20

31st May 2024

Elementary Algebra and Analysis.

Prop 1.17 Continued: Part (e): If $t \in f^{-1}(\bigcup_{\alpha \in J} E_\alpha)$ then, by definition, $f(t) \in \bigcup_{\alpha \in J} E_\alpha$, so $f(t)$ belongs to at least one E_α, $(\alpha \in J)$ say E_{α_0}. Thus, $t \in f^{-1}(E_{\alpha_0})$ and hence $t \in \bigcup_{\alpha \in J} f^{-1}(E_\alpha)$. $\therefore f^{-1}(\bigcup_{\alpha \in J} E_\alpha)$ is a subset of $\bigcup_{\alpha \in J} f^{-1}(E_\alpha)$. Conversely, if $t \in \bigcup_{\alpha \in J} f^{-1}(E_\alpha)$ then $t \in f^{-1}(E_{\alpha_0})$ for some α_0 in the index set J. Hence $f(t) \in E_{\alpha_0}$ and so $f(t) \in \bigcup_{\alpha \in J} E_\alpha$. Thus $t \in f^{-1}(\bigcup_{\alpha \in J} E_\alpha)$. Hence $\bigcup_{\alpha \in J} f^{-1}(E_\alpha)$ is also a subset of $f^{-1}(\bigcup_{\alpha \in J} E_\alpha)$. So, we have: $f^{-1}(\bigcup_{\alpha \in J} E_\alpha) = \bigcup_{\alpha \in J} f^{-1}(E_\alpha)$. Finally, let us prove that $f^{-1}(\bigcap_{\alpha \in J} E_\alpha) = \bigcap_{\alpha \in J} f^{-1}(E_\alpha)$. Now, if $t \in f^{-1}(\bigcap_{\alpha \in J} E_\alpha)$ then $f(t)$ lies in $\bigcap_{\alpha \in J} E_\alpha$ and so $f(t) \in E_\alpha \ \forall \alpha \in J$. Hence $t \in f^{-1}(E_\alpha) \ \forall \alpha \in J$. So, $t \in \bigcap_{\alpha \in J} f^{-1}(E_\alpha)$. Thus, the inclusion $f^{-1}(\bigcap_{\alpha \in J} E_\alpha) \subset \bigcap_{\alpha \in J} f^{-1}(E_\alpha)$ holds. Conversely, if $t \in \bigcap_{\alpha \in J} f^{-1}(E_\alpha)$ then $t \in f^{-1}(E_\alpha) \ \forall \alpha \in J$. Hence $f(t) \in E_\alpha \ \forall \alpha \in J$. So, $f(t) \in \bigcap_{\alpha \in J} E_\alpha$. Hence $t \in f^{-1}(\bigcap_{\alpha \in J} E_\alpha)$. So, the reverse inclusion $\bigcap_{\alpha \in J} f^{-1}(E_\alpha) \subset f^{-1}(\bigcap_{\alpha \in J} E_\alpha)$ holds. $\therefore f^{-1}(\bigcap_{\alpha \in J} E_\alpha) = \bigcap_{\alpha \in J} f^{-1}(E_\alpha)$ holds. The proof of (e) is complete.

Exercises: 1. Let f be given by the set $A = \{(-1,0), (2,1), (1,1), (3,4), (-4,2)\}$. Is f a function from $X = \{-1, 2, 1, 3, -4\}$ into $Y = \{0, 1, 4, 2, 3\}$? If it is a function, is it (i) injective? (ii) surjective? (iii) Does it have an inverse? Find (a) $f(E)$ where $E = \{2, 1, 3\}$ (b) $f^{-1}(W)$ where $W = \{1, 3\}$, and (c) $f^{-1}(V)$ where $V = \{0, 4, 2\}$. 2. Does the set $A = \{(2,1), (1,1), (3,4), (-4,0)\}$ define a function f from $X = \{-1, 2, 1, 3, -4\}$ into $Y = \{0, 1, 4, 2, 3\}$? Give reasons for your answer. 3. Prove that $f(x) = x^3$, $(x \in \mathbb{R})$ from the set $\mathbb{R}$ of all real numbers into $\mathbb{R}$ is bijective with the inverse

$$f^{-1}(y) = \begin{cases} -|y|^{1/3} & \text{if } y < 0, \\ y^{1/3} & \text{if } y \geq 0 \end{cases}$$

Definition 1.18: Let X and Y be non-empty sets. We define the projection map p of $X \times Y$ on X to be the function whose value at each point (a,b) of $X \times Y$ is a. Similarly, the projection map q of $X \times Y$ on Y is the function whose value at each point (a,b) of $X \times Y$ is b.

Proposition 1.19: An injective mapping f from a finite set X into itself is also surjective.

Proof: If $a \in X$, we must find $b \in X$ such that 1.19.1: $a = f(b)$. Consider the effect of performing f repeatedly. Let us write f^2 for $f \circ f$ and generally abbreviate $f \circ f \circ \ldots \circ f$, that is f^n is n copies of f. In the sequence: $a, f'(a), f^2(a), \ldots f^j(a), \ldots (j \geq 0)$ of elements of X, there must be repetitions of f is injective and X is finite, so assume that 1.19.2: $f^r(a) = f^m(a)$ (say). Since f is injective for x, y in X, $f(y) = f(x)$ implies $y = x$, so we may cancel f in 1.19.2. If we do this m times, we get 1.19.3: $f^{r-m}(a) = a$. Thus 1.19.1 holds with b as $f^{r-m-1}(a)$. This proves Prop 1.19

Definition 2.1: By a relation on a non-empty set X, we mean a correspondence of X with itself; that is, a function $\mathcal{P}$ from X into X. We say that $\mathcal{P}$ is (i) reflexive if every $x \in X$, $(x,x) \in \mathcal{P}$, that is x is related to x by $\mathcal{P}$ (written also as $x\mathcal{P}x$), (ii) symmetric if for all points u, v of X, $u\mathcal{P}v$ implies $v\mathcal{P}u$ (i.e. $(u,v) \in \mathcal{P}$ implies $(v,u) \in \mathcal{P}$); (iii) transitive if for all points u, v, w of X $u\mathcal{P}v$ and $v\mathcal{P}w$ imply $u\mathcal{P}w$.

Note. $(s,t) \in \mathcal{P}$ means s is related to t under the relation $\mathcal{P}$. In symbols, $s\mathcal{P}t$.

Definition 2.2: By an equivalence relation on a given non-empty set X, we mean a relation $\mathcal{P}$ on X which is reflexive, symmetric and transitive. Theorem 2.3: Every equivalence relation $\mathcal{P}$ on a non-empty set A (note: separate the points of A into distinct (i.e. disjoint) classes (called equivalence classes) A_α's, and every partition (or division) (A_x) of A into disjoint subsets gives rise to an equivalence relation.

21 Proof: First Part. For any $x \in A$, group together all the elements equivalent to x into an equivalence class A_x, that is, we put 2.3.1: $A_x = \{y \in A : x\rho y\}$. Since ρ is reflexive, $x \in A_x$. We claim that any two sets A_x, A_y either are disjoint or coincide. Suppose that A_x and A_y are not disjoint: we must prove that $A_x = A_y$. Since $A_x \cap A_y$ is not empty, $\exists\, w \in A_x \cap A_y$, hence $x\rho w$ and ~~$y\rho w$~~. Thus $w\rho y$, (as ρ is symmetric). Hence $x\rho y$ because ρ is transitive. Now, if $t \in A_y$ then $y\rho t$, hence $x\rho t$, (as ρ is transitive). Thus $t \in A_x$. $\therefore A_y \subset A_x$. Interchanging the roles played by A_x and A_y, we have the reverse inclusion: $A_x \subset A_y$. So, $A_x = A_y$. Thus the different blocks A_x's provide a division of A into non-empty subsets, any two of which are disjoint.

Second Part: Let (A_x) be a partition of A into disjoint subsets. Now, each point x of A lies in just one A_x, since A_x's are disjoint. Define ρ by $x\rho y$ if, and only if, x and y lie in the same A_x. Then $x\rho x$ for each $x \in A$, since every $x \in A$ belongs to only one A_x. Now, if x, y are in A and $x\rho y$ then by the definition of ρ, y and x belong to A_{x_0} for some x_0, hence $y\rho x$. Finally, if x, y, z are in A, and $x\rho y$ and $y\rho z$ ~~[illegible]~~ then x and y are in the same set, say ~~A_{x^*} and both y and z~~ are in the same set, say $A_{x^{**}}$. Since $y \in A_{x^{**}}$, we must have that $A_{x^*} = A_{x^{**}}$. So, x and z lie in the same set, hence $x\rho z$. $\therefore \rho$ is an equivalence relation.

Def 2.4: 1. A relation ρ between the elements of a non-empty set A is said to be an ordering of A (or, an order in A) if it is reflexive, transitive, and satisfies the antisymmetric law 2.4.1: for all points x, y of A, $x\rho y$ and $y\rho x$ imply $x = y$. 2. A set equipped with an ordering is called an ordered set. 3. An ordering ρ of a given non-empty set A is said to be (or, is called) a linear ordering if 2.4.2: for all points x, y of A either $x\rho y$ or $y\rho x$ holds. 4. A set equipped with a relation ρ that is reflexive and transitive is called a preordered set.

Exercises: Which of the following relations is (i) transitive, (ii) an equivalence relation (iii) a preordering (iv) an ordering, (v) a linear ordering? (a) On $\mathbb{Z}$ (the integers), $x\rho y$ if, and only if, $x - y = 2q$ for some $q \in \mathbb{Z}$, (b) On $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 1\}$, $x\rho y$ if and only if $x = my$ for some $m \in \mathbb{Z}^+$, (c) On $\mathbb{R}$ (the reals), $x\rho y$ if and only if $x \leq y$, (d) On $\mathbb{R}$, $x\rho y$ if and only if $x < y$, (e) On $\mathbb{Z}^X$ (the collection of all subsets of X), X a given non-empty set, $u\rho v$ if and only if u is a subset of v, (f) On $\mathbb{Z}$, $x\rho y$ if and only if $x = y$ (g) On $\mathbb{R}$, $x\rho y$ if and only if $x = y + z$ for some $z \in \mathbb{R}$.