1 1.4 Minimal Polynomials, Slyvester's Equation

题目1. Considering the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

解答.

(1) Find a matrix B such that $BAB^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, thus $BAB^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$

Let

$$B = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, B^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

$$\therefore BAB^{-1} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_{11} & A_{12} + XA_{11} - XA_{22} \\ 0 & A_{22} \end{bmatrix}$$

Therefore

$$A_{12} + XA_{11} - XA_{22} = 0$$

$$A_{12} = XA_{22} - XA_{11} (\text{sylvestor's equation})$$

A_{11} and A_{22} have no common eigenvalues, thus, there exists a unique solution $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} -2a + 2c & 2d - 5a - 3b \\ -2c & -5c - 3d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{cases} a = -2 \\ b = 31/9 \\ c = -3/2 \\ d = 7/6 \end{cases}$$

$$\therefore B = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & \frac{31}{9} \\ 0 & 1 & -\frac{3}{2} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2) Find a basis for the subspace $V_3 + V_4$, where V_λ is the eigenspace of A for the eigenvalue λ

The eigenvalue of BAB^{-1} is equal to the eigenvalue of A

$$BAB^{-1} - 3I = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Eigenvector $\vec{v_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $V_3 = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

$$BAB^{-1} - 4I = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvector $\vec{v_2} = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix}$, $V_4 = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix}\right\}$

$\because BAB^{-1}v = \lambda v \Rightarrow A(B^{-1}v) = \lambda(B^{-1}v))$

$\vec{v_1'} = B^{-1}\vec{v_1} = \begin{pmatrix} 2 \\ \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}$, $\vec{v_2'} = B^{-1}\vec{v_2} = \begin{pmatrix} \frac{59}{9} \\ \frac{19}{3} \\ 5 \\ 1 \end{pmatrix}$

$\therefore$ Basis for subspace $V_3 + V_4$ is $\left\{\begin{pmatrix} 2 \\ \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{59}{9} \\ \frac{19}{3} \\ 5 \\ 1 \end{pmatrix}\right\}$

题目2. Suppose we have complex matrix $A = \begin{bmatrix} B & I \\ & B \end{bmatrix}$. We know the characteristic polynomial of A is just the square of the character polynomial of B. Is the minimal polynomial of A the square of minimal polynomial of B

解答. (1) $\begin{bmatrix} X & 0 & I & 0 \\ 0 & Y & 0 & I \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & Y \end{bmatrix}$

$$\begin{bmatrix} X & 0 & I & 0 \\ 0 & 0 & X & 0 \\ 0 & Y & 0 & I \\ 0 & 0 & 0 & Y \end{bmatrix}$$

$$\begin{bmatrix} X & I & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & Y & I \\ 0 & 0 & 0 & Y \end{bmatrix}$$

therefore

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X & 0 & I & 0 \\ 0 & Y & 0 & I \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & Y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X & I & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & Y & I \\ 0 & 0 & 0 & Y \end{bmatrix}$$

, proving that the two matrices are similar

(2) $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, B^2 = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$

Let B's minimal polynomial be $f(x) = a_2x^2 + a_1x + a_0$

$\begin{cases} 9a_2 + 3a_1 + a_0 = 0 \\ 16a_2 + 4a_1 + a_0 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = \frac{1}{2}a_0 \\ a_1 = -\frac{7}{12}a_0 \end{cases}$

Therefore $B's$ minimal polynomial is $f(x) = x^2 - 7x + 12$

$A = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ $\therefore$ A is similar to XY in (1), therefore, we know that A is similar to the matrix,

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

which is also a Jordan Canonical Form of A. Thus, the minimal polynomial of A is $f(x) = (x-3)^2(x-4)^2 = x^4 - 14x^3 + 73x^2 - 168x + 144$

(3)

$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{rank}(A) = 2$

$A^2 = \begin{bmatrix} B^2 & 2B \\ 0 & B^2 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{rank}(A^2) = 1$

$A^3 = \begin{bmatrix} B^3 & 2B^2 \\ 0 & B^3 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \operatorname{rank}(A^3) = 0$

$A^4 = 0$, therefore the size of A's Jordan blocks are $1 \times 1, 3 \times 3$

$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(4) $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\therefore$ the minimal polynomial for B is $f(x) = x^2$

$\Rightarrow A = \begin{bmatrix} B & I \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow J.C.F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\therefore$ the minimal polynomial for A is $f(x) = x^3$

(5) When $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \min B = x^2 - 7x + 12, \min A = x^4 - 14x^3 + 73x^2 - 168x + 144$

When $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \min B = x^2, \min A = x^3$

Thus, when matrix B is non-singular, the minimal polynomial of matrix A is the (minimal of polynomial of $B)^2$

题目3. In class we see that for Sylvester's equations $AX - XB = C$, if A, B, have no common eigenvalue, then there is always a unique solution. What if A, B have common eigenvalues?
Let us take an extreme case, and assume that $A = B$.
So we are looking at an equation $AX - XA = C$ for constant $n \times n$ matrices A, C.
Let V be the space of $n \times n$ matrices and consider the linear map $L : V \to V$ such that $L(X) = AX - XA$

解答.

(1) $X = kA, k \in N^*$ has infinite solutions for $L(X) = 0$

(2) $L(XY) = AXY - XYA$

$\Rightarrow L(X)Y + XL(Y) = (AX - XA)Y + X(AY - YA)$

$\Rightarrow AXY - XAY + XAY - XYA$

$\Rightarrow AXY - XYA \therefore L(XY) = L(X)Y + XL(Y)$

(3) [DON'T KNOW]

(4) If $p(X)$ is X's minimal polynomial, when $L(X) = I$, then $L(p(X)) = L(X)p'(X) = Ip'(X) = p'(X)$

$\therefore p'(X) = 0$, which contradicts $p(X)$ being X's minimal polynomial, thus $L(X) = I$ has no solutions

(5) Let $A = PBP^{-1}, B = \operatorname{diag}(\lambda_1, \dots, \lambda_n), AX - XA = 0 \Rightarrow PBP^{-1}X - XPBP^{-1} = 0$

$\Rightarrow BP^{-1}X - P^{-1}XPBP = 0$

$\Rightarrow BP^{-1}XP - P^{-1}XPB = 0$

Let $Y = P^{-1}XP, BY - YB = 0$

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & & & \vdots \\ y_{n1} & \cdots & \cdots & y_{nn} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 y_{11} & \lambda_1 y_{12} & \cdots & \lambda_1 y_{1n} \\ \vdots & & & \vdots \\ \lambda_n y_{n1} & \cdots & \cdots & \lambda_n y_{nn} \end{bmatrix} = \begin{bmatrix} \lambda_1 y_{11} & \lambda_2 y_{12} & \cdots & \lambda_n y_{1n} \\ \vdots & & & \vdots \\ \lambda_1 y_{n1} & \cdots & \cdots & \lambda_n y_{nn} \end{bmatrix} =$$

Thus, the free elements exist on on the diagonal line.

$\therefore \dim \ker L = n$

(6) Let $A = (a_{ij}), i \le 3, j \le 3, X = (x_{ij}), i \le 3, j \le 3$

The diagonal of $AX = \begin{cases} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} \\ a_{21}x_{12} + a_{22}x_{22} + a_{23}x_{32} \\ a_{31}x_{13} + a_{32}x_{23} + a_{33}x_{33} \end{cases}$

Diagonal of $XA = \begin{cases} a_{11}x_{11} + a_{21}x_{12} + a_{31}x_{13} \\ a_{12}x_{21} + a_{22}x_{22} + a_{32}x_{23} \\ a_{13}x_{31} + a_{23}x_{32} + a_{33}x_{33} \end{cases}$

$$\begin{cases} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} = a_{11}x_{11} + a_{21}x_{12} + a_{31}x_{13} \\ a_{21}x_{12} + a_{22}x_{22} + a_{23}x_{32} = a_{12}x_{21} + a_{22}x_{22} + a_{32}x_{23} \\ a_{31}x_{13} + a_{32}x_{23} + a_{33}x_{33} = a_{13}x_{31} + a_{23}x_{32} + a_{33}x_{33} \end{cases}$$

$$\Rightarrow \begin{cases} a_{12} = a_{21} = a_{31} = a_{13} = 0 \\ a_{21} = a_{12} = a_{23} = a_{32} = 0 \\ a_{13} = a_{31} = a_{23} = a_{32} = 0 \end{cases}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$