24th May 2024. MTH 201(210)

Elementary Algebra and Analysis

Proof (of prop 1.15): 'if' part Suppose f is bijective. We are required to prove that there is a function g from Y into X such that $f(g(t)) = t$ for every point t of Y and that for every point x of X, $g(f(x)) = x$, that is, that the inverse of f exists (i.e. $g = f^{-1}$ exists).

Define h from Y into X as follows: $\forall x \in X$, let $h(f(x)) = x$. This is a function because for every $y \in Y$, $\exists u \in X$ such that $y = f(u)$ because f is surjective and since f is injective, u is unique. Then $f(h(y)) = y$ $\forall y \in Y$ because $f(h(y)) = f(h(f(u))) = f(u) = y$.

So, f has an inverse.

'Only if' part suppose that f has an inverse; that is, $\exists$ g a function from Y into X such that $\forall y \in Y$, $f(g(y)) = y$ and $\forall x \in X$, $g(f(x)) = x$. We are required to prove that f is bijective. Now, if x', x'' are in X with $f(x') = f(x'')$ then $g(f(x')) = g(f(x''))$, hence $x' = x''$, so f is injective. Also, if $y \in Y$ then $f(g(y)) = y$, and $g(y) \in X$ because g is a function from Y into X... f is also surjective.

The proof of prop 1.15 is complete

Note. if f^{-1} exists, it is also bijective because $f^{-1}(y') = f^{-1}(y'')$, (y', y'' in Y) implies $f(f^{-1}(y')) = f(f^{-1}(y''))$ $\therefore y' = y''$, so f^{-1} is injective. Also, if $x \in X$ then $f^{-1}(f(x)) = x$, and since $f(x)$ is in Y, f^{-1} is surjective.

Remark: $(f^{-1})^{-1} = f$ when f^{-1} exists because if f is a function from X into Y, (X, Y non-empty sets) then when f^{-1} exists, f^{-1} is a function from Y into X satisfying the conditions: $f^{-1}(f(x)) = x$ $\forall x \in X$ and $f(f^{-1}(y)) = y$ $\forall y \in Y$. So, with f as f^{-1} and g as f in our definition 1.13 of the inverse of a function we have that $(f^{-1})^{-1} = f$.

Definition 1.16:

1. If f is a function from X into Y and g is a function from Y into Z, (X, Y, Z non-empty sets), the composition of g, denoted $g \circ f$, is defined as a function h from X into

Z given by 1.16.1 : $\forall x \in X, h(x) = g(f(x))$

Note: $g\circ f \neq f\circ g$ in general. Example 1: Let $x \xmapsto{f} x - \left(\frac{x^2}{2}\right)$, and $t \xmapsto{g} 3t$, (x, t real numbers).

Then: $f(1) = 1 - \left(\frac{1^2}{2}\right) = \frac{1}{2}$

$g(f(1)) = g\left(\frac{1}{2}\right) = \frac{3}{2}$, and

$g(1) = 3$,

So $f(g(1)) = f(3) = 3 - \left(\frac{(3)^2}{2}\right) = 3 - \frac{9}{2} = -1\frac{1}{2}$.

So, $(g\circ f)(1) \neq (f\circ g)(1)$.

So $g\circ f \neq f\circ g$ in this example.

2. For any non-empty A, we shall denote by i_A the identity function on A defined by 1.16.2:

$i_A(x) = x \ \forall x \in A$

It is a function from A into A given by 1.16.2

Note: (i) If f is a function from X into Y, (X, Y non-empty set), then

$f\circ i_X = f$ and $i_Y \circ f = f$.

(ii) If f is bijective then $f^{-1}\circ f = i_X$ and $f\circ f^{-1} = i_Y$

(iii) If f and g are bijective where f is a function from X into Y and g is a function from Y into Z, (X, Y, Z non-empty sets) then $g\circ f$ is bijective, and $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.

3. Suppose X_1 contains X_2 and that f is a function

from X_1 into Y, and g is a function from X_2 into Y, (X_1, X_2, Y non-empty sets). If $\forall x \in X_2$ $f(x) = g(x)$ [1.16.3] i.e $f = g$ on X_2, then we say that g is a restriction of to X_2 and that f is an extension of g to X_1. So, the restriction of f to X_2 is defined by 1.16.3.

We shall denote it by $g = f|_{X_2}$

4. If f_1, f_2 are real-valued (or complex-valued) functions defined on a set X, then the sum $f_1 + f_2$ and the the product $f_1 f_2$ are defined by 1.16.4: $\forall x \in X$, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and $\forall x \in X$, $f_1 f_2(x) = f_1(x) f_2(x)$. They are also real-valued (or complex-valued) functions with domain X.

Proposition 1.17

Let f be a function from X into Y, (X, Y non-empty sets).

(a) For every subset A of X,

1.17.1: $A \subset f^{-1}(f(A))$ holds and the reverse inclusion holds if and only if f is injective.

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(b) For every subset E of Y.

1.17.2: $f(f^{-1}(E)) \subset E$ holds and the reverse inclusion holds if and only if f is surjective.

(c) For any collection $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ of subsets A_α's of a non-empty set X indexed by a non-empty set I

1.17.3: $f\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} f(A_\alpha)$ holds.

(d) For any collection $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ of subsets A_α's of a nonempty set X indexed by a non-empty set I, the inclusion 1.17.4

1.17.4: $f\left(\bigcap_{\alpha \in I} A_\alpha\right) \subset \bigcap_{\alpha \in I} f(A_\alpha)$ holds, and

the reverse inclusion holds when and only when f is injective.

(e) For any subsets E_α's of the non-empty set T, indexed by a non-empty set J,

1.17.5 : $f^{-1}\left(\bigcup_{\alpha \in J} E_\alpha\right) = \bigcup_{\alpha \in J} f^{-1}(E_\alpha)$, and

1.17.6: $f^{-1}\left(\bigcap_{\alpha \in J} E_\alpha\right) = \bigcap_{\alpha \in J} f^{-1}(E_\alpha)$ hold.

Proof: Part (a)

If $t \in A$ then $f(t) \in f(A)$, hence $t \in f^{-1}(f(A))$, as $f^{-1}(f(A)) = \{w : f(w) \in f(A)\}$.

'if' part

Suppose f is injective. We are required to prove that $f^{-1}(f(A)) \subset A$. If $t \in f^{-1}(f(A))$ then $f(t) \in f(A)$, hence:

$f(t) = f(u)$, for some point u of A, so $t = u$ (as f is injective).

Hence $t \in A$. So, $f^{-1}(f(A)) \subset A$.

'only if' part.

Suppose that $f^{-1}(f(A)) \subset A$ holds for every subset A of X. Suppose $t \in X$. Then for each point s of $X \setminus \{t\}$, $f(s) \notin f(\{t\})$, for if $s \in f^{-1}(f(\{t\}))$ then $s \in \{t\}$ a contradiction.

So, for all points s, t of X with $s \neq t$ we have that $f(s) \neq f(t)$,

$\therefore f$ is injective.

Note: If $x \xrightarrow{f} x^2$ $\forall x \in \mathbb{Z}$, ($\mathbb{Z}$ the set of all integers) i.e f is a function from $\mathbb{Z}$ into $\mathbb{Z}$ given by $f(x) = x^2$ $\forall x \in \mathbb{Z}$.

Then, with A as $\{1\}$, we have:

$$f^{-1}(f(A)) = f^{-1}(f(\{1\})).$$
$$= f^{-1}(\{1\})$$

$= \{x : f(x) \in \{1\}\}$

$= \{x : f(x) = 1\}$

$= \{x \in \mathbb{Z} : x^2 = 1\}$

$= \{1, -1\}$.

So, $f^{-1}(f(A)) = \{-1, 1\}$ is not a subset of A, (as $A = \{1\}$).

Proof: Part (b)

If w is in $f(f^{-1}(E))$ then by definition, $w = f(t)$ where $t \in f^{-1}(E)$. Hence $f(t) \in E$, so w

$\therefore$ $f(f^{-1}(E)) \subset E$ holds

'if' part

Suppose f is surjective. If $w \in E$, $(E \subset Y)$ then $w = f(t)$ for some $t \in X$, hence $t \in f^{-1}(E)$ $\therefore$ $w \in f(f^{-1}(E))$ $\therefore$ $E \subset f(f^{-1}(E))$ holds in this case.

'only if' part.

Suppose $E \subset f(f^{-1}(E))$ holds for every subset E of Y. Then $E = f(f^{-1}(E))$ holds.