Theorm 7 (Functions that Agree at all but one point)

Let c be a real number and $f(x) = g(x)$ for $x \neq c$ in an open interval containing c. If the limit of $g(x)$ exist as $x \to c$, then the limit of $f(x)$ also exist and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

PROOF: Assume that the limit of $g(x)$ as $x \to c$ is L. Then by definition for each $\varepsilon > 0$ there exist a $\delta > 0$ such that $|g(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

However, since $f(x) = g(x)$ for all x in the open interval other than $x = c$, it follows that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

Thus, we conclude that limit of $f(x)$ as $x \to c$ is also L.

Example → show that the function $f(x) = \frac{x^2 - 1}{x - 1}$ and $g(x) = x^2 + x + 1$ have the same values for all x other than $x = 1$

Soln

By Factorizing the numerator of F, we have

$$f(x) = \frac{x^3 - 1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$$

Thus if $x \neq 1$, we can cancel like factors to obtain

$$f(x) = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = x^2+x+1 = g(x), x \neq 1$$

THEOREM 2: Same Base Units

If b and c are real number and n is an integer then the ff are true.

① $\lim_{x \to c} b = b$ ② $\lim_{x \to c} x = c$ ③ $\lim_{x \to c} x^n = c^n$

PROOF: For 2 we need to show that for each $\varepsilon > 0$ there exist a $\delta > 0$ such that

$|x - c| < \varepsilon$ whenever $0 < |x-c| < \delta$

Since the right-handed inequality is the stricter version of the left handed one we simply choose $\delta = \varepsilon$

④ $\lim_{x \to 2} 3 = 3$ ⑤ $\lim_{x \to 2} x^2 = 2^2 = 4$

Example applies to theorm 2

~~Example: Find $\lim_{x\to 2} (4x^2+3)$~~

Solution

THEOREM 3: limit of a Polynomial Function

If p is a Polynomial Function and C is a real number then

$$\lim_{x\to c} P(x) = P(c)$$

PROOF: let the polynomial Function P be given by

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

By repeated application of the Sum and Scalar multiple

$$\lim_{x\to c} P(x) = a_n\left[\lim_{x\to c} x^n\right] + \cdots a_1\left[\lim_{x\to c} x\right] + \lim_{x\to 0} a_0$$

Now, using 1, 2, 3 of theorm of theorm 2 we have

$$\lim_{x\to c} P(x) = a_n c^n + \cdots a_1 c + a_0 = p(c)$$

Example: Find the limit of $(4x^2+3)$ $\lim_{x\to 2}$

$P(x) = 4x^2+3$ to simplify the value of P at $x=2$

$$\lim_{x\to 2} P(x) = p(2) = 4(2^2)+3$$
$$= 19.$$

THEORM 4: Limit of a Rational Function

If r is a rational function given by $r(x) = \frac{P(x)}{q(x)}$

and c is a real number such that $q(x) \neq 0$

then $\lim_{x\to c} r(x) = r(c) = \frac{P(c)}{q(c)}$

PROOF: By Theorm 3 we know that the polynomial function P and q we have

$$\lim_{x\to c} p(x) = p(c) \text{ and } \lim_{x\to c} q(x) = q(x)$$

Now, since $q(c) \neq 0$ we can apply limit law (c) (quotient) to conclude that

$$\lim_{x\to c} r(x) = \lim_{x\to c} \frac{P(x)}{q(x)} = \frac{\lim_{x\to c}(P(x))}{\lim_{x\to c}(q(x))} = \frac{P(c)}{q(c)} = r(c)$$

Note that the polynomial, rational and radical functions are three types of algebraic function.

E.g

Find the ff limit

$$\lim_{x \to 1} \frac{x^2+x+2}{x+1}$$

Soln

Since the denominator is not zero when $x=1$ we can apply theorem 4 to obtain

$$\lim_{x \to 1} \frac{x^2+x+2}{x+1} = \frac{1^2+1+2}{1+1} = \frac{4}{2} = 2$$

THEORM 5: Limit Involving Radicals

If c is a real number, n and m are positive integers and if f is a function whose limit exist at c, then the following properties are true

(i) $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ (ii) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$

(iii) $\lim_{x \to c} x^{m/n} = c^{m/n}$ (iv) $\lim_{x \to c} [(f(x))]^{m/n} = \left[\lim_{x \to c} f(x)\right]^{m/n}$

Example: The limit of a radical Function

Find the function of the limit

$$\lim_{x \to 3} \sqrt{2x^2 - 2}$$

Soln

$$\lim_{x \to 3} (2x^2 - 2) = 2(3^2) - 2 = 16 > 0$$

NO 2 of theorm 5

$$\lim_{x \to 3} \sqrt{2x^2 - 2} = \sqrt{\lim_{x \to 3} (2x^2 - 2)} = \sqrt{16} = 4$$

THEORM 6: Limit of a Trigonometric Function

If c is a real number then the following are true

(i) $\lim_{x \to c} \sin x = \sin c$ (i) $\lim_{x \to c} \cos x = \cos c$

(iii) $\lim_{x \to c} \tan x = \tan c$ (iv) $\lim_{x \to c} \cot x = \cot c$

(v) $\lim_{x \to c} \sec x = \sec c$ (v) $\lim_{x \to c} \cos x = \cos c$

Example

(1) By theorm 6 we have

$$\lim_{x \to 0} \sin x = \sin 0 = 0$$

② By theorm 6 and unit law 6, we have

$$\lim_{x \to \pi} (x \cos x) = \left[\lim_{x \to \pi} x\right]\left[\lim_{x \to \pi} \cos x\right]$$

$$= \pi \cos(\pi) = -\pi$$