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MTH 212 - Real Analysis

Example I

Find the ff limit using the above limit rules (laws)

(a) $\lim\limits_{x \to c} (x^3 + 4x^2 - 3)$ (b) $\lim\limits_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim\limits_{x \to 1} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

Solution

(a) $\lim\limits_{x \to c} (x^3) + \lim\limits_{x \to c} (4x^2) - \lim\limits_{x \to c} (3) = c^3 + 4c^2 - 3$

(b)
$$\frac{\lim\limits_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}}{\lim\limits_{x \to c} x^2 + 5} = \frac{\lim\limits_{x \to c} (x^4) + \lim\limits_{x \to c} (x^2) - \lim\limits_{x \to c} (1)}{\lim\limits_{x \to c} (x^2) + \lim\limits_{x \to c} (5)}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

(c)
$$\frac{\lim\limits_{x \to 1} (x^4) - \lim\limits_{x \to 1} (x^2) - \lim\limits_{x \to 1} (1)}{\lim\limits_{x \to 1} (x^2) + \lim\limits_{x \to 1} (5)} = \frac{1^4 + 1^2 - 1}{1^2 + 5}$$

$$= \frac{1}{6}$$

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Class Work

(a) $\lim_{x \to c} (2x+5)$ (b) $\lim_{t \to 6} 8(t-5)(t-7)$

(c) $\lim_{x \to 2} \frac{x+3}{x+6}$

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Solution.

(a) $\lim_{x \to c} (2x) + \lim_{x \to c} (5) = 2c + 5$

(b) $\left[\lim_{t \to 6} (8)\right] \cdot \left[\lim_{t \to 6} (t-5)\right] \cdot \left[\lim_{t \to 6} (t-7)\right] = (8) \cdot \left[\lim_{t \to 6} (t) - \lim_{t \to 6} (5)\right]$

$\cdot \left[\lim_{t \to 6} (t) - \lim_{t \to 6} (7)\right]$

$= 8 \cdot [6-5] \cdot [6-7]$

$= 8 \cdot [1][-1]$

$= -8$

(c) $\lim_{x \to 2} \frac{x+3}{x+6} = \frac{\lim_{x \to 2} (x+3)}{\lim_{x \to 2} (x+6)}$

$= \frac{\lim_{x \to 2} (x) + \lim_{x \to 2} (3)}{\lim_{x \to 2} (x) + \lim_{x \to 2} (6)}$

$= \frac{2+3}{2+6}$

$= \frac{5}{8}$

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Rational Function:

A rational function is a function that can be written as a ratio of two algebraic expressions. If a function is Considered rational and the denominator is not zero (0) the limit can be found by substitution.

Example 2

(a) $\lim_{x \to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}$

$= \frac{0}{6}$

$= 0$

It can be defined more formally as f(x) and g(x) are algebraic expressions and $g(x) \neq 0$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$

(b) Determine the limit

(i) $\lim_{x \to 3} (x^3 - 2x + 6)$ (ii) $\lim_{x \to 2} \left(\frac{x^3 - 3x + 6}{-x^2 + 15}\right)$

Solution

(i) $(3)^3 - 2(3) + 6 = \underline{\underline{27}}$

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(ii) $\lim\limits_{x \to 2} \frac{(2)^3 - 3(2) + 6}{(-2)^2 + 15} = \frac{8}{19}$

Class Work

(a) $\lim\limits_{x \to \infty} \frac{x^3 + 3x + 6}{x^5 + 2x^2 + 9}$ (b) $\lim\limits_{x \to +\infty} \frac{2x^2 - 2x + 3}{x^2 + 4x + 4}$

(c) $\lim\limits_{x \to -5} \frac{x^2 + 3x - 5}{x + 7}$ (d) $\lim\limits_{x \to 2} \frac{x + 3}{x + 6}$

Example 3

Find the limit if it exist

(a) $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$ (b) $\lim\limits_{x \to -3} \frac{x^2 + x - 6}{x + 3}$

(c) $\lim\limits_{x \to 0} |x|$

Solution

If $f(x) = \frac{x^2 - 1}{x - 1}$, then $f(1)$ does not exist

i.e $\lim\limits_{x \to 1} \frac{1^2 - 1}{1 - 1} = \frac{0}{0} \Rightarrow$ Direct substitution Fail

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$$\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = x+1 \Rightarrow \lim_{x\to 1}(1+1) = 2$$

(b) $$\lim_{x\to -3} \frac{x^2+x-6}{x+3} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$$

$$\frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}} \quad \lim_{x\to -3}(x-2) = (-3-2)$$

$$= -5$$

(c) $$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0^+}|x| = 0 = \lim_{x\to 0^-}|x| \Rightarrow \lim_{x\to 0}|x| = 0$$

Class Work

(1) $$\lim_{x\to 5} \frac{2x^2-7x-15}{x-5}$$ (2) $$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

Conjugate Method (Evaluating a Limit by Rationalizing the Numerator)

The conjugate, a binomial expression is the same expression with opposite middle sign

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for example the conjugate of $(\sqrt{x} - 5)$ is $(\sqrt{x} + 5)$. This is really useful if you have a radical in your limit. It is because the product of two conjugates containing radicals will itself contain no radical expression. i.e

$$(\sqrt{x} - 5)(\sqrt{x} + 5) = x - 25$$

Example: Evaluate

$$\lim_{x \to 5} \frac{\sqrt{x+11} - 4}{x - 5}$$

Solution

Direct substitution fails since the denominator becomes 0. $(x-5) \Rightarrow 5-5 \Rightarrow 0$

Rationalizing the numerator

1. $$\lim_{x \to 5} \left[\frac{\sqrt{x+11} - 4}{x - 5} \times \frac{\sqrt{x+11} + 4}{\sqrt{x+11} + 4} \right]$$

$$= \lim_{x \to 5} \left[\frac{(\sqrt{x+11} - 4)(\sqrt{x+11} + 4)}{(x-5)(\sqrt{x+11} + 4)} \right]$$

$$= \lim_{x \to 5} \left[\frac{(\sqrt{x+11})^2 - (4)^2}{(x-5)(\sqrt{x+11} + 4)} \right]$$

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$$= \lim_{x \to 5} \left[\frac{x + 11 - 16}{(x-5)(\sqrt{x+11} + 4)} \right]$$

$$= \lim_{x \to 5} \left[\frac{x - 5}{(x-5)(\sqrt{x+11} + 4)} \right]$$

$$= \lim_{x \to 5} \left[\frac{1}{\sqrt{x+11} + 4} \right]$$

$$= \frac{1}{\sqrt{5+11} + 4}$$

$$= \frac{1}{\sqrt{16} + 4}$$

$$= \frac{1}{4+4}$$

$$= \frac{1}{8}$$

2. $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

Direct subutituting fails since the denominator becomes 0. $x = 0$

Rationalizing the numerator

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$$\lim_{x\to 0}\left[\frac{\sqrt{x+1}-1}{x}\times\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right]$$

$$=\lim_{x\to 0}\left[\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}\right]$$

$$=\lim_{x\to 0}\left[\frac{(\sqrt{x+1})^2-(1)^2}{x(\sqrt{x+1}+1)}\right]$$

$$=\lim_{x\to 0}\left[\frac{x+1-1}{x(\sqrt{x+1}+1)}\right]$$

$$=\lim_{x\to 0}\left[\frac{x}{x(\sqrt{x+1}+1)}\right]$$

$$=\lim_{x\to 0}\left[\frac{1}{\sqrt{x+1}+1}\right]$$

$$=\frac{1}{\sqrt{0+1}+1}$$

$$=\frac{1}{\sqrt{1}+1}$$

$$=\frac{1}{1+1}$$

$$=\frac{1}{2}$$

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Classwork

(a) $\lim_{x \to 2} \frac{x+2}{\sqrt{x+6} - 2}$

(b) $\lim_{x \to -1} \frac{\sqrt{x+10} - 3}{x+1}$