Problems on divergence

1) Find the divergence of $\vec{a} = \underbrace{2x}_{a_1}\hat{i} + \underbrace{y}_{a_2}\hat{j} + \underbrace{z^2}_{a_3}\hat{k}$ at the point $(1,1,1)$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \quad : \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla\cdot\vec{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

$$\nabla\cdot\vec{a} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2) = 2 + 1 + 2z = 3 + 2z$$

$$\nabla\cdot\vec{a}_{(1,1,1)} = 3+2(1) = 5$$

2) $\vec{v} = \underbrace{2x^2y}_{v_1}\,\hat{i} + \underbrace{xyz}_{v_2}\,\hat{j} - \underbrace{x^2y^2z}_{v_3}\,\hat{k}$

find $\nabla\cdot\vec{v}$ at $(2,-1,0)$

$$\nabla\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\nabla\cdot\vec{v} = 4xy + xz - 2x^2yz$$

$$(\nabla\cdot\vec{v})_{(2,-1,0)} = 4(2)(-1) + 2(0) - 2(2)^2(-1)(0) = -8$$

③ Find the divergence of

$\vec{F} = (2x - y^2)\hat{i} + (3z + x^2)\hat{j} + (4y - z^2)\hat{k}$

at $(1, 2, 3)$

$f_1 = 2x - y^2$; $\frac{\partial f_1}{\partial x} = 2$; $\left.\frac{\partial f}{\partial x}\right|_{(1,2,3)} = 2$

$f_2 = 3z + x^2$; $\frac{\partial f_2}{\partial y} = 0$; $\left.\frac{\partial f_2}{\partial y}\right|_{(1,2,3)} = 0$

$f_3 = 4y - z^2$; $\frac{\partial f_3}{\partial z} = -2z$; $\left.\frac{\partial f_3}{\partial z}\right|_{(1,2,3)} = -6$

$\nabla \cdot \vec{F}$

$= 2 + 0 - 6$

$= -4$ //

Problems on Curl

① Find $\nabla \times \vec{v}$ where

$$\vec{v} = \underbrace{2x}_{v_1}\,\hat{i} \underbrace{-y}_{v_2}\,\hat{j} + \underbrace{z^2}_{v_3}\,\hat{k}$$

at $(0, 1, -1)$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -y & z^2 \end{vmatrix}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -y & z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(-y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(2x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(2x)\right] = 0$$

② Find the curl of

$\vec{F} = (2x - y^2)\hat{i} + (3z + x^2)\hat{j} + (4y - z^2)\hat{k}$

at $(1, 2, 3)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y^2 & 3z + x^2 & 4y - z^2 \end{vmatrix} = \hat{i}(4 - 3) - \hat{j}(0 - 0) + \hat{k}(2x + 2y)$$

$$\text{Curl } \vec{F} = \hat{i} + \hat{k}(2x+2y)$$

$$\text{Curl } \vec{F}_{(1,2,3)} = \hat{i} + \hat{k}(2+4)$$

$$= \hat{i} + 6\hat{k}$$

Practice

If $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ & $\phi = x^2 + y^2 + z^2$

Compute $\nabla\phi$, $\text{div}\vec{F}$, $\text{Curl}\vec{F}$ & $\nabla^2\phi$