Stat 50, Short Answers to MT#2, Spring 2025

1. The bus travel times of a specific route in non-peak hours at a city is approximately normal with mean 27.0 minutes and standard deviation 4.5 minutes. There are 9 scheduled bus trips during non-peak hours in that route. The travel times are approximately independent, each with $N(27, 4.5^2)$ distribution approximately. Let $\bar{X}$ denote the average daily travel time of these 9 bus trips.

   (a) (2.5 pts) Determine the distribution of $\bar{X}$ and then compute $P(\bar{X} < 24)$ by also showing your work and the area under a suitable curve.

   Answer. As a linear combination of independent normally distributed random variables, $\bar{X} \sim N(27, 4.5^2/9)$ with standard deviation $\sigma_{\bar{X}} = 1.5$ min. Moreover, the z-score of $\bar{x} = 24$ is $z = \frac{24-27}{1.5} = -2$. So, if $Z \sim N(0,1)$, then $P(\bar{X} < 24) = P(Z < 2) = 0.0228$.

   (b) (1.5 pt) Determine the 98th percentile value of $\bar{X}$ with one-decimal place of rounding.

   Answer. From the z-table, we get $z = 2.05$ as an approximate 98th percentile z-score (a more precise value is 2.054). Then, $\bar{x} = 27 + (2.05)(1.5) = 30.075 \simeq 30.1$ min.

2. (2.5 pts) Let X be normally distributed with mean 1.1 and standard deviation 0.2. Moreover, let $Y = e^X$. Determine the median and the 90th percentile value of Y.

   Answer. The median of Y is $e^{\mu} = e^{1.1} = 3.004$. The 90th percentile z-score is roughly 1.28 or 1.282, as we applied several times in class. Then, $x = 1.1 + (1.282)(0.2) = 1.3564$ and $y = e^{1.3564} \simeq 3.882$.

3. The lifetimes of a certain component in a machine has an approximate exponential distribution with a mean of 0.8 years. When this component fails, it is immediately replaced with another such component with the same exponential lifetime distribution. Assume that the actual lifetimes of the components are independent of each other.

   (a) (2 pts) Suppose that a company has three extra components in addition to the one which is newly installed in the machine. Let W denote the total lifetime of these four components. Compute the mean and standard deviation of W, using two decimal places of rounding.

   Answer. Lifetime of each component has $Exp(1.25)$ distribution since $\lambda = 1/0.8 = 1.25$, and $W \sim \Gamma(4, 1.25)$ as a sum of four i.i.d. Exp(1.25) distributions. Then, $E[W] = 4(0.8) = 3.2$ years and $\sigma_W = \frac{\sqrt{4}}{1.25} = 1.6$ years.

   (b) (2 pts) Compute the probability that exactly three components will fail one after another within two years. Show your work.

   Answer. We can apply Poisson process with $\lambda = 1.25$ and $t = 2$. So, use $X \sim Poisson(2.5)$ to compute $P(X = 3)$: $e^{-2.5}(2.5)^3/3! = 0.21376$.

4. Assume that about 98.9% of the items produced in a large production facility satisfy the specifications of the manufacturer. The remaining items are scrapped. A random sample of 100 items are inspected.

   (a) (1.5 pts) Determine the mean and standard deviation of the number of scrapped items in such samples of size 100. Explicitly state which distribution is used for these computations, and why?

   Answer. We can think of this experiment as $n = 100$ independent Bernoulli trials each with $p = 0.011$ (probability of detecting and scrapping a defective item). If X denotes the total number of scrapped items, then $X \sim Bin(100, 0.011)$. So, $\mu_X = np = 100(0.011) = 1.1$ items and $\sigma = \sqrt{np(1-p)} = \sqrt{100(0.011)(0.989)} = 1.043$ items.

   (b) (2.5 pts) Compute the probability that at most two of these items is scrapped. Explicitly state which distribution is used for this computation, and why?

   Answer. We can either use $X \sim Bin(100, 0.011)$, or a suitable Poisson approximation, say $Y \sim Poisson(1.1)$ since n is large, p is small and $\lambda = np = 1.1$.

Using X, we have $P(X \le 2) = p_X(2) + p_X(1) + p_X(0)$ where $p_X(2) = \binom{100}{2}(0.011)^2(0.989)^{98} = 0.20259$, $p_X(1) = \binom{100}{1}(0.011)^1(0.989)^{99} = 0.36798$, and $p_X(0) = (0.989)^{100} = 0.33085$. Combining them, we get

$$P(X \le 2) = 0.20259 + 0.36798 + 0.33085 \simeq 0.9014.$$

Similarly, applying Poisson approximation, we compute the approximate probability

$$P(Y \le 2) = e^{-1.1}(1 + 1.1 + \frac{1.1^2}{2}) \simeq 0.9004.$$

(c) (2 pts) On a particular day, the items produced by a specific machine in the facility are inspected until four of them are found defective and scrapped. Determine the mean and standard deviation of the number of inspections done until four of them are scrapped. Explicitly state which distribution is used for this computation, and use one-decimal point of rounding.

Answer. Negative binomial distribution with $r = 4$ and $p = 0.011$. Mean is $\frac{r}{p} = 4/0.011 = 363.64 \simeq 363.6$ inspections with a standard deviation of $\frac{\sqrt{r(1-p)}}{p} = \frac{2\sqrt{0.989}}{0.011} = 180.82 \simeq 180.8$ inspections.

4. (3 pts) Out of 20 students in an upper division mathematics course, four of them are graduate students. If three students are selected at random from this class, compute the probability that at most one of them is a graduate student.

Answer. The sampling is done without replacement from a rather small finite population. So, we use hypergeometric distribution with parameters $N = 20$ (population size), $R = 4$ (graduate students) and $n = 3$ (sample size): $X \sim HG(20, 4, 3)$. We compute $P(X = 0) + P(X = 1)$ where

$P(X = 0) = \frac{\binom{4}{0}\binom{16}{3}}{\binom{20}{3}} = \frac{28}{57}$ and $P(X = 1) = \frac{\binom{4}{1}\binom{16}{2}}{\binom{20}{3}} = \frac{8}{19}$ so that

$$P(X \le 1) = \frac{28}{57} + \frac{8}{19} = \frac{52}{57} \simeq 0.9123.$$