Coal1. By defintion $v(\varnothing) = 0$. If Player I and Player II form a coalition then the game matrix is shown in Table 1.

	1, 1	1, 2	2, 1	2, 2
1	$(1, 3)$	$(-3, 5)$	$(4, 1)$	$(4, 12)$
2	$(1, 3)$	$(1, 5)$	$(3, 1)$	$(1, 5)$

Table 1: Player I and Player II Form a Coalition

Then the Security Levels

$$v(\{1, 2\}) = 5 \qquad v(\{3\}) = 1$$

If Player I and Player III form a coalition then the game matrix is shown in Table 2.

	1, 1	1, 2	2, 1	2, 2
1	$(2, 2)$	$(0, 4)$	$(2, 3)$	$(0, 4)$
2	$(6, -4)$	$(2, 4)$	$(5, 11)$	$(2, 4)$

Table 2: Player I and Player III Form a Coalition

Then the Security Levels

$$v(\{1, 3\}) = 4 \qquad v(\{2\}) = 2$$

If Player I and Player III form a coalition then the game matrix is shown in Table 3.

	1, 1	1, 2	2, 1	2, 2
1	$(1, 3)$	$(3, 1)$	$(-1, 3)$	$(3, 3)$
2	$(-1, 6)$	$(1, 3)$	$(7, 9)$	$(3, 3)$

Table 3: Player II and Player III Form a Coalition

Then the Security Levels

$$v(\{2, 3\}) = 3 \qquad v(\{1\}) = -1$$

If all 3 players form a coalition then the maximum payoff is

$$v(\{1,2,3\}) = 7 + 5 + 4 = 16$$

The values of the character function is shown in Table 4.

S	$\varnothing$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	-1	2	1	5	4	3	16

Table 4: The Character Function in Coal1.

Coal2. Let the imputation $x = (x_1, x_2, x_3)$ then x in the core suffices

$$x_1 + x_2 + x_3 = v(N) \qquad x_1 \geqslant v(\{1\}) \qquad x_2 \geqslant v(\{2\}) \qquad x_3 \geqslant v(\{3\})$$
$$x_1 + x_2 \geqslant v(\{1,2\}) \qquad x_1 + x_3 \geqslant v(\{1,3\}) \qquad x_2 + x_3 \geqslant v(\{2,3\})$$

Then the core is graphed in Figure 1.

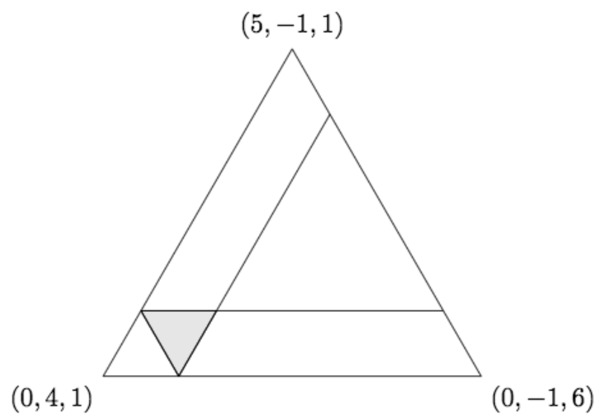

Figure 1: The Core of the Game in Coal2.

Its three vertices are $(0,3,2), (1,2,2), (1,3,1)$.

Coal3. Let the imputation $x = (x_1, x_2, x_3)$ then x in the core suffices

$$x_1 + x_2 + x_3 = 3 \qquad x_1, x_2, x_3 \geqslant 0 \qquad x_1 + x_2, x_1 + x_3, x_2 + x_3 \geqslant a$$

The set of conditions have solutions if $0 \leqslant a \leqslant 2$ and have no solutions if $2 < a \leqslant 3$. So the core is non-empty if $a \in [0,2]$.

Coal4. Let the imputation $x = (x_1, x_2, x_3)$ then x in the core suffices

$$x_1 + x_2 + x_3 = 10 \qquad x_1, x_2, x_3, x_1 + x_2 \geqslant 0 \qquad x_1 + x_3, x_2 + x_3 \geqslant 10$$

The only solution is $(0,0,10)$. So the core of this game is $\{(0,0,10)\}$.

Coal5. *Proof.* Suppose $x = (x_1, x_2, x_3)$ is in the core then

$$v(\{1,2,3\}) = x_1 + x_2 + x_3$$

By the stablity

$$\begin{aligned} v(\{1\}) + v(\{2\}) + v(\{3\}) &\leqslant x_1 + x_2 + x_3 \\ v(\{1,2\}) + v(\{3\}) &\leqslant (x_1 + x_2) + x_3 = x_1 + x_2 + x_3 \\ v(\{1,3\}) + v(\{2\}) &\leqslant (x_1 + x_3) + x_2 = x_1 + x_2 + x_3 \\ v(\{2,3\}) + v(\{1\}) &\leqslant (x_2 + x_3) + x_1 = x_1 + x_2 + x_3 \end{aligned}$$

and we have that

$$\begin{aligned} &\frac{1}{2}v(\{1,2\}) + \frac{1}{2}v(\{1,3\}) + \frac{1}{2}v(\{2,3\}) \\ \leqslant &\frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_1 + x_3) + \frac{1}{2}(x_2 + x_3) \\ = &x_1 + x_2 + x_3 \end{aligned}$$

□

Coal6. *Proof.* For any subsets $S_1, S_2 \subset N \setminus T$, since (N, v) is a convex game

$$\begin{aligned} &v_T(S_1 \cup S_2) + v_T(S_1 \cap S_2) \\ = &v((S_1 \cup S_2) \cup T) + v((S_1 \cup S_2) \cap T) - 2v(T) \\ = &v((S_1 \cup T) \cup (S_2 \cup T)) + v((S_1 \cup T) \cap (S_2 \cup T)) - 2v(T) \\ \geqslant &v(S_1 \cup T) + v(S_2 \cup T) - 2v(T) \\ = &v_T(S_1) + v_T(S_2) \end{aligned}$$

Since

$$v_T(\varnothing) = v(T) - v(T) = 0$$

When $S_1 \cap S_2 = \varnothing$ we have that

$$v_T(S_1 \cup S_2) \geqslant v_T(S_1) + v_T(S_2)$$

So v_T is a characteristic function on $N \setminus T$ and $(N \setminus T, v_T)$ is a convex game. □

Coal7. The game can be divided into simple games

$$v = w_{\{1\}} + w_{\{1,2\}} + \cdots + w_{\{1,\dots,n\}}$$

So the Shapley value is

$$\varphi_i(v) = \sum_{j=1}^{n} \varphi_i\left(w_{\{1,\dots,j\}}\right) = \sum_{j=i}^{n} \frac{1}{j} \qquad i = 1, \dots, n$$

Coal8. By definition

$$w(\varnothing) = v(\varnothing) = 0$$

For any non-empty distinct $S, T \subset N$ we have that $S, T \neq N$ then

$$w(S) + w(T) = v(S) + v(T) \leqslant v(S \cup T) \leqslant w(S \cup T)$$

So w satisfies the superadditivity property. Since $w = v + aw_N$

$$\varphi(w) = \varphi(v) + \frac{a}{n}(1, \dots, 1)$$

Coal9. Suppose the n players join the coalition in a random order. Then Player 1 has the same probability to be the jth player to join and contributes j where $j = 1, \dots, n$. So Player 1's expected contribution is

$$\varphi_1(v) = \sum_{j=1}^{n} \frac{1}{n} \cdot j = \frac{n+1}{2}$$

By the Symmetry Axiom, the Shapley value of the other players is

$$\varphi_i(v) = \frac{1}{n-1}\left(v(N) - \varphi_1(v)\right) = \frac{1}{2} \qquad i = 2, \dots, n$$

Coal10. Since $30 + 30 > 10 + 40$, a coalition is a winning one if and only if it contains at least 2 of 3 in II, III and IV, while I is a dummy. By the Symmetry Axiom and the Dummy Axiom, the Shapley value is

$$\varphi(v) = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Coal11. Suppose the n players join the coalition in a random order. Then Player 1 has the same probability to be the jth player to join where $j = 1, \ldots, n$ and contributes 1 only if $j < n$. So Player 1's expected contribution is

$$\varphi_1(v) = \sum_{j=1}^{n-1} \frac{1}{n} = \frac{n-1}{n}$$

By the Symmetry Axiom, the Shapley value of the other players is

$$\varphi_i(v) = \frac{1}{n-1}\left(v(N) - \varphi_1(v)\right) = \frac{1}{n(n-1)} \qquad i = 2, \ldots, n$$

Coal12. *Proof.* Since there are no veto players, For any $i = 1, \ldots, n$ we have that $v(N - \{i\}) = 1$ then $v(N) = 1$. Assume that $x = (x_1, \ldots, x_n)$ is in the core. By the group rationality

$$\sum_{i=j}^{n} x_j = 1 \implies \sum_{j \in N - \{i\}} x_j = 1 - x_i$$

By the stability, for any $i = 1, \ldots, n$

$$1 - x_i \geqslant v(N - \{i\}) = 1 \implies x_i \leqslant 0$$

Then $x_1 + \cdots + x_n \leqslant 0$, which leads a contradiction. So the core of this game is empty. □

Coal13. The Airport Game is equivalent to the following Road Game.

$$\text{Town} \xrightarrow{40} \text{A} \xrightarrow{10} \text{B} \xrightarrow{10} \text{C} \xrightarrow{10} \text{D} \xrightarrow{10} \text{E} \xrightarrow{10} \text{F}$$

By the Additivity Axiom, the Shapley value is shown in Table 1.

	A	B	C	D	E	F
$\varphi(v_1)$	$-40/6$	$-40/6$	$-40/6$	$-40/6$	$-40/6$	$-40/6$
$\varphi(v_2)$		$-10/5$	$-10/5$	$-10/5$	$-10/5$	$-10/5$
$\varphi(v_3)$			$-10/4$	$-10/4$	$-10/4$	$-10/4$
$\varphi(v_4)$				$-10/3$	$-10/3$	$-10/3$
$\varphi(v_5)$					$-10/2$	$-10/2$
$\varphi(v_6)$						$-10/1$
$\varphi(v)$	$-20/3$	$-26/3$	$-67/6$	$-29/2$	$-39/2$	$-59/2$

Table 1: Shapley value of the Road Game

14 Answer:

The nucleolus is ((v(1, 2) v(1)-v(2))/2, (v(1, 2)+v (2)-v(1))/2).

15 Answer:

	1	2	3
123	1	1	4
132	1	7	-2
213	2	0	4
231	3	0	3
312	3	7	-4
321	3	7	-4
	13/6	11/3	1/6

The Shapley value is (13/6, 11/3, 1/6).

S	e(x, S)	(2, 4, 0)	(2, 7/2, 1/2)
{1}	1-x_1	-1	-1
{2}	-x_2	-4	-7/2
{3}	-4-x_3	-4	-9/2
{1,2}	2-x_1-x_2=x_3-4	-4	-7/2
{1,3}	-1-x_1-x_3=x_2-7	-3	-7/2
{2,3}	3-x_2-x_3=x_1-3	-1	-1

The nucleolus is (2, 7/2, 1/2).

16 Answer:

S	v(S)	e(x, S)	(100/3, 200/3, 100)	(50, 50, 100)	(50, 75, 75)
{1}	0	-x_1	-100/3	-50	-50
{2}	0	-x_2	-200/3	-50	-75
{3}	0	-x_3	-100	-100	-75
{1,2}	0	-x_1-x_2=x_3-200	-100	-100	-125
{1,3}	0	-x_1-x_3=x_2-200	-400/3	-150	-125
{2,3}	100	100-x_2-x_3=x_1-100	-200/3	-50	-50

The nucleolus is (50, 75, 75).

17 Answer:

By symmetry axiom, we have x_2=⋯=x_n=t. We have x_1+(n-1)t=n. For set S_k, which has k elements. If 1∈S_k, e(x, S_k)=k-x_1-(k-1)t=(k-n)(1-t). Else, e(x, S_k)=-kt. Since {-k} = {k-n}, we can get 1-t=t. So, t=1/2 and x_1=(n+1)/2.

18 Answer:

v({1}) = val(A), v({2}) = val(B), v({1,2}) = max(a_{ij}+b_{ij}).

d_1=(x_2-val(B))/(x_1-val(A)), d_2=(x_1-val(A))/(x_2-val(B))

So, the Gately point satisfies d_1=d_2, meaning x_1-val(A) = x_2-val(B).

Since x_1+x_2= max(a_{ij}+b_{ij}),

x_1=(max(a_{ij}+b_{ij})+val(A)-val(B))/2 and x_2=(max(a_{ij}+b_{ij})-val(A)+val(B))/2.