Problem p.95

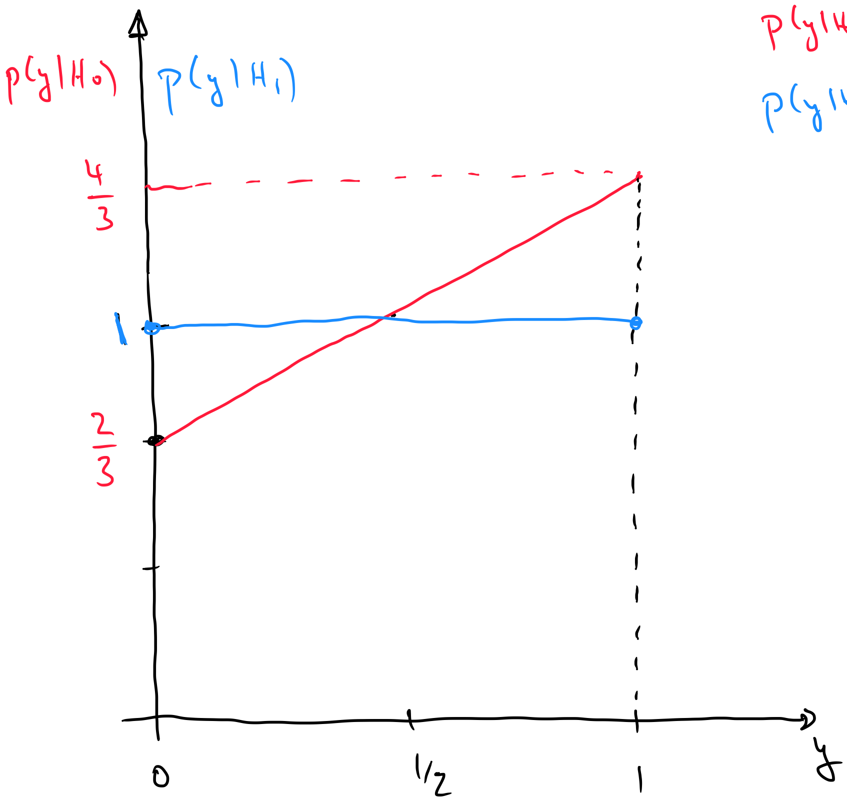
a) Bayes test

Sketch of $p(y|H_0)$ and $p(y|H_1)$

$p(y|H_0) = \frac{2}{3}(y+1)$

$p(y|H_1) = 1$

for $0 \le y \le 1$



Bayes rule:

$$\delta_B(y) = \begin{cases} 1 & \text{if } L(y) \ge \tau_B \\ 0 & \text{if } L(y) < \tau_B \end{cases}$$

with $\tau_B \Big|_{\substack{\text{uniform costs} \\ \text{equal priors}}} = \frac{\pi_0}{1-\pi_0}\Big|_{\pi_0 = 1-\pi_0 = 1/2} = 1$

and $L(y) = \dfrac{1}{\frac{2}{3}(y+1)} = \dfrac{3}{2(y+1)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$ for $0 \le y \le 1$

Find a rule to be applied to y rather than calculating $L(y)$ (to save signal processing power)

$$\Rightarrow \frac{3}{2(y+1)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \quad \Leftrightarrow \quad y \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{1}{2}$$

$$\Rightarrow \delta_B(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1/2 \\ 0 & \text{if } 1/2 < y \le 1 \end{cases} \leftarrow$$

Bayes risk:

$$r(\delta_B) = \pi_0 R_0(\delta_B) + (1-\pi_0) R_1(\delta_B)$$

uniform costs & equal priors:

$$R_0(\delta_B) = C_{10} \Pr\{\Gamma_1 | H_0\} = \underbrace{C_{10}}_{1} \int_0^{1/2} p(y|H_0)\, dy$$

$$= \int_0^{1/2} \frac{2}{3}(y+1)\, dy = \frac{1}{3} y^2 + \frac{2}{3} y \Big|_0^{1/2}$$

$$= \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

$$R_1(\delta_B) = C_{01} \Pr\{\Gamma_0 | H_1\} = \underbrace{C_{01}}_{1} \int_{1/2}^{1} dy = \frac{1}{2}$$

$$\pi_0 = \frac{1}{2} \Rightarrow r(\delta_B) = \frac{1}{2}\left[\frac{5}{12} + \frac{1}{2}\right] = \underline{\underline{\frac{11}{24}}} = 0.458\bar{3}$$

b) minimax test

For a given π_0, we have the optimum Bayes test

$$\delta_{\pi_0}(y) = \begin{cases} 1 & \text{if } \frac{3}{2(y+1)} \ge \frac{\pi_0}{1-\pi_0} \\ 0 & \text{"} \ \frac{3}{2(y+1)} < \frac{\pi_0}{1-\pi_0} \end{cases}$$

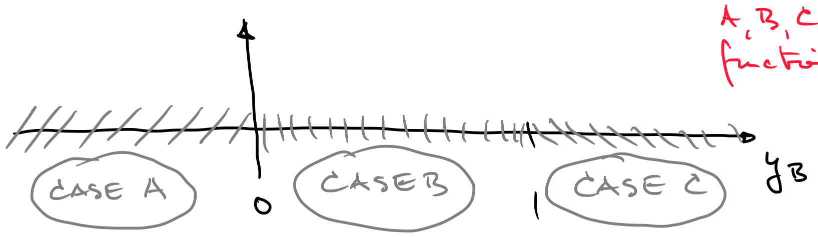
$$= \begin{cases} 1 & \text{if } y \le y_B \\ 0 & \text{"} \;\; y > y_B \end{cases} \quad \leftarrow \text{compare } y \text{ to } y_B$$

with y_B being defined by $\dfrac{3}{2(y_B+1)} = \dfrac{\pi_0}{1-\pi_0}$

$$\Rightarrow \boxed{y_B = y_B(\pi_0) = \frac{3}{2}\,\frac{1-\pi_0}{\pi_0} - 1 = \frac{3}{2\pi_0} - \frac{5}{2}} \iff \pi_0 = \frac{3}{2\left(y_B + \frac{5}{2}\right)}$$

insert below to characterize three cases A, B, C as a function of π_0

Distinguish 3 cases: A, B, C



Case A: $y_B \le 0 \Rightarrow \Gamma_0 = \{y \in [0,1]\},\ \Gamma_1 = \emptyset$

Case B: $0 < y_B < 1 \Rightarrow \Gamma_0 = \{y \in (y_B, 1]\}$

$\Gamma_1 = \{y \in [0, y_B]\}$

Case C: $y_B \ge 1 \Rightarrow \Gamma_0 = \emptyset,\ \Gamma_1 = \{y \in [0,1]\}$

Transform the cases to be characterized by π_0 by inserting $y_B = y_B(\pi_0)$ into the inequalities above:

$\Rightarrow$ case A: $y_B = \dfrac{3}{2\pi_0} - \dfrac{5}{2} \le 0 \iff \dfrac{3}{\pi_0} \le 5$

$$\Rightarrow \frac{3}{5} \le \pi_0 \le 1$$

case B: $0 < y_B = \frac{3}{2\pi_0} - \frac{5}{2} < 1 \iff \frac{3}{7} < \pi_0 < \frac{3}{5}$

case C: $y_B = \frac{3}{2\pi_0} - \frac{5}{2} \geq 1 \iff 0 \leq \pi_0 \leq \frac{3}{7}$

$\Rightarrow$ calculate Bayes risk for the three cases

case A: $\Gamma_0 = \{y \in [0,1]\}$, $\Gamma_1 = \phi$

$$\Rightarrow V(\pi_0) = \pi_0 R_0(\delta_{\pi_0}) + (1-\pi_0) R_1(\delta_{\pi_0})$$

$$= \pi_0 \underbrace{P(\Gamma_1 | H_0)}_{0 \text{ since } \Gamma_1 = \phi} + (1-\pi_0) \underbrace{P(\Gamma_0 | H_1)}_{1}$$

$$= 1 - \pi_0 \quad \text{for} \quad \boxed{\frac{3}{5} \leq \pi_0 \leq 1}$$

case B: $\Gamma_0 = \{y \in (y_B, 1]\}$, $\Gamma_1 = \{y \in [0, y_B]\}$

$$V(\pi_0) = \pi_0 P(\Gamma_1 | H_0) + (1-\pi_0) P(\Gamma_0 | H_1) \leftarrow$$

$$P(\Gamma_1 | H_0) = \int_0^{y_B} \frac{2}{3}(y+1)\, dy = \frac{y^2}{3} + \frac{2}{3} y \Big|_0^{y_B}$$

$$= \frac{y_B^2}{3} + \frac{2}{3} y_B = \frac{1}{3}\left(\frac{3}{2\pi_0} - \frac{5}{2}\right)^2 + \frac{2}{3}\left(\frac{3}{2\pi_0} - \frac{5}{2}\right)$$

$$= \frac{3}{4\pi_0^2} - \frac{3}{2\pi_0} + \frac{5}{12}$$

$$P(\Gamma_0 | H_1) = \int_{y_B}^{1} dy = 1 - y_B = 1 - \left(\frac{3}{2\pi_0} - \frac{5}{2}\right)$$

$$= \frac{7}{2} - \frac{3}{2\pi_0}$$

$$V(\pi_0) = \pi_0 \left(\frac{3}{4\pi_0^2} - \frac{3}{2\pi_0} + \frac{5}{12}\right) + (1-\pi_0)\left(\frac{7}{2} - \frac{3}{2\pi_0}\right)$$

$$= \frac{7}{2} - \frac{3}{4\pi_0} - \frac{37}{12}\pi_0 \quad \text{for} \quad \boxed{\frac{3}{7} < \pi_0 < \frac{3}{5}}$$

case C: $\Gamma_0 = \phi$, $\Gamma_1 = \{y \in [0,1]\}$

$$V(\pi_0) = \pi_0 \underbrace{P(\Gamma_1 | H_0)}_{=1} + (1-\pi_0)\underbrace{P(\Gamma_0 | H_1)}_{=0 \text{ since } \Gamma_0 = \phi}$$

$$= \pi_0 \quad \text{for} \quad \boxed{0 \le \pi_0 \le \frac{3}{7}}$$

$\Rightarrow$ all three cases together:

$$V(\pi_0) = \begin{cases} \pi_0 & \text{for} \quad 0 \le \pi_0 \le \frac{3}{7} \quad \text{CASE C} \\ \frac{7}{2} - \frac{3}{4\pi_0} - \frac{37}{12}\pi_0 & \text{"} \quad \frac{3}{7} < \pi_0 < \frac{3}{5} \quad \text{CASE B} \\ 1 - \pi_0 & \text{"} \quad \frac{3}{5} \le \pi_0 \le 1 \quad \text{CASE A} \end{cases}$$

Consistency checks:

$$V\left(\pi_0 = \frac{3}{7}\right)\Big|_{\text{case C}} = \frac{3}{7}$$

✓

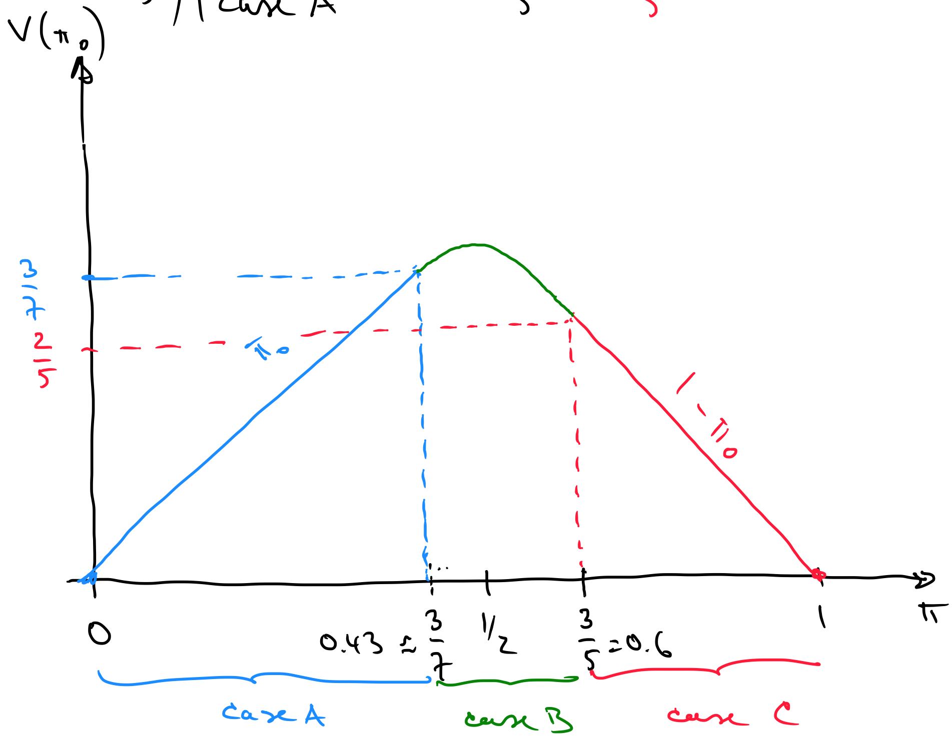
$$V\left(\pi_0 = \frac{3}{7}\right)\Big|_{\text{case B}} = \frac{7}{2} - \frac{\cancel{3}\cdot 7}{4\cdot \cancel{3}} - \frac{37}{12}\cdot\frac{3}{7}$$

$$= \frac{294 - 147 - 111}{84} = \frac{36}{84} = \frac{12}{28} = \frac{3}{7}$$

$$V\left(\pi_0 = \frac{3}{5}\right)\Big|_{\text{case B}} = \frac{7}{2} - \frac{3\cdot 5}{4\cdot 3} - \frac{37}{12}\cdot\frac{3}{5}$$

$$= \frac{210 - 75 - 111}{60} = \frac{24}{60} = \frac{2}{5}$$

$$V\left(\pi_0 = \frac{3}{5}\right)\Big|_{\text{case A}} = 1 - \frac{3}{5} = \frac{2}{5}$$



Apparently, the least favorable prior π_L is in the case B-interval so that we can determine π_L by the necessary condition

$$\frac{\partial}{\partial \pi} V(\pi_0)\Big|_{\substack{\pi_0 = \pi_L \\ \text{case B}}} = 0$$

$$\Rightarrow \frac{\partial}{\partial \pi_0}\left[\frac{7}{2} - \frac{3}{4\pi_0} - \frac{37}{12}\pi_0\right] = -\frac{37}{12} + \frac{3}{4\pi_0^2}\Big|_{\pi_0 = \pi_L} \overset{!}{=} 0$$

$$\Rightarrow \pi_L^2 = \frac{3}{\cancel{4}} \cdot \frac{\overset{3}{\cancel{12}}}{37} = \frac{9}{37} \Rightarrow \boxed{\pi_L = \frac{3}{\sqrt{37}} \approx 0.493}$$

$\Rightarrow$ minimax rule is given by $\delta(\pi_0)$ for $\pi_0 = \pi_L$ in case B

$$\Rightarrow \delta(\pi_L) = \begin{cases} 1 & \text{if } y \leq y_B(\pi_L) \\ 0 & \text{" } y > y_B(\pi_L) \end{cases}$$

$$= \begin{cases} 1 & \text{if } y \leq \frac{3}{2\pi_L} - \frac{5}{2} = \frac{1}{2}\left[\frac{3}{3/\sqrt{37}} - 5\right] \\ 0 & \text{" } y > \dots \end{cases}$$

$$= \begin{cases} 1 & \text{if } y \leq \frac{1}{2}\left(\sqrt{37} - 5\right) \approx 0.54 \\ 0 & \text{" } y > 0.54 \end{cases}$$

with minimax risk

$$V(\pi_L)\Big|_{\text{case B}} = \frac{7}{2} - \frac{3}{4\pi_L} - \frac{37}{12}\pi_L \Big|_{\pi_L = \frac{3}{\sqrt{37}}}$$

$$= \frac{7}{2} - \frac{\sqrt{37}}{4} - \frac{\sqrt{37}}{4} = \frac{7 - \sqrt{37}}{2} \approx 0.459$$

c) Neyman-Pearson rule

We have to find the false-alarm probability

$$P_F(\eta) = P_r\{L(y) \geq \eta \mid H_0\}$$

with $L(y) = \frac{3}{2(y+1)}$ for $0 \leq y \leq 1$

We can reformulate $L(y)$ as follows:

$$\delta_{NP}(y) = \begin{cases} 1 & \text{for } L(y) \geq \eta \\ 0 & \text{"} \quad L(y) < \eta \end{cases}$$

$$= \begin{cases} 1 & \text{for } \frac{3}{2(y+1)} \geq \eta \\ 0 & \text{"} \quad \frac{3}{2(y+1)} < \eta \end{cases}$$

$$= \begin{cases} 1 & \text{for } y \leq \frac{3}{2\eta} - 1 \\ 0 & \text{"} \quad y > \frac{3}{2\eta} - 1 \end{cases}$$

Apparently, $\Pr\{L(y) > \eta \mid H_0\} = \Pr\{L(y) \geq \eta \mid H_0\}$ and thus no randomization is required.

Obviously, we have for $0 \leq y \leq 1$

$$\frac{3}{4} \leq L(y) \leq \frac{3}{2}$$

Therefore, as in b), we have

case A': for $\eta \leq \frac{3}{4} \Rightarrow \Gamma_1 = [0, 1]$

case B': for $\frac{3}{4} < \eta < \frac{3}{2} \Rightarrow \Gamma_1 = \left[0, \frac{3}{2\eta} - 1\right]$

case C': for $\frac{3}{2} \leq \eta \Rightarrow \Gamma_1 = \phi$

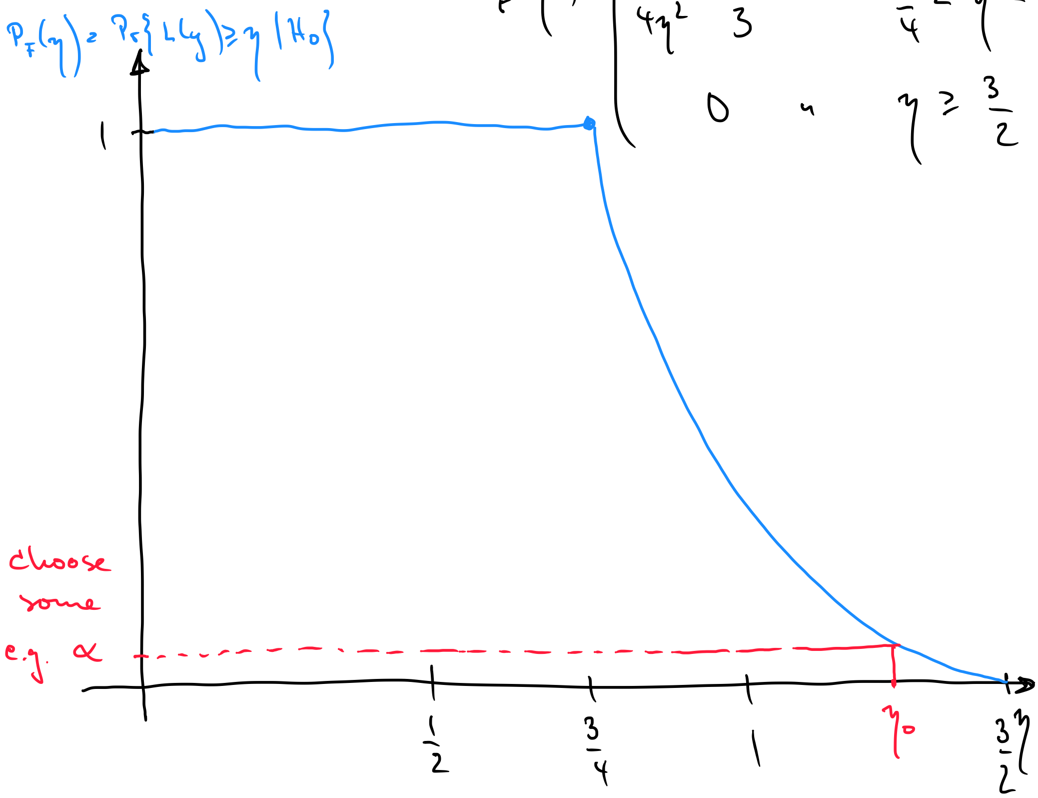
We obtain the false-alarm (FA) probability as a function of η according to

$$P_F(\eta) = \begin{cases} Pr\{ y \in [0,1] \mid H_0 \} = 1 & \text{for } \eta \le \frac{3}{4} \text{ (case A')} \\ \int_0^{\frac{3}{2\eta}-1} Pr(y \mid H_0)\, dy & \text{for } \frac{3}{4} < \eta < \frac{3}{2} \text{ (case B')} \\ Pr\{ \phi \} = 0 & \text{for } \eta \ge \frac{3}{2} \text{ (case C')} \end{cases}$$

$$\int_0^{\frac{3}{2\eta}-1} \frac{2}{3}(y+1)\, dy = \frac{y^2}{3} + \frac{2}{3} y \Big|_0^{\frac{3}{2\eta}-1} = \frac{1}{3}\left(\frac{3}{2\eta}-1\right)^2 + \frac{2}{3}\left(\frac{3}{2\eta}-1\right)$$

$$= \frac{1}{3}\left[\frac{9}{4\eta^2} - 1\right] = \frac{3}{4\eta^2} - \frac{1}{3}$$

$$\Rightarrow P_F(\eta) = \begin{cases} 1 & \text{for } \eta \le \frac{3}{4} \\ \frac{3}{4\eta^2} - \frac{1}{3} & \text{for } \frac{3}{4} < \eta < \frac{3}{2} \\ 0 & \text{for } \eta \ge \frac{3}{2} \end{cases}$$



In view of the missing randomization, we can find the threshold η_0 by (for $0 < \alpha < 1$)

$$\eta_0 = P_F^{-1}(\alpha)$$

In applications, typically α is in the range of $\alpha \in [0.01 \dots 0.05]$

$$\Rightarrow P_F(\eta_0) = \frac{3}{4\eta_0^2} - \frac{1}{3} \overset{!}{=} \alpha \quad \Rightarrow \eta_0^2 = \frac{3}{4} \cdot \frac{1}{\alpha + \frac{1}{3}}$$

$$\Rightarrow \eta_0 = \frac{3}{2} \frac{1}{\sqrt{3\alpha + 1}} .$$

As a result, the Neyman-Pearson rule is given by

$$\tilde{\delta}_{NP}(y) = \begin{cases} 1 & \text{if } \frac{3}{2(y+1)} \geq \frac{3}{2\sqrt{3\alpha+1}} \\ 0 & \text{if } \frac{3}{2(y+1)} < \frac{3}{2\sqrt{3\alpha+1}} \end{cases}$$

$$= \begin{cases} 1 & \text{if } y \leq \sqrt{1+3\alpha} - 1 \\ 0 & \text{if } y > \sqrt{1+3\alpha} - 1 \end{cases}$$

The detection probability can be found to be

$$P_D(\tilde{\delta}_{NP}) = \int_0^{\frac{3}{2\eta_0}-1} P(y|H_1)\, dy = \int_0^{\frac{3}{2\eta_0}-1} dy = \frac{3}{2\eta_0} - 1$$

$$= \frac{3}{2y_0(\alpha)} - 1 = \sqrt{1+3\alpha} - 1 .$$

Interestingly, the detection probability is identical to the threshold value for y.