1 1.2 Invariant Decomposition

题目1. Prove or find counter examples

1. For four subspaces, if any three of them are linearly independent, then the four subspaces are linearly independent

2. If subspaces V_1, V_2 are linearly independent, and V_1, V_3, V_4 are linearly independent, and V_2, V_3, V_4 are linearly independent, then all four subspaces are linearly independent

3. If V_1, V_2 are linearly independent, and V_3, V_4 are linearly independent, and $V_1 + V_2, V_3 + V_4$ are linearly independent, then all four subspaces are linearly independent

解答.

1. Let the subspaces be W, X, Y, Z are subspace of S where W has set of vectors $\vec{w_1}, \ldots, \vec{w_n}$ which follows the rules of subspaces (closed under addition and closed under scalar multiplication, and so on for X, Y, Z

Two subspaces are linear independent if and only if $W \cap X = \{0\}$. Which means that for a subspace to be linearly independent, it means no linear combination of one element from each of W, X is equal to 0 except for the trivial case. In other words, they are linearly independent if and only if $\text{span}(W) \cup \text{span}(X) = 0$

Thus, if three of the subspaces are linearly independent, then the union's of there span's is an empty intersection. However, just because three of the subspaces are linearly independent, it does not mean the fourth is. The span of the fourth, let's say Z, may intersect with the any of the previous four. We can use the following as a counter example

$$V_1 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, V_2 = \text{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, V_3 = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}, V_4 = \text{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

2. We can use the same example as 1.

3. If

$$W\text{basis} : W_1, \ldots W_p$$

$$X\text{basis} : X_1, \ldots X_q$$

$$Y\text{basis} : Y_1, \ldots Y_r$$

$$Z\text{basis} : Z_1, \ldots Z_s$$

And we know that W,X are linearly independent, and so are Y, Z, then

$$W + X\text{basis} : W_1, \ldots, W_p, X_1, \ldots, X_q$$

$$Y + Z\text{basis} : Y_1, \ldots, Y_r, Z_1, \ldots, Z_s$$

and if $W + X$, $Y + Z$ are linearly independent, then

$$\alpha_1 W_1 + \cdots + \alpha_p W_p + \beta_1 X_1 + \cdots \beta_q X_q + \lambda_1 Y_1 + \cdots + \lambda_r Y_r + \zeta_1 Z_1 + \cdots + \zeta_s Z_s = 0$$

which means all the coefficients are equal to zero. Which states that four vectors are linearly independent.

题目2. Let V be the space of $n \times n$ real matrices. Let $T : V \to V$ be the transpose operations. Find the non-trivial T-invariant decomposition of V and find the corresponding block form of T

解答. Let us define e_{ij} to be a matrix with a value of 1 at entry (i, j) and zero elsewhere. This is the basis for the space $M_{n\times n}$.

Thus the transpose operation on that space is as follows:

$$T(e_{ij}) = e_{ji}$$

Thus, we need to come up with arbitrary ordering of $\{e_{ij}\}$. Lets say that we have the matrix in $\mathbb{R}^2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A possible b asis for the vector space is as such.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

Where the standard can be represented as $S = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, so the Matrix A can also be written as $aE_{11} + bE_{12} + cE_{21} + dE_{22}$, which can be represented as vector

$$v = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Then the linear mapping T acts on the elements of the basis, giving us the assoicated matrix of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And we can get a block form from this matrix.

题目3. Let $p(x)$ be any polynomial, and define $p(A)$ in the obvious manner. Prove

1. If $AB = BA$, show that $\mathrm{Ker}(B), \mathrm{Ran}(B)$ are both A-invariant subspaces.
2. Prove that $Ap(A) = p(A)A$
3. Conclude that $N_\infty(A - \lambda I), R_\infty(A - \lambda I)$ are both A invariant for any $\lambda \in \mathbb{C}$

解答.

(1.1) $x \in \mathrm{Ker}(B), Bx = 0, ABx = B(Ax) = 0$, therefore, $Ax \in \mathrm{Ker}(B)$

(1.2) $x \in \mathrm{Ran}(B), \exists y, x = By, Ax = ABy = B(Ay)$, therefore, $Ax \in \mathrm{Ran}(B)$

(2) $A \cdot A^k = A^k \cdot A, AI = IA$, thefore, $A \cdot P(A), P(A) \cdot A$

(3.1) $x \in N_\infty(A - \lambda I), \therefore \exists (A - \lambda I)^x = 0$

$\therefore (A - \lambda I)^k(Ak) = A((A - \lambda I)^k x) = 0$

$Ax \in N_{\infty(A-\lambda I)}$

(3.2) $x \in R_\infty(A - \lambda I), \therefore \forall k, \exists y_k, (A - \lambda I)^k y_k = x$

$\therefore Ax = A(A - \lambda I)^y_k = (A - \lambda I)^k(Ay_k)$

And if we let $y'_k = Ay_k, \forall k, \exists y'_k, (A - \lambda I)^k y'_k = Ax$

题目4. Note that any linear map must have at least one eigenvector. Fix any two $n \times n$ matrices, A, B. Suppose $AB = BA$

1. If W is an A invariant subspace, show that A has an eigenvector in W
2. Show that $\mathrm{Ker}(A - \lambda I)$ is always B-invariant for all $\lambda \in \mathbb{C}$
3. Show that A, B has a common eigenvector.

解答.

(1) Because W is an A invarient space, and if we $V_1 = (V_1 \cdots V_t)$ be the basis of W and if

$$A = X \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_t \end{pmatrix} X^{-1}, X = (v_1, \cdots v_t)$$

$v_2, \cdots, v_t$ are other A invarient subspaces, Therefore A_1 must have eigenvectorm and thus $\vec{x}$ is an eigenvector of A in W.

(2) $x \in \mathrm{Ker}(A - \lambda I)$

$$\Rightarrow (A - \lambda I)x = 0$$

$$\Rightarrow (A - \lambda I)Bx = B(A - \lambda I)x = 0$$

$$\Rightarrow Bx \in \mathrm{Ker}(A - \lambda I)$$

(3) Because $\mathrm{Ker}(A - \lambda I)$ is a B-invariant subspace

$\Rightarrow$ B has an eigen vector in $\mathrm{Ker}(A - \lambda I)$

$\Rightarrow Bx = \lambda_o x, x \in \mathrm{Ker}(A - \lambda I)$

$\Rightarrow (A - \lambda I)x = 0 \Rightarrow Ax = \lambda x$

x is the common eigenvector of both A and B