28th May 2024

Continuation from last class

Proof of prop 1.17 continued: Part (b)

'if' part proved!

'only if' part (continued)

... Then, for every subset E of Y, $E = f(f^{-1}(E))$ holds.

Let $\omega \in Y$. Then $\omega = f(t)$ where $t \in f^{-1}(Y)$. Now, $t \in f^{-1}(Y)$ implies $f(t) \in Y$, hence $t \in X$.

$\therefore Y = f(X)$.

Hence f is surjective.

The proof of b is complete.

Note: That the reverse inclusion

$$E \subset f(f^{-1}(E))$$

may not hold in general.

For example: let $f(x) = x^2 \; \forall x \in \mathbb{Z}$, ($\mathbb{Z}$ the set of integers),

then $f^{-1}(\{1,-2\}) = \{x \in \mathbb{Z} : f(x) \in \{1,-2\}\}$

$= \{1,-1\}$

so $f(f^{-1}\{1,-2\}) = f(\{1,-1\})$

$= \{1\}$

and so $f(f^{-1}(\{1,-2\})) \not\supset \{1,-2\}$.

Notice that this function is not surjective,

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CamScanner

28th May 2024

(since $f(x) \geq 0 \; \forall x \in \mathbb{Z}$)

i.e f does not map $\mathbb{Z}$ onto $\mathbb{Z}$

Part c

If $w \in f\left(\bigcup_{\alpha \in I} A_\alpha\right)$ then $w = f(t)$ where $t \in \bigcup_{\alpha \in I} A_\alpha$, so $t \in A_{\alpha_0}$ for some $\alpha_0 \in I$.

Hence $w = f(t)$ belongs to $f(A_{\alpha_0})$.

So, $w \in \bigcup_{\alpha \in I} f(A_\alpha)$, (as $\alpha_0 \in I$)

$\therefore f\left(\bigcup_{\alpha \in I} A_\alpha\right) \subset \bigcup_{\alpha \in I} f(A_\alpha)$

To prove the reverse inclusion, suppose that $w \in \bigcup_{\alpha \in I} f(A_\alpha)$, then $w \in f(A_{\alpha_0})$ for some $\alpha_0 \in I$.

Thus $w = f(t)$, where $t \in A_{\alpha_0}$ since $\alpha_0 \in I$, $t \in \bigcup_{\alpha \in I} A_\alpha$ and hence:

$w = f(t)$ belongs to $f\left(\bigcup_{\alpha \in I} A_\alpha\right)$.

So, $\bigcup_{\alpha \in I} f(A_\alpha) \subset f\left(\bigcup_{\alpha \in I} A_\alpha\right)$.

Hence $\bigcup_{\alpha \in I} f(A_\alpha) = f\left(\bigcup_{\alpha \in I} A_\alpha\right)$. This proves part c

Part d

If $w \in f\left(\bigcap_{\alpha \in I} A_\alpha\right)$ then $w = f(t)$ where $t \in \bigcap_{\alpha \in I} A_\alpha$, so $t \in A_\alpha \; \forall \alpha \in I$, hence $f(t) \in f(A_\alpha) \; \forall \alpha \in I$. So:

$w \in \bigcap_{\alpha \in I} f(A_\alpha)$. $\therefore f\left(\bigcap_{\alpha \in I} A_\alpha\right) \subset \bigcap_{\alpha \in I} f(A_\alpha)$

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28th May 2024

Let us prove the second part of d.

'if' part (when f is injective):

Suppose f is injective. We are required to prove that the reverse inclusion holds.

If $w \in \bigcap_{\alpha \in I} f(A_\alpha)$, then $w \in f(A_\alpha)$ $\forall \alpha \in I$.

So, $w = f(t)$ where $t \in A_\alpha$ $\forall \alpha \in I$

because of a one-to-one so,

$t \in \bigcap_{\alpha \in I} A_\alpha$

Hence $f(t) \in f(\bigcap_{\alpha \in I} A_\alpha)$.

$\therefore w \in f(\bigcap_{\alpha \in I} A_\alpha)$.

So, $\bigcap_{\alpha \in I} f(A_\alpha) \subset f(\bigcap_{\alpha \in I} A_\alpha)$.

'only if' part (Only when f is injective)

Suppose that the reverse ~~in~~ inclusion.

Then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

holds for all subsets A_1, A_2 of X.

If $x \in X$ then for every point u of $\{x\}^c$

$f(u) \in f(\{x\}^c)$.

So, $f(u) \in f(X \setminus \{x\})$

i.e $f(u) \in f(X \cap \{x\}^c)$,

so $f(u) \in f(X) \cap f(\{x\}^c)$.

Hence $f(u) \in f(\{x\}^c)$.

Therefore $f(u) \neq f(x)$

28th May 2024

So f is injective

Note:

With $x \overset{f}{\longmapsto} x^2 - 1$,

($X = Y = \mathbb{Z}$, $\mathbb{Z}$ = the set of all integers),

$I = \{1, 2\}$, $A_1 = \{-1, 1, 6\}$ and $A_2 = \{1, -6, -3, 0\}$,

then $f(A_1 \cap A_2) = f(\{1\}) = \{0\}$

and $f(A_1) = \{0, 35\}$,

$f(A_2) = \{0, 35, 8, -1\}$

so $f(A_1) \cap f(A_2) = \{0, 35\}$

$\therefore f(A_1 \cap A_2) \not\supset f(A_1) \cap f(A_2)$