Elementary Algebra And Analysis

11th June 2024, Tuesday.

For example, On $\mathbb{Z} = \{0, \pm 1, \pm 2 \ldots\}$ define $a * b = a+b$ for all integers a and b. Then $\mathbb{Z}$ is a group w.r.t addition '+' of integers. But $\mathbb{Z}$ is not a group w.r.t multiplication '·' of integers, since for instance $2 \in \mathbb{Z}$

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but z has no inverse, so the inverse law does not hold in this case.

Note: The identity law hold w.r.t multiplication, here $e_{\mathbb{Z}}$ is the integer 1.

(vi) A Monoid is defined as a semigroup with an identity element. that is, it is a non-empty set that is equipped with an associative binary operation '*' satisfying the identity law.

e.g $\mathbb{Z}$ is a monoid with respect to multiplication '.' .

~~The set~~ $\mathbb{Z}^+ = \mathbb{N}$ of all integers $n \geq 1$ (also called the set of all natural numbers) is a semigroup w.r.t addition '+' but it is not a monoid under '+', since it has no additive identity element, as $0 \notin \mathbb{N}$ but $\mathbb{N}$ is a monoid w.r.t multiplication with multiplicative identity element, the integer 1.

(vii) A subset A of non-empty set X which is equipped with a binary

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operation '*' is said to be closed with respect to '*' if $\forall u, v$ in A, $u * v$ belongs to A

Notation: A set A, which is equipped with a binary operation '*' ~~is said to be closed with respect to '*' if $\forall u, v$ in A~~ will be written as the couple $(A, *)$.

(viii) A subgroup of a group $(G, *)$ is defined as a subset A of G which is a group with respect '*'. That is, A is closed w.r.t '*' and contains the identity element e_G and with each element $x \in A$, the inverse element (i.e the element described in the inverse law), denoted additively by $-x$ and multiplicatively by x^{-1}, belongs to A.

Proposition 2.15

(a) There is only one element e satisfying the identity law in a monoid (or a group) called the identity element.

(b) There is only one element b for each element a of a group $(G, *)$ such that

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$a*b = b*a = e$ (called the inverse element of a), denoted additively by $-a$ and multiplicatively by a^{-1}.

Proof

Part(a):

Suppose e' also satisfies the identity law. Then $e' = e*e' = e'*e$ and also $e*e' = e'*e$ $\therefore e' = e$

Part(b):

Let $a \in G$. If $\exists\, b', b''$ of G such that $a*b' = b'*a = e$ and $a*b'' = b''*a = e$, then $b' = b'*e$
$= b'*(a*b'') = (b'*a)*b'' =$ (as '*' is associative)
$e*b'' = b''$.

Definition 2.16:

(i) A ring is defined as a non-empty set R equipped with two binary operations denoted by the symbols '+' and '·' such that R is a group with respect to '+' and R is commutative w.r.t '+', and such that $(R, \cdot)$ is a semigroup for which both the right and left distributive laws.

2.16.1:

$\forall\, a, b, c$ in R

$a\cdot(b+c) = (a\cdot c) + (b\cdot c)$ holds (left distributive law).

Notation: It is denoted by the triple $(R, +, \cdot)$.