ESS1. *Proof.* Let $p = (0.25, 0.75, 0)$ then

$$Ap = \begin{bmatrix} 5 & 7 & 2 \\ 8 & 6 & 5 \\ 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.75 \\ 0 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 6.5 \\ 6.25 \end{bmatrix}$$

So p is a BR to p. Let $x = (t, -t, 0)$ then

$$x^{\mathrm{T}} Ax = \begin{bmatrix} t & -t & 0 \end{bmatrix} \begin{bmatrix} 5 & 7 & 2 \\ 8 & 6 & 5 \\ 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = -4t^2 \leqslant 0$$

The equality holds only if $t = 0$. So p is an ESS. □

ESS2. *Proof.* Let $p = (0.75, 0, 0.25, 0, 0)$ then

$$Ap = \begin{bmatrix} 0 & -1 & 3 & 3 & 3 \\ -1 & 0 & 3 & 3 & 3 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0 \\ 0.25 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0 \\ 0.75 \\ 0.5 \\ 0.5 \end{bmatrix}$$

So p is a BR to p. Let $x = (t, 0, -t, 0, 0)$ then

$$x^{\mathrm{T}} Ax = \begin{bmatrix} t & 0 & -t & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 3 & 3 & 3 \\ -1 & 0 & 3 & 3 & 3 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 0 \\ -t \\ 0 \\ 0 \end{bmatrix} = -4t^2 \leqslant 0$$

The equality holds only if $t = 0$. So p is an ESS. □

ESS3. Since p is the BR to p and $p_i > 0$ for all i, p is an equalizing strategy.

(i) Let $E = (1, 1)$ then

$$Ap \parallel E \implies p \parallel A^* E = (1, 3) \implies p = (0.25, 0.75)$$

Let $x = (t, -t)$ then

$$x^{\mathrm{T}} A x = \begin{bmatrix} t & -t \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} t \\ -t \end{bmatrix} = 4t^2 \geqslant 0$$

So p is not an ESS.

(ii) Let $E = (1, 1, 1)$ then

$$Ap \parallel E \implies p \parallel A^* E = (1, 3, 2) \implies p = (1/6, 1/2, 1/3)$$

Let $x = (s, t, -s-t)$ then

$$x^{\mathrm{T}} A x = -3s^2 - 6st - 5t^2 = -3\,(s+t)^2 - 2t^2 \leqslant 0$$

The equality holds only if $s, t = 0$. So p is an ESS.

ESS4. Since there is no diagonally dominant column, there is no pure ESS.

(i) Let the submatrix

$$A_1 = \begin{bmatrix} 6 & 5 \\ 8 & 4 \end{bmatrix}$$

Let $E = (1, 1)$ and p_1 be the equalizing strategy of A_1 then

$$A_1 p \parallel E \implies p_1 \parallel A_1^* E = (1, 2) \implies p = (1/3, 2/3)$$

Let $p = (0, 1/3, 2/3)$ then

$$Ap = \begin{bmatrix} 5 & 7 & 2 \\ 8 & 6 & 5 \\ 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 16/3 \\ 16/3 \end{bmatrix}$$

So p is a BR to p. Let $x = (t, -t)$ then

$$x^{\mathrm{T}} A_1 x = \begin{bmatrix} t & -t \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} t \\ -t \end{bmatrix} = -3t^2 \leqslant 0$$

The equality holds only if $t = 0$. So p is an ESS.

(ii) Let the submatrix

$$A_2 = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$$

Let $x = (t, -t)$ then

$$x^{\mathrm{T}} A_2 x = \begin{bmatrix} t & -t \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} t \\ -t \end{bmatrix} = 7t^2 \geqslant 0$$

So there is no ESS that uses only Row 1 and Row 3.

ESS5. *Proof.* By definition

$$w = \sum_i p_i w_i = \sum_i \left(p_i \sum_j a_{ij} p_j \right) = \sum_{i,j} p_i a_{ij} p_j$$

We have that

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = p_i \left(\sum_j a_{ij} p_j - \sum_{k,j} p_k a_{kj} p_j \right) = p_i \left(w_i - w \right)$$

$$\sum_i p_i \left(w_i - w \right) = \sum_i p_i w_i - w \sum_i p_i = w - w \cdot 1 = 0$$

Since A is symmetry, $a_{ij} = a_{ji}$, then

$$\begin{aligned}
&\frac{\mathrm{d}w}{\mathrm{d}t} \\
&= \sum_{i,j} \frac{\mathrm{d}p_i}{\mathrm{d}t} a_{ij} p_j + \sum_{i,j} p_i a_{ij} \frac{\mathrm{d}p_j}{\mathrm{d}t} \\
&= \sum_{i,j} \frac{\mathrm{d}p_i}{\mathrm{d}t} a_{ij} p_j + \sum_{i,j} \frac{\mathrm{d}p_i}{\mathrm{d}t} a_{ji} p_j \\
&= \sum_{i,j} \frac{\mathrm{d}p_i}{\mathrm{d}t} a_{ij} p_j + \sum_{i,j} \frac{\mathrm{d}p_i}{\mathrm{d}t} a_{ij} p_j \\
&= 2 \sum_{i,j} \frac{\mathrm{d}p_i}{\mathrm{d}t} a_{ij} p_j \\
&= 2 \sum_i \left(\frac{\mathrm{d}p_i}{\mathrm{d}t} \sum_j a_{ij} p_j \right) \\
&= 2 \sum_i p_i \left(w_i - w \right) w_i \\
&= 2 \sum_i p_i \left(w_i - w \right) w_i - 2w \cdot 0 \\
&= 2 \sum_i p_i \left(w_i - w \right) w_i - 2w \sum_i p_i \left(w_i - w \right) \\
&= 2 \sum_i p_i \left(w_i - w \right)^2 \\
&\geqslant 0
\end{aligned}$$

□