1 More Tensors

题目1. Squeezing a Ping Pong

解答. 1.

$\vec{p_1} = -\vec{e_1}$

$\vec{p_2} = -\vec{e_2}$

$\vec{p_3} = +\frac{\sqrt{2}}{2}\vec{e_1} + \frac{\sqrt{2}}{2}\vec{e_2}$

$T = \vec{p_1} \otimes \vec{f_1} + \vec{p_2} \otimes \vec{f_2} + \vec{p_3} \otimes \vec{f_3}$

Let $\begin{cases} \vec{e_1} \otimes \vec{e_1} = e_{11} \\ \vec{e_2} \otimes \vec{e_2} = e_{22} \\ \vec{e_1} \otimes \vec{e_2} = e_{12} \\ \vec{e_2} \otimes \vec{e_1} = e_{21} \end{cases}$

$$\begin{aligned} T &= -e_{11} - e_{22} - \left(\frac{\sqrt{2}}{2}\vec{e_1} + \frac{\sqrt{2}}{2}\vec{e_2}\right) \otimes (\vec{e_1} + \vec{e_2}) \\ &= -\left(\frac{\sqrt{2}}{2} + 1\right) e_{11} - \left(\frac{\sqrt{2}}{2} + 1\right) e_{22} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} e_{21} \end{aligned}$$

解答. 2.

Let $\vec{x} = \cos\theta\vec{e_1} + \sin\theta\vec{e_2}$

$\Rightarrow \vec{y} = \cos(\theta + \frac{\sqrt{2}}{2})\vec{e_1} + \sin(\theta + \frac{\sqrt{2}}{2})\vec{e_2} = -\sin\theta\vec{e_1} + \cos\theta\vec{e_2}$

$\Rightarrow \vec{y} = \cos(\theta - \frac{\pi}{2})\vec{e_1} + \sin(\theta - \frac{\pi}{2})\vec{e_2} = \sin\theta\vec{e_1} - \cos\theta\vec{e_2}$

For both y,

$\vec{x} \otimes \vec{x} = \cos^2\theta e_{11} + \sin^2\theta e_{22} + \cos\theta\sin\theta e_{12} + \sin\theta\cos\theta e_{21}$

$\vec{y} \otimes \vec{y} = \sin^2\theta e_{11} + \cos^2\theta e_{22} - \sin\theta\cos\theta e_{12} - \sin\theta\cos\theta e_{21}$

Let $T = k_1\vec{x} \otimes \vec{x} + k_2\vec{y} \otimes \vec{y}$

$\Rightarrow k_1\cos^2\theta + k_2\sin^2\theta = k_1\sin^2\theta + k_2\cos^2\theta$

Either $k_1 = k_2$ or $\cos^2\theta = \sin^2\theta$

Case 1: $k_1 = k_2$ has no solution, therefore

$$\begin{cases} \vec{x} = \frac{\sqrt{2}}{2}\vec{e_1} + \frac{\sqrt{2}}{2}\vec{e_2} \\ \vec{y} = \frac{\sqrt{2}}{2}\vec{e_1} - \frac{\sqrt{2}}{2}\vec{e_2} \end{cases} \quad \text{or} \quad \begin{cases} \vec{x} = -\frac{\sqrt{2}}{2}\vec{e_1} - \frac{\sqrt{2}}{2}\vec{e_2} \\ \vec{y} = \pm\left(\frac{\sqrt{2}}{2}\vec{e_1} - \frac{\sqrt{2}}{2}\vec{e_2}\right) \end{cases}$$

Case 2: $\cos^2\theta = \sin^2\theta$

$$\begin{cases} \vec{x} = \frac{\sqrt{2}}{2}\vec{e_1} + \frac{\sqrt{2}}{2}\vec{e_2} \\ \vec{y} = \frac{\sqrt{2}}{2}\vec{e_1} - \frac{\sqrt{2}}{2}\vec{e_2} \end{cases} \text{ or } \begin{cases} \vec{x} = \frac{\sqrt{2}}{2}\left(\vec{e_1} + \vec{e_2}\right) \\ \vec{y} = \frac{\sqrt{2}}{2}\left(-\vec{e_1} + \vec{e_2}\right) \end{cases} \text{ or } \begin{cases} \vec{x} = -\frac{\sqrt{2}}{2}\left(\vec{e_1} + \vec{e_2}\right) \\ \vec{y} = \pm\frac{\sqrt{2}}{2}\left(\vec{e_1} - \vec{e_2}\right) \end{cases}$$

解答. 3.

Short axis: direction $\vec{x}, \frac{\sqrt{2}}{2}\left(\vec{e_1} + \vec{e_2}\right)$

Long axis: direction $\vec{x}, \frac{\sqrt{2}}{2}\left(\vec{e_1} - \vec{e_2}\right)$

解答. 4.

If a circle is squeezed perpendicularly $\vec{p}$ and $\vec{f}$ would be collinear, let T be

$$T = \sum_{i=1}^{n} \vec{p_i} \otimes \vec{f_i} = \sum_{i=1}^{n} \vec{p_i} \otimes -k_i\vec{p_i} = -\sum_{i=1}^{n} k_i\vec{p_i} \otimes \vec{p_i}$$

$$\vec{p_i} = a_i\vec{e_i} + b_i\vec{e_2}, \sqrt{a_i^2 + b_i^2} = 1$$

Which gives us

$$T = -\sum_{i=1}^{n} k_i\left(a_i\vec{e_1} + b_i\vec{e_2}\right) \otimes \left(a_i\vec{e_1} + b_i\vec{e_2}\right) = -\sum_{i=1}^{n} k_i\left(a_i^2 e_{11} + b_i^2 e_{22} + a_i b_i e_{12} + a_i b_i e_{21}\right)$$

In matrix form

$$\begin{bmatrix} \sum_{i=1}^{n} k_i a_i^2 & \sum_{i=1}^{n} k_i a_i b_i \\ \sum_{i=1}^{n} k_i a_i b_i & \sum_{i=1}^{n} k_i b_i^2 \end{bmatrix}$$

Calculating the determinate gives us

$$\lambda_1\lambda_2 = |T| = \left[\sum_{i=1}^{n} k_i b_i^2 \cdot \sum_{i=1}^{n} k_i a_i - \sum_{i=1}^{n} k_i a_i b_i \cdot \sum_{i=1}^{n} k_i a_i b_i\right] \geq 0$$

$$\lambda_1\lambda_2 \leq 0 \Rightarrow \lambda_1 \leq 0, \lambda_2 \leq 0$$

$\therefore T$ is a negative sum definite

题目2. Change of Basis

解答. 1.

$\alpha_\beta(v_\beta) = \alpha_c(v_c)$

$\Leftrightarrow \alpha_\beta(v_\beta) = \alpha_c(Mv_\beta)$

$\Leftrightarrow \alpha_\beta(v_\beta) = (\alpha_c M)v_\beta$

$\therefore \alpha_c M = \alpha_\beta$

$\therefore \alpha_c = \alpha_\beta M^{-1}$

解答. 2.

$$T(v,w)=\sum_{i,j} x_{ij}\left(b_i^* v\right)\left(b_j^* w\right)=\sum_{i,j} v_i w_j$$

T_β is matrix with entries $x_{i,j}, v_\beta = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, w_\beta = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

$$v_\beta^T T_\beta W_\beta = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} w_1 & w_2 & \vdots & w_n \end{pmatrix} = \sum_{ij} x_{ij} v_i w_j$$

$\therefore T(v,w)=v_\beta^T T_\beta W_\beta$

解答. 3.

Since $T(v,w)$ is independent of basis

$v_\beta^T T_\beta w_\beta = v_c^T T_c w_c$

$v_\beta^T T_\beta w_\beta = (Mv_\beta)^T T_c (Mv_\beta)$

$T_\beta = M^T T_c M$

$T_c = (M^T)^{-1} T_\beta M^{-1}$

解答. 4.

$T(\alpha,\beta)=\alpha_\beta T_\beta \beta_\beta^T = \alpha_c T_c \beta_c^T$

$a_c = \alpha_\beta M^{-1}$

$\beta_c = \beta_\beta M^{-1}$

$\alpha_\beta T_\beta \beta_\beta^T = \alpha_\beta M^{-1} T_c (\beta_\beta M^{-1})^T$

$\Rightarrow T_\beta = M^{-1} T_c (M^{-1})^T$

$\Rightarrow T_c = M T_\beta M^T$

解答. 5.

$T(\alpha,\beta)=\alpha_\beta T_\beta v_\beta = \alpha_\beta M^{-1} T_c M v_\beta$

$T_\beta = M^{-1} T_c M$

$T_c = M T_\beta M^{-1}$

题目3. Why should the “gradient” be a row vector?

解答. Gradient of f is $\partial f = 2xdx + 2ydy + 2zdz$

$v_{old} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $v_{new} = Mv_{old} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$$\Rightarrow \begin{cases} x = u + w - v \\ y = v - w \\ z = w \end{cases}$$

Therefore

$$f(v_{old}) = x^2 + y^2 = z^2 = (u = w - v)^2 + (v - w)^2 + w^2 = u^2 + 2v^2 + 3w^2 + 2vw - 2uv - 4wv$$

$$= f_{new}(v_{new}) = f_{new}(\begin{pmatrix} u \\ v \\ w \end{pmatrix})$$

Therefore

$$f_{new}(v_{new}) = f_{new}\left(\begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) = u^2 = 2v^2 + 3w^2 + 2uw - 2uv - 4wv$$

The gradient of f_{new} is $df_{new} = (2u + 2w - 2v)du + (4v - 2u - 4w)dv + (6w + 2u - 4v)dw$

$\nabla f_{new}(a,b,c) = \begin{pmatrix} 2a + 2c - 2b \\ -2a + 4b - 4c \\ 2a - 4b + 6c \end{pmatrix}$

$(M^{-1})^T = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{-T} = \begin{bmatrix} 1 & -1 & 1 \\ & 1 & -1 \\ & & 1 \end{bmatrix}^T$

$(M^{-1})^T \nabla(x,y,z) = \begin{pmatrix} 1 & -1 & 1 \\ & 1 & -1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ & 1 & -1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 2a - 2b + 2c \\ 2b - 2c \\ 2c \end{pmatrix}$

$= \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2a - 2b + 2c \\ 2b - 2c \\ 2c \end{pmatrix} = \begin{pmatrix} 2a + 2c - 2b \\ -2a + 4b - 4c \\ 2a - 4b + 6c \end{pmatrix}$

Therefore

$$f_{new}(a,b,c) = (M^{-1})^T(\nabla f(x,y,z))$$

when $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$