LINEAR PROGRAMMING PROBLEM (LPP)

A Linear programming problem deals with optimising (maximising or minimising) a function. It constitutes three parts: objective function, decision variable, and constraints.

The functions which need to be optimised are known as Objective functions.

The variables whose values determine the solution of the given problem are called decision variables of the problem.

The set of simultaneous linear equations or inequalities that the problem is subject to are known as constraints.

Rules

1. There must be a well-defined Objective to achieve (maximise or minimise).
2. There is only a finite number of decision variables
3. At least a few of the resources must be in limited supply, which gives rise to constraints.
4. All the elements should be quantifiable. i.e All the decision variables should assume only non-negative values
5. Both the given objective function and constraints must be linear equations or inequalities
6. There must be alternative course of the line of action to choose from

Programming problems

These are a class of problems that determine the optimal allocation of limited resources to meet given objectives

Mathematical Programming

If the objective and constraints of an optimization problem are given as mathematical functions and functional relations, it is called Mathematical programming and is generally given as

Max. or Min. $f = f(x)$

Subject to $g_i(x) \leq$ or $\geq b_i$ —— (*)

where $\underline{x} = (x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, m$.

If $g_i = 0$ and $b_i = 0$ $\forall i = 1, 2, \ldots, m$, then (*) is called an unconstrained mathematical programming problem

LPP

Here the objective functions $f(x)$ and constraint $g_i(x)$, $i = 1, 2, \ldots, m$ are all linear functions. Hence a LPP is defined as

Max. (or min) $r = C_1x_1 + C_2x_2 + C_3x_3 + \ldots + C_nx_n$

Subject to : $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \leq = \geq b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \leq = \geq b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n \leq = \geq b_3$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \leq = \geq b_m$

$x_1, x_2, x_3, \ldots, x_n \geq 0$

where a_{ij}, b_i, c_j $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ are constants.

In Matrix form we have

$$\text{Max. (or min)} \; r = C^T \underline{x}$$

$$\text{Subject to: } A\underline{x} \, (\leq, =, \geq) \, \underline{b}$$

where

$\underline{C}$ is a column vector, $\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$, $\underline{x}$ is a column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\underline{b}$ is a column vector $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

A is an $m \times n$ matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & a_{m2} \cdots & a_{mn} \end{pmatrix}$

Another form of presenting a LPP is

$$\text{Max. or Min. } f = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to } \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i \leq, \geq b_i$$

$$x_j \geq 0, \; j = 1, 2, \ldots, n$$

Note: only one of the signs $(\leq, =, \geq)$ is applicable to each of the constraint.

Transformation of Real Life Problems into LPP.

To formulate a LP model for a real life problem, we adopt the following four steps:

(i) Define the input variables say x_j, $j = 1, 2, \ldots, n$

(ii) Determine the quantity to be maximized or minimized (optimized) and express it as a linear mathematical function. This constitutes the Objective function

(iii) Identify all stated requirements restrictions and limitations and express them as linear mathematical functions. These constitute the constraints.

(iv) Identify all other hidden conditions that are clearly stated in the real-life problem but are obvious from the real life situation that is being modeled. Express these also in mathematical functions. this will also be a part of the constraints.

Assumptions of the LPP

LPP are based on four (4) main assumptions as follows

1. Proportionality:

Individual activities are considered independent of the others.

2. Additivity

This assumes that there are interactions between any of the activities.

3. Divisibility

This assumes that activity units can be divided into any fractional levels, so that non-integer values for the decision variables are allowed.

4. Certainty:

This assumes that all the parameters of the model (a_{ij}, b, c values) are known constants.

Application

Example 1

ABC paints produces two types of paint using two basic raw materials X and Y. The maximum quantities of raw materials X and Y available per week are 10 tonnes and 8 tonnes respectively. This is shown below

Raw material	Paint type one	Paint type two
X	2	1
Y	1	2

It has been observed that the weekly demand for the first type of paint cannot exceed that of the second type by more than 2 tonnes. Also, the maximum demand for paint type one cannot exceed 2 tonnes per week. The profit per tonne of paint type 1 and paint 2 are ₦200 and ₦300 respectively. How many tonnes of paint type 1 and 2 would the company produce to maximize gross profit? Determine the gross profit.

Solution

Let the number of tonnes of paint type 1 to be produced be x_1 and that of paint type 2 be x_2. The Objective function is the gross profit and is denoted by P. Hence

$$P = 200x_1 + 300x_2$$

The constraints are

$$2x_1 + x_2 \leq 10$$
$$x_1 + 2x_2 \leq 8$$
$$x_1 - x_2 \leq 2$$
$$x_1 \leq 2$$

Hence, the problem is

$$\text{Max. } P = 200x_1 + 300x_2$$

Subject to
$$2x_1 + x_2 \leq 10$$
$$x_1 + 2x_2 \leq 8$$
$$x_1 - x_2 \leq 2$$
$$x_1 \leq 2$$
$$x_1, x_2 \geq 0.$$

Example 2.

A factory manufactures two types of products S and T and sells them at a profit of \$2 on type S and \$3 on type T. Each product is processed on two machines M_1 and M_2. Type S requires 1 minute of processing time on M_1 and 2 minutes on M_2. Type T requires 1 minute M_2. Machine M_1 is available for not more than 6 hours 40 minutes while Machine M_2 is available for 10 hours during any working day. Formulate the problem as an LPP so as to maximise the profit.

Soln

Let the factory decide to produce x_1 units of produce S and x_2 units of product T to maximise its profit.

To produce these units of type S and type T products it requires $x_1 + x_2$ processing minutes on M_1 and $2x_1 + x_2$ processing minutes on M_2. Since machine M_1 is available for maximum 6 hours 40 minutes and M_2 is available for maximum 10 hours doing any working day, the constraints are

$$x_1 + x_2 \leq 400$$

$$2x_1 + x_2 \leq 600$$

and

$$x_1, x_2 \geq 0$$

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Since the profit from type S is \$2 and the profit from type T is \$3 the total profit is $2x_1 + 3x_2$. As the objective is to maximise the profit the objective function is to maximise $Z = 2x_1 + 3x_2$.

The complete formulation of The LPP is
Maximize $Z = 2x_1 + 3x_2$ subject to the constraint

$$x_1 + x_2 \leq 400$$

$$2x_1 + x_2 \leq 600$$

and

$$x_1, x_2 \geq 0.$$

Graphical Method

Definition of Terms

1. Feasible solution

All the solutions of a LPP that satisfy all the constraints

2. Feasible Region

This is the set of all feasible solutions

3. Optimal solution

This is the 'best' feasible solution to the LPP. it is the solution that satisfies both the objective functions and the constraints

4. Optimal value

This is the value of the objective function corresponding to an optimal solution. This is obtained by substituting the optimal solution

5. Unique optimal solution

If there is only one feasible solution that satisfies both the objective function and constraints (that is an optimal solution) such an optimal solution is said to be a unique optimal solution

6. Multiple optimal solution

a. If there are more than one optimal solution to an LPP, the LPP is said to have multiple or Alternative optimal solution.

The graphical method is used only when the LPP involves the variables x_1 and x_2 or x and y. To solve a LPP using the graph, we note that a straight line is the shortest distance between two points.. Hence, we need two points to draw the line representing a constraint.

Steps.

1. Consider all the constraints to be equations
2. For each constraint find the value of x_1 (or x) when x_2 (or y) $= 0$ and values of x_2 (or y) when x_1 (or x) $= 0$.
3. Rule your graph sheet to indicate the x and y axes.
4. Mark in the points obtained in 2 above and join them with a ruler
5. Indicate the direction of each of the lines drawn by considering whether the constraints are $\leq$ or $\geq$. Hence determine the feasible region
6. Find the feasible solution from the feasible region. This is located at the intersection of two lines
7. Determine the feasible solution that satisfies all the constraints and gives the maximum or minimum value of

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the objective function (depending on the problems you are solving). This will be the optimal solution.

8. Obtain the optimum value by substituting your result of step 6 in the objective function