1 Tangent vectors and Cotangent Vectors

题目1. For a symmetric positive-definite matrix A, and define $(v, w) = v^T Aw$

解答. 1. Show that this is an inner product

To prove that the symmetric positive-definite matrix A is an inner product, we need to prove for three cases

$$\begin{cases} 1.\text{Bilinear} \\ 2.\text{Symmetric} \\ 3.\text{Positive-Definite} \end{cases}$$

1. Bilinear

$$(kv, w) = (kv)^T Aw = k(v^T Aw) = k(v, w)$$

$$(v, kw) = v^T A(kw) = v^T A(kw) = kv^T Aw = k(v, w)$$

$$(u + v, w) = (u + v)^T Aw = u^T Aw + v^T Aw = (u, w) + (v, w)$$

$$(u, v + w) = u^T A(v + w) = u^T Av + u^T Aw = (u, v) + (u, w)$$

2. Symmetric

$$(u, w) = v^T Aw$$

$\because v^T Aw$ is a scalar

$$v^T Aw = (v^T aw)^T = w^T A^T v = (w, v)$$

$$\therefore (v, w) = (w, v)$$

3. Positive Definite

$\because A$ is positive definite

$$(u, v) = v^T Av \geq 0, \forall v$$

and is equal to 0 if $v = 0$

解答. 2. The Reiz map (inverse of the bra) from V^* to V would send a row vector v^T to what?

The bra of V is a linear map $\langle v, - \rangle$ such that $\begin{cases} V \mapsto R \\ W \mapsto v^T Aw \end{cases}$

The bra map of v is $\begin{cases} V \mapsto V^* \\ v \mapsto v^T A \end{cases}$

and since the Rieze map is the inverse of the bra map, the Riez Map is $\begin{cases} V^* \mapsto V \\ v^T A \mapsto v \end{cases}$

Let $v^T A = w^T \Rightarrow v = (w^T A^{-1})^T = A^{-T} w = A^{-1} w$

$\therefore$ the Riez map is $w^T \mapsto A^{-1}w$, the Riez map sends the row vector v^T to the column vector $A^{-1}v$

解答. 3. The bra map from V to $V*$ would send a vector v to what?

The bra map sends vector v to vector $v^T A$

解答. 4. The dual of the Riez map from $V*$ to V would send a row vector v^T to what?

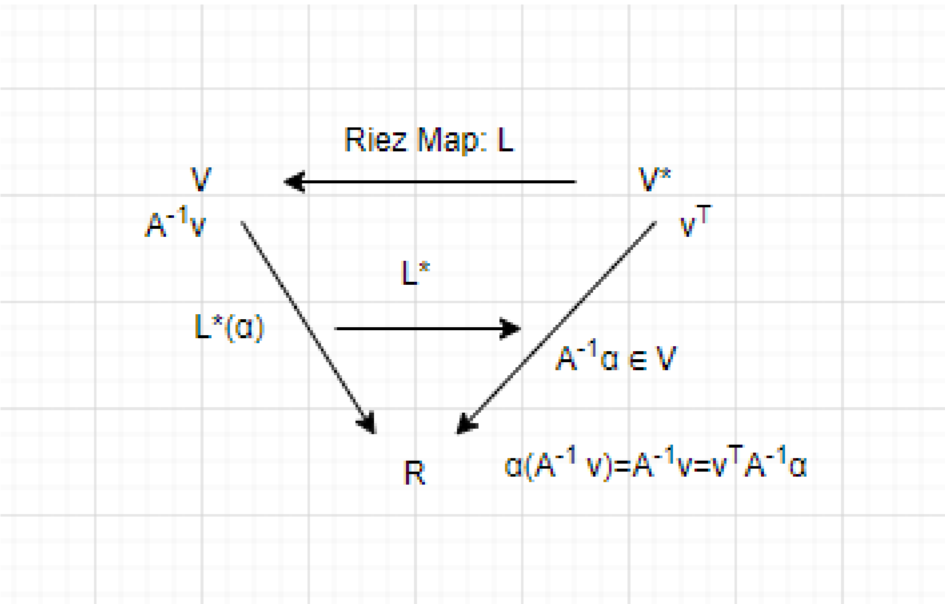
Riez map $L : V* \to V$

Let $\alpha \in V^*$

Dual map of Riez map: $\alpha \to \alpha \circ L = L^*(\alpha)$

L^* sends α to $A^{-1}\alpha$

$\therefore$ the dual map of Riez map sends $v^T to A^{-1} v$



题目2. What is a derivative?

解答. 1. Constant functions in V must be sent to zero by all derivations at any point.

Let f be any constant function in V, dual vector $v_1 \in V^*$ is derivation at $\vec{p}_1 \in M, \vec{p}$ is any point in M

To prove, $v(f) = 0$, since v_1 is a derivation at $\vec{p} \in M$

$\forall g \in V, \exists v_1(fg) = f(\vec{p})(g) + g(\vec{p})v_1(f)$

WLOG, we can pick another constant $g \in V$, at $\vec{p}_2 \in M$, there is a new derivation v_2

$$v_1(fg) = f(\vec{p}_1)v_1(g) + g(\vec{p}_1)v_1(f) = \text{const}_1 \cdot v_1(g) + \text{const}_2 \cdot v_1(f) \tag{1}$$

$$v_2(fg) = f(\vec{p_2})v_2(g) + g(\vec{p_2})v_2(f) = \text{const}_1 \cdot v_2(g) + \text{const}_2 \cdot v_2(f) \tag{2}$$

$$(1) - (2) = v_1(fg) - v_2(fg) = \text{const}_1 \left[v_1(g) - v_2(g)\right] + \text{const}_2 \left[v_1(f) - v_2(f)\right]$$

$\because$ v_1, v_2 are dual vectors

$$(v_1 - v_2)(fg) = \text{const}_1(v_1 - v_2)(g) + \text{const}_2(v_1 - v_2)(f)$$

$$\Rightarrow (v_1 - v_2)(fg - \text{const}_1 g - \text{const}_2 f) = 0$$

$$\Rightarrow v_1 = v_2$$

$\because$ $v_1 = v_2$, find a new function t such that, $(1) \to (3), (2) \to (4)$

$$v_1(fg) = f(\vec{p_1})v_1(t) + t(\vec{p_1})v_1(f) \tag{3}$$

$$v_2(fg) = f(\vec{p_2})v_2(t) + t(\vec{p_2})v_2(f) \tag{4}$$

$$(3) - (4) \Rightarrow \left[t(p_1) - t(p_2)\right] v(f)$$

$$v(f) = 0$$

$\therefore$ constant unctions in V must be sent to zero by derivations at any point

解答. 2.

Let $x - p_1 = g_1$ be a function in V

$\therefore$ we have

$$\begin{aligned} v((x-p_1)f) &= v(g_1 f) \\ &= v(g_1)f(\vec{p}) + v(f)g_1(\vec{p}) \\ &= f(\vec{p})v(x-p_1) + v(f)(p_1 - p_1) \\ &= f(\vec{p})v(x-p_1) \\ &= f(\vec{p_1} \left[v(x) - v(p_1)\right]) \end{aligned}$$

$\because$ p_1 is a constant function, we know $v(p_1) = 0$

$$v(x-p_1)(f) = f(\vec{p})v(x)$$

$\therefore$

$$v((y-p_2)f) = f(\vec{p})v(y)$$

$$v((z-p_3)f) = f(\vec{p})v(z)$$

解答. 3.

Since a, b, c are nonnegative integers and $a + b + c > 1$, they can't be 0 at the same time

Let $a \geq 1$, assume $f\vec{p} = (x-p_1)^{a-1}(y-p_2)^b(z-p_3)^c$

Then $v((x-p_1)^a(y-p_2)^b(z-p_3)^c = v((x-p_1)f)$ From problem 2, we know that

$$\begin{aligned} v(x-p_1)f &= f(\vec{p})v(x) \\ &= (p_1-p_1)^{a-1}(p_2-p_2)^b(p_3-p_3)^c v(x) \\ &= 0^{a+b+C-1}v(x) \end{aligned}$$

$\because a+b+c-1 > 0,)^{a+b+c-1}v(x) = 0$

$\therefore v((x-p_1)^a(x-p_2)^b(x-p_3)^c) = 0$

解答. 4.

Since f is an anyaltic function, we can first calculate the Taylor expansion of f at $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

$$f(x,y,z) = f(\vec{p}) + \frac{\partial f}{\partial x}(\vec{p})(x-p_1) + \frac{\partial f}{\partial y}(\vec{p})(y-p_2) + \frac{\partial f}{\partial z}(\vec{p})(z-p_3) + 0$$

$$= v(f(\vec{p_1})) + \frac{\partial f}{\partial x}(\vec{p})v(x) - \frac{\partial f}{\partial x}(\vec{p})v(\vec{p_1}) + \frac{\partial f}{\partial y}(\vec{p})v(y) - \frac{\partial f}{\partial y}(\vec{p})v(\vec{p_2}) + \frac{\partial f}{\partial z}(\vec{p})v(z) - \frac{\partial f}{\partial z}(\vec{p})v(\vec{p_3})$$

$$= \frac{\partial f}{\partial x}(\vec{p})v(x) + \frac{\partial f}{\partial y}(\vec{p})v(y) + \frac{\partial f}{\partial z}(\vec{p})v(z)$$

解答. 5.

$$\begin{aligned} v(f) &= \frac{\partial f}{\partial x}(\vec{p})v(x) + \frac{\partial f}{\partial y}(\vec{p})v(y) + \frac{\partial f}{\partial z}(\vec{p})v(z) \\ &= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right](\vec{p}) \begin{pmatrix} v(x) \\ v(y) \\ y(z) \end{pmatrix} \\ &= \nabla_v \to (f), \vec{v} = \begin{pmatrix} v(x) \\ v(y) \\ v(z) \end{pmatrix} \end{aligned}$$

题目3. What is a vector field?

解答. 1.

$\because x$ is a vector field

$$X_p(fg) = (X(fg)(\vec{p}))$$

$$X(fg) = fX(g) + gX(f)$$

$$X_{\vec{p}}(fg) = (X(fg))(\vec{p}) = [fX(g) = gX(f)]\,(\vec{p})$$

$$= (fX(g))(\vec{p}) + (gX(f))(\vec{p}) = f(\vec{p})X(g) + g(\vec{p})X(f))$$

$\therefore x$ follows the leibniz rule and is x is a deivatation at $\vec{p}$

解答. 2.

$\forall \vec{p}\ X(f)(\vec{p}) = X_{\vec{p}}(f)$

$X_{\vec{p}}$ is the derivation of any function at $\vec{p}$

$X_{\vec{p}}$ evaluate the tiny change of f with point $\vec{p}$'s velocity

$X_{\vec{p}} : V \mapsto R$

$\partial f\vec{p}$ evaluates $X_{\vec{p}}$ to number $\mathbb{R}$ $X_{\vec{p}}(f) = \partial f\vec{p}(X_{\vec{p}})$

$\vec{p}$ is arbitrary point

$X(f) = \partial f(x)$

$\partial f(x) = X(f)$

解答. 3.

Since XY are a vector field

$$X(fg) = fX(g) + gX(f)$$

$$Y(fg) = fX(g) + gX(f)$$

LHS $= (X \circ Y - Y \circ X)(fg) = X(fY(g) + gY(f)) - Y(fX(g) + gX(f))$

$= X(f)Y(g) + f(X \circ Y)(g) + X(g)Y(f) + g(X \circ Y)(f) - [Y(f)X(g) + f(Y \circ X)(g) + Y(g)X(f) + g(Y \circ X)(f)]$

RHS $= (X \circ Y - Y \circ X)(g) + g(X \circ Y - Y \circ X)(f)$

$= f(X \circ Y)(g) - f(Y \circ X)(g) + g(X \circ Y)(f) - g(Y \circ X)(f)$

Since

$$X(f)Y(g) = Y(g)X(f)$$

$$Y(f)X(g) = X(g)Y(f)$$

$$\therefore LHS = RHS$$

$\therefore X \circ Y - Y \circ X$ satisfies the leibniz rule

$\therefore X \circ Y - Y \circ X$ is always a vector field

解答. 4.

If A,B are skew-symmetric, then $A^T = -A, B^T = -B$

$$\begin{aligned}(AB - BA)^T &= (AB)^T - (BA)^T = B^T A^T - A^T B^T \\ &= (-B)(-A) + (-A)(-B) \\ &= BA - AB \\ &= -(AB - BA)\end{aligned}$$

$\therefore AB - BA$ is skew symmetric