problem p.86: Proof of Nyquist theorem

Matched filter output

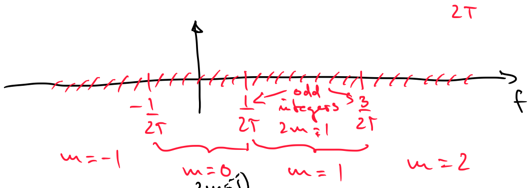
$$x(t) = \int_{\mathbb{R}} X(f)\, e^{j2\pi f t}\, df$$

$$\downarrow$$

$$x(nT) = \int_{\mathbb{R}} X(f)\, e^{j2\pi f nT}\, df$$

$$\downarrow$$

split up integral into disjoint intervals of length $\frac{1}{T}$ $\Rightarrow$ $\displaystyle\int_{-\infty}^{\infty} \ldots df = \sum_{m\in\mathbb{Z}} \int_{\frac{2m-1}{2T}}^{\frac{2m+1}{2T}} \ldots df$



$$\Rightarrow x(nT) = \sum_{m\in\mathbb{Z}} \int_{\frac{2m-1}{2T}}^{\frac{2m+1}{2T}} X(f)\, e^{j2\pi \boxed{f nT}}\, df$$

Subst.: $\boxed{f = v + \frac{m}{T}}$ $\left(\text{e.g. } \frac{2m\pm1}{2T} = v + \frac{m}{T} \Leftrightarrow v = \pm\frac{1}{2}\right)$

$$= \sum_{m\in\mathbb{Z}} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X\left(v + \frac{m}{T}\right) e^{j2\pi v nT}\, dv$$

$$= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \underbrace{\left[\sum_{m\in\mathbb{Z}} X\left(f + \frac{m}{T}\right)\right]}_{B(f)} e^{j2\pi f nT}\, df$$

$$= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} B(f)\, e^{j2\pi \overset{\downarrow}{n} T}\, df$$

with $B(f) = \sum_{m\in\mathbb{Z}} X\left(f + \frac{m}{T}\right)$ being periodic since

$$B\left(f + \frac{k}{T}\right)\Big|_{k\in\mathbb{Z}} = \sum_{m\in\mathbb{Z}} X\left(f + \frac{k}{T} + \frac{m}{T}\right)$$

$$= \sum_{m\in\mathbb{Z}} X\left(f + \frac{k+m}{T}\right)$$

reindexing the sum by $\varkappa = \underset{\in\mathbb{Z}}{k+m}$

$$= \sum_{\varkappa \in \mathbb{Z}} X\left(f + \frac{\varkappa}{T}\right) = B(f) \quad \text{Q.E.D.}$$

$\Rightarrow$ since $B(f)$ is periodic in f with period $F = \frac{1}{T}$, we can expand it into a Fourier series according to

$$\rightarrow B(f) = \sum_{n \in \mathbb{Z}} b_n e^{j 2\pi n \frac{f}{F}} = \sum_{n \in \mathbb{Z}} b_n e^{j 2\pi n f T} \quad (**)$$

$$\text{and } b_n = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} B(f) e^{-j 2\pi n f T} df$$

From the blue and purple expressions

$$\Rightarrow \boxed{b_n} = T\, x(-nT) \qquad (*)$$

"No ISI condition" $\Leftrightarrow x(nT) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{else} \end{cases}$

Plug into $(*)$

"No ISI" $\Leftrightarrow b_n = \begin{cases} T & \text{for } n = 0 \\ 0 & \text{else} \end{cases}$

Plug into $(**)$

$$\Rightarrow B(f) = b_0 = T \qquad (***)$$

Since $B(f) = \sum_{m \in \mathbb{Z}} X\left(f + \frac{m}{T}\right) \qquad (****)$

$\Rightarrow$ equate the RHS (right-hand side) of $(***)$ and LHS (left-hand side) of $(****)$

$\Rightarrow$ Nyquist condition for zero ISI in the frequency domain

$$\sum_{m \in \mathbb{Z}} X\left(f + \frac{m}{T}\right) = T \Leftrightarrow x(nT) = \delta_n$$

Equivalence relation !

Problem p.91: Proof of the RC pulse satisfying the Nyquist criterion / theorem

① For $0 \leq |f| \leq \frac{1-\beta}{2T}$ and $|f| > \frac{1+\beta}{2T}$

$\Rightarrow X(f) = T$ overlaps with $X(f) = 0$

$\Rightarrow$ for intervals $|f| \leq \frac{1-\beta}{2T}$ and

$\frac{1+\beta}{2T} < |f| \leq \frac{1}{T}$ apparently $X(f) = T$

② Consider for $f \in \left[\frac{1-\beta}{2T}, \frac{1+\beta}{2T}\right]$

$$X_{rc}(f) + X_{rc}\left(f - \frac{1}{T}\right)$$

$$= \frac{T}{2}\left[1 + \cos\left(\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right)\right]$$

$$+ \frac{T}{2}\left[1 + \cos\left(\frac{\pi T}{\beta}\left[-\left(f - \frac{1}{T}\right)\right] - \frac{1-\beta}{2T}\right)\right]$$

$$= T + \frac{T}{2}\left[\cos\left(\frac{\pi T}{\beta} f - \frac{(1-\beta)\pi}{2\beta}\right) + \cos\left(-\frac{\pi T f}{\beta} + \frac{\pi}{\beta} - \frac{\pi}{2\beta}(1-\beta)\right)\right]$$

$$= T + \frac{T}{2}\left[\cos\left(\frac{\pi T f}{\beta} - \frac{\pi}{2\beta} + \frac{\pi}{2}\right) + \cos\left(-\frac{\pi T f}{\beta} + \frac{\pi}{\beta} + \frac{\pi}{2} - \frac{\pi}{2\beta}\right)\right]$$

$\updownarrow$ $\cos(x) = \cos(-x)$

$$= T + \frac{T}{2}\left[\underbrace{\underbrace{\cos\left(\underbrace{\frac{\pi T f}{\beta} - \frac{\pi}{2\beta}}_{y} + \frac{\pi}{2}\right)}_{= -\sin y} + \underbrace{\cos\left(\underbrace{\frac{\pi T f}{\beta} - \frac{\pi}{2\beta}}_{y} - \frac{\pi}{2}\right)}_{\sin y}}_{=0}\right]$$

$$= \underline{\underline{T}}$$

$\Rightarrow$ considering $X(f)$ being an even function completes the proof for $-\frac{1}{T} \le f \le \frac{1}{T}$ $\square$.