Barg1. (i) The payoff matrices are

$$A = \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \quad A + B = \begin{bmatrix} 5 & 6 \\ 0 & 5 \end{bmatrix} \quad A - B = \begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix}$$

The Cooperative Strategy is $\langle \text{Row } 1, \text{Col } 2 \rangle$ and the payoff vector is $(5, 1)$ with $\sigma = 6$. $\langle \text{Row } 1, \text{Col } 1 \rangle$ is the only saddle point in $A - B$, so it is the Optimal Threat and the Disagreement Point is $(3, 2)$ with $\delta = 1$. Then the TU Cooperative Value is

$$\left(\frac{1}{2} (\sigma + \delta), \frac{1}{2} (\sigma - \delta) \right) = (3.5, 2.5)$$

The Side Payment is 1.5 from Player I to Player II.

(ii) The payoff matrices are

$$A = \begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 8 \\ 5 & 6 \end{bmatrix} \quad A + B = \begin{bmatrix} 11 & 10 \\ 9 & 11 \end{bmatrix} \quad A - B = \begin{bmatrix} 3 & -6 \\ -1 & -1 \end{bmatrix}$$

The Cooperative Strategies are $\langle \text{Row } 1, \text{Col } 1 \rangle$ and $\langle \text{Row } 2, \text{Col } 2 \rangle$. Each payoff vector is $(7, 4)$ or $(5, 6)$ with $\sigma = 11$. Since Col 1 is dominated by Col 2 in $A - B$, the Optimal Threat is $\langle \text{Row } 2, \text{Row } 2 \rangle$ and the Disagreement Point is $(5, 6)$ with $\delta = -1$. Then the TU Cooperative Value is

$$\left(\frac{1}{2} (\sigma + \delta), \frac{1}{2} (\sigma - \delta) \right) = (5, 6)$$

The Side Payment is 2 or 0 from Player I to Player II.

Barg2. *Proof.* Since the zero-sum game matrix $A - A^{\mathrm{T}}$ is skew-symmetric, the value of the game $\delta = 0$. So the payoff to Player I and Player II are equal in the TU Cooperative Value. $\square$

Barg3. (i) The game matrices are

$$A = \begin{bmatrix} 5 & 7 & 1 \\ 1 & 9 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 10 \\ 1 & -2 & 1 \end{bmatrix}$$

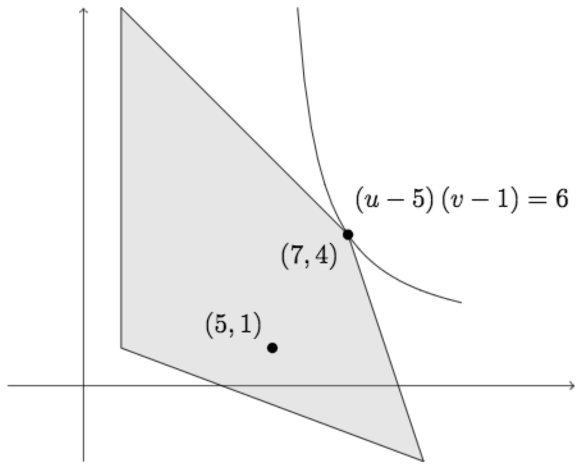


Figure 1: The Feasible Set in Barg3.(i)

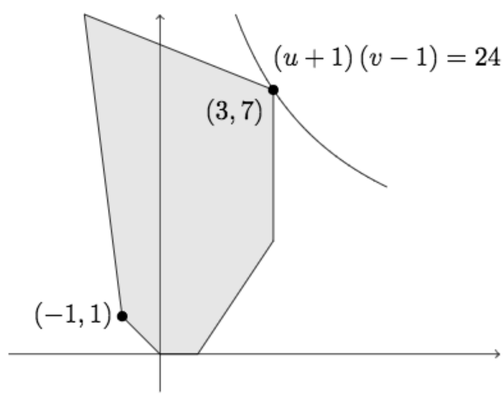


Figure 2: The Feasible Set in Barg3.(ii)

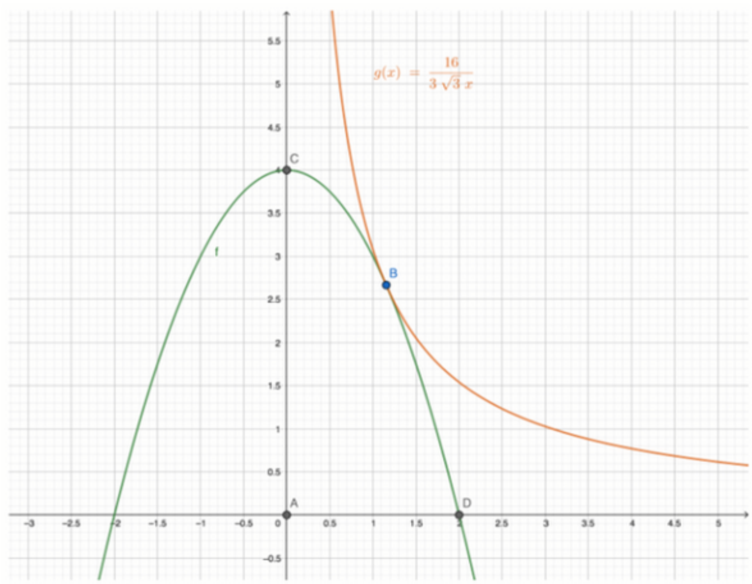
Since Col $1, 2$ are dominated by Col 3 in B, the security level of two players is $(5, 1)$. Let u, v be utility functions of Player I and Player II then then the feasible set is shown in Figure 1. The Nash Product $(u-5)(v-1)$ reaches its maximum 6 on the Pareto Line at $(7, 4)$, which means the cooperative strategy $\langle \text{Row } 1, \text{Col } 2 \rangle$.

(ii) The game matrices are

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 9 & 7 \end{bmatrix}$$

Let u, v be utility functions of Player I and Player II then the feasible set is shown in Figure 2. The Nash Product $(u+1)(v-1)$ reaches its maximum 24 on the Pareto Line at $(3, 7)$, which means the cooperative strategy $\langle \text{Row } 2, \text{Col } 3 \rangle$.

Barg4 Solution:



The Pareto front is $y = 4 - x^2, 0 \leq x \leq 2$.

$$\begin{aligned} w &= (4 - x^2 - 0) \times (x - 0) \\ &= 4x - x^3 \qquad 0 \leq x \leq 2 \\ \frac{dw}{dx} &= 4 - 3x^3 \\ \frac{dw}{dx} &= 0 \Rightarrow x = \frac{2}{\sqrt{3}} \end{aligned}$$

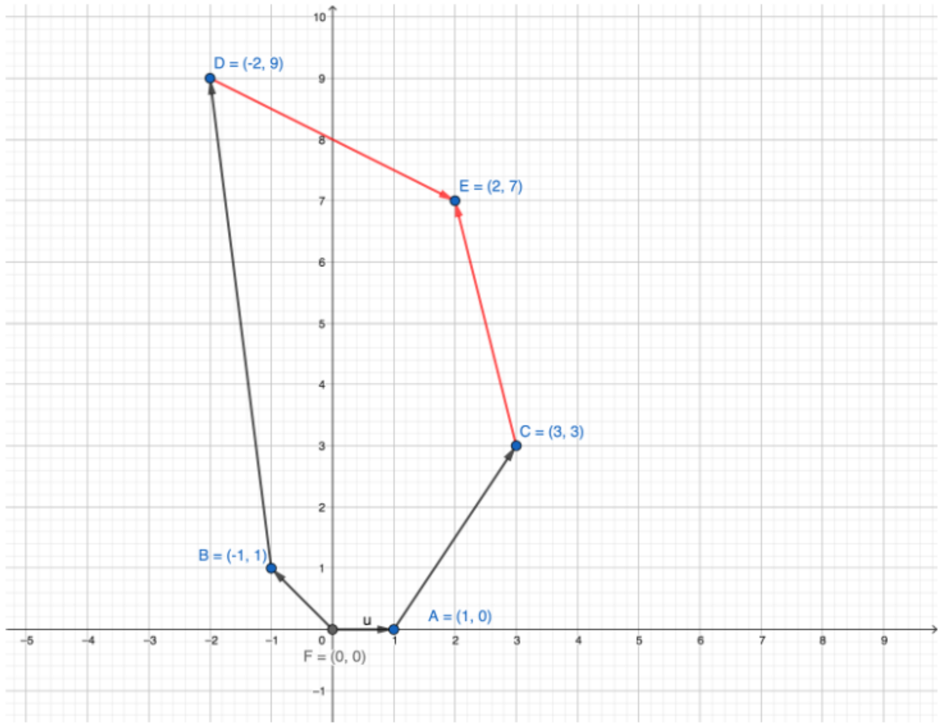
So the Nash Bargaining Solution is $(\frac{2}{\sqrt{3}}, \frac{8}{3})$.

Barg5 Proof:
After any linear transformation $F : \{u, v\} \to \{\alpha_1 u + \beta_1, \alpha_2 v + \beta_2\}$,where $\alpha_1 > 0, \alpha_2 > 0, \beta_1, \beta_2$ are given numbers.

$$\begin{aligned} r_1' &= \frac{(\alpha_1 v_1 + \beta_1) - (\alpha_1 w_1 + \beta_1)}{(\alpha_1 v_1 + \beta_1) - (\alpha_1 d_1 + \beta_1)} \\ &= \frac{\alpha_1(v_1 - w_1)}{\alpha_1(v_1 - d_1)} \\ &= \frac{v_1 - w_1}{v_1 - d_1} \\ r_2' &= \frac{(\alpha_2 w_2 + \beta_2) - (\alpha_2 v_2 + \beta_2)}{(\alpha_2 w_2 + \beta_2) - (\alpha_2 d_2 + \beta_2)} \\ &= \frac{\alpha_2(w_2 - v_2)}{\alpha_2(v_2 - d_2)} \\ &= \frac{w_2 - v_2}{w_2 - d_2} \end{aligned}$$

So $r_1 \geq r_2 \Leftrightarrow r_1' \geq r_2'$. The Axiom of Invariance Under Change of Location and Scale is valid for the bargaining solution obtained from the Zeuthen's Principle.

Barg6 Solution:
For matrix A, -1 is its saddle point, while for matrix -B, the same position is also a saddle point.So for any λ, the value of game $\lambda A - B$ is $-\lambda - 1$.



The Pareto fronts are segments DE and CE. So we should try $\lambda = \frac{1}{2}, 4$.
For $\lambda = \frac{1}{2}$:

$$\frac{1}{2}A + B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{9}{2} & 8 & 8 \end{pmatrix}$$

$$\sigma(\frac{1}{2}) = 8 \qquad \delta(\frac{1}{2}) = -\frac{3}{2}$$

$$\phi_1(\frac{1}{2}) = \frac{\sigma(\frac{1}{2}) + \delta(\frac{1}{2})}{2\lambda} = \frac{13}{2}$$

$$\phi_2(\frac{1}{2}) = \frac{\sigma(\frac{1}{2}) - \delta(\frac{1}{2})}{2} = \frac{19}{4}$$

$(\frac{13}{2}, \frac{19}{4})$ lies in the directed line DE but is outside the NTU-feasible set, so Player I pays Player II.
For $\lambda = 4$:

$$4A + B = \begin{pmatrix} 4 & -3 & 0 \\ 15 & 1 & 15 \end{pmatrix}$$

$$\sigma(4) = 15 \qquad \delta(4) = -5$$

$$\phi_1(4) = \frac{\sigma(4) + \delta(4)}{2\lambda} = \frac{5}{4}$$

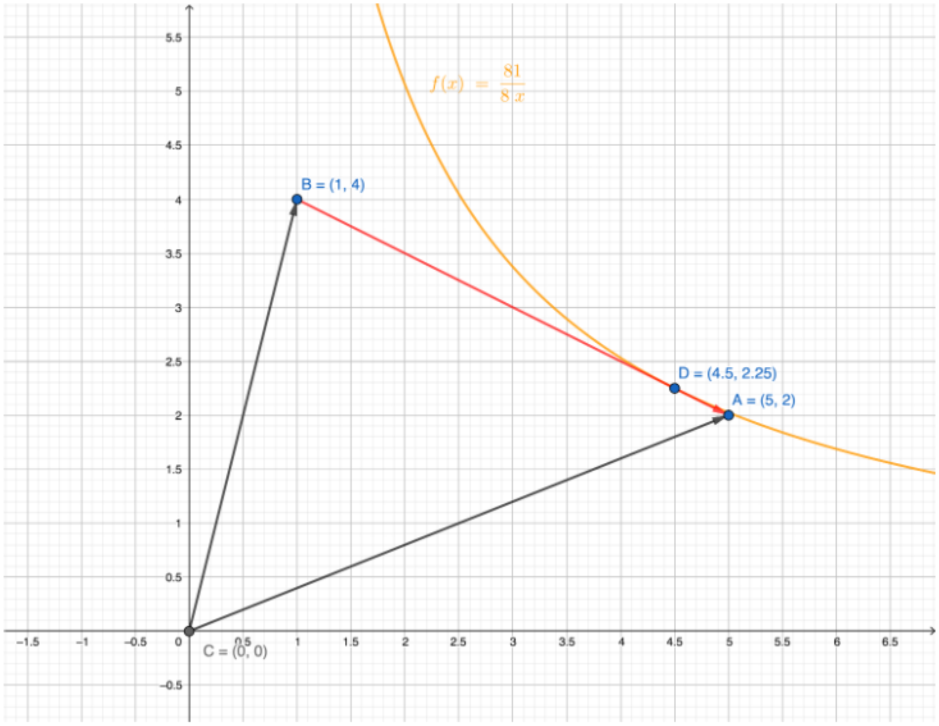
$$\phi_2(4) = \frac{\sigma(4) - \delta(4)}{2} = 10$$

$(\frac{5}{4}, 10)$ lies on the directed line CE but is outside the NTU-feasible set, so Player II pays Player I.
So the NTU value is (2,7) with Cooperative strategy $\langle Row2, Col3 \rangle$. The equilibrium exchange rate is between $\frac{1}{2}$ and 4.

Barg7 Solution:

(i) $\begin{pmatrix} (5,2) & (0,0) \\ (0,0) & (1,4) \end{pmatrix}$

The Pareto front is segment AB with slope $-\frac{1}{2}$. So we should try $\lambda = \frac{1}{2}$.



For $\lambda = \frac{1}{2}$:

$$\frac{1}{2}A + B = \begin{pmatrix} \frac{9}{2} & 0 \\ 0 & \frac{9}{2} \end{pmatrix}$$

$$\frac{1}{2}A - B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{7}{2} \end{pmatrix}$$

$$\sigma(\frac{1}{2}) = \frac{9}{2} \qquad \delta(2) = 0$$

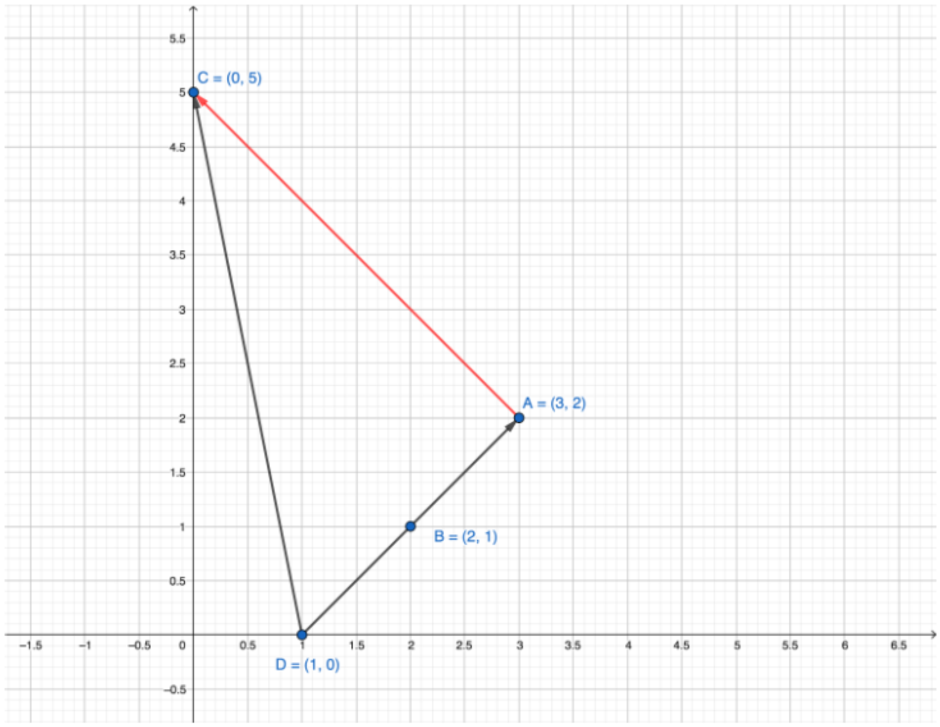
$$\phi_1(\frac{1}{2}) = \frac{\sigma(\frac{1}{2}) + \delta(\frac{1}{2})}{2\lambda} = \frac{9}{2}$$

$$\phi_2(\frac{1}{2}) = \frac{\sigma(\frac{1}{2}) - \delta(\frac{1}{2})}{2} = \frac{9}{4}$$

$(\frac{9}{2}, \frac{9}{4})$ is within the NTU-feasible set. So the NTU value is $(\frac{9}{2}, \frac{9}{4})$ with Cooperative strategy being $\frac{1}{8}\langle Row1, Col1\rangle + \frac{7}{8}\langle Row2, Col2\rangle$. The equilibrium exchange rate is $\lambda^* = \frac{1}{2}$.

(ii) $\begin{pmatrix} (3,2) & (0,5) \\ (2,1) & (1,0) \end{pmatrix}$

The Pareto front is segment AC with slope -1. So we should try $\lambda = 1$.



For $\lambda = 1$:

$$A + B = \begin{pmatrix} 5 & 5 \\ 3 & 1 \end{pmatrix}$$
$$A - B = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}$$
$$\sigma(1) = 5 \qquad \delta(2) = 1$$
$$\phi_1(1) = \frac{\sigma(1) + \delta(1)}{2\lambda} = 3$$
$$\phi_2(1) = \frac{\sigma(1) - \delta(1)}{2} = 2$$

$(3,2)$ is within the NTU-feasible set. So the NTU value is $(3,2)$ with Cooperative strategy being $\langle Row1, Col1 \rangle$. The equilibrium exchange rate is $\lambda^* = 1$.