed strategy

Recall that

- Equilibrium Principle: BR to each other
- Maximin Principle: Safety first

For Player I: Find p so that $\text{Min}_q p^T A q$ is largest. p is called the Safety Strategy or Optimal Strategy.

$\text{Min}_q p^T A q$ is the lower value.

For Player II: Find q so that $\text{Max}_p p^T A q$ is smallest. q is called the Safety Strategy or Optimal Strategy.

$\text{Max}_p p^T A q$ is the upper value.

Minimax Theorem: Maximin=Minimax

Value = Lower Value (Maximin) = Upper Value (Minimax)

Assuming Mnimax Theorem we can prove the following.

Theorem:

(Maximin Principle implies Equilibrium Principle)

Let p~ belong to X^*, q~ belong to Y^*, be safety strategies for Player I and II respectively.

Then, p~, q~ are best responses to each other i.e. <p~, q~> is an equilibrium pair.

Proof:

$p^{\sim}$ is a safety strategy for Player I means

$$\mathrm{Min}_q\, p^{\sim T}Aq = \mathrm{Max}_p \mathrm{Min}_q\, p^T Aq$$

$q^{\sim}$ is a safety strategy for Player II means

$$\mathrm{Max}_p\, p^T A q^{\sim} = \mathrm{Min}_q \mathrm{Max}_p\, p^T Aq$$

Then, we show in the following $p^{\sim}$ is a BR to $q^{\sim}$.

$$p^{\sim T}Aq^{\sim} \geqslant \mathrm{Min}_q\, p^{\sim T}Aq = \mathrm{Max}_p\mathrm{Min}_q\, p^TAq$$

$$= \mathrm{Min}_q\mathrm{Max}_p\, p^TAq = \mathrm{Max}_p\, p^TAq^{\sim}$$

$$\geqslant p^TAq^{\sim}, \text{ for any } p \text{ in } X^*$$

This completes the proof.

Similarly, $q^{\sim}$ is a BR to $p^{\sim}$.

Assuming Minimax Theorem we can prove the following.

Theorem (Equilibrium Principle implies Maximin Principle):

Let $p^{\sim}$, $q^{\sim}$ be best responses to each other i.e. $\langle p^{\sim}, q^{\sim} \rangle$ is an equilibrium pair.

Then, $p^{\sim}$ (in X^*), $q^{\sim}$ (in Y^*)are safety strategies for Player I and II respectively.

Proof:

As $p^{\sim}$ is a BR to $q^{\sim}$, we have

$$
\begin{aligned}
p^{\sim T}Aq^{\sim} &= \mathrm{Max}_p\, p^TAq^{\sim} \\
&\geqslant \mathrm{Max}_p\mathrm{Min}_q\, p^TAq = \mathrm{Min}_q\mathrm{Max}_p\, p^TAq \\
&\geqslant \mathrm{Min}_q\, p^{\sim T}Aq \\
&= p^{\sim T}Aq^{\sim} \quad (q^{\sim} \text{ is a BR to } p^{\sim})
\end{aligned}
$$

Thus, $\mathrm{Min}_q\, p^{\sim T}Aq = \mathrm{Max}_p\mathrm{Min}_q\, p^TAq$ and $p^{\sim}$ is a safety strategy for Player I.

Similarly, $q^{\sim}$ is a safety strategy for Player II.

Remark: Since the Equilibrium Principle is equivalent to the Safety Principle, we can solve 2-person 0-sum game using either principle whichever is easier.

Theorem: (Exchange Principle)

Let $<p_1, q_1>$, $<p_2, q_2>$ be equilibrium pairs. Then, $<p_1, q_2>$ is also an equilibrium pair.

Proof: By the equivalence of Maximin Principle and Equilibrium Principle, p_1, p_2 are safety strategies of Player I and, similarly, q_1, q_2 are safety strategies of Player II. Therefore,

$<p_1, q_2>$ is an equilibrium pair. This completes the proof.

Definition: p belong to X^* is called an equalizing strategy whenever A(p,q)=A(p,q') for any q, q' in Y^*.

q belong to Y^* is called an equalizing strategy whenever A(p',q)=A(p,q) for any p, p' in X^*.

Remark: To define an equalizing strategy we only need to know that it equalizes on the opponent's pure strategies.

Example (Odd or Even):

$$\begin{array}{|cc|} -2 & 3 \\ 3 & -4 \end{array}$$

Then, (7/12, 5/12), (7/12, 5/12) are equalizing startegies for Player I and Player II respectively.

Theorem: Suppose p, q are equalizing strategies for Player I, Player II respectively. Then, p, q are BR to each other and hence <p, q> is an equilibrium pair.

Proof: Obvious

Solutions to special games:

Magic Square Games:

A magic square game is an matrix game such that the row sums and the column sums of the payoff matrix are all equal.

Example:

$$\begin{pmatrix} 13 & 2 & 7 & 12 \\ 11 & 8 & 1 & 14 \\ 4 & 15 & 10 & 5 \\ 6 & 9 & 16 & 3 \end{pmatrix}$$

Then, <1/4,1/4,1/4,1/4> is an equalizing strategy for Player I and, <1/4,1/4,1/4,1/4> is an equalizing strategy for Player II.

Thus, <(1/4,1/4,1/4,1/4), (1/4,1/4,1/4,1/4)> is an equilibrium pair.