1 1.3 Jordan Canonical Form

题目1. Find a basis in the following vector space so that the linear map involved will be in Jordan normal form. Also find the Jordan normal form.

解答.

(1) $\because V$ is a real vector space, $V \subset \mathbb{R}^4$

Let $x = a + bi, y = c + di$

$$V \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ -b \\ b-c \\ b-d \end{bmatrix}, V = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Get Eigenvalues,

$$\det(XI - V) = \begin{vmatrix} x-1 & & 1 & \\ & x+1 & & \\ & -1 & x+1 & \\ & -1 & & x+1 \end{vmatrix} = (x+1)^3(x-1)$$

eigen	alg mult
1	1
-1	3

$$\mathrm{Ker}(V - I) = \mathrm{span}\left\{ \begin{pmatrix} (1 \\ 0 \\ 0 \\ 0) \end{pmatrix} \right\}, V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{Ker}(V + I)^2 = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0, \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathrm{Ker}(V + I) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$V_2 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \end{pmatrix}, (V+I)V_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \\ 4 \end{pmatrix}, V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore V = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & 1 & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 0 \end{pmatrix}^{-1}$$

(2)

$$V \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 4a \\ 3b+d \\ 2c \\ d+e \end{pmatrix}, V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\det(XI - V) = \begin{vmatrix} X & 0 & 0 & 0 & 0 \\ -4 & X & 0 & 0 & 0 \\ 0 & -3 & X & -1 & 0 \\ 0 & 0 & -2 & X & 0 \\ 0 & 0 & 0 & -1 & X-1 \end{vmatrix} = (x-1)x^2(x-\sqrt{2})(x+\sqrt{2})$$

eigen	alg mult
1	1
$\sqrt{2}$	1
$-\sqrt{2}$	1
0	2

$$\text{Ker}(V-I) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, V_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Ker}(V-\sqrt{2}I) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1-\sqrt{2} \\ \sqrt{2}-1 \\ 1 \end{pmatrix} \right\}, V_2 = \begin{pmatrix} 0 \\ 0 \\ 1-\frac{\sqrt{2}}{2} \\ \sqrt{2}-1 \\ 1 \end{pmatrix}$$

$$\mathrm{Ker}(V+\sqrt{2}I)=\mathrm{span}\left\{\begin{pmatrix}0\\0\\1+\sqrt{2}\\-1-\sqrt{2}\\1\end{pmatrix}\right\}, V_3=\begin{pmatrix}0\\0\\1+\frac{\sqrt{2}}{2}\\-1-\sqrt{2}\\1\end{pmatrix}$$

$$\mathrm{Ker}(V)=\mathrm{span}\left\{\begin{pmatrix}0\\1\\0\\-3\\3\end{pmatrix}\right\}, \mathrm{Ker}(V^2)=\mathrm{span}\left\{\begin{pmatrix}1\\-4\\-6\\0\\12\end{pmatrix},\begin{pmatrix}1\\0\\-6\\0\\12\end{pmatrix}\right\}$$

$$V_4=\begin{pmatrix}1\\0\\-6\\0\\12\end{pmatrix}, VV_4=\begin{pmatrix}0\\4\\0\\-12\\12\end{pmatrix}$$

$$\therefore V=\underbrace{\begin{pmatrix}0&0&0&0&1\\0&0&0&4&0\\0&1-\frac{\sqrt{2}}{2}&1+\frac{\sqrt{2}}{2}&0&-6\\0&\sqrt{2}-1&-1-\sqrt{2}&-12&0\\1&1&1&12&12\end{pmatrix}}_{X}\begin{pmatrix}1&&&&\\&\sqrt{2}&&&\\&&-\sqrt{2}&&\\&&&0&1\\&&&&0\end{pmatrix}\underbrace{\begin{pmatrix}0&0&0&0&1\\0&0&0&4&0\\0&1-\frac{\sqrt{2}}{2}&1+\frac{\sqrt{2}}{2}&0&-6\\0&\sqrt{2}-1&-1-\sqrt{2}&-12&0\\1&1&1&12&12\end{pmatrix}^{-1}}_{X^{-1}}$$

(3) $\det(XI-A)=(x^2-a_1a_4)(x^2-a_2a_3)$

Let $\sqrt{x}=\begin{cases}\sqrt{x} & x\geq 0\\ -i\sqrt{x} & x<0\end{cases}$

Case 1: $a_1a_4\neq a_2a_3$ and $a_1a_2\neq 0, a_2a_3\neq 0$

Eigenvalues are $\sqrt{a_1a_4}, -\sqrt{a_1a_4}, \sqrt{a_2a_3}, -\sqrt{a_2a_3}$

$$A=\begin{pmatrix}\sqrt{\frac{a_1}{a_4}}&-\sqrt{\frac{a_1}{a_4}}&0&0\\0&0&\sqrt{\frac{a_2}{a_3}}&-\sqrt{\frac{a_2}{a_3}}\\0&0&1&1\\1&1&0&0\end{pmatrix}\begin{pmatrix}\sqrt{a_1a_4}&&&\\&-\sqrt{a_1a_4}&&\\&&\sqrt{a_2a_3}&\\&&&-\sqrt{a_2a_3}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{a_1}{a_4}}&-\sqrt{\frac{a_1}{a_4}}&0&0\\0&0&\sqrt{\frac{a_2}{a_3}}&-\sqrt{\frac{a_2}{a_3}}\\0&0&1&1\\1&1&0&0\end{pmatrix}^{-1}$$

Case 2: $a_1a_4=a_2a_3\neq 0$

Eigenvalues are $\sqrt{a_1a_4}, -\sqrt{a_1a_4}$

$$A=\begin{pmatrix}\sqrt{\frac{a_1}{a_4}}&0&-\sqrt{\frac{a_1}{a_4}}&0\\0&\sqrt{\frac{a_2}{a_3}}&0&-\sqrt{\frac{a_2}{a_3}}\\0&1&0&1\\1&0&1&0\end{pmatrix}\begin{pmatrix}\sqrt{a_1a_4}&&&\\&\sqrt{a_1a_4}&&\\&&-\sqrt{a_1a_4}&\\&&&-\sqrt{a_1a_4}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{a_1}{a_4}}&0&-\sqrt{\frac{a_1}{a_4}}&0\\0&\sqrt{\frac{a_2}{a_3}}&0&-\sqrt{\frac{a_2}{a_3}}\\0&1&0&1\\1&0&1&0\end{pmatrix}^{-1}$$

Case 3: $a_1a_4=0, \bar{V}a_2a_3=0, a_1a_4\neq a_2a_3$

Eigenvalues are $0(2), \sqrt{a_2a_3}, -\sqrt{a_2a_3}$

1. If $a_1 = a_4 = 0$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{a2}{a3}} & -\sqrt{\frac{a2}{a3}} \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{a2}{a3}} & -\sqrt{\frac{a2}{a3}} \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1}$$

2. If $a_1 = 0, a_4 \neq 0$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{a2}{a3}} & -\sqrt{\frac{a2}{a3}} \\ 0 & 0 & 1 & 1 \\ a_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{a2}{a3}} & -\sqrt{\frac{a2}{a3}} \\ 0 & 0 & 1 & 1 \\ a_4 & 0 & 0 & 0 \end{pmatrix}^{-1}$$

Case 4: $a_1a_4 = a_2a_3 = 0$

1. If 4 of them are zero, $a_1 = a_2 = a_3 = a_4 = 0$

$$A = I0I^{-1}$$

2. If 3 of them are zero, $a_1 = a_2 = a_3 = 0, a_4 \neq 0$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a^4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a^4 & 0 & 0 & 0 \end{pmatrix}^{-1}$$

3. If 2 of them are zero, $a_1 = a_2 = 0, a_3, a_4 \neq 0$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_3 & 0 \\ a^4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_3 & 0 \\ a^4 & 0 & 0 & 0 \end{pmatrix}^{-1}$$

题目2. A partition of integer n is a way to write n as a sum of other positive integers.

解答.

(1) For the partition $n = a_1 + \cdots + a_k$.

If we look at each column, column a_i represents the length of a chain

If we look at each row, row j represents how many eigen vectors that we can find in $\mathrm{Ker}(A^j)$, but not in $\mathrm{Ker}(A^{j-1})$

Thus, it is equal to $\dim\ker(A^j) - \dim\ker(A^{j-1})$, therefore, the two dot diagrams are transpose of each other.

(2) The total number of dots that are in the first column and the first row is $2a_1 - 1$, which is an odd number

$\forall$ self-conjugat partitions the partition of distinct odd numbers is $n = 2a_1 - 1 + 2a_2 - 3 + \cdots + 2a_k - (k+1), k = n$, and $\forall$ distinct odd number partition n $= b_1 + \cdots + b_k$, where $a_1 = \frac{b_1+1}{2}, \ldots, a_k = \frac{b_k+2k-1}{2}$

(3) Because A is upper triangular and nilpotent

A takes the form $\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$ if none of a_ij is zero, and that A only has one eigenvalue of 0

$$\therefore \dim A = 3, \dim\ker A = 1$$

A^2 takes the form $\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim A^2 = \dim\ker A^2 = 2$

A^3 takes the form $\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim A^3 = 1, \dim\ker A^3 = 3$

$A^4 = 0, \dim\ker A^4 = 4 \Rightarrow$ the chain of A is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, A\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, A^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, A^3\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Thus A's Jordan canonical form has only one Jordan block, and the possibiliy is 100%