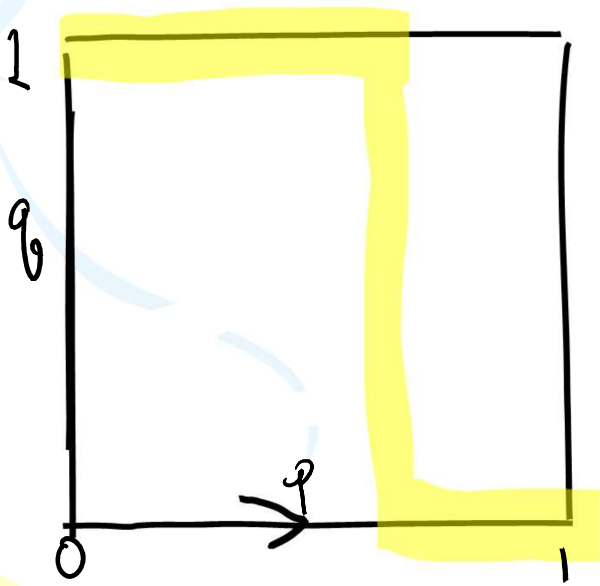
Tetraskelion Method to find all SE's for 2x2 games:

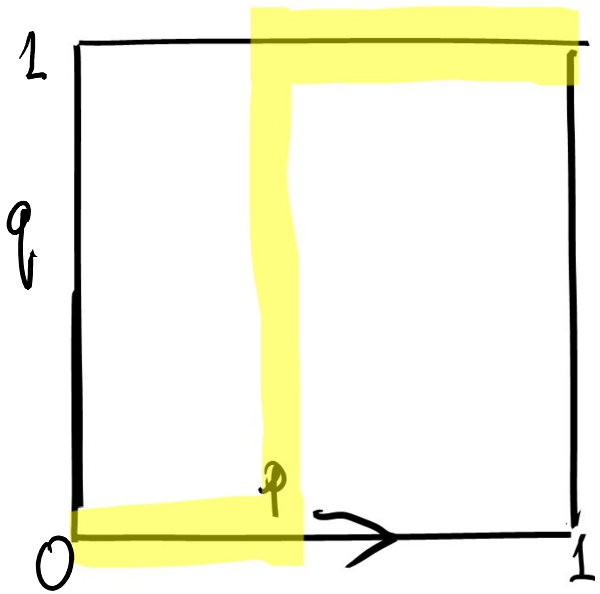
For 2x2 games, the players' sets of strategies are parameterized by the unit interval.

We can make use of this to devise a graphical method to find strategies that are best response to each other.

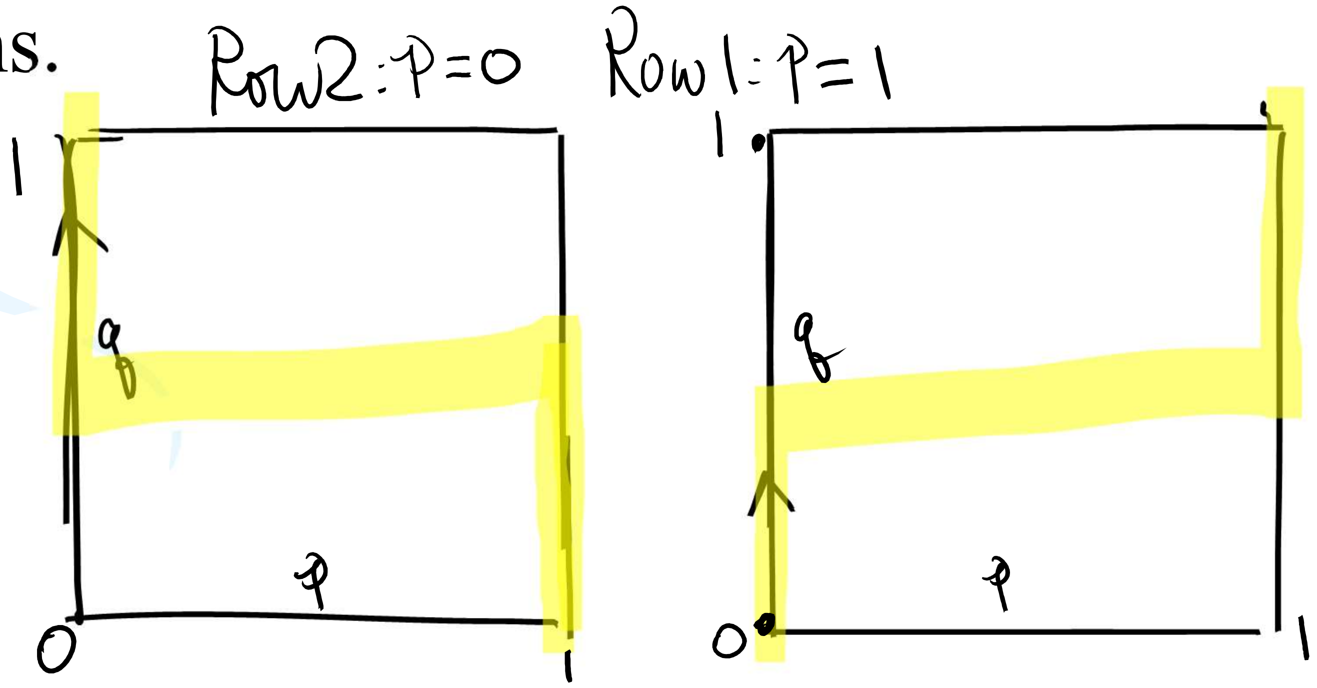
- For each mixed strategy (p, 1-p) of Player I, we use the method as in 0-zero sum games to find the best response columns.
- Plot this in the unit square as a graph parameterized by p, $0 \leqslant p \leqslant 1$, on x-axis.

Col 1: $q=1$, Col2: $q=0$





- For each mixed strategy (q, 1-q) of Player II, we use the method as in 0-zero sum games to find the best response rows.
- Plot this in the unit square in the above as a graph parameterized by q, $0 \leqslant q \leqslant 1$, on y-axis.
- The SE's are the intersections of the two graphs.



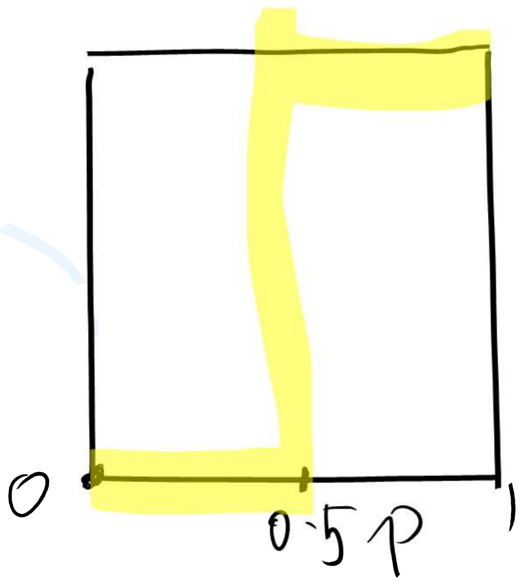
Tetraskelion Method to find SE for 2x2 games

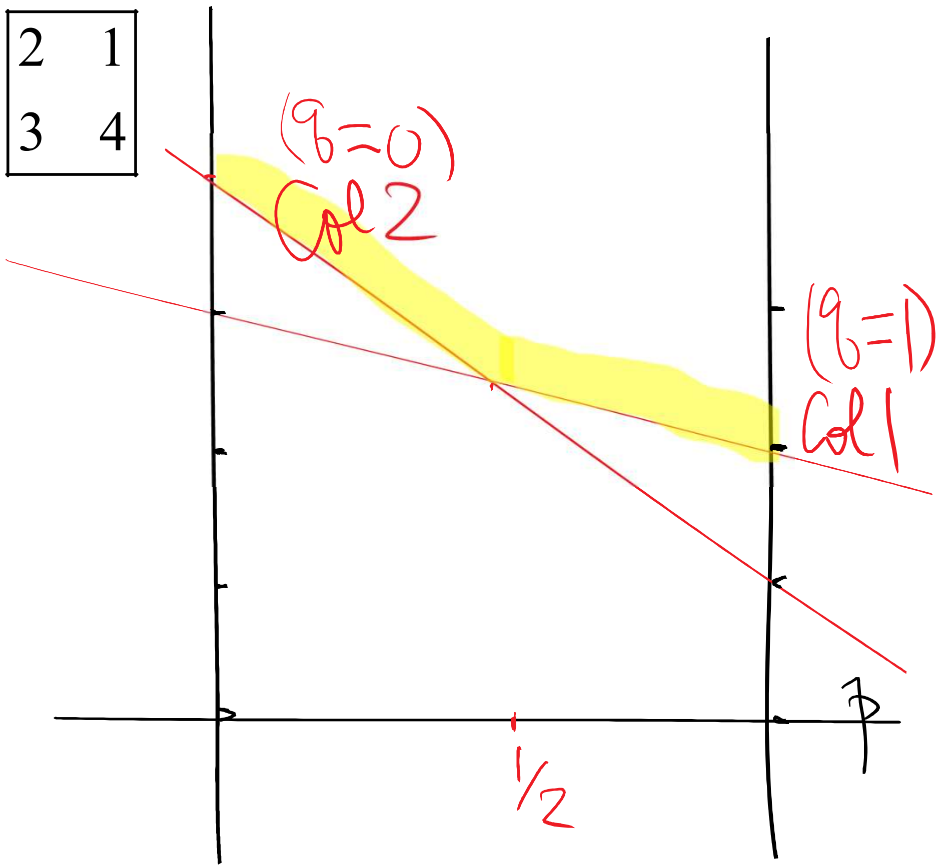
Example: Find all SE's for the following bimatrix game.

- For each mixed strategy (p, 1-p) of Player I, we use the method as in 0-zero sum games to find the best response columns on B.
- Plot this in the unit square in the above as a graph parameterized by p, $0 \leqslant p \leqslant 1$

$$\begin{bmatrix} 3,2 & 2,1 \\ 0,3 & 4,4 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

p	2	1
$1-p$	3	4

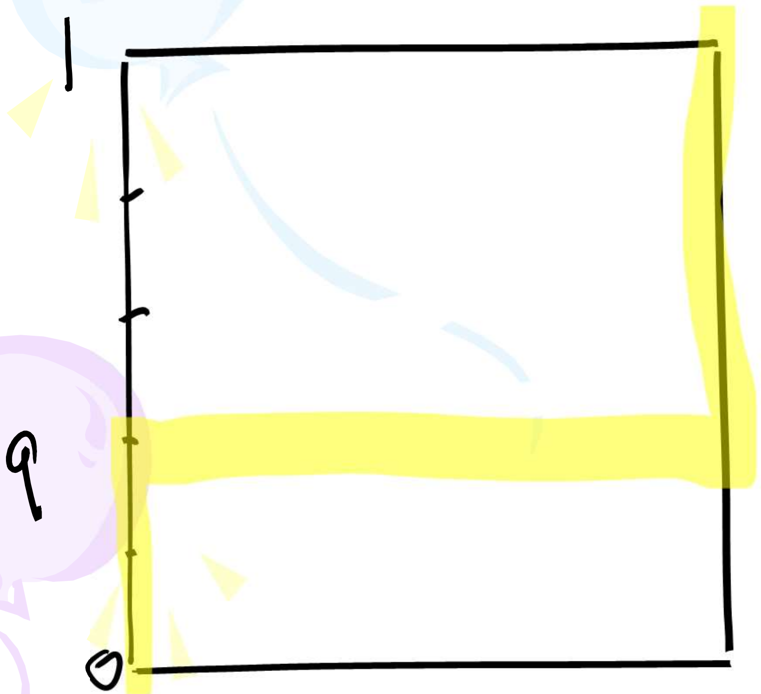


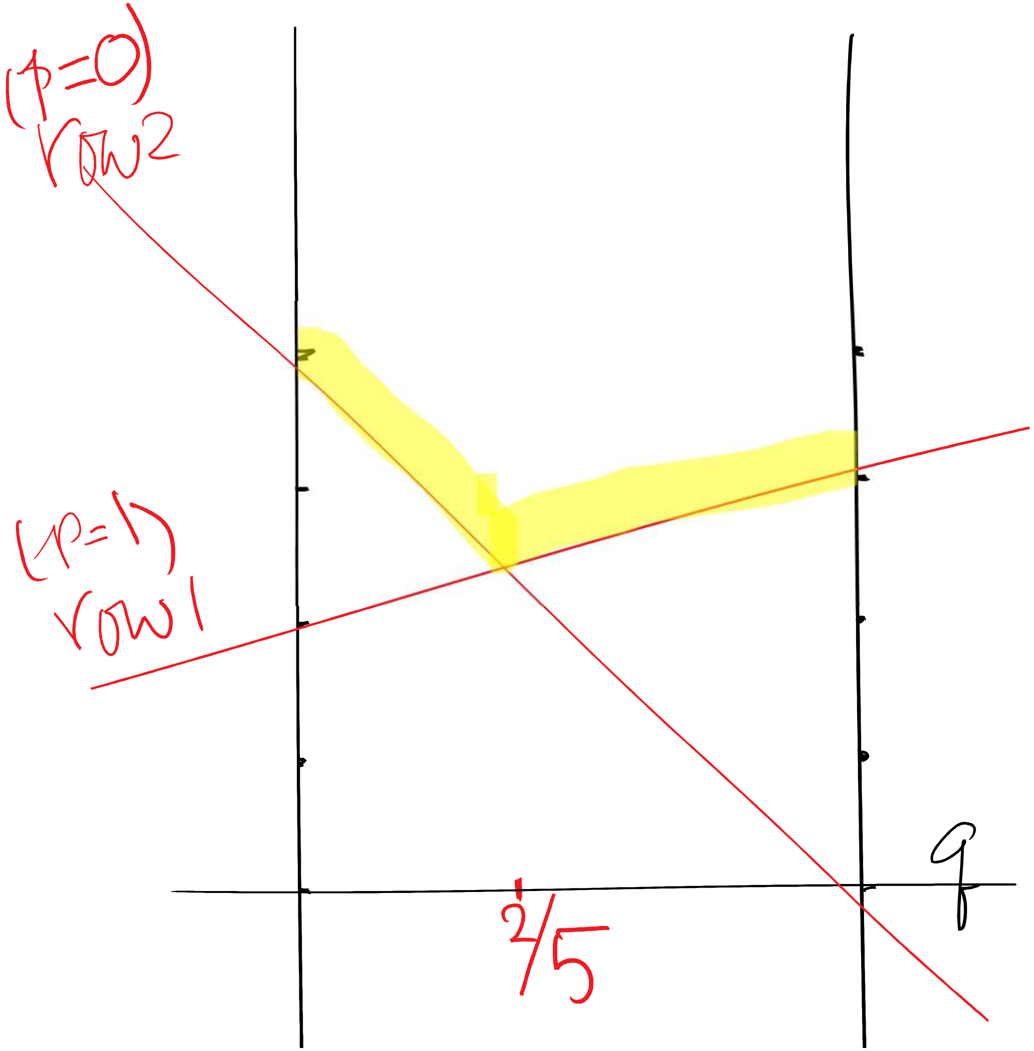


For each mixed strategy (q, 1-q) of Player II, we use the method as in 0-zero sum games to find the best response rows on A
Plot this in the unit square as a graph parameterized by q, 0⩽q ⩽1

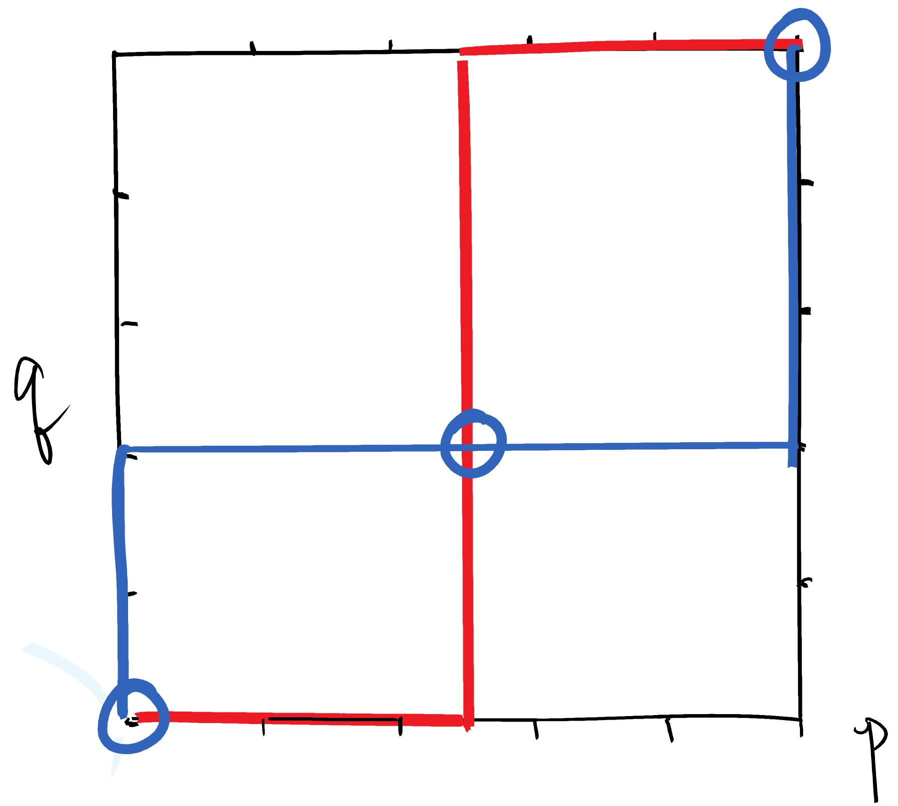
3, 2	2, 1
0, 3	4, 4

q	$1-q$
3	2
0	4





SE: <Row 2, Col 2>, <Row 1, Col 1>, <(0.5, 0.5), (0.4, 0.6)>

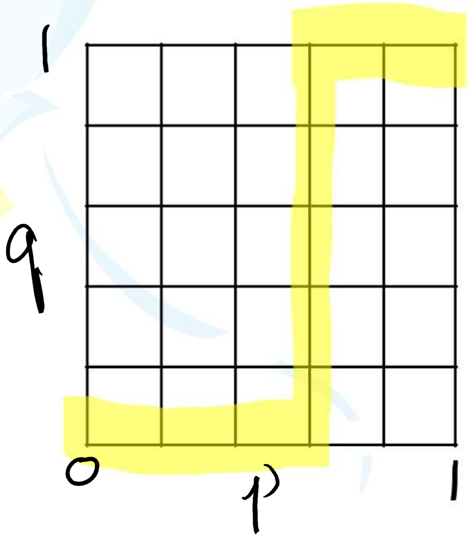


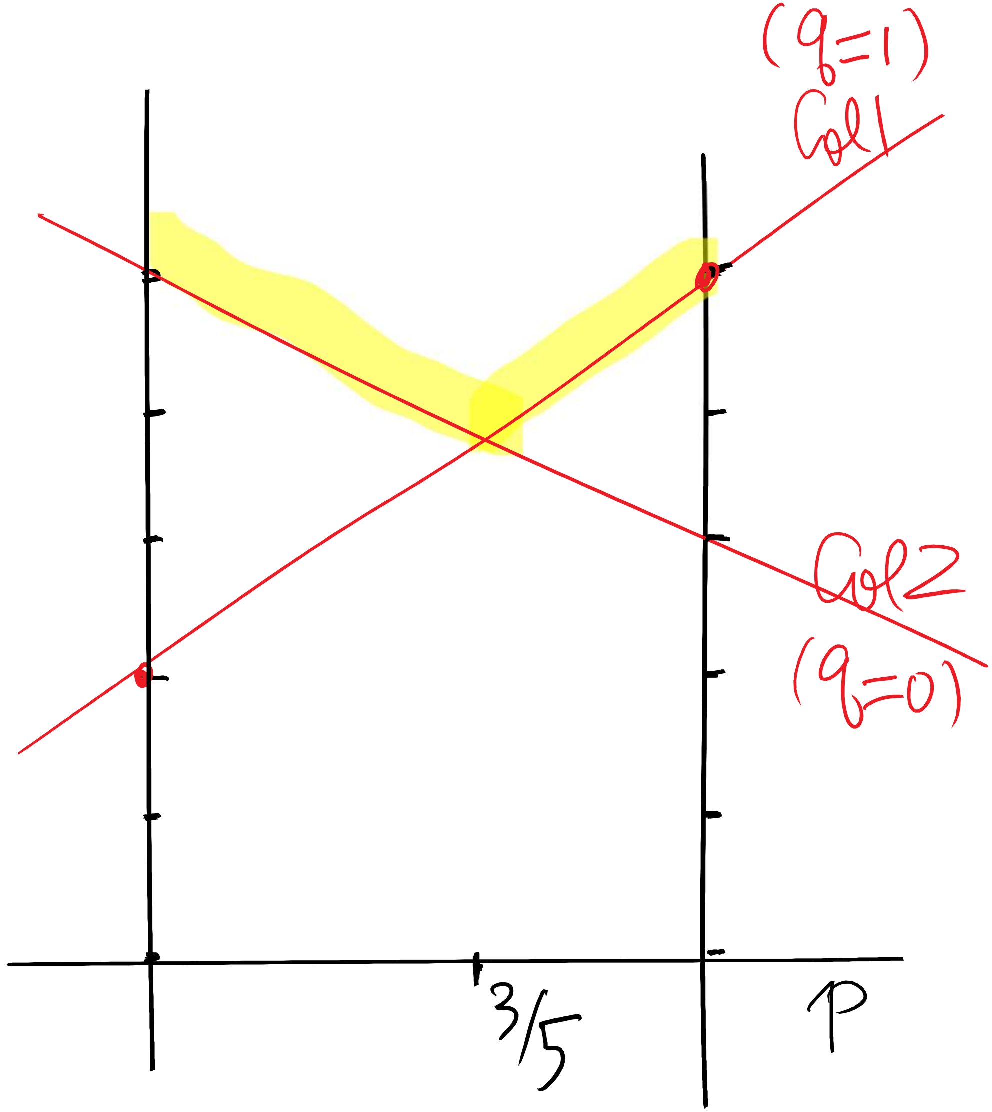
Example:

$(3,2)$	$(4,5)$	$(1,3)$
$(2,4)$	$(1,2)$	$(3,5)$

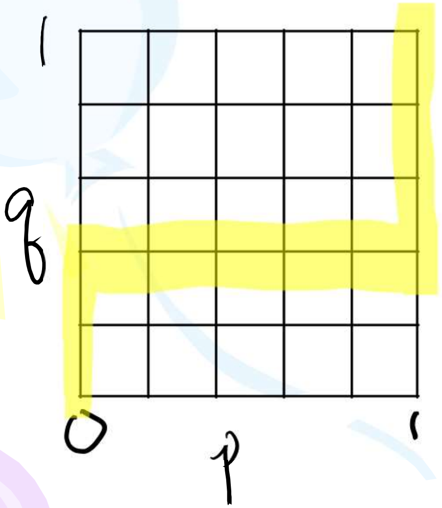
(4, 5)	(1, 3)
(1, 2)	(3, 5)

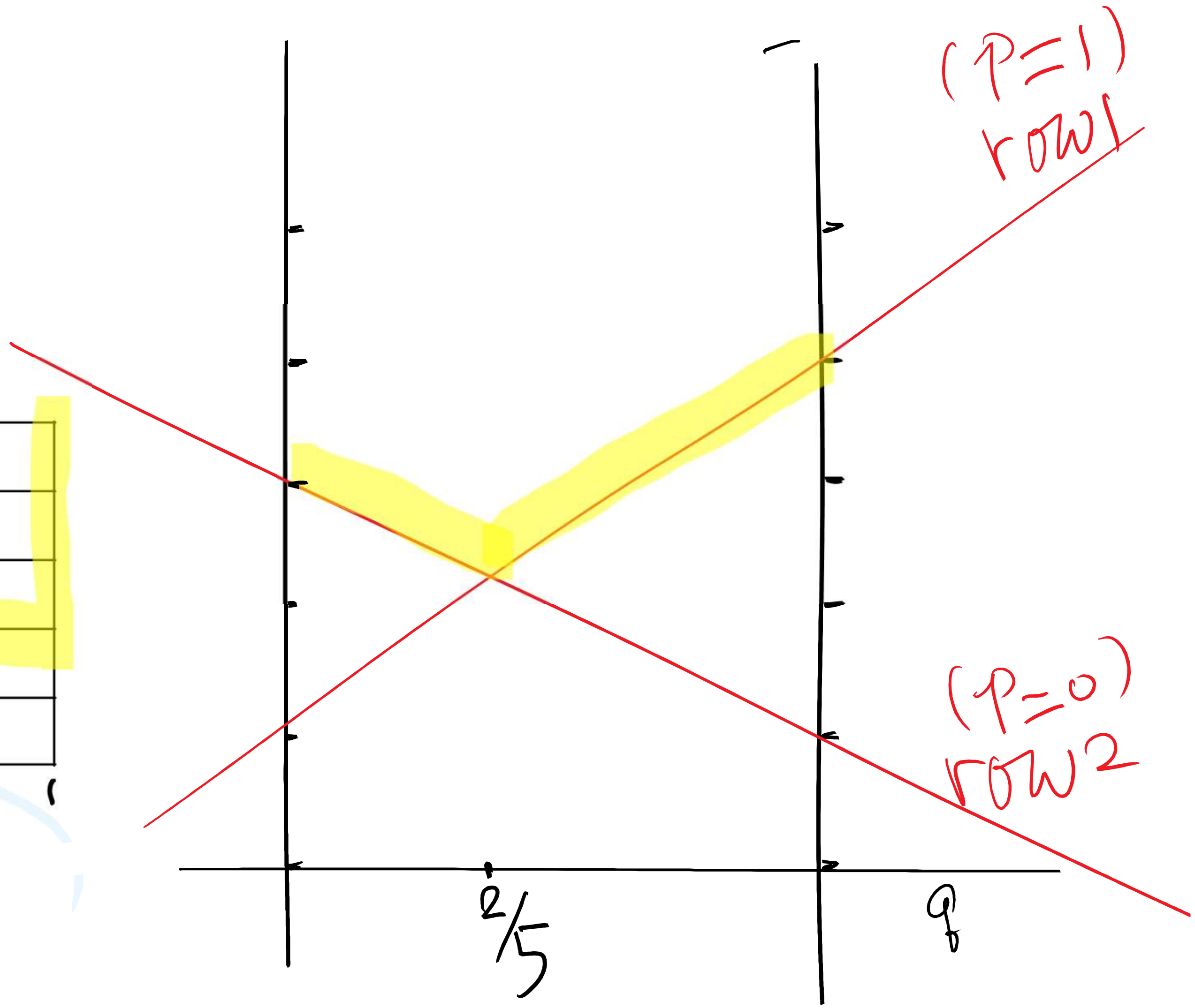
p	5	3
$1-p$	2	5





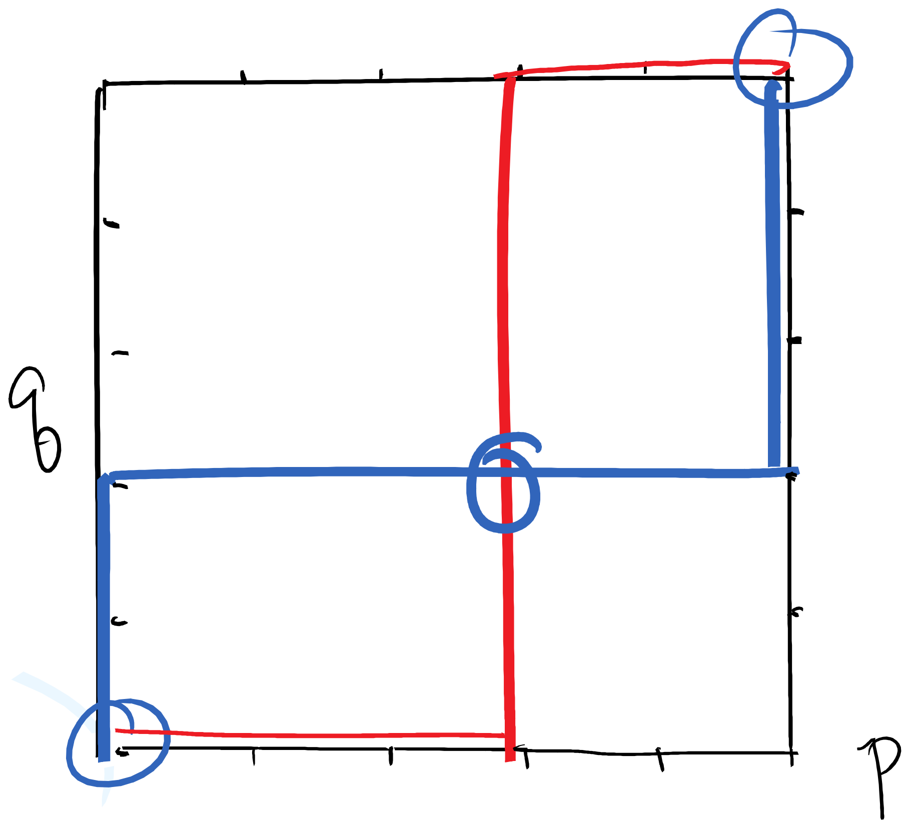
q	$1-q$
4	1
1	3





SE: <Row 2, Col 2>, <Row 1, Col 1>,

< (0.6, 0.4), (0, 0.4, 0.6)>



Duopoly

Example:

There are two firms sending out fishing boats every day to catch fish to sell in the market the same day. The day's catch of fish is brought into port by 9:00 am and must be sold the same day.

Each day is taken to be entirely separate in the sense that the two firms concern themselves with the profit of only the current day and engage in no longer-term planning at all.

When they set sail, each has already decided on the catch to bring back that day, and neither knows what the other has decided. Perhaps each has an educated guess, but neither has spies in the other's camp or any source of factual knowledge of the other's commitments.

Each of them, however, does know both cost functions, as well as the market demand function which is a single market price because the catches of the two firms are identical.

The decision variables of the firms are their output levels.

REAL WORLD EXAMPLES OF DUOPOLY:

汉译世界学术名著丛书

财富理论的数学原理的研究

〔法〕奥古斯丹·古诺 著

古诺与数理经济学

安东尼·奥古斯丹·古诺（Antoine Augustin Cournot），1801年8月28日生于法国上索恩省的格莱。早年就读于当地学校，并在贝桑松公立中学首次接受数学专门训练。1821年，他进巴黎的高等师范学校，继续研读数学。1834年，在里昂任数学教授，次年任格勒诺布尔地方高等专科学校的校长。1838年，他发表了《财富理论的数学原理的研究》（Recherches sur les principes mathématiques de la théorie des richesses），这也正是这本译著的法文原著作。同年，古诺应召赴巴黎任总督学。他在1838年被授予荣誉骑士团的爵位，并于1845年为荣誉勋位受勋者。1854年他成为第戎高等专科学校校长，但从1862年起就不再正式授课。从那时起直至去世，他一直忙于著述。他的《数学原理》（Principes Mathématiques）一书，注意者甚少，并不成功。1863年，他在《财富理论的原理》（Principes de la théorie des richesses）的标题下，以通俗的文字演算和解释上述著作，到1876年，又在《经济学说概论》（Revue sommaire des doctrines économiques）一书中作了进一步阐述。他于次年3月31日在巴黎去世。

古诺的数学著作，有1841年的《函数理论与微积分基础》，1847年的《代数与几何之间对应的根源与界限》，1843年的《机遇与概率理论的阐述》。在上述的最后一本书里，他阐述了将概率论

应用于统计学的方法。

古诺也有哲学方面的著述，例如，1861 年的《科学与历史学中基本思想的连贯性》和 1872 年的《对当代思想与事件发展过程的思考》。他还翻译过若干英文的数学著作，其中包括约翰·赫谢尔的《天文学》。他还编辑过两卷欧拉著的著名的《致君主》，等。

古诺一生的事迹，可读里亚德在《双月评论》(法文) 1877 年 7 月号的文章和《传记信息》(法文)。对古诺经济学方面著作的评论，可见帕尔格雷夫的《政治经济学辞典》(英文)；杰文斯《政治经济学理论》(英文)第二版的序言；瓦尔拉斯的《纯粹经济学要义》(法文)，奥斯庇兹和里本的《价格理论的研究》(德文)和马歇尔的《经济学原理》(英文)；维尔弗里杜·帕累托在《经济学家报》杂志 1892 年 1 月号上写过一篇文章《谈古诺利用数学论述政治经济学中的一个失误》(意大利文)，同一杂志的 1897 年 7 月号还登过一篇埃奇沃思的《关于纯垄断的理论》(意大利文)。本文作者即将在《经济学季刊》上，专门为既想详细领会古诺著作中的推理，又不太熟悉所需数学内容的读者，发表一篇评论与阐述《数学原理》的论文，作者也正在为同一目的撰写一本简明的微积分导论。

在将这本书译为英文时，译者(耶鲁大学 1879 年哲学学士，罗德岛皮斯代尔的纳撒尼尔·T. 培根先生) 致力于既保持法文的古朴韵味又尽可能地流畅。他还极其细心地推演了书中的数学推导，从而发现了大量令人吃惊的差错。大部分是印刷错误，也有一部分是原作者粗心所致，还有些则令人莫明所以。除去对论述有严重影响的两条之外，其余都已经纠正。一条是第 114 页上的算式(6)，另一条则是第 140 页的最后一个不等式。算式中的错误都

公元	中國大事	外國大事
1835清宣宗道光15	山西教民曹順等起義。 清政府規定西班牙、墨西哥貨幣兌換額。 英船侵入劉公島海面。	
1836清宣宗道光16	湖南武岡瑤人起義。	日本大鹽平八郎反對幕府統治的起義。 “正義者同盟”在巴黎建立。 英國工業危機（—1837年）。 英國倫敦“工人協會”建立。 英國倫敦大學創立。 南非洲布爾人大遷徙。
1837清宣宗道光17	洪秀全創立“拜上帝會”。 山東、福建教民起義。 四川涼山彝人起義。 鴉片輸入增加到三萬九千多箱。 太平軍將領陳玉成在世（—1862年）。	英國工人倫敦大會通過要求工人普選權的“國民憲章”。
1838清宣宗道光18	鴉片輸入增加到五萬多箱。 清政府派林則徐到廣東禁烟。	英國、阿富汗戰爭(—1842年)。 英、法强迫土耳其簽訂不平等條約。
1839清宣宗道光19	林則徐在虎門銷毀鴉片。	埃及、土耳其第二次戰爭（—1841年）。 英國憲章派第一次向國會請願。
1840清宣宗道光20	鴉片戰爭開始。英軍攻廣州被擊退。英軍攻陷定海，侵犯乍浦。	英國併吞新西蘭。 德意志工人運動家培培爾在世（—1913年）。 法國文學家左拉在世（—1902年）。
1841清宣宗道光21	英國侵略軍攻佔虎門，逼近廣州。平英團在廣州三元里打擊英軍。 英軍攻佔廈門、定海、寧波。	阿富汗喀布爾人民反對英國侵略的鬥爭。 捷克作曲家德沃夏克在世（—1904年）。 德意志費爾巴哈“基督教的本質”出版。
1842清宣宗道光22	英國侵略軍攻佔吳淞、鎮江，進攻南京。 “中英南京條約”簽訂。	“萊茵報”在科隆創辦，由馬克思主編。 恩格斯僑居英國曼徹斯特（—1844年）。 英國憲章派第二次向國會請願。

Cournot Duoploy

Cournot 1838

Two firms produce the same product.

Cost function for Firm 1: $C_1(q_1)=q_1$

Cost function for Firm 2: $C_2(q_2)=q_2$

q_1, q_2 are the quantities produced.

Price function: $P(q)=25-q$,

where q is the total amount in the market.

Cournot Duopoly

Monopoly:

How much the monopolist should produce?

Profit=$\mathcal{P}(q)=(25-q)q - q = 24q - q^2$

Find q to maximize profit.

Cournot Duopoly

$d/dq(24q-q^2)=24-2q=0$

Thus, $q=12$.

Total profit$=24 \times 12-144=144$

Cournot Duopoly

Duopoly: Firm 1 and Firm 2 will use quantity as strategy.

Profit to Firm 1: $\mathcal{P}_1(q_1)=(25-q)q_1-q_1$,

where $q=q_1+q_2$.

Profit to Firm 2: $\mathcal{P}_2(q_2)=(25-q)q_2-q_2$.

Firm 1 will find q_1 to maximize profit.

Firm 2 will find q_2 to maximize profit.

Cournot Duopoly

Firm 1 will solve:

$0=\partial/\partial q_1 \mathcal{P}_1=(25-q)-q_1-1=24-q_2-2q_1$

Firm 2 will solve:

$0=\partial/\partial q_2 \mathcal{P}_2=(25-q)-q_2-1=24-q_1-2q_2$

Both knows what the other will do.

The solution is then solving these two equations simultaneously.

Answer: $q_1=q_2=8$

Cournot Duopoly

Total profit for two firms is

(25-16)16-16=128

Recall that the monopolist gets 144.

Remark: Cournot already had the concept of BR to each other.

Question: Should Cournot get the Nobel Prize?

Stackelberg Leader and Follower

Suppose Firm 1 moves first.

Firm 1 announces the quantity q_1 (strategy) that it will produce.

Knowing this, Firm 2 will choose its quantity q_2 to maximize profit (as BR).

Stackelberg Leader and Follower

Firm 2 will solve:

$0=\partial/\partial q_2 \mathcal{P}_2 = 24-q_1-2q_2$

Then, $q_2= \frac{1}{2}(24-q_1)$

Firm 1 knows this, its profit is

$\mathcal{P}_1(q_1)=(25-q)q_1-q_1,$

where $q=q_1 + \frac{1}{2}(24-q_1)$

Stackelberg Leader and Follower

$\mathcal{P}_1(q_1)=(25-q_1-1/2\ (24-q_1))q_1-q_1$

$=12q_1-1/2\ q_1^{\ 2}$

$0=\ d/dq_1\ (\mathcal{P}_1(q_1))=12-q_1$

Then, $q_1=12$, $q_2=6$

Stackelberg Leader and Follower

Total profit=(25-18)18-18=108

Profit to Firm 1= 7x12-12=72

Profit to Firm 2=7x6-6=36

Firm 1 is called the Stackelberg Leader and Firm 2 called the follower.

Bertrand Duopoly

In economics, there is also a concept of Bertrand Equilibrium for duopoly.

In this case, the firms will choose price as strategies.

Potential Games

Potential Game and Congestion Game

Games with PSE's:

Potential Game:

We consider a finite n-person game in strategic form. Let $\{1, \ldots, n\}$ be the set of players. For Player i, $i=1, \ldots, n$, let X_i be its set of pure strategies and let

$u_i : X_1 \times \ldots \times X_n \rightarrow R$ be the payoff function.

A function $P: X_1 \times \ldots \times X_n \rightarrow R$, is called a potential function iff for any $x_1 \in X_1, \ldots x_{i-1} \in X_{i-1}$, $x, z \in X_i$, $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, we have

$$(*)\ u_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) - u_i(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$$
$$= P(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) - P(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$$

A game with a potential function is called a potential game.

Theorem (Rosenthal, 1973): Any potential game admits a PSE.

Proof: $P: X_1 \times \ldots \times X_n \to R$ is a function defined on a finite set. Let P achieves its maximum at $(x_1, \ldots, x_n)$. Then, this is a PSE because of (*).

$$
\begin{aligned}
(*)\ & u_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) - u_i(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \\
&= P(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) - P(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)
\end{aligned}
$$

Example 1: Consider the following bimatrix game

$[A, B] = \boxed{\begin{matrix} 1,1 & 9,0 \\ 0,9 & 6,6 \end{matrix}}$. Then, $A = \begin{pmatrix} 1 & 9 \\ 0 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 9 & 6 \end{pmatrix}$ are the payoff matrices for Player I and II respectively.

Let $P = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}$. Then, P is the potential function for this bimatrix game.

Example 2: (Battle of Sexes)

$$[A, B] = \boxed{\begin{matrix} 2,1 & -1,-1 \\ -1,-1 & 1,2 \end{matrix}}.$$

Let $P = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Then, P is the potential function.

A typical example of potential game is the Congestion Game.

Example:

Suppose Driver a has to go from point A to point C and Driver b has to go from point B to point D.

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ D & \rightarrow & C \end{array}$$

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "3"'] & B \arrow[d, "2"] \\ D \arrow[r, "4"'] & C \end{tikzcd}$$

AB is called road segment 1.
BC is called road segment 2.
AD is called road segment 3.
DC is called road segment 4.

Let $c_j(1)$, $c_j(2)$ denote the payoffs (negative of the cost) for one or tow drivers using the road segment respectively. In summary we have

$$\begin{array}{ccccc} & & c_1(1), c_1(2) & & \\ & A & \rightarrow & B & \\ c_3(1), c_3(2) & \downarrow & & \downarrow & c_2(1), c_2(2) \\ & D & \rightarrow & C & \\ & & c_4(1), c_4(2) & & \end{array}$$

The strategies for Driver a are {1,2} and {3,4} (denoted by Row 1, Row 2 respectively).

The strategies for Driver b are {1,3} and {2,4} (denoted by Column 1, Column 2 respectively).

$$
\begin{tikzcd}
A \arrow[r, "{c_1(1), c_1(2)}"] \arrow[d, swap, "{c_3(1), c_3(2)}"] & B \arrow[d, "{c_2(1), c_2(2)}"] \\
D \arrow[r, swap, "{c_4(1), c_4(2)}"] & C
\end{tikzcd}
$$

Then, the payoff bimatrix is

$$
[\mathrm{A},\mathrm{B}] = \begin{array}{c|cc|}
 & \{1,3\} & \{2,4\} \\
\{1,2\} & c_1(2)+c_2(1), c_1(2)+c_3(1) & c_2(2)+c_1(1), c_2(2)+c_4(1) \\
\{3,4\} & c_3(2)+c_4(1), c_3(2)+c_1(1) & c_4(2)+c_3(1), c_4(2)+c_2(1)
\end{array}
$$

$$[\mathrm{A,B}] = \begin{array}{c|cc|} & \{1,3\} & \{2,4\} \\ \hline \{1,2\} & c_1(2)+c_2(1), c_1(2)+c_3(1) & c_2(2)+c_1(1), c_2(2)+c_4(1) \\ \{3.4\} & c_3(2)+c_4(1), c_3(2)+c_1(1) & c_4(2)+c_3(1), c_4(2)+c_2(1) \\ \hline \end{array}$$

Now let

$$P = \begin{pmatrix} c_1(1)+c_1(2)+c_2(1)+c_3(1) & c_2(1)+c_2(2)+c_1(1)+c_4(1) \\ c_3(1)+c_3(2)+c_4(1)+c_1(1) & c_4(1)+c_4(2)+c_3(1)+c_2(1) \end{pmatrix}$$

Then, P is the potential for this congestion game and hence a PSE exists.

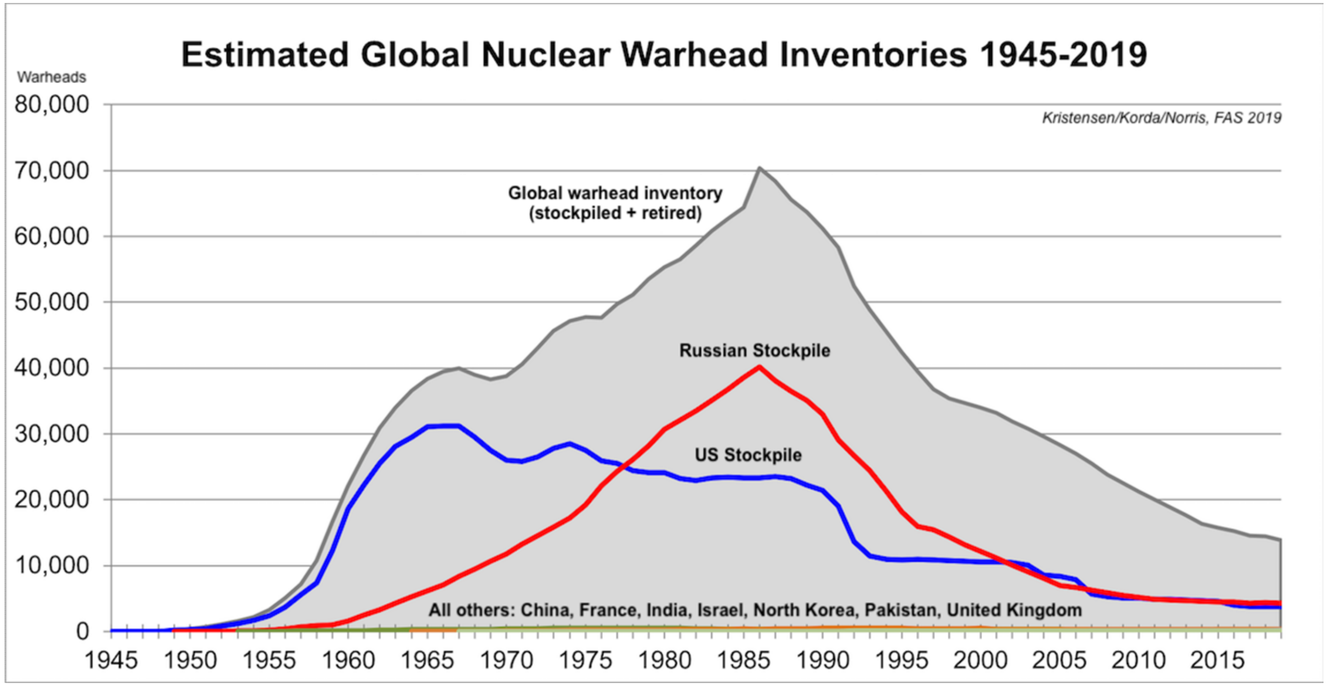
Mutually Assured Destruction

Background:

1st World War: 1914-1918

2nd World War started two decades after the 1st World War.

Cold War: 1947-1991



MAD (Mutually Assured Destruction)

Wikipedia

Mutually Assured Destruction, or mutual assured destruction (MAD), is a doctrine of military strategy and national security policy in which a full-scale use of high-yield weapons of mass destruction by two or more opposing sides would cause the complete annihilation of both the attacker and the defender. It is based on the theory of deterrence where the threat of using strong weapons against the enemy prevents the enemy's use of those same weapons. The strategy is a form of Nash equilibrium in which neither side, once armed, has any incentive to initiate a conflict or to disarm.

The MAD doctrine assumes that each side has enough nuclear weaponry to destroy the other side and that either side, if attacked for any reason by the other, would retaliate without fail with equal or greater force. The expected result is an immediate irreversible escalation of hostilities resulting in both combatants‘ mutual, total and assured destruction.

The doctrine requires that neither side construct shelters on a massive scale, such as there are in Switzerland. Switzerland would actually survive a nuclear attack surprisingly well. If the US were to construct a similar system of shelters, it would violate the MAD doctrine and destabilize the situation, because it would not need to fear the consequences of a Soviet second strike. The same principle is invoked against missile defense.

The doctrine further assumes that neither side will dare to launch a first strike because the other side will launch on warning (also called fail-deadly) or with surviving forces (a second strike), resulting in unacceptable losses for both parties.

Strategic Defense Initiative

Star War

A central instrument for putting pressure on the Soviet Union was Reagan's massive defense build-up, which raised defense spending from $134 billion in 1980 to $253 billion in 1989. This raised American defense spending to 7 percent of GDP, dramatically increasing the federal deficit. Yet in its efforts to keep up with the American defense build-up, the Soviet Union was compelled in the first half of the 1980s to raise the share of its defense spending from 22 percent to 27 percent of GDP, while it froze the production of civilian goods at 1980 levels.

Reagan’s most controversial defense initiative was SDI, the visionary project to create an anti-missile defense system that would remove the nuclear sword of Damocles from America’s homeland. Experts still disagree about the long-term feasibility of missile defense, some comparing it in substance to the Hollywood sci-fi blockbuster Star Wars. But the SDI’s main effect was to demonstrate U. S. technological superiority over the Soviet Union and its ability to expand the arms race into space. This helped convince the Soviet leadership under Gorbachev to throw in the towel and bid for a de-escalation of the arms race.

Game Theory and Nuclear Deterrence

Philip D. Straffin Jr

Winston Churchill:

"Safety will be the sturdy child of terror,
and survival the twin brother of annihilation."

The players are US and Russian. Each has two strategies Wait and Strike. It is assumed that if a country is struck and it will launch a retaliation strike or called a second strike. The following is the payoff matrix.

		Russia	
		Wait	*Strike*
US	*Wait*	(PU, PR)	(SU, FR)
	Strike	(FU, SR)	$(\, , \,)$

- PU: Peace payoff for US
- PR: Peace payoff for Russia
- FU: First strike payoff for US
- FR: First strike payoff for Russia
- SU: Second strike payoff for US
- SR: Second strike payoff for SR

		Russia	
		Wait	*Strike*
US	*Wait*	(PU, PR)	(SU, FR)
	Strike	(FU, SR)	$(\; , \;)$

Clearly, SU < FU $\ll$ PU.

SU FU PU

Similarly, SR < FR $\ll$ PR.

SR FR PR

Let p = the US estimate of the probability that Russia will STRIKE in any given time period.

Let q = the Russia estimate of the probability that US will STRIKE in any given time period.

The expected payoff to Russia if she plays WAIT is

$(1-q)PR + q\,SR$

Russia will get FR if she plays STRIKE.

Russia will WAIT if $(1-q)PR + q\,SR > FR$

It is equivalent to $q < (PR-FR)/(PR-SR)$

The above inequality will be true if q is not too large.

The first source of instability in nuclear deterrence is that being struck first is considered worse than being struck second, it is rational to launch a "preemptive first strike" if you believe your opponent is likely to strike.

The second source of instability is that even if the above inequality holds, it does not follow that there is no chance Russia will STRIKE.

However, the larger the difference is between the sides of this inequality, the more irrational it would be for Russia to STRIKE, and the greater care we could expect Russia to take to prevent an accidental strike.

$$q < (PR-FR)/PR-SR)$$

The US goal should be to make this difference as large as possible. The

ways to do this are to

$q < (PR\text{-}FR)/PR\text{-}SR)$

- raise the righthand side by making (PR-FR) almost as large as (PR-SR) i.e. making FR close to SR. This means assuring a strong retaliation against a Russia first strike, so that Russia gains little by striking first instead of second.
- lower q

Lowering q means being very careful not to make statements that imply US might consider strike first.

Also, as the situation is completely symmetric, p should be lowered.

We are then led to the final conclusion that US should be to:

(1) make SU and FU close by assuring strong retaliation against Russia first strike, and

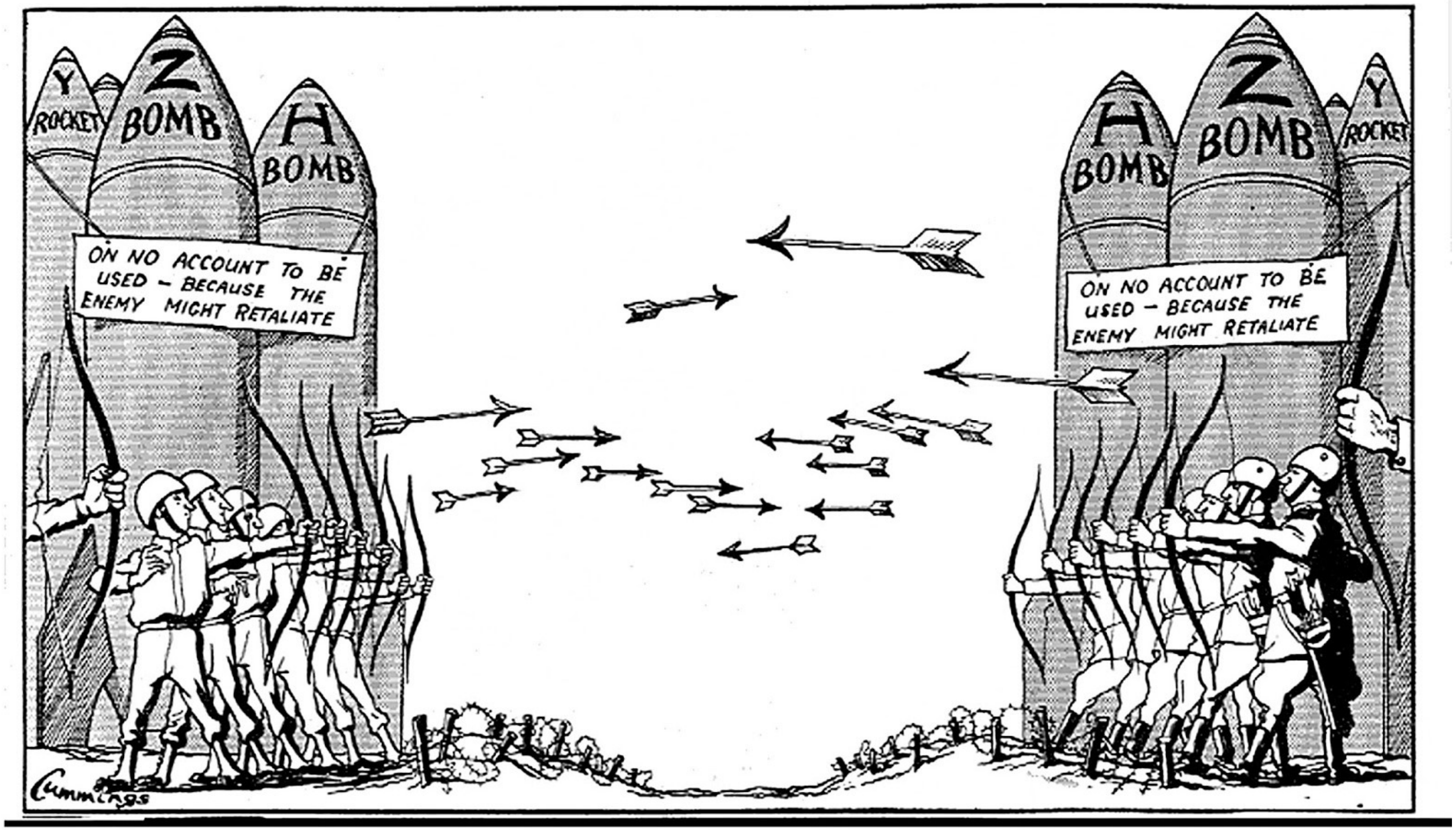
(2) make SR and FR close by assuring Russia could retaliate strongly against a first strike by US.

Goal (2) seems paradoxical but it is crucial. If Russia feels it cannot retaliate strongly against a first strike, it will conclude that a first strike by US is more likely.

To pursue goal (1) and (2), we should ideally like to have missiles that are invulnerable, so they cannot be destroyed by a Russia first strike, and inaccurate enough that they cannot destroy Russia missiles in a first strike.

If nuclear deterrence is to work, both sides must have a number of well-protected missiles, and those missiles should not be aimed at the other side's missiles. If they were, they would be effective only for a first strike. To be more effective for retaliation, they should be aimed at cities, and measures to protect cities should not be taken.

The logic of nuclear deterrence requires that we all be hostage to nuclear terror. We must aim missiles not at military forces, but at children.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