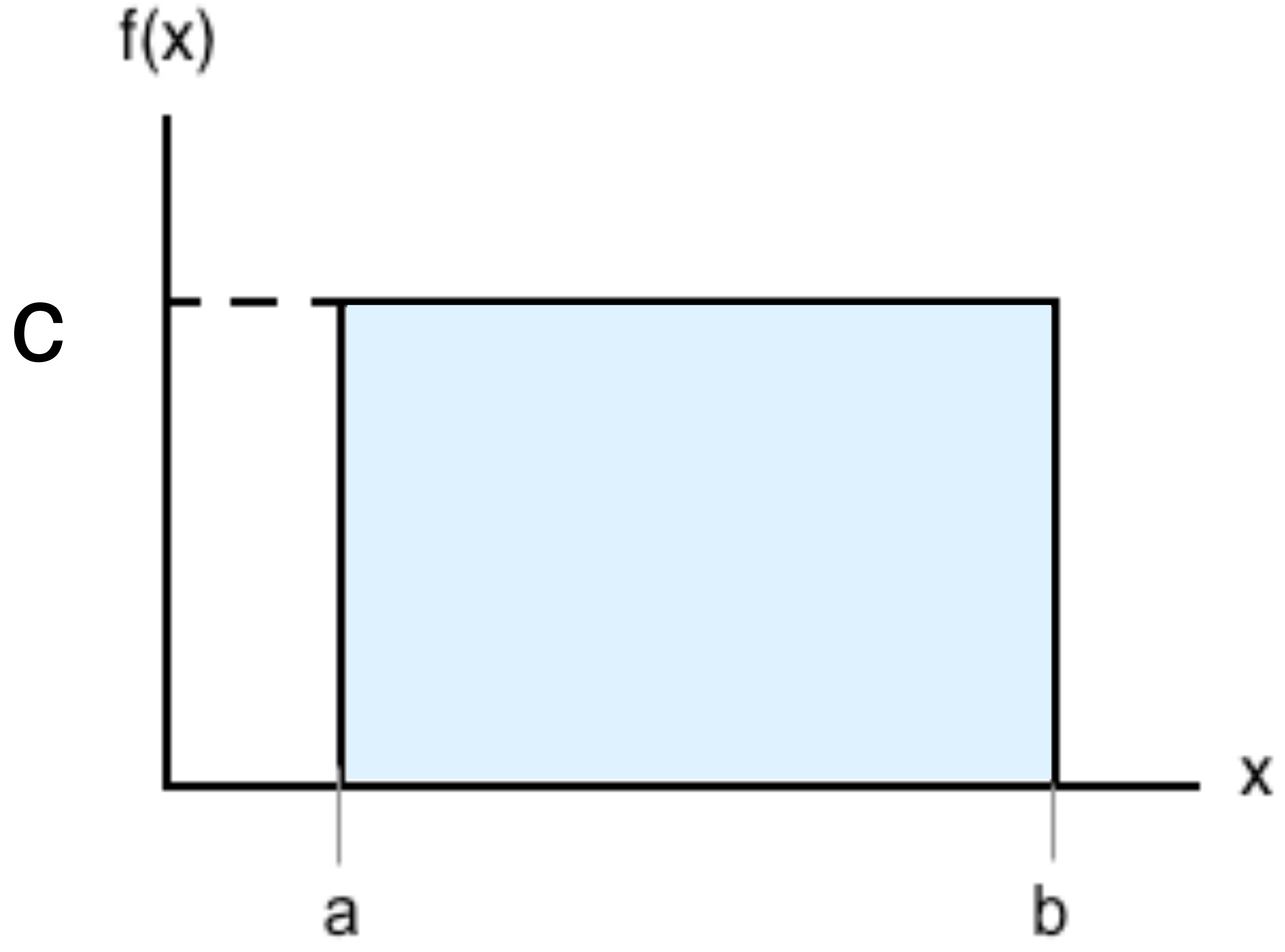

Uniform distribution

Definintuition

For $a<b$, the ***uniform*** distribution $U_{[a,b]}$ is constant inside $[a,b]$ and 0 outside

$$f(x) = \begin{cases} c & x \in [a,b] \quad \text{equally likely} \\ 0 & x \notin [a,b] \quad \text{never happen} \end{cases}$$



For $(\alpha, \beta) \subseteq [a,b]$

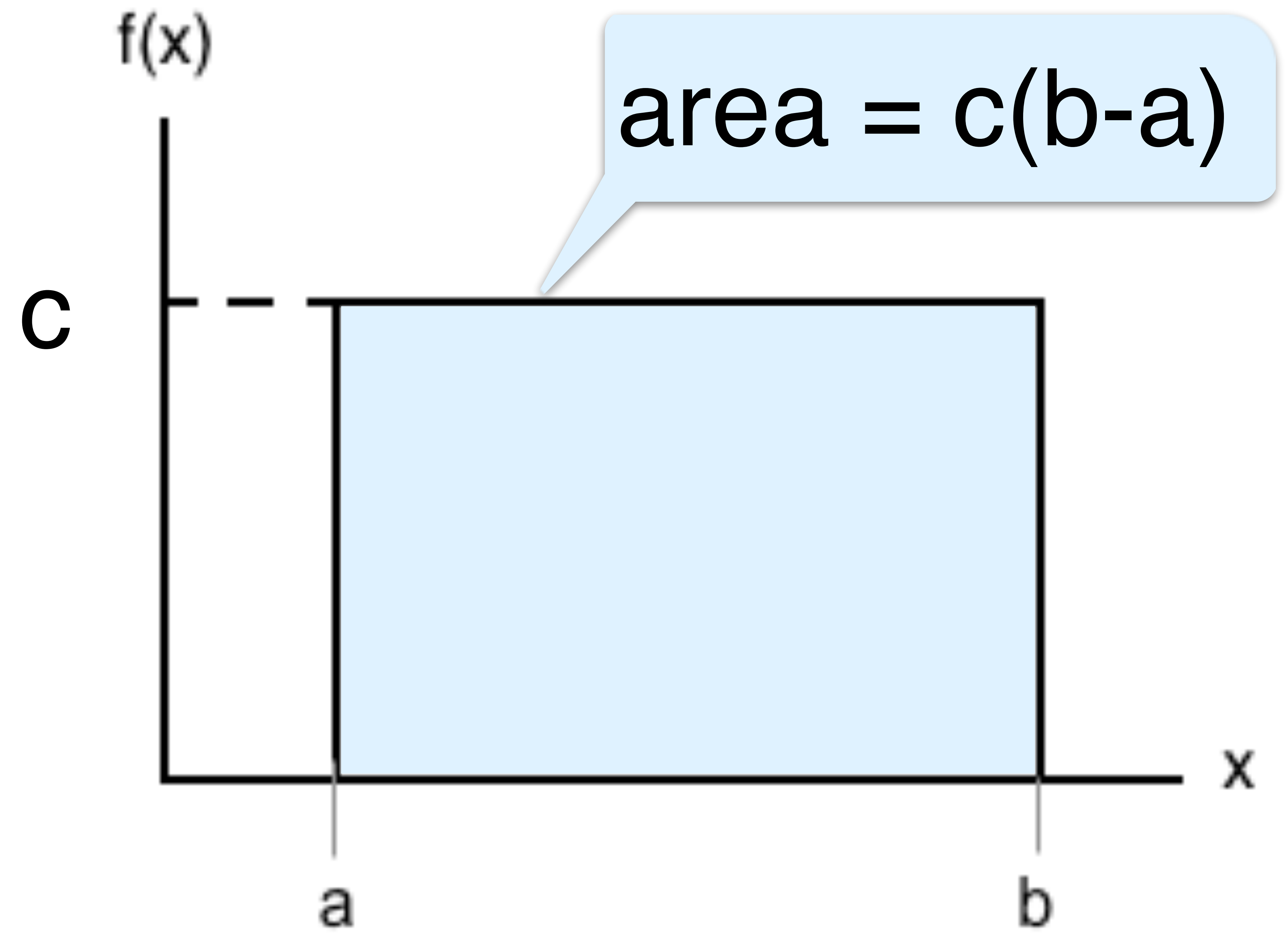
- Probability determined by, and is $\propto$ to, length $\beta-\alpha$
- Area under curve is always a rectangle
- Integrals are just height x width

c?

$$f(x) = \begin{cases} c = \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x)dx = c(b-a)$$

$$c = \frac{1}{b-a}$$



ΣWill It ADD?

YES IT ADDS!

Who’s Uniform

Departure times

Wait time for a bus

Location of chip defect

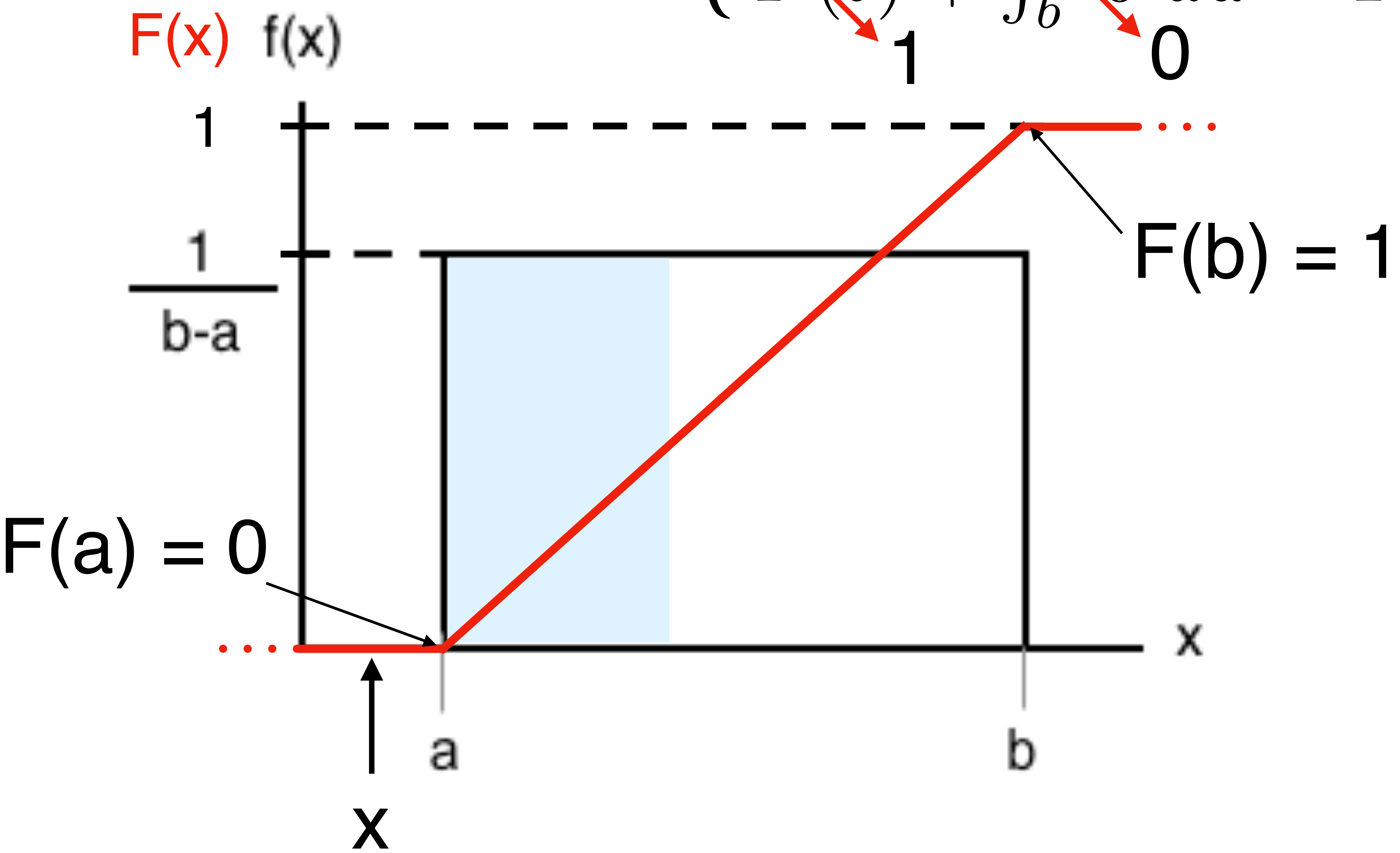
Location of a molecule in space

Considering a small area in time or space

Not so many…

CDF

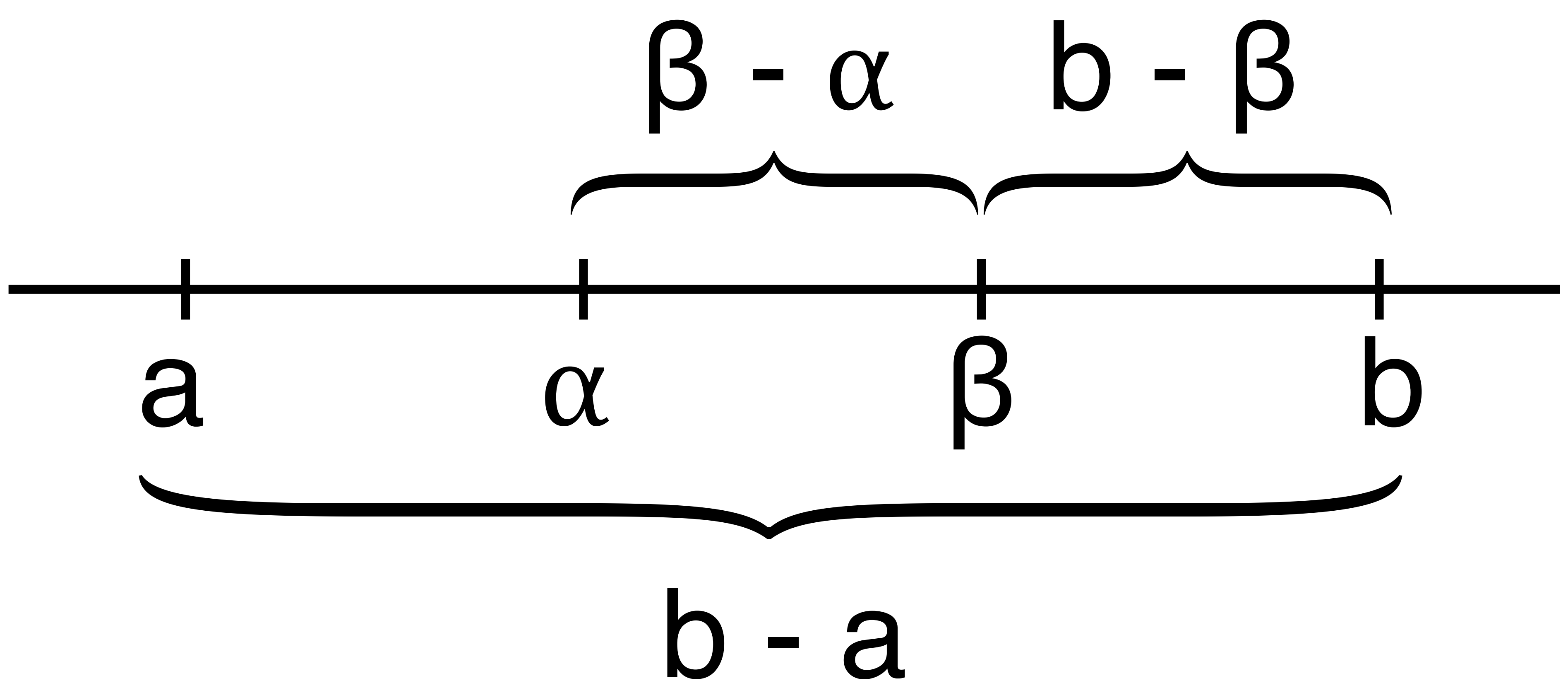
$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} \int_{-\infty}^{x} 0 \; du = 0 & x \leq a \\ \cancelto{0}{F(a)} + \int_a^x \frac{1}{b-a} du = \frac{x-a}{b-a} & a \leq x \leq b \\ \cancelto{1}{F(b)} + \cancelto{0}{\int_b^x 0 \; du} = 1 & x \geq b \end{cases}$$



Interval Probabilities

For a ≤ α ≤ β ≤ b

Interval	Probability
$(\alpha, \beta]$	$F(\beta) - F(\alpha) = \frac{\beta-a}{b-a} - \frac{\alpha-a}{b-a} = \frac{\beta-\alpha}{b-a}$
$[\beta, \infty)$	$F(\infty) - F(\beta) = 1 - \frac{\beta-a}{b-a} = \frac{b-\beta}{b-a}$
$\{\alpha\}$	$F(\alpha) - F(\alpha) = 0$



μ & σ

$X \sim U_{[0,1]}$ first $f(x) = 1, \quad 0 \le x \le 1$

$EX = \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$ Also by symmetry

$EX^2 = \int_0^1 x^2 \, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$V(X) = EX^2 - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$\sigma = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}} \approx 0.29$

Translation and Scaling

Uniformity preserved under translation and scaling

$X \sim U_{[0,\,1]}$ wolog $(a \neq 0)$

For any constants $a > 0$ and b, $Y \stackrel{\text{def}}{=} aX + b \sim U_{[b,\, a+b]}$

Range $0 \to b$ $1 \to a+b$

PDF $Y = aX + b \stackrel{\text{def}}{=} g(X)$

$$f_Y(y) = \left.\frac{f_X(x)}{|g'(x)|}\right|_{x=g^{-1}(y)} = \frac{1}{a}$$

Or: equal-length interval map to qual-length intervals

General μ & σ

$Y \sim U_{[a, b]}$

$Y = (b - a)X + a$

$EX = \frac{1}{2}$

$$EY = (b-a)EX + a = \frac{b-a}{2} + a = \frac{a+b}{2}$$

$V(X) = \frac{1}{12}$

$$V(Y) = V[(b-a)X + a] = (b-a)^2 V(X) = \frac{(b-a)^2}{12}$$

$$\sigma = \frac{b-a}{2\sqrt{3}} \approx 0.29(b-a)$$

Uniform Distributions

$U_{[a,b]}$ a<b

PDF $f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$

CDF $F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x \geq b \end{cases}$

$\mu = \frac{a+b}{2}$ $V = \frac{(b-a)^2}{12}$ $\sigma = \frac{b-a}{2\sqrt{3}}$

Exponential Distribution