Games in Coalitional Form

Many-Person TU Games

In many-person **cooperative games**, there are no restrictions on the agreements that may be reached among the players.

In addition, we assume that all payoffs are measured in the same units and that there is a **transferable utility** which allows **side payments** to be made among the players.

Side payments may be used as inducements for some players to use certain mutually beneficial strategies. Thus, there will be a tendency for players, whose objectives in the game are close, to form alliances or coalitions.

The structure given to the game by coalition formation is conveniently studied by reducing the game to a form in which coalitions play a central role.

第 一 回 宴桃園豪傑三結義 斬黃巾英雄首立功

話說天下大勢，分久必合，合久必分。周末七國分爭，並入於秦。及秦滅之後，楚漢分爭，又並入於漢。漢朝自高祖斬白蛇而起義，一統天下。後來光武中興，傳至獻帝，遂分爲三國。推其致亂之由，殆始於桓靈二帝。桓帝禁錮善類，崇信宦官。及桓帝崩，靈帝即位，大將軍竇武，太傅陳蕃，共相輔佐。時有宦官曹節等弄權，竇武、陳蕃謀誅之，作事不密，反爲所害。中涓自此愈橫。

建寧二年四月望日，帝御溫德殿，方陞座，殿角狂風驟起，只見一條大青蛇，從梁上飛將下來，蟠於椅上。帝驚倒，左右急救入宮，百官俱奔避。須臾，蛇不見了，忽然大雷大雨，加以冰雹，落到半夜方止，壞卻房屋無數。建寧四年二月，洛陽地震；又海水泛溢，沿海居民，盡被大浪捲入海中。光和元年，雌雞化雄。六月朔，黑氣十餘丈，飛入溫德殿中。秋七月，有虹見於玉堂；五原山岸，盡皆崩裂。種種不祥，非止一端。

帝下詔問羣臣以災異之由，議郎蔡邕上疏，以爲蜺墮雞化，乃婦寺干政之所致，言頗切直。帝覽奏歎息，因起更衣。曹節在後竊視，悉宣告左右；遂以他事陷邕於罪，放歸田里。後張讓、趙忠、封諝、段珪、曹節、侯覽、蹇碩、程曠、夏惲、郭勝十人朋比爲奸，號爲『十常侍』。帝尊信張讓，呼爲『阿父』，朝政日非，以致天下人心思亂，盜賊蜂起。

時鉅鹿郡有兄弟三人：一名張角，一名張寶，一名張梁。那張角本是個不第秀才。因入山採藥，遇一老人，碧眼童顏，手執藜杖，喚角至一洞中，以天書三卷授之曰：『此名太平要術。汝得之，當代天宣化，普救世人；若萌異心，必獲惡報。』角拜問姓名。老人曰：『吾乃南華老仙

金瓶梅

第一回

西門慶熱結十弟兄

Coalitional Form. Characteristic Functions.

Let $n \geq 2$ denote the number of players in the game, numbered from 1 to n, and let N denote the set of players,

$N = \{1, 2, . . . , n\}$.

A **coalition**, S, is defined to be a subset of N,

$S \subseteq N$, and the set of all coalitions is denoted by 2^N.

We also speak of the empty set, $\emptyset$, as a coalition, the **empty coalition**. The set N is also a coalition, called the **grand coalition**.

Example:

$N=\{1,\ldots,n\}$

If there are just two players, $n = 2$, then there are four coalition, $\{\emptyset, \{1\}, \{2\}, N\}$.

If there are 3 players, there are 8 coalitions, $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, N\}$.

For n players, the set of coalitions, 2^N, has 2^n elements.

Definition. The **coalitional form** of an n-person game is given by the pair (N, v),

where $N = \{1, 2, \ldots, n\}$ is the set of players and v is a real-valued function, called the **characteristic function** of the game, defined on the set, 2^N, of all coalitions (subsets of N), and satisfying

(i) $v(\emptyset) = 0$, and

(ii) (superadditivity) if S and T are disjoint coalitions

$(S \cap T = \emptyset)$, then

$v(S) + v(T) \leq v(S \cup T)$.

Remark:

The quantity $v(S)$ is a real number for each coalition $S \subseteq N$, which may be considered as the value, or worth, or power, of coalition S when its members act together as a unit.

Condition (i) says that the empty set has value zero, and (ii) says that the value of two disjoint coalitions is at least as great when they work together as when they work apart.

Note that many concepts in cooperative games do not require the assumption of super-additivity.

Example: A jazz band game

A singer (S), a pianist (P), and a drummer (D) are offered \$1,000 to play together by a night club owner. The owner would alternatively pay \$800 the singer-piano duo, \$650 the piano-drums duo, and \$300 the piano alone. Moreover, the singer-drums duo makes \$500 a night in one well located subway station, and the singer alone gets on average \$200 a night in a bar. The drum alone can make no profit.

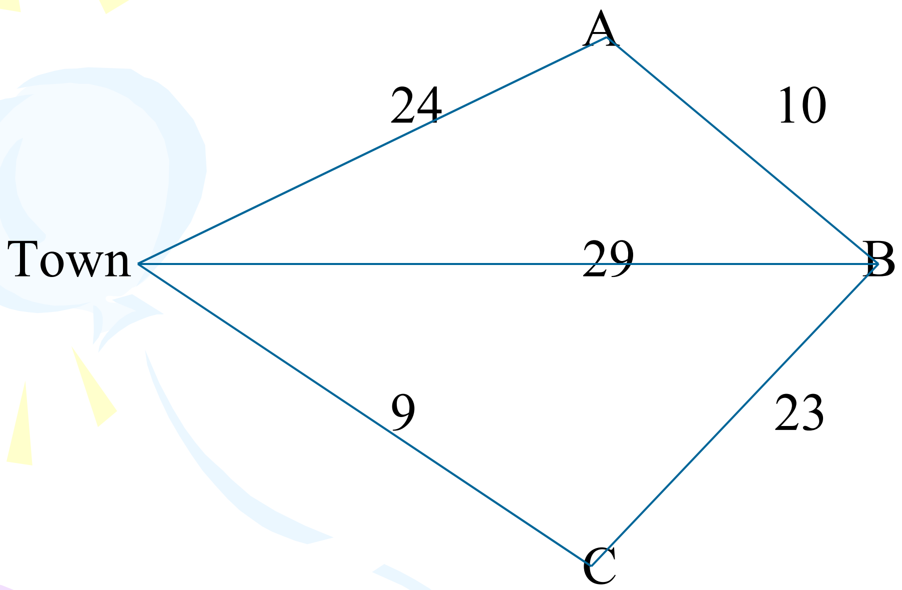
$N=\{S,P,D\}$, $v(\varnothing)=0$, $v(N)=1{,}000$

$v(\{S,P\})=800$, $v(\{P,D\})=650$, $v(\{S,D\})=500$,

$v(\{S\})=200$, $v(\{P\})=300$, $v\{\{D\})=0$

Example: Road Game

Three new farms require road access to a nearby town. The possible routes along which road segments can be built are shown in the following, together with their estimated costs. The benefits of being connected are assumed to be sufficiently great that each farmer would surely want to build a road for his own use, even if the others did not participate.



N ={A,B,C},v(Ø)=0, v(A)= -24, v(B)= -29, v(C)= -9,

V(A,B)= -34, v(B,C)= -32, v(A,C)= -33, v(N)= -42.

Example: Oil Market game

Country 1 has oil which it can use to run its transport system at a profit of a per barrel. Country 2 wants to buy the oil to use in its manufacturing industry, where it gives a profit of b per barrel, while Country 3 wants it for food manufacturing where the profit is c per barrel. $a<b\leq c$.

The characteristic function is:

$v(\emptyset)=0$,

$v(1)=a$, $v(2)=v(3)=v(2,3)=0$

$v(1,2)=b$, $v(1,3)=v(1,2,3)=c$.

Example (Glove Game): Player 1 and 2 possesses a left glove, Player 3 a right glove. A pair (left and right) has value 1, one glove has value 0.

Characteristic function:

$$v(\emptyset)=0,$$

$$v(1)=v(2)=v(3)=v(1,2)=0$$

$$v(1,3)=v(2,3)=v(1,2,3)=1.$$

Example: (An Assignment Game) Two house owners, A and B, are expecting to sell their houses to two potential buyers, C and D, each wanting to buy one house at most. Players A and B value their houses at 10 and 20 respectively, in some unspecified units. In the same units, Player C values A's house at 15 and B's house at 22, while Player D values A's house at 17 and B's house at 26.

Determine a characteristic function for the game.

Solution: First note that there is no value if buyers and sellers are not matched together.

Hence, v(A)=v(B)=v(A, B)=v(C)=v(D)=v(C,D)=0.

If A and C are matched, A will sell the house to C at his asking price of 10. C realizes a profit of 5. Hence, v(A, C)=5.

Similarly, v(A, D)=7, v(B, C)=2, v(B, D)=6.

When A, B and C form a coalition, C will buy the house from A getting a profit of 5.

v(A,B,C)= 5, Similarly, v(A,B,D)=7.

When A, C and D form a coalition, D gets to buy the house from A getting a profit of 7.

Hence, v(A,C,D)=7, v(B,C,D)=6.

When, A, B, C, D form a coalition, the maximum profit is achieved by C buying a house from A and D buying a house from B. The total profit is 5+6=11.

Thus, v(A,B,C,D)=11.

Example: Bankruptcy Game

A bankruptcy problem is an ordered pair (E, d), where $d=(d_1, d_2, \ldots, d_n)$ such that $d_i \geqq 0$, for all i,

and $0 \leqq E < d_1 + \ldots + d_n = D$

E is the estate which has to be divided among n creditors denoted by {1, …, n}, d_i is the claim of creditor i.

Based on (E, d), we define a characteristic function on subsets of N.

For any $S \subseteq N$, let

$d(S)=\sum_{i \in S} d_i$, total claim from creditors in S.

The marginal right m_i of each creditor i to be the amount that is not claimed by others, so

$m_i = (E-d(N \setminus \{i\}))_+ = \max\{E-d(N \setminus \{i\}), 0\}$.

Similarly, for any $S \subseteq N$ we define $v(S) = (E-d(N \setminus S))_+$.

Clearly, $v(\phi)=0$. We have to verify superadditivity.

Let S, T be two disjoint subsets of N.

We want to prove

(*) $v(S) + v(T) \leqq v(S \cup T)$ for $S \cap T = \phi$.

Proof：(The proof is straightforward. It can be omitted.)

Note that, $v(S \cup T) = (E - d(N \setminus (S \cup T)))_{+}$

$v(S) = (E - d(N \setminus S))_{+} = (E - d(N \setminus (S \cup T)) - d(T))_{+}$

$v(T) = (E - d(N \setminus T))_{+} = (E - d(N \setminus (S \cup T) - d(S))_{+}$

If either v(S) or v(T) equals to 0 then (*) is valid trivially.

Suppose both are not 0.

Then,

$v(S)=(E-d(N\setminus S))_+ = E - d(N\setminus(S\cup T)) - d(T)$

$v(T)=(E-d(N\setminus T))_+ = E - d(N\setminus(S\cup T) - d(S)$

$v(S)+v(T)=2(E - d(N\setminus(S\cup T)) - d(T) - d(S).$

Then, (*) follows from our assumption that

$E< d(N)= d(N\setminus(S\cup T)) + d(T) + d(S).$

This completes the proof that (N, v) is a game in coalition form, called the Bankruptcy Game.

Relation to Strategic Form.

Transforming a game from strategic form to coalitional form entails specifying the value, $v(S)$, for each coalition $S \in 2^N$.

We define $v(S)$ for each $S \in 2^N$ as the **value** of the 2-person zero-sum game obtained when the coalition S acts as one player and the complementary coalition, $S^C = N - S$, acts as the other player, and where the payoff to S is

$\sum_{i \in S} u_i(x_1, \ldots, x_n)$, the total of the payoffs to the players in S .

Remark: The value, v(S), is the analogue of the safety level. It represents the total amount that coalition S can guarantee for itself, even if the members of S^C gang up against it, and have as their only object to keep the sum of the payoffs to members of S as small as possible.

This is a lower bound to the payoff S should receive because it assumes that the members of S^C ignore what possible payoffs they might receive as a result of their actions.

To see that v is a characteristic function, note that Condition (i) holds, since the empty sum is zero.

To see that (ii) holds, note that if **s** is a set of strategies for S that guarantees them v(S), and **t** is a set of strategies for T that guarantees them v(T), then the set of strategies (**s**,**t**) guarantees $S \cup T$ at least $v(S) + v(T)$. Perhaps other joint strategies can guarantee even more, so certainly,

$v(S \cup T) \geq v(S) + v(T)$.

Example: Let [A, B] be a bimatrix game between Player I and II. Reduce it to coalition form and find its characteristic function.

Solution: $N=\{I, II\}$, $v(\varnothing)=0$, $v(N)=\text{Max}\{a_{ij}+b_{ij}\}$,

$v(I)=\text{Val}(A)=$ Safety level of I

$v(II)=\text{Val}(B^T)=$Safety level of II

Example: This is an example of reducing to coalition form of a three-person game with Players I, II, and II with two pure strategies each.

If I chooses 1:

		III: 1	III: 2
II	1	(0,3,1)	(2,1,1)
	2	(4,2,3)	(1,0,0)

If I chooses 2:

		III: 1	III: 2
II	1	(1,0,0)	(1,1,1)
	2	(0,0,1)	(0,1,1)

$v(\mathrm{I}) = ?$

II & III

		1,1	1,2	2,1	2,2
I	1	0	$\cancel{2}$	$\cancel{4}$	1
	2	1	$\cancel{1}$	$\cancel{0}$	0

$v(\mathrm{I}) = \frac{1}{2}$

Example: This is an example of reducing to coalition form of a three-person game with Players I, II, and II with two pure strategies each.

If I chooses 1:

$$
\begin{array}{cc|cc|}
 & & \multicolumn{2}{c}{\text{III}} \\
 & & 1 & 2 \\
\hline
\text{II} & 1 & (0,3,1) & (2,1,1) \\
 & 2 & (4,2,3) & (1,0,0) \\
\hline
\end{array}
$$

If I chooses 2:

$$
\begin{array}{cc|cc|}
 & & \multicolumn{2}{c}{\text{III}} \\
 & & 1 & 2 \\
\hline
\text{II} & 1 & (1,0,0) & (1,1,1) \\
 & 2 & (0,0,1) & (0,1,1) \\
\hline
\end{array}
$$

$v(\text{I}, \text{III}) = ?$

$$
\begin{array}{cc|cc|}
 & & \multicolumn{2}{c}{\text{II}} \\
 & & 1 & 2 \\
\hline
 & 1,1 & 1 & 7 \\
\text{I}\,\&\,\text{III} & 1,2 & 3 & 1 \\
 & 2,1 & \cancel{1} & \cancel{1} \\
 & 2,2 & \cancel{2} & \cancel{1} \\
\hline
\end{array}
$$

$v(\text{I}, \text{III}) = \frac{5}{2}$

Theorem: Given any game in coalition form (N, v), one can find a strategic form game whose reduction to coalition form has v as its characteristic function.

Proof: (proof can be omitted)One way of constructing a strategic form game whose reduction to coalitional form has a given characteristic function, v, is as follows.

The strategy space X_i for Player i is taken to be the set of all coalitions that contain i:

$X_i = \{S \in 2^N : i \in S\}$. Then the strategies for Player i is to players to form coalitions.

Payoff to Player i is the minimum amount, $v(\{i\})$, unless all members of the coalition, S_i, chosen by Player i, choose the same players to form coalition as player i has, in which case the coalition S_i is given its value $v(S_i)$ which it then splits among its members evenly.

Thus the payoff function u_i is

$u_i\ (S_1, \ldots, S_n) = v(S_i)/|S_i|$ if $S_j = S_i$ for all $j \in S_i$

$u_i(S_1, \ldots, S_n) = v(\{i\})$ otherwise,

where $|S_i|$ represents the number of members of the coalition S_i.

Clearly, a coalition S can guarantee itself v(S) simply by having each member of S select S as his coalition of choice. Moreover, since v is superadditive, the coalition S cannot guarantee more for itself by having its members form subcoalitions.

This completes the proof.

Imputations and the Core

In cooperative games, it is to the joint benefit of the players to form the grand coalition, N, since by superadditivity the amount received, v(N), is as large as the total amount received by any disjoint set of coalitions they could form.

It is reasonable to suppose that "rational" players will agree to form the grand coalition and receive v(N). The problem is then to agree on how this amount should be split among the players.

One of the possible properties of an agreement on a fair division, that it be stable in the sense that

no coalition should have the desire and power to upset the agreement.

Such divisions of the total return are called points of the core, a central notion of game theory in economics.

Imputations.

A payoff vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ means that player i is to receive x_i.

The first desirable property of an imputation is that the total amount received by the players should be v(N).

Definition. A payoff v ector, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, is said to be **group rational** or **efficient** if

$x_1+\ldots+x_n = v(N)$.

No player could be expected to agree to receive less than that player could obtain acting alone.

Definition. A payoff vector, **x**, is said to be **individually rational** if $x_i \geq v(\{i\})$

for all $i = 1, \ldots, n$.

Definition. An **imputation** is a payoff vector that is group rational and individually rational. The set of imputations may be written

$\{\mathbf{x} = (x_1, \ldots, x_n) : \sum_{i \in N} x_i = v(N)$, and $x_i \geq v(\{i\})$ for all $i \in N\}$.

Remark: The set of imputations is a convex set.

Theorem: The set of all imputations is a nonempty convex set.

Proof:

Note that $v(\{1\}) + \ldots + v(\{n\}) \leq v(N)$.

The set of imputations is then the convex hull of the n points X_i , $i \in \{1,\ldots,n\}$, defined as follows.

The j^{th} –component of X_i equals to $v(\{j\})$ for $j \neq i$,
the i^{th} –component of X_i equals to $v(N) - \sum_{j \neq i} v(\{j\})$.

Example: Consider the game with characteristic function v given by

$v(\emptyset)=0$, $v(1)=1$, $v(2)=0$, $v(3)=1$,

$v(1,2)=4$, $v(1,3)=3$, $v(2,3)=5$,

$v(1,2,3)=8$.

Then, the set of imputations is the set

$\{(x_1,x_2,x_3): x_1+x_2+x_3=8,\ x_1\geq 1,\ x_2\geq 0,\ x_3\geq 1.\}$

This is a triangle in 3-space with vertices (7,0,1), (1,6,1), (1,0,7).

The Core.

If there exists a coalition, S, whose total return from **x** is less than what that coalition can achieve acting by itself, that is, if $\Sigma_{i \in S} x_i < v(S)$, then there will be a tendency for coalition S to form and upset the proposed **x.** Such an imputation has an inherent instability.

Definition. An imputation **x** is said to be **unstable through a coalition** S if

$v(S) > \Sigma_{i \in S} x_i$. We say **x** is **unstable** if there is a coalition S such that **x** is unstable through S, and we say **x** is **stable** otherwise.

Definition. The set, C, of stable imputations is called the **core**,

$$C = \{\mathbf{x} = (x_1, \ldots, x_n) : \Sigma_{i \in N}\, x_i = v(N),\ \Sigma_{i \in S}\, x_i \geq v(S), \text{ for all } S \subset N\}.$$

The core is convex and can consist of many points; but the core can also be empty. It may be impossible to satisfy all the coalitions at the same time. One may take the size of the core as a measure of stability, or of how likely it is that a negotiated agreement is prone to be upset.

Example: Give the following bimatrix game, find the core.

$$\begin{pmatrix} (2,6) & (10,5) \\ (4,8) & (0,0) \end{pmatrix}$$

Solution: v(Ø)=0, v(1,2)=15, v(1)=10/3, v(2)=6.

The core is the set in the plane

$\{(x_1,x_2)\colon x_1+x_2=15,\ x_1\geq 10/3,\ x_2\geq 6.\}$.

This is the segment joining (10/3, 35/3) and (9,6).

Example: Consider the game with characteristic function v given by

$v(\emptyset)=0$, $v(1)=1$, $v(2)=0$, $v(3)=1$,

$v(1,2)=4$, $v(1,3)=3$, $v(2,3)=5$,

$v(1,2,3)=8$.

Find its core.

Solution:

The set of imputations is

all (x_1,x_2,x_3) such that $x_1+x_2+x_3=8$,

and $x_1 \geqq 1$, $x_2 \geqq 0$, $x_3 \geqq 1$

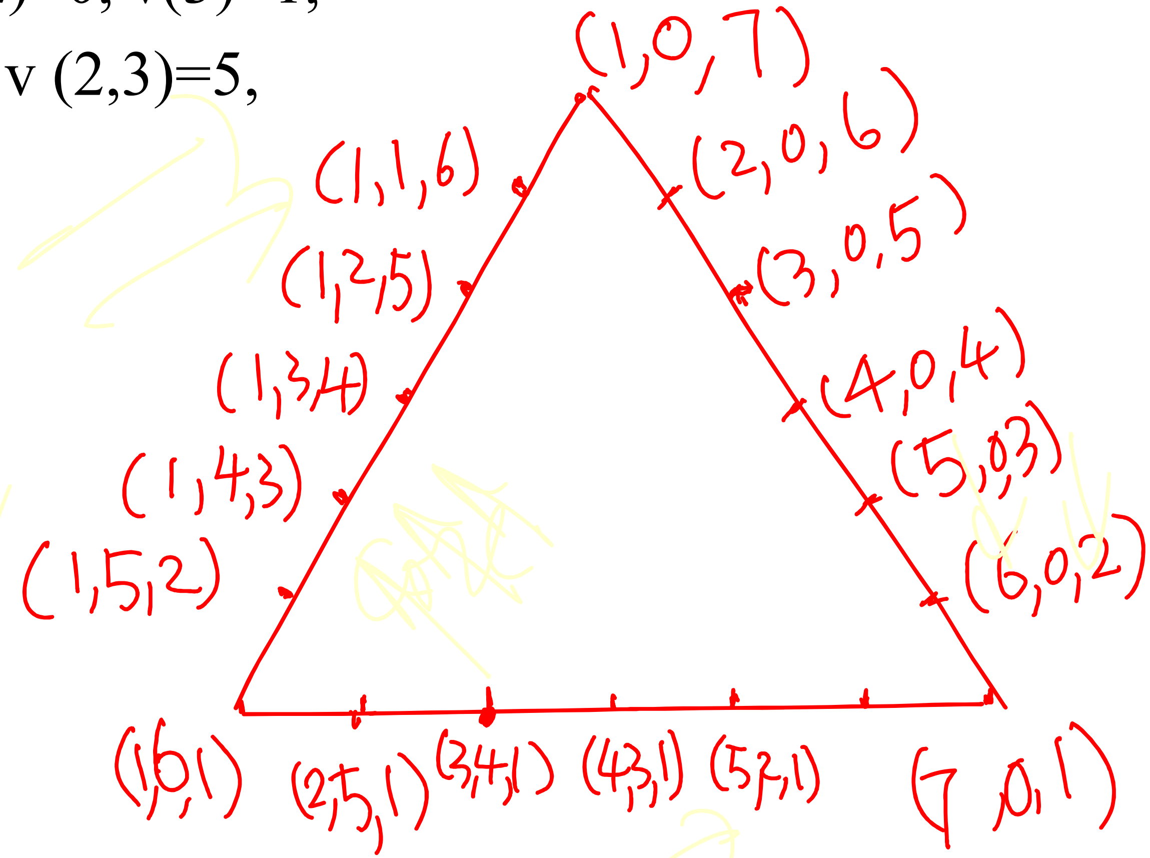
Example: Consider the game with characteristic function v given by

$v(\emptyset)=0$, $v(1)=1$, $v(2)=0$, $v(3)=1$,

$v(1,2)=4$, $v(1,3)=3$, $v(2,3)=5$,

$v(1,2,3)=8$.

Find its core.



Example: Consider the game with characteristic function v given by

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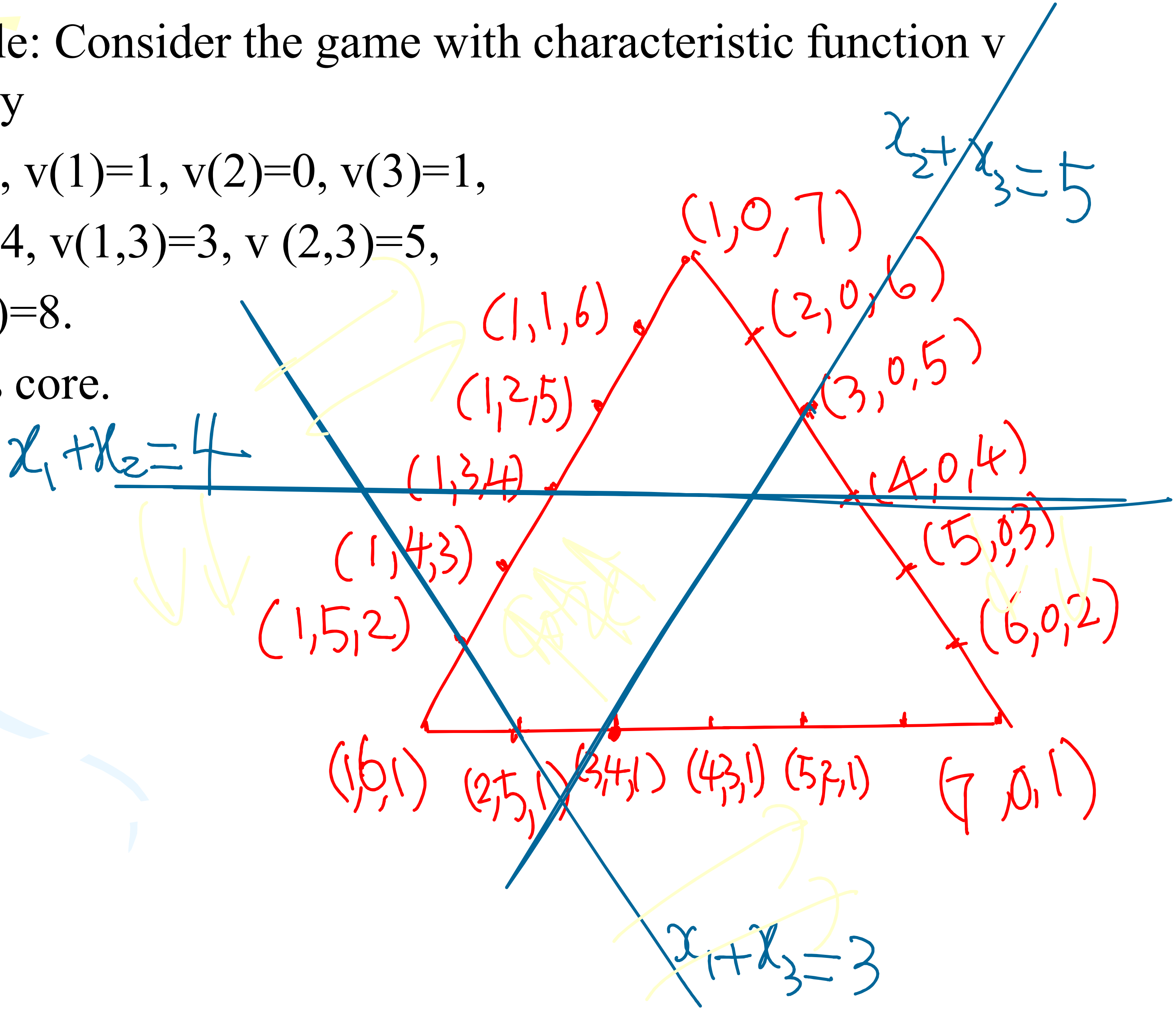
v(1,2,3)=8.

Find its core.



Example: A certain object d'art is worth a_i dollars to Player i for i=1, 2, 3. We assume $a_1<a_2<a_3$, so Player 3 values the object most.

But Player 1 owns this object so $v(1)=a_1$. Player 2 and 3 by themselves can do nothing, so $v(2)=0$, $v(3)=0$, and $v(2,3)=0$.

If Player 1 and 2 come together, the joint worth is a_2, so $v(1,2)=a_2$.

Similarly, $v(1,3)=a_3$. If all three get together, the object is still only worth a_3, so $v(N)=a_3$.

Let us find the core of this game.

The core consists of all (x_1,x_2,x_3) satisfying

$x_1 \geq a_1$, $x_2 \geq 0$, $x_3 \geq 0$,

$x_1+x_2 \geq a_2$, $x_1+x_3 \geq a_3$, $x_2+x_3 \geq 0$,

$x_1+x_2+x_3=a_3$.

Then, $x_2=a_3-x_1-x_3 \leq 0$. Thus, $x_2=0$. Then, $x_1 \geq a_2$, $x_1+x_3=a_3$.

The core is then the set $\{(x,0,a_3-x): a_2 \leq x \leq a_3\}$.

Therefore, the object will be purchased by Player 3 from Player 1 at a prize x between a_2 and a_3. Player 1 will get x dollars. Player 2 plays no active role in this, but his presence jacks up the price above a_2.

Constant-Sum Games.

Definition. A game in coalitional form is said to be **constant-sum**, if $v(S)+v(S^C) = v(N)$

for all coalitions $S \in 2^N$. It is said to be **zero-sum** if, in addition, $v(N) = 0$.

Example: The coalition form of an n-person zero-sum game is a zero-sum game in coalition form.

Essential Games.

Definition. A game in coalitional form is said to be **inessential** if

Σ_i v({i}) =v(N), and **essential** if Σ_i v({i}) ≠v(N).

If a game is inessential, then the unique imputation is **x** = (v({1}), . . . , v({n})), which may be considered the "solution" of the game.

Theorem 1. The core of an essential n-person constant-sum game is empty.

Proof. Let x be an imputation.

Since the game is essential, we have $\Sigma_{i \in N} v(\{i\}) < v(N)$.

Then there must be a player, k, such that $x_k > v(\{k\})$, for otherwise

$v(N) = \Sigma_{i \in N} x_i \leq \Sigma_{i \in N} v(\{i\}) < v(N)$.

Since the game is constant-sum, we have $v(N - \{k\}) + v(\{k\}) = v(N)$.

But then, x must be unstable through the coalition $N - \{k\}$, because

$\Sigma_{i \neq k} x_i = \Sigma_{i \in N} x_i - x_k < v(N) - v(\{k\}) = v(N - \{k\})$.

We will introduce a class of games with nonempty core.

Definition: A game in coalition form (N, v) is convex iff for any coalitions $S, T \subseteq N$,

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T)$$

We first show that the core of a convex game is nonempty by displaying explicitly an imputation that lies in the core.

Theorem: Let (N, v) be a convex game.

Denote N={1, … ,n}. Define

$x_1 = v(1)$

$x_2 = v(1,2) – v(1)$

$\vdots$

$\vdots$

$x_n = v(1,2,\ldots,n) – v(1,2,\ldots,n-1)$

Then, $x = (x_1,\ldots,x_n)$ is an imputation and is in the core of (N, v). Consequently, the core of a convex game is nonempty.

Remark: The formula

$x_1 = v(1)$

$x_2 = v(1,2) – v\ (1)$

$\vdots$

$x_n = v(1,2,\ldots,n) – v(1,2,\ldots,n\text{-}1)$

represents the marginal contribution of each player when they join the coalition one by one according to the order [123…n].

Players can join the coalition in different order.

Let π be a permutation of $\{1,2,\ldots,n\}$. Then,

Let x^{π} be the imputation representing the marginal contributions of each player when they join the coalition one by one according to the order

$[\pi(1)\pi(2)\ \pi(3)\ldots\pi(n)]$.

Example: Given $v(\emptyset)=0$, $v(1)=1$, $v(2)=0$, $v(3)=1$, $v(1,2)=4$, $v(1,3)=3$, $v(2,3)=5$, $v(1,2,3)=8$.

Then, $x = (1, 3, 4)$. For the permutation

$\pi(1)=2$, $\pi(2)=1$, $\pi(3)=3$. We have $x^{\pi}=(4, 0, 4)$.

Proof of Theorem: We first show that $x = (x_1,\ldots,x_n)$ is an imputation.

(i) Efficiency: $\sum_i x_i = v(N)$ by definition of x.

(ii) Individual Rationality: By superadditivity of characteristic function,

$v(1,\ldots, i\text{-}1) + v(i) \leqq v(1,\ldots,i)$

Hence, $v(i) \leqq x_i$.

This completes the proof that x is an imputation.

Then, we proceed to show x is in the core.

First, we prove the following Lemma.

Lemma: For a convex game (N, v), let $S \subseteq T$. Let $R \cap T = \phi$. Then,

$$v(S \cup R) - v(S) \leqq v(T \cup R) - v(T)$$

Remark: For a convex game, marginal contribution will be bigger by joining a "larger" coalition.

Proof:

$$v(S \cup R) + v(T) \leqq v(S \cup R \cup T) + v((S \cup R) \cap T)$$
$$= v(T \cup R) + v(S).$$

Hence, $v(S \cup R) - v(S) \leqq v(T \cup R) - v(T)$.

For the inequality

$$v(S \cup R) - v(S) \leqq v(T \cup R) - v(T),$$

we let $S = \{i_1, \ldots, i_{j-1}\}$, $i_1 < \ldots < i_{j-1} < i_j$,

$T = \{1, 2, \ldots, i_j - 1\}$, $R = \{i_j\}$.

Then, we get

$$v(i_1, \ldots, i_j) - v(i_1, \ldots, i_{j-1}) \leqq v(1, \ldots, i_j) - v(1, \ldots, i_j - 1)$$

Let $S = \{i_1, \ldots, i_k\}$, $i_1 < \ldots < i_k$. We then compute

$$x(S) = \sum_{i \in S} x_i$$
$$= [v(1,\ldots,i_1) - v(1, \ldots, i_1\text{-}1)] + [v(1,\ldots,i_2) - v(1,\ldots i_2\text{-}1)]$$
$$+ \ldots + [v(1,\ldots i_k) - v(1,\ldots,i_k\text{-}1)]$$

For a general term in the square bracket in the above, we have shown

$$v(1,\ldots,i_j) - v(1,\ldots,i_j - 1) \geqq v(i_1,\ldots,i_j) - v(i_1,\ldots,i_{j-1})$$

Thus,

$$x(S) = \sum_{i \in S} x_i$$

$$\geqq [v(\{i_1\}) - v(\phi)] + [v(\{i_1, i_2\}) - v(\{i_1\})] + \ldots$$

$$+[\ v(\{i_1, \ldots, i_j\}) - v(\{i_1, \ldots, i_{j-1}\})] + \ldots +$$

$$+[\ v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\})]$$

$$= v(\{i_1, \ldots, i_k\}) = v(S)$$

This completes the proof of the stability condition. Hence, x lies in the core of (N, v).

Example: Production Game

- Player: $N=\{1,2,\ldots,n\}$
- Commodity: $C=\{C_1,\ldots,C_q\}$
- Goods: $G=\{G_1,\ldots,G_m\}$
- Player i holds b_{i1} quantity of C_1, … ,b_{iq} quantity of C_q
- The commodities are worthless in themselves, except that they can be used to produce goods $G_1,\ldots,G_m$, which can then be sold at fixed market prices.
- To produce one unit of G_l it requires

  a_{il} unit of C_1, … , a_{ql} unit of C_q.
- One unit of G_j can be sold for a price of p_j dollars.

When a coalition $S \subseteq N$ forms, its members will pool their commodities to produce x_i unit of G_i, $1 \leqq i \leqq m$, so as to maximize the total market sale of their products.

Thus the characteristic function $v(S)$ is given by the solution of the following linear programming problem.

Maximize $\sum_i p_i x_i$

Subject to

$$\sum_i a_{ki} x_i \leqq b_k(S),\ k=1,\ \ldots,q$$

$$x_i \geqq 0,\ \ i=1,\ \ldots\ ,m.$$

where $b_k(S)=\sum_{j\in S} b_{jk}$, $k=1,\ \ldots,q$

It is then easy to see that (N, v) is a game in coalition form.

Claim: (N, v) is a convex game.

Proof of claim: (proof can be omitted)

Let S, T$\subseteq$N. It suffices to check the case when $S \cap T \neq \phi$. Without loss of generality, we assume $S \cup T=\{1,\ldots,r\}$, $S=\{1,\ldots,h\}$, $T=\{t,\ldots,r\}$, $S \cap T=\{t+1,\ldots,h\}$.

Let the solution for $v(S)$ to be $(x_1,\ldots,x_h)$, the solution for $v(T)$ to be $(y_t,\ldots,y_r)$.

Then,

$v(S) + v(T) = \sum_i p_i x_i + \sum_j p_j y_j$.

Note that $(x_1,\ldots,x_h, y_{h+1},\ldots,y_t)$ satisfies the constraints for the linear programming problem for $v(S\cup T)$.Thus,

$(*)\ v(S\cup T) \geqq (p_1x_1 +\ldots+ p_hx_h) + (p_{h+1}y_{h+1} +\ldots+ p_ty_t)$

Also, $(y_t,\ldots,y_h)$ satisfies the constraints for the linear programming problem for $v(S\cap T)$. Thus,

$(**)\ v(S\cap T) \geqq (p_ty_t +\ldots+ p_hy_h)$.

Combining (*) and (**), we get

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T),$$

the condition for a convex game.

Theorem: The core of the production game is nonempty.

Remark: The Market games, introduced by Shapley and Shubik is a more elaborated model of production game. It is also a convex game and hence has a nonempty core.

Theorem: A bankruptcy game is a convex game.

Proof: (proof can be omitted) Let (N, v) be a bankruptcy game derived from the bankruptcy problem (E, d), where

$d=(d_1,\ldots,d_n)$, $E < (d_1+\ldots+d_n)$.

Let S, T $\subseteq$N. Then,

$v(S) = (E-d(N\setminus S))_+$

$v(T) = (E-d(N\setminus T))_+$

$v(S\cup T) = (E-d(N\setminus S\cup T))_+$

$= (E-d(N\setminus S\cup T))_+$

$v(S\cap T) = (E-d(N\setminus S\cap T))_+$

To show the game is convex we need to verify

(*) $v(S) + v(T) \leqq v(S \cup T) + v(S \cap T)$.

Note that if either $v(S)$ or $v(T)=0$, then (*)is valid.

Now assume both are positive. Then,

$v(S) + v(T) = E - d(N \setminus S) + E - d(N \setminus T)$

$v(S \cup T) = E - d(N \setminus S \cup T) = E - d((N \setminus S) \cap (N \setminus T))$

$v(S \cap T) = (E - d(N \setminus S \cap T))_+ = (E - d((N \setminus S) \cup (N \setminus T)))_+$

(i) $v(S \cap T) > 0$

Then, $v(S \cap T) = E - d((N \setminus S) \cup (N \setminus T))$

$v(S \cup T) + v(S \cap T) =$

$2E - (d((N \setminus S) \cap (N \setminus T)) + d((N \setminus S) \cup (N \setminus T)))$

$= 2E - (d(N \setminus S) + d(N \setminus T)) = v(S) + v(T)$

(ii) $v(S\cap T)=(E-d(N\setminus S\cap T))_{+}=0$

Thus, $E \leqq d((N\setminus S)\cup(N\setminus T))$

$$=d(N\setminus S) + d(N\setminus T)-d((N\setminus S)\cap(N\setminus T))$$

Hence,

$$(E-d(N\setminus S))+(E-d((N\setminus T))\leqq E-d((N\setminus S)\cap(N\setminus T))$$

$$=v((S\cup T))$$

In particular, we have

$v(S) + v(T) \leqq v(S\cup T) + v(S\cap T)$.

This completes the proof of the Theorem.

Corollary: The core of a bankruptcy game is nonempty.