MTH 212 24th May 2024.

Absolute Value Theorem.

For the sequence $\{a_n\}$ if $\lim_{n\to\infty} |a_n| = 0$

then $\lim_{n\to\infty} a_n = 0$

Proof: Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$. Since both of these sequences converges to $0$ and since $-|a_n| \le a_n \le |a_n|$ We can apply the squeeze theorem and conclude that $\{a_n\}$ converges to $0$.

 May 24th 2024.

Def. of Monotonic Sequence: A sequence $\{a_n\}$ is monotonic if its terms are non-decreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or if its terms are non-increasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$$

Monotone Convergence Theorem.

A monotone sequence (either non-increasing or non-decreasing) that is bounded converges.

A Bounded sequence. [Definition.]

~~The terms of a sequence sets~~ A sequence $\{a_n\}$ if there is a positive real number $N$ such that the absolute value of $|a_n| \leq M$ for all $n$

Convergence of Series of a Real Number.

A series of real numbers $\sum_{n=1}^{\infty} a_n$ converges to sum $S$ if the sequence of partial sum $S_N = \sum_{n=1}^{N} a_n$ converges to $S$. This means that for every $\varepsilon > 0$, $\exists\, n \in \mathbb{N} \mid \forall\, m \geq n$, the partial sum $S_m$ satisfies $|S_m - S| < \varepsilon$. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if the limit of the sequence of partial

251 May 24th 2024.

sums exist and is finite. e.g The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$. In this case,

the S of the series is $S = \frac{1}{1-r}$

N.B: If the sequence of partial sum $(S_n)$ converges to S, then we say that the series $\sum_{n=0}^{\infty} a_n$ converges also if the sum $(S_n)$ diverges, then we say that the series also diverges.

Some Important Tests For Series Convergence.

The nth term test.

If $\lim_{n \to \infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges

If $\lim_{n \to \infty} a_n = 0$, the test is inconclusive.

- Direct Comparison Test: Compare $\sum a_n$ with the known series $\sum b_n$: (i) If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If $0 \leq b_n \leq a_n$ and $\sum b_n$ diverges, then $\{a_n\}$ diverges.

- Limit Comparison test: Compare $\sum a_n$ with $\sum b_n$ by considering the limit

252 May

- If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where $c$ is a finite positive number, both series either converges or diverge together

(3) Ratio Test: If $\lim_{n \to \infty} \left| \frac{a_n}{a_n} \right| = L$

If $L < 1$, the series converges
If $L > 1$ the series diverges
If $L = 1$ the test is inconclusive

(4) Root Test: If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$

If $L < 1$, the series converges
If $L > 1$, the series diverges
If $L = 1$, the test is inconclusive

5. Integral Test → let $a_n$ $f(n)$ for a Continous, positive decreasing function $f(x)$. Then the Series $\sum_{n=1}^{\infty} a_n$ converges iff $a_n$ only if the Improper Integral $\int_1^{\infty} f(x)\,dx$ converges

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## ④ Special Series

Geometric Series $\sum_{n=0}^{\infty} ar^n$ converges

if $|r|<1$ and diverges if $|r| \geq 1$

P-Series $\sum_{n=1}^{\infty} \frac{1}{n^P}$ converges if $P>1$ and

diverges if $P \leq 1$

## ⑤ Alternating Series Test (Leibniz's Test)

An alternating Series

$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ (where $a_n \geq 0$) converges

if $a_n$ is monotonically decreasing and

$\lim_{n \to \infty} a_n = 0$

## ⑥ Absolute and Conditional Convergence

A Series $\sum_{n=1}^{\infty} a_n$ is absolutely

Convergent if $\sum_{n=1}^{\infty} |a_n|$ Converges

A series that converges but does not converge absolutely is conditionally convergent

i.e $\sum a_n$ is conditionally convergent

if $\sum a_n$ converges but $\sum |a_n|$ diverges