博弈论 hw14

Coa7. Consider the n-person game with players 1, 2, ..., n, whose characteristic function satisfies

$$v(S) = k, \text{ if } \{1, \ldots, k\} \subseteq S, \text{but } k+1 \notin S, k = 1, 2, 3...$$

and

$$v(S) = 0$$

otherwise. For example, $v(\{2,3,5\}) = 0, v(\{1,3,4,6\}) = 1$ and $v(\{1,2,3,5,6,7\}) = 3$. Find the Shapely value.

For player $k+1 \notin S$, Player $k+n, n = 1,2,3$ is a dummy as it can never contribute, thus $\phi_k(w_s) = 0$ for $k+n, n = 1,2,3$. For players $1, \ldots k \subseteq S$, their contributions are symmetrical. Thus, $\phi_k(w_s) = k/|S|$

Coal8. Let (N, v) be a game in coalition form. Define a new set function w on subsets of N such that $w(S) = v(S)$ for any subject $S \subseteq N, S \neq N$ and $w(N) = v(N) + a$, where a is a positive number. Show that w satisfies the super additivity property. Then, express the Shapely value of w in terms of that of v.

Shapley's axiom of super additivity states that if u and v are characteristic functions, then $\phi(u+v) = \phi(u) + \phi(v)$. By the principle of super additivity, that in a simple games, every subset of a losing coalition is losing, and every superset of winning coalition is winning. Thus, $w(N)$ satisfies the super additivity principle because it is still winning after adding a

---

Coal9. Suppose

$$v(S) = \begin{cases} |S| & \text{if } 1 \in S \\ 0 & \text{otherwise} \end{cases}$$

Find the Shapely value

This is a dictator game, for player $i \neq 1$ they are all dummy players because they neither affect nor harms the coalition that they join. In other words, the only impactful player in this game is player 1, where when player $i = 1$, then the game equals $\phi_i(w_s) = |S|/|S|$ for when $i = 1 \in S$, and $\phi_i(w_s) = 0$ for when $i \neq 1 \in S$

Coal10. For the weighted majority game of 4 players with

$$\begin{matrix} I & II & III & IV \\ 10 & 30 & 30 & 40 \end{matrix}$$

shares. Find the Shapley value.

$v(\emptyset) = 0, v(I) = 10, v(II) = 30, v(III) = 30, v(IV) = 40$

In this weighted majority game, the characteristic functions satisfies

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q \\ 0 & \text{otherwise} \end{cases}$$

where non-negative value w_i is the number of shares that player i has and q, the quota, is twice the average number of shares a player has $q = (\frac{1}{2}) \sum_{i \in N} w_i$

The quota for this game would be $(10 + 30 + 30 + 50)/2 = 60$

A coalition is a set of players that join forces to vote together. If there are four players P_1, P_2, P_3, P_4, then the possible coalitions are as such:
$\{P_1\}, \{P_2\}, \{P_3\}, \{P_4\}, \{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}, \{P_1, P_2, P_3\}, \{P_1, P_2, P_4\}, \{P_1, P_3, P_4\}, \{P_2, P_3, P_4\}$
, and $\{P_1, P_2, P_3, P_4\}$ To see which of these coalitions are winning coalitions, we have to find out whether the sum of the weights in each coalition are greater then the quota.

The winners are

Coalition	Weight	Win or Lose
$\{P_2, P_4\}$	80	Win
$\{P_3, P_4\}$	80	Win
$\{P_1, P_3, P_4\}$	90	Win
$\{P_1, P_2, P_3\}$	70	Win
$\{P_1, P_2, P_4\}$	90	Win
$\{P_2, P_3, P_4\}$	110	Win
$\{P_1, P_2, P_3, P_4\}$	120	Win

P_1 is critical player 1 times

P_2 is critical player 3 times

P_3 is critical player 3 times

P_4 is critical player 5 times

Banzhaf Power Index, $T = 12, P_1 = 1/12, P_2 = 1/4, P_3 = 1/4, P_4 = 5/12$

The Shapely power index can be found using sequential coalitions by determining the pivotal player in each sequential coalition.

There are N participants, so that means there are $N!$ sequential coalitions, and we have to find out determine how many times each player is a pivotal player. The answers that we get is the same as Banzhaf's Power Index

---

Coal11. Find the Shapely value of the weighted majority game with n players in which player 1 has $2n-3$ shares and player 2 to n have 2 shares each.

P_1 has $2n-3$ shares

$\sum_{i=2}^{n} P_i = 2(n-1)$ shares

Thus the quota to reach is $(2n-3+2(n-1))/2 = \frac{4n-5}{2}$

In this case, P_1 is a critical player in all coalitions that he is in except in the one that P_1 is in by himself, since $2n-3 < 2n-5/2$, the right hand side being the quota. In a coalition with P_1 and at least one other player, they form a winning coalition $2n-3+2(n-1) > 2n-5/2, n = 2, 3, \ldots$.

On the other hand, another winning condition is when players $2-n$ form a coalition, they will also win, as $2(n-1) > 2n-5/2$. In this coalition, they are all critical players as if anyone player where to leave, the coalition would become a losing coalition.

Thus, since there are N players, there are $N!$ sequential coalitions, and P_1 is a critical player in all but one of the coalitions, his power is $\frac{N!-1}{N!}$, where as $P_n, n = 2, \ldots, n$, each player is a critical player exactly twice, when it is just itself and P_1 in a coalition, and when players $2-n$ are all in a coalition .

Coal12. In a simple Game, (N, v) a player i, is said to be a veto player, if $v(N - \{i\}) = 0$. Show that the core is empty if there are no veto players.

A game (N, v) is a simple if for every coalition S is a proper subset of N, either v(S) = 0, or $v(S) = 1$. Assuming there are no veto players, we need to prove that the core is empty.

We have $v(N \backslash \{i\}) = 1, \forall i \in N$, and thus

$$\sum_{i \in N} v(N \backslash \{i\}) = |N|$$

but

$$\sum_{i \in N} \sum_{j \in N \setminus \{i\}} x_j = (|N|-1) \sum_{j \in N} = |N|-1$$

for any allocation x, but this contradicts coalitional rationality.

Coal13. Calculate the Shapley value of the Airport Game in the case of six pilots whose aircraft, all different, require progressively longer runaways costing 40, 50, 60, 70, 80, 90 units respectively

$v(I) = 40, v(II) = 50, v(III) = 60, v(IV) = 70, v(V) = 80, v(VI) = 90$

We need to define the cost vector $c = (40, 50, 60, 70, 80, 90)$ through

$$C(S) = \max_{i \in S} c_i, \forall S \subseteq N$$

possible games are (I), (II), (III), (IV), (V), (VI), (I, II), (I, III) and so on. There will be $2^N - 1$ games. In this case, there are 63 games,

then the associated cost game is

$$cv = (40, 50, 60, 50, 60, 70, 80, 90, \dots)$$

1 40's, 2 50s, 4 60's, 8 70's, 16 80's, 32 90's.

6 with 1 player, 16 with 2 players, 20 with 3 players, 15 with 4 players, 6 with 5 players, 1 with 6 players

The Shapley's formula is noted as

$$\phi = \sum \frac{|S|!(|N| - |S| - 1)!}{|N|!} * (v(Su\{i\}) - v(S))$$

From this, the Shapley value is computed as

$$sh_{cv} = (6.68, 8.69, 11.17, 14.5, 19.5, 29.5)$$

Personal Notes

The Shapley Value

core is useful as a measure of stability, presents a set of imputations without distinguishing one point of the set as preferable to another, also the core may be empty

assign to each game in coalitional form a UNIQUE vector of payoffs, called the Shapley value where

- the ith entry of the value vector may be considered as a measure of the value or power of the ith player in the game

value function ϕ, $\phi_i(v)$ represents the worth or value of player i in the game with characteristic function v

imputation - represent (as being done, caused, or possessed by someone, attribute)

Number of Coalitions for N players is $2^N - 1$

sequential coalition - when the order that players join a coalition matters

- if a player joins the coalition, and the coalition changes from a losing to winning coalition, then that player is known as pivotal player
- number of sequential coalitions of N players is $N!$