1 Dual Stuff

题目1. Let V be the space of real polynomials of degree less than n. So dim $V = n$. Then for each $a \in \mathbb{R}$, the evaluation ev_a is a dual vector.

解答. 1. Under the basis of $1, x, \dots, x^{n-1}$, the polynomial $p(x) = k_o + k_1 x + k_2 x^2 + \dots + k_{n-1}x^{n-1}$ can be represented as $\begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_m \end{pmatrix} \Rightarrow$

$$L : \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_m \end{pmatrix} \mapsto \begin{pmatrix} p(a_1) \\ p(a_2) \\ \vdots \\ p(a_m) \end{pmatrix} = \begin{pmatrix} k_0 + k_1 a_1 + k_2 a_1^2 + \dots + k_{n-1} a_1^{n-1} \\ k_0 + k_1 a_2 + k_2 a_2^2 + \dots + k_{n-1} a_2^{n-1} \\ \vdots \\ k_0 + k_1 a_n + k_n a_2^2 + \dots + k_{n-1} a_n^{n-1} \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_m \end{pmatrix}$$

Thus the matrix for L under this basis is

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$

which happens to be the Vandermonder Matrix!

解答. 2.

$$|L| = \prod_{1 \le j < i \le n} (a_i - a_j)$$

$$\begin{cases} |L| \neq 0 & \text{iff } a_1, a_2, \dots, a_n \text{ are distinct} \\ L \text{ is invertable} & \text{iff } a_1, a_2, \dots, a_n \text{ are distinct} \end{cases}$$

解答. 3. Firstly, $ev_{a_1}, ev_{a_2}, \dots, ev_{an} \in V^*$ and dim $V^* = n$

$\therefore$ we only need to prove that $ev_{a_1}, ev_{a_2}, \dots, ev_{an}$ is in fact independent

Let $b_1 ev_{a_1} + b_2 ev_{a_2} + \dots + b_n ev_{an} = (0, 0, \cdots, 0)$

$\Rightarrow b_1 p(a_1) + b_2 p(a_2) + \dots + b_n p(a_n) = (0, 0, \cdots, 0)$

$$\Rightarrow (b_1, b_2, \cdots, b_n) \begin{pmatrix} p(a_1) \\ p(a_2) \\ \vdots \\ p(a_n) \end{pmatrix} = (0, 0, \cdots, 0)$$

$\Rightarrow (b_1, b_2, \cdots, b_n)L = (0, 0, \cdots, 0)$

$\therefore L$ is in fact invertable, and thus $b_1 = b_2 = \cdots = b_n = 0 \Rightarrow ev_{a_1}, ev_{a_2}, \ldots, ev_{an}$ is linear independent

$\because ev_{a_1}, ev_{a_2}, \ldots, ev_{an} \Rightarrow$ we must have $b_1 = b_2 = \cdots = b_n = 0 \Rightarrow L$ is invertable

$\therefore ev_{a_1}, ev_{a_2}, \ldots, ev_{an}$ form a basis for V^* iff all $a_1, a_2, \cdots a_n$ are distinct

解答. 4. Let $a_n = -1, a_2 = 0, a_3 = 1$

$$L = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, L^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$Lp_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow p_{-1} = L^{-1}x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow p_{-1} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$\therefore p_{-1}(x) = -\frac{1}{2}x + \frac{1}{2}x^2$

$\therefore p_o(x) = -x^2 + 1$

$\therefore p_1(x) = \frac{1}{2}x^2 + \frac{1}{2}x$

解答. 5. Let $b_2ev_2 + b_{-1}ev_{-1} + b_0ev_0 + b_1ev_1 + b_2ev_2 = (0, 0, 0, 0)$

$$\therefore \begin{pmatrix} b_{-2} & b_{-1} & b_0 & b_1 b_2 \end{pmatrix} \begin{pmatrix} p(-2) \\ p(-1) \\ p(0) \\ p(1) \\ p(2) \end{pmatrix} = (0, 0, 0, 0)$$

$$\therefore \begin{pmatrix} b_{-2} & b_{-1} & b_0 & b_1 b_2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} = (0, 0, 0, 0)$$

$\therefore b_{-2} = -1, b_{-1} = 4, b_0 = -6, b_1 = 4, b_2 = -1$

$\therefore -ev_{-2} + 4ev_{-1} - 6ev_0 + 4ev_1 - ev_2 = 0$

题目2. Let V be the space of real polynomials of degree less than 3. Which of the following is a dual vector? Prove it or show why not.

解答. 1.Let $p(x)=a_0+a_1x+a_2x^2$, $(x+1)p(x)=a_0+(a_0+a_1)x+(a_1+a_2)x^2+a_2x^3$

$$\therefore p\mapsto ev_5((x+1)p(x))\Leftrightarrow\begin{pmatrix}a_0\\a_1\\a+2\end{pmatrix}\mapsto 6a_0+30a_1+150a_2$$

$\therefore$ it is equal to the row vector $\begin{pmatrix}6&30&15\end{pmatrix}$

$\therefore$ map is a dual vector

解答. 2. Let $p(x)=a_0+a_1x+a_2x^2$, $\lim_{x\to\infty}\frac{p(x)}{x}=\lim_{x\to\infty}(a_1+a_2x)$

$$\therefore p\mapsto\lim_{x\to\infty}\frac{p(x)}{x}\Leftrightarrow\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}\mapsto a_1+a_2x$$

$\therefore$ it cannot equal a constant row vector

$\therefore$ the map is not a dual vector

解答. 3. Let $p(x)=a_0+a_1x+a_2x^2$, $\lim_{x\to\infty}\frac{p(x)}{x^2}=a_2$

$$\therefore p\mapsto\lim_{x\to\infty}\frac{p(x)}{x^2}\Leftrightarrow\begin{pmatrix}a_0&a_1&a_2\end{pmatrix}\mapsto a_2$$

$\therefore$ is is equal to row vector $\begin{pmatrix}0&0&1\end{pmatrix}$

$\therefore$ so this is a dual vector

解答. 4. Let $p(x)=a_0+a_1x+a_2x^2$, $p(3)p'(4)=(a_0+3a_1+9a_2)(a_1+2(4)a_2)=a_0a_1+8a_0a_4+3a_1^2+24a_1a_4+0a_1a_2+7_2a_2a_4$

$$\therefore p\mapsto p(3)p'(4)\Leftrightarrow\begin{pmatrix}a_0\\a_0\\a_2\end{pmatrix}\mapsto a_0a_1+8a_0a_4+3a_1^2+24a_1a_4+0a_1a_2+7_2a_2a_4$$

$\therefore$ it cannot be equal to a row veector $\therefore$ the map is not a dual vector

解答. 5. Let $p(x)=a_0+a_1x+a_2x^2$, $\deg=\begin{cases}2 & a_2\neq0\\1 & a_2=0\text{ and }a_1\neq0\\0 & a_1=a_2=0\end{cases}$

$$p\mapsto\deg(p)\Leftrightarrow\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}\mapsto\begin{cases}2 & a_2\neq0\\1 & a_2=0\text{ and }a_1\neq0\\0 & a_1=a_2=0\end{cases}$$

$\therefore$ it is not a linear map since $L\begin{pmatrix}0\\1\\0\end{pmatrix}+L\begin{pmatrix}1\\0\\0\end{pmatrix}=1+2\neq L\left(\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}=2\right)$

$\therefore$ it is not a dual vector

题目3. For the differentiable function: ...

解答. Let $v=\begin{pmatrix}x\\y\end{pmatrix}, p=\begin{pmatrix}x_0\\y_0\end{pmatrix}$

$$\nabla_v f=\lim_{t\to 0}\frac{f\left(\begin{pmatrix}x_0\\y_0\end{pmatrix}+t\begin{pmatrix}x\\y\end{pmatrix}\right)-f\left(\begin{pmatrix}x_0\\y_0\end{pmatrix}\right)}{t}$$

Which is equivalent to the map

$$\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}\mapsto\left(\frac{\partial f}{\partial x}\Big|_{x_0},\frac{\partial f}{\partial y}\Big|_{y_0}\right)\begin{pmatrix}x\\y\end{pmatrix}$$

$\therefore$ it is equal to the row vector $\left(\frac{\partial f}{\partial x}\Big|_{x_0},\frac{\partial f}{\partial y}\Big|_{y_0}\right)$

$\therefore$ it is dual vector $\therefore$ coordinates under standard basis, $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ is $\left(\frac{\partial f}{\partial x}\Big|_{x_0},\frac{\partial f}{\partial y}\Big|_{y_0}\right)$

题目4. Consider a linear map $L:V\to W$

解答. 1.

The domain of L^* is in W^*, so $\mathrm{Ker}(L^*)$ is collection of dual vectors in W^*

$\therefore$ $\mathrm{Ker}(L^*)$ is a collection of dual vectors in W^* that kills $\mathrm{Ran}(L)$

解答. 2.

The codomain of L^* is in V^*, so $\mathrm{Ran}(L^*)$ is collection of dual vectors in V^*

$\therefore$ $\mathrm{Ran}(L^*)$ is a collection of dual vectors in V^* that kills $\mathrm{Ker}(L)$