One of the main features of cooperative games is that the players have freedom to choose a joint strategy. This allows any probability mixture of the payoff vectors to be achieved.

The set of payoff vectors that the players can achieve if they cooperate is called the Cooperative Feasible Set.

.

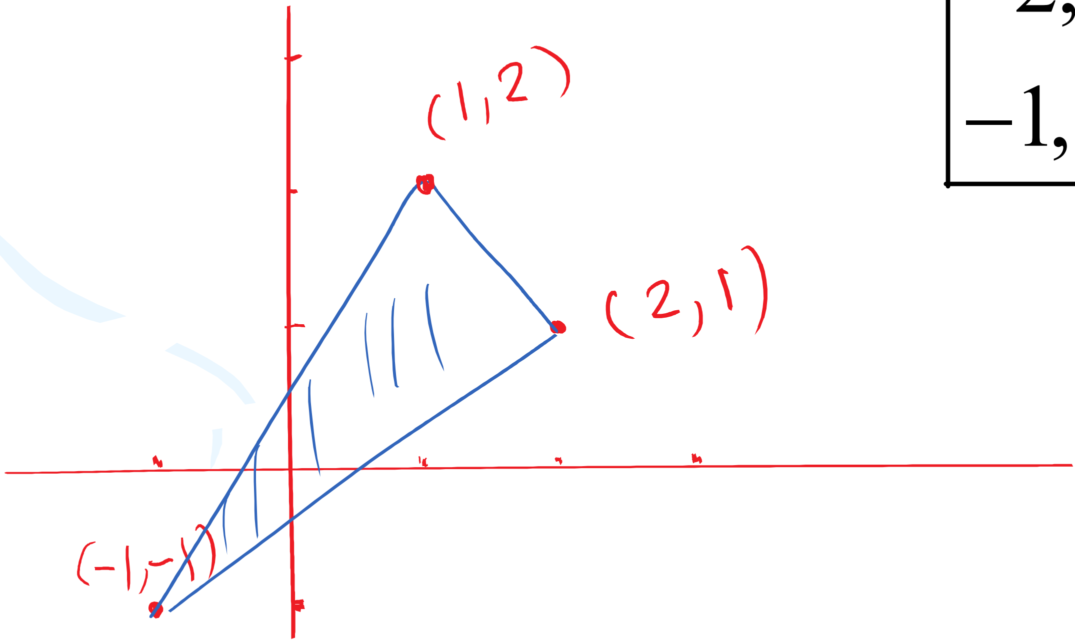
When players cooperate in a bimatrix game with matrices [**A**,**B**], they may agree to achieve a payoff vector of any of the mn points, (a_{ij}, b_{ij}) for $i = 1, \ldots, m$ and $j = 1. \ldots, n$. They may also agree to any probability mixture of these points.

Therefore, the set of all such payoff vectors is the convex hull these mn points. Without a transferable utility, this is all that can be achieved.

Definition. The NTU feasible set is the convex hull of the mn points, (a_{ij}, b_{ij}) for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Example: Plot the NTU feasible set of Battle of sexes.

2,1	−1,−1
−1,−1	1,2



The distinguishing feature of the TU case is that the players may make **side payments** of utility as part of the agreement.

By making a side payment, the payoff vector (a_{ij}, b_{ij}) can be changed to $(a_{ij} + s, b_{ij} - s)$.

If the number s is positive, this represents a payment from Player II to Player I. If s is negative, the side payment is from Player I to Player II.

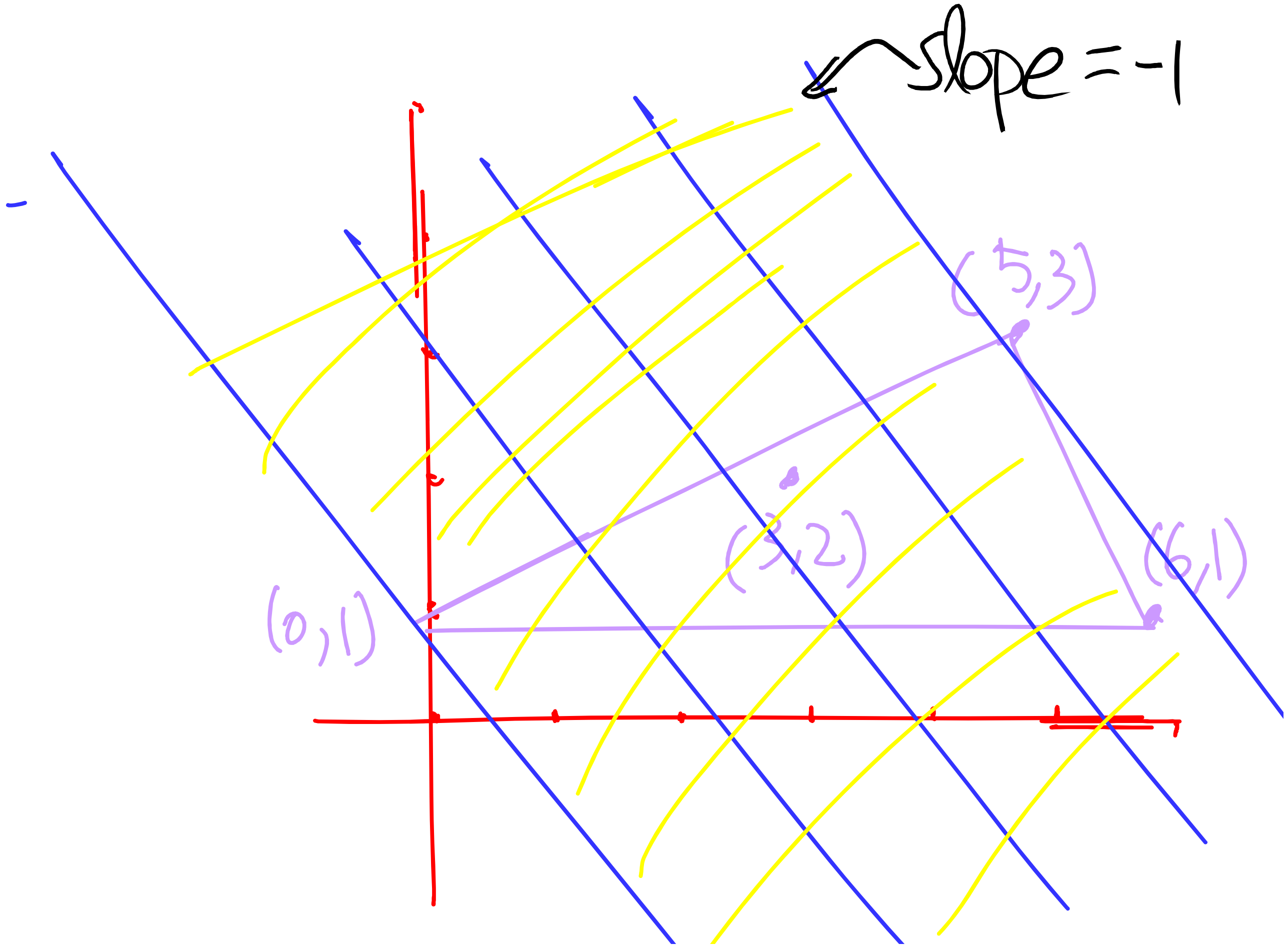
Thus the whole line of slope -1 through the point (a_{ij}, b_{ij}) is part of the TU feasible set.

Definition. The TU feasible set is the convex hull of the set of vectors of the form

$(a_{ij} + s, b_{ij} - s)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ and for arbitrary real numbers s.

Example:

3, 2	5, 3
6, 1	0, 1



If an agreement is reached in a cooperative game, be it a TU or an NTU game, it may be expected to be such that no player can be made better off without making at least one other player worse off. Such an outcome is said to be Pareto optimal.

Definition. A feasible payoff vector, (v_1, v_2), is said to be **Pareto optimal** if for any feasible payoff vector (v_1', v_2') such that

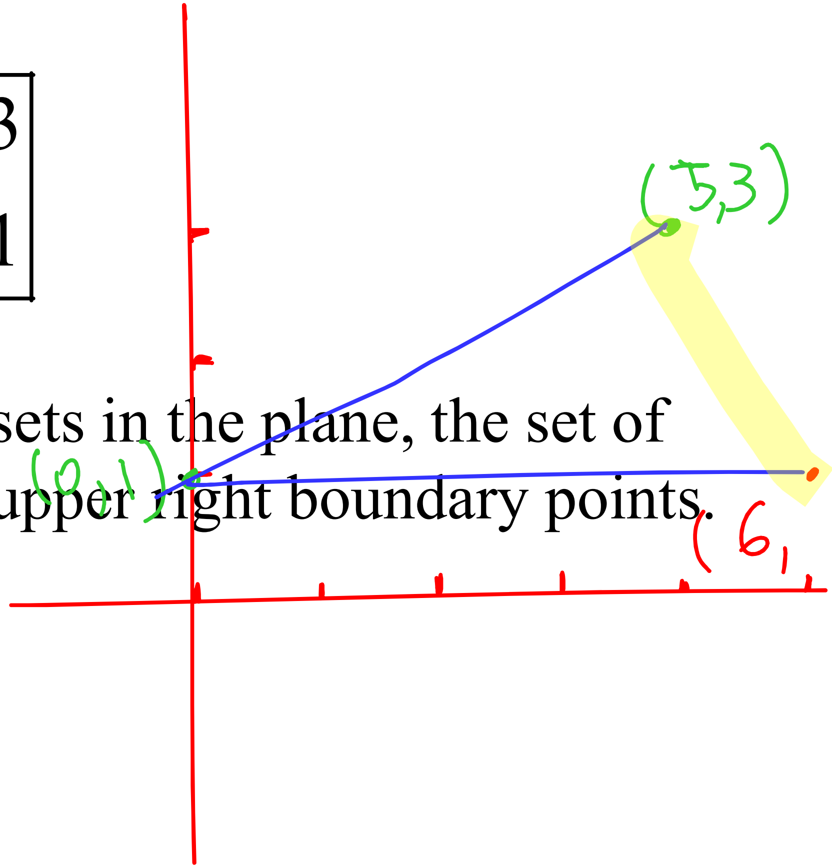
$$v_1' \geq v_1 \text{ and } v_2' \geq v_2 \text{ implies } (v_1', v_2') = (v_1, v_2).$$

Remark: A position that has a win-win move is not Pareto optimal.

Example: Find the Pareto optimal set of the NTU feasible set of the following game.

3, 2	5, 3
6, 1	0, 1

For more general convex feasible sets in the plane, the set of Pareto optimal points is the set of upper right boundary points.



Cooperative Games with Transferable Utility.

We assume that the players are "rational" in the sense that, given a choice between two possible outcomes of differing personal utility, each player will select the one with the higher utility.

The TU-Problem:

In the model of the game, we assume there is a period of pre-play negotiation, during which the players meet to discuss the possibility of choosing a joint strategy together with some possible side payment to induce cooperation.

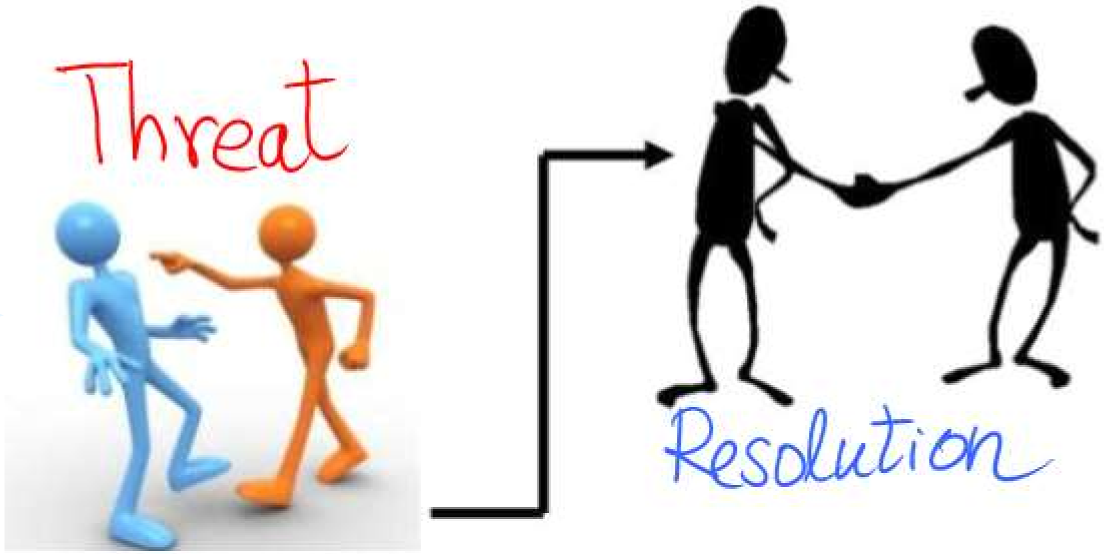
They also discuss what will happen if they cannot come to an agreement; each may threaten to use some unilateral strategy that is **bad** for the opponent.

If they do come to an agreement, it may be assumed that the payoff vector is Pareto optimal.

In this model of the pre-play negotiation, the threats and counter-threats may be made and remade until time to make a decision. Ultimately the players announce what threats they will carry out if agreement is not reached. It is assumed that if agreement is not reached, the players will leave the negotiation table and carry out their threats.

However, being rational players, they will certainly reach agreement, since this gives a higher utility. The threats are only a formal method of arriving at a reasonable amount for the side payment, if any, from one player to the other.

The TU problem then is to choose the threats and the proposed side payment judiciously.



The players use threats to influence the choice of the final payoff vector. The problem is how do the threats influence the final payoff vector, and how should the players choose their threat strategies?

For two-person TU-games, there is a very convincing answer.

The TU Solution: If the players come to an agreement, then rationality implies

that they will agree to play to achieve the largest possible total payoff, call it σ,

$\sigma = \max \{(a_{ij} + b_{ij}): i, j\}$ as the payoff to be divided between them.

That is they will jointly agree to use some row i_0 and column j_0 such that $a_{i0j0} + b_{i0j0} = \sigma$. Such a joint choice (i_0, j_0) is called their **cooperative strategy**.

Question: Should they share evenly on σ ?

Suppose now that the players have selected their **threat strategies**, say **p** for Player I and **q** for Player II.

Then if agreement is not reached, Player I receives $\mathbf{p}^T\mathbf{A}\mathbf{q}$ and Player II receives $\mathbf{p}^T\mathbf{B}\mathbf{q}$. The resulting payoff vector,

$\mathbf{D} = \mathbf{D}(\mathbf{p}, \mathbf{q}) =$

$(\mathbf{p}^T\mathbf{A}\mathbf{q}, \mathbf{p}^T\mathbf{B}\mathbf{q}) = (D_1,D_2)$

This point is in the NTU feasible set and is called the **disagreement point** or **threat point**.

Once the disagreement point is determined, the players must agree on a point (x, y) on the line $x+y = \sigma$ to be used as the cooperative solution.

Player I will accept no less than D_1 and Player II will accept no less than D_2 since these can be achieved if no agreement is reached.

But once the disagreement point has been determined, the game becomes symmetric.

They will split evenly the excess of σ over $D_1 + D_2$. This is called the Egalitarian Principle.

Each will get $\frac{1}{2}(\sigma - (D_1 + D_2))$ in addition to what they will get from the disagreement point.

The resolution point is then

$\boldsymbol{\phi} = (\phi_1, \phi_2)$

$= (½\,(\sigma + D_1 - D_2\,),\ ½\,(\sigma - D_1 + D_2\ )\,)$

To select the threat optimally, Player I wants to maximize $D_1 - D_2$ and Player II wants to minimize it.

This is in fact a zero-sum game with matrix $\mathbf{A}-\mathbf{B}$:

$D_1 - D_2 = \mathbf{p}^T\mathbf{A}\mathbf{q} - \mathbf{p}^T\mathbf{B}\mathbf{q} = p^T(A - \mathbf{B})\mathbf{q}$.

Let $\mathbf{p}^*$ and $\mathbf{q}^*$ denote optimal strategies of the 0-sum game $\mathbf{A} - \mathbf{B}$ for Players I and II respectively, and let δ denote the value,

$$\delta = \mathrm{Val}(\mathbf{A} - \mathbf{B}) = \mathbf{p}^{*T} (\mathbf{A} - \mathbf{B})\mathbf{q}^*.$$

When these strategies are used, the disagreement point or threat point becomes

$\mathbf{D} = (D_1, D_2) = \mathbf{D}(\mathbf{p}^*, \mathbf{q}^*)$.

Since $\delta = \mathbf{p}^{*T} \mathbf{A}\mathbf{q}^* - \mathbf{p}^{*T} \mathbf{B}\mathbf{q}^* = D_1 - D_2$,

we have as the TU solution:

$\boldsymbol{\phi} = (\phi_1, \phi_2)$

$= (\tfrac{1}{2} (\sigma + D_1 - D_2), \tfrac{1}{2} (\sigma - D_1 + D_2)) = (\tfrac{1}{2} (\sigma + \delta), \tfrac{1}{2} (\sigma - \delta))$

To achieve the payoff ϕ^*, this requires a side payment of $(\sigma+\delta)/2 - a_{i0\ j0}$ from Player II to Player I. If this quantity is negative, the payment of $a_{i0\ j0} - (\sigma+\delta)/2$ goes from Player I to Player II.

Remark: The slope of the line joining

(D_1,D_2) to $(\frac{1}{2}(\sigma + D_1 - D_2), \frac{1}{2}(\sigma - D_1 + D_2))$ is 1. It is perpendicular to the line $x + y = \sigma$.

Therefore, the line passing through (D_1,D_2) with slope=1 intersects the line $x + y = \sigma$ at the resolution point (ϕ_1, ϕ_2) .

Example: Find the TU value, the associated side payment and the optimal threats of the following bimatrix game.

$$\begin{pmatrix} 3,2 & 5,3 \\ 6,1 & 0,1 \end{pmatrix}$$

$\sigma=8$

$$A-B=\begin{bmatrix}1 & 2\\ 5 & -1\end{bmatrix}$$

$\delta=11/7$,

Optimal threat Strategies: $p^* =(6/7, 1/7)$,

$q^*=(3/7, 4/7)$

ϕ = TU Resolution Point or TU value$=(\phi_1, \phi_2)$

$= (½(\sigma+ D_1 - D_2), ½(\sigma- D_1 + D_2))$

$=(335/70, 225/70)$

Cooperative strategy=<Row 1, Column 2>

Side payment=I to pay 15/70 to II

Example: Find the TU value, the associated side payment and the optimal threats of the following bimatrix game.

$$\begin{array}{|cc|} 4,2 & 1,3 \\ 2,3 & 4,1 \end{array}$$

$\sigma=6$

$$A - B = \boxed{\begin{matrix} 2 & -2 \\ -1 & 3 \end{matrix}}$$

$\delta=1/2$,

Optimal threat Strategies: $p^* =(1/2, 1/2)$,

$q^*=(5/8, 3/8)$

ϕ = TU Resolution Point or TU value$=(\phi_1, \phi_2)$

$= (\frac{1}{2}(\sigma+ D_1 - D_2), \frac{1}{2}(\sigma- D_1 + D_2))$

$=(13/4, 11/4)$

Cooperative strategy=<Row 1, Column 1>

Side payment=I to pay ¾ to II

Example: Find the TU value, the associated side payment and the optimal threats of the following bimatrix game.

$$\begin{bmatrix} 1,-1 & 4,1 & 6,4 & 6,1 \\ 1,-3 & 8,7 & 5,-1 & 3,3 \end{bmatrix}$$

Solution: First note that σ=8+7=15. Cooperative strategy is <Row 2, Col 2>. The Threat Game is

$$A-B=\begin{bmatrix} 2 & 3 & 2 & 5 \\ 4 & 1 & 6 & 0 \end{bmatrix}$$

this was solved in the lectures on 2-person 0-sumgame. We got δ=5/2 and

Threat strategy = <(¾, ¼), (½ , ½, 0, 0)>.

TU-solution $(\sigma+\delta/2, \sigma-\delta/2) = (35/4, 25/4)$.

The side payment is ¾ from Player II to Player I.

To sum up, we have

Cooperative strategy = <Row 2, Col 2>

Threat strategy = <(¾, ¼), (½ , ½, 0, 0)>

Disagreement point = (9/4, ½)

TU resolution point $= (\sigma+\delta/2, \sigma-\delta/2) = (35/4, 25/4)$

Side payment = ¾ from Player II to Player I

It is worthwhile to note that there may be more than one possible cooperative strategy yielding σ as the sum of the payoffs. The side payment depends on which one is used.

Also there may be more than one possible disagreement point because there may be more than one pair of optimal strategies for the game

A − **B**.

However, all such disagreement points must be on the same 45^0 line. To see this suppose (D_1,D_2) and (D'_1, D'_2) are two threat points. Then, $val(A-B)=D_1 - D_2 = D'_1 - D'_2$. Hence,

$D_1 - D'_1 = D_2 - D'_2$ Thus, the line joining them has slope equal to 1.

Since the TU resolution point ϕ is the intersection of the line $x + y = \sigma$ with the line with slope 1 through the disagreement point, all disagreement points give rise to the same TU-value.

Example: Find the TU value, the cooperative strategy, the associated side payment and the optimal threats for the following bimatrix game.

$$\begin{pmatrix} 8,3 & -1,-1 & 6,4 \\ 2,4 & 7,4 & 5,4 \end{pmatrix}$$

$\sigma = 11$

$$A - B = \begin{bmatrix} 5 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

Using the graphical method, one sees that the three lines corresponding to the Col 1, Col 2, Col 3 meet at (0.5, 1.5). Therefore, $\delta=1.5$.

Optimal threat strategy for I: $p^* =(0.5, 0.5)$,

Optimal threat strategies for II: $q^*=(q_1, q_2, 1-q_1 - q_2)$ such that

$5q_1+2(1-q_1-q_2) = -2q_1+3 q_2 + (1-q_1-q_2)$, $1\geq q_1, q_2\geq 0$, $1\geq q_1 + q_2\geq 0$.

Then, $q_2=1/4+3/2 q_1$ and $1\geq q_1, q_2\geq 0$, $1\geq q_1 + q_2\geq 0$.

. The optimal threat strategy for II is then $q^*=(q_1, q_2, 1-q_1 - q_2)$ where $q_2=1/4 +3/2 q_1$, and $3/10 \geq q_1\geq 0$.

ϕ = TU Resolution Point=(ϕ_1, ϕ_2)
$= (½(\sigma+\delta), ½(\sigma- \delta))=(6.25, 4.75)$

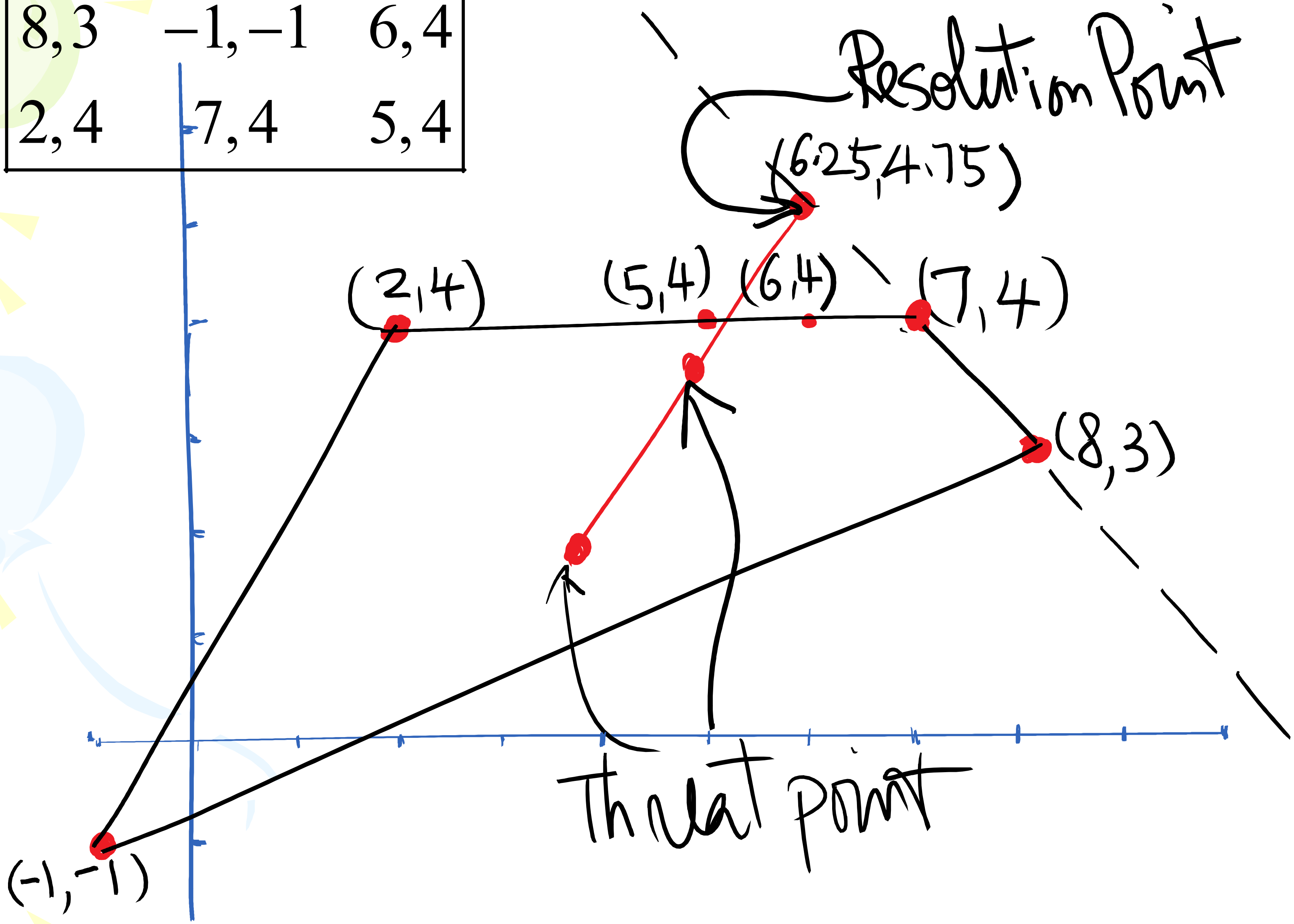
Cooperative strategy=<Row 1, Column 1>

Side payment=I to pay 1.75 to II,

or <Row 2, Col 2> and I to pay II 0.75.

Note that $p^{*T}(A-B)q^*=1.5$. The line joining $(p^{*T}Aq, p^{*T}Bq)$ to the resolution point (6.25, 4.75) has slope 1.

8, 3	−1, −1	6, 4
2, 4	7, 4	5, 4



Remark: Since the players split evenly the excess of σ over $D_1 + D_2$. The line joining the optimal threat point to the resolution point, called the sharing line has slope equals to 1. On the other hand, the side payment is transferred along the line with slope equals to -1. This is called the utility transfer line. The absolute value of the slopes of these two lines are reciprocal to each other.

Cooperative Games with Non-Transferable Utility.

We now consider games in which side payments are forbidden. It may be assumed that the utility scales of the players are measured in non-comparable units.

The players may argue, threaten, and come to a binding agreement as before, but there is no monetary unit with which the players can agree to make side payments. The players may barter goods that they own, but this must be done within the game and reflected in the bimatrix of the game.

The Bargaining Problem

Two individuals have before them several possible contractual agreements. Both have interests in reaching agreement but their interests are not entirely identical. What “will be” the agreed contract, assuming that both parties behave rationally?

THE BARGAINING PROBLEM[1]

BY JOHN F. NASH, JR.

A new treatment is presented of a classical economic problem, one which occurs in many forms, as bargaining, bilateral monopoly, etc. It may also be regarded as a nonzero-sum two-person game. In this treatment a few general assumptions are made concerning the behavior of a single individual and of a group of two individuals in certain economic environments. From these, the solution (in the sense of this paper) of the classical problem may be obtained. In the terms of game theory, values are found for the game.

INTRODUCTION

A TWO-PERSON bargaining situation involves two individuals who have the opportunity to collaborate for mutual benefit in more than one way. In the simpler case, which is the one considered in this paper, no action taken by one of the individuals without the consent of the other can affect the well-being of the other one.

The economic situations of monopoly versus monopsony, of state trading between two nations, and of negotiation between employer and labor union may be regarded as bargaining problems. It is the purpose of this paper to give a theoretical discussion of this problem and to obtain a definite "solution"—making, of course, certain idealizations in order to do so. A "solution" here means a determination of the amount of satisfaction each individual should expect to get from the situation, or, rather, a determination of how much it should be worth to each of these individuals to have this opportunity to bargain.

This is the classical problem of exchange and, more specifically, of bilateral monopoly as treated by Cournot, Bowley, Tintner, Fellner, and others. A different approach is suggested by von Neumann and Morgenstern in *Theory of Games and Economic Behavior*[2] which permits the identification of this typical exchange situation with a nonzero sum two-person game.

In general terms, we idealize the bargaining problem by assuming that the two individuals are highly rational, that each can accurately compare his desires for various things, that they are equal in bargaining skill, and that each has full knowledge of the tastes and preferences of the other.

[1] The author wishes to acknowledge the assistance of Professors von Neumann and Morgenstern who read the original form of the paper and gave helpful advice as to the presentation.

[2] John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior*, Princeton: Princeton University Press, 1944 (Second Edition, 1947), pp. 15–31.

Example: (John Nash, The bargaining Problem, Econometrica 18 (1950))

Let us suppose that two intelligent individuals, Bill and Jack, are in a position where they may barter goods but have no money with which to facilitate exchange. Further, let us assume for simplicity that the utility to either individual of a portion of the total number of goods involved is the sum of utilities to him of the individual goods in that portion. We give below a table of goods possessed by each individual with the utility of each to each individual. The utility functions used for the two individuals are, of course, to be regarded as arbitrary.

Bill's goods	Utility to Bill	Utility to Jack
book	2	4
whip	2	2
ball	2	1
bat	2	2
box	4	1

Jack's goods		
pen	10	1
toy	4	1
knife	6	2
hat	2	2

Nash Bargaining Model.

A **bargaining problem** is given by (S, u*, v*) such that

(i) S is a compact (i.e. bounded and closed), convex set, in the plane.

One is to think of S as the set of vector payoffs achievable by the players if they agree to cooperate. It is the analogue of the NTU-feasible set, although it is somewhat more general in that it does not have to be a polyhedral set. It could be a circle or an ellipse, for example. We refer to S as the **NTU-feasible set**

(ii) $(u^*, v^*) \in S$, is called the **threat point, disagreement point** or **status-quo point.**

The bargaining players have to decide on a point in S. The status-quo point, (u^*, v^*), is the settlement when negotiation breaks down.

Solution to a bargaining problem:

Given an NTU-feasible set, S, and a threat point or status quo point, $(u^*, v^*) \in S$, the problem is to find a point,
$(u^\#, v^\#) = \mathbf{f}(S, u^*, v^*)$, to be considered a "fair and reasonable outcome" or "solution" of the game for an arbitrary compact convex set S and point
$(u^*, v^*) \in S$.

Nash Solution to all bargaining problems:

In the approach of Nash, "fair and reasonable" is defined by a few axioms. Then it is shown that these axioms lead to a unique solution,
denoted as $\mathbf{f}(S, u^*, v^*)$ for any (S, u^*, v^*).

Nash Axioms for f$(S, u^*, v^*) = (u^\#, v^\#)$**.**

(1) **Feasibility.** $(u^\#, v^\#) \in S$.

(2) **Pareto Optimality.** $(u^\#, v^\#)$ is Pareto optimal in S.

(3) **Symmetry.** If S is symmetric about the 45^0 line $u = v$, and if $u^* = v^*$,

then $u^\# = v^\#$.

(4) **Independence of irrelevant alternatives.** If T is a closed convex subset of S, and if the threat point

$(u^*, v^*) \in T$ and $\mathbf{f}(S, u^*, v^*) = (u^\#, v^\#) \in T$,

then $\mathbf{f}(T, u^*, v^*) = (u^\#, v^\#)$.

(5) **Invariance under change of location and scale.** If

$T = \{(u\_,v\_) : u\_ = \alpha_1 u + \beta_1,\ v\_ = \alpha_2 v + \beta_2 \text{ for } (u, v) \in S\}$,

where $\alpha_1 > 0$, $\alpha_2 > 0$, β_1, and β_2 are given numbers, then

$\mathbf{f}(T, \alpha_1 u^* + \beta_1, \alpha_2 v^* + \beta_2) = (\alpha_1 u^\# + \beta_1, \alpha_2 v^\# + \beta_2)$.

Theorem. There exists a unique function **f** satisfying the Nash axioms. Moreover, if there exists a point $(u, v) \in S$ such that $u > u^*$ and $v > v^*$, then $\mathbf{f}(S, u^*, v^*)$ is that point of S that maximizes $(u - u*)(v - v*)$ among points of S such that $u \geq u^*$ and $v \geq v^*$.

$(u - u^*)(v - v^*)$ is called the Nash product.

We only give a proof of the interesting case that there exists a point $(u, v) \in S$ such that $u > u^*$ and $v > v^*$.

Proof:

We will show that the point maximizing the Nash product is the unique solution satisfying the five Nash axioms.

Existence:

We show $(u^\#, v^\#)$ in $S_+ = \{(u, v) \in S : u \geq u^*, v \geq v^* \}$ maximizing the Nash product indeed satisfies the Nash axioms.

The first 4 axioms are very easy to verify.

To check the fifth axiom, note that

$$(\alpha_1 u + \beta_1 - \alpha_1 u^* - \beta_1)(\alpha_2 v + \beta_2 - \alpha_2 v^* - \beta_2)$$
$$= \alpha_1 \alpha_2 (u - u^*)(v - v^*)$$

If $(u - u^*)(v - v^*)$ maximizes the Nash product, then so does $(\alpha_1 u + \beta_1 - \alpha_1 u^* - \beta_1)(\alpha_2 v + \beta_2 - \alpha_2 v^* - \beta_2)$.

Uniqueness:

For an arbitrary closed convex set S and $(u^*, v^*) \in S$, let $(u^\wedge, v^\wedge)$ be the point of

S_+ that maximizes $(u - u^*)(v - v^*)$. We will show that $(u^\wedge, v^\wedge) = f(S, u^*, v^*)$.

Define α_1, β_1, α_2, and β_2 so that under the change of scale and location transformation :

$(u, v) \rightarrow (\alpha_1 u + \beta_1, \alpha_2 v + \beta_2)$

(u^*, v^*) goes to $(0, 0)$ and $(u^\wedge, v^\wedge)$ goes to $(1, 1)$. Then,

$(1, 1)$ maximizes the Nash product too. Therefore, to simplify the notations we might as well assume from the very beginning that $(u^*, v^*) = (0, 0)$ and $(u^\wedge, v^\wedge) = (1, 1)$.

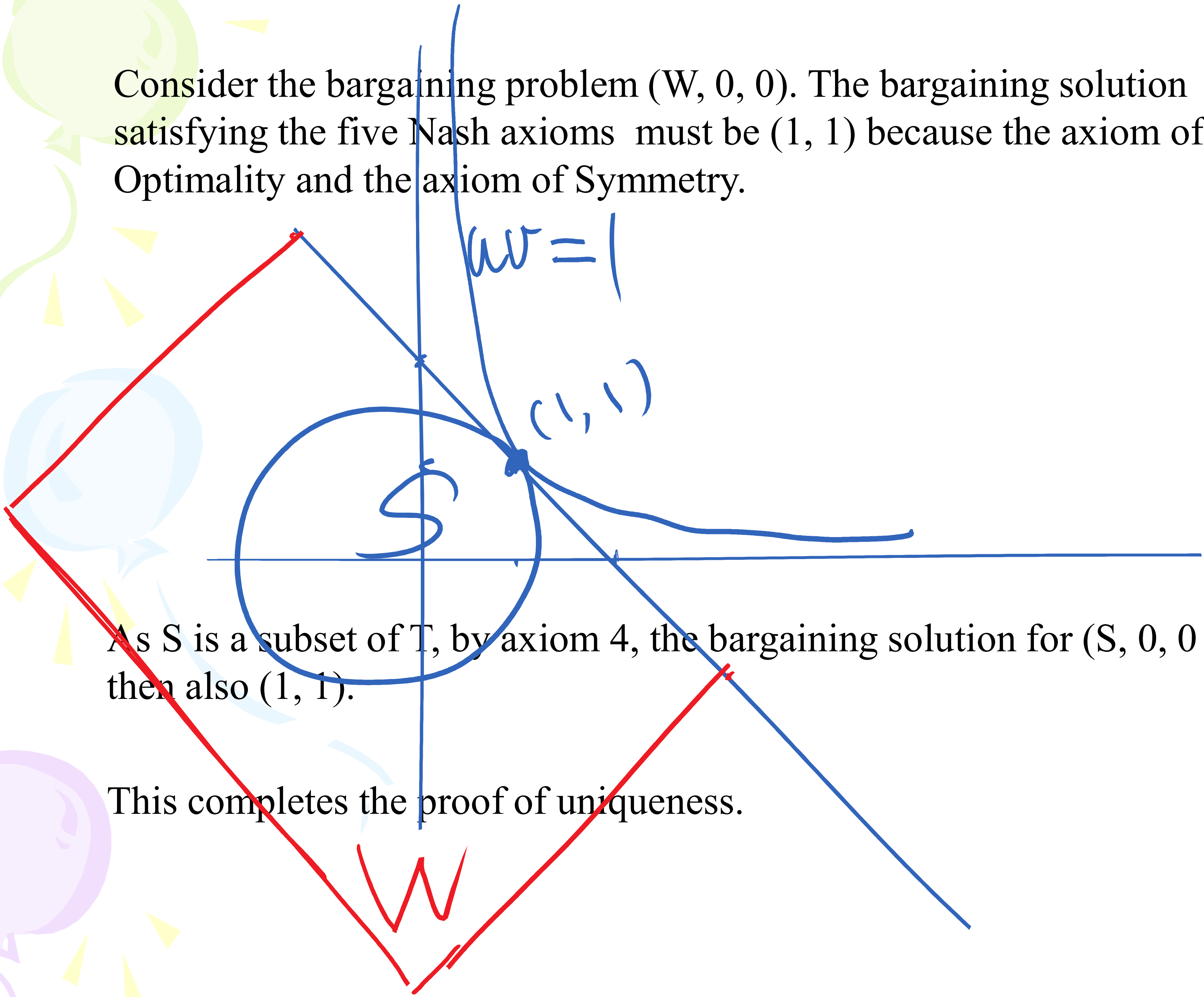
Note that the gradient vector of the function uv at the point $(1, 1)$ is the vector $(1, 1)$. $u+v=2$ is then the tangent line to $uv=1$ at $(1,1)$. We claim that S lies entirely in $\{(u, v) : u + v \leq 2\}$.

Indeed, let (r,s) be a point in S such that $r + s > 2$. Then, the line segment joining $(1,1)$ and (r, s) lies entirely in S because S is convex. Note that $(1, 1) \cdot (r-1, s-1) > 0$. Therefore, the function uv is larger than 1 at some point of the line segment. This is a contradiction!

Therefore, S is a subset of $\{(u, v) : u + v \leq 2\}$.

Then, enclose S by a symmetric convex set W which also lies in $\{(u, v) : u + v \leq 2\}$.

Consider the bargaining problem (W, 0, 0). The bargaining solution satisfying the five Nash axioms must be (1, 1) because the axiom of Pareto Optimality and the axiom of Symmetry.



As S is a subset of T, by axiom 4, the bargaining solution for (S, 0, 0) is then also (1, 1).

This completes the proof of uniqueness.

Remark: It is clear that the Nash Bargaining solution depends on the choice of the disagreement point or threat point. In a subsequent paper, Nash studied the problem of choosing the threat point optimally so that one does not need to specify the threat point externally.

Example:

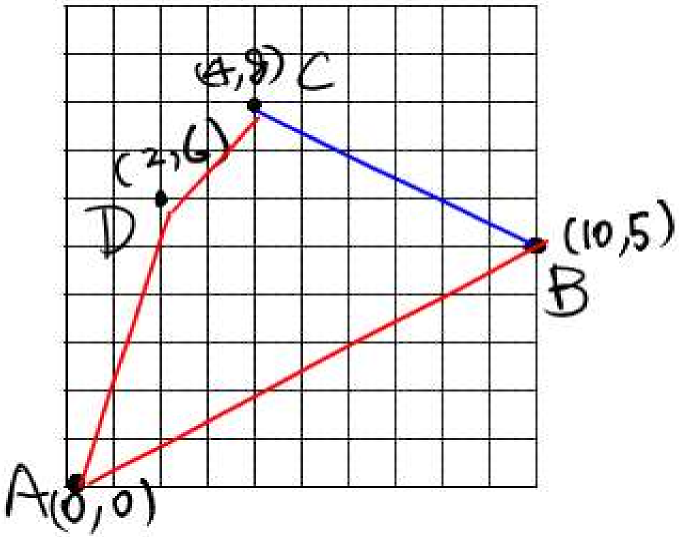
Find the NTU solution of the following bimatrix game with the given status quo point.

2,6	10,5
4,8	0,0

SQ=(0,0)

Let A=(0, 0), B=(10, 5), C=(4, 8), D=(2, 6).

Then, the NTU feasible set is the quadrilateral ABCD. By Pareto Optimality, the resolution point is on the segment BC.



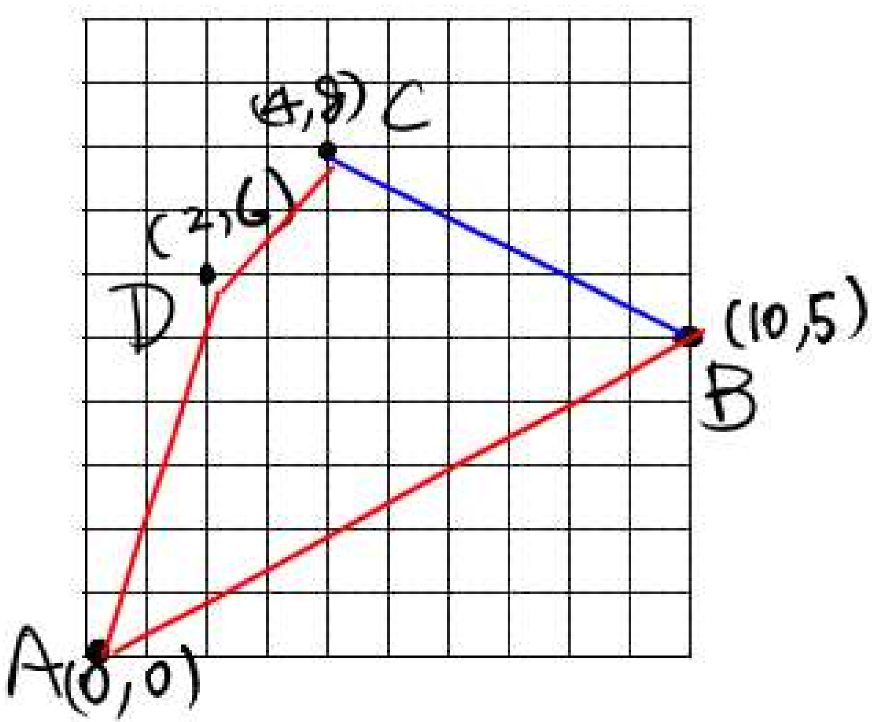
We can parameterize segment BC as

$t \mapsto (1-t)(10, 5) + t(4, 8) = (10-6t, 5+3t),\ 0 \leqq t \leqq 1.$

The Nash product on BC is then the function

$t \mapsto (10-6t)(5+3t) = 50-18t^2,\ 0 \leqq t \leqq 1$. The max of this function occurs at t=0 i.e. at B.

The NTU value is then (10, 5). The cooperative strategy is <Row1, Col2>.



Example:

Find the NTU solution of the following bimatrix game with the given status quo point.

2,6	10,5
4,8	0,0

SQ=security levels

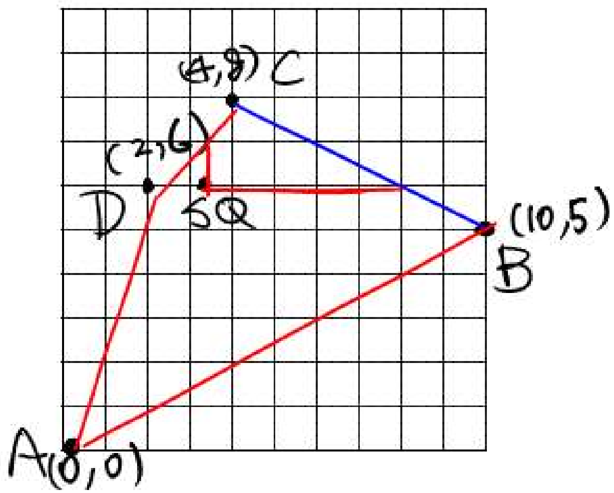
$$A=\begin{pmatrix} 2 & 10 \\ 4 & 0 \end{pmatrix},\quad B=\begin{pmatrix} 6 & 5 \\ 8 & 0 \end{pmatrix}$$

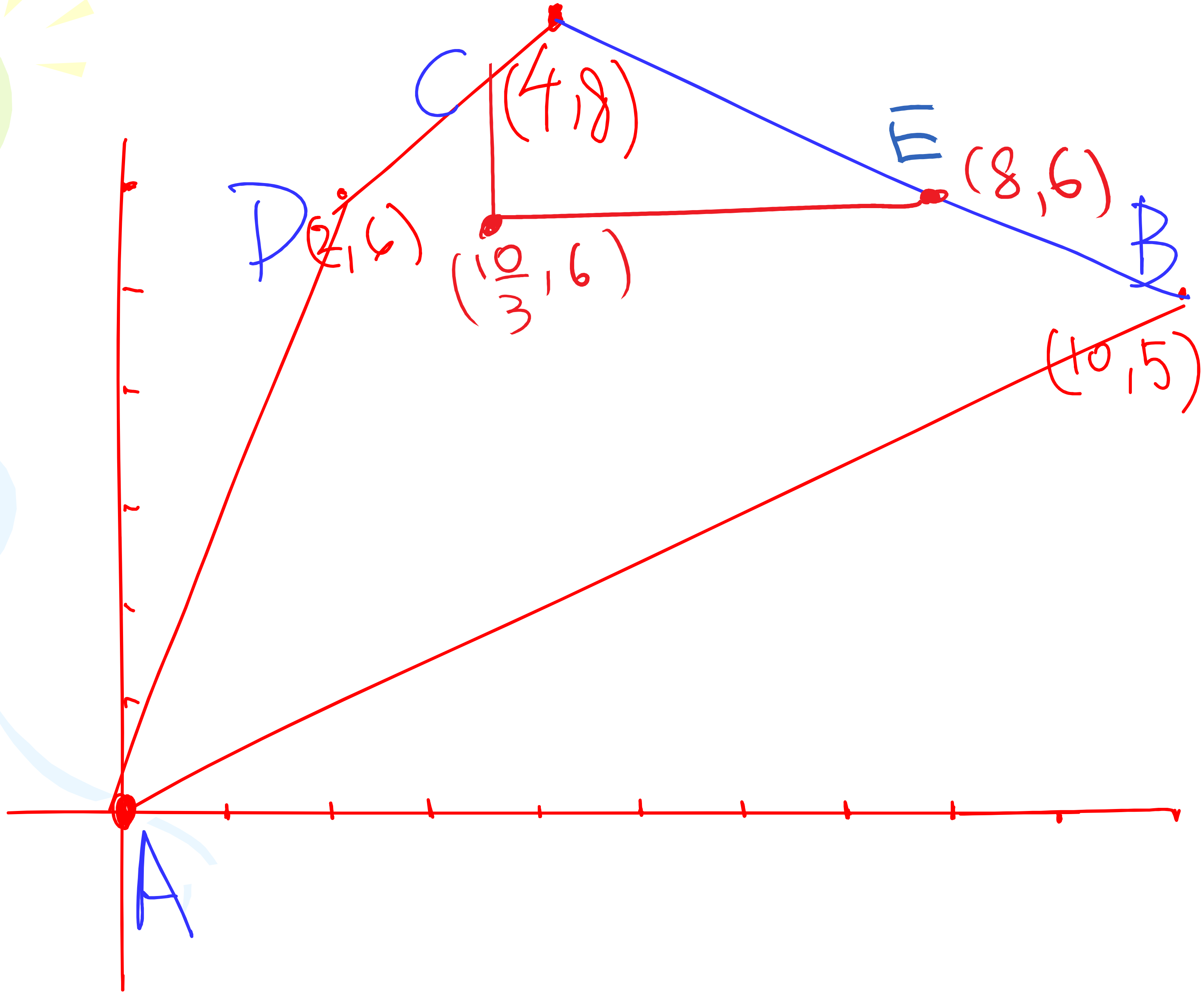
As in the previous example, the NTU feasible set is ABCD.

SQ=(val(A), val(B^T)) = (10/3, 6).

As SQ=(10/3, 6) , the Axiom of Pareto Optimality implies that the resolution point locates in the intersection of

{(x,y): x≥10/3, y≥6} and BC.

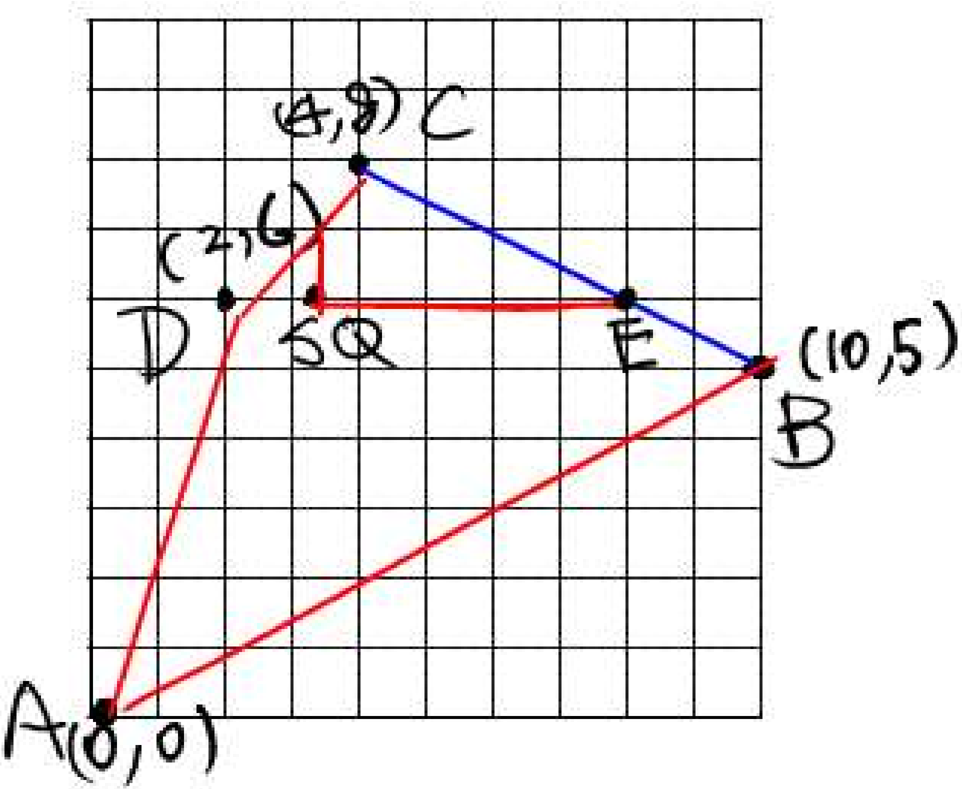




C
(4,8)
E
(8,6)
B
D
(2,6)
(10/3,6)
(10,5)
A

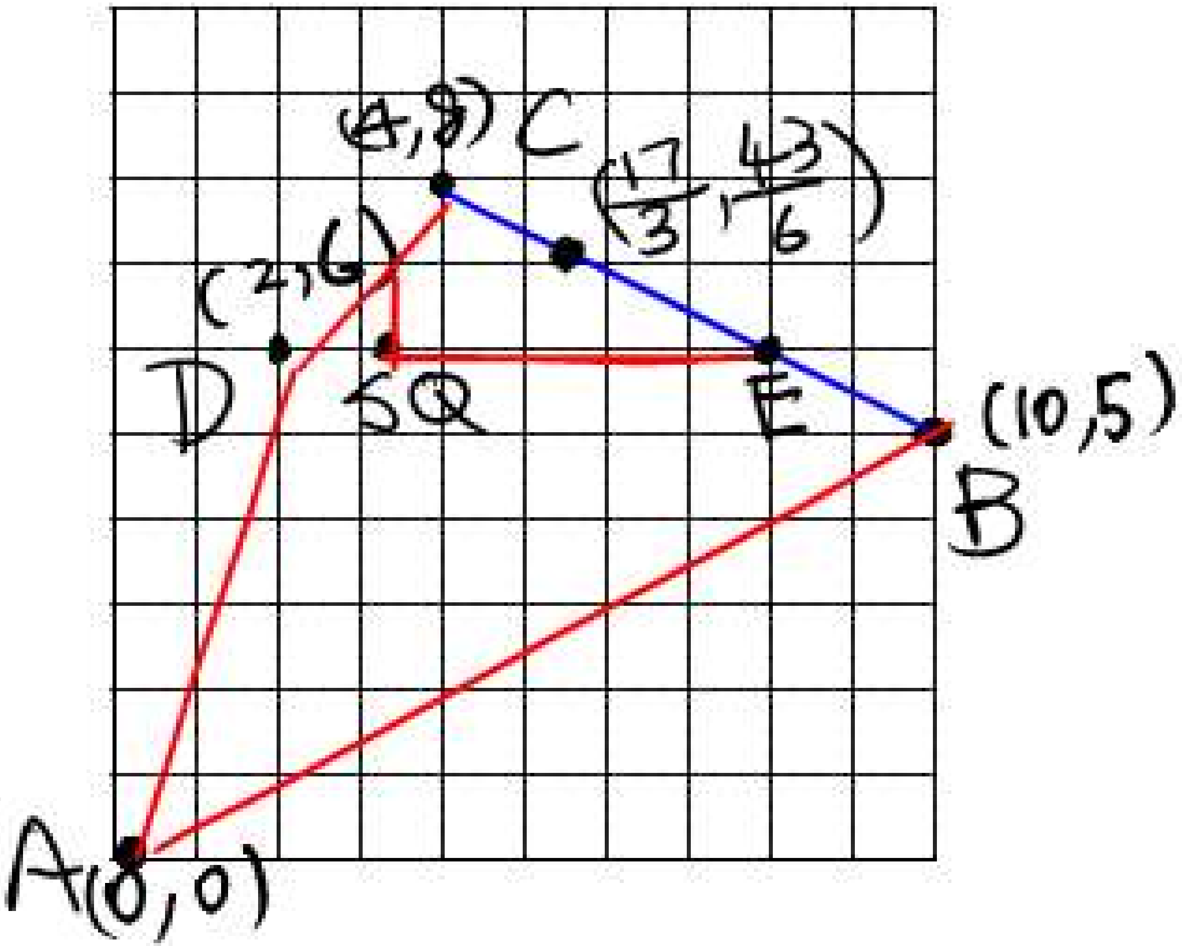
Let E=(8,6) be the intersection of y=6 and BC. The resolution point must lie in the segment EC.

EC is parameterized as
$t \mapsto (1-t)(8,6) + t(4,8) = (8-4t, 6+2t)$. The Nash product function is then $t \mapsto 2t(14/3 - 4t)$, $0 \leqq t \leqq 1$. The max occurs at $t=7/12$. The resolution point is then $(17/3, 43/6)$.



Since (17/3, 43/6)=5/18 (10, 5) +13/18 (4, 8),

the cooperative strategy is to play <Row 1, Col 2> with prob=5/18 and play <Row 2, Col 1> with prob=13/18.



Remark: In the previous example, SQ= (10/3, 6) and the NTU resolution point is (17/3, 43/6). The slope of the line joining the resolution point and the SQ point, called the sharing line, is ½.

In this case, the resolution point lies in the line segment joining B to C. In this case the line joining B and C is called the utility transfer line.

The slope of the line joining B=(10, 5), C=(4,8) is

(8-5)/(4-10)= -1/2.

Therefore, slope of sharing line = - slope of utility transfer line

in this example. In fact, this is true in general.

Remark: Let $(x(t), y(t))$ be a curve with parameter t. The Nash Product w.r.t the Disagreement Point, (a, b) is $(x(t) - a)(y(t) - b)$. Its derivative w.r.t. t is

$$x'(t)(y(t)-b) + y'(t)(x(t)-a).$$

If $t=t_0$ the Nash Product is a critical point, then

$$x'(t_0)(y(t_0)-b) + y'(t_0)(x(t_0)-a)=0$$

This says that the absolute value of the slope of the tangent $(x'(t_0), y'(t_0))$ to the curve at t_0 equals to the absolute values the slope of the line joining the disagreement point to the resolution point.

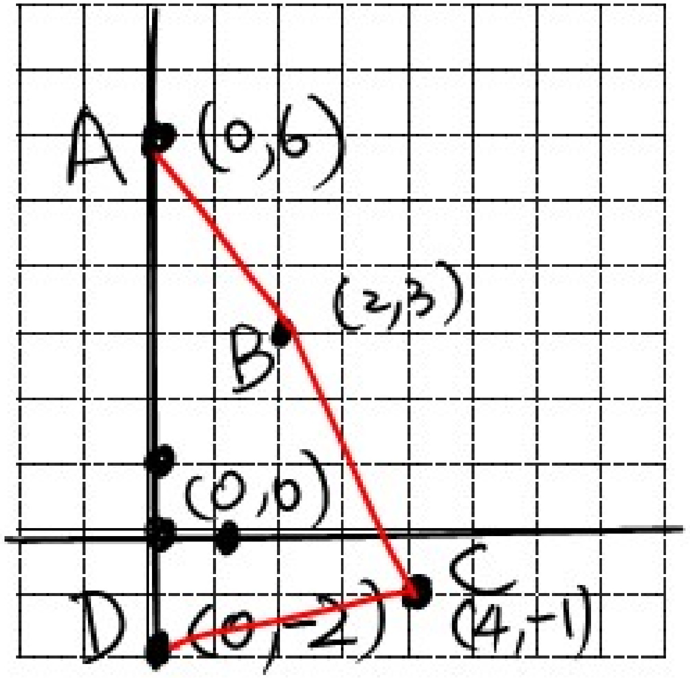
Example:

Find the NTU solution of the following bimatrix game with the given status quo point.

$$\begin{bmatrix} (2,3) & (0,-2) \\ (0,6) & (4,-1) \\ (1,0) & (0,1) \end{bmatrix} \qquad SQ=(0,0)$$

Let A=(0,6)), B=(2,3), C=(4, -1), D=(-2, 0).

Then, the NTU feasible set is the quadrilateral ABCD. By Pareto Optimality, the resolution point is on the segments AB, BC', where C' is the intersection of BC with the x-axis.

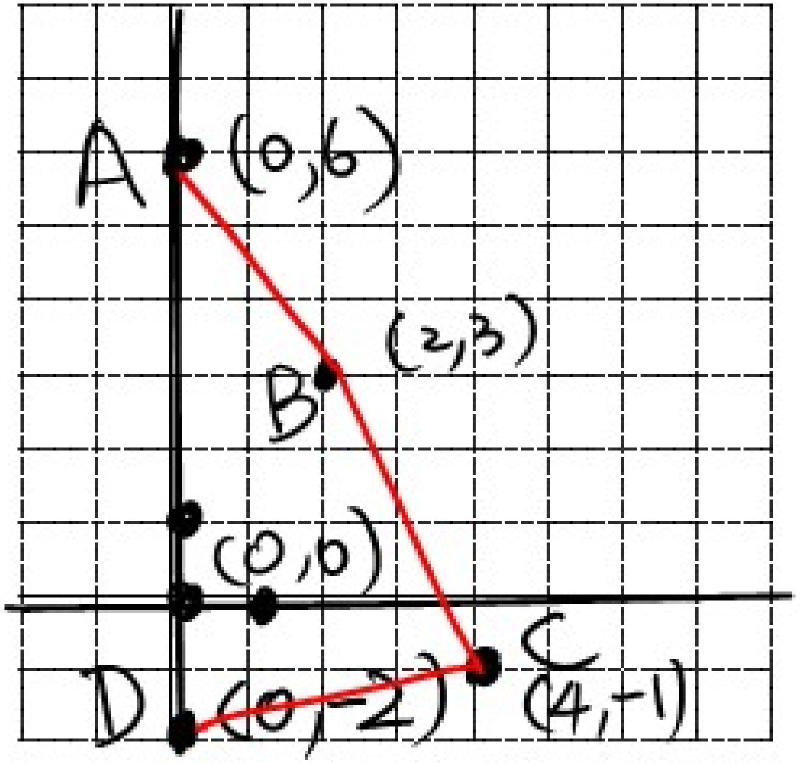


Segment AB:

We can parameterize segment AB as
$t \mapsto (1-t)(-1, 6) + t(2, 3) = (-1+3t, 6-3t),\ 0 \leqq t \leqq 1.$

The Nash product on AB is then the function
$t \mapsto (-1+3t)(6-3t) = -6+21t-9t^2,\ 0 \leqq t \leqq 1.$

The max of this function occurs at t=1. The maximum is 6.

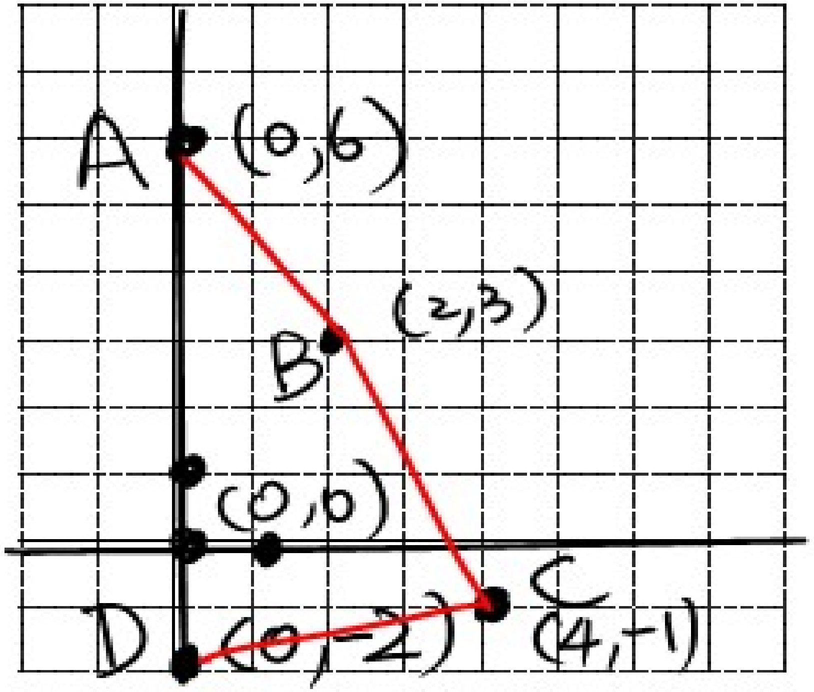


Segment BC':

We can parameterize segment BC as
$t \mapsto (1-t)(2, 3) + t(4, -1) = (2+2t, 3-4t)$, $0 \leqq t \leqq 1$. The intersection of BC with x-axis is when $3-4t=0$, i.e. $t=3/4$. Therefore, $C'=(7/2, 0)$

The Nash product on BC' is then the function

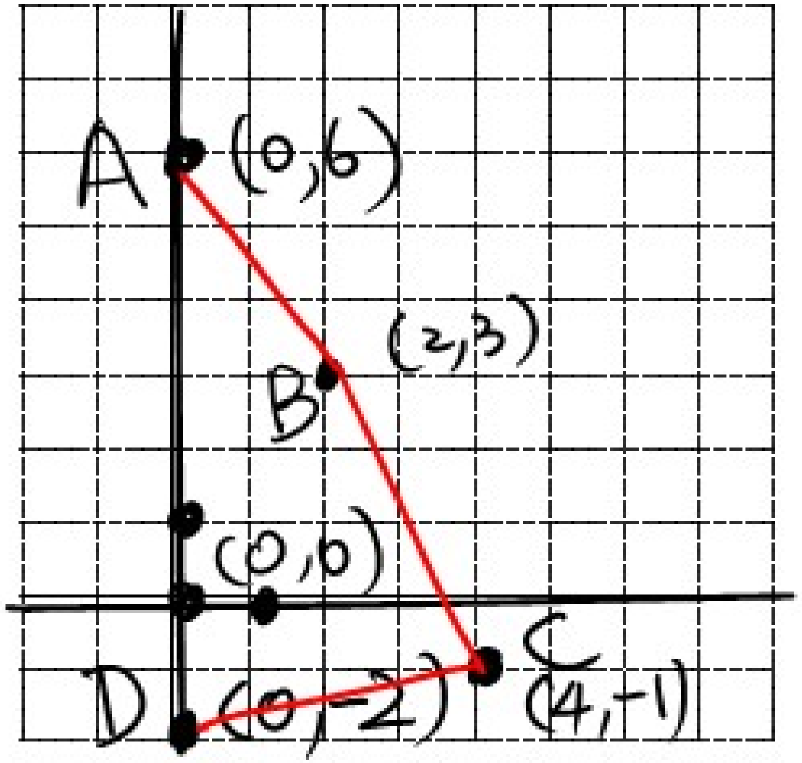
$t \mapsto (2+2t)(3-4t) = 6-2t-8t^2$, $0 \leqq t \leqq 3/4$. This function is decreasing on $0 \leqq t \leqq 3/4$The max of this function occurs at $t=0$. The maximum is 6.



Combining two cases, the maximum of the Nash product on the part of Pareto optimal boundary with x≥0, y≥ 0, occurs at (2, 3).

Therefore, the resolution point or the NTU value is (2, 3). The cooperative strategy is

<Row 1, Col 1>.



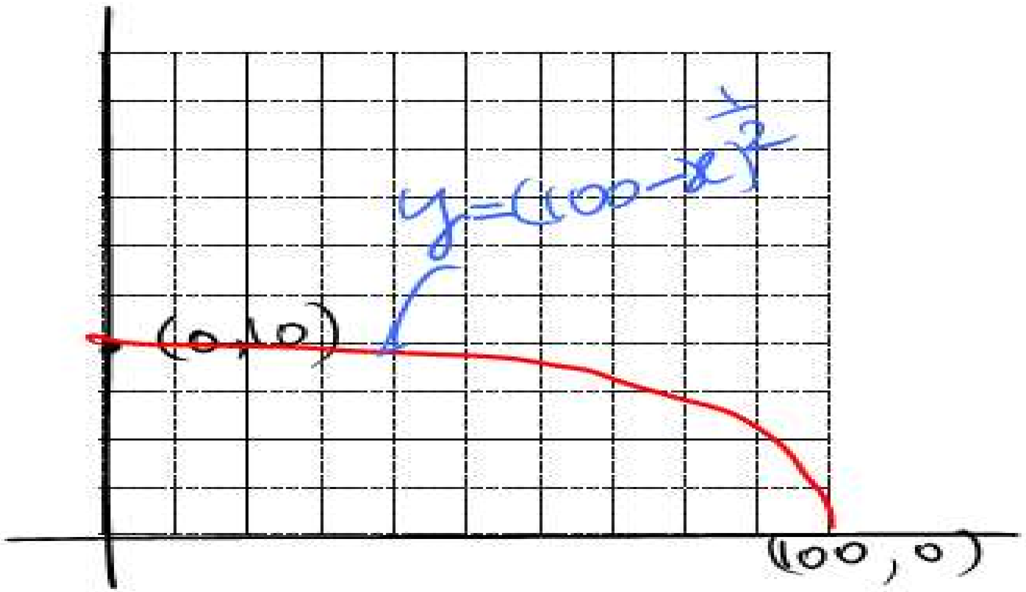
Example: (Divide the Dollar) Suppose two players are to divide a potential profit of \$100. If the players come to an agreement, they then divide the money accordingly. If they fail to reach an agreement, both receive nothing. Suppose that Player I's utility from \$x is x, and Player II's utility from \$x is $(x)^{1/2}$. Find the Nash Bargaining solution.

Solution: Let S be the region bounded by the three curves,

$\{(x, 0): 0 \le x \le 100\}$, $\{(0, y): 0 \le y \le 10\}$,

$\{(x, (100-x)^{1/2}) : 0 \le x \le 100\}$.

Then, the bargaining problem is (S, (0, 0)), i.e. the status quo point is (0, 0).



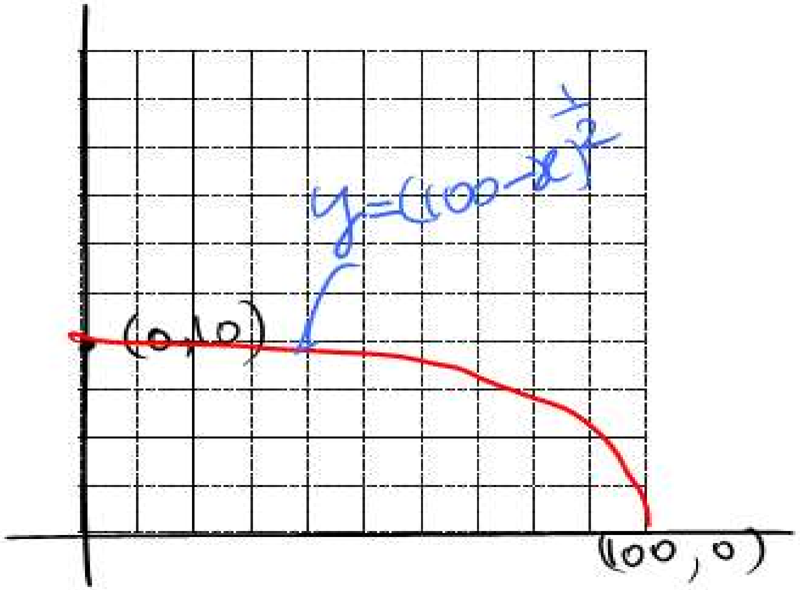
The Pareto optimal boundary is the curve

$\{(x, (100-x)^{1/2}) : 0 \leq x \leq 100 \}$.

The Nash product is then $x(100-x)^{1/2}$ for $0 \leq x \leq 100$. To find its maximum, we solve $0 = (100-x)^{1/2} - x/(2 (100-x)^{1/2})$.

We get $x = 200/3$.

The NTU solution is for Player I to get 200/3 and Player II to get 100/3.



Example (Kalai-Smorodinsky):

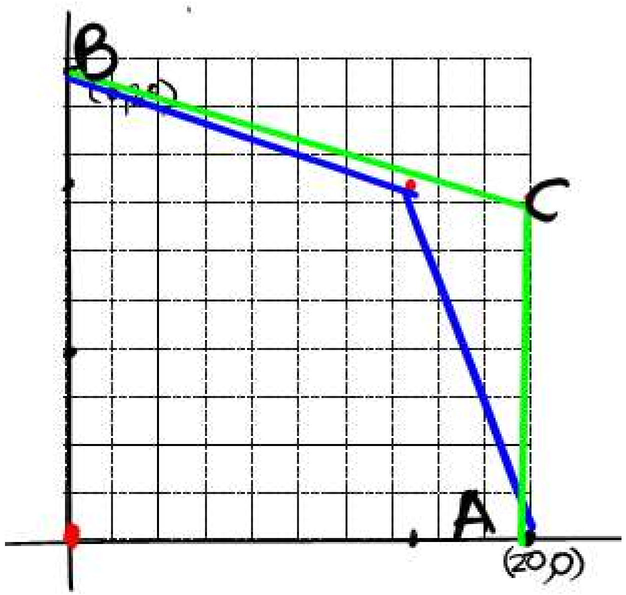
Let Q be the quadrilateral with vertices at (0, 0), (20, 0), (0, 20), (15, 15).

Let P be the quadrilateral with vertices at (0, 0), (20, 0), (0, 20), (20, 14).

Note $Q \subseteq P$.

(i) Find the Nash bargaining solution of (Q, (0, 0)).

Solution: (15, 15) by symmetry.



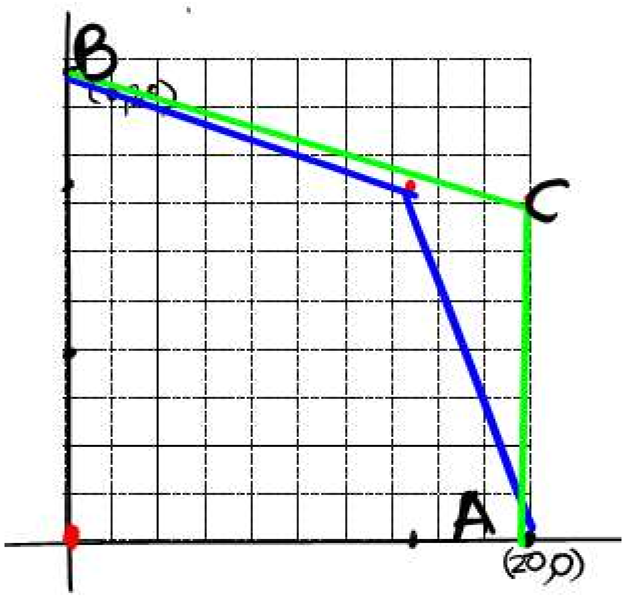
(ii) Find the Nash bargaining solution of (P, (0, 0)).

Solution: Let A= (20, 0), B=(0, 20), C=(20, 14). The Pareto optimal boundary consists of segments AC, CB.

On AC, the maximum of the Nash product occurs at (20, 14).

On CB, lets parametrize CB as (1-t)(20, 14) + t(0, 20)=(20-20t, 14 + 6t), for t between 0 and 1. The Nash product is $40(7 – 4t – 3t^2)$. This is a decreasing function of t. Hence, maximum occurs at t=0, i.e. at (20, 14).

The Nash bargaining solution is then (20, 14).



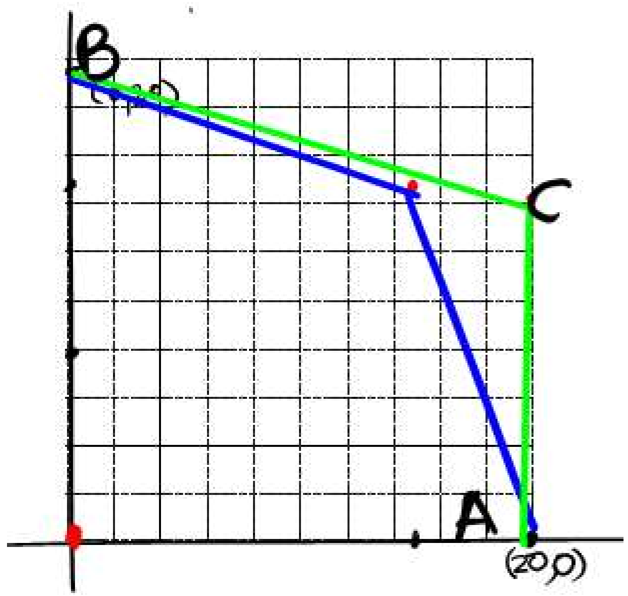
Discussion:

According to Nash Bargaining Axioms

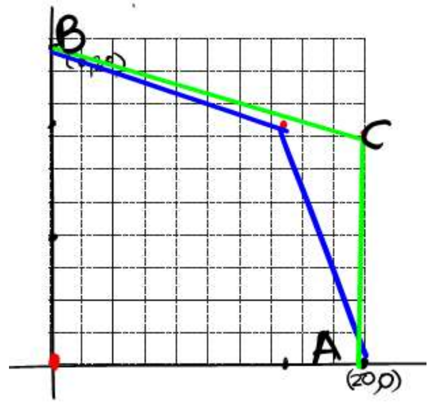
for (Q, (0, 0)), Player I receives 15, Player II receives 15,

for (P, (0, 0)), Player I receives 20, Player II receives 14.

Note that Q is contained in P i.e more feasible outcomes are added to Q. However, for any feasible outcomes which will give Player I more, there are feasible outcomes to give Player II more. It turns out that the Nash Bargaining Axioms give Player I more but give Player II less. Is this FAIR?



Kalai-Smorodinsky solution: For any bargaining problem $(S, (d_1, d_2))$, define The Utopia Points I_1, I_2 for each player.



$I_1 = \max \{u: \text{for some } v,\ (u, v) \in S,\ d_1 \leq u,\ d_2 \leq v\}$,

$I_2 = \max \{v: \text{for some } u,\ (u, v) \in S,\ d_1 \leq u,\ d_2 \leq v\}$

The Kalai-Smorodinsky solution (u_1, u_2), a compromise between the utopia and the disagreement points, is on the Pareto optimal boundary such that $(u_1 - d_1)/(I_1 - u_1) = (u_2 - d_2) / (I_2 - u_2)$

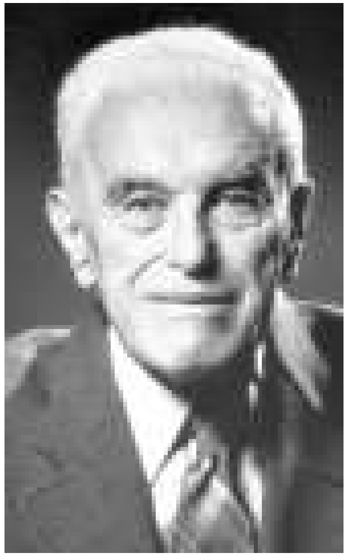
The Kalai-Smorodinsky solution for $(P, (0, 0))$ is then the intersection of segment BC with the line x-y=0 as $I_1 = I_2 = 20$.

The solution is then (200/13, 200/13),

i.e. Player receives 200/13, Player II receives 200/13.

Zeuthen's Principle (adopted from "Bargaining" by John Harsanyi, 1994 Nobel Prize in Economics)

Suppose that, at a given stage of the bargaining process, Player 1's last offer corresponded to the payoff vector $v=(v_1, v_2)$ whereas Player 2's last offer corresponded to the payoff vector $w=(w_1, w_2)$ such that v and w belong to the Pareto Optimal boundary of the cooperative feasible set, with $v_1>w_1$ but $v_2<w_2$.



Which player will have to make the next concession? Zeuthen tries to answer the question as follows.

A player will be more willing to risk conflict if he/she does not have much to lose in case the negotiation fails. In contrast, a player is less willing to risk conflict when he/she has more to lose.

If Player 1 accepts his opponent's last offer w, then he will get payoff w_1. On the other hand, if he insists on his own last offer v, then he will obtain either payoff v_1 or will obtain the disagreement point payoff d_1 i.e. Player 2 sticks to his last offer.

Suppose Player 1 assigns probability p to negotiation breaking down and probability (1-p) for offer being accepted. Then, he must conclude that

by insisting on his last offer v and by making no further concession he will get the expected payoff $(1-p)v_1+pd_1$,

whereas by being conciliatory and accepting Player 2's last offer w he will get w_1.

Therefore, Player 1 can rationally stick to his last offer only if

$$(1\text{-}p)\ v_1 + pd_1 \geq w_1 \text{ i.e. if } p \leq (v_1\text{-}w_1)/(v_1\text{-}d_1) = r_1.$$

Similarly, if Player 2 assigns probability q to the hypothesis that Player 1 will stick to his last offer v (bargaining process ends in disagreement), then Player 2 can rationally stick to his own last offer w only if

$$(1\text{-}q)\ w_2 + qd_2 \geq v_2 \text{ i.e. if } q \leq (w_2\text{-}v_2)/(w_2\text{-}d_2) = r_2.$$

Then, r_1, r_2 are the highest probability of a disagreement that Player 1 and Player 2, respectively, would face rather than accept the last offer of the other player, where

$$r_1 = (v_1 - w_1) / (v_1 - d_1)$$

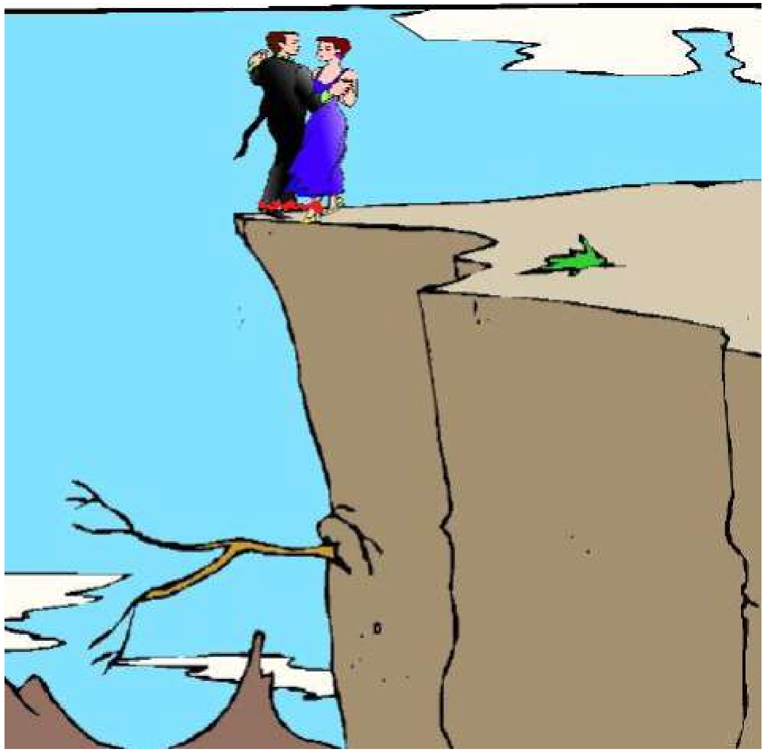
$$r_2 = (w_2 - v_2) / (w_2 - d_2)$$

We will call r_1, r_2 the two players' risk limits expressed as ratio of difference in payoffs.

In an article celebrating Schelling's Nobel Prize for Economics Michael Kinsley, Washington Post Op Ed Columnist and former student of Schelling's, summarizes the professor's Reorientation of Game Theory as follows:

"[Y]ou're standing at the edge of a cliff, chained by the ankle to someone else. You'll be released, and one of you will get a large prize, as soon as the other gives in. How do you persuade the other guy to give in, when the only method at your disposal -- threatening to push him off the cliff -- would doom you both?"

"Answer: You start dancing, closer and closer to the edge. That way, you don't have to convince him that you would do something totally irrational: plunge him and yourself off the cliff. You just have to convince him that you are prepared to take a higher risk than he is of accidentally falling off the cliff. If you can do that, you win."



Zeuthen suggests that the next concession must always come from the player with a smaller risk limit (except that if the two players' risk limits are equal then both of them must make concessions to avoid a conflict). We call this suggestion Zeuthen's Principle.

The following Theorem shows comparison of risk limits is equivalent to comparison of Nash products.

Theorem: If the two players follow Zeuthen's Principle then the next concession will always made by the player whose last offer is associated with a smaller Nash product (unless both are associated with equal Nash product, in which case both of them have to make concessions).

Proof: This is straightforward from the following.

$$r_1 \geq r_2 \quad \text{iff} \quad (v_1 - w_1)/(v_1 - d_1) \geq (w_2 - v_2)/(w_2 - d_2)$$

$$\text{iff } [(v_1 - d_1) - (w_1 - d_1)]/(v_1 - d_1) \geq [(w_2 - d_2) - (v_2 - d_2)]/(w_2 - d_2)$$

$$\text{iff} \quad 1 - (w_1 - d_1)/(v_1 - d_1) \geq 1 - (v_2 - d_2)/(w_2 - d_2)$$

$$\text{iff} \quad (v_2 - d_2)/(w_2 - d_2) \geq (w_1 - d_1)/(v_1 - d_1)$$

$$\text{iff} \quad (v_1 - d_1)(v_2 - d_2) \geq (w_1 - d_1)(w_2 - d_2)$$

Corollary: If the two players act according to Zeuthen’s Principle then they will tend to reach the Nash solution as their agreement point.

Proof: By the Theorem at each stage of the bargaining process the offer point with the smaller Nash product will be replaced by a new offer point with a higher Nash product. The process will end at the payoff vector with the highest Nash product.

Rubinstein Bargaining Model

In 1982, Ariel Rubinstein made a breakthrough in his paper entitled "Perfect Equilibrium in a Bargaining Model".

Rubinstein Bargaining Model:

- Two players to divide a pie of size 1.
- Players make offers alternatively: Player I makes offer first. He is called the Odd Player, he/she will make offers at odd rounds. Player II is called the Even Player. At each round, the corresponding player will make an offer. If the other player accepts, the game ends. Otherwise, the game moves to the next round.
- Discount factor: A discount factor of δ, $0<\delta<1$, will apply when moving onto the next round.
- Infinite horizon: The game will play on indefinitely if there is no agreement.

Rubinstein proved the existence of a unique subgame perfect strategy (PPSE) in his 1982 paper.

Sketch of solution:

If there is a SE, as any round looks the same, the offer should be the same.

Odd will offer (x_1, x_2), $x_1 + x_2 = 1$, such that

Odd offers x_2 to Even, Odd will not accept anything less than x_1.

Even will offer (y_1, y_2), $y_1 + y_2 = 1$, such that

Even offers y_1 to Odd, Even will not accept anything less than y_2.

Odd knowing Even 's strategy, will offer Even δy_2 and keeping $(1- \delta y_2)$.

In the previous round, Even knowing what Odd will do in the next round, will offer Odd $\delta(1- \delta y_2)$, keeping

$[1- \delta(1- \delta y_2)]$.

From a Backward Induction reasoning, In order for Even to be indifferent in moving onto the next round, we should have

$\delta[1- \delta(1- \delta y_2)] = \delta y_2$. Then, we get $y_2 = 1/(1+\delta)$.

Hence, the solution is $\delta/(1+\delta)$ to Odd and $1/(1+\delta)$ to Even. This is a subgame perfect strategy.