博弈论 hw14

Coal14. Let (N, v) be a 2-person game in coalition form and v is the characteristic function. Find the nucleolus.

The set $N = \{1, 2\}$ of 2 players and the characteristic function v, which maps every subset of S of N as a coalition, to a nonnegative number $v(S)$, called the value of coalition S. For are 2 player game, there will be three coalitions. Like in any n player game, $v(\emptyset) = 0$. Then there is $v(1), v(2)$ and $v(1, 2)$. In order to find the nucleolus, we would start off with the pro rata payoff, then we would find the largest excess. Minimize that excess and continue until we can no longer do so.

Coal15. Find the Shapely value and the nucleolus of the game with characteristic function

$$v(\emptyset) = 0,$$
$$v(\{1\}) = 1, v(\{2\}) = 0, v(\{3\}) = -4$$
$$v(\{1, 2\}) = 2, v(\{1, 3\}) = -1, v(\{2, 3\}) = 3$$
$$v(\{1, 2, 3\}) = 6$$

An efficient game would be $x_1 + x_2 + x_3 = 6$, if our first guess is a pro rata payoff vector of

S	$v(S)$	$e(x, S)$
1	1	$1 - x_1$
2	0	$0 - x_2$
3	-4	$-4 - x_3$
12	2	$2 - x_1 - x_2 = x_3 - 4$
13	-1	$-1 - x_1 - x_3 = x_2 - 7$
23	3	$3 - x_2 - x_3 = x_2 - 3$
123	6	$6 - x_1 - x_2 - x_3 = 0$

The Shapely value of this game is noted as

$\phi_1 = 13/6$

$\phi_2 = 22/6$

$\phi_3 = 1/6$

Coal16. A man has three wives whose marriage contract specify that in case of his death, they receive 100, 200, 300 respectively respectively. When the man dies, leaving an estate of 200, find the nucleolus starting from the pro rata allocation.

By the pro rate allocation, we allocate based on percentages. The first of the wives (A) would get $100/600$ or $1/6$ of the money, the second (B) of the wives would get $200/600$, or $1/3$, and the third (C) of the wives would get $300/600$ or $1/2$ of the money. Thus, when the man dies and leaves an estate of 200. The first of the wives would get $200/6$, the second $200/3$, and the third $200/2$.

Let's compare this allocation with those suggested by the Shapely value and the nucleolus. First, we must decide on the characteristic functions.

By definition, $v(\emptyset)=0$, and $v(ABC)=200$, $A, B,$ nor C are guaranteed to receive anything since the other two could receive the whole amount, thus, $v(A)=v(B)=v(C)=0$, Similarly $v(A,B)=0,\ v(B,C)=100,\ v(A,C)=0$

In order to find the nucleolus, let $x=(x_1,x_2,x_3)$ be an efficient allocation, that is $x_1+x_2+x_3=200$

Consider the first arbitrary point, such as the pro rata point $(200/6, 200/3, 200/2)$, the excesses are

S	$v(S)$	$e(x,S)$	$(200/6, 200/3, 200/2)$
A	0	$0-x_1$	$-200/6$
B	0	$0-x_2$	$-200/3$
C	0	$0-x_3$	$-200/2$
AB	0	$0-x_1-x_2$	$-200/2$
BC	100	$100-x_2-x_3$	$-200/3$
AC	0	$0-x_1-x_3$	$-500/3$

The largest of these numbers is $-200/6$, corresponding to the coalition A, this coalition will claim that every other coalition is doing better than it is.

We can try to improve on things for this coalition by making x_1 larger, or x_2+x_3 smaller. Thus we need to decrease the excess for A, but at the same time, BC will increase at the same rate so the excess of the two will meet in the middle at -50, when $x_1=50$. It is clear that no choice of x can make the maximum excess smaller than -50 since at least one of the coalitions A or BC will have excess of at least -50, thus $x_1=50$ is the first component of nucleolus..

Next we have to find $x_2+x_3=200-50=150$. We choose them to make the next largest excess smaller.

S	$v(S)$	$e(x,S)$	$(200/6, 200/3, 200/2)$	$(50, 200/3, 250/3)$
A	0	$0-x_1$	$-200/6$	-50
B	0	$0-x_2$	$-200/3$	$-200/3$
C	0	$0-x_3$	$-200/2$	$-250/3$
AB	0	$0-x_1-x_2=x_3-200$	$-200/2$	$-250/3$
BC	100	$100-x_2-x_3=x_1-100$	$-200/3$	-150
AC	0	$0-x_1-x_3=x_2-200$	$-400/3$	$-400/3$

We can see that the next largest excess after $-200/6, -50$ is $-200/3$, corresponding to coalition B. To make this smaller, we increase x_2 and decrease x_1+x_3. Thus we need to decrease x_3. If B and C change at the same rate. They will meet in the middle, arriving at 75.

And $x_2+x_3=150$, we get that $x_2=x_3=75$

S	$v(S)$	$e(x, S)$	$(200/6, 200/3, 200/2)$	$(50, 200/3, 250/3)$	$(50, 75, 75)$
A	0	$0 - x_1$	$-200/6$	-50	-50
B	0	$0 - x_2$	$-200/3$	$-200/3$	-75
C	0	$0 - x_3$	$-200/2$	$-250/3$	-75
AB	0	$0 - x_1 - x_2 = x_3 - 200$	$-200/2$	$-350/3$	-125
BC	100	$100 - x_1 - x_3 = x_2 - 100$	$-200/3$	$-100/3$	-25
AC	0	$0 - x_2 - x_3 = x_1 - 200$	$-400/3$	$-450/3$	-150

The shapely value for this case would be

$\phi_A = 100/3$

$\phi_B = 250/3$

$\phi_C = 250/3$

Coal17

Suppose $v(S) = \begin{cases} |S| & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$

Find the nucleolus (Hint: use the symmetry axiom)

$(1/|S|, 2/|S|, \dots, n/|S|), i = 1, \dots, n$

Coal18. Let $[A, B]$ be a bimatrix game. Find the Gately point for the game in coalition form derived from $[A, B]$

The Gately point is the imputation such that it minimizes $maxd_i : 1 \leq i \leq n$ among all imputations.

Personal Notes

find imputation $x = (x_1, \dots, x_n)$ that minimizes the worst inequity, in other words, ask each coalition S how dissatisfied it is with the proposed imputation x and we try to minimize the maximum dissatisfaction

x measures the amount (size of inequity) by which coalition S falls short of its potential $v(S)$

- imputation x is in the core if and only if all its excess are negative or zero
- first look at coalitions S whose excess, for a fixed allocation is largest, then adjust x to make this largest excess smaller, then repeat for the next largest until no more adjustments can be made, the imputation is known as the **nucleolus**

pro rata proportional

Theorem: The nucleus of a game in coalition form exists and is unique. The nucleolus is a group rational, individually rational, and satisfies the symmetry axiom and the dummy axiom. If the core is not empty, the nucleolus is in the core.