题目1. Let $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we have a function $f(x) = x|x|$. Note that as a real function, $f(x)$ is everywhere differentiable.

解答.

(1) Let $A_t = \begin{bmatrix} 1 & 1 \\ 0 & 1+t \end{bmatrix}$. Note that $\lim A_t = J$. Find $\lim f(A_t)$

$$A_t = \begin{bmatrix} 1 & 1 \\ 0 & 1+t \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1+t \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{t} \\ 0 & 1 \end{bmatrix}$$

$$f(A_t) = \begin{bmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(1) & 0 \\ 0 & f(1+t) \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{t} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} f(1) & \frac{f(1+t)-f(1)}{t} \\ 0 & f(1+t) \end{bmatrix}$$

$$\lim f(A_t) = \begin{bmatrix} f(1) & f'(t) \\ 0 & f(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(2) Let $A_t = \begin{bmatrix} 1 & 1 \\ -t^2 & 1 \end{bmatrix}$. Note that $\lim A_t = J$. Find $\lim f(A_t)$. Is $f(J)$ well defined?

$$A_t = \begin{bmatrix} 1 & 1 \\ -t^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -it & it \end{bmatrix} \begin{bmatrix} 1-it & 0 \\ 0 & 1+it \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{i}{2t} \\ \frac{1}{2} & \frac{i}{2t} \end{bmatrix}$$

$$f(A_t) = \begin{bmatrix} 1 & 1 \\ -it & it \end{bmatrix} \begin{bmatrix} f(1-it) & 0 \\ 0 & f(1+it) \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{i}{2t} \\ \frac{1}{2} & \frac{i}{2t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(f(1-it)+f(1+it)) & \frac{1}{2it}(f(1+it)-f(1-it)) \\ \frac{it}{2}(f(1+it)-f(1-it)) & \frac{1}{2}(f(1-it)+f(1+it)) \end{bmatrix}$$

$$\lim f(A_t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\because \lim f(A_t) \neq f(\lim A_t)$, $\therefore f(J)$ is not well-defined

题目2. Compute the following

解答.

(1) Find the derivative of $\sin(tA)$ as function of t

$$\sin(tA) = tA - \frac{(tA)^3}{3!} + \frac{(tA)^5}{5!} \cdots$$

$$\frac{d}{dt}\sin(tA) = A - \frac{(tA)^2}{2!} + \frac{(tA)^4}{4!} \cdots = A\cos(At)$$

(2) For the forumla $f\left(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}\right) = \begin{bmatrix} f(2A) & B \\ & f(2A) \end{bmatrix}$, what is the block matrix B in terms of f and A?

For the taylor expansion at $\begin{bmatrix} 2A & 0 \\ 0 & 2A \end{bmatrix}$, $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^2 = 0$

$$\therefore f\left(\begin{bmatrix} 2A & A \\ 0 & 2A \end{bmatrix}\right) = f\left(\begin{bmatrix} 2A & 0 \\ 0 & 2A \end{bmatrix}\right) + f\left(\begin{bmatrix} 2A & \\ & 2A \end{bmatrix} \cdot \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right)$$

$$\Rightarrow \begin{bmatrix} f(2A) & 0 \\ 0 & f(2A) \end{bmatrix} + \begin{bmatrix} 0 & f'(2A)\cdot A \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f(2A) & f'(2A\cdot A) \\ & f(2A) \end{bmatrix}$$

$\therefore B = f'(2A) \cdot A$

(3) Prove or find counter example: The derivative to $f(A+tB)$ as a differentiable function of t at $t=0$ is $f'(A)B$

Let $f(x) = x^2$

$\Rightarrow f(A+tB) = A^2 + t(AB+BA) + t^2B^2$

$\Rightarrow t=0$, $\frac{d}{dt}f(A+tB) = AB+BA$

But we know that $f'(A)B = 2AB$ when $AB \neq BA$

$\therefore f'(A)B \neq \frac{d}{dt}f(A+tB)$

题目3. Suppose $AB = BA$, in previous homework, we see that this implies that A, B must have a common eigenvalue

解答.

(1) Show that we can find invertible X_1, such that $X_1AX_1^{-1} = \begin{bmatrix} a_1 & * \\ & A_1 \end{bmatrix}$, $X_1BX_1^{-1} = \begin{bmatrix} b_1 & * \\ & B_1 \end{bmatrix}$, and that $A_1B_1 = B_1A_1$

Let the common eigenvector be $\vec{v}_0$

$$A\vec{v}_0 = a_0\vec{v}_0, B\vec{v}_0 = b_0\vec{v}_0$$

$$X_1^{-1} = \begin{pmatrix} a_1* \\ A_1 \end{pmatrix} = \begin{pmatrix} a_1\vec{v}_1 & * \end{pmatrix}, A_{X_1^{-1}} = \begin{pmatrix} A\vec{v}_1 \cdots A\vec{v}_n \end{pmatrix} = \begin{pmatrix} a_1\vec{v}_1 & * \end{pmatrix}$$

Let $\vec{v}_1 = \vec{v}_0, a_1 = a_0$

$$X_1AX_1^{-1} = \begin{bmatrix} a_1 & * \\ & A_1 \end{bmatrix}, X_1BX_1^{-1} = \begin{bmatrix} b_1 & * \\ & B_1 \end{bmatrix}$$

$$X_1ABX_1{}^{-1} = \begin{bmatrix} a_1 & * \\ & A_1 \end{bmatrix} \begin{bmatrix} b_1 & * \\ & B_1 \end{bmatrix} = \begin{bmatrix} a_1b_1 & * \\ & A_1B_1 \end{bmatrix}$$

$$= X_1BAX_1{}^{-1} = \begin{bmatrix} a_1b_1 & 0 \\ & A_1B_1 \end{bmatrix}$$

$\therefore A_1B_1 = B_1A_1$

(2) Show that A,B can be simultaneously triangularized

$\because A_1B_1 = B_1A_1$, we can find

$$X_2AX_2{}^{-1} = \begin{bmatrix} a_2 & * \\ & A_2 \end{bmatrix}, X_2BX_2{}^{-1} = \begin{bmatrix} b_2 & * \\ & B_2 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 & 0 \\ 0 & X_2' \end{bmatrix}$$

$$X_2X_1AX_1{}^{-1}X_2{}^{-1} = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix}, X_2X_1BX_1{}^{-1}X_2{}^{-1} = \begin{bmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{bmatrix}$$

$\because$ we repeat this process for all X_n, $\therefore AB$ can be simultaneously triangularized