Vector operators / Vector Calculus

29/01/21

① Gradient of scalar quantity :-

If $\phi(r) = \phi(x, y, z)$ is a scalar field in 3-dimension, then its gradient at any point is defined in cartesian co-ordinates as

$$\boxed{\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}}$$ - vector

Here $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \psi}{\partial z} \longrightarrow$ Partial derivatives of ϕ w.r.t. to x, y, z seperately

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}} \longrightarrow \text{diff. operator}$$

2) The divergence of vector field

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ its divergence at any point is defined as,

$$\boxed{\text{div}(\vec{a}) = \nabla \cdot \vec{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}} - \text{Scalar}$$

(3) The Curl of vector field

The curl of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is defined as,

$$\text{curl}\,\vec{a} = \nabla \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} \rightarrow \text{vector quantity}$$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \quad (\text{vector}) \qquad \phi - \text{Scalar}$$

$$\text{div}\,\vec{a} = \nabla\cdot\vec{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \quad (\text{Scalar}) \qquad \vec{a} \rightarrow \text{vector}$$

$$\text{Curl}\,\vec{a} = \nabla\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} - (\text{vector}) \qquad \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Problems on grad, div & curl

1. Find gradient of $\phi = x + y + z$

Sol: $$\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\phi = x + y + z$$

$$\frac{\partial \phi}{\partial x} = 1 + 0 + 0 = 1$$

$$\frac{\partial \phi}{\partial y} = 0 + 1 + 0 = 1$$

$$\frac{\partial \phi}{\partial z} = 0 + 0 + 1 = 1$$

$$\nabla \phi = 1\hat{i} + 1\hat{j} + 1\hat{k}$$

$$\nabla \phi = \hat{i} + \hat{j} + \hat{k}$$

$$|\nabla \phi| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \text{ units}$$

(2) Find $\nabla \phi$ & $|\nabla \phi|$ at $(0, 1, 4)$ for $\phi = 2x^2 + y^3 - z$

$$\phi = 2x^2 + y^3 - z$$

$$\frac{\partial \phi}{\partial x} = 4x + 0 - 0 = 4x$$

$$\frac{\partial \phi}{\partial y} = 0 + 3y^2 - 0 = 3y^2$$

$$\frac{\partial \phi}{\partial z} = 0 + 0 - 1 = -1$$

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\boxed{\nabla \phi = 4x\hat{i} + 3y^2\hat{j} - \hat{k}}$$

$\nabla \phi$ at $(\underset{x}{0}, \underset{y}{1}, \underset{z}{4})$

$$\nabla \phi_{(0,1,4)} = 0\hat{i} + 3\hat{j} - \hat{k} = 3\hat{j} - \hat{k}$$

$$|\nabla \phi| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10} \text{ unit}$$

(3) If $\phi = 3x^2yz^2$ find $\nabla\phi$ & $|\nabla\phi|$ at $(1,-1,4)$

$$\phi = 3x^2yz^2$$

$$\frac{\partial\phi}{\partial x} = (3yz^2)\frac{\partial}{\partial x}(x^2) = 6xyz^2$$

$$\frac{\partial\phi}{\partial y} = (3x^2z^2)\frac{\partial}{\partial y}(y) = 3x^2z^2$$

$$\frac{\partial\phi}{\partial z} = (3x^2y)\frac{\partial}{\partial z}(z^2) = 6x^2yz$$

$$\frac{\partial\phi}{\partial x}_{(1,-1,4)} = 6(1)(-1)(4)^2 = -96$$

$$\frac{\partial\phi}{\partial y}_{(1,-1,4)} = 3(1)^2(4)^2 = 48$$

$$\frac{\partial\phi}{\partial z}_{(1,-1,4)} = 6(1)^2(-1)(4) = -24$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi_{(1,-1,4)} = -96\hat{i} + 48\hat{j} - 24\hat{k}$$

$$|\nabla\phi| = \sqrt{96^2 + 48^2 + 24^2}$$

$$= \sqrt{9216 + 2304 + 576} = \sqrt{12,096}$$

$$|\nabla\phi| = 109.98 \simeq \underline{\underline{110}}$$