Two Person 0-Sum Games (Part I)

Two-Person Zero-Sum Games

The individual most closely associated with the creation of the theory of games is John von Neumann, one of the greatest mathematicians of last century. Although others preceded him in formulating a theory of games - notably Émile Borel –

it was von Neumann who published in 1928 the paper that laid the foundation for the theory of two-person zero-sum games,

i.e. games with only two players in which one player wins what the other player loses. We will refer to the players as Player I and Player II.

Example: Odd or Even

Players I and II simultaneously call out one of the numbers one or two.

Player I’s name is Odd; he wins if the sum of the numbers is odd. Player II’s name is Even; she wins if the sum of the numbers is even.

The amount paid to the winner by the loser is always the sum of the numbers in dollars.

To build a mathematical model describing the example, the following are the essential information.

*** There are two players: Player I, Player II**

*** For each player there is a set of actions to choose from.**

Player I: {1, 2}

Player II: {1, 2}

*** Associated to the choice of action of Player I and Player II, there is a payoff from Player II to Player I.**

Strategic Form of 2-person 0-sum games

The simplest mathematical description of a game is the strategic form. For a two-person zero-sum game, the payoff function of Player II is the negative of the payoff of Player I, so we may restrict attention to the single payoff function of Player I.

Definition: The strategic form, or normal form, of a two-person zero-sum game is given by a triplet (X, Y, A), where

(1) X is a nonempty set, the set of strategies of Player I

(2) Y is a nonempty set, the set of strategies of Player II

(3) A is a real-valued function defined on $X \times Y$.

(Thus, $A(x, y)$ is a real number for every $x \in X$ and every $y \in Y$.)

The interpretation is as follows. Simultaneously, Player I chooses $x \in X$ and Player II chooses $y \in Y$, each unaware of the choice of the other.

Then their choices are made known and I wins the amount $A(x, y)$ from II.

If A is negative, I pays the absolute value of this amount to II.

Thus, A (x, y) represents the winnings of I and the losses of II.

Matrix Games

A finite two-person zero-sum game in strategic form, (X, Y, A), is sometimes called a matrix game because the essential information of the game, the payoff function A, can be represented by a matrix.

If $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$, then by the game matrix or payoff matrix we mean the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij} = A(x_i, y_j)$,

In this form, Player I chooses a row, Player II chooses a column, and II pays I the entry in the chosen row and column.

Note that the entries of the matrix are the winnings of the row chooser and losses of the column chooser.

Remark: For a matrix game, we will also refer Player I as the Row chooser, and Player II as the Column chooser. (Some book calls the Row chooser as Rose or Rowena, and the Column chooser as Colins).

Examples:

Scissors-Rock-Paper

Players I and II simultaneously display one of the three objects: scissors, rock or paper.

If they both choose the same object to display, there is no payoff. If they choose different objects, then scissors win over paper (scissors cut paper), rock wins over scissors (rock breaks scissors), and paper wins over rock (paper covers rock). If the payoff upon winning or losing is one unit, then the matrix of the game is as follows.

	S	R	P
S	0	-1	1
R	1	0	-1
P	-1	1	0

Matching Pennies:

Two players simultaneously choose heads or tails. Player I wins if the choices match and Player II wins otherwise. If the payoff upon winning or losing is one unit, then the payoff matrix of the game is as follow

$$\begin{array}{c|cc} & H & T \\ \hline H & 1 & -1 \\ T & -1 & 1 \end{array}$$

Simplified Morra:
Each of two players show one finger or two fingers, and simultaneously guesses how many fingers the other player will show. If both players guess correctly, or both players guess incorrectly, there is no payoff. If just one player guesses correctly, that player wins a payoff equal to the total of fingers shown by both players. If we let (x,y) be the strategy of showing x finger and guessing y finger, then the payoff matrix of the game is as below.

	(1, 1)	(1, 2)	(2, 1)	(2, 2)
(1, 1)	0	2	−3	0
(1, 2)	−2	0	0	3
(2, 1)	3	0	0	−4
(2, 2)	0	−3	4	0

Odd or Even

Players I and II simultaneously call out one of the numbers one or two. Player I's name is Odd; he wins if the sum of the numbers if odd. Player II's name is Even; she wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars. To put this game in strategic form we must specify X, Y and A. Here we may choose X = {1, 2}, Y = {1, 2}, and A as given in the following table.

A(x, y) = I's winnings = II's losses.

	1	2
1	–2	3
2	3	–4

Question: How will the game play out?

This is not an easy question to answer. What are the principles guiding players' choices?

The first general principle in solving games in strategic form is that players will discard inferior (dominated) strategies.

Removing Dominated Strategies.

Definition. We say the i^{th} row of a matrix $A = (a_{ij})$ dominates the k^{th} row if

$a_{ij} \geq a_{kj}$ for all j. We say the i^{th} row of A strictly dominates the k^{th} row if $a_{ij} > a_{kj}$ for all j.

Similarly, the j^{th} column of A dominates (strictly dominates) the k^{th} column if $a_{ij} \leq a_{ik}$ (resp. $a_{ij} < a_{ik}$) for all i.

Anything Player I can achieve using a dominated row can be achieved at least as well using the row that dominates it. Hence dominated rows may be deleted from the matrix. A similar argument shows that dominated columns may be removed.

We may iterate this procedure and successively remove several rows and columns. (Examples to be given later)

Example from a play: ***Orphan of the House Zhao*** (趙氏孤兒)

The main story is that the Duke of Zhao was overthrown by one of his officers. The new ruler thus sought to kill every one of the Zhao family, ending the old emperor's bloodline. The Duke of Zhao's wife gave birth to a baby boy at that time, and gave the baby to her doctor to hide from the new emperor. The mother was killed but the baby escaped with the doctor.

The new emperor then learned about the Zhao baby's presence and sought to kill the baby. He forcefully took every one of the village's infants and declared that if nobody would confirm which baby is of the Zhao bloodline, then he would kill every one of the village's infants to ensure the death of the Zhao Orphan.

The doctor also had a newborn baby. To save the royal bloodline is his top priority, he turned in his own son to the new emperor pretending it was Zhao Orphan.

The doctor’s wife was very upset and went mad. As the Zhao Orphan grew, the doctor sent him to be raised next to the new emperor and eventually told him of the truth. The Zhao Orphan then killed the new ruler, became the new Duke of Zhao.

The strategy of sacrificing his own son dominates the strategy of being silent and doing nothing.

Example: **Battle of Bismarck Sea**

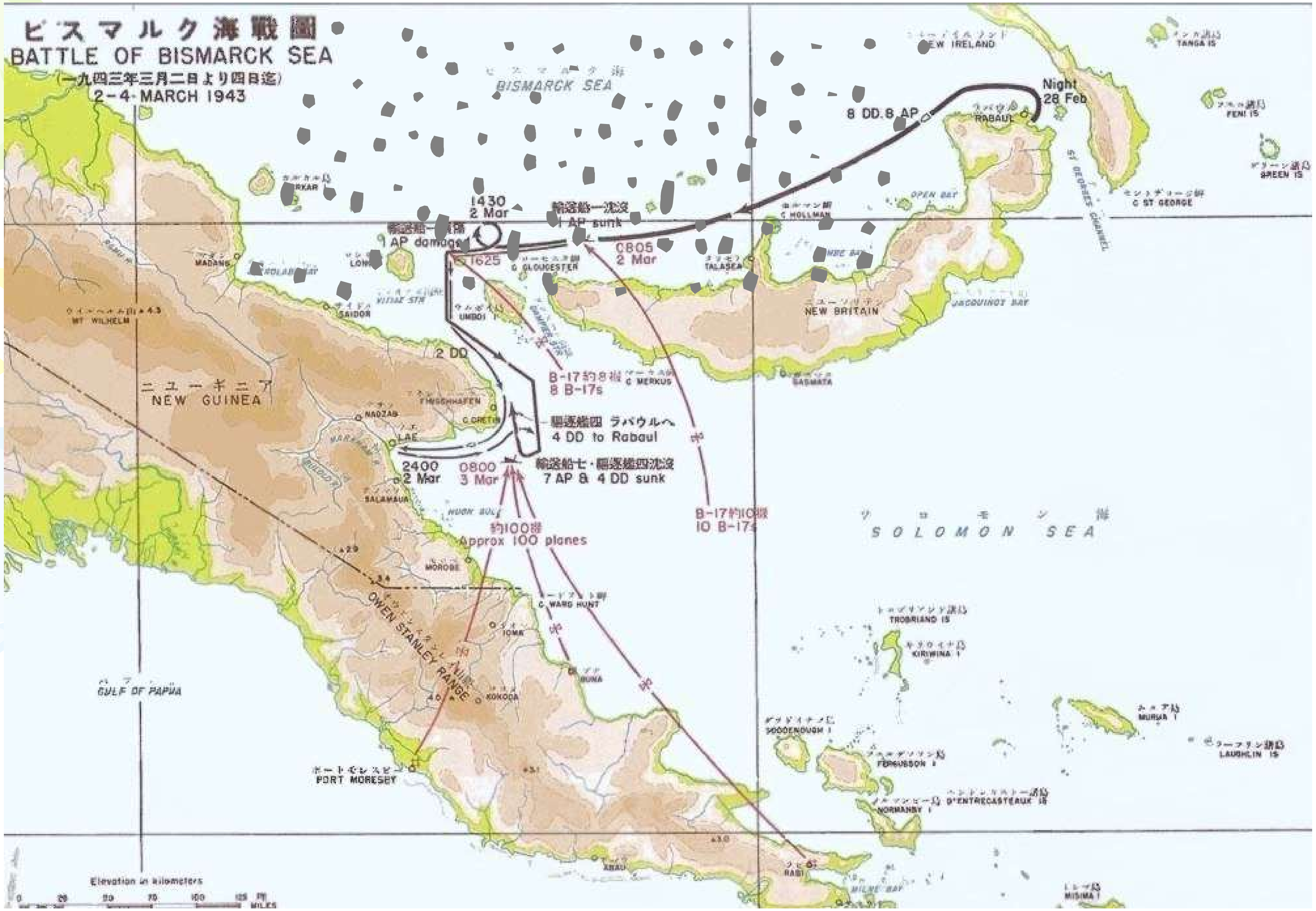
In the critical stages of the struggle for New Guinea, intelligence reports indicated that the Japanese would move a troop and supply convoy from the port at the eastern tip of New Britain to Lae, which lies just west of New Britain or New Guinea. It could travel north of New Britain, where poor visibility was almost certain, or south of the Island, where the weather would be clear; in either case, the trip would take three days.

General Kenney had the choice of concentrating the bulk of his reconnaissance aircraft on the north route or the south. Once sighted, the convoy could be bombed until its arrival at Lae. In days of bombing time, Kenney's staff estimated the following outcomes for the various choices:

	N	S
N	2	2
S	1	3

For this game the second column is dominated by the first column. The Japanese will remove the second column from his consideration. Kenney, knowing the Japanese's removal of the second column, will then play the first row.

Hence, <N, N> is the outcome of this game.



ビスマルク海戦圖
BATTLE OF BISMARCK SEA
(一九四三年三月二日より四日迄)
2-4 MARCH 1943
BISMARCK SEA
NEW IRELAND
Night
28 Feb
8 DD. 8 AP
RABAUL
1430
2 Mar
1 AP damaged
1 AP sunk
0805
2 Mar
1625
NEW BRITAIN
NEW GUINEA
2 DD
B-17約8機
8 B-17s
4 DD to Rabaul
7 AP & 4 DD sunk
2400
2 Mar
0800
3 Mar
約100機
Approx 100 planes
B-17約10機
10 B-17s
SOLOMON SEA
OWEN STANLEY RANGE
GULF OF PAPUA
PORT MORESBY
LAE
SALAMAUA
MADANG
FINSCHHAFEN
Elevation in kilometers

[6]

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MILITARY DECISION AND GAME THEORY*

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The United States military doctrine of decision prescribes that a commander select the course of action which offers the greatest promise of success in view of the enemy's capabilities of opposing him. This paper analyzes two battle decisions of World War II, and develops the analogy between existing military doctrine and the 'theory of games' proposed by von Neumann. Current U. S. doctrine is conservative. The techniques of game theory permit analysis of the risk involved if the commander deviates from current doctrine to base his decision on his estimate of what his enemy intends to do rather than on what his enemy is capable of doing. The idea of 'mixed strategies' presents more difficulties but may be useful, particularly for command decisions for small military organizations.

VON NEUMANN and Morgenstern point out[1] that in the early stages of the development of a new theory, application serves to corroborate theory. The theory of games has been most fully developed for the two-person situation, the conflict of two opposing individuals or groups. Almost all battle decisions involve two opposing military forces. Moreover, the student of game theory need not analyze numerous battles to learn the military philosophy of decision. The doctrine has been formalized and is available in military texts.[2]

MILITARY-DECISION DOCTRINE

A military commander may approach decision with either of two philosophies. He may select his course of action on the basis of his estimate of what his enemy *is able to do* to oppose him. Or, he may make his selection on the basis of his estimate of what his enemy *is going to do*. The former is a doctrine of decision based on enemy capabilities; the latter, on enemy intentions.

The doctrine of decision of the armed forces of the United States is a doctrine based on enemy capabilities. A commander is enjoined to select

*The basic concepts of this paper were developed by the author, then a colonel in the U. S. Air Force, while a student at the Air War College, 1949–1950.

the course of action which offers the greatest promise of success in view of the enemy capabilities. The process of decision, as approved by the Joint Chiefs of Staff and taught in all service schools, is formalized in a five-step analysis called the *Estimate of the Situation*. These steps are illustrated in the following analysis of an actual World War II battle situation.

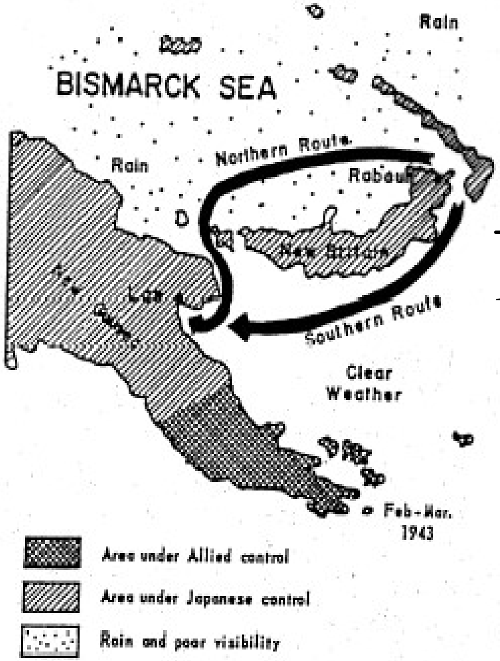


FIG. 1. *The Rabaul-Lae Convoy Situation.* The problem is the distribution of reconnaissance to locate a convoy which may sail by either one of two routes.

THE RABAUL-LAE CONVOY SITUATION

General Kenney was Commander of the Allied Air Forces in the Southwest Pacific Area. The struggle for New Guinea reached a critical stage in February 1943. Intelligence reports indicated a Japanese troop and supply convoy was assembling at Rabaul (see Fig. 1). Lae was expected to be the unloading point. With this general background Kenney proceeded to make his five-step Estimate of the Situation.

	N	S
N	2	2
S	1	3

Step 1. The Mission

General MacArthur as Supreme Commander had ordered Kenney to intercept and inflict maximum destruction on the convoy. This then was Kenney's mission.

Step 2. Situation and Courses of Action

The situation as outlined above was generally known. One new critical factor was pointed out by Kenney's staff. Rain and poor visibility were predicted for the area north of New Britain. Visibility south of the island would be good.

The Japanese commander had two choices for routing his convoy from Rabaul to Lae. He could sail north of New Britain, or he could go south of that island. Either route required three days.

Kenney considered two courses of action, as he discusses in his memoirs.[3] He could concentrate most of his reconnaissance aircraft either along the northern route where visibility would be poor, or along the southern route where clear weather was predicted. Mobility being one of the great advantages of air power, his bombing force could strike the convoy on either route once it was spotted.

Step 3. Analysis of the Opposing Courses of Action

With each commander having two alternative courses of action, four possible conflicts could ensue. These conflicts are pictured in Fig. 2.

Step 4. Comparison of Available Courses of Action

If Kenney concentrated on the northern route, he ensured one of the two battles of the top row of sketches. However, he alone could not determine which one of these two battles in the top row would result from his decision. Similarly, if Kenney concentrated on the southern route, he ensured one of the battles of the lower row. In the same manner, the Japanese commander could not select a particular battle, but could by his decision assure that the battle would be one of those pictured in the left column or one of those in the right column.

Kenney sought a battle which would provide the maximum opportunity for bombing the convoy. The Japanese commander desired the minimum exposure to bombing. But neither commander could determine the battle which would result from his own decision. Each commander had full and independent freedom to select either one of his alternative strategies. He had to do so with full realization of his opponent's freedom of choice. The particular battle which resulted would be determined by the two independent decisions.

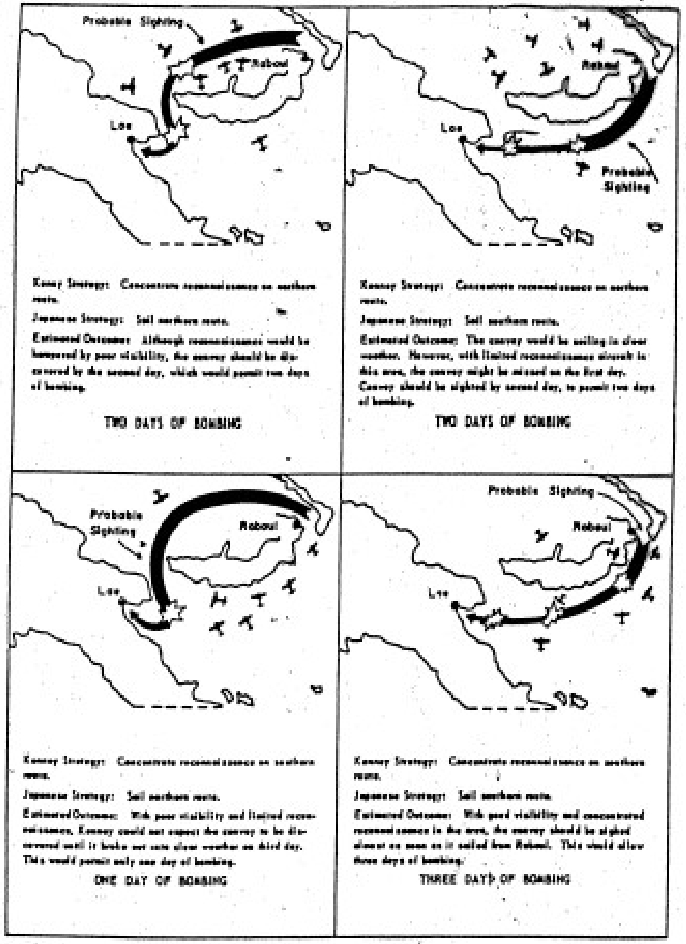


FIG. 2. *Possible Battles for the Rabaul-Lae Convoy Situation.* Four different engagements of forces may result from the interaction of Kenney's two strategies with the two Japanese strategies. Neither commander alone can determine which particular battle will result.

	N	S
N	2	2
S	1	3

The U. S. doctrine of decision—the doctrine that a commander base his action on his estimate of what the enemy is capable of doing to oppose him—dictated that Kenney select the course of action which offered the greatest promise of success in view of all of the enemy capabilities. If Kenney concentrated his reconnaissance on the northern route, he could expect two days of bombing regardless of his enemy's decision. If Kenney selected his other strategy, he must accept the possibility of a less favorable outcome.

Step 5. The Decision

Kenney concentrated his reconnaissance aircraft on the northern route.

Discussion

Let us assume that the Japanese commander used a similar philosophy of decision, basing his decision on his enemy's capabilities. Considering the four battles as sketched, the Japanese commander could select either the left or the right column, but could not select the row. If he sailed the northern route, he exposed the convoy to a maximum of two days of bombing. If he sailed the southern route, the convoy might be subjected to three days of bombing. Since he sought minimum exposure to bombing, he should select the northern route.

These two independent choices were the actual decisions which led to the conflict known in history as the Battle of the Bismarck Sea. Kenney concentrated his reconnaissance on the northern route; the Japanese convoy sailed the northern route; the convoy was sighted approximately one day after it sailed; and Allied bombing started shortly thereafter. Although the Battle of the Bismarck Sea ended in a disastrous defeat for the Japanese, we cannot say the Japanese commander erred in his decision. A similar convoy had reached Lae with minor losses two months earlier. The need was critical, and the Japanese were prepared to pay a high price. They did not know that Kenney had modified a number of his aircraft for low-level bombing and had perfected a deadly technique. The U. S. victory was the result of careful planning, thorough training, resolute execution, and tactical surprise of a new weapon—not of error in the Japanese decision.

Those familiar with game thoery will recognize that the Rabaul-Lae situation presents all the features of a two-person game. The two commanders have independent choices of action, and these interact to determine a particular battle. The Bismarck-Sea battle exposed the Japanese convoy to a certain number of days of bombing. The 'game' situation

Example: Penalty Kick in Soccer

Ball takes 0.2 second to get to the goal line

Strategies for Kicker: Natural Side (right-footed kicks left, left-footed kicks right), Opposite Side

Strategies for Goalie: Correct Side (for righted-footed kicker goes right, for left-footed kicker goes left), Middle or Wrong Side.

Collect data from elite soccer leagues from France and Italy.

Kicker	Goalie: Correct side	Goalie: Middle or wrong side
Natural side ("left")	63.6 percent	94.4 percent
Opposite side ("right")	43.7 percent	89.3 percent

In general, we may not be able to remove any strategy or after the removal of some strategies the game matrix is still quite big. The Principle of Removal of Dominated Strategies can only help us to simplify the game matrix somewhat.

Definition: Let (X, Y, A) be a matrix game. Let $x \in X$ be a strategy of Player I. Then, $y \in Y$ is called a Best Response to x if $A(x, y) \leq A(x, y')$ for any $y' \in Y$.

Equivalently, Let $y \in Y$ be a strategy of Player II. Then, $x \in X$ is called a Best Response to y if $A(x, y) \geqq A(x', y)$ for any $x' \in X$.

Example: For P of Player I, BR of Player II is S. For R of Player II, BR of Player I is P of Player I.

	S	R	P
S	0	−1	1
R	1	0	−1
P	−1	1	0

We may apply the following two principles to analyze the game.

Equilibrium Principle: Best Responses to each other
做到最好

This principle involves the interactions of the players.

Maximin Principle: Safety First
安全第一

Under this principle, each player only concerns his/her own payoff.

BR to each other in pure strategies

Saddle points (Pure Strategic Equilibrium, PSE, for 2-person 0-sum games)

If some entry a_{ij} of the matrix **A** has the property that

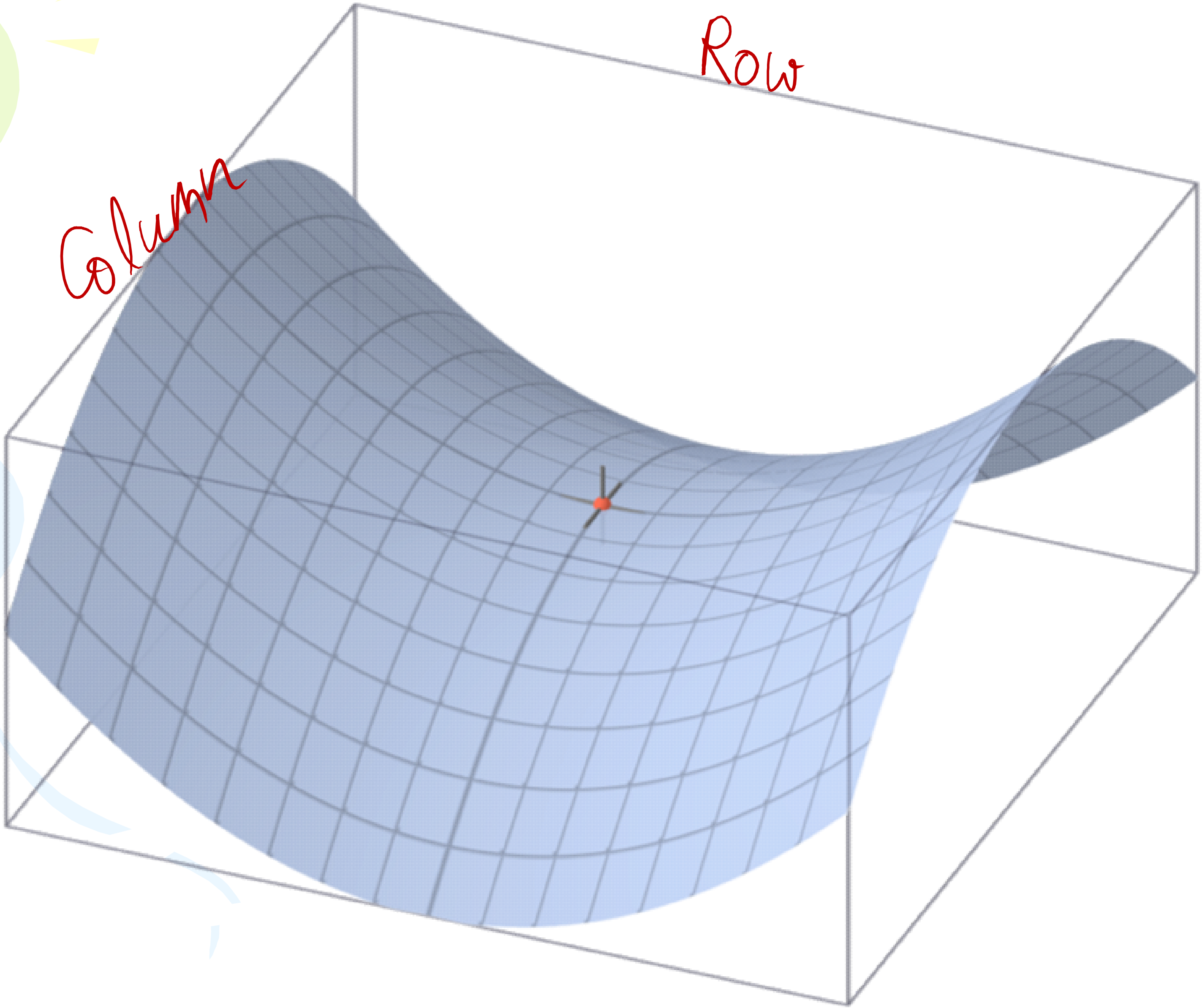
(1) a_{ij} is the minimum of the i^{th} row, and

(2) a_{ij} is the maximum of the j^{th} column,

then we say a_{ij} is a saddle point. If a_{ij} is a saddle point, then Player I can then win at least a_{ij} by choosing row i, and Player II can keep her loss to at most a_{ij} by choosing column j.

<Row i, Column j> is then an equilibrium pair,

i.e. BR to each other.


Row
Column

Example: For the following matrix, the entry at the 2nd row and 2nd column is a saddle point, i.e. minimum in its row and maximum in its column.

$$A = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{pmatrix}$$

For large $m \times n$ matrices it is tedious to check each entry of the matrix to see if it has the saddle point property. We can use the following labeling algorithm to find saddle points.

Labelling Algorithm:

1. Go through the game matrix row by row. Put a star on the entry that is the minimum of its row.
2. Go through the game matrix column by column. Put a star on the entry that is the maximum of its column.
3. The entries with two stars are saddle points.

Example: In the following matrix A, there is no saddle point.

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix}$$

However, if 2 in the position a_{12} were changed to 1, then we have matrix B.

$$B = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix}$$

Here, the minimum of the 4^{th} row and the maximum of the 2^{nd} column occurs at the same location. Therefore, b_{42} is a saddle point.

Remark: A game matrix may have several saddle points. However, the values of the saddle points are all equal (exercise). In this case, we can call this the value of the game and it is well-defined.

For the game of Odd and Even, the game matrix has no saddle point.

$$\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

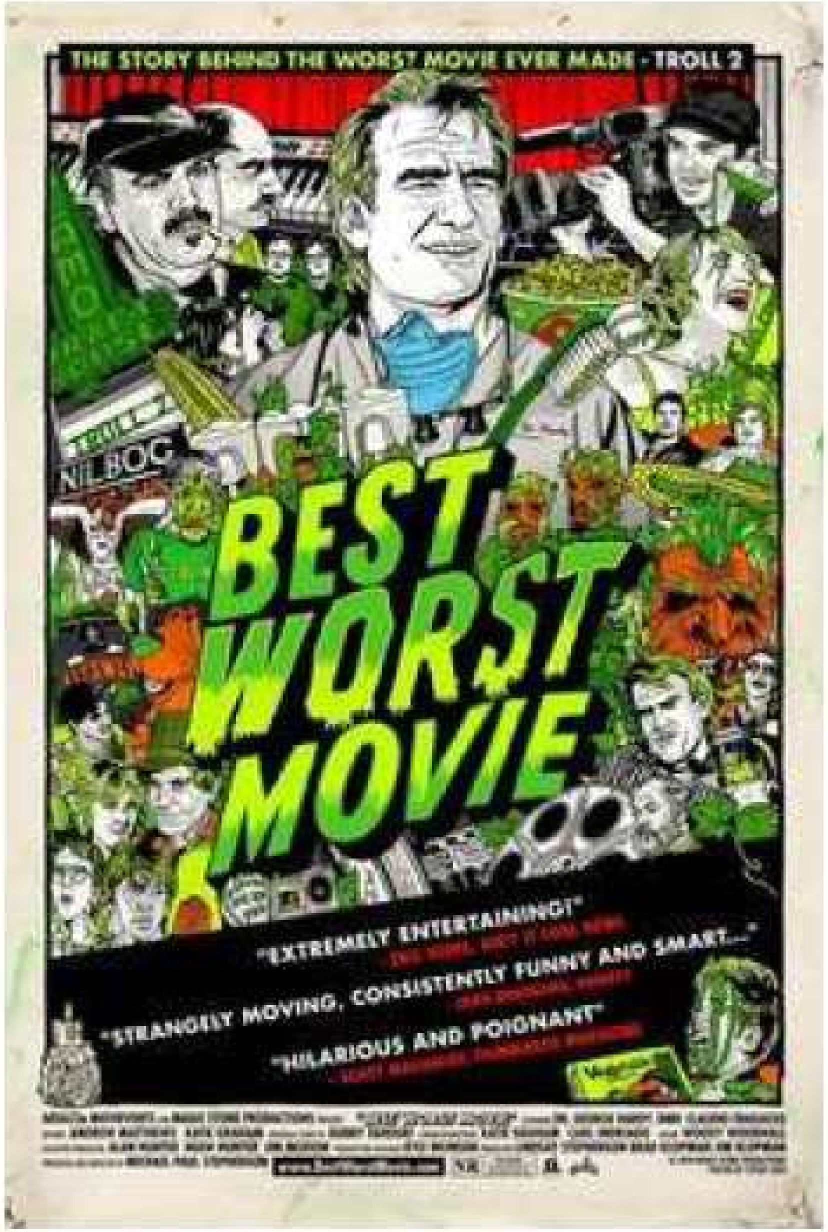
Therefore, we will try Maximin Principle (Safety First). It may give us a new point of view to analyze the problem.

Maximin Principle means to find the "risk" for each strategy and then find the strategy, called the safety strategy, with minimum risk.

Using this "safety strategy", one can guarantee to get at least a certain amount of payoff.

Remark: Maximin Principle is an important decision rule in Ethics (e.g. John Rawls' Theory of Justice).

Example of Maximin:



$$\boxed{\begin{matrix} -2 & 3 \\ 3 & -4 \end{matrix}}$$

If Player I uses Row 1, the worst payoff is -2.

$$\underset{j}{\mathrm{Min}}\, A(x_1, y_j) = -2$$

If Player I uses Row 2, the worst payoff is -4.

$$\underset{\mathrm{j}}{\mathrm{Min}}\, A(x_2, y_j) = -4$$

Therefore, the "best" of the worst case is -2.

$$\underset{\mathrm{i}}{\mathrm{Max}}\, \underset{\mathrm{j}}{\mathrm{Min}}\, A(x_i, y_j) = -2$$

It is achieved by Row 1. If Player I uses Row 1, he can guarantee to get a payoff of -2.

$$\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

Adopting the Maximin (Safety) Principle for Player II, we will talk about Minimax.

$$\underset{j}{\text{Min}}\,\underset{i}{\text{Max}}\; A(x_i, y_j) = 3$$

It can be achieved by using Column 1 or Column 2.

Can we do better? $\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$

Suppose Player I flips a fair coin to decide his strategy. If Head appears, he uses Row 1. If Tail appears, he uses Row 2.

Player is expected to get $(-2)\times(0.5)+3\times(0.5)=0.5$, if Player II uses Column 1.

Player I is expected to get $(3)\times(0.5)+(-4)\times(0.5)=-0.5$, if Player II uses Column 2.

Therefore, the worst payoff for this randomized strategy is -0.5.

We should randomize our strategy!

Mixed Strategies

Consider a finite 2-person zero-sum game, (X, Y, A), with $m \times n$ matrix, **A**.

Let us take the strategy space X to be the first m integers,

$X = \{1, 2, . . ., m\}$, and similarly, $Y = \{1, 2, . . . , n\}$.

A mixed strategy for Player I may be represented by a column vector, $(p_1 , p_2 , . . . , p_m)^T$ of probabilities that add up to 1.

Similarly, a mixed strategy for Player II is an n-tuple
$\mathbf{q} = (q_1 , q_2 , . . . , q_n)^T$.

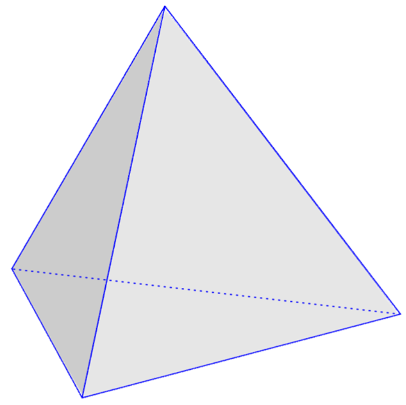
The sets of mixed strategies of players I and II will be denoted respectively by X*, Y*.

$X^* = \{\mathbf{p} = (p_1 , . . . , p_m)^T : p_i \geq 0$, for $i = 1, . . . , m$ and $p_1 +\ldots+ p_m =1\}$

$Y^* = \{\mathbf{q} = (q_1 , . . . , q_n)^T : q_j \geq 0$, for $j = 1, . . . , n$ and $q_1 +\ldots+ q_n =1\}$

$\mathbf{p} = (p_1, . . . , p_m)^T$ means playing Row 1 with probability p_1 , playing Row 2 with probability p_2 ,…, playing Row m with probability p_m.

The m-dimensional unit vector $\mathbf{e}_k \in X^*$ with a one at the k^{th} component and zeros elsewhere may be identified with the pure strategy of choosing row k.



Remark: X*, Y* are compact convex sets such that the vertices correspond to pure strategies.

Example: When m=2,

$X^* = \{p = (p_1 , p_2)^T : p_i \geq 0$, for $i = 1, 2$ and $p_1 + p_2 =1\}$.

We can then write

$X^* = \{p = (p , 1-p)^T : 1 \geqq p \geqq 0\}$.

Therefore, X^* is parametrized by the unit interval.

Suppose Player I is using $(p_1,\ldots,p_m)^T$.

The payoff to Player I when Player II uses Column j is $\Sigma_i\, p_i a_{ij}$.

Suppose Player II is using $(q_1,\ldots q_n)^T$, i.e. using Column j with probability q_j. The payoff to Player I is then

$$\Sigma_j\,(\Sigma_i\, p_i a_{ij}) q_j = \sum_i \sum_j p_i a_{ij} q_j$$

Extension of payoff to mixed strategies:

We may consider the set of Player I's pure strategies, X, to be a subset of X*.

Similarly, Y may be considered to be a subset of Y*.

We could if we like consider the game (X, Y, A) in which the players are allowed to use mixed strategies as a new game

(X*, Y*,A), where A(**p**, **q**) = $\mathbf{p}^{\mathbf{T}}\mathbf{Aq} = \sum_i \sum_j p_i a_{ij} q_j$

Remark:

In this extension, we have made a rather subtle assumption. We assumed that when a player uses a mixed strategy, he is only interested in his average return. He does not care about his maximum possible winnings or losses — only the average.

This is actually a rather drastic assumption.

The main justification for this assumption comes from **utility theory**.

The basic premise of utility theory is that one should evaluate a payoff by its utility to the player rather than on its numerical monetary value. Utility theory is one of the fundamental contributions of von Neumann and Morgenstern.

Remark: There are some philosophical issues about using mixed strategies.

Do we really use mixed strategies in real life?

Safety Strategies:

For each $p \in X^*$, the worst payoff is $\underset{q \in Y^*}{\mathrm{Min}}\, p^T Aq$

The minimum is achieved at a pure strategy of Player II.

Lemma: For each fixed p, the minimum of A(p, q) as a function of q on Y* is achieved at a vertex of Y*, or, equivalently, at a pure strategy.

Proof: Let $(p_1,\ldots,p_m)^T$ be a given strategy in X^*. Then, for any $(q_1,\ldots,q_n)^T \varepsilon Y^*$, $A(p, q)=\Sigma_{i,j}\, p_i\, a_{ij}\, q_j = \Sigma_j\, (\Sigma_i\, p_i\, a_{ij})q_j$, a linear function of $q_1,\ldots,q_n$.

Note that $(d/dq_i)A(p,q)= \Sigma_i\, p_i\, a_{ij}$, a constant!

When we minimize $A(p, q)$ as a function of q, the optimal value must then occur at a vertex of Y^*.

Remark: If the minimum of $A(p, q)$ is attained at q. Then q is a BR to p.

Example:

Given $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \end{pmatrix}$

Suppose the Row Chooser (Player I) plays Row 1 with probability 0.4, and Row 2 with probability 0.6. Find the BR of the Column Chooser (Player II).

Solution:

$$\begin{array}{cc} 0.4 & \\ 0.6 & \end{array}\begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \end{pmatrix}$$

$$\begin{array}{ccc} -0.8 & 1.4 & 1.2 \\ \text{BR} & & \end{array}$$

Thus,

$$\operatorname*{Min}_{q \in Y^*} p^T A q = \operatorname{Min}(p^T A e_1, \ldots, p^T A e_n)$$

The function $p \mapsto \operatorname*{Min}_{q \in Y^*} p^T A q$ is then a continuous function on X^*.

As X^* is a compact set, there exists $\tilde{p} \in X^*$ such that

the maximum of the function $p \mapsto \operatorname*{Min}_{q \in Y^*} p^T A q$ is attained at $\tilde{p}$.

Theorem : There exists $\tilde{p} \in X^*$ such that

$$\operatorname*{Min}_{q \in Y^*} \tilde{p}^T A q = \operatorname*{Max}_{p \in X^*} \operatorname*{Min}_{q \in Y^*} p^T A q.$$

$\tilde{p}$ is called the safety strategy of Player I and

$$\underline{V} = \operatorname*{Min}_{q \in Y^*} \tilde{p}^T A q = \operatorname*{Max}_{p \in X^*} \operatorname*{Min}_{q \in Y^*} p^T A q$$

is called the lower value of the game.

Remark: If Player I uses his safety strategy then he/she is guaranteed to get at least $\underline{V}$ no matter what Player II does.

A similar theorem holds for Player II in the form of MinMax.

Theorem : There exists $\tilde{q} \in Y^*$ such that

$$\operatorname*{Max}_{p \in X^*} p^T A \tilde{q} = \operatorname*{Min}_{q \in Y^*} \operatorname*{Max}_{p \in X^*} p^T A q = \overline{V}$$

$\tilde{q}$ is called the safety strategy of Player II and

$\overline{V}$ is called the upper value of the game.

Remark: If Player II uses his/her safety strategy then he/she is guaranteed to lose at most $\overline{V}$ no matter what Player I does.

Theorem : $\underline{V} \le \overline{V}$

Proof :

Let $\tilde{p}, \tilde{q}$ be safety strategy of Player I and II respectively.

$$\underline{V} = \underset{q \in Y^*}{\text{Min}}\, \tilde{p}^T A q \le \tilde{p}^T A \tilde{q} \le \underset{p \in X^*}{\text{Max}}\, p^T A \tilde{q} = \overline{V}$$

Remark: If $\underline{V} < \overline{V}$, then both players may try to deviate from their safety strategies.

MiniMax Theorem (John von Neumann, 1928)

$$\operatorname*{Min}_{q\in Y^*} \operatorname*{Max}_{p\in X^*} p^T Aq = \operatorname*{Max}_{p\in X^*} \operatorname*{Min}_{q\in Y^*} p^T Aq$$

$i.e.\ \underline{V} = \overline{V}.$

Then, $V = \underline{V} = \overline{V}$ is called the value of the game.

Remark: Because of the Minimax Theorem, the safety strategies are also called the optimal strategies.