The Midterm

题目1. Quaternion

解答.

$$a\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+b\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}+c\begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix}+d\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

where $a, b, c, d \in \mathbb{R}$

1. Show that this gives a model of the quaternions as well. i.e, they statisfy $i^2 = j^2 = k^2 = ijk = -1$

$$i=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\Rightarrow i^2=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}^2=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

$$j=\begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix}\Rightarrow j^2=\begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix}^2=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

$$k=\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\Rightarrow k^2=\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}^2=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

$$ijk=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix}\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

$\therefore$ this also gives a model of the quaternions as well

2. Prove Hamilton Product

$\because L_q : \mathbb{R}^4 \to \mathbb{R}^4 \mapsto L_q(a + bi + cj + dk) = (r + xi + yj + zk)(a + bi + cj + dk)$

$\because [v_1, v_2, v_3, v_4]$ be standard basis for $\mathbb{R}^4 \Rightarrow$

$L_q v_1 = (r + xi + yj + zk)(1 + 0i + 0j + 0k) = r + xi + yj + zk$

$L_q v_2 = (r + xi + yj + zk)(0 + 0i + 0j + 0k) = -x + ri + +zj - yk$

$L_q v_3 = (r + xi + yj + zk)(0 + 0i + 1j + 0k) = -y - zi + rj + xk$

$L_q v_4 = (r + xi + yj + zk)(0 + 0i + 0j + 1k) = -z + yi - xj + rk$

$$L_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix}$$

Using the same logic, we can get R_q

$$R_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & z & -y \\ y & -z & r & x \\ z & y & -x & r \end{bmatrix}$$

$$L_r R_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix} \begin{bmatrix} r & -x & -y & -z \\ x & r & z & -y \\ y & -z & r & x \\ z & y & -x & r \end{bmatrix}$$

$$= \begin{bmatrix} r^2 - x^2 - y^2 - z^2 & -2rx & -2ry & -2rz \\ 2rx & r^2 - x^2 + y^2 + z^2 & -2xy & -2xz \\ 2rx & -2xy & r^2 + x^2 - y^2 + z^2 & -2yz \\ 2rz & -2xz & -2yz & r^2 + x^2 + y^2 - z^2 \end{bmatrix}$$

$$R_q L_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & z & -y \\ y & -z & r & x \\ z & y & -x & r \end{bmatrix} \begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix}$$

$$= \begin{bmatrix} r^2 - x^2 - y^2 - z^2 & -2rx & -2ry & -2rz \\ 2rx & r^2 - x^2 + y^2 + z^2 & -2xy & -2xz \\ 2rx & -2xy & r^2 + x^2 - y^2 + z^2 & -2yz \\ 2rz & -2xz & -2yz & r^2 + x^2 + y^2 - z^2 \end{bmatrix}$$

$\therefore L_q R_q = R_q L_q$

3.

$$L_q R_{\bar{q}} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & -c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 + c^2 + d^2 & ab - ab - cd + cd & ac + bd - ac - bd & ad - bc + bc - ad \\ ab - ab + cd - cd & b^2 - a^2 - d^2 - c^2 & bc - ad - ad + bc & bd + ac + bd + ac \\ ac - bd - ac + bd & bc + ad + ad + bc & c^2 - d^2 + a^2 - b^2 & cd + cd - ab - ab \\ ad + bc - cb - ad & bd - ac + bd - ac & cd + cd + ab + ab & d^2 - c^2 - b^2 + a^2 \end{bmatrix}$$

$\because q\bar{q} = (a+bi+cj+dk)(a-bi-cj-dk) = a^2+b^2+c^2+d^2+0 = 1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 0 & 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 0 & 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$$

Let $q = r+xi+yj+zk$, where it's matrix representation is

$$\begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix}$$

It's conjugate $\bar{q} = r-xi-yj-zk$ would have the matrix representation

$$\begin{bmatrix} r & x & y & z \\ -x & r & z & -y \\ -y & z & r & -x \\ -z & y & -x & r \end{bmatrix}$$

Which is the transpose of it's conjugate form. Thus, when $Q =$

$$\begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{bmatrix}$$

we can show that Q is orthogonal if and only if $L_qR_{\bar{q}}$ is also orthogonal

$$\Rightarrow L_qR_{\bar{q}} = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}, (L_qR_{\bar{q}})^T(L_qR_{\bar{q}}) = \begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^TQ \end{bmatrix}$$

$\because q$ is a unit quaternion, the matrix representation of $q\bar{q}$ has to be equal to the identiy matrix.

$$\Rightarrow (L_qR_{\bar{q}})^T(L_qR_{\bar{q}}) = R_{\bar{q}}^TL_q^TL_qR_{\bar{q}} = R_qR_{\bar{q}} = I$$

$\therefore Q$ is orthognal

题目2. Drazin Inverse and Differential Equation

解答.

1. Prove that $\begin{bmatrix} R^{-1} & 0 \\ 0 & 0 \end{bmatrix} = X \begin{bmatrix} (R')^{-1} & 0 \\ 0 & 0 \end{bmatrix} X^{-1}$

$$\Rightarrow \begin{bmatrix} R^{-1} & 0 \\ 0 & 0 \end{bmatrix} X = X \begin{bmatrix} (R')^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Let X be $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} R^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} (R')^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} R^{-1}x_1 & R^{-1}x_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1(R')^{-1} & 0 \\ x_3(R')^{-1} & 0 \end{bmatrix}$$

Thus we need to prove the two cases

$$\begin{cases} R^{-1}x_1 = x_1(R')^{-1} & (1) \\ R^{-1}x_2 = 0 \qquad\qquad x_2 = 0 & (2) \end{cases}$$

Prove (1),

$$\because \begin{bmatrix} R & \\ & N \end{bmatrix} = X \begin{bmatrix} R' & \\ & N' \end{bmatrix} X^{-1}$$

$$\Rightarrow \begin{bmatrix} R & \\ & N \end{bmatrix} X = X \begin{bmatrix} R' & \\ & N' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} R & \\ & N \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} R' & \\ & N' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Rx_1 & Rx_2 \\ Nx_3 & Nx_4 \end{bmatrix} = \begin{bmatrix} x_1R' & x_2N' \\ x_3R' & x_4N' \end{bmatrix}$$

$\therefore Rx_1 = x_1R'$

$$\therefore R^{-1}x_1 = x_1(R')^{-1}$$

Prove (2)

$$Rx_2 = X_2N'$$

$$Rx_2N' = X_2(N')^2$$

$$R^2X_2 = X_2(N')^2$$

$\therefore R^kx_2 = x_2(N')^k \because N'$ is nilpotent, $R^kx_2 = 0, x_2 = 0$

Since conditions (1) and (2) are both proved, we get the equation to be true.

2. Show that $AA^{(D)} = A^{(D)}A$, $A^{(D)}AA^{(D)} = A^{(D)}$, and $A^{(D)}A^{k+1} = A^k$ where k is the smallest integer such that $\ker(A^k) = \ker(A^{k+1})$

$$AA^{(D)} = X \begin{bmatrix} A_R & \\ & A_N \end{bmatrix} X^{-1} X \begin{bmatrix} A_R^{-1} & \\ & 0 \end{bmatrix} X^{-1} = X \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^{-1}$$

$$A^{(D)}A = X \begin{bmatrix} A_R^{-1} & \\ & 0 \end{bmatrix} X^{-1} X \begin{bmatrix} A_R & \\ & A_N \end{bmatrix} X^{-1} = X \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^{-1}$$

$\therefore AA^{(D)} = A^{(D)}A$

$$A^{(D)}AA^{(D)} = X\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^{-1}X\begin{bmatrix} A_R^{-1} & \\ & 0 \end{bmatrix} X^{-1} = X\begin{bmatrix} A_R^{-1} & \\ & 0 \end{bmatrix} X^{-1} = A^{(D)}$$

$\therefore A^{(D)}AA^{(D)} = A^{(D)}$

$$A^{(D)}A^{k+1} - A^k = X\begin{bmatrix} A_R^{-1} & \\ & 0 \end{bmatrix} X^{-1}X\begin{bmatrix} A_R^{k+1} & \\ & A_N^{k+1} \end{bmatrix} X^{-1} - X\begin{bmatrix} A_R^{k} & \\ & A_N^{k} \end{bmatrix} X^{-1}$$

$$= X\begin{bmatrix} A_R^{k} & \\ & 0 \end{bmatrix} X^{-1} - X\begin{bmatrix} A_R^{k} & \\ & A_N^{k} \end{bmatrix} X^{-1}$$

$\because A_N$ is nilpotent

$$= X\begin{bmatrix} A_R^{k} & \\ & 0 \end{bmatrix} X^{-1} - X\begin{bmatrix} A_R^{k} & \\ & 0 \end{bmatrix} X^{-1} = 0$$

3. Calculate $(ab^*)^{(D)}$ for non-zero vectors $a, b \in C_n$

$\because (b^*a)$ is an inner product, the result equals a constant value.

$$A = (ab^*) = X\begin{bmatrix} A_R & \\ & A_N \end{bmatrix} X^{-1}$$

$$A = X\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} X^{-1}$$

And it's trace

$$\operatorname{tr}\left(\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}\right) = \operatorname{tr}(A) = \operatorname{tr}(ab^*) = \operatorname{tr}(b^*a) = b^*a$$

so we can guess that

$$A = X\begin{bmatrix} b^*a & 0 \\ 0 & 0 \end{bmatrix} X^{-1}$$

$$\because A^{(D)} = X\begin{bmatrix} A_k^{-1} & 0 \\ 0 & 0 \end{bmatrix} X^{-1}$$

$$\therefore A^{(D)} = X\begin{bmatrix} \frac{1}{b^*a} & 0 \\ 0 & 0 \end{bmatrix} X^{-1} = \frac{1}{(b^*a)^2}A$$

$$A = (ab^*)$$

$$\therefore (ab^*)^{(D)} = \frac{(ab^*)}{(b^*a)^2}$$

4. For fixed A, show that we can find a polynomial $p(x)$ such that $A(D) = p(A)$

5. If $AB = BA$, show that $e^{-A^{(D)}Bt}AA^{(D)}v_0$ is a solution to $Av' + Bv = 0$ for any constant vector v_0.

题目3. Sherman-Morrison-Woodbury Form

解答.

1. Write $(I_m - AB)^{-1}$ as the sum of a series of matrices.

$$(I_m - AB)^{-1} = I_m + AB + ABAB + \cdots = \sum_{i=0}^{+\infty}(AB)^i$$

2. Deduce the formula $(I_m - AB)^{-1} = I_m + A(I_n - BA)^{-1}B$ when $I_m - AB$ and $I_n - BA$ are invertible and all eigenvalues of AB and BA have absolute value less than 1.

$$(I_m - AB)^{-1} = \sum_{i=0}^{+\infty}(AB)^i = I_m + \sum_{i=1}^{+\infty}(AB)^i = I_m + A\left(\sum_{i=1}^{+\infty}(BA)^{i-1}\right)B = I_m + A(I_n - BA)^{-1}B$$

3. Prove that $Ap(BA) = p(AB)A$ for all polynomials $p(x)$

Let $p(x) = \sum_{i=0}^{n} a_i x^i$

$$Ap(BA) = A\left(\sum_{i=0}^{n} a_i(BA)^i\right) = \sum_{i=0}^{n} a_i A(BA)^i = \sum_{i=0}^{n} a_i(BA)^i A = p(AB)A$$

4. For any well-defined $f(A)$, there must exist a polynomial $p(x)$ such that $f(A) = p(A)$

Let $A =$

$$A = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

where X, Y are any two square matrix.

$$\Rightarrow f(A) = \begin{bmatrix} f(X) & \\ & f(Y) \end{bmatrix} = p(A) = \begin{bmatrix} p(X) & \\ & p(Y) \end{bmatrix}$$

$\therefore f(X) = p(X), f(Y) = p(Y)$

$\therefore$ there exists a polynomial $p(X)$ such that $p(X) = f(X), p(Y) = f(Y)$ 5. Show that $Af(BA) = f(AB)A$ as long as $f(AB)$ and $f(BA)$ are defined

$\because$ f is defined on AB and BA, there exists a p such that

$$p(AB) = f(AB)$$

$$p(BA) = f(BA)$$

$\therefore$

$$Af(BA) = Ap(BA) = p(AB)A = f(AB)A$$

6. Verify that $f(AB) = I_m + Af(BA)B$ using the identity above.

$$I_m + Af(BA)B = I_m + f(AB)AB = I_m + \left(\sum_{i=0}^{+\infty}(AB)^i(AB)\right) = \sum_{i=0}^{+\infty}(AB)^i = f(AB)$$

题目4. Equations of Matrices

解答.

1. Show that the solutions to the Sylvester's equation $NX - XN = 0$ are exactly the polynomials of N

Let $X =$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$XN = \begin{bmatrix} 0 & a_{11} & \cdots & a_{1(n-1)} \\ 0 & a_{21} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n1} & \cdots & a_{n(n-1)} \end{bmatrix}, NX = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

It is clear to see that

$$\begin{cases} a_{i1} = 0 & i = 2, 3, \cdots, n \\ a_{1j} = 0 & j = 1, 2, \cdots, n-1 \\ a_{ij} = a_{(i-1)(j-1)} \end{cases}$$

Thus, the matrix X is an upper triangular matrix where the elements on the every upper diagonal is equal. This means it is a linear combination of powers of nilpotent matrices.

2. Show that $Y, Y - I, (Y - I)^2, \ldots, (Y - I)^{n-1}$ are linearly independent in the space of matrices, and that they span the space of matrices made of polynomials of N

$$Y = e^N = \sum_{k=0}^{+\infty} \frac{1}{k!} N^k = \sum_{k=0}^{n-1} \frac{1}{k!} N^k$$

$$Y - I = \sum_{k=1}^{+\infty} \frac{1}{k!} N^k$$

$$(Y - I)^k = \sum_{i_1=1}^{n-1} \cdots \sum_{i_k=1}^{n-1} \frac{1}{i_1! \cdots i_k!} N^{i_1 + \cdots + i_k - k}$$

Let us assume that $Y, Y - I, (Y - I)^2, \ldots, (Y - I)^{n-1}$ are not linearly independent matrices

Let

$$Y = \sum_{k=1}^{n-1} a_k (Y - I)^k$$

which deduces

$$I + \sum_{k=1}^{n-1} \frac{1}{k!} N^k = Y = \sum_{k=1}^{n-1} a_k N^k \sum_{i_1=1}^{n-1} \cdots \sum_{i_k=1}^{n-1} \frac{1}{i_1! \cdots i_k!} N^{i_1 + \cdots + i_k - k}$$

However, this shows that I can not be expressed as a linear combinations of other matrixes, which contradicts our assumption. The same logic can be extended for $(Y-I)^k$

Let

$$(Y-I)^j = \sum_{k=j+1}^{n-1} a_k (Y-I)^k$$

$$(N^j)\sum_{i_1=1}^{n-1}\cdots\sum_{i_j=1}^{n-1}\frac{1}{i_1!\cdots i_j!}N^{i_1+\cdots+i_j-j} = (Y-I)^j = \sum_{k=j+1}^{n-1} a_k N^k \sum_{i_1=1}^{n-1}\cdots\sum_{i_k=1}^{n-1}\frac{1}{i_1!\cdots i_k!}N^{i_1+\cdots+i_k-k}$$

However, this shows that the elements of N^j component cannot be expressed as a linear combination of the matrices $(Y-I)^{j+1}, \cdot, (Y-I)^{n-1}$ and so on, which also contradicts our assumption.

$\therefore Y, Y-I, (Y-I)^2, \ldots, (Y-I)^{n-1}$ are linearly independent in the space of matrices

$\because N^n = 0, N^{n-1} \neq 0$ the span of solution space is n, and thus these matrices form the basis of the solution space of X.

3. Find all solutions X to the matrix equation $e^X = e^N$

$$\because e^N = I + N + \frac{N^2}{2!} + \cdots + = \sum_{n=0}^{\infty}\frac{N^n}{n!}$$

$\because N$ is a nilpotent jordan block, then e^N is an upper triangular with 1 on the main diagonal, so it is invertable.

$$\therefore e^{-N}e^X = e^X e^{-N} = I$$

Therefore X and N commute, and

$$X = N + 2\pi k i I, k \in \mathbb{Z}$$

4. Find real matrices A, B such that $AB \neq BA$ but $e^A = e^B$

We can use the fact that an exponent of a diagonal matrix is simply the exponent of the elements along the diagonal. Let $A = \begin{bmatrix} i\pi & 0 \\ 0 & -i\pi \end{bmatrix}, B = \begin{bmatrix} i\pi & 1 \\ 0 & -i\pi \end{bmatrix}$

$$AB = \begin{bmatrix} i\pi & 0 \\ 0 & -i\pi \end{bmatrix}\begin{bmatrix} i\pi & 1 \\ 0 & -i\pi \end{bmatrix} = \begin{bmatrix} -\pi^2 & i\pi \\ 0 & -\pi^2 \end{bmatrix}$$

$$BA = \begin{bmatrix} i\pi & 1 \\ 0 & -i\pi \end{bmatrix}\begin{bmatrix} i\pi & 0 \\ 0 & -i\pi \end{bmatrix} = \begin{bmatrix} -\pi^2 & -i\pi \\ 0 & -\pi^2 \end{bmatrix}$$

$$e^A = \begin{bmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e^B = \begin{bmatrix} 1 & \frac{i}{2\pi} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i\pi & 0 \\ 0 & -i\pi \end{bmatrix}\begin{bmatrix} 1 & -\frac{i}{2\pi} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{i}{2\pi} \\ 0 & 1 \end{bmatrix} e^{\begin{bmatrix} i\pi & 0 \\ 0 & -i\pi \end{bmatrix}} \begin{bmatrix} 1 & -\frac{i}{2\pi} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{i}{2\pi} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{i}{2\pi} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\therefore AB \neq BA, e^A = e^B$

5. Prove that there is no solution X to the equation $\sin X = \begin{bmatrix} 1 & 1996 \\ 0 & 1 \end{bmatrix}$

$\because \sin^2 x + \cos^2 x = 1$

$$\sin^2 X + \cos^2 X = I$$

$\therefore$

$$\cos^2 X = I - \sin^2 X \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1996 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3992 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3992 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \cos^2 x$ is nilpotent, so is $\cos x$, which is a contradiction because that means the top right element of the matrix (-3992) should actually be zero.

题目5. Newton's Method

解答.

1. Show that if X_n has no purely imaginary eigenvalue, then X_{n+1} has no purely imaginary eigenvalue

Let λ_b be any one of the eigenvalues of X_n, where v is the eigenvector of the matrix X_n

$$X_n v = \lambda_n v$$

$$X_n^{-1} v = \frac{v}{\lambda}$$

$\therefore$ Both X_{n+1} and X_n have the eigenvector v

$$\because X_{n+1} = \frac{1}{2}\left(X_n + X_n^{-1}\right), X_{n+1} v = \lambda_{n+1} v$$

$$\therefore \lambda_{n+1} = \frac{1}{2}\left(\lambda_n + \frac{1}{\lambda_n}\right)$$

Let $\lambda_b = x + iy, x \neq 0$

$$\lambda_{n+1} = \frac{1}{2}\left(x + iy + \frac{1}{x+iy}\right) = \frac{1}{2}\left(\frac{(x+iy)(x+iy)(x-iy)}{(x+iy)(x-iy)} + \frac{(x-iy)}{(x+iy)(x-iy)}\right)$$

$$= \frac{x(x^2+y^2+1)}{2(x^2+y^2)} + i\frac{y(x^2+y^2-1)}{2(x^2+y^2)}$$

Which means that X_{n+1} also has no purely imaginary eigenvalue

2. If A is 1×1, and not purely imaginary, show that X_n does indeed converge to $\text{sign}(A)$

$\because \frac{f(x)-1}{f(x)+1} = (\frac{x-1}{x+1})^2$ where $f(x) = \frac{1}{2}\left(x + \frac{1}{x}\right)$

$$\frac{X_{n+1}-1}{X_{n+1}+1} = \frac{\frac{1}{2}(X_n + X_n^{-1}) - 1}{\frac{1}{2}(X_n + X_n^{-1}) + 1}\left(\frac{2}{2}\right) = \frac{X_n + X_n^{-1} - 2}{X_n + X_n^{-1} + 2} = \frac{(X_n - 1)^2}{(X_n + 1)^2}$$

$\because X_0 = A$

$$= \frac{(A-1)^{2n}}{(A+1)^{2n}} \Rightarrow X_n = \frac{\frac{(A-1)^{2n}}{(A+1)^{2n}} - 1}{\frac{(A-1)^{2n}}{(A+1)^{2n}} + 1} = \frac{1 - \frac{(A+1)^{2n}}{(A-1)^{2n}}}{1 + \frac{(A+1)^{2n}}{(A-1)^{2n}}}$$

Let $\frac{1-\frac{(A+1)^{2n}}{(A-1)^{2n}}}{1+\frac{(A+1)^{2n}}{(A-1)^{2n}}}$ be (f) and let $\frac{\frac{(A-1)^{2n}}{(A+1)^{2n}}-1}{\frac{(A-1)^{2n}}{(A+1)^{2n}}+1}$ be (g)

$$\begin{cases} \lim\limits_{n\to+\infty} X_n = \lim\limits_{n\to+\infty} (f) = \frac{1-0}{1+0} = 1 & A > 0, |A+1|^{2^n} < |A-1|^{2^n} \\ \lim\limits_{n\to+\infty} X_n = \lim\limits_{n\to+\infty} (g) = \frac{0-1}{0+1} = -1 & A < 0, |A+1|^{2^n} > |A-1|^{2^n} \end{cases}$$

$\therefore$ for $A_{1\times1}$, $\lim\limits_{n\to+\infty} X_n = \text{sign(A)}$

3. If A is diagonalizable and has no purely imaginary eigenvalue, show that X_n indeed coverge to sign(A)

$\because A$ is diagonalizable, then X_n is also diagonalizable. Thus

$$\because X_n = BAB^{-1}$$

$$\therefore \lim_{m\to+\infty} X_m = B \lim_{n\to+\infty} \begin{bmatrix} \lambda_{1m} & & & \\ & \lambda_{2m} & & \\ & & \ddots & \\ & & & \lambda_{nm} \end{bmatrix}_{n\times m} B^{-1}$$

If $A = B\ \text{diag}(\lambda_{10}, \ldots, \lambda n0) B^{-1}$

$$\Rightarrow B \lim_{n\to+\infty} \begin{bmatrix} \lambda_{1m} & & & \\ & \lambda_{2m} & & \\ & & \ddots & \\ & & & \lambda_{nm} \end{bmatrix} B^{-1} = B \lim_{n\to+\infty} \begin{bmatrix} \text{sign}\lambda_{10} & & & \\ & \text{sign}\lambda_{20} & & \\ & & \ddots & \\ & & & \text{sign}\lambda_{n0} \end{bmatrix} B^{-1} = \text{sign}A$$

4. Suppose A is an $n \times n$ Jordan block with eigenvalue 1. Show that $X_{n-1} = I$.

From the result question 2, we can see describe X_n as

$$X_n = [I - (A+I)^{2^n}(A-I)^{-2^n}][I + (A+I)^{2^n}(A-I)^{-2^n}]^{-1}$$

Let $A = I + N$, then

$$X_{n-1} = [I - (I+N+I)^{2^{n-1}}(I+N-I)^{-2^{n-1}}][I + (I+N+I)^{2^{n-1}}(I+N-I)^{-2^{n-1}}]^{-1}$$

$$= [I - N^n(2I+N)^{-2^{n-1}}][I - N^n(2I+N)^{-2^{n-1}}]^{-1} = I$$

题目6. Real Mobius Transformation

解答.

1. Show that $f_A(x) \circ f_B = f_{AB}$

Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$f_A(f_B(x)) = \frac{a_1\left(\frac{a_2x+b_2}{c_2x+d_2}\right)+b_1}{c_1\left(\frac{a_2x+b_2}{c_2x+d_2}\right)+d_1} = \frac{a_1a_2x+a_1b_2+b_1c_2x+b_1d_2}{c_1a_2x+c_1b_2+d_1c_2x+d_1d_2} = \frac{(a_1a_2+b_1c_2)x+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)x+(c_1b_2+d_1d_2)}$$

$$AB = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{bmatrix}$$

$\therefore f_A \circ f_B = f_{AB}$

2. $k \in \mathbb{R} - \{0\}$

$$f_{kA}(x) = \frac{(ka_1)x+(kb_1)}{(kc_1)x+(kd_1)} = \frac{a_1x+b_1}{c_1x+d_1} = f_A(x)$$

Conversly, if $f_A(x) = f_B(x) \Rightarrow$

$$\frac{a_1x+b_1}{c_1x+d_1} = \frac{a_2x+b_2}{c_2x+d_2} \Rightarrow a_1c_2x^2+(a_1d_2+b_1c_2)x+b_1d_2 = a_2c_1x^2+(b_2c_1+d_1a_2)x+b_2d_1$$

Since x is an arbitary value, let $k_1 = a_1c_2 = a_2c_1, k_2 = b_1d_2 = b_2d_1$, then

$$a_2 = \frac{k_1}{c_1}, b_2 = \frac{k_2}{d_1}, c_2 = \frac{k_1}{a_1}, d_2 = \frac{k_2}{b_1}$$

$$\Rightarrow a_1d_2+b_2c_2 = d_1a_2+b_2c_1 \Rightarrow \frac{a_1}{b_1}k_2+\frac{b_1}{a_1}k_1 = \frac{d_1}{c_1}k_1+\frac{c_1}{d_1}k_2 \Rightarrow \frac{k_1}{k_2} = \frac{a_1c_1}{b_1d_1}$$

$$\frac{a_1c_1}{b_1d_1} = \frac{a_1c_2}{b_1d_2} \Rightarrow \frac{c_1}{c_2} = \frac{d_1}{d_2}, \frac{a_1c_1}{b_1d_1} = \frac{a_1c_2}{b_2d_1} \Rightarrow \frac{b_1}{b_2} = \frac{c_1}{c_2}\frac{a_1c_1}{b_1d_1} = \frac{a_2c_1}{b_2d_1} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{d_1} = \frac{c_1}{c_2} = \frac{d_1}{d_2} = A = kB$$

3.

$$f_A\left(\frac{x}{y}\right) = \frac{a_1\left(\frac{x}{y}\right)+b_1}{c_1\left(\frac{x}{y}\right)+d_1} = \frac{a_1x+b_1y}{c_1x+d_1y}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1x+b_1y \\ c_1x+d_1y \end{bmatrix}$$

$\therefore A$ and f_A can be used interchangably

4. Show that there are only four kinds of Mobius transformations

If x is the fixed point of the Mobius transformation f_A and $x \in \mathbb{R}$, then

$$\frac{ax+b}{cx+d} = x \Leftrightarrow cx^2-(a-d)x-b = 0$$

$\therefore f_A$ may have every point fixed, two points fixed, one point fixed, or have no fixed point.

$(a) f_A$ has two fixed points

$$\begin{cases} (a-d)^2+4bc>0 & c\neq 0 \\ a\neq d & c=0 \end{cases}$$

$(b) f_A$ has one single fixed point

$$\begin{cases} (a-d)^2+4bc>0 & c\neq 0 \\ a=d\oplus b\neq 0 & c=0 \end{cases}$$

$(c) f_A$ has no fixed point

$$\begin{cases} (a-d)^2+4bc<0\oplus c\neq 0 \end{cases}$$

$(d) f_A$ everyone is fixed

$$\begin{cases} b=c=0 \\ a=d\neq 0 \end{cases}$$