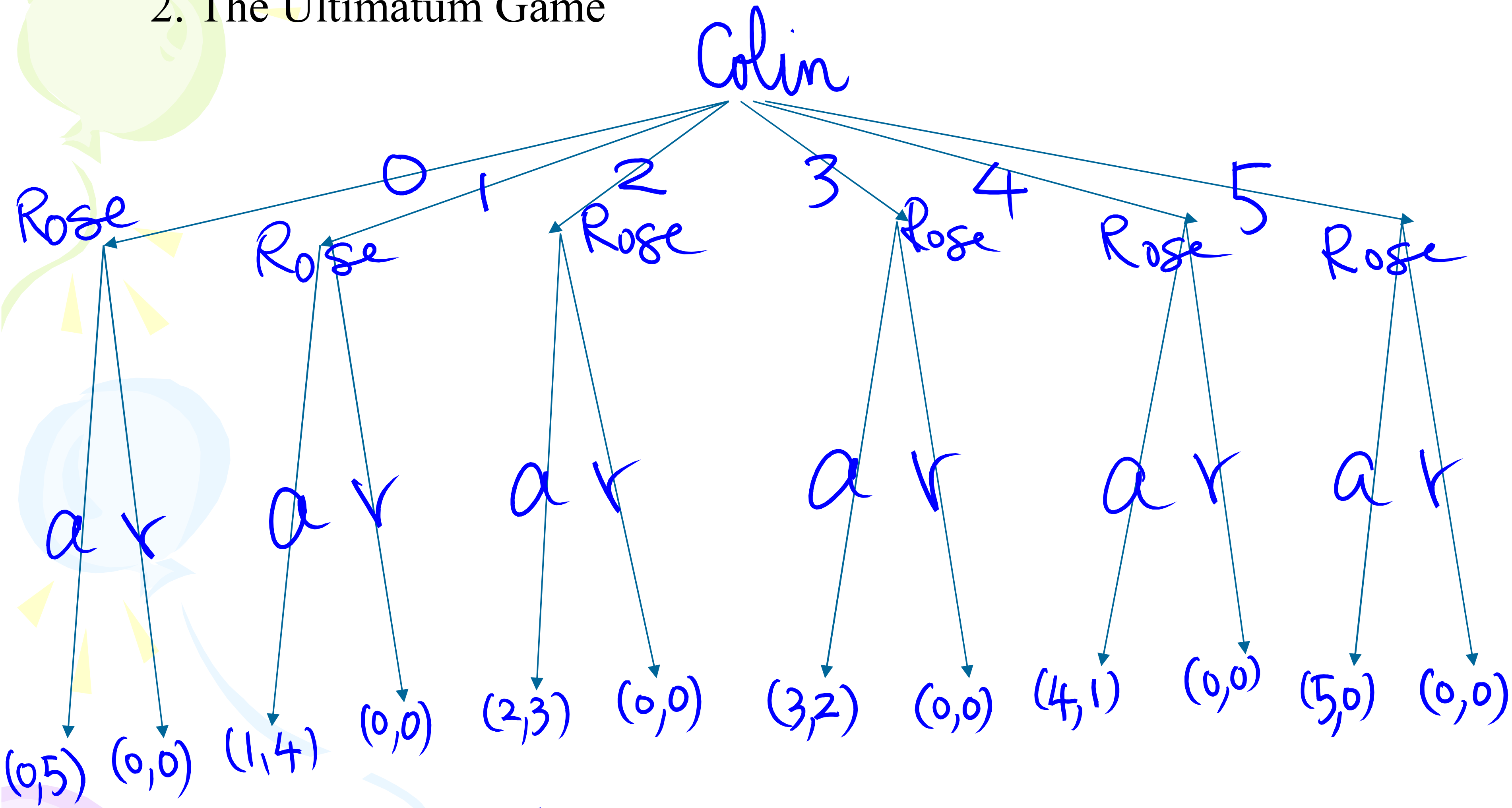
2. The Ultimatum Game: There are 5 gold coins for Colin and Rose to divide. Colin will offer Rose from 0 to 5 coins. If Rose rejects the offer, each will get nothing.

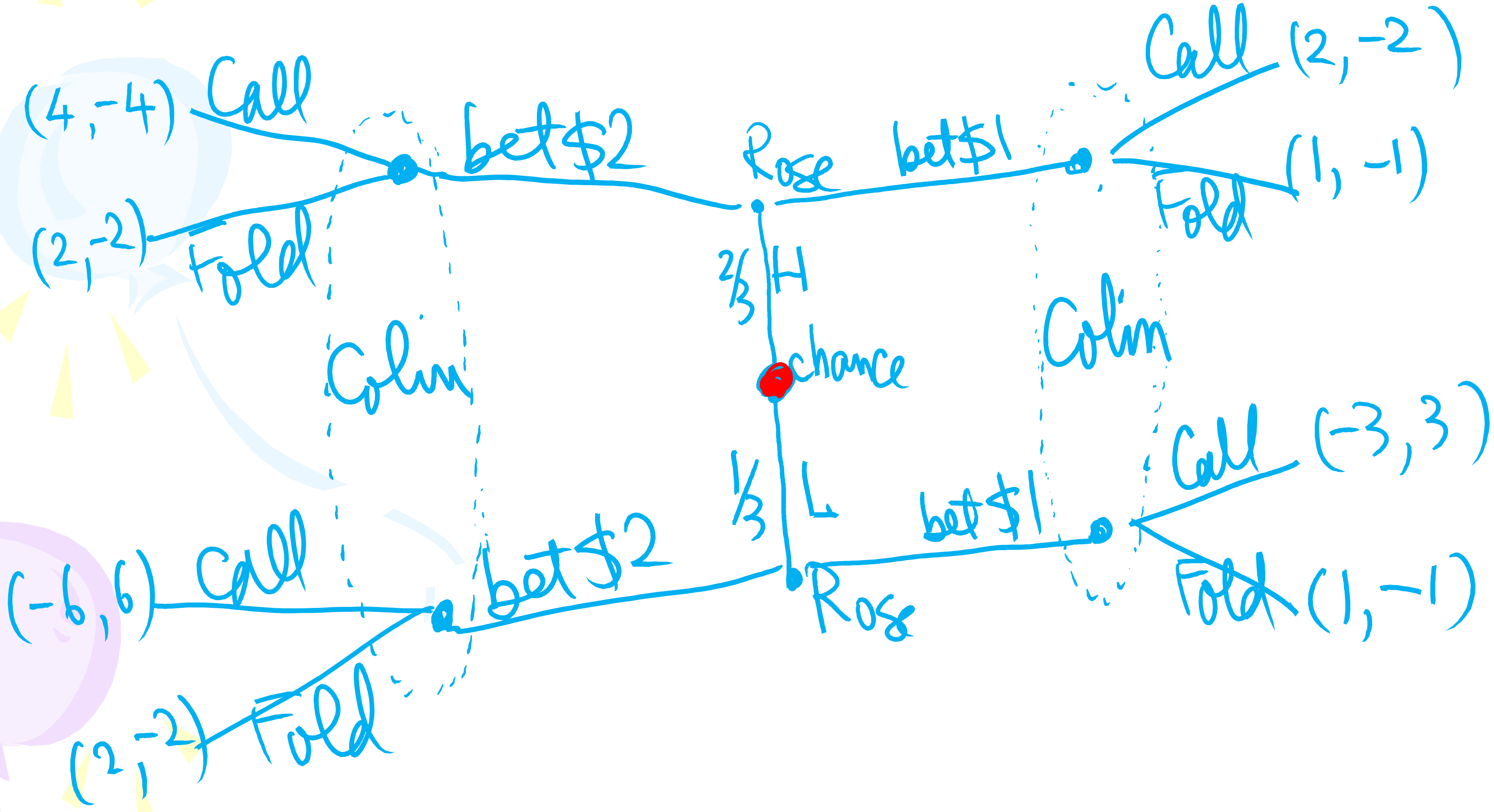
2. The Ultimatum Game



a = accept , r = reject

3. Two cards marked H (High) and one card marked L (Low) are placed in a hat. Rose draw a card and inspects it. She may then bet \$1 or \$2. Colin not knowing Rose's card but is informed of Rose's bet may either fold or call. If Colin folds, he pays Rose the amount of her bet. If Colin calls he receives from Rose three times of the amount of her bet if Rose's card is L. If Rose's card is H, Colin pays Rose two times of the amount of Rose's bet.

3. Two cards marked H (High) and one card marked L (Low) are placed in a hat. Rose draw a card and inspects it. She may then bet $1 or $2. Colin not knowing Rose's card but is informed of Rose's bet may either fold or call. If Colin folds, he pays Rose the amount of her bet. If Colin calls he receives from Rose three times of the amount of her bet if Rose's card is L. If Rose's card is H, Colin pays Rose two times of the amount of Rose's bet.



Example 4:

Basic Endgame in Poker.

- Both players put 1 dollar, called the ante, in the center of the table.
- The money in the center of the table, so far two dollars, is called the pot.
- Then Player I is dealt a card from a deck. It is a winning card with probability ¼ and a losing card with probability 3/4.

- Both players put 1 dollar, called the ante, in the center of the table.
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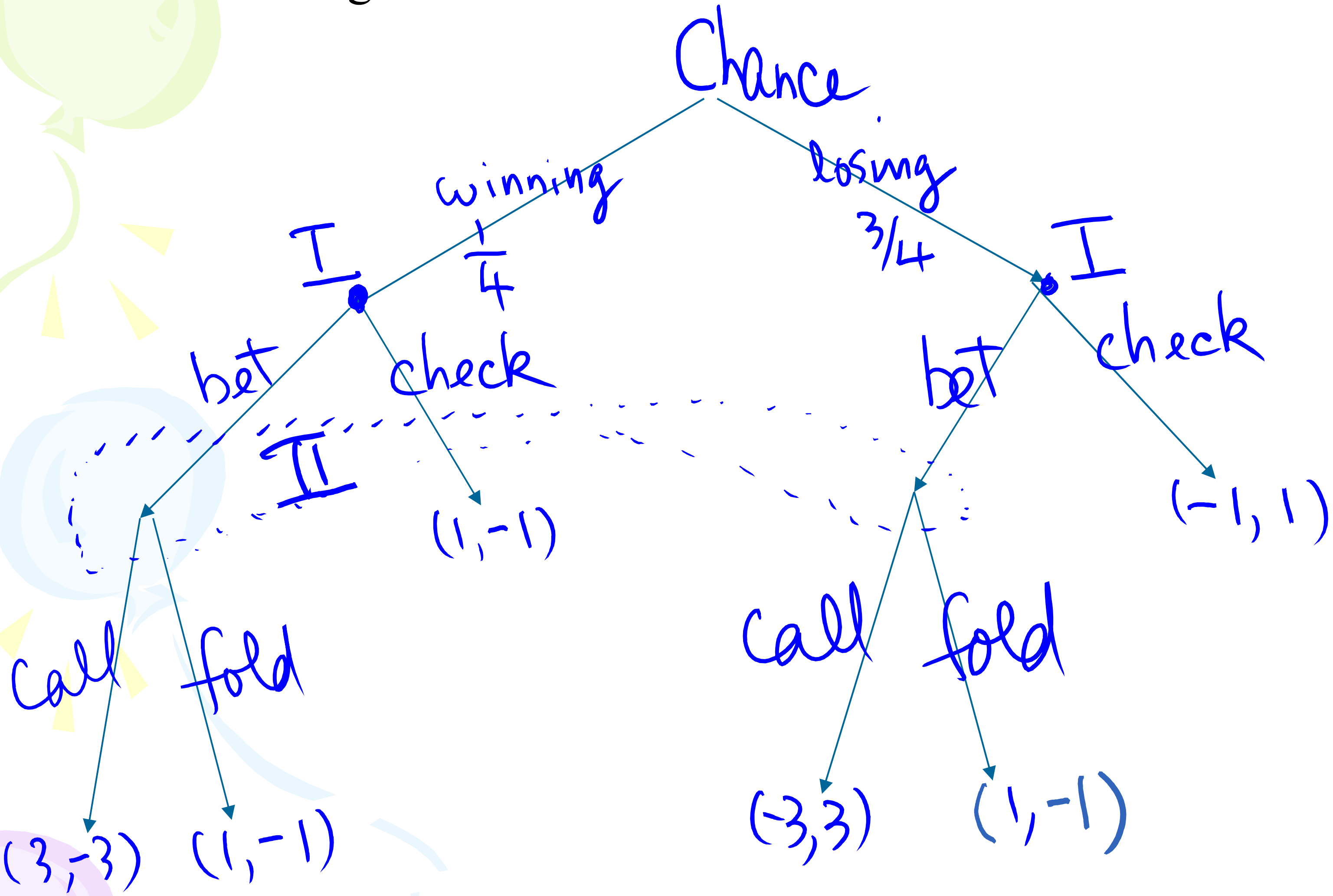
- Then Player I is dealt a card from a deck. It is a winning card with probability ¼ and a losing card with probability 3/4.
- Player I sees this card but keeps it hidden from Player II. (Player II does not get a card.)
- Player I then checks or bets.
- If he checks, then his card is inspected; if he has a winning card he wins the pot and hence wins the 1 dollar ante from II, and otherwise he loses the 1 dollar ante to II.
- If I bets, he puts 2 dollars more into the pot.

- Both players put 1 dollar, called the ante, in the center of the table.
- The money in the center of the table, so far two dollars, is called the pot.
- Then Player I is dealt a card from a deck. It is a winning card with probability ¼ and a losing card with probability 3/4.
- Player I sees this card but keeps it hidden from Player II. (Player II does not get a card.)
- Player I then checks or bets:
- If he checks, then his card is inspected; if he has a winning card he wins the pot and hence wins the 1 dollar ante from II, and otherwise he loses the 1 dollar ante to II.
- If I bets, he puts 2 dollars more into the pot.
- Then Player II – not knowing what card Player I has – must fold or call.
- If she folds, she loses the 1 dollar ante to I no matter what card I has.
- If II calls, she adds 2 dollars to the pot. Then Player I's card is exposed and I wins 3 dollars (the ante plus the bet) from II if he has a winning card, and loses 3 dollars to II otherwise.

- Basic Endgame in Poker:



Reduction of a Game in Extensive Form to Strategic Form.

Pure strategy. A pure strategy is a player’s complete plan for playing the game. It should cover every contingency.

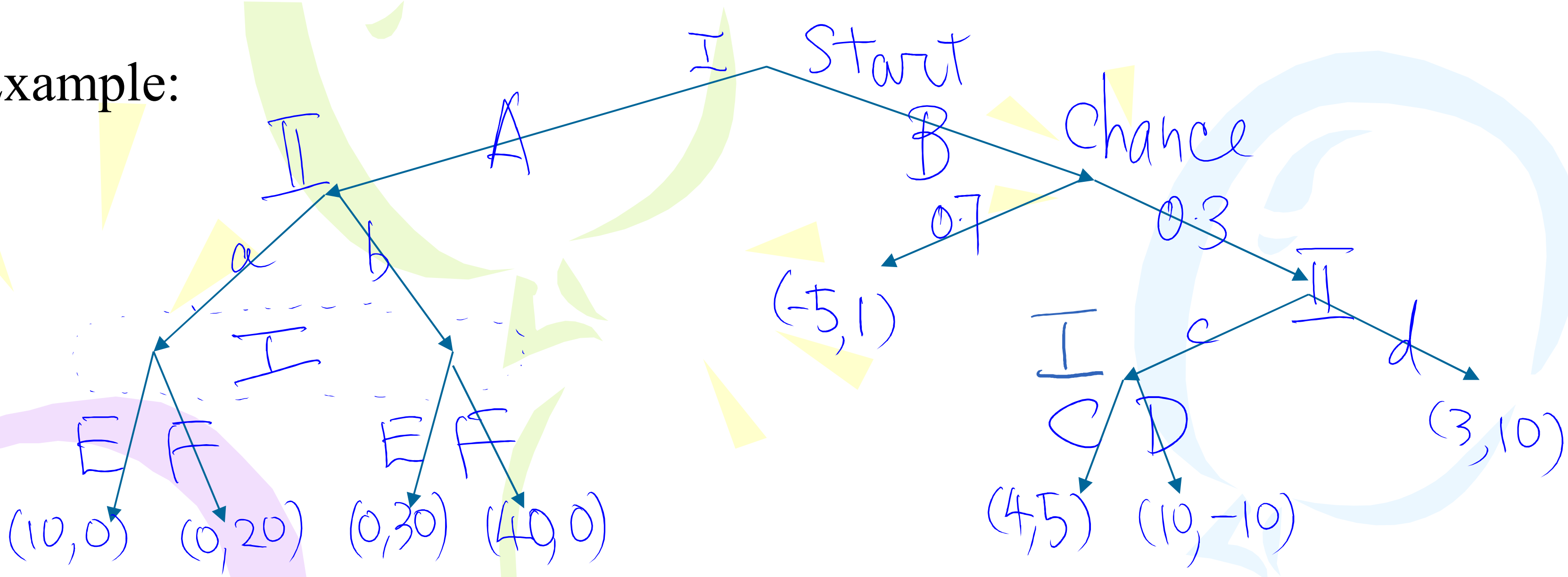
A pure strategy for a Player is a rule that tells him exactly what move to make in each of his information sets. It should specify a particular edge leading out from each information set.

锦囊妙计

诸葛亮授予跟刘备前往招亲的赵云三个锦囊妙计

- 第一个锦囊：见乔国老，并把刘备娶亲的事情搞得东吴人尽皆知；
- 第二个锦囊：用谎言（曹操打荆州）骗泡在温柔乡里的刘备回去；
- 第三个锦囊：让孙夫人摆平东吴的追兵，她是孙权妹妹，东吴将领惧她三分。

Example:



Player I has 3 information sets. The set of edges coming out from the information sets are{A,B}, {E,F}, {C,D}.

Player II has two information sets. The set of edges coming out from the information sets are {a,b}, {c,d}.

Set of Pure Strategies for I: {A,B}x{C,D}x{E.F}

={ACE, ACF, ADE, ADF, BCE, BCF, BDE, BDF}

Set of Pure Strategies for II: {a,b}x{c,d}={ac,ad,bc,bd}

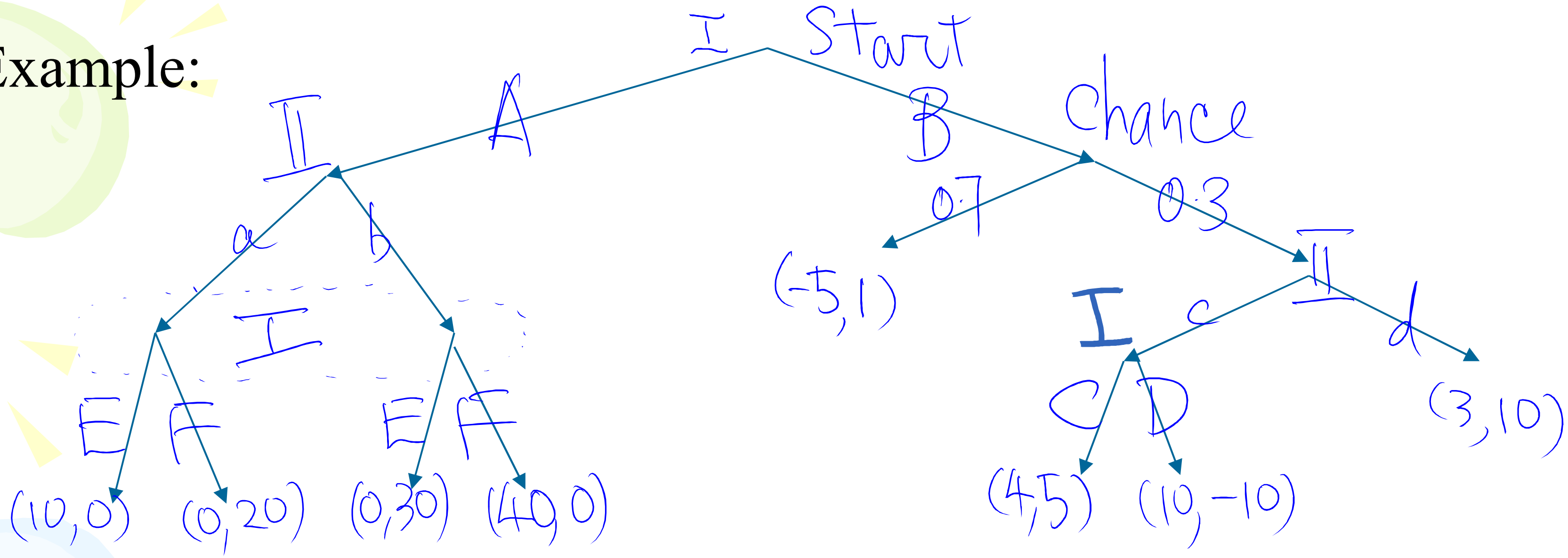
Remark: Even for a simple game, there may be a large number of pure strategies.

Example: (Two move chess) There are 20 possible moves for Player I and 20 possible moves for Player II. How many pure strategies are there for Player II?

Reduced pure strategy: Specification of choices at all information sets except those that are eliminated by the previous moves.

Remark: It is not too easy to find the set of reduced pure strategies. Also we need to use full pure strategies for another important concept.

Example:



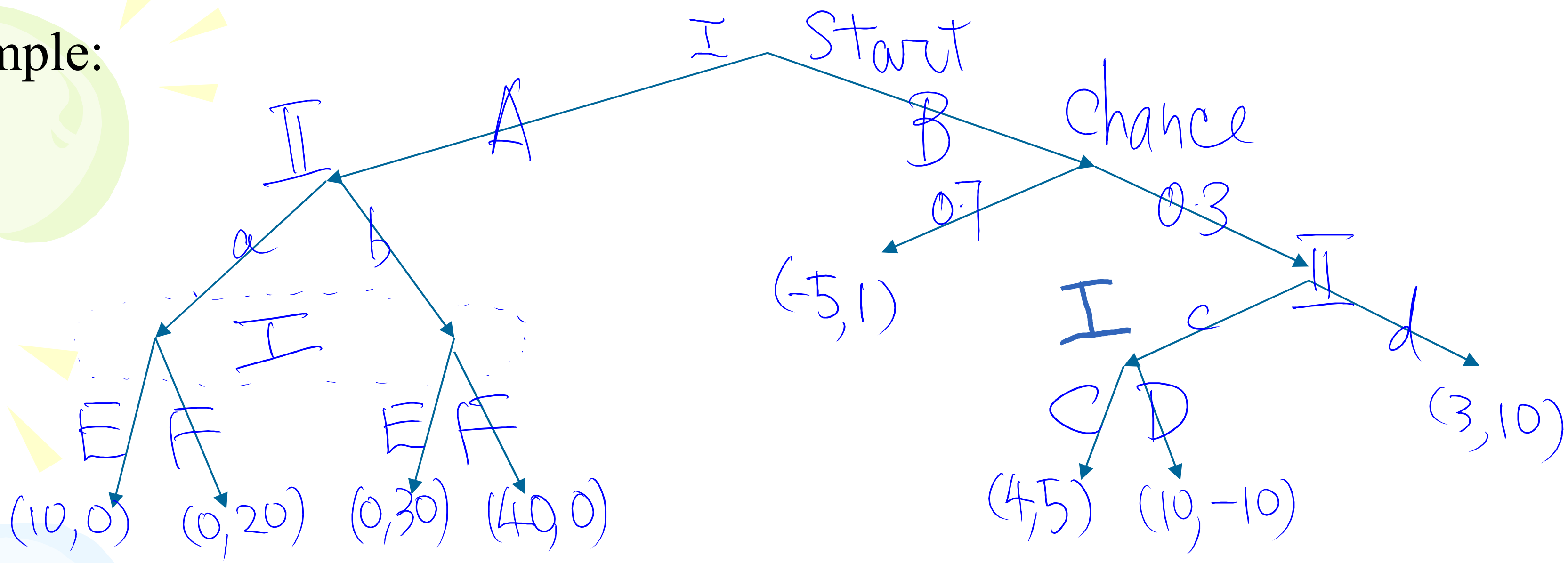
Set of Reduced Pure Strategies for I: {AE, AF, BC, BD}

Set of Reduced Pure Strategies for II: {ac,ad,bc,bd}

Now that we have a set of pure strategies for each player, we need to find the payoffs to put the game in strategic form.

Random payoffs. The actual outcome of the game for given pure strategies of the players depends on the chance moves selected, and is therefore a random quantity. We represent random payoffs by their expected values.

Example:



Suppose Player I uses BCF and Player II uses ac.

Then, the payoff will be (-5, 1) with probability 0.7 and (4,5) with probability 0.3. Then, the expected payoff is

$$0.7(-5,1) + 0.3(4,5) = (-2.3, 2.2)$$

Remark: In representing the random payoffs by their averages, we are making a rather subtle assumption. We are saying that receiving \$5 outright is equivalent to receiving \$10 with probability 0.5. The proper setting for this concept is Utility Theory developed by von Neumann and Morgenstern.

There is a concept of rationality as utility maximization.

《史记·货殖列传》

“天下熙熙皆为利来,

天下攘攘皆为利往”

孟子见梁惠王

王曰：“叟不远千里而来，亦将有利吾国夫？”

孟子对曰：“王何必曰利？亦有仁义而已矣。

王曰『何以利吾國』？大夫曰『何以利吾家』？

士庶人曰『何以利吾身』？上下交征利而國危矣。

萬乘之國弒其君者，必千乘之家；千乘之國弒其君者，必百乘之家。萬取千焉，千取百焉，不為不多矣。苟為後义而先利，不奪不饜。

未有仁而遺其親者也，未有义而後其君者也。王亦曰仁义而已矣，何必曰利？」”

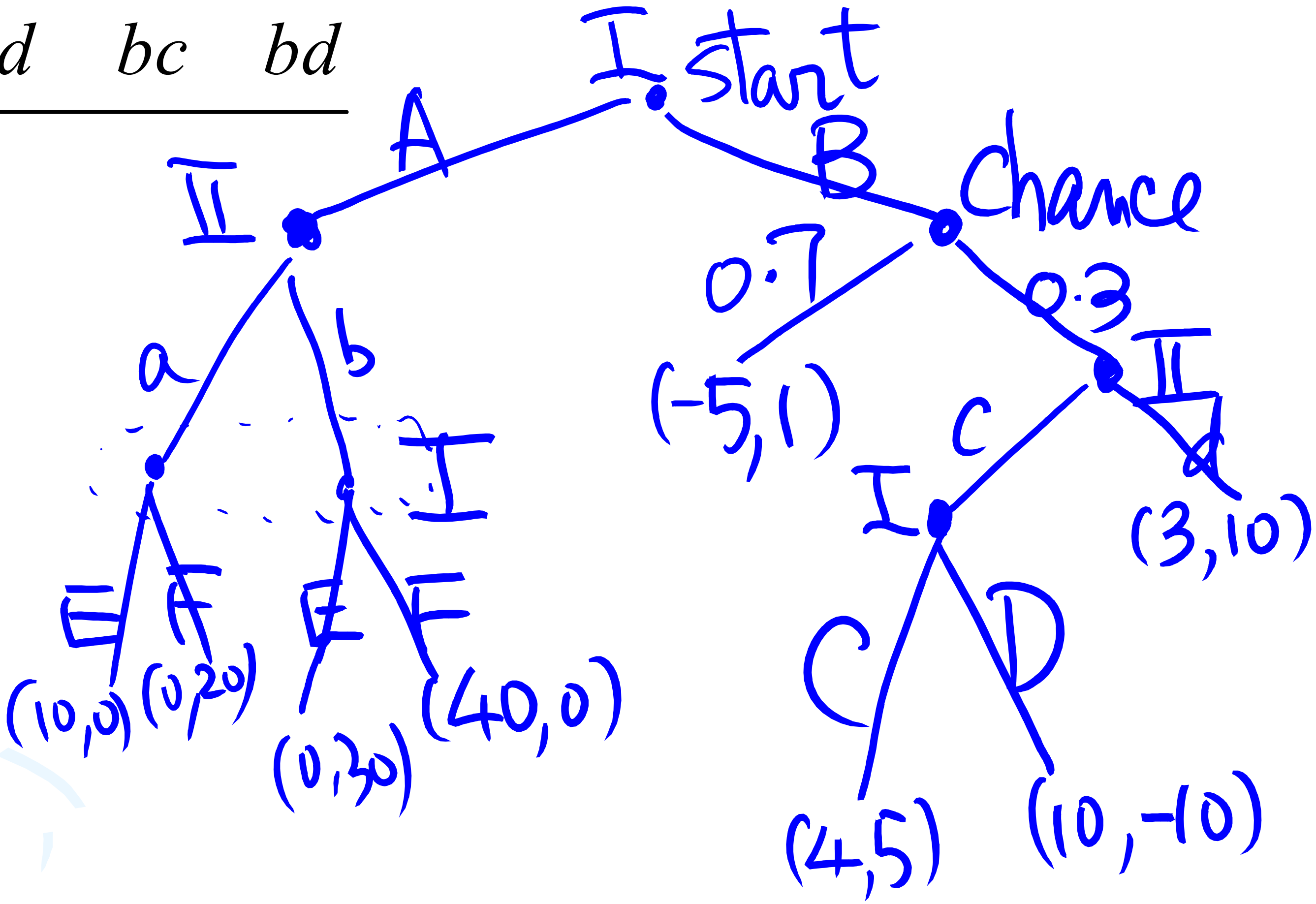
Game in strategic form (normal form):

- Set of players: $\{1,\ldots,n\}$
- A set of pure strategies X_i for player i, $i=1,\ldots,n$.
- Payoff function for the i^{th} player, $i=1,\ldots,n$.

$u_i : X_1 \times \ldots \times X_n \to \mathbb{R}$

Example: Payoff function tabulated in bimatrix form:

	ac	*ad*	*bc*	*bd*
ACE				
ACF				
ADE				
ADF				
BCE				
BCF				
BDE				
BDF				



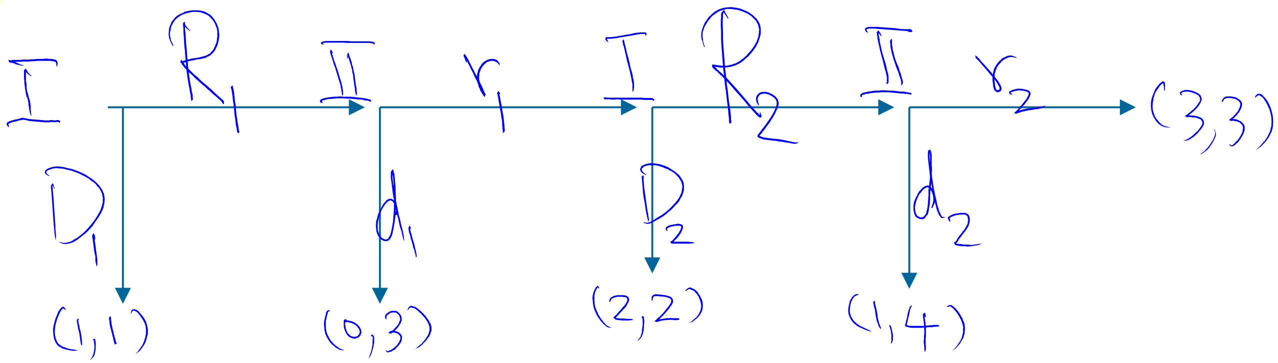
Set of reduced strategies for I: $\{AE, AF, BC, BD\}$

Set of reduced strategies for II: $\{ac, ad, bc, bd\}$

Straegic Form:

	ac	ad	bc	bd
AE	$(10, 0)$	$(10, 0)$	$(0, 30)$	$(0, 30)$
AF	$(0, 20)$	$(0, 20)$	$(40, 0)$	$(40, 0)$
BC	$(-2.3, 2.2)$	$(-2.6, 3.7)$	$(-2.3, 2.2)$	$(-2.6, 3.7)$
BD	$(-0.5, -2.3)$	$(-2.6, 3.7)$	$(-0.5, -2.3)$	$(-2.6, 3.7)$

Example:



Strategic Form:

	r_1r_2	r_1d_2	d_1r_2	d_1d_2
R_1R_2	(3,3)	(1,4)	(0,3)	(0,3)
R_1D_2	(2,2)	(2,2)	(0,3)	(0,3)
D_1R_2	(1,1)	(1,1)	(1,1)	(1,1)
D_1D_2	(1,1)	(1,1)	(1,1)	(1,1)

Remark: A Kuhn Tree gives complete description of the game. A pure strategy is a complete plan to play the game, it should cover all contingency i.e. giving specific instruction to which move (edge) to choose at each information set. There are usually a large number of pure strategies.

On can then think of each pure strategy as a book of instructions. For each information set, there is a page in the book stating on what choice to make at that information set. The set of pure strategies is then a library of such books.

A mixed strategy is a probability distribution on the books in the library. According to a certain randomization device having the prescribed probability distribution, the player is instructed to choose a particular book (pure strategy) in the library (set of pure strategies).

Solution Concept: Pure Strategic Equilibrium

Definition. A vector of Pure Strategy choices $(x_1, x_2, \ldots, x_n)$ with $x_i \in X_i$ for $i = 1, \ldots, n$ is said to be a Pure Strategic Equilibrium, or PSE for short, if for all $i = 1, 2, \ldots, n$, and for all $x \in X_i$,

$$u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \geq u_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n). \quad (*)$$

Equation (*) says that if the players other than player i use their indicated strategies, then the best player i can do is to use x_i. Such a pure strategy choice of player i is called a **best response** to the strategy choices of the other players.

It is self enforcing!

Remark: We will extend this definition to mixed strategies. The formulation is basically the same.

The notion of strategic equilibrium may be stated: a particular selection of strategy choices of the players forms a PSE if each player is using a best response to the strategy choices of the other players.

Remark: In the above analysis, we used the basic assumption of Common Knowledge.

David Lewis:

A fact is common knowledge if everyone knows it, everyone knows that everyone knows it, everyone knows that everyone knows that everyone knows it,..., and so on ad infinitum.

你知，我知，你知我知，我知你知,你知我知你知........

- Common knowledge is a phenomenon which underwrites much of social life. In order to communicate or otherwise coordinate their behavior successfully, individuals typically require mutual or common understandings or background knowledge.

- In *A Treatise of Human Nature*, Hume argued that a necessary condition for coordinated activity was that agents all know what behavior to expect from one another. Without the requisite mutual knowledge, Hume maintained, mutually beneficial social conventions would disappear.

Example: The Department Store (Thomas Schelling, The Strategy of Conflict)

When a man loses his wife in a department store without any prior understanding on where to meet if they separated, the chances are good that they will find each other. It is likely that each will think of some obvious place to meet, so obvious that each will be sure that it is "obvious" to both of them. One does not simply predict where the other will go, which is wherever the first predicts the second to predict the first to go, and so ad infinitum. Not " What would I do if I were she?" but "What would I do if I were she wondering what she would do if she were wondering what I would do if I were she…?"

Example: The Clumsy Waiter (Stanford Encyclopedia of Philosophy)

A waiter serving dinner slips, and spills gravy on a guest’s white silk evening gown. The guest glares at the waiter, and the waiter declares “ I’m sorry. It was my fault.”

Why did the waiter say that he was at fault? He knew that he was at fault, and he knew that she knew he was at fault observed from the guest’s angry expression . However, the sorry waiter wanted assurance that the guest knew that he knew that he was at fault. By saying openly that he was at fault, the waiter knew that the guest knew what he wanted her to know, namely, that he knew he was at fault. Note that the waiter’s declaration established at least three levels of nested knowledge.

Certain assumptions are implicit in the preceding story. In particular, the waiter must know that the guest knows he has spoken the truth, and that she can draw the desired conclusion from what he says in this context.

More fundamentally, the waiter must know that if he announces "It was my fault" to the guest, she will interpret his intended meaning correctly and will infer what his making this announcement ordinarily implies in this context. This in turn implies that the guest must know that if the waiter announces "It was my fault" in this context, then the waiter indeed knows he is at fault. Then, on account of his announcement, the waiter knows that the guest knows that he knows he was at fault. The waiter's announcement was meant to generate higher-order levels of knowledge of a fact each already knew.

Example on Common Knowledge from movie:

Princess Bride（公主新娘）

Our hero Westley, in the guise of the Dread Pirate Roberts, confronts his foe-for-the-moment, the Sicilian, Vizzini. Westley challenges him to a Battle of Wits. Two glasses are placed on the table, each containing wine and one purportedly containing poison. The challenge, simply, is to select the glass that does not lead to immediate death. Vizzini then makes argument along the line of common knowledge to guess which glass is poisonous.

Example on Common Knowledge:

Once upon a time a time an evil King decided to grant sadistic amnesty to a large group of prisoners, who were kept incommunicado in the dungeons. The King placed a hat on each prisoner; two of these hats were red, the rest white. The King summoned the prisoners and commanded them not to look upward. Thus each prisoner could see the hat of every one of his fellow prisoners, but not his own. The King spoke thus: “Most of you are wearing white hats, but at least one of you is wearing a red hat. Every day from now on you will be brought to here from your solitary confinement. The day that you guess correctly the color of the hat that you are wearing is the day you will go free. If you guess incorrectly, you will be instantly beheaded.”

How many days would it take the two red-hatted prisoners to infer, rationally, the color of their hats?

Example on Common Knowledge:

An honest father tells his two sons that he has placed 10^n dollars in one envelope, and 10^{n+1} dollars in the other, where n is chosen with equal probability among integers between 1 to 6. The father randomly hands each son an envelope. The first son looks inside and finds $10,000. He calculates that the other envelope contains either $1,000 or $100,000 with equal probability. The expected amount in the other envelope is then $50,500. The second son finds only $1,000 in his envelope. Again, he calculates that the expected amount of the other envelope is $5,050. The father privately asks each son whether he would be willing to pay $1 to switch envelopes. Both son say yes. The father then tells each son what his brother said and repeats the question. Again, both say yes. The father relays the brothers' answers and ask each a third time. Again both say yes. But if the father relays the answer and ask a fourth time, the son with $1,000 will say yes, but the son with $10,000 will say no.

Why?

Application of Common Knowledge

Agreeing to Disagree

In 1976, Robert Aumann wrote a classic paper called Agreeing To Disagree. The main theorem essentially said that

People with the same priors cannot agree to disagree.

(Two Bayesian Rationalists having the same prior and common knowledge of each other's posteriors must agree on the same posterior.)

Labeling Algorithm for Finding All PSE's for 2-person Games

- Put an asterisk after each of Player I's payoffs that is a maximum of its column.
- Put an asterisk after each of Player II's payoffs that is a maximum of its row.
- Then any entry of the matrix at which both I's and II's payoffs have asterisks is a PSE, and conversely.

Example: Prisoner's Dilemma

	Confess	Silent
Confess	$(-9^*, -9^*)$	$(0^*, -10)$
Silent	$(-10, 0^*)$	$(-1, -1)$

Bad News: Many games have no PSE's.

Example:

$$\boxed{\begin{array}{cc} (-2, 2^*) & (3^*, -3) \\ (3^*, -3) & (-4, 4^*) \end{array}}$$

More Bad News: There may be more than one PSE.

$(0^*, 2^*)$	$(0, 2^*)$
$(-1, -1)$	$(1^*, 1^*)$

Example of PSE of n-person games:

- 4000 drivers drive from A to B.
- Each driver has two possibilities, through U or through R.

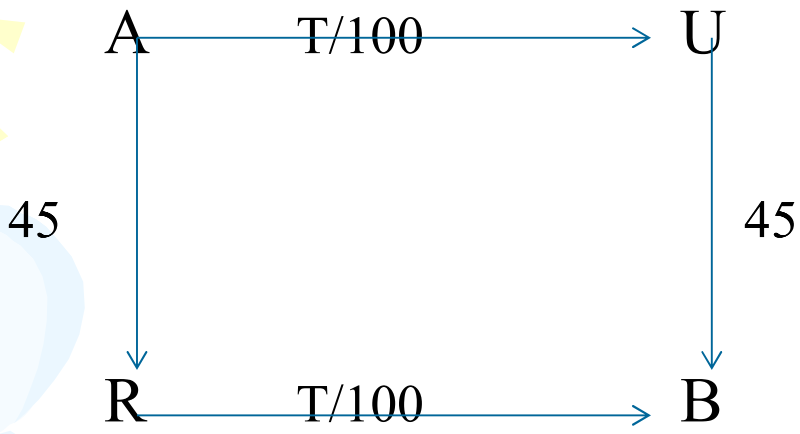
$$
\begin{tikzcd}
A \arrow[r, "T/100"] \arrow[d, "45"'] & U \arrow[d, "45"] \\
R \arrow[r, "T/100"] & B
\end{tikzcd}
$$

where T is the number of drivers using this road segment.

This is a 4000-person game such that each player has two strategies, U or R.

Example of PSE for n-person game:

- 4000 drivers drive from A to B.
- Each driver has two possibilities, through U or through R.



where T is the number of drivers using this road segment.

This is a 4000-person game such that each player has two strategies, U or R.

PSE: 2,000 choosing U and 2,000 choosing R.

Now, the average travel time per driver is 2,000/100 +45=65.

Proof:

Given that a player knows that 2,000 are using U and 1,999 are using R. This player will work to find out his/her BR. If he/she uses R, then the travel time is 2000/100 + 45 =65.

If he/she uses U, the travel time is 2001/100 + 45 = 65.01.

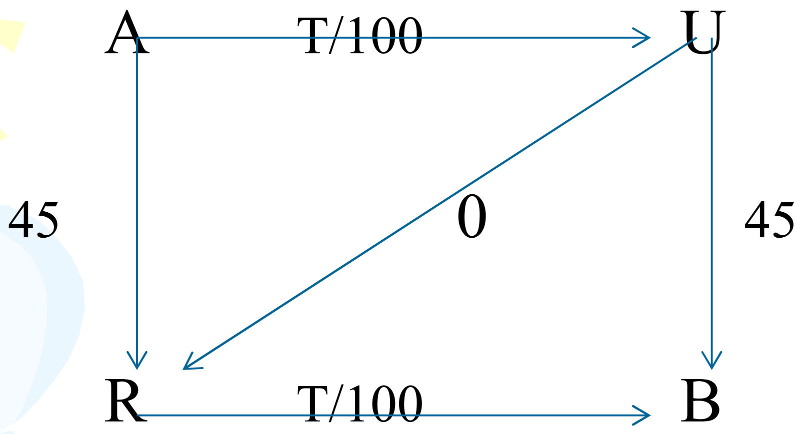
Therefore, using R is the BR.

Similarly, when a player knows that 1,999 are using U and 2,000 are using R, the BR for he/she is U.

Therefore, 2,000 using U and 2,000 using R is a PSE because none will deviate from the strategy he/she is using.

Add a “super fast” road from U to R.

Then, each driver has three strategies: A-U-B, A-R-B, A-U-R-B.



PSE: Note that when all 4,000 use the same A-U or R-B, the travel time is 40 that is less than 45. All 4,000 drivers choosing A-U-R-B is a PSE (exercise).

Now the average travel time for each driver is then

4,000/100 + 4,000/100= 80!

This is called the Braess Paradox.

This phenomenon is real(Wikipedia):

In Seoul, South Korea, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheongyecheon restoration project.

In Stuggart, Germany after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.

In 1990 the closing of 42nd street in New York City reduced the amount of congestion in the area.

In 2008 Youn, Gastner and Jeong demonstrated specific routes in Boston, New York City and London where this might actually occur and pointed out roads that could be closed to reduce predicted travel time.

The result of not charging toll fees.

Games of perfect information always have at least one PSE that may be found by the method of backward induction.

Method of Backward Induction: Starting from any terminal vertex and trace back to the vertex leading to it. The player at this vertex will discard those edges with lower payoff. Then, treat this vertex as a terminal vertex and repeat the process. Then, we get a path from the root to a terminal vertex

Theorem: The path obtained by the method of backward induction defines a PSE.

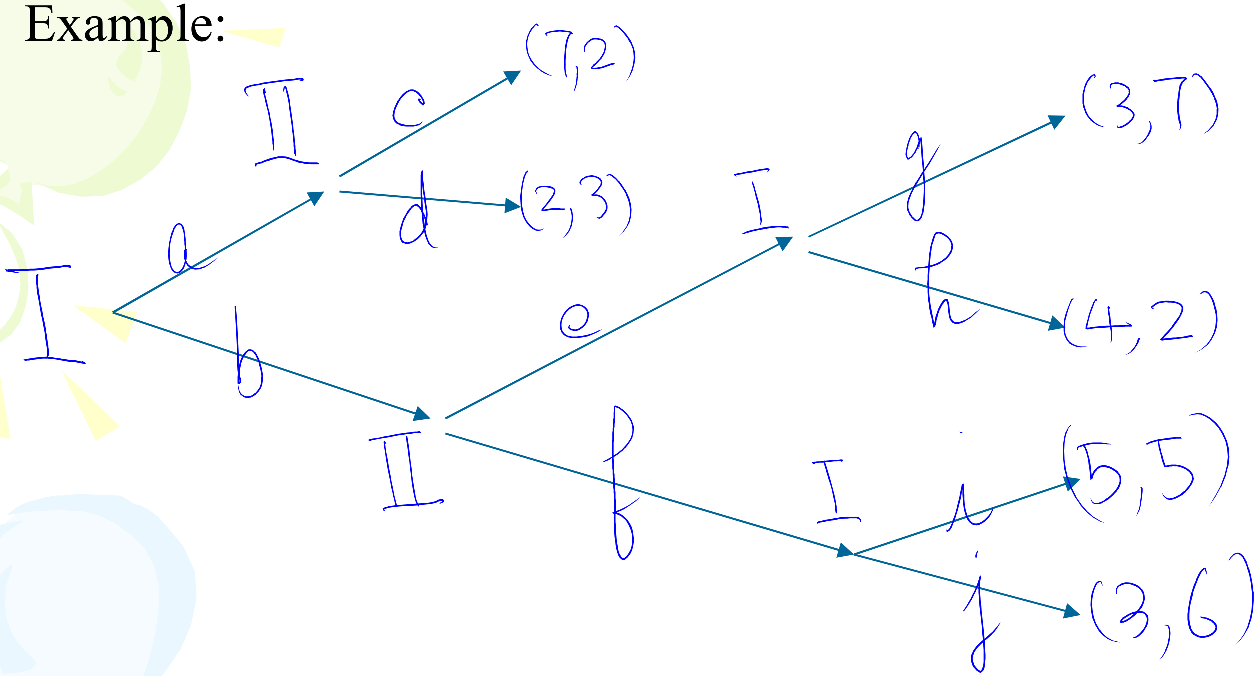
Soren Kierkegaard (1813-1855)

"Life can only be understood backwards;

but it must be lived forwards."

PROBLEM CHILD #3 - "THE GOOD OL' DAYS"

Example:


I
II
II
I
I
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b
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h
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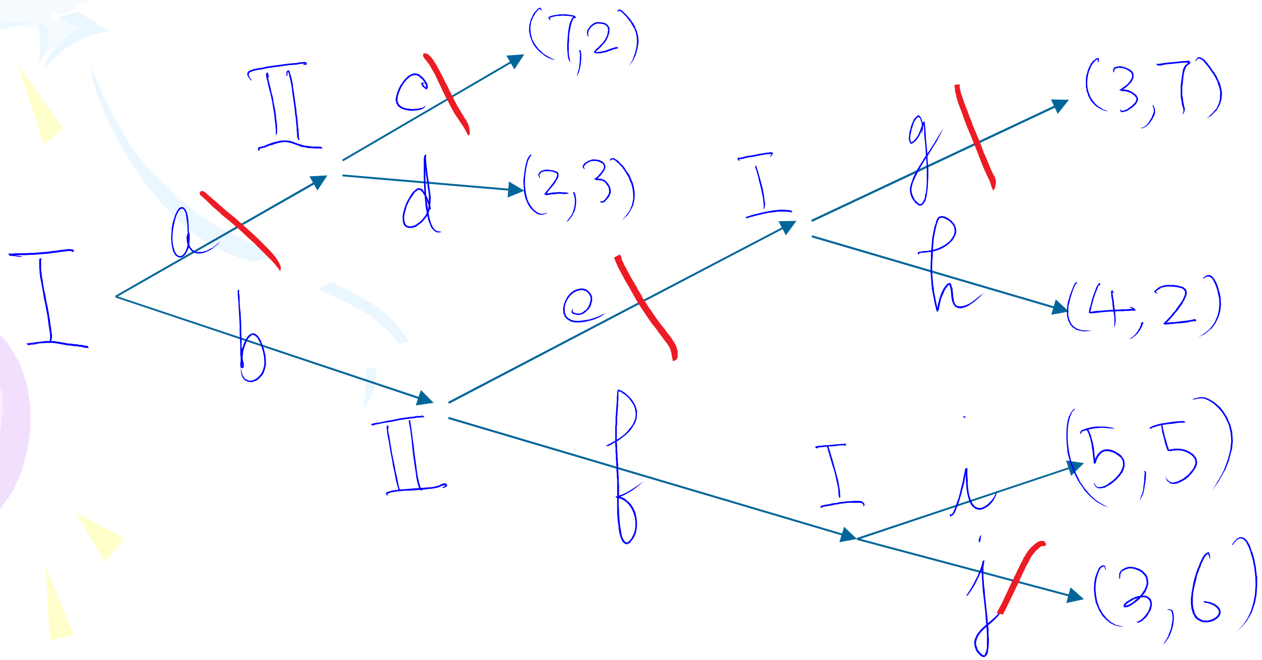
(3,7)
(4,2)
(5,5)
(3,6)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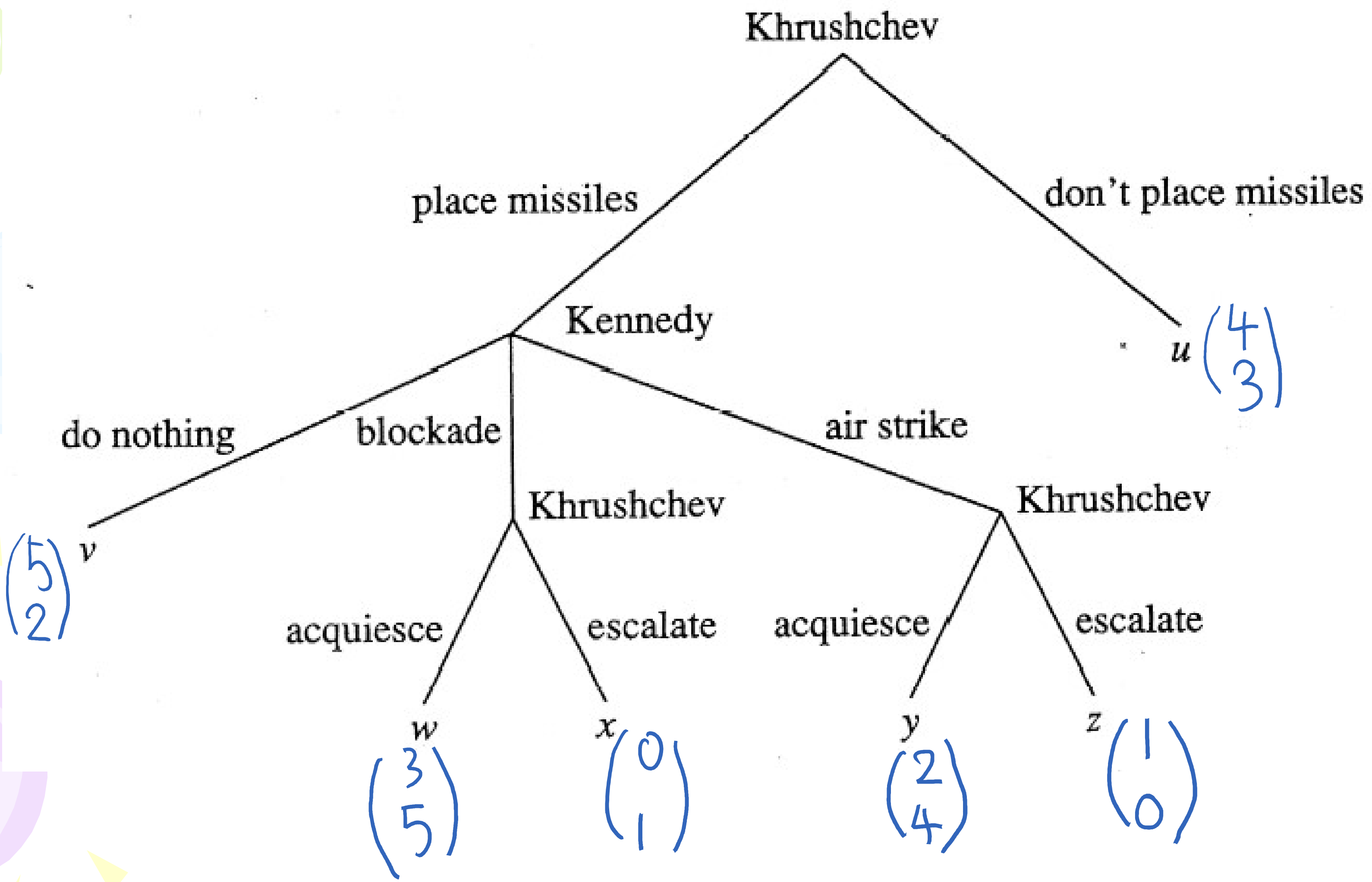
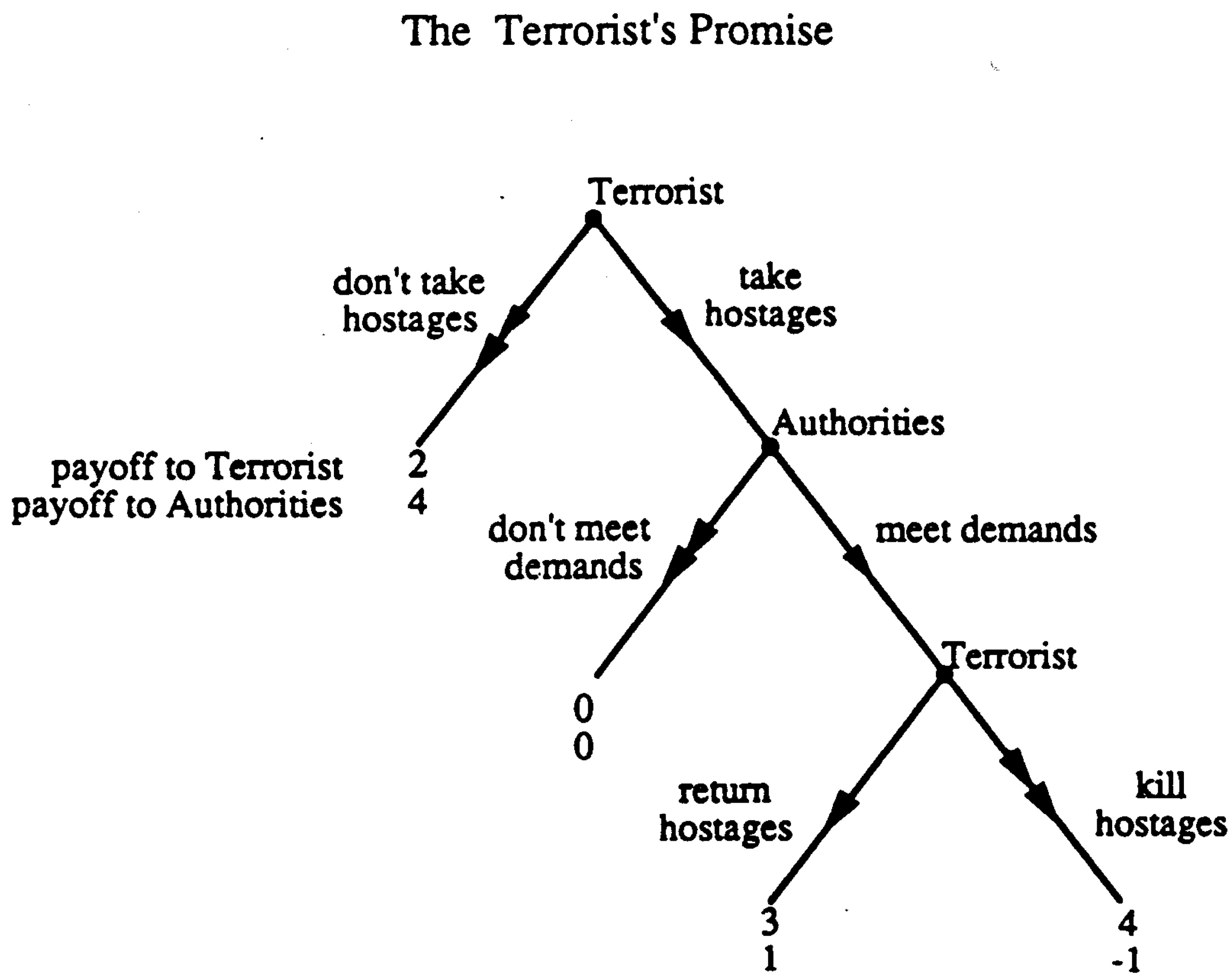
(7,2)
(2,3)
(3,7)
(4,2)
(5,5)
(3,6)

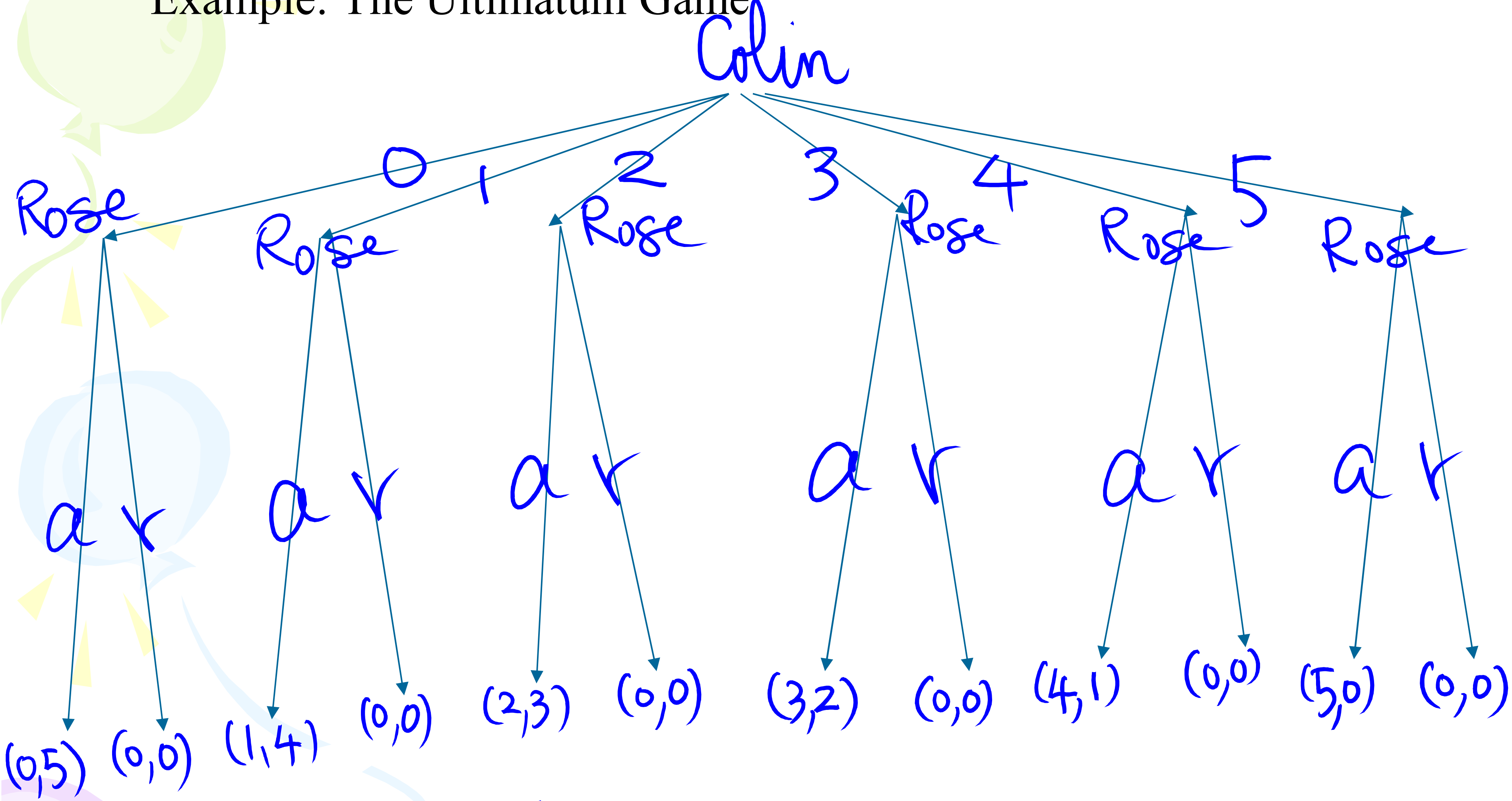
Example: (backward induction)



Example: (backward induction)



Example: The Ultimatum Game



a = accept , r = reject

Zermelo's Theorem

Theorem (Zermelo ,1912): In chess either white can force a win, or balck can force a win, or both can force at least a draw.

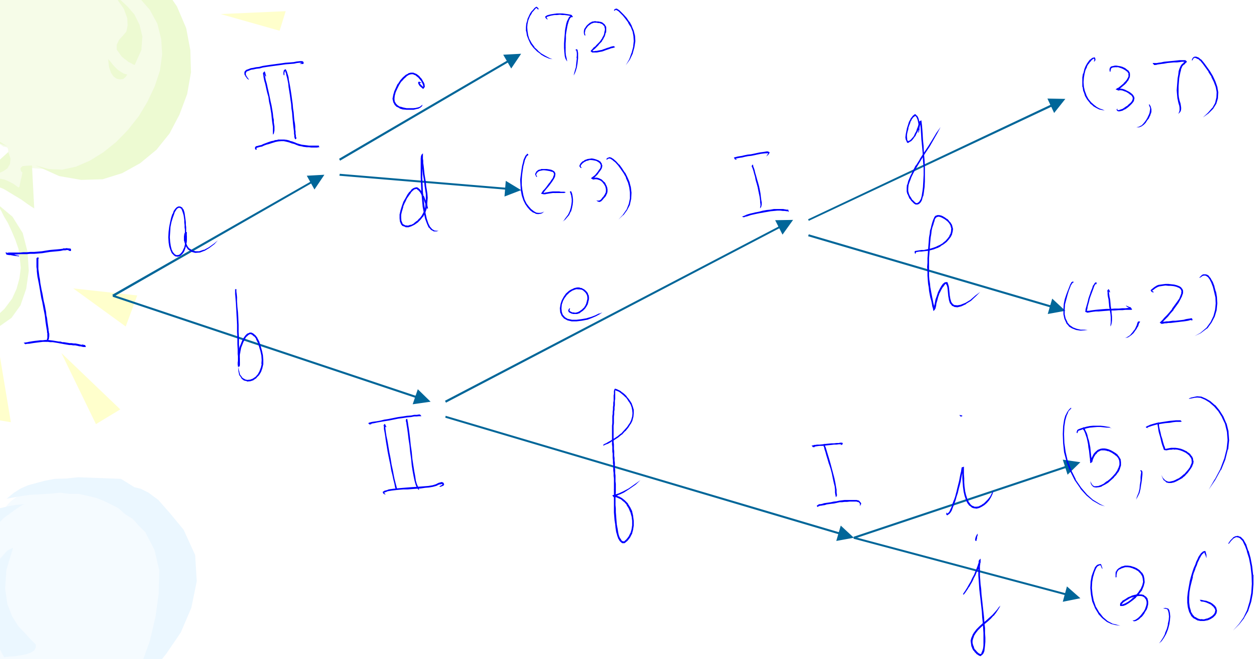
Sketch of Proof:

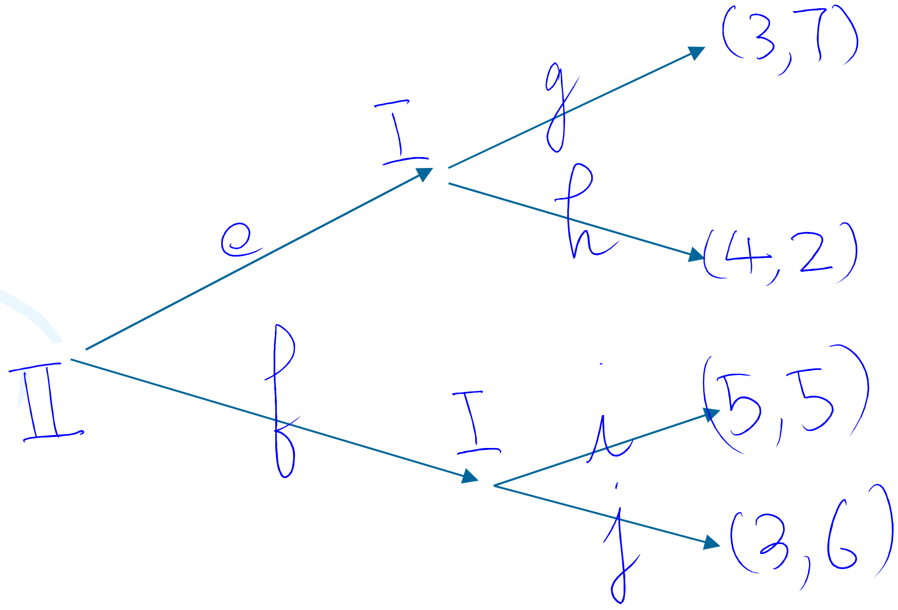
Draw the Kuhn Tree for the game of Chess (You will need a lot of paper!). This is a Kuhn tree for a game of perfect information!

Then, apply the method of Backward Induction to this Kuhn Tree (It will take a lot of time!).

In fact, the PSE obtained by the method of backward induction satisfies stronger properties so that it is called a perfect pure strategy equilibrium.

Definition: A subgame of a game presented in extensive form is obtained by taking a vertex in the Kuhn tree and all the edges and paths originated from this vertex.


I
a
b
II
c
d
(7,2)
(2,3)
II
e
f
I
g
h
(3,7)
(4,2)
I
i
j
(5,5)
(3,6)


II
e
f
I
g
h
(3,7)
(4,2)
I
i
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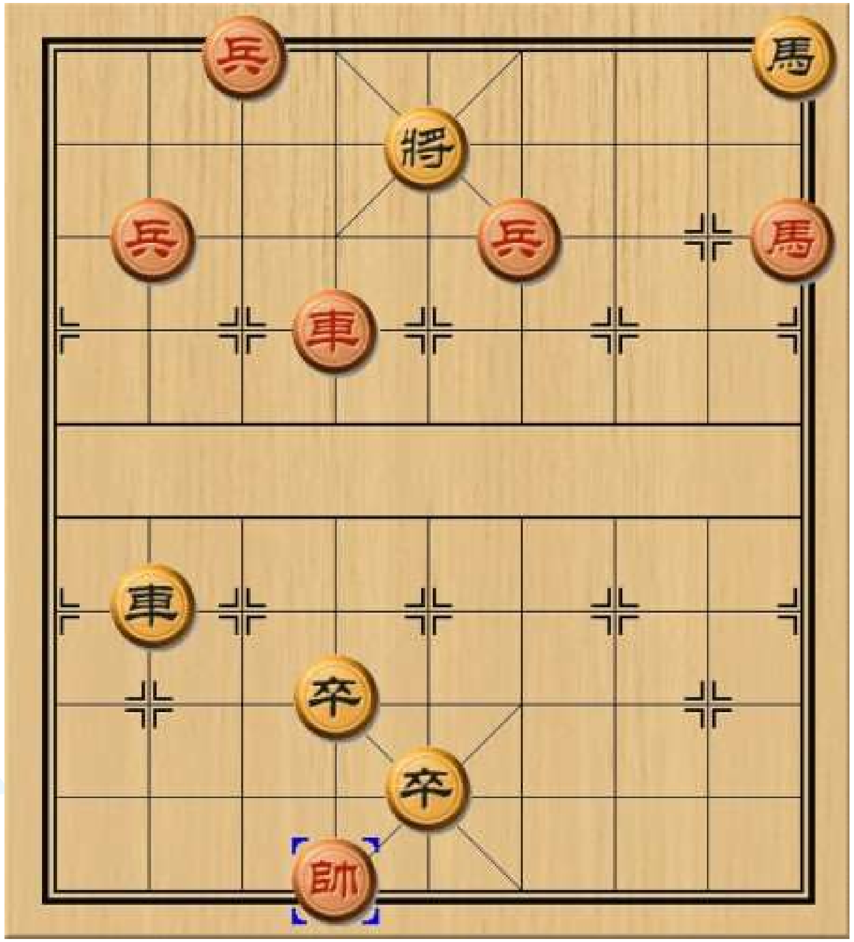
(5,5)
(3,6)

Definition: A PSE of a game in extensive form is called a Perfect Pure Strategy Equilibrium (PPSE) if it is a PSE for all subgames.

Theorem: The path obtained by the method of backward induction defines a PPSE.

Subgame of the game of chess

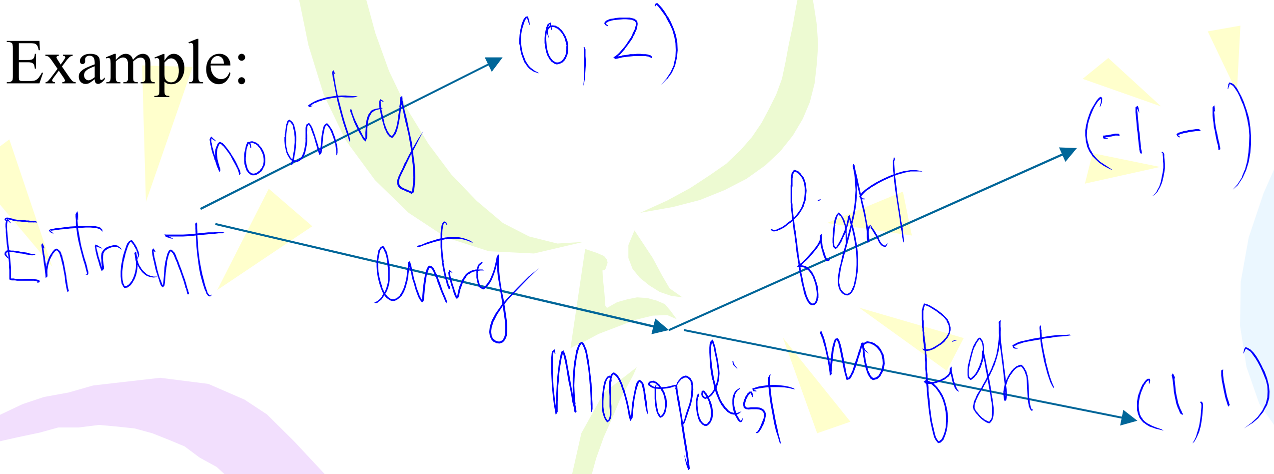


Remark: Subgame perfect implies that when making choices, a player look forward and assumes that the choice that will subsequently be made by himself and by others will be rational. Threats which would be irrational to carry through are ruled out. It is precisely this kind of forward-looking rationality that is most suited to economic applications.

Example: Incredible Threats

A threat made by a player in a game in extensive form which would not be in the best interest for the player to carry out.

Example:

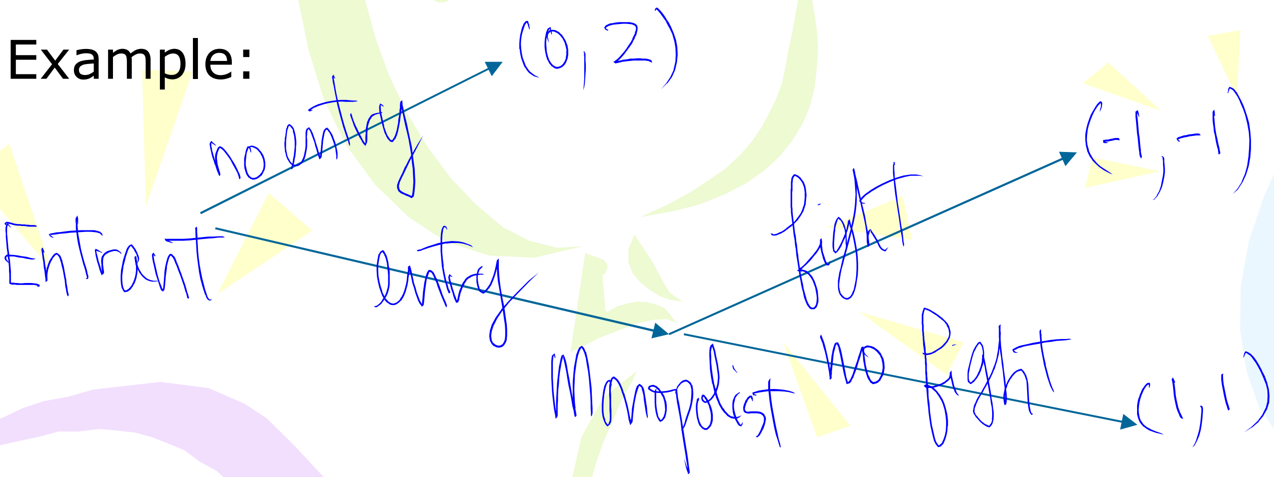


Example:

Reduction to Strategic Form and finding all PSE's.

		Monopolist	
		fight	no fight
Entrant	no entry	(0^*,2^*)	(0,2^*)
	entry	(-1,-1)	(1^*,1^*)

Example:



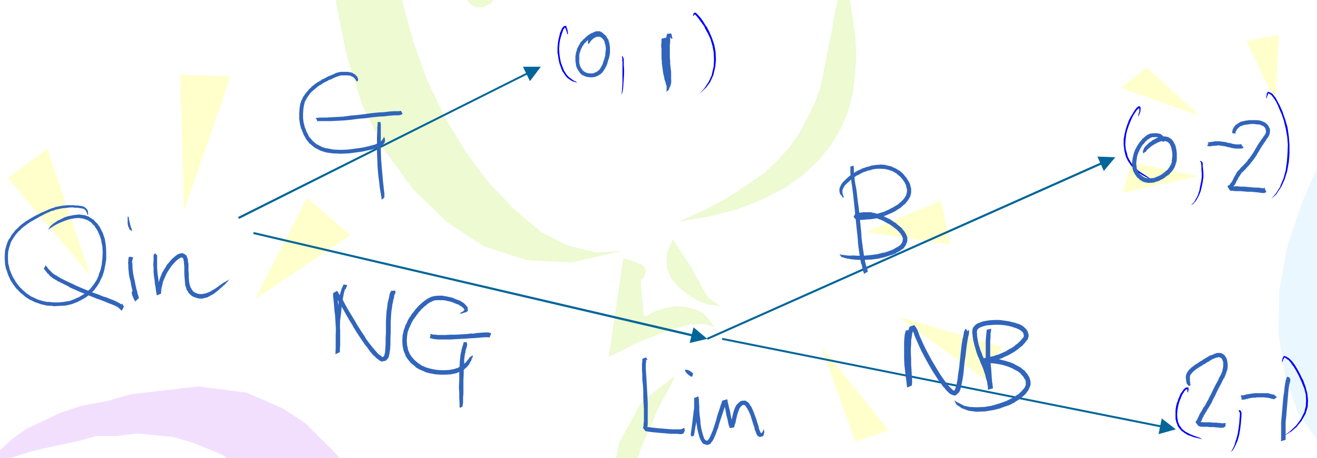
Strategic Form

		Monopolist	
		fight	no fight
Entrant	no entry	(0 *,2 *)	(0,2 *)
	entry	(-1,-1)	(1 *,1 *)

Incredible Threats

完璧归赵
史记卷八十一　廉颇蔺相如列传

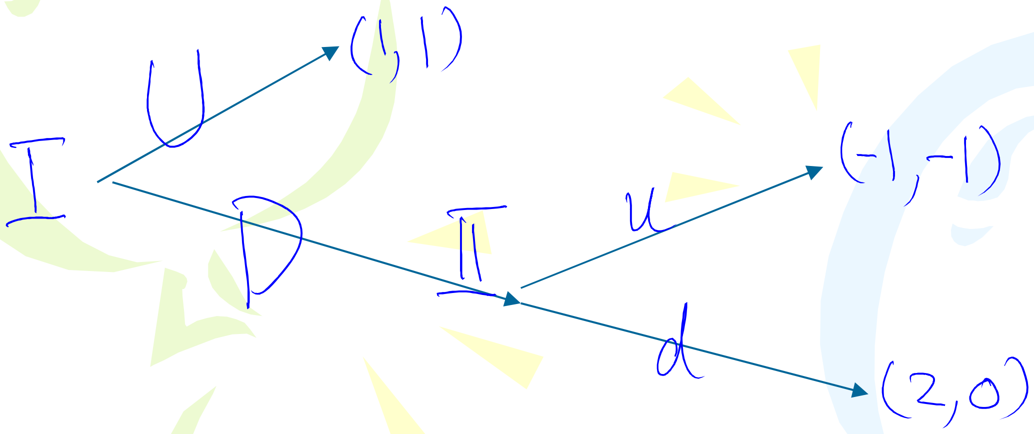
秦王坐章台见相如，相如奉璧奏秦王。秦王大喜，传以示美人及左右，左右皆呼万岁。相如视秦王无意偿赵城，乃前曰：“璧有瑕，请指示王。”王授璧，相如因持璧却立，倚柱，怒发上冲冠，谓秦王曰：“大王欲得璧，使人发书至赵王，赵王悉召群臣议，皆曰‘秦贪，负其强，以空言求璧，偿城恐不可得’。议不欲予秦璧。臣以为布衣之交尚不相欺，况大国乎！且以一璧之故逆强秦之欢，不可。于是赵王乃斋戒五日，使臣奉璧，拜送书于庭。何者？严大国之威以修敬也。今臣至，大王见臣列观，礼节甚倨；得璧，传之美人，以戏弄臣。臣观大王无意偿赵王城邑，故臣复取璧。大王必欲急臣，臣头今与璧俱碎于柱矣！”相如持其璧睨柱，欲以击柱。秦王恐其破璧，乃辞谢固请，召有司案图，指从此以往十五都予赵。相如度秦王特以诈详为予赵城，实不可得，乃谓秦王曰：“和氏璧，天下所共传宝也，赵王恐，不敢不献。赵王送璧时，斋戒五日，今大王亦宜斋戒五日，设九宾于廷，臣乃敢上璧。”秦王度之，终不可强夺，遂许斋五日，舍相如广成传。相如度秦王虽斋，决负约不偿城，乃使其从者衣褐，怀其璧，从径道亡，归璧于赵。



Strategic Form

		Lin	
		Break	Not Break
Qin	Not Give	(0 ,-2)	(2,-1)
	Give	(0,1)	(0 ,1)

Example:



Strategic Form

	u	d
U	(1,1)	(1,1)
D	(-1,-1)	(2,0)

Remark on Incredible Threats:

PSE allow players to make noncredible threats provided they never have to carry them out. Decisions on the equilibrium path are driven in part by what the players expect will happen off the equilibrium path.