The Babylonian Talmud is the compilation of ancient law and tradition set down during the first five centuries A.D. which serves as the basis of Jewish religious, criminal and civil law. One problem discussed in the Talmud is the so called marriage contract problem:

A man has three wives whose marriage contracts specify that in the case of his death they receive 100, 200 and 300 respectively. The Talmud gives apparently contradictory recommendations. While the man dies leaving an estate of only 100, the Talmud recommends equal division. However, if the estate is worth 300 it recommends proportional division (50,100,150), while for an estate of 200, its recommendation of (50,75,75) is a complete mystery. This particular Mishna has baffled Talmudic scholars for two millennia. In 1985, it was recognized that the Talmud anticipates the modern theory of cooperative games. Each solution corresponds to the nucleolus of an appropriately defined game.

A famous Mishna (The oldest authoritative postbiblical collection and codification of Jewish oral laws.) in the Talmud (Baba Metzia 2a) states the following:

Two hold a garment; one claims it all, the other claims half. Then the one is awarded ¾, the other ¼.

The Nucleolus

Key idea:

Given characteristic function, v, and try to find an imputation **x** = $(x_1, \ldots, x_n)$ that minimizes the worst inequity. That is, we ask each coalition S how dissatisfied it is with the proposed imputation **x** and we try to minimize the maximum dissatisfaction.

Question: How to measure inequity or dissatisfaction?

論語 季氏

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Definition of Excess: As a measure of the inequity of an imputation **x** for a coalition S is defined as the excess,

$e(\mathbf{x}, S) = v(S) - \sum_{j\in S} x_j$,

Remark:

- The excess measures the amount (the size of the inequity) by which coalition S falls short of its potential v(S) in the allocation **x (**不幸福指数）.
- Since the core is defined as the set of imputations such that

$\Sigma_{i\in S} x_i \geq v(S)$ for all coalitions S, we immediately have that an imputation **x** is in the core if and only if all its excesses are negative or zero.

Josh Billings:

The wheel that squeaks the loudest is the one that gets the grease.

The concept of the nucleolus:

On the principle that the one who yells loudest gets served first, we look first at those coalitions S whose excess, for a fixed allocation **x**, is the largest.

Then we adjust **x**, if possible, to make this largest excess smaller.

When the largest excess has been made as small as possible, we concentrate on the next largest excess, and adjust **x** to make it as small as possible, and so on. When we cannot improve further, the imputation we get is called the nucleolus.

Example 1. The Bankruptcy Game. (O' Niell (1982)) A small company goes bankrupt owing money to three creditors. The company owes creditor A \$10,000 , creditor B \$20,000 , and creditor C \$30,000. If the company has only \$36,000 to cover these debts, how should the money be divided among the creditors?

A **pro rata** split of the money would lead to the allocation of \$6000 for A, \$12,000 for B, and \$18,000 for C, denoted by $\mathbf{x} = (6, 12, 18)$ in thousands of dollars. We shall compare this allocation with those suggested by the Shapley value and the nucleolus.

First, we must decide on a characteristic function to represent this game. Of course we will have $v(\emptyset) = 0$ from the definition of characteristic function, and $v(ABC) = 36$ measured in thousands of dollars,

By himself, A is not guaranteed to receive anything since the other two could receive the whole amount; thus we take $v(A) = 0$.

Similarly, $v(B) = 0$.

Creditor C is assured of receiving at least \$6000, since even if A and B receive the total amount of their claim, namely \$30,000, that will leave \$36,000 - \$30,000 = \$6000 for C. Thus we take $v(C) = 6$.

Similarly, we find $v(AB) = 6$, $v(AC) = 16$,

and $v(BC) = 26$.

To find the nucleolus of this game,

let $\mathbf{x} = (x_1, x_2, x_3)$ be an efficient allocation (that is, let $x_1+x_2+x_3 = 36$).

Consider first an arbitrary point, say the pro rata point

(6, 12, 18). The excesses are e(x,A)=-6, e(x,B)=-12, e(x,C)=-12, e(x,AB)=-12, e(x,AC)=-8, e(x,BC)=-4.

S	$v(S)$	$e(x,S)$	$(6,12,18)$
A	0	$-x_1$	-6
B	0	$-x_2$	-12
C	6	$6-x_3$	-12
AB	6	$6-x_1-x_2=x_3-30$	-12
AC	16	$16-x_1-x_3=x_2-20$	-8
BC	26	$26-x_2-x_3=x_1-10$	-4

The largest of these numbers is −4 corresponding to the coalition BC. This coalition will claim that every other coalition is doing better than it is.

So we try to improve on things for this coalition by making x_2+x_3 larger, or, equivalently, x_1 smaller (since $x_1 = 36-x_2-x_3$).

But as we decrease the excess for BC, the excess for A will increase at the same rate and so these excesses will meet at -5, when $x_1 = 5$. It is clear that no choice of $\mathbf{x}$ can make the maximum excess smaller than -5 since at least one of the coalitions A or BC will have excess at least -5. Hence, $x_1 = 5$ is the first component of the nucleolus.

Though x_1 is fixed, we still have x_2 and x_3 to vary subject to $x_2 + x_3 = 36 - 5 = 31$, and we choose them to make the next largest excess smaller. We choose the point **x** = (5, 12, 19) as the next guess, because we see that the next largest excess after −4, − 5 is −8 corresponding to coalition AC.

To make this smaller, we must increase x_3 (decrease x_2).

S	$v(S)$	$e(x,S)$	$(6,12,18)$	$(5,12,19)$
A	0	$-x_1$	−6	−5
B	0	$-x_2$	−12	−12
C	6	$6-x_3$	−12	−13
AB	6	$6-x_1-x_2=x_3-30$	−12	−11
AC	16	$16-x_1-x_3=x_2-20$	−8	−8
BC	26	$26-x_2-x_3=x_1-10$	−4	−5

But as we do so, the excesses for coalitions B and AB increase at the same rate. Since the excess for coalition AB starts closer to −8 we find x_2 and x_3 so that

e(**x**, AB) = e(**x**, AC).

This occurs at $x_2 = 10.5$ and $x_3 = 20.5$. The nucleolus is therefore

(5, 10.5, 20.5). It is in the core.

S	$v(S)$	$e(x,S)$	$(6,12,18)$	$(5,12,19)$	$(5,10.5,20.5)$
A	0	$-x_1$	-6	-5	-5
B	0	$-x_2$	-12	-12	-10.5
C	6	$6-x_3$	-12	-13	-14.5
AB	6	$6-x_1-x_2=x_3-30$	-12	-11	-9.5
AC	16	$16-x_1-x_3=x_2-20$	-8	-8	-9.5
BC	26	$26-x_2-x_3=x_1-10$	-4	-5	-5

Note that the Shapley value for this case is in the following.

φ_A = (1/3)(0)+ (1/6)(6)+ (1/6)(10)+(1/3)(10) = 6

φ_B = (1/3)(0)+(1/6)(6)+(1/6)(20)+(1/3)(20) = 11

φ_C =(1/3)(6)+(1/6)(16)+(1/6)(26)+(1/3)(30)= 19

To define more precisely the concept of the nucleolus of a game with characteristic function v, we define an ordering on vectors that reflects the notion of smaller maximum excess as given in the previous example.

Define $\mathcal{O}(x)$ as the vector of excesses arranged in decreasing (non-increasing) order for $\{e(x, S) : S \subseteq N, S \neq N, S \neq \phi\}$.

As an example, let $\mathbf{x} = (6, 12, 18)$.

Then, $\mathcal{O}(x) = (-4, -6, -8, -12, -12, -12)$. On the vectors $\mathcal{O}(x)$ we use the lexicographic ordering.

We say a vector $\mathbf{y} = (y_1, \ldots, y_k)$ is lexicographically less than a vector $\mathbf{z} = (z_1, \ldots, z_k)$, and write $\mathbf{y} <_L \mathbf{z}$, if $y_1 < z_1$, or $y_1 = z_1$ and $y_2 < z_2$, or $y_1 = z_1$, $y_2 = z_2$ and $y_3 < z_3, \ldots$, or $y_1 = z_1, \ldots, y_{k-1} = z_{k-1}$ and $y_k < z_k$.

S	$v(S)$	$e(x,S)$	$(6,12,18)$
A	0	$-x_1$	-6
B	0	$-x_2$	-12
C	6	$6-x_3$	-12
AB	6	$6-x_1-x_2=x_3-30$	-12
AC	16	$16-x_1-x_3=x_2-20$	-8
BC	26	$26-x_2-x_3=x_1-10$	-4

The nucleolus is an imutation that minimizes $\mathcal{O}(x)$ in the lexicographic ordering.

Definition. Let (N, v) be a game in coalition form.
Let $X = \{x : \Sigma_j\, x_j = v(N),\ x_i \geqq v\{i\},\ i=1,\ldots,n\}$ be the set of imputations. We say that a vector
$\nu \in X$ is a nucleolus if for every $x \in X$ we have $\mathcal{O}(\nu) \leq_L \mathcal{O}(x)$.

Properties of the Nucleolus. The main properties of the nucleolus are stated in the following theorem.

Theorem : The nucleolus of a game in coalitional form exists and is unique. The nucleolus is **group rational**, **individually rational**, and satisfies the **symmetry axiom** and the **dummy axiom**. If the core is not empty, the **nucleolus is in the core**.

Remark: In contrast to the Shapley value, the nucleolus will be in the core provided the core is not empty.

Since the nucleolus satisfies the first three axioms of the Shapley value, it does not satisfy the additivity axiom.

We will do some examples before we sketch the proof of the Theorem.

Computation of the Nucleolus.

The nucleolus is more difficult to compute than the Shapley value.

The first step of finding the nucleolus is to find a vector

$\mathbf{x} = (x_1, \ldots, x_n)$ that minimizes the maximum of the excesses

$e(\mathbf{x}, S)$ over all S subject to

$\Sigma x_j = v(N)$.

This problem of minimizing the maximum of a collection of linear functions subject to a linear constraint is easily converted to a linear programming problem and can thus be solved by the simplex method, for example.

After this is done, one may have to solve a second linear programming problem to minimize the next largest excess, and so on.

Example: A famous Mishna in the Talmud (Baba Metzia 2a) states the following:

Two hold a garment; one claims it all, the other claims half. Then the one is awarded ¾, the other ¼.

Show that the Talmud solution is the nucleolus.

Solution:

Let A, B be the two players and A is the one who claims the whole garment. Then, the characteristic function is

v(Ø)=0, v(AB)=1, v(A)=1/2, v(B)=0. Let x=(x_1, x_2) is the payoff vector. Then,

S	$v(S)$	$e(x,S)$
A	$\frac{1}{2}$	$\frac{1}{2}-x_1$
B	0	$-x_2$

Our first guess is a pro rata payoff vector (2/3, 1/3). Then,

S	$v(S)$	$e(x,S)$	$(\frac{2}{3},\frac{1}{3})$
A	$\frac{1}{2}$	$\frac{1}{2}-x_1$	$-\frac{1}{6}$
B	0	$-x_2$	$-\frac{1}{3}$

-1/6 is the largest excess. We want to increase x_1 to make it smaller. However, we need to keep $x_1+x_2=1$. When x_1 increases, the excess $-x_2$ also increases. The best that we can do is to balance both terms. Then, we solve $x_1+x_2=1$ and $\frac{1}{2}-x_1=-x_2$.

The solutions are $x_1=3/4$, $x_2=1/4$. This is the nucleolus.

S	$v(S)$	$e(x,S)$	$(\frac{2}{3},\frac{1}{3})$	$(\frac{3}{4},\frac{1}{4})$
A	$\frac{1}{2}$	$\frac{1}{2}-x_1$	$-\frac{1}{6}$	$-\frac{1}{4}$
B	0	$-x_2$	$-\frac{1}{3}$	$-\frac{1}{4}$

The Babylonian Talmud is the compilation of ancient law and tradition set down during the first five centuries A.D. which serves as the basis of Jewish religious, criminal and civil law. One problem discussed in the Talmud is the so called marriage contract problem: a man has three wives whose marriage contracts specify that in the case of this death they receive 100, 200 and 300 respectively. The Talmud gives apparently contradictory recommendations. While the man dies leaving an estate of only 100, the Talmud recommends equal division. However, if the estate is worth 300 it recommends proportional division (50,100,150), while for an estate of 200, its recommendation of (50,75,75) is a complete mystery. This particular Mishna has baffled Talmudic scholars for two millennia. In 1985, it was recognised that the Talmud anticipates the modern theory of cooperative games. Each solution corresponds to the nucleolus of an appropriately defined game.

Example: The following is from the 2000-year old Babylonian Talmud.

Three creditors are owed debts of 100, 200, and 300. The estate has a worth of 100. The division of the estate proposed in the Talmud is (100/3, 100/3, 100/3).

Show that this is the nucleolus.

Solution: Let the creditors be A, B, C. Then, v(Ø)=0, v(A)=0, v(B)=0, v(C)=0, v(AB)=0, v(AC)=0, v(BC)=0, v(ABC)=100.

S	$v(S)$	$e(x,S)$
A	0	$-x_1$
B	0	$-x_2$
C	0	$-x_3$
AB	0	$-x_1-x_2=x_3-100$
AC	0	$-x_1-x_3=x_2-100$
BC	0	$-x_2-x_3=x_1-100$

Our first guess is a pro rata division

(100/6, 200/6, 300/6).Then, the largest excess is e(x, A)=-100/6.

We increase x_1 by getting from x_3, x_2 remains fixed. Then, we solve the equations

$-x_1=-x_3$, and $x_1+x_3=400/6$. We get $x_1=100/3$, $x_3=100/3$

S	$v(S)$	$e(x,S)$	$(\frac{100}{6},\frac{200}{6},\frac{300}{6})$
A	0	$-x_1$	$\frac{-100}{6}$
B	0	$-x_2$	$\frac{-200}{6}$
C	0	$-x_3$	$\frac{-300}{6}$
AB	0	$-x_1-x_2=x_3-100$	$\frac{-300}{6}$
AC	0	$-x_1-x_3=x_2-100$	$\frac{-400}{6}$
BC	0	$-x_2-x_3=x_1-100$	$\frac{-500}{6}$

From the last column, we see that we cannot decrease any excess without raising the other. Then, (100/3, 100/3, 100/3) is the nucleolus.

S	$v(S)$	$e(x,S)$	$(\frac{100}{6},\frac{200}{6},\frac{300}{6})$	$(\frac{100}{3},\frac{100}{3},\frac{100}{3})$
A	0	$-x_1$	$\frac{-100}{6}$	$\frac{-100}{3}$
B	0	$-x_2$	$\frac{-200}{6}$	$\frac{-100}{3}$
C	0	$-x_3$	$\frac{-300}{6}$	$\frac{-100}{3}$
AB	0	$-x_1-x_2=x_3-100$	$\frac{-300}{6}$	$\frac{-200}{3}$
AC	0	$-x_1-x_3=x_2-100$	$\frac{-400}{6}$	$\frac{-200}{3}$
BC	0	$-x_2-x_3=x_1-100$	$\frac{-500}{6}$	$\frac{-200}{3}$

Example: Given v(Ø)=0, v(A)=0, v(B)=0, v(C)=0, v(AB)=60, v(AC)=80, v(BC)=100, v(ABC)=105. Starting with allocation (20, 35, 50), find the nucleolus.

S	$v(S)$	$e(x,S)$
A	0	$-x_1$
B	0	$-x_2$
C	0	$-x_3$
AB	60	$60-x_1-x_2=x_3-45$
AC	80	$80-x_1-x_3=x_2-25$
BC	100	$100-x_2-x_3=x_1-5$

Our first guess is (20, 35, 50). Then, we have the following. The largest excess is e(x, BC)=15.

We decrease x_1 and give it to x_3, x_2 remains unchanged. We solve equations $x_1-5=x_3-45$, $x_1+x_3=70$. Thus, $x_1=15$, $x_3=55$.

S	$v(S)$	$e(x,S)$	$(20,35,50)$
A	0	$-x_1$	-20
B	0	$-x_2$	-35
C	0	$-x_3$	-50
AB	60	$60-x_1-x_2=x_3-45$	5
AC	80	$80-x_1-x_3=x_2-25$	10
BC	100	$100-x_2-x_3=x_1-5$	15

Note that the last entries of the last column tell us that we cannot make any adjustment anymore.

Therefore, (15, 35, 55) is the nucleolus. Also, note that this does not lie in the core.

S	$v(S)$	$e(x,S)$	$(20,35,50)$	$(15,35,55)$
A	0	$-x_1$	–20	–15
B	0	$-x_2$	–35	–35
C	0	$-x_3$	–50	–55
AB	60	$60-x_1-x_2=x_3-45$	5	10
AC	80	$80-x_1-x_3=x_2-25$	10	10
BC	100	$100-x_2-x_3=x_1-5$	15	10

Example: Given v(Ø)=0, v(A)=0, v(B)=0, v(C)=0, v(AB)=4, v(AC)=0, v(BC)=3, v(ABC)=6. Starting with (2, 3, 1), find the nucleolus.

S	$v(S)$	$e(x,S)$
A	0	$-x_1$
B	0	$-x_2$
C	0	$-x_3$
AB	4	$4-x_1-x_2=x_3-2$
AC	0	$-x_1-x_3=x_2-6$
BC	3	$3-x_2-x_3=x_1-3$

Our first guess is (2, 3, 1). Then, we get the following. The largest excess occurs at e(x, C), e(x, AB), e(x, BC).

For their formula, we will keep x_3 fixed, decrease x_1 and give it to x_2. We will solve $x_1-3=-x_1$. Then $x_1=1.5$, $x_2=3.5$.

S	$v(S)$	$e(x,S)$	$(2,3,1)$
A	0	$-x_1$	-2
B	0	$-x_2$	-3
C	0	$-x_3$	-1
AB	4	$4-x_1-x_2=x_3-2$	-1
AC	0	$-x_1-x_3=x_2-6$	-3
BC	3	$3-x_2-x_3=x_1-3$	-1

The largest excess is e(x,C)=e(x,AB)=-1.

From the formula for e(x,C) and e(x,AB), we see that we cannot decrease it anymore. The next largest is e(x,A)=e(x,BC)=-1.5. Again, we cannot decrease it. Once x_3, x_1 are fixed, x_2 is also fixed. Thus, (1.5, 3.5, 1) is the nucleolus. It is in the core.

S	$v(S)$	$e(x,S)$	$(2,3,1)$	$(1.5,3.5,1)$
A	0	$-x_1$	-2	-1.5
B	0	$-x_2$	-3	-3.5
C	0	$-x_3$	-1	-1
AB	4	$4-x_1-x_2=x_3-2$	-1	-1
AC	0	$-x_1-x_3=x_2-6$	-3	-2.5
BC	3	$3-x_2-x_3=x_1-3$	-1	-1.5

Example: Given v(Ø)=0, v(A)=0, v(B)=0, v(C)=6, v(AB)=6, v(AC)=16, v(BC)=26, v(ABC)=36. Starting with

(12, 12, 12) find the nucleolus.

Solution: Our initial guess is (12, 12, 12).

The largest excess is e(x, BC)=2. We decrease x_1 and give it all to x_3. We solve for e(x A)=e(x, BC) and get x_1=5.

Hence, x_3=19.

S	$v(S)$	$e(x,S)$	$(12,12,12)$
A	0	$-x_1$	-12
B	0	$-x_2$	-12
C	6	$6-x_3$	-6
AB	6	$6-x_1-x_2=x_3-30$	-18
AC	16	$16-x_1-x_3=x_2-20$	-8
BC	26	$26-x_2-x_3=x_1-10$	2

The largest excess for (5,12,19) occurs at e(x,BC) and e(x, A). However, we cannot adjust x_1 anymore. Hence, we should look at the next to the largest excess that occurs at e(x, AC). We decrease x_2 and give it to x_3. We solve for e(x,AC)=e(x,AB).

S	$v(S)$	$e(x,S)$	(12,12,12)	(5,12,19)
A	0	$-x_1$	-12	-5
B	0	$-x_2$	-12	-12
C	6	$6-x_3$	-6	-13
AB	6	$6-x_1-x_2=x_3-30$	-18	-11
AC	16	$16-x_1-x_3=x_2-20$	-8	-8
BC	26	$26-x_2-x_3=x_1-10$	2	-5

We solve $x_2-20=x_3-30$, $x_2+x_3=31$. We get $x_2=10.5$, $x_3=20.5$.

We cannot improve further.

Thus, (5, 10.5, 20.5) is the nucleolus. It is in the core.

S	$v(S)$	$e(x,S)$	$(12,12,12)$	$(5,12,19)$	$(5,10.5,20.5)$
A	0	$-x_1$	-12	-5	-5
B	0	$-x_2$	-12	-12	-10.5
C	6	$6-x_3$	-6	-13	-14.5
AB	6	$6-x_1-x_2=x_3-30$	-18	-11	-9.5
AC	16	$16-x_1-x_3=x_2-20$	-8	-8	-9.5
BC	26	$26-x_2-x_3=x_1-10$	2	-5	-5

Now we return to the proof of the Theorem.

We first recall the main statement of the Theorem in the following.

Theorem : The nucleolus of a game in coalitional form exists and is unique. The nucleolus is **group rational**, **individually rational**, and satisfies the **symmetry axiom** and the **dummy axiom**. If the core is not empty, the nucleolus **is in the core**.

Sketch of the Proof of Theorem: (proof can be omitted)

Existence: The set of imputations is a compact convex set. The function of sending an imputation x to $\mathcal{O}(x)$ is a continuous map. Then, we can prove the existence rather easily.

Uniqueness: Let x, y be two imputation such that $\mathcal{O}(x)$, $\mathcal{O}(y)$ are minimal w.r.t. $\leq_L$. Suppose $x \neq y$, we will derive a contradiction. Clearly, $\mathcal{O}(x)=\mathcal{O}(y)$. Let $S_1, S_2, \ldots, S_m$, $m=(2^n-2)$, be the coalitions that appear in $\mathcal{O}(x)$ with $e(x, S_1)$ being the first entry and $e(x, S_m)$ the last entry in $\mathcal{O}(x)$.

Note that $e(x, S_i) = e(y, S_i)$ for all i implies $x=y$.

Let k be the first positive integer such that $e(x, S_k) \neq e(y, S_k)$.

Then,

(i) $e(x, S_k) > e(y, S_k)$.

(ii) For $j>k$, $e(x, S_k) \geqq e(x, S_j)$.

(iii) For $j>k$, $e(x, S_k) \geqq e(y, S_j)$

Now consider the imputation $z=(x+y)/2$.

We have $e(z, S_1)=e(x, S_1), \ldots, e(z, S_{k-1})=e(x, S_{k-1})$.

$e(z, S_k) = e(x, S_k)/2 + e(y, S_k)/2 < e(x, S_k)$

For $j>k$, $e(z, S_j) = e(x, S_j)/2 + e(y, S_j)/2 \leqq e(x, S_k)$.

Hence, $\mathcal{O}(z) <_L \mathcal{O}(x)$. This is a contradiction. Hence, $x = y$.

The proof of uniqueness is then complete.

Symmetry: The Symmetry Axiom is a consequence of the following stronger property.

Theorem: (Individual Monotonicity): Let i, j be distinct elements in N.
Suppose that for any $S \subseteq N$ such that $i, j \notin S$, we have $v(S \cup \{i\}) \geqq v(S \cup \{j\})$.

Let $(x_1, \ldots, x_n)$ be the nucleolus allocation.
Then, $x_i \geqq x_j$.

Proof: Let $S_1, S_2, \ldots, S_m$, $m = (2^n - 2)$, be the coalitions such that $\mathcal{O}(x) = (e(x, S_1), \ldots, e(x, S_m))$.

Suppose the conclusion of the Theorem is not true i.e. $x_i < x_j$. We will then adjust the payoff to Player i and Player j to get an imputation $y=(y_1, \ldots, y_n)$ such that the vector of excess w.r.t $y = (y_1, \ldots, y_n)$ arranged in decreasing order is smaller than $\mathcal{O}(x)$. This is a contradiction.

Observation:

For T disjoint from $\{i, j\}$, $e(x, T\cup\{i\}) > e(x, T\cup\{j\})$.

Suppose k is the first positive integer such that S_k contains i or j but not both. By the observation above, S_k contains i.

Now consider an imputation $y=(y_1, \ldots, y_n)$ such that

$y_k = x_k$ for $k \neq i, j$, $y_i = x_i + t$, $y_j = x_j - t$,
where t is a small positive number.

Note that $e(x, T)=e(y, T)$ unless one and only one element of $\{i,j\}$ lies in T.

For S_k, we have $e(y, S_k) < e(x, S_k)$ for $t > 0$. We see that when t is a sufficiently small positive number $\mathcal{O}(y) \leqq_L \mathcal{O}(x)$. This is a contradiction that x is the nucleolus.

Dummy:

Suppose Player 1 is a dummy player. Let $x = (x_1, \ldots, x_n)$ be an imputation and $\mathcal{O}(x)$ is minimal w.r.t. $\leqq_L$.

Suppose $x_1 \neq 0$.

Let $S_1, S_2, \ldots, S_m$, $m = (2^n - 2)$, be the coalitions such that

$\mathcal{O}(x) = (e(x, S_1), \ldots, e(x, S_m))$.

Note that $e(x, \{1\}) = -x_1$ and $e(x, N \setminus \{1\}) = x_1$.

(i) $x_1 > 0$

For S not containing 1, $e(x, S) > e(x, S \cup \{1\})$.

Let k be the first positive integer such that $1 \in S_k$.

Clearly, $k > 1$ and $e(x, S_k) < e(x, S_1)$.

Then, set $z = (z_1, \ldots, z_n)$ such that

$z_1 = (1\text{-}t)x_1$, $z_i = x_i + tx_1/n\text{-}1$ for some t in (0,1). Then, z is an imputation.

For any $S \subseteq N$, $1 \notin S$, we have $e(z, S) < e(x, S)$.

For any $S \subseteq N$, $1 \in S$, we have $e(z, S) > e(x, S)$.

When t is sufficiently small, we have

$e(z, S_1) > \max\{e(z, S): 1 \in S\}$.

For such t, we get $\mathcal{O}(z) <_L \mathcal{O}(x)$.

This contradicts that x is the minimal element w.r.t. $\leqq_L$.

(i) $x_1<0$

Recall that $S_1, S_2, \ldots, S_m$, $m = (2^n - 2)$, are the coalitions such that

$\mathcal{O}(x) = (e(x, S_1), \ldots, e(x, S_m))$.

Note that $e(x, \{1\}) = -x_1$ and $e(x, N \setminus \{1\}) = x_1$.

For S not containing 1, $e(x, S) < e(x, S \cup \{1\})$.

Let k be the first positive integer such that $1 \notin S_k$.

Clearly, $k>1$ and $e(x, S_k) < e(x, S_1)$.

Then, set $z = (z_1, \ldots, z_n)$ such that

$z_1 = (1-t)x_1$, $z_i = x_i + tx_1/n-1$ for some t in (0,1). Then, z is an imputation.

For any $S \subseteq N$, $1 \notin S$, we have $e(z, S) > e(x, S)$.

For any $S \subseteq N$, $1 \in S$, we have $e(z, S) < e(x, S)$.

When t is sufficiently small, we have

$e(z, S_k) < \max\{e(z, S) : 1 \in S\}$.

For such t, we get

Hence, $\mathcal{O}(z) <_L \mathcal{O}(x)$. This contradicts that x is the minimal element w.r.t. $\leqq_L$.

Example: Find the Nucleolus of the weighted voting game such that N = {1, 2, 3, 4}, Player 1 has 1 vote, Player 2, 3, and 4 each has 2 votes. 4 votes to win.

Solution: Player 1 is a dummy and will get 0 in the nucleolus. Player 2, 3, 4 are symmetrical. Hence the nucleolus imputation is (0, 1/3, 1/3, 1/3).

Gately Point

Let (N, v) be a game in coalition form. Let $x = (x_1, \ldots, x_n)$ be an imputation. For $S \subseteq N$,

$e(x, S) = v(S) - \sum_{i \in S} x_i$ measure the inequality of the imputation x to the coalition S.

- $e(x, S)$ then measures the amount the coalition S will lose if they quit.

When Player i threatens to quit, the credibility of his threat is measured by the Propensity of Player i to disrupt the grand coalition defined as follows.

$$d_i = \frac{-e(x, N \setminus \{i\})}{-e(x,\{i\})}$$

d_i is the ratio of the loss of the rest of the coalition against the loss of Player i.

Definition: The Gately point is the imputation such that it minimizes max$\{d_i : 1 \leqq i \leqq n\}$among all imputations.

Therefore, the Gately point is the imputation such that

The maximum of

$$\frac{-e(x, N \setminus \{1\})}{-e(x,\{1\})}, \ldots, \frac{-e(x, N \setminus \{n\})}{-e(x,\{n\})}$$

is smallest.

Note that

$$\frac{-e(x, N\setminus\{i\})}{-e(x,\{i\})} = \left(\sum_{j\neq i} x_j - v(N\setminus\{i\})\right)/(x_i - v(\{i\}))$$

$$= (v(N) - x_i - v(N\setminus\{i\}))/(x_i - v(\{i\}))$$

$$= (v(N) - v(N\setminus\{i\}) - v(\{i\}) + v(\{i\}) - x_i)/(x_i - v(\{i\}))$$

$$= (v(N) - v(N\setminus\{i\}) - v(\{i\}))/(x_i - v(\{i\})) - 1$$

Using this expression for the propensity to disrupt, it is easy to prove the following characterization of the Gately point.

Theorem: At the Gately point we have

$$\frac{-e(x, N\setminus\{1\})}{-e(x,\{1\})} = \cdots = \frac{-e(x, N\setminus\{n\})}{-e(x,\{n\})}$$

Proof: We may assume the game is essential.

Let x be a Gately point. Then, the maximum of

$$(*) \quad \frac{-e(x, N\setminus\{1\})}{-e(x,\{1\})}, \ldots, \frac{-e(x, N\setminus\{n\})}{-e(x,\{n\})}$$

is smallest.

Suppose the assertion of the Theorem is not true. Then, terms in (*) are not all equal. Let

$\dfrac{-e(x, N\setminus\{1\})}{-e(x,\{1\})}$ be the smallest in (*).

Note that

$$\frac{-e(x, N\setminus\{i\})}{-e(x,\{i\})} = (v(N) - v(N\setminus\{i\}) - v(\{i\}))/(x_i - v(\{i\})) - 1$$

By supperadditivity ,

$v(N) - v(N\setminus\{i\}) - v(\{i\})) \geqq 0.$

Therefore, $x_1 - v(\{1\}) = \in > 0.$

Define an imputation z for $0<t<1$ such that

$z_1=x_1 - t\in$, $z_2=x_2 + t\in/(n-1)$, … , $z_n=x_n + t\in/(n-1)$.

It is clear that when t is small enough the maximum of

$\frac{-e(z, N\diagdown\{1\})}{-e(z,\{1\})}, \ldots, \frac{-e(z, N\diagdown\{n\})}{-e(z,\{n\})}$ is smaller than that for x.

This is a contradiction and hence completes the proof of the Theorem.

Example: Let N={1,2,3}. Define v such that $v(1)=v(2)=v(3)=0$, $v(1,2)=4$, $v(1,3)=0$, $v(2,3)=3$, $V(1,2,3)=6$. Find a Gately point.

Solution: We look for an imputation $x=(x_1, x_2, x_3)$ such that $d_1=d_2=d_3$ where

$$d_1 = (v(1,2,3)-v(1)-v(2,3))/(x_1 - v(1)) - 1 = (3/x_1) - 1$$

$$d_2 = (v(1,2,3)-v(2)-v(1,3))/(x_2 - v(2)) - 1 = (6/x_2) - 1$$

$$d_3 = (v(1,2,3)-v(3)-v(1,2))/(x_3 - v(3)) - 1 = (2/x_3) - 1$$

Hence, we have to search for an imputation such that

$\frac{3}{x_1} = \frac{6}{x_2} = \frac{2}{x_3}$, $x_1 + x_2 + x_3 = 6$.

We get $x_1 = 18/11$, $x_2 = 36/11$, $x_3 = 12/11$.

It is easy that this satisfies the individual rationality axiom and hence (18/11, 36/11, 12/11) is a Gately point.

Remark: Gately point is not too difficult to find. However, it may not be unique as illustrated from the following example.

Example: Let N={1, 2, 3} and v be the characteristic function corresponding to the majority voting game for N.

Then, v(1,2,3)=1, v(1)=v(2)=v(3)=0, v(1,2)=v(1,3)=v(2,3)=1.

Then,

$d_1 = d_2 = d_3 = -1$ for any imputation x. In other words, all imputations are Gately point.

Remark: In early 1940's, engineers in the Tennessee Water Project already had the concept of Gately point.

Game Theory and the Tennessee Valley Authority

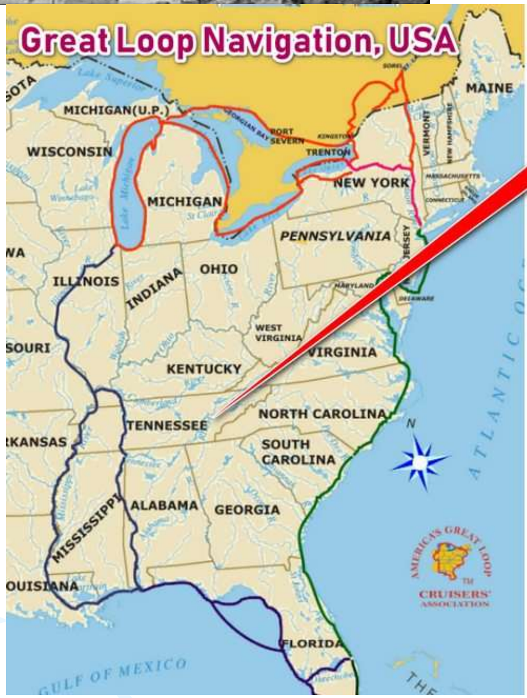
By P.D. Straffin, J.P. Heaney (1980)

The TVA was a response to the Great Depression. It was a major regional development project created by an Act of US Congress to stimulate activity in the mid-south US.

The major purposes to be served were improved navigation, flood control, provision of electric power.

TVA Act mandated that the cost of the projects be specifically allocated to the purposes involved.

TVA engineers and consultants devoted considerable time and thoughts to a number of cost allocation methods in the late 1930's. These methods foreshadowed modern ideas in Game Theory.



Tennessee valley:
Dams & locks help inland navigation

Cost Allocation Problem:

Cost-sharing Game:

$N=\{1,\ldots n\}$= set of n purposes among which costs be allocated.

For any subset S of N, C(S)= the cost of a project designed for the purposes in S only.

C(i) is called the Alternate Cost = cost of an alternate project to serve purpose i only.

We assume subaditivity for the cost function C.

Let S, T be disjoint. Then, subaditivity says

$$C(S \cup T) \leq C(S) + C(T).$$

Remark: C(S) is not a characteristic function for a game in coalition form.

We will like to find a cost allocation vector $(c_1,\ldots c_n)$ where c_i is the cost allocated to the ith purpose.

Clearly, we assume

$$c_1 + \ldots + c_n = C(N),\ c_i \geq 0.$$

TVA engineers figured out the core principle for cost allocation.

Ransmeier (1942)

The method should have a reasonable logical basis. It should not result in charging any objective with a greater investment than would suffice for its development at an alternate single purpose site. ($c_i \leq C(i)$). Finally, it should not charge any two or more objectives with a greater investment than would suffice for alternate dual or multiple purpose investment. ($\sum_{i \in S} c_i \leq C(S)$)

The TVA engineers proposed many methods.

1. They will charge each project its separable cost ($s_i = C(N) - C(N\text{-}i)$).

2. The remaining part called the non-separable cost

($NSC = C(N) - (s_1 + \ldots + s_n)$) will be shared by some methods.

Tennessee Valley Water Project:

1 = Navigation, 2 = Flood control, 3 = Power

N={1, 2, 3}

Subset S	C(S) (cost in thousands of dollars)
∅	0
{1}	163,520
{2}	140,826
{3}	250,096
{1,2}	301,607
{1,3}	378,821
{2,3}	367,370
{1,2,3}	412,584

s_1=412,584 − 367,370=45,214
s_2=412,584 − 378,821=33,763
s_3=412,584 − 301,607=110,977

Total separable cost=45,214 + 33,763 + 110,977=189,954

NSC=412,584 − 189,954= 222,630

Cost allocation for Purpose i = s_i + i's share of NSC

How to share NSC (222,630)?

There are two methods to share the NSC.

I. Equal Allocation of NSC:

$$c_i = s_i + NSC/n$$

II. Alternate Cost Avoided Method (ACA, Martin Glaeser 1938)

ACA method is to assign NSC to each purpose in proportion to the

Alternate Cost Avoided by Purpose i: $C(i) - s_i$

Total Alternate Cost Avoided = $\sum_j (C(j) - s_j)$

ACA Method: $c_i = s_i + [(C(i) - s_i)/ \sum_j (C(j) - s_j)]$ NSC

This is exactly the Gately Point for an appropriate characteristic function.

Therefore, TVA engineers already had the idea of modern Game Theory almost half century ago.