1 Chapter 1: Topics in Linear Algebra

题目1. We would like to find a real $n \times n$ matrix A such that $A^2 - I$

1. For each even number n, find a real solution

2. If odd n, show that there is no real solution

解答.

We can use an example to prove that when n is even, there exists a matrics whose square equals the negative identity. For example, the matrix,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

To prove that no real solution exists when n is odd, we can use proof by contradiction. The key in solving this problem is using the determinant of the matrices.

Let A, B be $n \times n$ matrices and c be scalar. Then we have

1. $\det(AB) = \det(A)\det(B)$

2. $\det(cA) = c^n \det(A)$

Assuming that we have A such that $A^2 = -I$, we know that if two matrices are similar, then their determiant is equal, thus we obtain

$$\det(A^2) = \det(-I)$$

$$\det(A^2) = (-1)^n \det(I) = -1$$

because n is odd and $\det(I) = 1$

Since A is a real matrix, the determinant of A is also real. Thus, the solution to $\det(A)^2 = -1$ is impossible. Hence there is no such real A that satisfies the assumption.

题目2. Suppose $A^2 = -I$ for a real $n \times n$ matrix A. For each vector $v \in \mathbb{R}^n$, we write iv to mean Av. For any $n \times n$ matrix B, we say it is complex linear if $B(kv) = kBv$ for any complex number $k \in \mathbb{C}$

解答. If A^2 = -I. That means A is composed with itself on the identity map on $\mathbb{R}$. That means that the effect of applying A twice is the same as multiplication by -1. This is similar to multiplication by imaginary unit i. Thus complex scalar multiplication can be defined as

$$(x + iy)v = xv + yA(v)$$

1. In order to prove that B is complex linear if and only if AB=BA, we have to prove two things.

(a) When B is complex linear, $\Rightarrow AB = BA$

(b) If $AB = BA$, then B is complex linear.

Proof (a): We know that B is complex linear, therefore $\forall k \in \mathbb{C}$

$$B(kv) = kB(v)$$

$$\Rightarrow B((a + bi)v) = a + bi(Bv)$$

$$\Rightarrow \forall a, b \in \mathbb{R}, aBv + bB(iv) = a + bi(Bv)$$

$$\Rightarrow \forall a, b \in \mathbb{R}, aBv + bBAv = a + bA(Bv)$$

Proof (b):

2. Let $A =$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, X = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

Since we know that property $AB = BA$, we can show that X does not have to be complex linear

3. We can pick C_1, C_2 such that

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

题目3. If V is an abstract vector space over C, then for each vector v and each $k \in C$, obviously kv is well defined. But as a result, for each vector v and each $k \in \mathbb{R}$, kv must be well-defined. So any complex vector space must be a real vector space (but NOT vice versa).

解答.

1. The map C is real linear.

$$\forall k, a, b \in \mathbb{R}, c = a + bi$$

$$k = i, kc' = b + ai, (kc)' = -b - ai, kc' \neq (kc)'$$

$$\Rightarrow c(kv) \neq k(cv)$$

Therefore map cannot be complex linear and only real linear. But, can be real linear.

2. $\mathbb{C}$ linear implies $\mathbb{R}$ linear because if C is linear, then when $C = a$, and $a \in \mathbb{R}$

3. $\mathbb{R}$ basis $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}$, real dimension 4

$\mathbb{C}$ basis $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, complex dimension 2

4. $\mathbb{C}$ linear independence implies $\mathbb{R}$ linear independence because

$$\forall c_1, \dots, c_k \in \mathbb{C}, \sum (a_j + b_j i)v = 0, \therefore a_j = 0, j = 1, \dots, k$$

5. $\mathbb{C}$ spanning implies $\mathbb{R}$ spanning because the basis of the real span is a subset of the complex span

题目4. Adapted from Gilbert Strang 9.3.11-15. Take the permutation matrix

解答. A permutation matrix simply manipulates the rows. Sending $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ to $\begin{bmatrix} b \\ c \\ d \\ a \end{bmatrix}$ would require permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

And F_4 Fourier matrix is

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

1.

$$P \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, P \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}$$

2. $PF_4 = F_4 D$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{bmatrix}$$

where $a_1 = 1, a_2 = i, a_3 = -1, a_4 = -i$

Eigenvalues and eigen vectors of P

The characteristic equation is $(P - \lambda I) = 0$ Where we get that $\det(P - \lambda I)$ is

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{bmatrix}\right) = \lambda^4 - 1 = 0$$

Thus the solutions $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i$

Sub into the characteristic polynomial and solve using reduced row echelon form to get eigen vectors. The eigenvectors for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ -1 \\ i \\ 1 \end{bmatrix}$$

respectively.

3.

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \end{bmatrix}$$

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} c_0 + ic_1 - c_2 - ic_3 \\ c_3 + ic_0 - c_1 - ic_2 \\ c_2 + ic_3 - c_0 - ic_1 \\ c_1 + ic_2 - c_3 - ic_0 \end{bmatrix}$$

4. Write C as a polynomial of P. Find the eigenvalues and eigen vectors of C

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, P^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = f(p) = C_0P^4 + C_1P + C_2P^2 + C_3P^3$$

$$\lambda_C = C_0 + C_1\lambda + C_2\lambda^2 + C_3\lambda^3$$

$$\lambda_{C_1} = C_0 + C_1 + C_2 + C_3, x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_{C_1} = C_0 - iC_1 - C_2 + iC_3, x_2 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

$$\lambda_{C_1} = C_0 - C_1 - C_2 - C_3, x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_{C_1} = C_0 + iC_1 - C_2 - iC_3, x_4 = \begin{bmatrix} 1 \\ i \\ 1 \\ -i \end{bmatrix}$$