1 1.2 最大公因子和小公倍数

题目4. 设$a, b, c \in \mathbb{Z}, a \neq 0$, 则$a \mid bc$ 当且仅当 $\frac{a}{(a,b)} \mid c$

解答. Let $d = (a, b)$, then, $1 = (\frac{a}{d}, \frac{b}{d})$, and thus

$$\frac{a}{d} \mid \frac{b}{d}c$$

Since we know that $1 = (\frac{a}{d}, \frac{b}{d})$, we can say that $\frac{a}{d} \mid c \Rightarrow \frac{a}{a,b}$

题目5. m和n是互素的正整数：证明：

(1) 对于每个整数 $a, (a, mn) = (a, m)(a, n)$

(2) mn的每个正因子d均可惟一地表示成$d = d_1 d_2$, 其中 d_1和d_2分别为m和n地正因子。

解答.

(1) $(a, mn) = (a, m)(\frac{a}{(a,m)}, \frac{mn}{(a,m)})$

And since $\frac{a}{(a,m)}$ and $\frac{m}{(a,m)}$ are coprime

$$= (a, m)(\frac{a}{(a, m)}, n)$$

And because $(m, n) = 1$

$$= (a, m)(a, n)$$

(2)

题目6. 设n为正整数，a,b是不全为零的整数，证明：
(1)$(a^n,b^n)=(a,b)^n$
(2) 若a和b是互素的正整数，$ab=c^n, c\in\mathbb{Z}$，则a和b都是正整数的n次方幂，事实上，$a=(a,c)^n, b=(b,c)^n$

解答.
(1) We can use the definition of primes to solve this question. First let $a=p_1^{\alpha_1}\cdots p_k^{a_k}, b=p_1^{\beta_1}\cdots p_k^{\beta_k}$, where $p_1,\ldots,p_k$ are all primes, and α_i,β_i are non-negative integers, $i=1,\ldots,k$
Thus

$$a^n=p_1^{n\alpha_1}\cdots p_k^{na_k}$$

$$b^n=p_1^{n\beta_1}\cdots p_k^{n\beta_k}$$

Let us say that $(a^n,b^n)=p_1^{\gamma_1}\cdots p_k^{\gamma_k}$, where

$$\gamma_i=\min(n\alpha_i,n\beta_i)=n\min(\alpha_i,\beta_i), i=1,2\ldots k$$

Thus, $(a^n,b^n)=p_1^{\gamma_1}\cdots p_k^{\gamma_k}=(p_i^{\min(\alpha_i,\beta_i)}\cdots p_k^{\min(\alpha_k,\beta_k)})^n=(a,b)^n$
(2) Let

$$a=p_1^{\alpha_1}\cdots p_k^{a_k}, b=p_1^{\beta_1}\cdots p_s^{\beta_s}$$

Becuase $(a,b)=1$, then $ab=p_1^{\alpha_1}\cdots p_k^{a_k}p_1^{\beta_1}\cdots p_s^{\beta_s}$ is a unique solution. Then let

$$c=p_1^{c_1}\cdots p_k^{c_k}q_1^{d_1}\cdots q_s^{d_s}$$

Then

$$p_1^{\alpha_1}\cdots p_k^{a_k}p_1^{\beta_1}\cdots p_s^{\beta_s}=ab=c^n=p_1^{nc_1}\cdots p_k^{nc_k}q_1^{nd_1}\cdots q_s^{nd_s}$$

If $\alpha_i=nc_i$, $\beta_j=nd_j,(i=1,\ldots k, j=1,\ldots s)$, we get that $a=(p_1^{c_1}\cdots p_k^{c_k})^n$ and $b=(q_1^{c_1}\cdots q_s^{c_s})^n$.
Therefore, a and b are an integer to the nth power

题目9. 用辗转相除法求963和957地最大公因子，并求出方程$963x+657y=(963,657)$ 的全部整数解

解答. Using Euclid's Algorithm, we can derive the 全部疏解. First we use the foward notation.

$$963=1(657)+306$$

$$657 = 2(306) + 45$$

$$306 = 6(45) + 36$$

$$45 = 1(36) + 9$$

$$36 = 4(9) + 0$$

So, the GCD is (9) We get the equaltion $963/9x + 657/9y = 9/9 \Rightarrow 107x + 73y = 1$ Then we use reverse euclid's algorithm.

$$9 = 45 - 36$$

$$9 = (657 - 2(306)) - (306 - 6(45))$$

$$9 = (657 - 2(963 - 657)) - (963 - 657 - 6(657 - 2(306)))$$

$$9 = (657 - 2(963 - 657)) - (963 - 657 - 6(657 - 2(963 - 657)))$$

$$= 22(657) - 15(963) \Rightarrow 963 \times -15 + 657 \times 22 = 9$$

THerefore, we have found an answer $(x_0, y_0) = (-15, 22)$ to get the equation for 全部整数解

$$\begin{cases} x = -15 + 657t \\ y = 22 - 963t \end{cases}$$

题目10. 求下列方程的全部整数解

(1)$6x + 20y - 15x = 23$

(2)$25x + 13y + 7x = 2$

题目12. 设$f(x) = x^n + a_1x^n - 1 + \cdots + a_{n-1}x + a_n$ 是首项系数为1的整系数多项式，则$f(x)$的每个有理数必为整数

题目13. 说m和n为正整数，则在$n, 2n, \ldots, mn$这m个数当中恰有(m, n)个是m的倍数。

题目16. 设m和n是互素的非零整数，证明：对每个整数a，如果$m \mid a, n \mid a$,则 $mn \mid a$

解答. Since $m \mid a$, $a = bm$ for some integer b, then if take $n \mid a$, which means $n \mid bm$, and since $(m, n) = 1$ (are coprime). We know that n must divide b. And therefore if $n|b$, where $b = nj$ for some integer j, $a = mnj$. So $mn \mid a$.

2 1.3 惟一分解定理

题目3. 设a, b, c均为正整数，证明
(1) $(a, [b, c]) = [(a, b), (a, c)]$
(2) $[a, (b, c)] = ([a, b][b, c])$

解答. (1) We can use the universal gcd laws. First we can use $[x, y] = xy/(x, y)$ to elimante all LCMs. Then we can use the properties of gcd to break it down

$$(a, [b, c]) = [(a, b), (a, c)]$$

$$\Rightarrow (a, \frac{b}{(b, c)}) = \frac{(a, b)(a, c)}{(a, b, c)}$$

$$\Rightarrow (a, b, c)(a(b, c), bc) = (a, b)(a, c)(b, c)$$

$$= (aab, aac, abb, abc, acc, bbc, bbc)$$

(2) Using a similar laws as above

$$[a, (b, c)] = ([a, b][b, c])$$

$$\Rightarrow \frac{a(b, c)}{(a, b, c)} = a(\frac{b}{(a, b)}, \frac{c}{(a, c)})$$

题目4. 整数n叫作无平方因子，是指不存在整数$m \geq 2$，是的$m^2 \mid n$, 证明

解答. (1) Here we have to prove that n is a square free integer if and only if $n = 1$ or it is a product of different primes.
Lets suppose $n = p_1^{\alpha_1} \cdot p_k^{\alpha_k}$ where p_i are prime numbers and α_i are integers $i = 1, \ldots, k$
Proof By Contradiction: If n is a square free number, then it has prime numbers that are the

same.

$n = p_1^{\alpha_1} \cdot p_k^{\alpha_k}$ where there is a pair $p_i = p_j$, If it has prime numbers that are the same, then we get a value where $a_i = 2$. Therefore there is a square number. And thus, it contradicts that n is a square free number.

(2) Here we have to prove that every number n can be represented as the product of a square number and a square free number.