1. Suppose $A = (a_{ij})$ is a matrix game and that a_{ij} is a saddle point. Show that Row i, Col j are safety strategies for player 1 and player 2 (aka optimal strategies)

Maxmin Principle for safety strategies
$$\begin{cases} \text{player 1: Find } p \text{ so that } \operatorname{Min}_q p^T A q \text{ is the largest} \\ \quad p \text{ is the safety strategy} \\ \text{player 2: Find } q \text{ so that } \operatorname{Max}_q p^T A q \text{ is the smallest} \\ \quad q \text{ is the safety strategy} \end{cases}$$

Definition of saddle point:

a_{ij} is the minimum of ith row

a_{ij} is the maximum of jth col

Saddle point occurs when the maximum of row equals the minimum of column, or in other words

(player 1) Max Min = Min Max, thus, we need (p, q) such that these occur giving us the row and column that are the safety strategies of player 1 and player 2 respectively

2. Solve the game with matrix $\begin{pmatrix} 0 & 2 \\ t & 1 \end{pmatrix}$ for any arbitrary number t. Draw the graph of $v(t)$, the value of the game, as a function of t

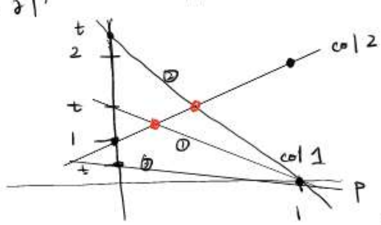
player2 uses col 1 : payoff : $(1-p)t$

col 2 : payoff : $2p + (1-p)$

$$\begin{array}{c} p \\ (1-p) \end{array} \begin{pmatrix} 0 & 2 \\ t & 1 \end{pmatrix} \qquad \left\{ \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right\}$$

min is lowest point of graph

max is highest point of graph



① $1 < t < 2 \Rightarrow p = \dfrac{|t-1|}{|t-1| + |0-2|}$ $\quad 1-p = \dfrac{|0-2|}{|t-1| + |0-2|}$

② $t \geq 2 \nearrow$

③ $t \leq 1$ No Intersection

Column 2 dominates column 1

Thus you derive row 1, column 2 to be saddle point

$$p = \frac{|c-d|}{|c-d| + |a-b|}$$

$$1-p \quad \frac{|a-b|}{|c-d| + |a-b|}$$

3. p_1, p_2 optimal strategies for row player of matrix game

if $0 \leq t \leq 1$, then $tp_1 + (1-t)p_2$ is also an optimal strategy

This is a similar proof to the fourth question on homework 1

we proved that for $p_1, p_2 \in C$, $\lambda p_1 + (1-\lambda) p_2 \in C$, $\lambda \in [0,1]$ is convex set and BR

If the convex set is a BR, then it can also be known as optimal strategy

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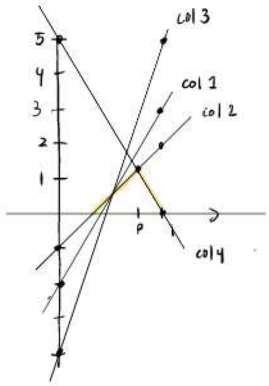
4. solve the following games

i)
$$\begin{array}{c|cccc} & 0 & & 0 & \\ p & 3 & 2 & 5 & 0 \\ 1-p & -2 & 1 & -4 & 5 \end{array}$$

$3p - 2(1-p)$ col 1
• $2p + (1-p)$ col 2
$5p - 4(1-p)$ col 3
• $5(1-p)$ col 4



col 2 & col 4

$2p + (1-p) = 5(1-p)$

$p + 1 = 5 - 3p$

$6p = 4$, $p = \frac{2}{3}$, $1-p = \frac{1}{3}$

$$\begin{array}{c|cccc} & 0 & q & 0 & 1-q \\ \frac{2}{3} & 3 & 2 & 5 & 0 \\ \frac{1}{3} & -2 & 1 & -4 & 5 \end{array}$$

$$\begin{array}{c|cc} & q & 1-q \\ \frac{2}{3} & 2 & 0 \\ \frac{1}{3} & 1 & 5 \end{array}$$

$2q = q + 5(1-q)$

$q = 5 - 5q$

$6q = 5$

$q = \frac{5}{6}$, $1-q = \frac{1}{6}$

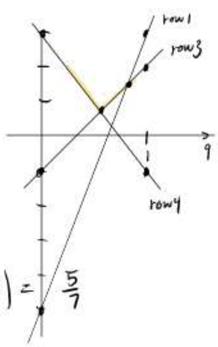
player 1 safety
$\left(\frac{4}{6}, \frac{2}{6}\right)$

player 2 BR
$\left(0, \frac{5}{6}, 0, \frac{1}{6}\right)$

value $= \frac{4}{6} \times 2 + \frac{2}{6} + 1 = \frac{10}{6}$

ii)
$$\begin{array}{c|cc} & q & 1-q \\ 0 & \cancel{3} & \cancel{-5} \\ & \cancel{1} & \cancel{-4} \\ & 2 & -1 \\ & -1 & 3 \end{array}$$ dom

$3q - 5(1-q)$ row 1
$2q - 1(1-q)$ row 3
$-q + 3(1-q)$ row 4



Row 3 Row 4
$2q - 1(1-q) = -q + 3(1-q)$

$3q - 1 = -4q + 3$

$7q = 4$, $q = \frac{4}{7}$, $1-q = \frac{3}{7}$

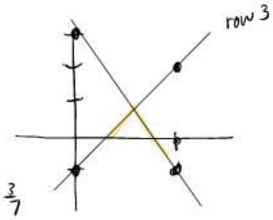
higher val
$2\left(\frac{4}{7}\right) - \left(\frac{3}{7}\right) = \frac{5}{7}$

$$\begin{array}{c|cc} & 4/7 & 3/7 \\ 0 & 3 & -5 \\ 0 & 1 & -4 \\ \frac{4}{7} & 2 & -1 \\ \frac{3}{7} & -1 & 3 \end{array}$$

row 3 row 4
$2p - (1-p) = -p + 3(1-p)$

$3p - 1 = -4p + 3$

$7p = 4$ $p = \frac{4}{7}$, $1-p = \frac{3}{7}$



lower $= 2\left(\frac{4}{7}\right) - \left(\frac{3}{7}\right) = \frac{5}{7}$

Value of Game is $\frac{5}{7}$

4. Reduce by domination

row i dominates k^{th} row if $a_{ij} \geq a_{kj}$ for all j
col j dominates k^{th} col if $a_{ij} \leq a_{ik}$ for all i

a)
dom
$$\begin{vmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{vmatrix}$$

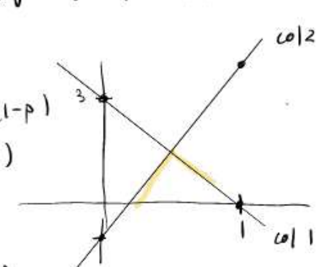
$$\begin{matrix} 4 & 1 & 0 \\ 3 & 2 & -1 \\ -1 & 4 & 3 \\ \cancel{-2} & \cancel{1} & \cancel{2} \end{matrix}$$ dom

dom
$$\begin{matrix} 4 & 1 & 0 \\ 3 & 2 & -1 \\ -1 & 4 & 3 \end{matrix}$$

$$\begin{matrix} 4 & 0 \\ \cancel{3} & \cancel{-1} \\ -1 & 3 \end{matrix}$$ dom

$$\begin{array}{c|cc} & q & 1-q \\ p & 4 & 0 \\ 1-p & -1 & 3 \end{array}$$

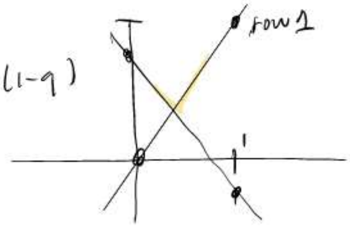
col 1 $4p - (1-p)$
col 2 $3(1-p)$



$4p - (1-p) = 3(1-p)$

$5p - 1 = 3 - 3p$ $8p = 4$ $p = \frac{4}{8}$, $1-p = \frac{4}{8}$

row1 $4q$
row2 $-q+3(1-q)$



$4q = -q+3(1-q)$
$5q = 3-3q$
$8q = 3 \quad q = \frac{3}{8} \quad 1-q = \frac{5}{8}$

Player 1 $\left(\frac{4}{8}, \frac{4}{8}\right)$ value $= \frac{4}{8}\times 4 + \frac{4}{8}\cdot -1 =$
Player 2 BR $\left(\frac{3}{8}, \frac{5}{8}\right)$

$\underline{\frac{12}{8}}$

dom

b) $\begin{pmatrix} 10 & 0 & 7 & \cancel{1} \\ 2 & 6 & 4 & \cancel{7} \\ 6 & 3 & 3 & \cancel{5} \end{pmatrix}$

$\begin{matrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ \cancel{6} & \cancel{3} & \cancel{3} \end{matrix}$

$\frac{5}{10}(10 \quad 0 \quad 7) + \frac{5}{10}(2 \quad 6 \quad 4)$
$(5 \quad 0 \quad \frac{35}{10} \qquad \frac{10}{10} + \frac{30}{10} + \frac{20}{10}$
$\left(\frac{60}{10}, \frac{30}{10}, \frac{45}{10}\right) \geq (6,3,3) \Rightarrow$ reduce row 3

$\begin{matrix} 10 & 0 & \cancel{7} \\ 2 & 6 & \cancel{4} \end{matrix}$

$\frac{5}{10}\begin{pmatrix}10\\2\end{pmatrix} + \frac{5}{10}\begin{pmatrix}0\\6\end{pmatrix}$
$\begin{pmatrix}5\\1\end{pmatrix} + \begin{pmatrix}0\\3\end{pmatrix} \leq \begin{pmatrix}7\\4\end{pmatrix} \rightarrow$ reduce col 3

$$\begin{array}{c|cc} & q & 1-q \\ \hline p & 10 & 0 \\ 1-p & 2 & 6 \end{array}$$

$10p + 2(1-p) = 6(1-p)$
$8p+2 = 6-6p$
$14p = 4 \quad p = \frac{4}{14} \quad 1-p = \frac{10}{14}$

$10q = 2q + 6(1-q)$
$8q = 6-6q$
$14q = 6$
$q = \frac{6}{14}, \quad 1-q = \frac{8}{14}$

$$\begin{array}{c|cccc} & \frac{6}{14} & \frac{8}{14} & 0 & 0 \\ \hline \frac{4}{14} & 10 & 0 & 7 & 1 \\ 0 & 2 & 6 & 4 & 7 \\ \frac{10}{14} & 6 & 3 & 3 & 5 \end{array}$$

$V = \frac{4}{14}\times 10 + \frac{10}{14}\times 6 = \frac{100}{14}$