One of the central themes in Games in Coalition Form is to find an appropriate payoff vector, $(x_1,\ldots,x_n)$. Our basic assumption is that the payoff vector must be group rational as well as individually rational, i.e. $(x_1,\ldots,x_n)$ must be an imputation.

If $(x_1,\ldots,x_n)$ is interpreted as the payoff to each people in a society, then the properties of the payoff vector is a central topic in philosophy and in political science.

φ is called a value function if φ assigns any game in coalition form (N, v) an imputation of (N, v) i.e.

$\varphi(v) = (\varphi_1(v), \ldots, \varphi_t(v))$ is an imputation for (N, v).

Example: Me-First value

φ is called a Me-First value if $\varphi_T(v)$ is the largest among all $\varphi_i(v)$, $1 \leq i \leq T$, for any v.

Question: Is Me-First value good?

Question:

What is a Win-Win (合作共赢) value?

What is a fair value?

What is Justice?

Aristotle (384-322 BC) believes that justice consists in giving people what they deserve.

Justice, in the tradition of Aristotle, means treating individuals in accordance with their deserts, treating equals equally, and treating unequals unequally.

It is an important question in philosophy and political theory on the question of a just distribution of resources in the society.

We assume that $(x_1,\ldots,x_n)$ is an imputation.

What other properties that $(x_1,\ldots,x_n)$ should possess besides group and individual rationality?

Does a solution with the desirable properties exist?

When a solution exists, we want to know whether it is unique. In real life situation, a unique solution is desirable.

We will discuss solution concepts such as Shapley value and Nucleolus.

The Shapley Value

The concept of the core is useful as a measure of stability. As a solution concept, it presents a set of imputations without distinguishing one point of the set as preferable to another. Also the core may be empty.

The Shapley Value is to assign to each game in coalitional form a UNIQUE vector of payoffs, called the Shapley Value. The i^{th} entry of the value vector may be considered as a measure of the value or power of the i^{th} player in the game.

Alternatively, the value vector may be thought of as an arbitration outcome of the game decided upon by some fair and impartial arbiter. The central "value concept" in game theory is the one proposed by Shapley in 1953.

Value Functions. The Shapley Axioms.

When the arbiter requires the players to form the grand coalition, the key question is then how to split the worth of the grand coalition, v(N), among the players fairly.

Shapley approached this problem by axiomatizing the concept of fairness.

A **value function**, $\boldsymbol{\varphi}$, is function that assigns to each possible set function of an n-person game, v, an n-tuple, $\boldsymbol{\varphi}(v) = (\varphi_1(v), \varphi_2(v), \ldots, \varphi_n(v))$ of real numbers.

$\varphi_i(v)$ represents the worth or value of player i in the game with characteristic function v.

The axioms of fairness are placed on the function, $\boldsymbol{\varphi}$.

The Shapley Axioms for $\varphi(v)$:

1. **Efficiency.** $\sum_i \varphi_i(v) = v(N)$.
2. **Symmetry.** If i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S not containing i and j, then $\varphi_i(v) = \varphi_j(v)$.
3. **Dummy Axiom.** If i is such that $v(S) = v(S \cup \{i\})$ for every coalition S not containing i, then $\varphi_i(v) = 0$.
4. **Additivity.** If u and v are characteristic functions, then $\boldsymbol{\varphi}(u + v) = \boldsymbol{\varphi}(u) + \boldsymbol{\varphi}(v)$.

Remark:

- Axiom 1 is group rationality, that the total value of the players is the value of the grand coalition.
- The second axiom says that if the characteristic function is symmetric in players i and j, then the values assigned to i and j should be equal.
- The third axiom says that if player i is a dummy in the sense that he neither helps nor harms any coalition he may join, then his value should be zero.
- The strongest axiom is number 4. It reflects the feeling that the arbitrated value of two games played at the same time should be the sum of the arbitrated values of the games if they are played at different times. It should be noted that if u and v are characteristic functions, then so is u + v.
- All the axioms can be postulated on a general set function. Superadditivity is not needed.

Example: Let $S \subseteq N$. Define a characteristic function w_S such that

$w_S(T) = 1$ if $S \subseteq T$,

$w_S(T) = 0$ otherwise.

Find $\varphi(w_S)$ using the Shapley axioms.

Solution: For $i \notin S$, Player i is a dummy as it can never contributes.

Thus, $\varphi_i(w_S)=0$ for $i \notin S$.

For $i, j \in S$, their contributions are symmetrical.

As $\sum_{i \in N} \varphi_i(w_S)=1= \sum_{i \in S} \varphi_i(w_S)$, we see that

$\varphi_i(w_S)=0$ for $i \notin S$,

$\varphi_i(w_S)=1/|S|$, for $i \in S$.

Theorem: There exists a unique function **φ** on the set of all set functions satisfying the Shapley axioms.

Proof:

Uniqueness: Suppose **φ** satisfies the Shapley axioms.

For a given nonempty set $S \subseteq N$, let w_S be defined for all $T \subseteq N$ such that

$w_S(T) = 1$ if $S \subseteq T$,

$w_S(T) = 0$ otherwise.

From axiom 3, $\varphi_i(w_S) = 0$ if i does not lie in S.

From axiom 2, if both i and j are in S,

then $\varphi_i(w_S) = \varphi_j(w_S)$.

From axiom 1, $\sum_i \varphi_i (w_S) = w_S(N) = 1$,

so that $\varphi_i (w_S) = 1/|S|$ for all $i \in S$.

Similarly,

$\varphi_i (cw_S) = c/|S|$ for $i \in S$

$\varphi_i (cw_S) = 0$ for i does not lie in S.

Claim: Any v may be written as $v = \sum_{S \subseteq N} c_S w_S$

Proof: For all $T \subseteq N$, define c_T inductively according to the number of elements in T.

Let $c_\emptyset = 0$.

Then, set

$c_T = v(T) - \sum_{S \subsetneq T} c_S$

Then,

$\sum_{S\subseteq N} c_S\, w_S\,(T) = \sum_{S\subseteq T} c_S = c_T + \sum_{S\subsetneq T} c_S = v(T).$

Now we have shown $v = \sum_{S\subseteq N} c_S\, w_S$ for any v.

Axiom 4 says that if a value function exists, it must be

$\varphi_i(v) = \sum_{S\subseteq N\ i\in S} [c_S / |S|]$

This completes the proof of uniqueness by noting that c_S is independent of φ.

Remark: Note that the superadditivity of v is not needed in this proof. The linear space of set functions has dimension equal to 2^n. The above shows that $\{w_S\}$ forms a basis.

Example: Given $v(\emptyset)=0$, $v(1)=1$, $v(2)=0$, $v(3)=1$, $v(1,2)=4$, $v(1,3)=3$, $v(2,3)=5$, $v(1,2,3)=8$. Find the Shapley value of v.

Solution: We find c_S for $S \subseteq N$.

$c_{\{1\}}=v(1)=1$, $c_{\{2\}}=v(2)=0$, $c_{\{3\}}=v(3)=1$.

$c_{\{1,2\}}=v(1,2)-c_{\{1\}}-c_{\{2\}}=4-1-0=3$.

$c_{\{1,3\}}=v(1,3)-c_{\{1\}}-c_{\{3\}}=3-1-1=1$.

$c_{\{2,3\}}=v(2,3)-c_{\{2\}}-c_{\{3\}}=5-0-1=4$.

$$c_{\{1,2,3\}}=v(1,2,3)-c_{\{1,2\}}-c_{\{1,3\}}-c_{\{2,3\}}-c_{\{1\}}-c_{\{2\}}-c_{\{3\}}$$
$$=8-3-1-4-1-0-1=-2$$

Then,

$$v=w_{\{1\}}+w_{\{3\}}+3w_{\{1,2\}}+w_{\{1,3\}}+4w_{\{2,3\}}-2w_{\{1,2,3\}}$$

Since $v = w_{\{1\}} + w_{\{3\}} + 3w_{\{1,2\}} + w_{\{1,3\}} + 4w_{\{2,3\}} - 2w_{\{1,2,3\}}$

$\varphi_1(v) = 1+0+3\times 1/2 + 1/2 + 0 - 2\times 1/3 = 7/3$

$\varphi_2(v) = 0+0+3\times 1/2 + 0 + 4\times 1/2 - 2\times 1/3 = 17/6$

$\varphi_3(v) = 0+1+0 + 1/2 + 4\times 1/2 - 2\times 1/3 = 17/6$

Existence: (1st Proof) **Proof by $\mathbf{v = \sum_{S \subseteq N} c_S w_S}$**

In the existence proof, we express v as a sum of set functions $c_S w_S$. Since we can figure out the payoff vector satisfying the Shapley axioms for each set function $c_S w_S$, we can then find the value for v using the Additivity Axiom in the following.

$\varphi_i(v) = \sum_{S \subseteq N\ i \in S} [c_S / |S|]$

We then show that this formula satisfies the Shapley Axioms (proof can be omitted).

Additivity Axiom: This is automatic from its definition.

Dummy Axiom:

Suppose i is a dummy, i.e. $v(T \cup \{i\}) = v(T)$, for any T not containing i.

Then, we have

$\sum_{S\subseteq T} c_S = v(T) = v(T\cup\{i\}) = \sum_{S\subseteq T} c_S + \sum_{S \subseteq T} c_{S\cup\{i\}}$

Thus, $0=\sum_{S \subseteq T} c_{S\cup\{i\}}$ for any T not containing i. Using this we can prove by induction on the number of elements of T to show that $c_{T\cup\{i\}} = 0$ for any T not containing i.

This is clearly true when T is empty.

Suppose it is true for S containing not more than k elements.

Now suppose T has k+1 elements. Note that

$0=\sum_{S\subseteq T} c_{S\cup\{i\}} = 0 + \ldots + 0 + c_{T\cup\{i\}}$ by using the inductive assumption.

$c_{T\cup\{i\}} = 0$ for any T not containing i implies that

$\varphi_i(v) = \sum_{S\subseteq N\ i\in S} [c_S / |S|] = 0$. This completes the proof of the Dummy Axiom.

Symmetry Axiom: Suppose for any subset T not containing i and j we have $v(T\cup\{i\}) = v(T\cup\{j\})$.

Thus,

$v(T\cup\{i\})=\sum_{S\subseteq T} c_S + \sum_{S\subseteq T} c_{S\cup\{i\}}$

$=v(T\cup\{j\})=\sum_{S\subseteq T} c_S + \sum_{S\subseteq T} c_{S\cup\{j\}}$

We then get $\sum_{S\subseteq T} c_{S\cup\{i\}} = \sum_{S\subseteq T} c_{S\cup\{j\}}$ for any T not containing i and j. As in the proof of the Dummy Axiom, we can prove by induction on the number of elements of T to get

$c_{S\cup\{i\}} = c_{S\cup\{j\}}$ for any S not containing i and j.

$$
\begin{aligned}
\text{Now } \varphi_i(v) &= \sum_{S\subseteq N\ i\in S} [c_S /|S|] = \\
&= \sum_{S\subseteq N\ i,j\in S} [c_S /|S|] + \sum_{S\subseteq N\ i\in S, j\notin S} [c_S /|S|] \\
&= \sum_{S\subseteq N\ i,j\in S} [c_S /|S|] + \sum_{S\subseteq N\ i\notin S, j\in S} [c_S /|S|] \\
&= \varphi_j(v)
\end{aligned}
$$

Efficiency Axiom:

Now note that $\sum \varphi_i (v) = \sum_{S\subseteq N} \; c_S$

From $v = \sum_{S\subseteq N} \; c_S w_S$, we get $v(N) = \sum_{S\subseteq N} \; c_S$

This completes the proof of the Efficiency Axiom.

Existence: (2nd proof) **Proof by an explicit formula of the Shapley Value.**

Suppose we form the grand coalition by entering the players into this coalition one at a time.

As each player enters the coalition, he receives the amount by which his entry **increases** the value of the coalition he enters.

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The amount a player receives by this scheme depends on the order in which the players are entered.

The Shapley value is just the average payoff to the players if the players are entered in completely random order.

Figuring out the formula:

Suppose we choose a random order of the players with all $n!$ orders (permutations) of the players equally likely. Then we enter the players according to this order.

When player i enters, he forms coalition S (that is, if he finds $S - \{i\}$ there already), he contributes the amount $[v(S) - v(S - \{i\})]$.

The probability that when i enters he will find coalition $S - \{i\}$ there already is

$(|S| - 1)!(n - |S|)!/n!$.

- The numerator is number of these permutations in which the

$|S| - 1$ members of $S - \{i\}$ come first ($(|S| - 1)!$ ways), then player i, and then the remaining $n - |S|$ players

($(n - |S|)!$ ways).

- The denominator is the total number of permutations of the n players.

Claim: The following value satisfies the Shapley axioms.

$\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$, where for $i = 1, \ldots, n$,

$$\varphi_i(v) = \sum_{S \subseteq N, i \in S} [(|S|-1)!(n-|S|)!]\,[v(S)-v(S-\{i\})]\,/n!$$

Remark: Let x= $(x_1, \ldots, x_n)$ such that

$x_1 = v(1)$

$x_2 = v(1,2) - v(1)$

$\vdots$

$\vdots$

$x_n = v(1,2,\ldots,n) - v(1,2,\ldots,n-1)$

This is the marginal contribution of each player when they join the coalition one by one according to the order [123…n].

Let π be a permutation of $\{1,\ldots,n\}$. Let x^π be the payoff vector when the players join the coalition according to the order $[\pi(1)\ldots\pi(n)]$.

Then, the average of x^π over all permutations is the Shapley value.

Proof of Claim:

Axioms 2, 3, and 4 are easy to check directly.

Axiom 1 follows from the above interpretation of the formula, since in each realization of forming the grand coalition, exactly v(N) is distributed to the players. Hence, the average amount given to the players is also v(N).

This completes the existence proof by an explicit formula.

It is useful to rephrase the theorem on Shapley value in terms of the explicit formula.

Theorem (Shapley Value): There exists a unique value function satisfying the Shapley axioms. Indeed, the Shapley value is given by an explicit formula.

$$\varphi_i(v) = \sum_{S \subseteq N, i \in S} [(|S|-1)!(n-|S|)!]\,[v(S)-v(S-\{i\})]\,/n!\ ,$$

for $i = 1, \ldots, n$.

Remark: The Shapley value gives us a unique payoff vector that satisfies the Efficiency, Symmetry, Dummy, Additivity axioms.

Question: Is the Shapley value individually rational, i.e. is $\varphi_i(v) \geq v(\{i\})$ when v is a characteristic function?

Answer: **Yes!** The proof easily follows from the formula for $\varphi_i(v)$.

Example: Given v(Ø)=0, v(1)=1, v(2)=0, v(3)=1, v(1,2)=4, v(1,3)=3, v(2,3)=5, v(1,2,3)=8. Find the Shapley value of v. Does it belong to the core?

Solution:

	1	2	3
123	1	3	4
132	1	5	2
213	4	0	4
231	3	0	5
312	2	5	1
321	3	4	1
	$14/6$	$17/6$	$17/6$

Since

$v(1,2) \leqq 14/6 + 17/6$

$v(1,3) \leqq 14/6 + 17/6$

$v(2,3) \leqq 17/6 + 17/6$

it belongs to the core.

Question: Does the Shapley value necessarily lie in the core if the core is not empty?

Answer: **No!**

The following is a counter example. v(Ø)=0, v(1)=0, v(2)=0, v(3)=0, v(1,2)=6, v(1,3)=6, v(2,3)=0, v(1,2,3)=6.

The core is not empty as (6,0,0) lies in the core.

The Shapley value is (4, 1, 1). It is unstable through the coalition {1, 2}.

	1	2	3
123	0	6	0
132	0	0	6
213	6	0	0
231	6	0	0
312	6	0	0
321	6	0	0
	4	1	1

Example: A certain object d'art is worth a_i dollars to Player i for i=1, 2, 3. We assume $a_1<a_2<a_3$, so Player 3 values the object most.

But Player 1 owns this object so $v(1)=a_1$. Player 2 and 3 by themselves can do nothing, so $v(2)=0$, $v(3)=0$, and $v(2,3)=0$.

If Player 1 and 2 come together, the join worth is a_2, so $v(1,2)=a_2$.

Similarly, $v(1,3)=a_3$. If all three get together, the object is still only worth a_3, so $v(1,2,3)=a_3$. Find the Shapley value.

Solution: The Shapley value is not in the core!!

	1	2	3
123	a_1	a_2-a_1	a_3-a_2
132	a_1	0	a_3-a_1
213	a_2	0	a_3-a_2
231	a_3	0	0
312	a_3	0	0
321	a_3	0	0
	$(2a_1+a_2+3a_3)/6$	$a_2-a_1/6$	$3a_3-2a_2-a_1/6$

We have shown that Shapley value may not lie in the core even when the core is nonempty. Convex games possess the nice property that the Shapley value lies in the core.

Theorem: The Shapley value of a convex game lies in the core.

Proof: Let (N, v) be a convex game.

When N is labelled as 1,..n, we have shown that the imputation

$x_1 = v(1)$

$x_2 = v(1,2) - v(1)$

:

:

$x_n = v(1,2,\ldots,n) - v(1,2,\ldots,n-1)$

lies in the core.

Clearly, the result is still true for any re-labeling of $\{1,\ldots n\}$.

Let π be a permutation of $\{1,\ldots n\}$ then

$x_{\pi(1)} = v(\pi(1))$

$x_{\pi(2)} = v(\pi(1), \pi(2)) - v(\pi(1))$

:

:

$x_{\pi(n)} = v(\pi((1), \pi(2),\ldots, \pi(n)) - v(\pi(1), \pi(2),\ldots, \pi(n-1))$.

This gives rise to the payoff vector x^{π}.

It also lies in the core.

As the core is convex, taking the average of n! of them it still lies in the core. Note that this average of n! imputations is exactly the Shapley value. This completes the proof that the Shapley value of a convex game lies in the core.

Corollary: The Shapley value of the Production Games and the Bankruptcy Games lies in the core.

Simple Games. The Shapley-Shubik Power Index.

The Shapley value has an important application in modeling the power of members of voting games. This application was developed by Shapley and Shubik in 1954 and the measure is now known as the Shapley-Shubik Power Index.

Players are members of legislature or members of the board of directors of a corporation, etc. In such games, a proposed bill or decision is either passed or rejected. Those subsets of the players that can pass bills without outside help are called winning coalitions while those that cannot are called losing coalitions.

In all such games, we may take the value of a winning coalition to be 1 and the value of a losing coalition to be 0. Such games are called simple games.

A method for evaluating the distribution of power in a committee system

Lloyd S. Shapley and Martin Shubik

In the following paper we offer a method for the *a priori* evaluation of the division of power among the various bodies and members of a legislature or committee system. The method is based on a technique of the mathematical theory of games, applied to what are known there as "simple games" and "weighted majority games." [1] We apply it here to a number of illustrative cases, including the United States Congress, and discuss some of its formal properties.

The designing of the size and type of a legislative body is a process that may continue for many years, with frequent revisions and modifications aimed at reflecting changes in the social structure of the country; we may cite the role of the House of Lords in England as an example. The effect of a revision usually cannot be gauged in advance except in the roughest terms; it can easily happen that the mathematical structure of a voting system conceals a bias in power distribution unsuspected and unintended by the authors of the revision. How, for example, is one to predict the legree of protection which a proposed system affords to minority interests? Can a consistent criterion for "fair representation" be found?[2] It is difficult even to *describe* the net effect of a double representation system such as is found in the U.S. Congress (i.e., by states and by population), without attempting to deduce it *a priori*. The method of measuring "power" which we present in this paper is intended as a first step in the attack on these problems.

Our definition of the power of an individual member depends on the chance he has of being critical to the success of a winning coalition. It is easy to see, for example, that the chairman of a board consisting of an even number of members (including himself) has no power if he is allowed to

THE CONCEPT OF POWER

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What is "power"? Most people have an intuitive notion of what it means. But scientists have not yet formulated a statement of the concept of power that is rigorous enough to be of use in the systematic study of this important social phenomenon. Power is here defined in terms of a relation between people, and is expressed in simple symbolic notation. From this definition is developed a statement of power comparability, or the relative degree of power held by two or more persons. With these concepts it is possible for example, to rank members of the United States Senate according to their "power" over legislation on foreign policy and on tax and fiscal policy.

THAT some people have more power than others is one of the most palpable facts of human existence. Because of this, the concept of power is as ancient and ubiquitous as any that social theory can boast. If these assertions needed any documentation, one could set up an endless parade of great names from Plato and Aristotle through Machiavelli and Hobbes to Pareto and Weber to demonstrate that a large number of seminal social theorists have devoted a good deal of attention to power and the phenomena associated with it. Doubtless it would be easy to show, too, how the word and its synonyms are everywhere embedded in the language of civilized peoples, often in subtly different ways: power, influence, control, pouvoir, puissance, Macht, Herrschaft, Gewalt, imperium, potestas, auctoritas, potentia, etc.

I shall spare the reader the fruits and myself the labor of such a demonstration. Reflecting on the appeal to authority that might be made does, however, arouse two suspicions: First (following the axiom that where there is smoke there is fire), if so many people at so many different times have felt the need to attach the label power, or something like it, to some Thing they believe they have observed, one is tempted to suppose that the Thing must exist; and not only exist, but exist in a form capable of being studied more or less systematically. The second and more cynical suspicion is that a Thing to which people attach many labels with subtly or grossly different meanings in many different cultures and times is probably not a Thing at all but many Things; there are students of the subject, although I do not recall any who have had the temerity to say so in print, who think that because of this the whole study of "power" is a bottomless swamp.

Paradoxical as it may sound, it is probably too early to know whether these critics are right. For, curiously enough, the systematic study of power is very recent, precisely because it is only lately that serious attempts have been made to formulate the concept rigorously enough for systematic study.[1] If we take as our criterion for the efficiency of a scientific concept its usability in a theoretical system that possesses a high degree

[1] By demonstrating the importance of concepts such as power and influence, particularly in political analysis, and by insisting upon rigorous conceptual clarity, Harold Lasswell has had a seminal influence. Cf. especially Reference 3. A similar approach will be found in References 6, 7, 8, 10. For the approach of the present article I owe a particularly heavy debt to March, with whom I had countless profitable discussions during a year we both spent as fellows at the Center for Advanced Study in the Behavioral Sciences. I have drawn freely not only on our joint work but on his own published and unpublished writings on the

SOME AMBIGUITIES IN THE NOTION OF POWER*

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The notion of power is often said to be central to the analysis of politics. But while that analysis is a very ancient activity, the conceptual clarification of the notion of power has been undertaken only in the past generation. The reason for this discrepancy I leave to the historians of political ideas. In this introduction I merely observe that the clarification has not proceeded as far as is needed, so that we are still not at all sure of what we are talking about when we use the term. Nevertheless there is light ahead, owing especially to some formal definitions that have been offered in recent years by Shapley and Shubik, March, Dahl, Cartwright, and Karlsson. By reason of the formality of these definitions the issues of meaning have been more sharply delineated than was previously possible. Hence we have reached the point, I believe, where we may confront definitions with each other and specify precisely how they differ. In so doing we may be able to resolve some of the ambiguities remaining in the concept of power. In that hope this essay is written.

But first a personal remark: most contemporary criticism of political theory is directed, unfortunately, at the so-called giants of the past. In such an enterprise, it is not personally embarrassing—indeed it is academically fashionable and intellectually trivial—to explain where Plato went wrong or what Rousseau meant. What political theory needs, however, is criticism of contemporary theory, for this is the theory that is important in guiding political research. But such criticism may be personally embarrassing, especially when, as in this instance, it is directed at the work of men whom I regard as at the very forefront of the social sciences. I want to make it clear, therefore, that (a) I regard the theories I discuss as a great advance, one which I have in the past struggled to make and failed and (b) I utter criticism not captiously but in the spirit of contributing to the dialectic of understanding.

I. FIVE FORMAL DEFINITIONS OF "POWER"

I start with a simple statement of the basic elements of each of the five definitions, ignoring most of the subtleties of each writer's interpretations, and usually using the symbols preferred by the authors. I have also offered verbal translations of the formal definitions, translations which exhibit, I suppose, all the characteristic pitfalls of translations generally.

Shapley, a mathematician who developed his notions originally to discuss the value of n-person games, was aided in applying it to social world by an economist, Shubik.[1] Their definition relates only to the power resulting from the right to vote in a system where voting, and only voting, determines outcomes:

$$P_i = \frac{m(i)}{n!},$$

where P is the power to determine outcomes in a voting body for a participant, i, in a set of participants: $\{1, 2, \ldots, n\}$ where $m(i)$ is the number of times i is in the pivotal position and where *pivotal position* is defined thus: when the rules define q votes as winning,

$$\frac{n+1}{2} \leq q \leq n \quad \text{or} \quad \frac{n}{2} + 1 \leq q \leq n,$$

the pivot position is the qth position in an ordered sequence of the votes. (Note that there are $n!$ ordered sequences or permutations of n things.)

Manifestly,

$$\sum_{i=1}^{n} P_i = 1.$$

In words, the Shapley-Shubik definition may be stated thus: the power of a voter to determine an outcome in a voting body is the ratio of (a) the number of possible times the voter may be in a pivotal position in an ordered sequence, to (b) the number of ordered sequences possible, *i.e.*, $n!$. What this measures is thus the participant's chance to be the last added member of a minimal winning coalition, a position that is highly attractive presumably because

* I thank Professors Robert Dahl, William Flanagan, Carl Hempel, and Dennis Sullivan for criticisms helpful in improving the argument of this paper. An earlier version was delivered at the Annual Meeting of the American Political Science Association, New York City, September 1963.

[1] L. S. Shapley and Martin Shubik, "A Method for Evaluating the Distribution of Power in a Committee System," this REVIEW, Vol. 48 (1954), pp. 787–92; L. S. Shapley, "A Value for N-Person Games," *Annals of Mathematics Study No. 28* (Princeton, 1953), pp. 307–17 and "Simple Games," *Behavioral Science*, Vol. 7 (1962), pp. 59–66.

The First Power Index

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An interesting puzzle in the history of social choice theory is why the discoveries of Borda and Condorcet in the 1780s had no immediate intellectual influence and were mostly forgotten until Duncan Black, himself puzzled about the oblivion, looked into the history of the mathematical theory of voting (Black 1958, p. 184). One explanation is that the French writers sowed on poor spoil (the French Academy), made more barren still by the subsequent Revolution. But the fact that Condorcet and Borda wrote at all indicates a reasonably wide intellectual interest in voting (then a newly blossoming institution). If this interest was indeed wide, then an explanation based on French politics is not persuasive. In fact there was wide interest and the purpose of this brief note is to call attention to an early expression – also in the 1780s – of the main intuition involved in power indices, that is, the intuition that in weighted voting games a voter's chance to win is not proportional to weight.

This expression comes from Luther Martin, a delegate from Maryland to the Constitutional Convention in Philadelphia in 1787. Among the fifty-five delegates Martin was one of only five or six true Antifederalists, that is, defenders of provincial political establishments against the (mostly Federal) framers. In his report to the Maryland legislature he criticized the Constitution at length, as well as the philosophy and motives of the framers. His criticism was published serially in the newspapers and as a pamphlet, *The Genuine Information, Delivered to the Legislature of the State of Maryland Relative to the Proceedings of the General Convention Lately Held at Philadelphia (1788)*. It has often been reprinted but is nowadays seldom read. Partly, the reason for this neglect is that the *Genuine Information* is a less adequate source for the debates than is Madison's *Notes of Debates in the Federal Convention of 1787* and is less intellectually impressive than the *Federalist Papers* or even other Antifederalist writings like the *Letters of a Federal Farmer*. Partly, also, the reason for the neglect is that Martin was extremely prolix and repetitive. Nevertheless, he was a clever and thoughtful man, as the extract here shows.

Shapley and Shubik (1954) and Straffin (1977) remark that one main use of power indices is to analyze constitutions. This was Martin's purpose. As a defender of small states, he assumed that large states would combine against the small when the new Constitution came in force. Consequently, he was impelled to ask what would happen if, in the House of Representatives, members from each state voted together. This imagined situation is thus a weighted voting game and it is appropriate to analyze it in terms of a power index.

The weights were: Virginia: 10; Massachusetts: 8; Pennsylvania: 8; New York: 6; Maryland: 6; Connecticut: 5; North Carolina: 5; South Carolina: 5; New Jersey: 4; New Hampshire: 3; Georgia: 3; Rhode Island: 1; and Delaware: 1.

Martin's use of the game model was:

"... even if the States who had most inhabitants ought to have a greater number of delegates, yet the *number of delegates* ought not to be in *exact proportion* to the *number of inhabitants*, because the influence and power of those States whose delegates are numerous, will be *greater* when compared to the influence and power of the other States, than the *proportion* which the numbers of delegates bear to each other; as for instance, though Delaware has *one* delegate, and Virginia but *ten*, yet Virginia has *more* than *ten times* as much *power* and *influence* in the government as Delaware; ... further ...-Virginia, as the system is now reported, by uniting with her the delegates of *four* other States, can carry a question *against* the sense and interest of *eight* States by *sixty-four* different combinations; the *four* States voting with Virginia, being every time *so far different* as not to be composed of the *same four*; whereas the State of Delaware can only, by uniting four other States with her, carry a measure against the sense of eight States by *two* different combinations – a mathematical proof that the State of *Virginia* has *thirty-two* times greater chance of carrying a measure against the sense of eight States than *Delaware*, although Virginia has only *ten times* as many delegates: (Kaminski and Saladino, III, 1984, p. 349; Farrand, III, 1965, p. 198; and Storing, II, 1981, p. 50).[1]

Evidently, in speaking of combinations, Martin had in mind the calculation later formalized as the Banzhaf power index (1965), which now, perhaps, ought to be called the Martin-Banzhaf index.[2]

As it turned out, members of the state delegations in the House have not systematically voted together, nor have the large and small states been pitted against each other. In both cases, party, region, and economic interest have been much more compelling points of difference than mere geography. So Martin's model turned out to be politically irrelevant and was forgotten. Nevertheless, he deserves credit for his calculation and intuition.

And they are indeed worth remembering. For one thing, they add to Black's puzzle about the century of oblivion for Condorcet. More importantly, they show that power indices are not merely mathematicians' fancies but obvious categories of thought for practical politicians.

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1 Maurice Salles and Philip Straffin have both pointed out to me that Martin miscounted. There are sixty-nine combinations by which Virginia can win with four other states

2 Straffin has also suggested that Martin's calculation is closer in spirit to the Deegan-Packel index (1979) than Banzhaf, since Martin did not count pivots. But Martin and Banzhaf do share a count of combinations as well as the basic intuition that the chance to win differs from the weight

Definition. A game (N, v) is **simple** if for every coalition $S \subset N$, either $v(S) = 0$ or $v(S) = 1$.

In a simple game, a coalition S is said to be a **winning** coalition if $v(S) = 1$ and a **losing** coalition if $v(S) = 0$.

So in a simple game every coalition is either winning or losing. It follows from superadditivity of v that in simple games every subset of a losing coalition is losing, and every superset of a winning coalition is winning.

For simple games, $[v(S) - v(S - \{i\})]$ is always zero or one. It is zero if $v(S)$ and $v(S - \{i\})$ are both zero or both one, and it is one otherwise.

Therefore we may remove $[v(S) - v(S - \{i\})]$ from the formula provided

S are winning with i and losing without i.

$$\varphi_i(v) = \sum_{S \text{ winning}, S-\{i\} \text{ losing}} (|S| - 1)!(n - |S|)!/n! .$$

Examples:

(1) the **majority rule game** where $v(S) = 1$ if $|S| > n/2$, and $v(S) = 0$ otherwise;

(2) the **unanimity game** where $v(S) = 1$ if $S = N$ and $v(S) = 0$ otherwise; and

(3) the **dictator game** where $v(S) = 1$ if $1 \in S$ and $v(S) = 0$ otherwise.

Example: **weighted voting games**.

They are defined by a characteristic function of the form

$v(S) = 1$ if $\Sigma_{i\in S}\, w_i > q$

$v(S) = 0$ if $\Sigma_{i\in S}\, w_i \leq q$

,

for some non-negative numbers w_i, called the weights, and some positive number q, called the quota.

If $q = (1/2) \sum_{i\in N} w_i$, this is called a weighted majority game.

Example: Suppose there are 4 players, I, II, III, IV. Player I has 1 vote, Player II, III, IV each has 2 votes. 4 votes are required to win. Find the power index of each player.

Solution: Since Player I can never pivot, his power index is 0. By the Symmetry axiom, the power index for Player II, III, IV are equal. Hence, each then has power 1/3.

Measurement of Power in Political Systems

L. S. Shapley

1. Introduction

We begin with a motivating example. County governments in New York State are headed by Boards of Supervisors. Typically each municipality in a county has one seat, though a larger city may have two or more. But the supervisorial districts are usually quite unequal in population, and an effort is made to equalize citizen representation throughout the county by giving the individual Supervisors different numbers of votes in council. Table 1 shows the situation in Nassau County in 1964:

Table 1.

District	Population	%	No. of Votes	%
Hempstead 1	728,625	57.1	31	27.0
Hempstead 2			31	27.0
Oyster Bay	285,545	22.4	28	24.3
North Hempstead	213,335	16.7	21	18.3
Long Beach	25,654	2.0	2	1.7
Glen Cove	22,752	1.8	2	1.7
Totals	1,275,801		115	

Under this system, a majority of 58 out of 115 votes is needed to pass a measure. But an inspection of the numerical possibilities reveals that the three weakest members of the board actually have no voting power at all. Indeed, even their combined total of 25 votes is never enough to tip the scales. The assigned voting weights (when everyone is present and voting) might just as well be (31, 31, 28; 0, 0, 0)--or (1, 1, 1, 0, 0, 0) for that matter.

This example shows that numerical voting weights may not translate into political power in the obvious way, and points up the need for a

Shareholding Structure of AIIB:

Vote Type	% of Total Votes	Total Votes	Vote per Member	China (Largest PFM)	Maldives (Smallest PFM)
Basic votes	**12**	138,510	2,430	2,430	2,430
Share votes	85	981,514	Varies	297,804	72
Founding Member votes	3	34,200	**600**	600	600
Total	**100**	1,154,224	varies	300,834 (26.1%)	3,102 (0.3%)

Example: Consider the game with players I II, III, and IV, having 10, 20, 30, and 40 shares of stock respectively, in a corporation. Decisions require approval by a majority (more than 50%) of the shares. This is a weighted majority game with weights

$w_1 = 10$, $w_2 = 20$, $w_3 = 30$ and $w_4 = 40$ and with quota $q = 50$.

Example: Find the power indices of the following weighted majority game.

$w_1 = 10$, $w_2 = 20$, $w_3 = 30$ and $w_4 = 40$.

Solution: To avoid confusion, we denote the players by I, II, III, IV. We first figure out the minimal winning coalitions. They are {I,II,III}, {II,IV}, {III,IV}. Then, from the minimal winning coalitions, we can figure out when will each player pivot and then find the power index.

I to pivot: [II, III, I]

Power index: 2!/4!=1/12

II to pivot: [I,III,II], [IV, II], [I, IV, II]

Power index: (2! + 2! + 2!)/4!=1/4

III to pivot: [I,II,III], [IV,III], [I,IV,III]

Power index: (2! + 2! + 2!)/4!=1/4

IV to pivot: [II,IV], [I,II,IV], [III,II,IV], [III,IV], [I,III,IV]

Power index: (2! + 2! +2! + 2! + 2!)/4!=5/12

Application to Politics

Example (UN Security Council): The League of Nations was formed after WWI to mediate peaceful solutions of conflicts in the world. At that time, countries big or small all had one vote. WWII broke out within two decades of the formation of the League of Nations. The United Nation was established in 1945. Based on the lesson of the League of Nations, the UN Security Council consisted of five permanent members and six non-permanent members. From 1946 to 1965, to pass a motion, all five permanent members and any two non-permanent members had to agree. Find the power indices for the members in the Security Council.

Solution: By the Symmetry axiom, all P (permanent member) have the same power index. Similarly, all N (Non-permanent member) have the same power index.

Let's compute the power index for N.

For N to pivot, it has to be the 7^{th} member to join and before its joining, all five P have to be on board as in the following.

PPPPPN N NNNN

7^{th}

To count the number of cases for N to pivot, we let this N to sit at the 7^{th} place. We select one seat from the first six seat to put another N in. There are 6 ways to do this. Once this seat is selected, we can put in the 5 P's and then the remaining 5 N's. Therefore, there are 6x5!x5! ways for N to pivot. The Shapley value for N is 6x5!x5!/11!=1/462.

The total power for N is then 6/462. The total power for P is 456/462. Each P then has power 1/5 x 456/462.

The power ratio is P:N=91.2:1

Example (UN Security Council 1965): In 1965,having complaints that the permanent members have too much power, UN reformed the composition of the Security Council. The reform was to have 5 P and 10 N. 11 votes are needed to win which must include all P.

To compute the power index for N, we argue as before to count the number of cases for N to pivot.

PPPPPNNNNN N NNNN

11^{th}

The number of cases is C_5^{10} x5!x9!. When we divide this number by 15! we get the power index of N.

The power ratio is: P:N=105.25:1

Current List of Non-Permanent Members:

- Belgium (https://newyorkun.diplomatie.belgium.be/) (2020)
- Dominican Republic (https://consejoseguridad.mirex.gob.do/) (2020)
- Estonia (https://un.mfa.ee/) (2021)
- Germany (https://new-york-un.diplo.de/un-en) (2020)
- Indonesia (https://www.kemlu.go.id/newyork-un/en/default.aspx) (2020)
- Niger (2021)
- Saint Vincent and the Grenadines (http://svg-un.org/) (2021)
- South Africa (https://www.southafrica-newyork.net/pmun/) (2020)
- Tunisia (https://www.diplomatie.gov.tn/en/nc/mission/etranger/mission-permanente-de-tunisie-aupres-des-nations-unies-a-new-york-etats-unis/) (2021)
- Viet Nam (https://vnmission-newyork.mofa.gov.vn/en-us/) (2021)

Example (Legislature in the US):In the United States, the House of Representatives have 435 members, and the senate has 101 members. To pass a bill, one needs

(i) 218 representatives, 51 senators, and the President.

Or

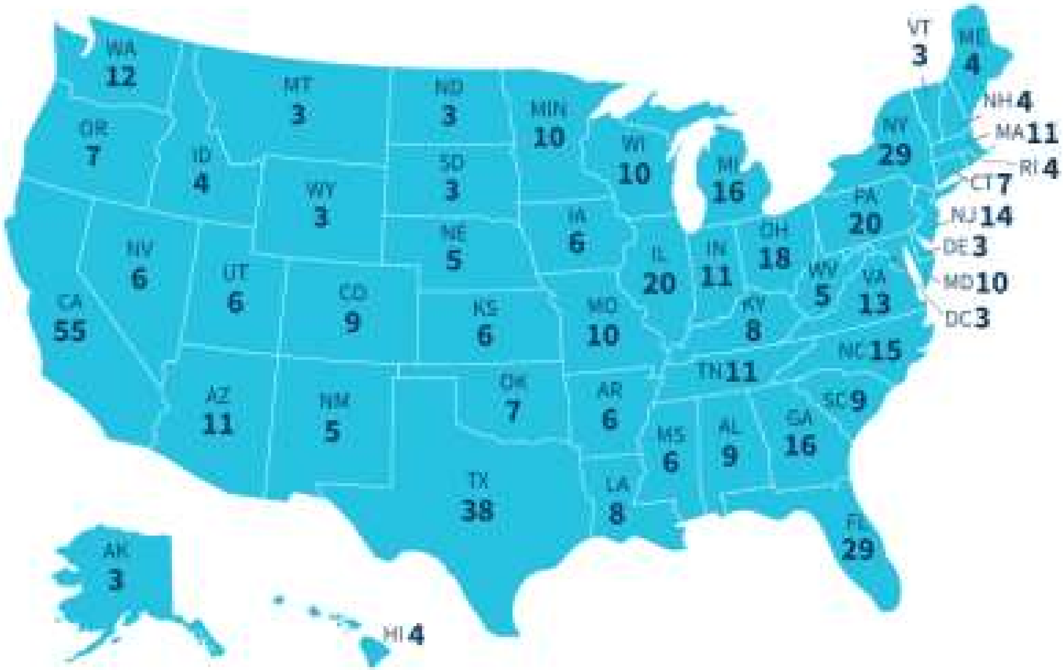
(ii) 290 representatives, 68 senators.

The power ratio is

President: Senator: Representative= 870:4.3:1

Example (US Presidential Election Game):

In the US, presidential election is not by popular votes. It is by the Electoral College formed by electors from each state. Each state has a certain number of electors in the Electoral College. One has to get 270 votes from the Electoral College to be elected president of USA. A presidential candidate who wins the popular vote in a certain state will grab all the electors from that state (Winner Takes All). Find and compare the power index for voter in each state.



ONE MAN, 3.312 VOTES: A MATHEMATICAL ANALYSIS OF THE ELECTORAL COLLEGE

JOHN F. BANZHAF III†

*The significant standard for measuring . . . voting power, as **Mr.** Banzhaf points out, is . . . his [voting member's] "ability ***, by his vote, to affect the passage or defeat of a measure". . . .*

* * * *

In order to measure the mathematical voting power . . . it would be necessary to have the opinions of experts based upon computer analyses.[1]

I. INTRODUCTION

IN THE WAKE of the Supreme Court's reapportionment decisions Congress is now seriously considering several proposals to abolish the Electoral College and to replace it with one of a number of alternative plans.[2] The ensuing discussion of this issue has been heated, and even the United States Chamber of Commerce and the American Bar Association have taken public stands.[3] Sentiment for some change in

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1. Iannucci v. Board of Supervisors, 20 N.Y.2d 244, 251–53, 229 N.E.2d 195, 198–99, 282 N.Y.S.2d 502, 507–09 (1967).

2. For the actual bills and an explanation of the proposed amendments see pp. 317–22 *infra*.

One interesting application of the reapportionment decisions in this area was a suit by the State of Delaware in the United States Supreme Court against all of the other states. Delaware v. New York, 385 U.S. 895 (1966). Although Delaware recognized that the entire Electoral College system could be modified only by constitutional amendment, it asked the Court to outlaw the unit-vote or "winner take all" system of casting electoral votes by which all of a state's votes, by law, go to the candidate receiving the greatest number of popular votes. It claimed that the effect of that system was to disadvantage voters in the less populous states. The arguments were based in part on the author's techniques. See Motion for Leave to File Complaint, Complaint and Brief at 83–85, Delaware v. New York, 385 U.S. 895 (1966). The Supreme Court declined to hear the case, apparently on procedural grounds.

This analysis supports Delaware's claim that its citizens have far less than average voting power but for reasons other than those stated in the complaint and brief. However it also shows that two remedies tentatively proposed — a proportional or a district system — would only create new and even greater inequalities.

3. The Chamber of Commerce of the United States, following a policy referendum of its member organizations, Jan. 31 [1966] announced it favored abolishing the electoral college and shifting to either a nationwide popular vote or a district system of choosing Presidential electors. The final vote of the Chamber members for approving the new policy position was 3,877 (91.5 percent) in favor and 362 (8.5 percent) opposed.

1966 CONGRESSIONAL QUARTERLY ALMANAC 496.

A special Commission on Electoral College Reform of the American Bar Association recommended in January 1967 that the President and Vice President be elected as a team by a national popular vote and that the Electoral College be abolished. The Commission terms the present system "archaic, undemocratic, complex

Example (Airport game): A group of pilots want to share the cost of an emergency landing strip. They fly different types of planes, which require different runway lengths. Find the Shapley values.

Pilots	*aircraft*	*runway*	cost
A, B, C, D	*type* 1	1000	50
E, F	*type* 2	1200	60
G	*type* 3	1500	80
H, I, J	*type* 4	1900	110

Example (Airport game): A group of pilots want to share the cost of an emergency landing strip. They fly different types of planes, which require different runway lengths. Find the Shapley values.

Pilots	aircraft	runway	cost
A,B,C,D	type 1	1000	50
E,F	type 2	1200	60
G	type 3	1500	80
H,I,J	type 4	1900	110

This is equivalent to the sum of the following Road Games.

Original game Γ_0 : $\Gamma_0 = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

$$\Gamma_0 \quad \text{TOWN} \xrightarrow{50} \text{A,B,C,D} \xrightarrow{10} \text{E,F} \xrightarrow{20} \text{G} \xrightarrow{30} \text{H,I,J}$$

$$\Gamma_1 \quad \text{TOWN} \xrightarrow{50} \text{A,B,C,D} \xrightarrow{0} \text{E,F} \xrightarrow{0} \text{G} \xrightarrow{0} \text{H,I,J}$$

$$\Gamma_2 \quad \text{TOWN} \xrightarrow{0} \text{A,B,C,D} \xrightarrow{10} \text{E,F} \xrightarrow{0} \text{G} \xrightarrow{0} \text{H,I,J}$$

$$\Gamma_3 \quad \text{TOWN} \xrightarrow{0} \text{A,B,C,D} \xrightarrow{0} \text{E,F} \xrightarrow{20} \text{G} \xrightarrow{0} \text{H,I,J}$$

$$\Gamma_4 \quad \text{TOWN} \xrightarrow{0} \text{A,B,C,D} \xrightarrow{0} \text{E,F} \xrightarrow{0} \text{G} \xrightarrow{30} \text{H,I,J}$$

Therefore, by Additivity axiom

$$\varphi(\Gamma_0) = \varphi(\Gamma_1) + \varphi(\Gamma_2) + \varphi(\Gamma_3) + \varphi(\Gamma_4)$$

	A	B	C	D	E	F	G	H	I	J
$\varphi(\Gamma_1)$	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5
$\varphi(\Gamma_2)$	0	0	0	0	$-10/6$	$-10/6$	$-10/6$	$-10/6$	$-10/6$	$-10/6$
$\varphi(\Gamma_3)$	0	0	0	0	0	0	-5	-5	-5	-5
$\varphi(\Gamma_4)$	0	0	0	0	0	0	0	-10	-10	-10
$\varphi(\Gamma_0)$	-5	-5	-5	-5	$-40/6$	$-40/6$	$-70/6$	$-130/6$	$-130/6$	$-130/6$

Theorem: (Individual Monotonicity)Let i, j be distinct elements in N.
Suppose that for any $S \subseteqq N$ such that $i, j \notin S$, we have
$v(S \cup \{i\}) \geqq v(S \cup \{j\})$.
Then, $\varphi_i (v) \geqq \varphi_j (v)$.

Proof: Note that

$$\varphi_i(v) = \sum_{S \subseteq N, i \notin S} [(|S|)!(n-|S|-1)!] [v(S \cup \{i\})-v(S)] \ /n!$$

$$\varphi_j(v) = \sum_{S \subseteq N, j \notin S} [(|S|)!(n-|S|-1)!] [v(S \cup \{j\})-v(S)] \ /n!$$

In $\varphi_i(v)$, we split the sum into two parts (i) S, $i, j \notin S$, (ii)$S \cup \{j\}$, $i, j \notin S$.
In $\varphi_j(v)$, we split the sum into two parts (i) S, $i, j \notin S$, (ii)$S \cup \{i\}$, $i, j \notin S$.

For S, $i, j \notin S$, $v(S \cup \{i\})-v(S)] \geqq v(S \cup \{j\})-v(S)]$,

$$v(S \cup \{i\} \cup \{j\})-v(S \cup \{j\})] \geqq v(S \cup \{j\} \cup \{i\})-v(S \cup \{i\})].$$

Hence the result.

Strong Monotonicity Axiom: A value function φ is said to be strongly monotonic if for any two set functions v, w such that whenever

$v(S \cup \{i\}) - v(S) \geqq w(S \cup \{i\}) - w(S)$ for any S not containing i,

we must have

$\varphi_i(v) \geqq \varphi_i(w)$.

Theorem: (Peyton Young): Shapley value is the unique set function satisfying the Efficiency Axiom, Symmetry Axiom and Strong Monotonicity Axiom.

Remark: Monotonicity is a desirable property for resource allocation (or cost allocation) schemes. Then, the Shapley value is an appealing solution concept for these problems.

Sketch of Proof:

Suppose we have a value function ψ satisfying the Efficiency Axiom, Symmetry Axiom and Strong Monotonicity Axiom. We will show that it coincides with the Shapley value

Recall that every set function v can be expressed as

$v = \sum_{S \subseteq N} c_S v_S$, where v_S is the set function such that $v_S(T)=1$ if $S \subseteq T$, $v_S(T)=0$ otherwise.

The Shapley value φ then assigns to Player i
$\varphi_i(v_S)=1/|S|$ for $i \in S$, $\varphi_i(v_S)=0$ for $i \notin S$.

We first show that the Dummy Axiom is valid for ψ.

Indeed, suppose $v(S\cup\{i\})-v(S)=0$ for any S not containing i.
Let η be the set function such that $\eta(S)=0$ for all S.
Then, $v(S\cup\{i\})-v(S)\geqq \eta(S\cup\{i\})- \eta(S)$, and
$\eta(S\cup\{i\})- \eta(S) \geqq v(S\cup\{i\})-v(S)$ for all S not containing i.
Hence, $\psi_i(v)= \psi_i(\eta)$. Now $\psi_i(\eta)=0$ because of the Efficiency Axiom and the Symmetry Axiom. Thus, $\psi_i(v)=0$. The Dummy Axiom is valid for ψ.

We will prove by induction on the number of nonzero terms in the summation $v= \sum_{S\subseteq N} c_S v_S$ that $\varphi_i(v)= \psi_i(v)$ for all i.

If there is one term in v i.e. $v = cv_S$. All members in S are then symmetrical and all members outside S are dummies. Couple these with the Efficiency Axiom we get $\varphi_i(v) = \psi_i(v)$ for all i.

Suppose the assertion is valid when v has most k-1 terms in the summation.

Let $S_{1, \ldots ,} S_k \subseteq N$ and $v = \sum_i c_{Si} v_{Si}$.

Let $R = S_1 \cap \ldots \cap S_k$. For $i \notin R$, we may suppose $i \notin S_k$.

Let w be obtained from v by deleting the term involving S_k.

Then, w has at most k-1 terms and the inductive hypothesis can be applied.

In other words, $\psi_i(w)=\varphi_i(w)$ for all i. As φ is the Shapley value, we have $\varphi_i(w)=\varphi_i(v)$, for $i \notin R$. Note that

$v(S\cup\{i\})-v(S)= w(S\cup\{i\})-w(S)$, for all S not containing i.

Hence, the Monotonicity Axiom implies that

$\psi_i(w)= \psi_i(v)$ for $i \notin R$. Hence, $\psi_i(v)=\psi_i(w)=\varphi_i(w)=\varphi_i(v)$, for $i \notin R$.

As for $i,j \in R$, we have

$v(S\cup\{i\})-v(S)=v(S\cup\{j\})-v(S)$ for all S not containing i and j.

They are then symmetrical. Since φ and ψ both satisfy the Efficiency Axiom, Symmetry Axiom and agree on $N \setminus R$, they should agree on R too. Thus, $\varphi_i(v)= \psi_i(v)$ for all i.

This completes the inductive procedures and hence ψ is the Shapley value φ as asserted.