Variance

Distribution Properties

Deterministic functions of distribution

Not random

⬅ Long-term average

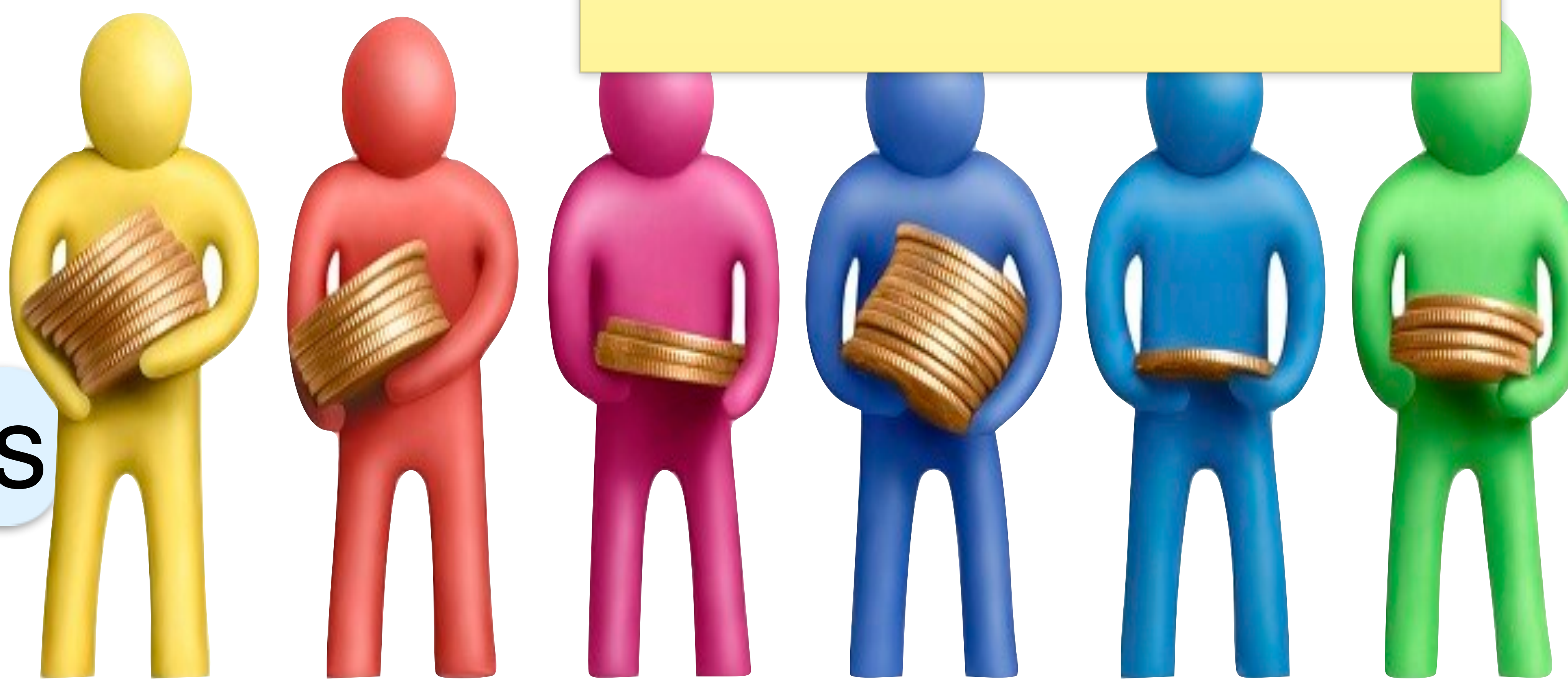
Expectation EX, μ

Die: μ=3.5

➡ Consistency

Variation from the mean

Money Matters



Two companies, each with 1,000 employees

Both same mean salary: $100K But

C1: Every employee makes $100K

100M total

C2: Every employee makes $1, CEO $99,999,001

100M-999

Which will you join?

Same mean Very different distributions

Mean ain't all Variation matters!

Difference from Mean

X r.v. with mean μ

How much X differs from μ on average?

Candidate

E | X - μ |

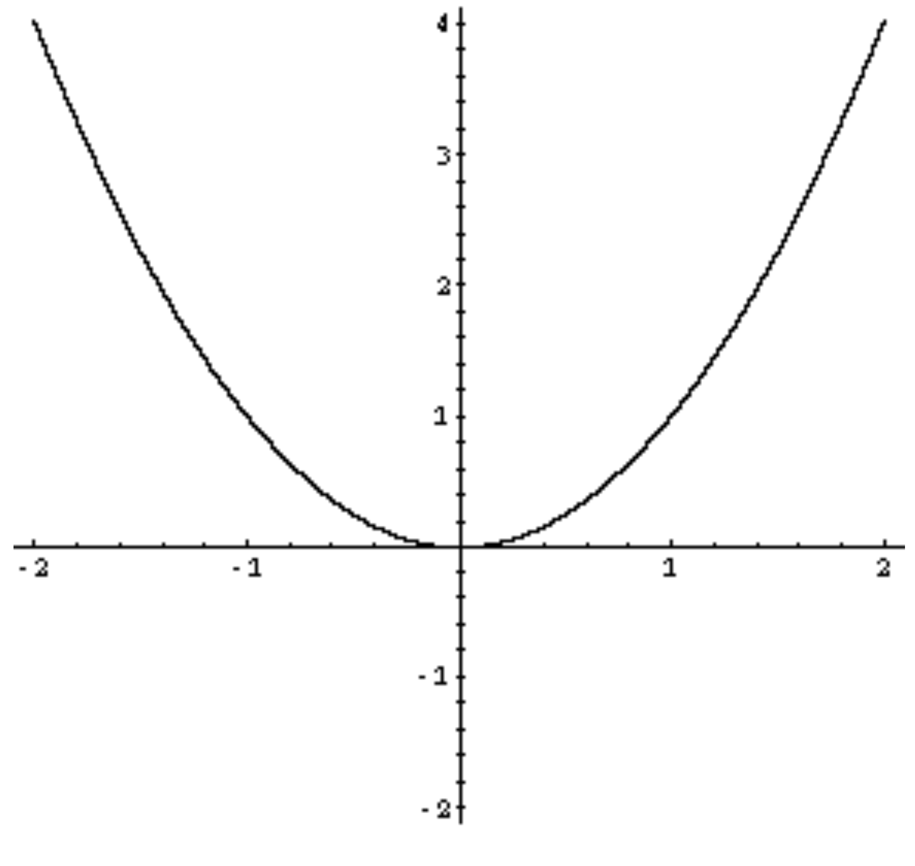
Mean absolute difference

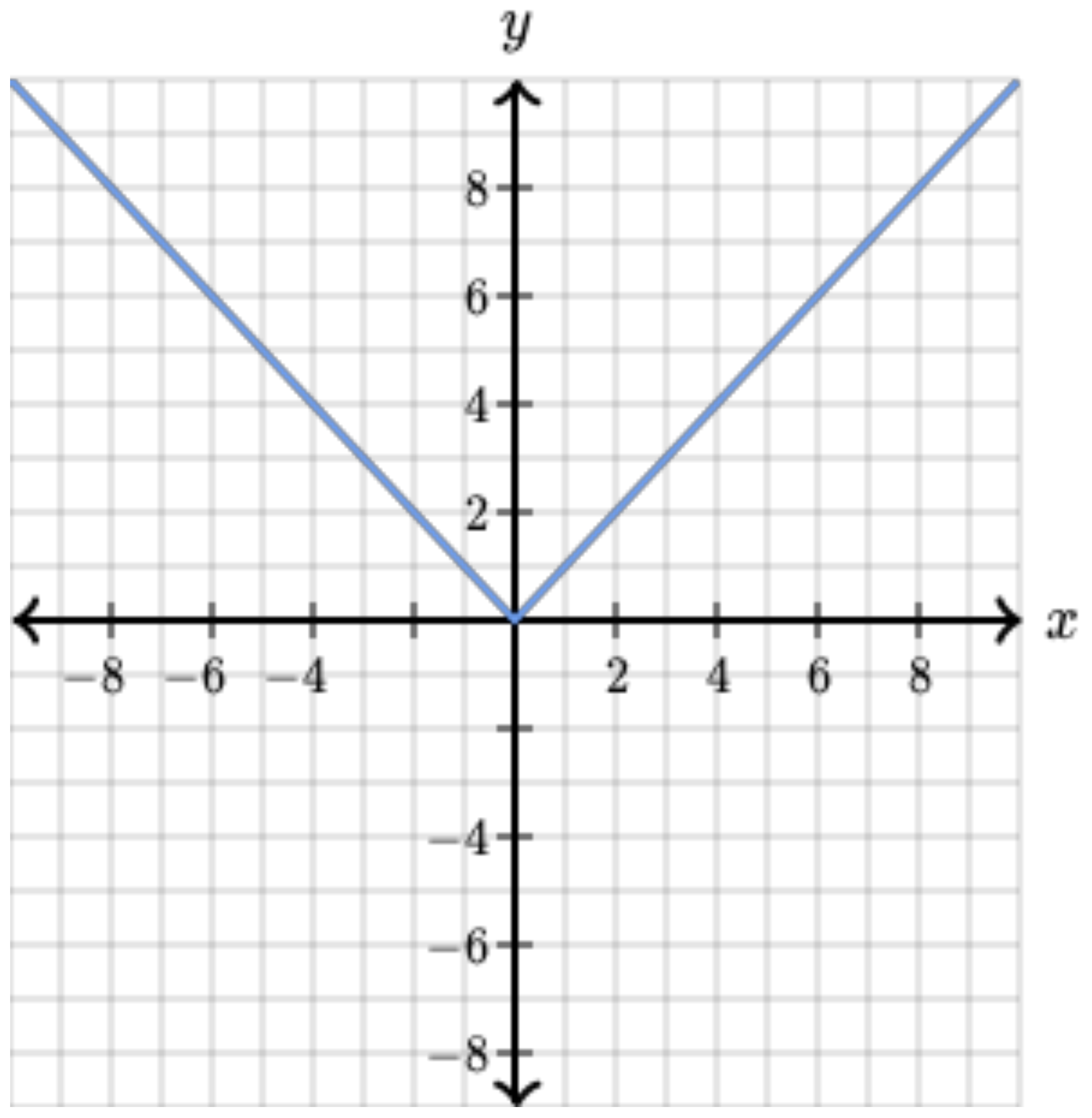
Not commonly used

Absolute value function hard to analyze

Instead

$E(X - \mu)^2$





Variance

Expected squared difference between X and its mean

$$V(X) = E\,[\,(X - \mu)^2\,] \quad = E\,(X - \mu)^2$$

Standard deviation

$$\sigma_X = +\sqrt{V(X)}$$

Constants

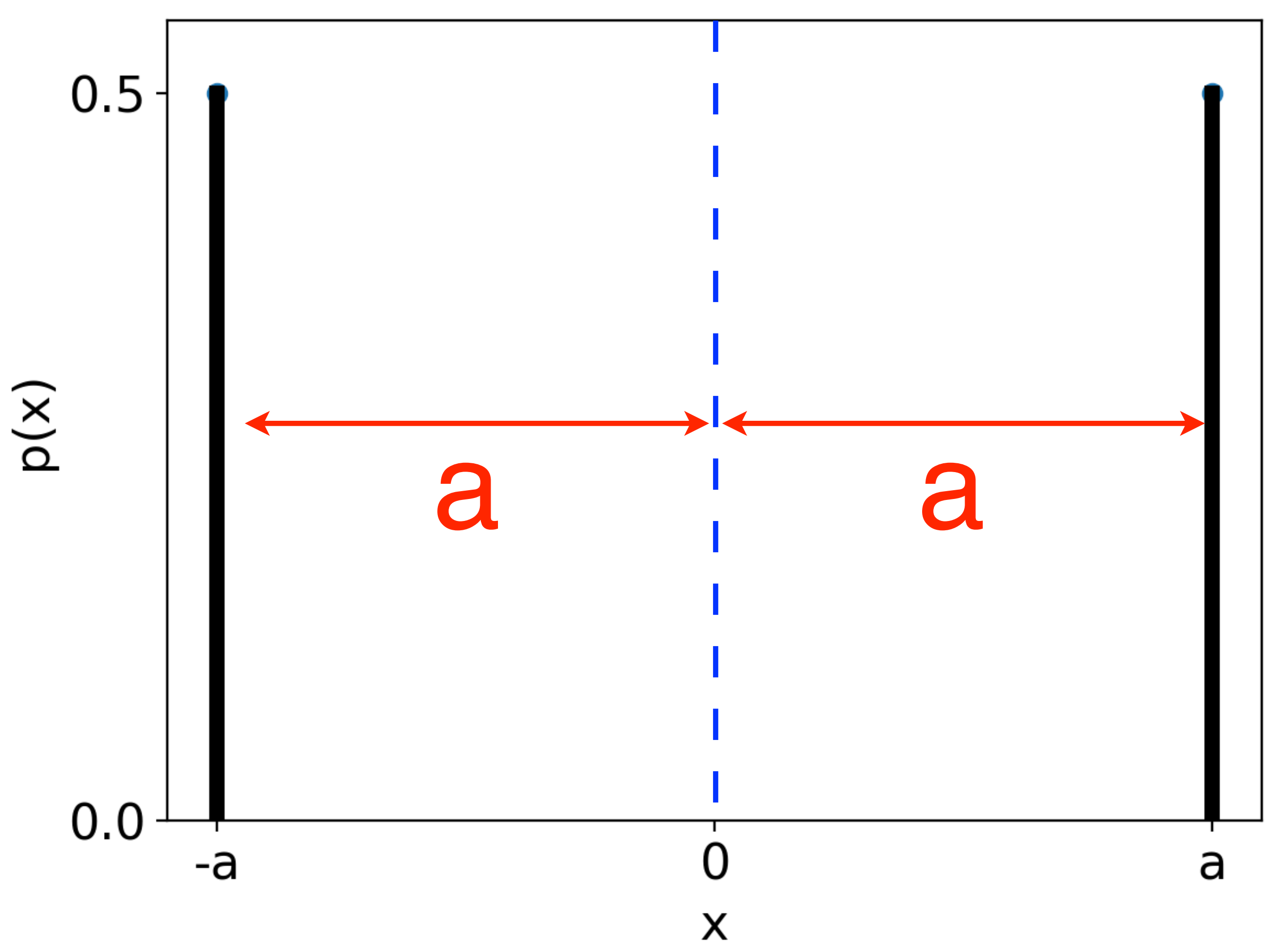
Properties of distribution

Examples

x	p_x	$x - \mu$	$(x - \mu)^2$
$-a$	½	$-a$	a^2
a	½	a	a^2

$\mu = 0$

$\mu = 0$



$V(X) = ½ \cdot a^2 + ½ \cdot a^2 = a^2$

X^2 is always a^2

$(X - \mu)^2 = a^2$ always

$\sigma_X = a$

"average" distance from mean

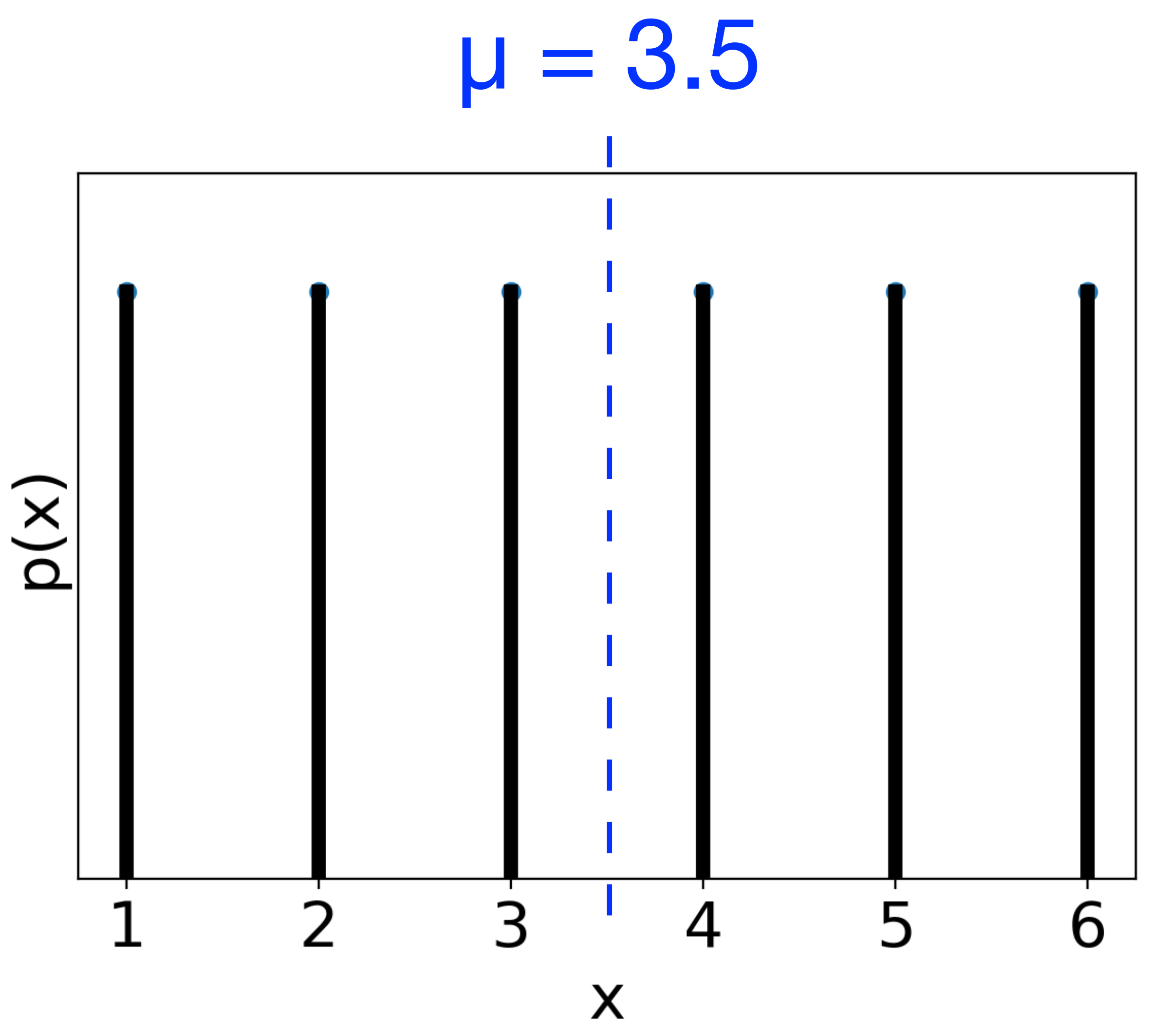
Fair Die

$\mu = 3.5$

$$V(X) = E(X - \mu)^2 = \frac{2(6.25 + 2.25 + 0.25)}{6} = \frac{8.75}{3} = 2.92..$$

$$\sigma = \sqrt{2.92\ldots} = 1.71\ldots$$

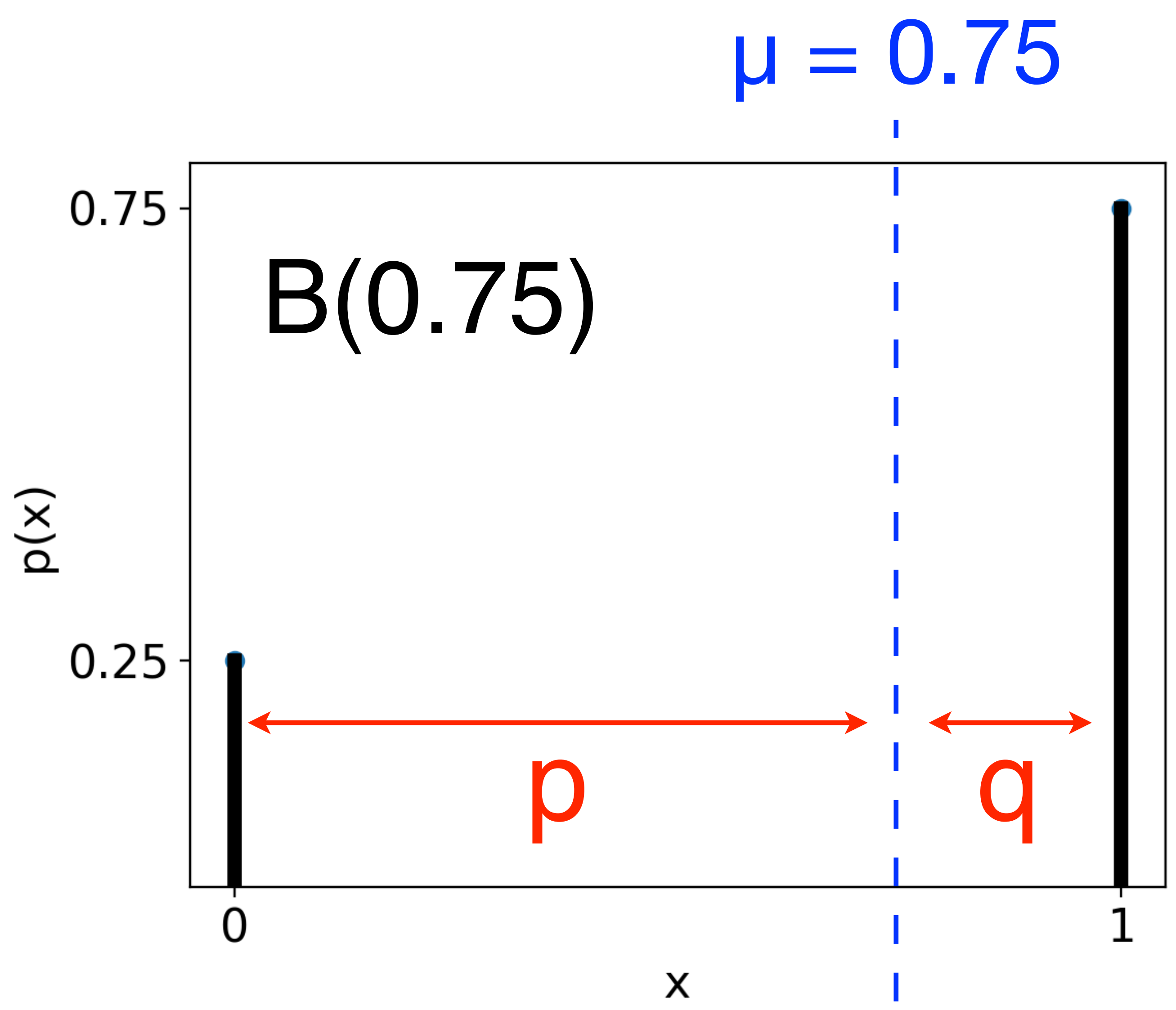
x	p_x	$x - \mu$	$(x - \mu)^2$
1	⅙	-2.5	6.25
2	⅙	-1.5	2.25
3	⅙	-0.5	0.25
4	⅙	0.5	0.25
5	⅙	1.5	2.25
6	⅙	2.5	6.25



Bernoulli p

x	p_x	x - μ	$(x - \mu)^2$
0	q	0-p = -p	p^2
1	p	1-p = q	q^2

$\mu = p$



$$V(X) = q \cdot p^2 + p \cdot q^2 = pq(p+q) = pq$$

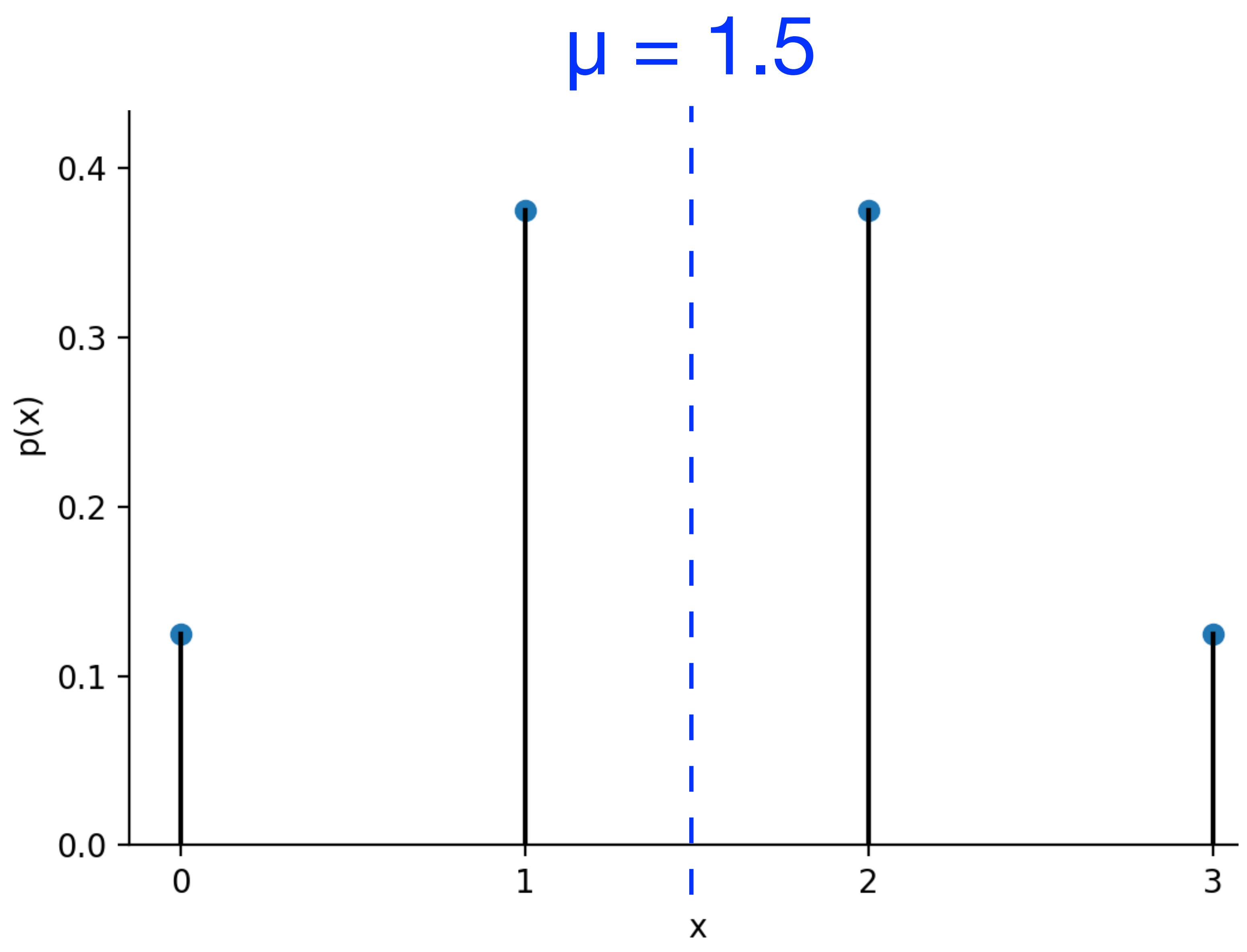
3 Coins

Toss 3 fair coins

X = # heads

x	p_x	$x - \mu$	$(x - \mu)^2$
0	⅛	-1.5	2.25
1	⅜	-0.5	0.25
2	⅜	0.5	0.25
3	⅛	1.5	2.25

$\mu = 1.5$



$$V = 2(\tfrac{1}{8} \cdot 2.25 + \tfrac{3}{8} \cdot 0.25) = \tfrac{1}{4}(2.25 + 0.75) = \tfrac{3}{4}$$

$$\sigma = \sqrt{3}/2$$

Soon: simpler derivation

Observations

$V(X) = E(X - \mu)^2$

$$0 \leq V(X) \leq \max (X-\mu)^2$$

=
X is a
constant

=
X constant or
takes two values
with equal prob.

$$0 \leq \quad \sigma_X \quad \leq \max |X-\mu|$$

$V(X) = EX^2 - \mu^2$

$$V(X) \leq E(X^2)$$

Properties

How simple modification affect V and σ

Addition	(translation)	$X + b$
Multiplication	(scaling)	$a \cdot X$
+ & x	(affine transformation)	$aX + b$

Addition

X - random variable

b - constant (e.g. 2)

$\mu_{x+b} = \mu_x + b$

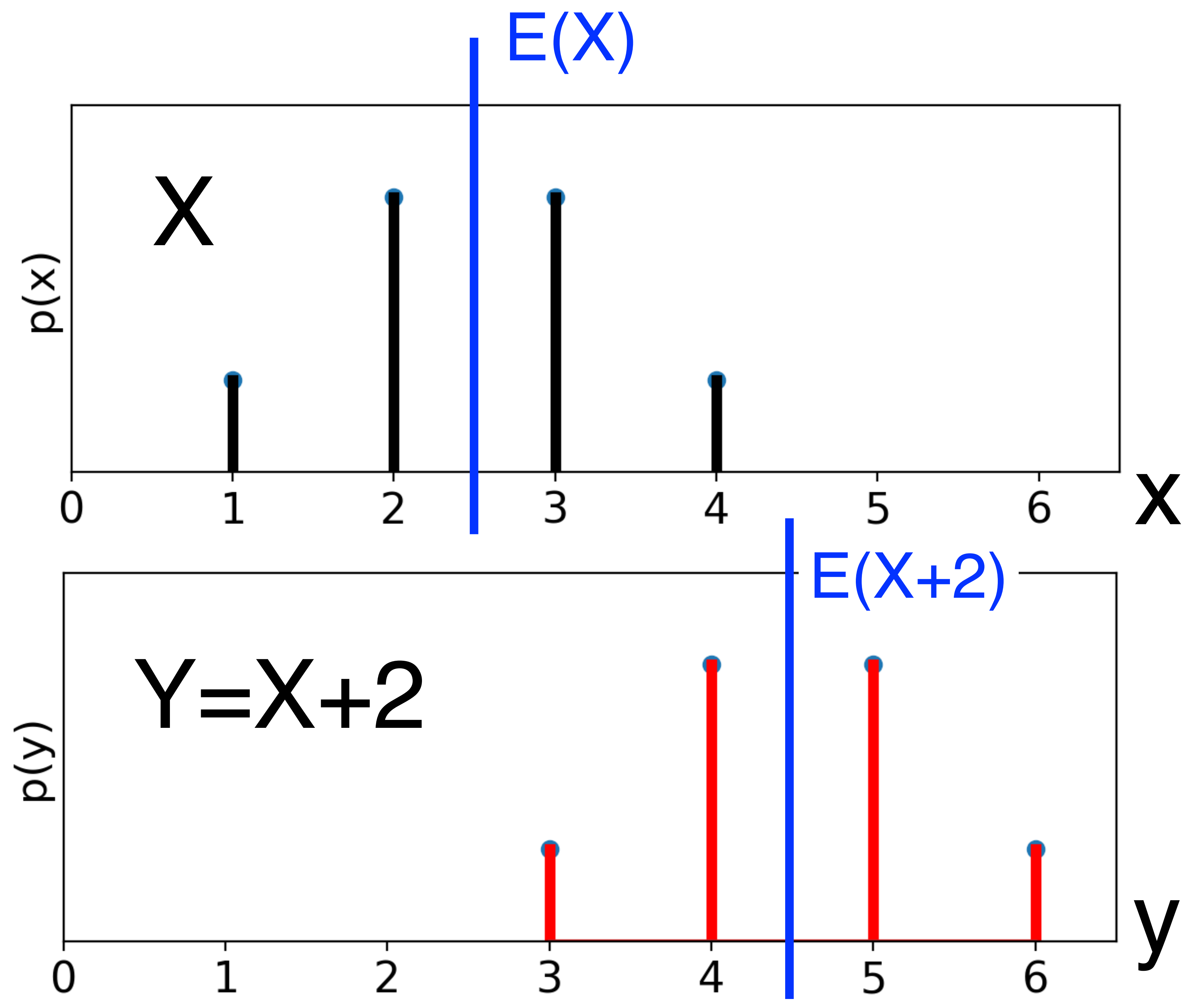
Linearity of expectation

$V(X + b) = E[(X + b - \mu_{x+b})^2]$

$= E[(X + b - \mu_x - b)^2]$

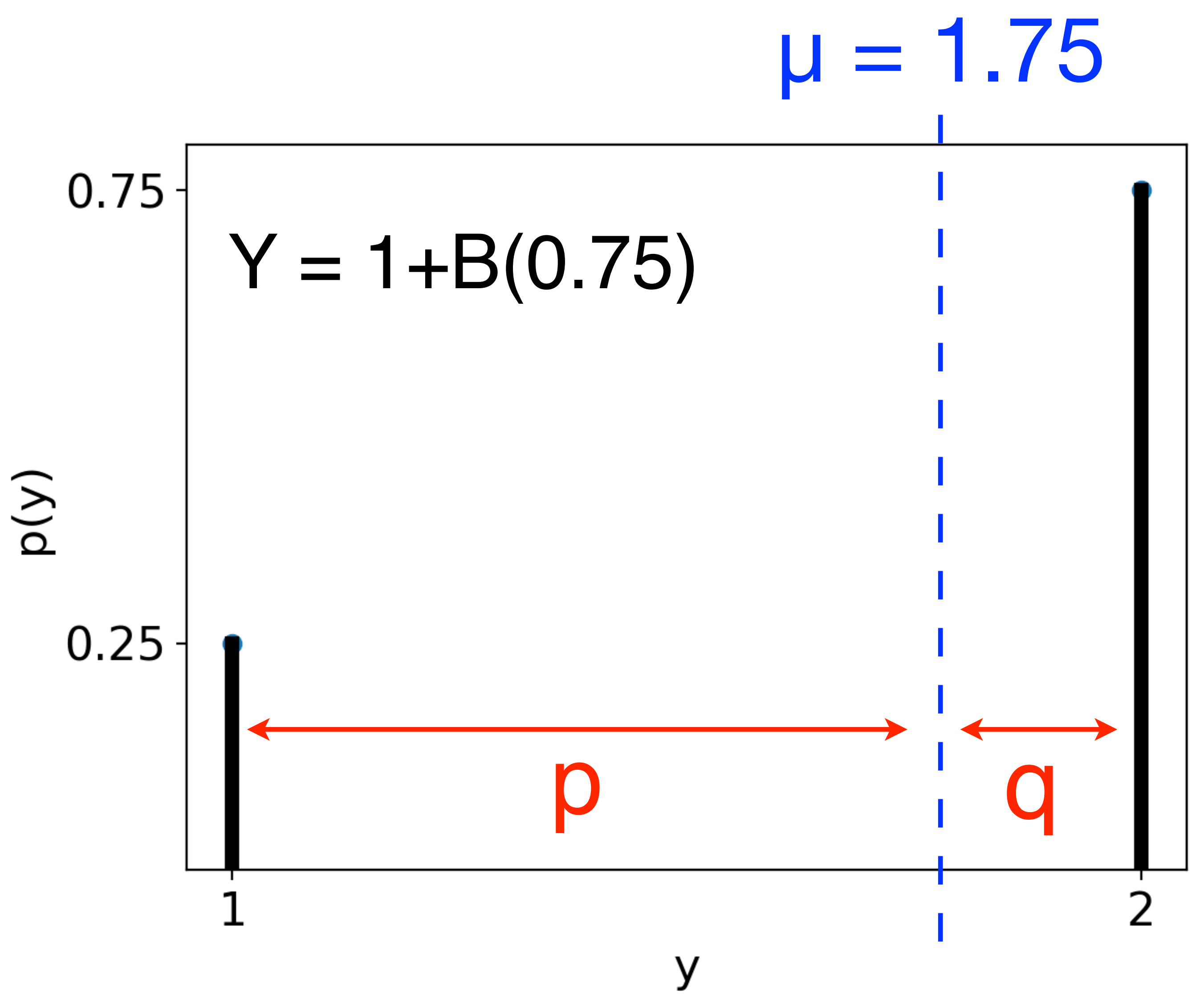
$= E(X - \mu_x)^2$

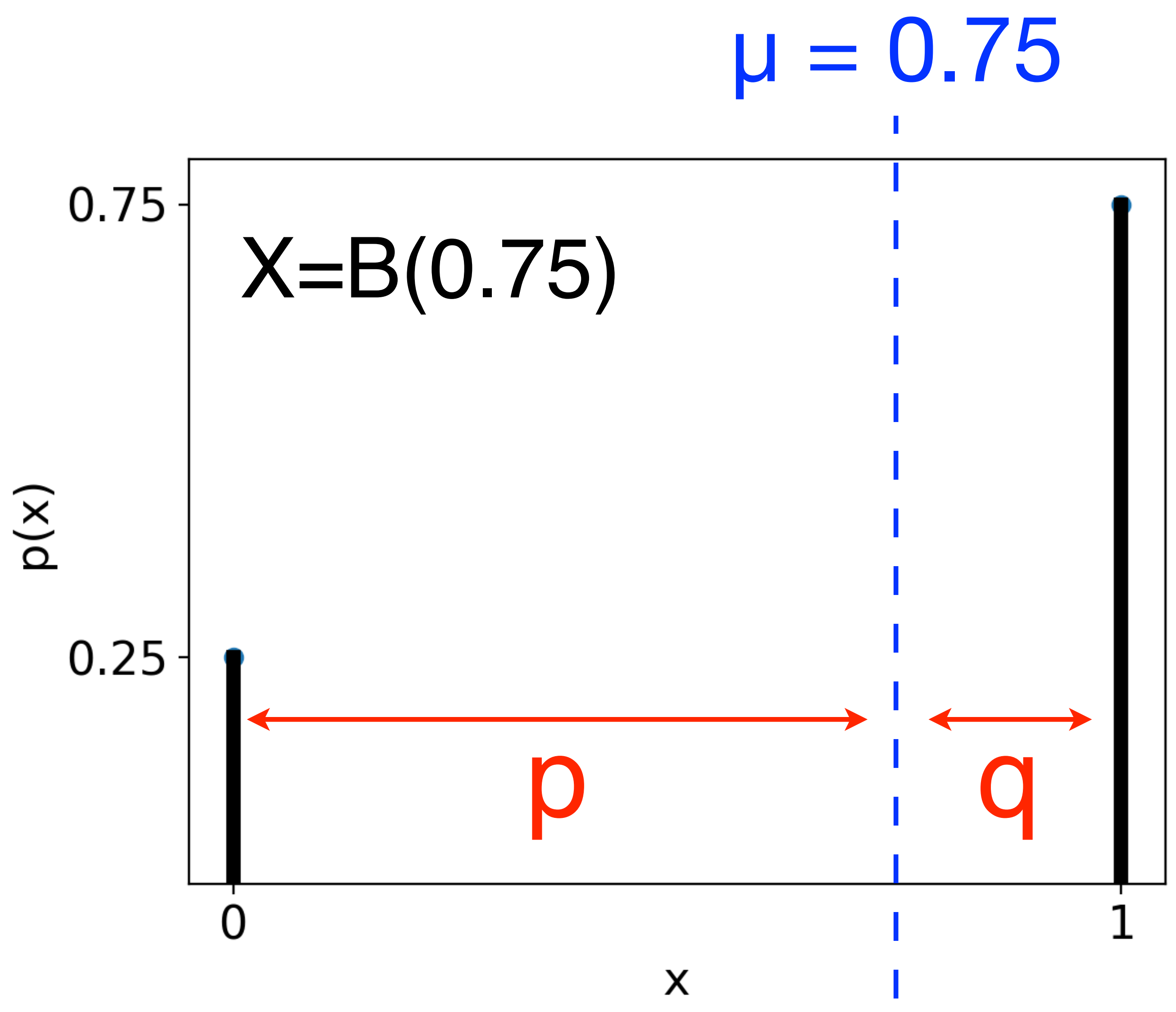
$= V(X)$



Translated B(p)

$X \sim B(p)$ $V(X) = p(1-p)$





$Y = X + 1$

y	p_y
1	$1-p$
2	p

$\mu_y = 1 + p$ (linearity of expectations)

$$V(Y) = E(Y - \mu_y)^2 = (1 - p)(1 - 1 - p)^2 + p(2 - 1 - p)^2$$

$$= (1 - p)p^2 + p(1 - p)^2 = p(1 - p)(p + 1 - p) = p(1 - p)$$

$$= V(X)$$ ✅

Scaling

$$V(aX) = E(aX - \mu_{ax})^2$$

$$\mu_{ax} = a\mu_X$$

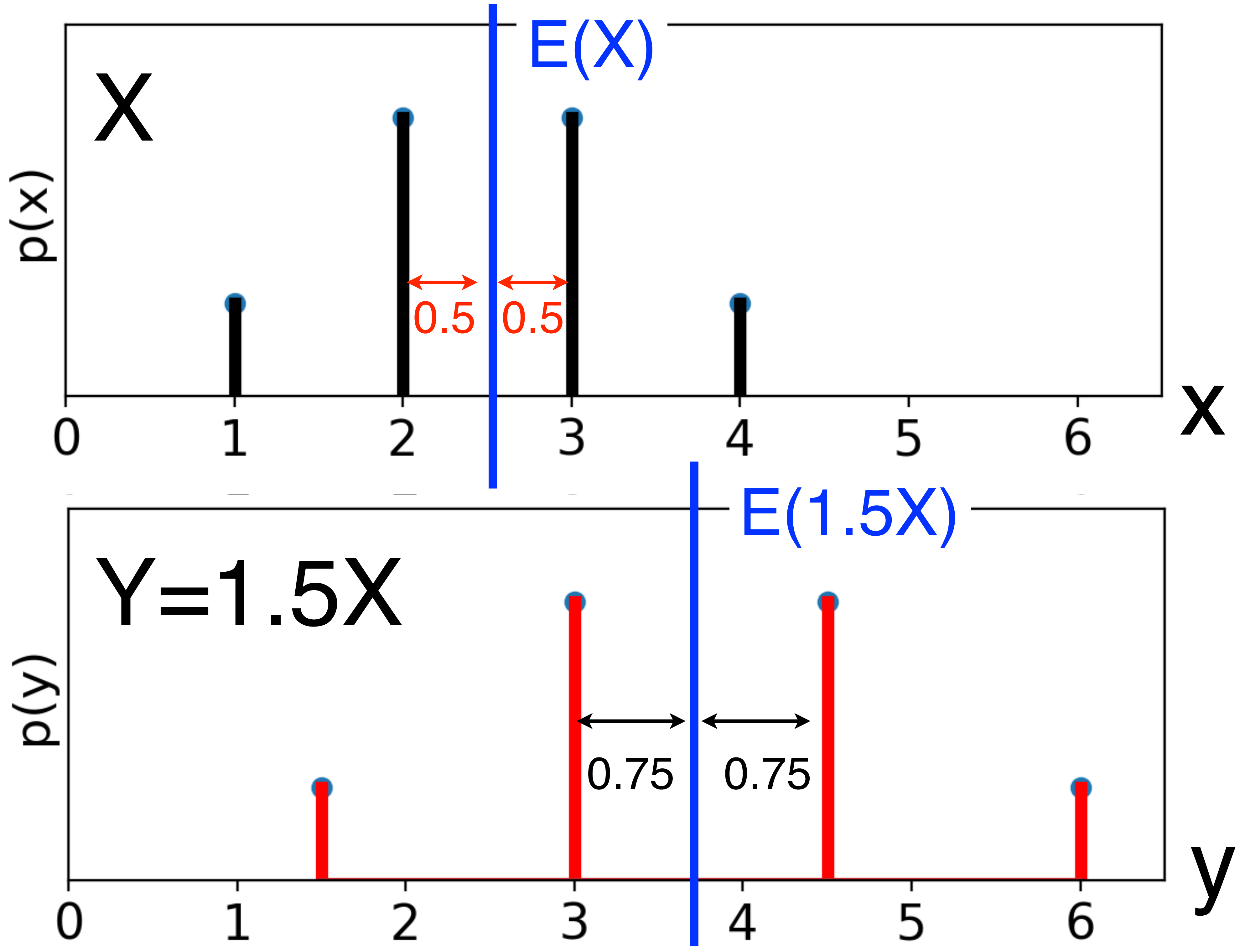
$$= E(aX - a\mu_x)^2$$

$$= E[a^2(X - \mu_x)^2]$$

$$= a^2 E(X - \mu_x)^2$$

$$= a^2 V(X)$$

Difference from mean grew by a^2



$$\sigma_{aX} = \sqrt{V(aX)} = \sqrt{a^2 V(X)} = |a|\sigma_x$$

"Average" difference from mean grew by a factor of $|a|$

Affine Transformation

$$V(aX + b) = V(aX) = a^2V(X)$$

$$\sigma_{ax+b} = |a|\sigma_x$$

This Lecture: Variance

Next: Two Variables