Assignment Solution on 0-sum games

0-Sum1. Show that if a matrix game A has two saddle points then their values must be equal.

(Remark: The value of a saddle point is defined to be the value of the game. This exercise shows the consistency of this definition.)

Solution:

Suppose a_{ij}, a_{kl} are saddle points. Then, $a_{ij} \geq a_{kj} \geq a_{kl} \geq a_{il} \geq a_{ij}$. Hence, the four terms are all equal.

0-Sum2. Show that if a 2x2 matrix game has a saddle point then it either has a dominated row or a dominated column.

(Remark: This result only works for 2x2 game. It is easy to show this result is not true for bigger games.)

Solution: Without loss of generality, we assume that for $\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ a is a saddle point. Thus, $a \leq b, d \leq a$. Now, either $d \leq c$ or $d > c$. If $d \leq c$, then the first column dominates the second column. Suppose $d > c$. Then, $b > c$ by transitivity. Thus, the first row dominates the second column in this case.

0-Sum3. Given the matrix game $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -1 \end{pmatrix}$, (i) Find a BR to Player I's strategy of (0.5, 0.5), (ii) Find the set all BR to (0.5, 0.5), (iii) Find a BR to Player II's strategy of (1/3, 1/3, 1/3).

(Remark: This is just a simple exercise getting you familiar with the concept of BR.)

Solution:

$$\begin{array}{cc} 0.5 & \multirow{2}{*}{$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -1 \end{pmatrix}$} \\ 0.5 & \end{array}$$

$$\begin{array}{ccc} 2 & 2 & 1 \\ \text{BR} & \text{BR} & \end{array}$$

$$\begin{array}{cc} \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} & \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -1 \end{pmatrix} & \begin{array}{l} 2 \ \text{BR} \\ 4/3 \end{array} \end{array}$$

0-Sum4. Given a matrix game A , let p be a mixed strategy of Player I. Show that the set of BR to p is a convex set. (A set C is called a convex set if $p,q \in C$ implies $\lambda p + (1-\lambda)q \in C$ for any $\lambda \in [0,1]$.)

(Remark: Convexity is a good property. You will see many similar results in this course.)

Solution:
Suppose q_1, q_2 are BR to p i.e. $p^T A q_1 = \max_q p^T A q = p^T A q_2$.

Hence, for any $\lambda \in [0,1]$,

$$p^T A(\lambda q_1 + (1-\lambda) q_2) = p^T A(\lambda q_1 + (1-\lambda) q_2) = \lambda p^T A q_1 + (1-\lambda) p^T A q_2 = \max_q p^T A q$$

Hence, $\lambda q_1 + (1-\lambda) q_2$ is a BR to p .

0-Sum5. Suppose $A = (a_{kl})$ is a matrix game and that a_{ij} is a saddle point. Show that Row i, Col j are safety strategies for Player I and Player II respectively.

(Remark: We will prove this result in full generality.)

Solution:
We will show that Rowi is a safety strategy for Player I by showing that it achieves the maximin. In fact, let $p=(p_1,\cdots p_m)^T$ is a mixed strategy of Player I. Consider $\min_q p^T Aq$. The minimum occurs at a pure strategy, a column, of Player II. Then, $\min_q p^T Aq = \min(\sum_r p_r a_{r1} \cdots, \sum_r p_r a_{rn}) \le \sum_r p_r a_{rj} \le a_{ij}$.
Note that $\min_q p^T Aq = a_{ij}$ when p is Rowi. Therefore, $\max_p \min_q p^T Aq = a_{ij}$. This show that Rowi is a safety strategy of Player I.
The proof that Columnj is a safety strategy of Player II follows similar arguments.

0-Sum6.

Solve the game with matrix $\begin{pmatrix} 0 & 2 \\ t & 1 \end{pmatrix}$ for any arbitrary number t . Draw the graph of $v(t)$, the value of the game, as a function of t , for $-\infty < t < \infty$.

Solution:

If $t<0$, the strategy pair $\langle 1,1 \rangle$ is a saddle-point, and the value is $v(t)=0$. If $0 \le t \le 1$, the strategy pair (2,1) is a saddle-point, and the value is $v(t)=t.If t>1$, there is no saddle-point; I's optimal strategy is $((t-1)/(t+1), 2(t+1))$, II's optimal strategy is $(1/(t+1), t/(t+1))$, and the value is $v(t)=2t/(t+1)$.

0-Sum7.

Suppose that $p_{1,}\ p_2$ are optimal strategies for the row player of a matrix game. Prove that if $0 \le t \le 1$ then $tp_1+(1-t)p_2$ is also an optimal strategy for the row player.

0-Sum8.

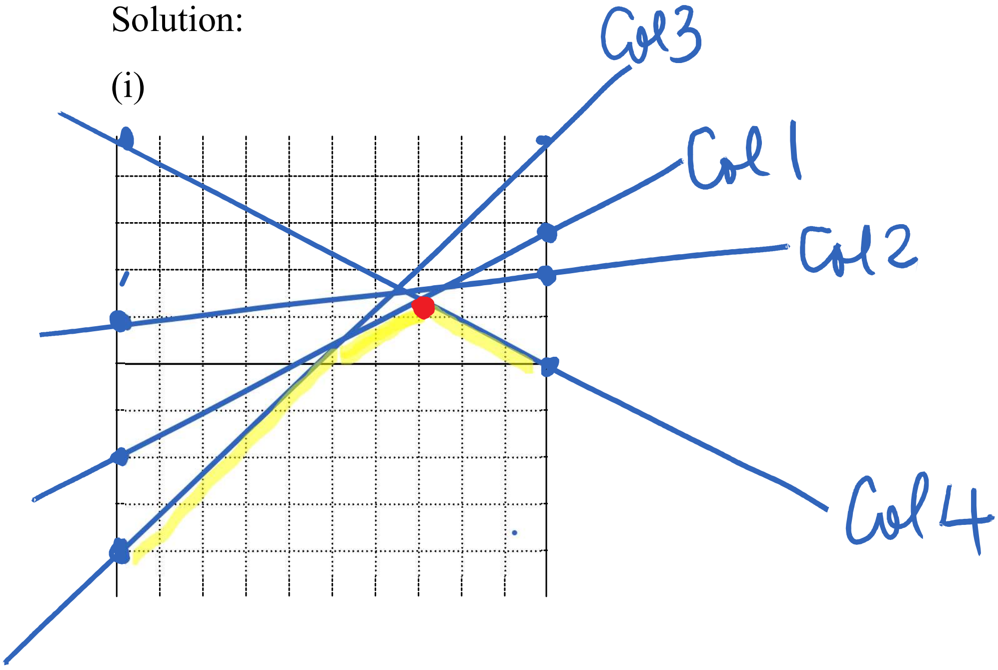
Solve the following games.

(i) $\begin{array}{|cccc|}\hline 3 & 2 & 5 & 0 \\ -2 & 1 & -4 & 5 \\ \hline\end{array}$

(ii) $\begin{array}{|cc|}\hline 3 & -5 \\ 1 & -4 \\ 2 & -1 \\ -1 & 3 \\ \hline\end{array}$

Solution:

(i)

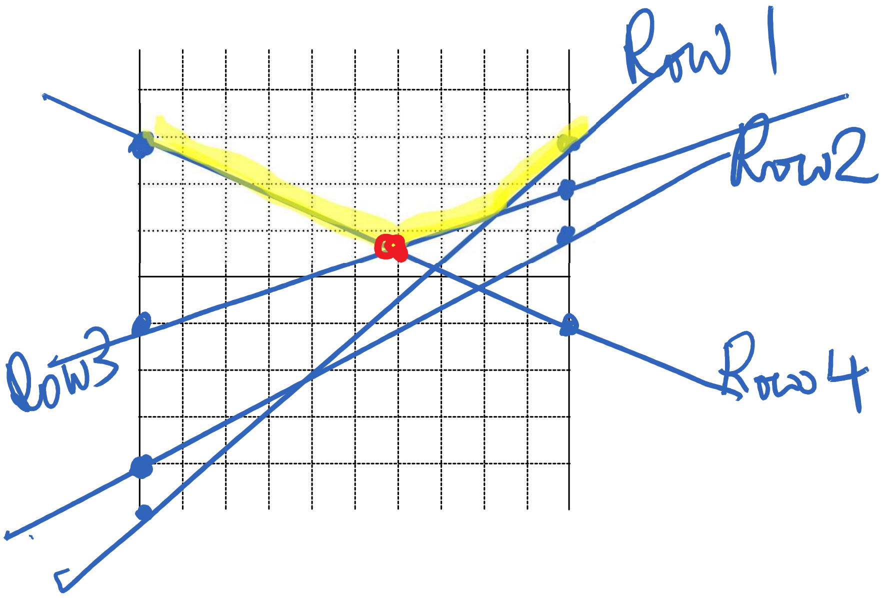


The Maximin occurs at the intersection of Col1 and Col4. To compute p, we consider the 2x2 game $\begin{array}{c|cc|}\cline{2-3} 7/10 & 3 & 0 \\ 3/10 & -2 & 5 \\ \cline{2-3}\end{array}$ and get p =7/10. Hence, the safety strategy for Player I is (7/10, 3/10). For Player II's BR, Col2, Col3 have to be deleted. Then, considering the 2x2 game $\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \hline \multicolumn{1}{|c}{3} & \multicolumn{1}{c|}{0} \\ \multicolumn{1}{|c}{-2} & \multicolumn{1}{c|}{5} \\ \hline\end{array}$ we get the (1/2,0,0,1/2) as the BR to (7/10, 3/10) and (7/10,3/10) is the BR to (1/2,0,0,1/2).

Answer: Safety strategy for I=(7/10, 3/10), Safety strategy for II=(1/2,0,0,1/2).

Value = 3/2

(ii)



The Minimax occurs at the intersection of Row3 and Row4. Considering the 2x2 game $\begin{array}{|cc|}\hline 2 & -1 \\ -1 & 3 \\ \hline\end{array}$, we get q=4/7.

Answer: Safety strategy for Player II is (4/7,3/7).

Safety strategy for Player I is (0,0,4/7,3/7). Value=5/7.

0-Sum9.

Reduce by domination to 2x2 games and solve.

(a) $\begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 10 & 0 & 7 & 1 \\ 2 & 6 & 4 & 7 \\ 6 & 3 & 3 & 5 \end{pmatrix}$.

Solution:

a) Column 2 dominates column 1; then row 3 dominates row 4; then column 4 dominates column 3; then row 1 dominates row 2. The resulting submatrix consists of row 1 and 3 vs. columns 2 and 4. Solving this 2 by 2 game and moving back to the original game we find that value is 3/2, I's optimal strategy is $p(1/2,0,1/2,0)$ and II's optimal strategy is $q=(0,3/8,0,5/8)$.

(b) Column 2 dominates column 4; then (1/2)row 1+ (1/2)row 2 dominates row 3; then (1/2)col 1+(1/2)col 2 dominates col 3. The resulting 2 by 2 game is easily solved.. Moving back to the original game we find that the value is 30/7, I's optimal strategy is (2/7,5/7,0) and II's optimal strategy is (3/7,4/7,0,0).

0-Sum10.

For the following matrix games, reduce by domination and solve.

(i)

10	3	7	3
2	6	4	7
6	2	3	5

(ii)

1	1	0	0	0
0	0	1	0	1
0	1	0	1	0
1	0	0	0	1
0	0	0	1	1

(iii)

1	1	0	0	0	0	0
0	1	1	0	0	0	0
0	0	1	1	0	0	0
0	0	0	1	1	0	0
0	0	0	0	1	1	0
0	0	0	0	0	1	1

Solution:

(i) We delete the 4th column as it is dominated by the 2nd column. Then, for the resulting 3x3 game, we can delete the 3rd row because it is dominated by 0.5 times 1st row plus 0.5 times 2nd row. For the resulting 2x3 game, the 3rd

column is dominated by 0.5 times 1st column plus 0.5 times 2nd column. We then have a 2x2 game $\begin{array}{|cc|} 10 & 3 \\ 2 & 6 \end{array}$.

Answer: Safety strategy for Player I is (4/11,7/11,0,0)
Safety strategy for Player II is (3/11,8/11,0,0).
Value = 54/11.

(ii) 5th column is deleted as it is dominated by the 4th column. 3rd row is then deleted as it is dominated by the 1st row. First column is deleted as it is dominated by the 2nd column. Then we get a 3x3 diagonal game.

Answer: Safety strategy for Player I is (1/3,1/3,0,1/3)

Saftey strategy for Player II is (0,1/3,1/3,1/3,0).

Value =1. (Note that the safety strategies are not unique. We only give one answer here.)

(iii) 6th column is deleted as it is dominated by the 7th. 5th rwo is deleted as it is dominated by the 4th. 4th column is deleted as it is dominated by the 5th. 1st column is deleted as it is deleted by the 2nd. 1st and 3rd row are deleted as they are dominated by the 2nd. 2nd column is deleted as it is dominated by the 3rd. We then get a 3x3 diagonal game.

Answer: Safety strategy for Player I is (0,1/3,0,1/3,0,1/3).

Safety strategy for Player II is (0,0,1/3,0,1/3,0,1/3).

Value = 1.

0-Sum11.

Solve the following symmetric games.

(i) $\begin{array}{|ccc|} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{array}$

(ii)

0	−2	2
2	0	−3
−2	3	0

Solution:

(i) Let (x,y,z) be a safety strategy for Player I. Then, it satisfies the following set of inequalities.

$(x,y,z) \cdot (0,-1,2) \geqq 0$

$(x,y,z) \cdot (1,0,-3) \geqq 0$

$(x,y,z) \cdot (-2,3,0) \geqq 0$

Thus, $z \geqq y/2$, $x \geqq 3z$, $y \geqq 2x/3$.

So $x \geqq 3z \geqq 3y/2 \geqq x$. Thus, $x=3z=3y/2$. $x=1/2$, $y=1/3$, $z=1/6$.

Answer: (1/2,1/3,1/6) is a safety strategy for Player I and Player II.

(ii) Let (x,y,z) be a safety strategy for Player I. Then, it satisfies the following set of inequalities.

$(x,y,z) \cdot (0,2,-2) \geqq 0$

$(x,y,z) \cdot (-2,0,3) \geqq 0$

$(x,y,z) \cdot (2,-3,0) \geqq 0$

So $y \geqq z$, $z \geqq 2x/3$, $x \geqq 3y/2$. Thus, $x \geqq 3y/2 \geqq 3z/2 \geqq x$.

Hence, $x=3y/2=3z/2$. $x=3/7$, $y=2/7$, $z=2/7$.

Answer (3/7,2/7,2/7) is a safety strategy for Player I and Player II.

0-Sum12.

For the following matrix game (0.2, 0.6, 0.2) is a safety strategy of Player I.

12	-35	-2	-2	64	8
0	6	-11	20	0	-6
-7	7	25	-3	-74	10

Find the value of the game and a safety strategy of Player II.

Solution:

Note that the payoffs to Player I when Player II uses Col1,Col2, Col3, Col4, Col4, Col5, Col6 are 1, -2, -2, 11, -2, 0 respectively.

As Maximin achieves at safety strategy, we see that Value=2. Safety strategy for Player II should only involve Col2, Col3, Col5.

Let (0,x,y,0,z,0) be a safety strategy for Player II. Then, we must have

-35x-2y+64z=6x-11y=7x+25y-74z=-2.

We can solve these 3 equations in 3 unknowns to get x+0.4, y=0.4, z=0.2.

Answer: Safety strategy for Player II is (0,0.4.0.4,00.2,0). Value = -2.

0-Sum 13. For the following matrix game, it is given that $(52/143, 50/143, 41/143)$ is an optimal strategy for Player I.

$$\begin{pmatrix} 0 & 5 & -2 \\ -3 & 0 & 4 \\ 6 & -4 & 0 \end{pmatrix}$$ Find an optimal strategy for Player II.

Solution:

The payoff to Player I when Player II uses Col1, Col2, Col3 are all equal to 96/143.

Let (x,y,z) be a safety strategy for Player II. Then, x, y, z satisfy the following set of equations.

$5y-2z=-3x+4z=6x-4y=96/143$.

Answer: The safety strategy for Player II is (44/143, 42/143, 57/143).

0-Sum 14. Let A be an $n \times n$ matrix game and that A is invertible. Suppose further that Player II has an optimal strategy which is also an equalizing strategy. Find a formula of the value in terms of A.

(This is basically an exercise in Linear Algebra.)

Solution:

Let q be an optimal strategy for Player II. Then, $Aq=v(1,1,\ldots,1)^T$. Note that v is not equal to 0 because A is invertible. As $q=vA^{-1}(1,\ldots,1)^T$ and $(1,\ldots,1)^T \cdot q=1$, we have

$$v = (1,\ldots,1)A^{-1}(1,\ldots,1)^T$$

0-Sum15.

For the game Matching Pennies, write it in the form of a linear programming problem as in the lecture.