1 Tensor Calculations

题目1. Kronecker Product

解答. 1.

$\forall \vec{V}_1, \vec{V}_2 \in V, \vec{W}_1, \vec{W}_2 \in W, K \in \mathbb{R}, \mathbb{C}$

$(X \otimes Y)(\vec{V}_1, \vec{W}_1) + (X \otimes Y)(\vec{V}_2, \vec{W}_1) = X\vec{V}_1 \otimes Y\vec{W}_1 + X\vec{V}_2 \otimes Y\vec{W}_1 = X(\vec{V}_1, \vec{V}_2) \otimes Y\vec{W}_1 = (X \otimes Y)(\vec{V}_1 + \vec{V}_2, \vec{W}_1)$

同理 $(X \otimes Y)(\vec{V}_1, \vec{W}_1) + (X \otimes Y)(\vec{V}_1, \vec{W}_2) = X\vec{V}_1 \otimes Y\vec{W}_1 + XV_2 \otimes Y\vec{W}_2 = X\vec{V}_1 \otimes Y(\vec{W}_1) + \vec{W}_2 = (X \otimes Y)(\vec{V}_1, \vec{W}_1 + \vec{W}_2)$

$(X \otimes Y)(KV_1, W_1) = (KX\vec{V}_1) \otimes (Y\vec{W}_1) = K(X\vec{V}_1) \otimes (YW_1) = K(X \otimes Y)(V_1, W_1)$

$(X \otimes Y)(V_1, KW_1) = (X\vec{V}_1) \otimes (KY\vec{W}_1) = K(X\vec{V}_1) \otimes (Y\vec{W}_1) = K(X \otimes Y)(\vec{V}_1, \vec{W}_1)$

$\therefore X \otimes Y$ is bilinear

解答. 2.

Assume that $(X \otimes Y)(V_i, W_j) = (\lambda_i V_i) \otimes (\mu_j W_j) = \lambda_i \mu_j (V_i \otimes W_j)$

so $\lambda_i \mu_j$ is an eigenvalue of $(X \otimes Y)$, λ_i is eigenvalue of X, μ_j is eigenvalue of Y

thus trace$(X \otimes Y) = \sum$ all eigenvalues of$(X \otimes Y) = \sum_{j=1}^{m} \sum_{i=1}^{n} (\lambda_i \lambda_j)$

$$= \sum_{i=1}^{N} \lambda_i \cdot \sum_{j=1}^{m} \lambda_j = (\text{trace}X)(\text{trace}Y)$$

题目2. Quantum Entaglement

解答. $\mathbb{R}^n \otimes (\mathbb{R}^m)^* \to \mathbb{R}$ is an element of $\mathbb{R} \otimes (\mathbb{R}^n \otimes (\mathbb{R}^n)^*)^*$, i.e $\mathbb{R} \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n$

i.e. $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$

trace $= \sum_{i,j} (a_{ij} \vec{e}_i \otimes \vec{e}_j)(\vec{e}_i, \vec{e}_j$ is the basis of $\mathbb{R}^n)$

For an rank on matrix, $M = \vec{V}a$, trace$(M) = \sum a_{ij}(e_i v \otimes e_j \alpha)$

So trace$(V\alpha) = \sum a_{ij} V_i \alpha_j$

Meanwhile trace$(V\alpha) = \text{trace} \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \left(\begin{pmatrix} \alpha_1 & \cdots & \alpha_n \end{pmatrix} \right) = \sum_{i=1}^{n} V_i \alpha_i = V_1\alpha_1 + V_2\alpha_2 + \cdots + V_n\alpha_n$ Thus

$$a_i j = \begin{cases} 1 & \text{if} i = j \\ 0 & \text{if} i \neq j \end{cases}$$

So the entries of tensor trace are a_{ij}

解答. 1.

Symmetric: $(V_1 \otimes W_1, V_2 \otimes W_2) = (V_1, V_2)(W_1, W_2), (V_2 \otimes W_2, V_1 \otimes W_1) = (V_2, V_1)(W_2, W_1)$

$(V_1, V_2) = (V_2, V_1), (W_1, W_2) = (W_2, W_1)(HA, HB)$ is inner product space

$\therefore (V_1 \otimes W_1, V_2 \otimes W_2) = (V_2 \otimes W_2, V_1 \otimes W_1)$

Positive definite: $\because (V_1, V_1) \geq 0, (W_1, W_1) \geq 0$ and $(V_1, V_1) = 0 = (W_1, W_1)$ iff $V_1 = \vec{O}, W_1 = \vec{O}$

$\therefore (V_1 \otimes W_1, V_1 \otimes W_1) = (V_1, V_1)(W_1, W_1) > 0$ iff $V_1 = \vec{O}, W_1 = \vec{O} \Leftrightarrow \vec{U}_1 \otimes \vec{W}_1 = 0$

解答. 2.

$\alpha(e_1 \otimes e_1) + b(e_2 \otimes e_2)$

$\because$ We can write it in the form $\alpha(e_1 \otimes e_1) + b(e_2 \otimes e_2)$, rank ≤ 2

Or it can be written as $\vec{V} = (V_1\vec{e}_1 + V_2\vec{e}_2), \vec{W} = W_1\vec{e}_1 + W_2\vec{e}_2$, $\vec{V} \otimes \vec{W}$ form.

Then $\vec{V} \otimes \vec{W} = V_1W_1\vec{e}_1 \otimes \vec{e}_1 + V_2W_2\vec{e}_2 \otimes \vec{e}_2 + V_1W_2\vec{e}_1 \otimes \vec{e}_2 + V_2W_1\vec{e}_2 \otimes \vec{e}_1 = a\vec{e}_1 \otimes \vec{e}_1 + b\vec{e}_2 \otimes \vec{e}_2$

$V_1W_1V_2W_2 = ab \neq= 0$, according to $V_1W_1 = a, V_2W_2 = b$

$V_1W_1V_2W_2 = 0$, according to $V_1W_2 = 0, V_2W_1 = 0$

解答. 3.

$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \vec{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, W = \vec{V} \otimes \vec{W}$

$(W, L \otimes I_B(W)) = (\vec{V} \otimes \vec{W}, \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix} \otimes \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}) = (V_1^2 - V_2^2)(W_1^2 + W_2^2) = a$

$(W, I_A \otimes L(W)) = (\vec{V} \otimes \vec{W}, \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \otimes \begin{bmatrix} W_1 \\ -W_2 \end{bmatrix}) = (V_1^2 + V_2^2)(W_1^2 - W_2^2) = b$

$V_1 = 1, V_2 = 0, W_1 = \sqrt{\frac{a+b}{2}}, W_2 = \sqrt{\frac{a-b}{2}}, (a \geq b)$

$V_1 = \sqrt{\frac{a+b}{2}}, V_2 = \sqrt{\frac{b-a}{2}}, W_1 = 1, W_2, (a < b)$

解答. 4.

$(W, L \otimes I_B(W)) = (W, \alpha e_1 \otimes e_2 + b((-e_1) \otimes e_2))$

$(W, I_A \otimes L(W)) = (W, \alpha e_1 \otimes e_2 + be_1 \otimes (-e_2))$

$\Rightarrow = (W, \alpha, ae_1 \otimes e_2 - be_1 \otimes e_2)$

$\therefore$ observing A is identical to observing B