1 Multilinear Maps

题目1. Elementary Layer Operations for Tensors

解答. 1.

(i, j, k) entry of 3D matrix $\alpha \otimes \beta \otimes \lambda$

$$\begin{aligned}
\alpha \otimes \beta \otimes \lambda &= \alpha \otimes \beta \otimes \lambda \left(\begin{pmatrix} 0 \\ \vdots \\ i \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ j \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ k \\ \vdots \\ 0 \end{pmatrix} \right) \\
&= \left[\alpha \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} i, \right] \left[\beta \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} j, \right] \left[\lambda \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} k \right] \\
&= \alpha i \beta j \lambda k
\end{aligned}$$

解答. 2.

All 3D matrices are sum of λ rank-one 3D matrices

$$A = \alpha_1 \otimes \beta_1 \otimes \lambda_1 + \alpha_2 \otimes \beta_2 \otimes \lambda_3 + \cdots + \alpha_r \otimes \beta_r \otimes \lambda_r$$

$$M_3 : A \mapsto (\alpha_1 E) \otimes \beta_1 \otimes \lambda_1 + (\alpha_2 E) \otimes \beta_2 \otimes \lambda_2 + \cdots + (\alpha_r E) \otimes \beta_r \otimes \lambda_r$$

$$\alpha_1(E_1 + E_2) \otimes B_1 \otimes \lambda_1 + \alpha_2(E_1 + E_2) \otimes \beta_2 \otimes \lambda_2 + \cdots + \alpha_r(E_1 + E_2) \otimes \beta_r \otimes \lambda_r$$

$$= \alpha_1 E_1 \otimes \beta_1 \otimes \lambda_1 + \alpha_2 E_1 \otimes \beta_2 \otimes \lambda_2 + \cdots + \alpha_r E_1 \otimes \beta_r \otimes \lambda_r$$

$$+ \alpha_1 E_2 \otimes \beta_1 \otimes \lambda_1 + \alpha_2 E_2 \otimes \beta_2 \otimes \lambda_2 + \cdots + \alpha_r E_2 \otimes \beta_r \otimes \lambda_r$$

$\because$

$$\alpha_1(kE) \otimes \beta_1 \otimes \lambda_1 + \alpha_2(kE) \otimes \beta_2 \otimes \lambda_2 + \cdots + \alpha_r(kE) \otimes \beta_r \otimes \lambda_r$$

$$= k \left[\lambda_1 E \otimes \beta_1 \otimes \lambda_1 + \alpha_2 E \otimes \beta_2 \otimes \lambda_2 + \cdots + \alpha_r \alpha_E \otimes \beta_r \otimes \lambda_r \right]$$

$\therefore$ M_E is a linear map and layer operation is extended to a linear map M_E

解答. 3.

Let A be a rank r 3D tensor, $B = M_E(A)$

We need to prove that $rank(B) = r$ Since A is a rank r 3D tensor, A can be expressed as

$$A = \alpha_1 \otimes \beta_1 \otimes \lambda_1 + \alpha_2 \otimes \beta_2 \otimes \lambda_2 + \cdots + \alpha_r \otimes \beta_r \lambda_r$$

$$B = M_E(A) = (\alpha_1 E) \otimes \beta_1 \otimes \lambda_1 + (\alpha_2(E)) \otimes \beta_2 \otimes \lambda_2 + \cdots + (\alpha_r E) \otimes \beta_r \lambda_r$$

$\therefore rank(B) = rank(A)$ and $rank(B) \leq r$

Thus, if $rank(B) = p < r$, then $B = \alpha_1 \otimes \beta_1 \otimes \lambda_1 + \cdots + \lambda_p \otimes \beta_p \otimes \lambda_p$

Since $B = M_E(A)$, then $A = M_E^{-1}(B)$

$$A = (\lambda_r E^{-1} \otimes \beta_1 \otimes \lambda_1 + \cdots + \alpha_p E^{-1} \otimes \beta_p \otimes \lambda_p)$$

where $rank(A) \leq p < r$, which is contradicting

$\therefore rank(B) = r$

解答. 4.

If a 3D matrix has rank $\lambda_1, \lambda_1 < r$, then it can be expressed as the sum of r_1 rank-one tensor

$$M = \alpha_1 \otimes \beta_1 \otimes \lambda_1 + \alpha_2 \otimes \beta_2 \otimes \lambda_2 + \cdots + \alpha_{r_1} \otimes \beta_{r_1} \otimes \lambda_{r_1}$$

So, the i^{th} value of the horizontal layer is

$$M_i = \lambda_{1i}(\alpha_1 \otimes \beta_1) + \lambda_{2i}(\alpha_2 \otimes \beta_2) + \cdots + \lambda_{r_1 i}(\alpha_n \otimes \beta_{r1})$$

$\therefore$

All layers of M can expressed as sum of r_1 rank one matrix

$rank(M_i) \leq r_1 < r \forall i$

Which contradicts our known conditions, therefore 3D matrix has at least rank r

解答. 5.

A 2D layer matrix has rank 2, $rank(M) \geq 2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$M = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\because rank(M) \leq 2 \therefore rank(M) = 2$

题目2. Let M be a $3\times 3\times 3$ "3D matrix" whose (i,j,k) entry is $i+j+k$. We interpret this as a multilinear map $M:\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$

解答. 1.

Horizontal layer 1 of M is $A_1 = \begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}$

Horizontal layer 2 of M is $A_2 = \begin{bmatrix}4&5&6\\5&6&7\\6&7&8\end{bmatrix}$

Horizontal layer 3 of M is $A_3 = \begin{bmatrix}5&6&7\\6&7&8\\7&8&9\end{bmatrix}$

Since M sends (u,v,w) to $\left[u^TA_1v, u^TA_2v, u^TA_3v\right]w$

So M sends (u,v,w) to $\left[v^TA_1v, v^TA_2v, v^TA_3v\right]v$

$v^TA_1v = (x,y,z)\begin{pmatrix}3&4&5\\4&5&6\\5&6&7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 3x^2+5y^2+7z^2+8xy+10xz+12yz$

$v^TA_2v = (x,y,z)\begin{pmatrix}4&5&6\\5&6&7\\6&7&8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 4x^2+6y^2+8z^2+10xy+12xz+14yz$

$v^TA_3v = (x,y,z)\begin{pmatrix}5&6&7\\6&7&8\\7&8&9\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 5x^2+7y^2+9z^2+12xy+14xz+16yz$

$\therefore$

$$\begin{aligned}\left[v^TA_1v, v^TA_2v, v^TA_3v\right]\begin{pmatrix}x\\y\\z\end{pmatrix} &= x(3x^2+5y^2+7z^2+8xy+10xz+12yz)\\ &+ y(4x^2+6y^2+8z^2+10xy+12xy+14yz)\\ &+ z(5x^2+7y^2+9z^2+12xy+14xz+16yz)\end{aligned}$$

$$= 3x^5+6y^5+9z^3+12x^2y+15xy^2+21xz^2+15x^2z+22yz^2+21y^2z+36xyz$$

解答. 2.

$M:(v_1,v_2,v_3)\mapsto M(v_1,v_2,v_3)$

$M' : (v_1, v_2, v_3) \mapsto M'(\sigma(v_1), \sigma(v_2), \sigma(v_3))$

M' is attained by exchanging the x, y, z direction of M, follwoing the order of the σ map.

Since we know the entry of M',

$$f(i) + f(j) + f(k) = i + j + k$$

$\therefore$, the (i, j, k) entry of the multilinear map M^σ is always equal to $i + j + k$

$i \in \{1, 2, 3\}, j \in \{1, 2, 3\}, k \in \{1, 2, 3\}$

M^σ is always equal to M when σ is an identity map

$M(v_1, v_2, v_3) = M(v_{\sigma(1)}, v_{\sigma(2)}, v_3\sigma(3))$

解答. 3.

Horizontal layer 1 of M, $A_1 = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$

$rank(A_1) = 2$

From the previous answer we know $rank(M) \geq rank(A_1) = 2$

$$\begin{aligned} M &= \begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right] \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

So M can be decomposed as three $rank$ 1 tensors, $\therefore 2 \leq rank(M) \leq 3$