

Expectation of Modified Variables

(Law of The Unconscious Statistician)

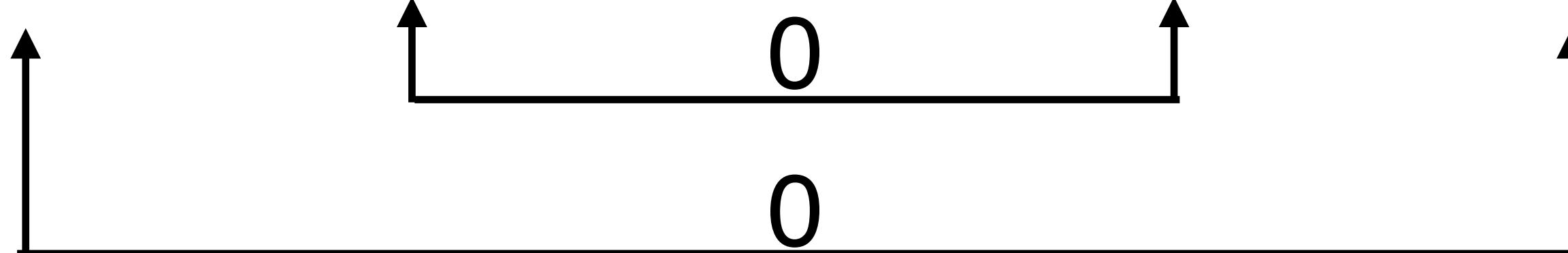


Expectation Reminder

x	-2	-1	0	1	2
p(x)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$$E(X) = \sum_x p(x) \cdot x$$

$$= -2 \cdot \frac{1}{5} + -1 \cdot \frac{1}{5} + 0 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{5} = 0$$



“By Symmetry”

Expectation of a Square

X	x	-2	-1	0	1	2
p(x)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	

$Y = X^2$

	y	0	1	4
p(y)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	

$$P(Y = 0) = P(X^2 = 0) = P(X = 0) = \frac{1}{5}$$

$$P(Y = 1) = P(X^2 = 1) = P(X \in \{-1, 1\}) = \frac{2}{5}$$

$$P(Y = 4) = P(X^2 = 4) = P(X \in \{-2, 2\}) = \frac{2}{5}$$

$$E(Y) = \frac{1}{5} \cdot 0 + \frac{2}{5} \cdot 1 + \frac{2}{5} \cdot 4 = \frac{10}{5} = 2$$

Alternative Formulation

$$\begin{aligned} E(Y) &= \sum_y y \cdot P(Y=y) \\ &= \sum_y y \cdot P(X \in g^{-1}(y)) \\ &= \sum_y y \sum_{x \in g^{-1}(y)} p(x) \\ &= \sum_y \sum_{x \in g^{-1}(y)} y \cdot p(x) \\ &= \sum_y \sum_{x \in g^{-1}(y)} g(x) \cdot p(x) \\ &= \sum_x g(x) \cdot p(x) \end{aligned}$$



Example

Visualize

Square Again

X	x	-2	-1	0	1	2
X	p(x)	1/5	1/5	1/5	1/5	1/5

$Y = X^2$

	y	0	1	4
	p(y)	1/5	2/5	2/5

$$E(Y) = \sum_{y=0,1,4} y \cdot p(Y=y) = \frac{1}{5} \cdot 0 + \frac{2}{5} \cdot 1 + \frac{2}{5} \cdot 4 = 10/5 = 2$$

$$\begin{aligned} E(Y) &= \sum_x x^2 \cdot p(x) \\ &= (-2)^2 \cdot \frac{1}{5} + (-1)^2 \cdot \frac{1}{5} + 0^2 \cdot \frac{1}{5} + 1^2 \cdot \frac{1}{5} + 2^2 \cdot \frac{1}{5} \\ &= \frac{4}{5} + \frac{1}{5} + \frac{1}{5} + \frac{4}{5} = 2 \end{aligned}$$

Visualization

$$E(Y) = \sum_y y \cdot P(Y=y)$$

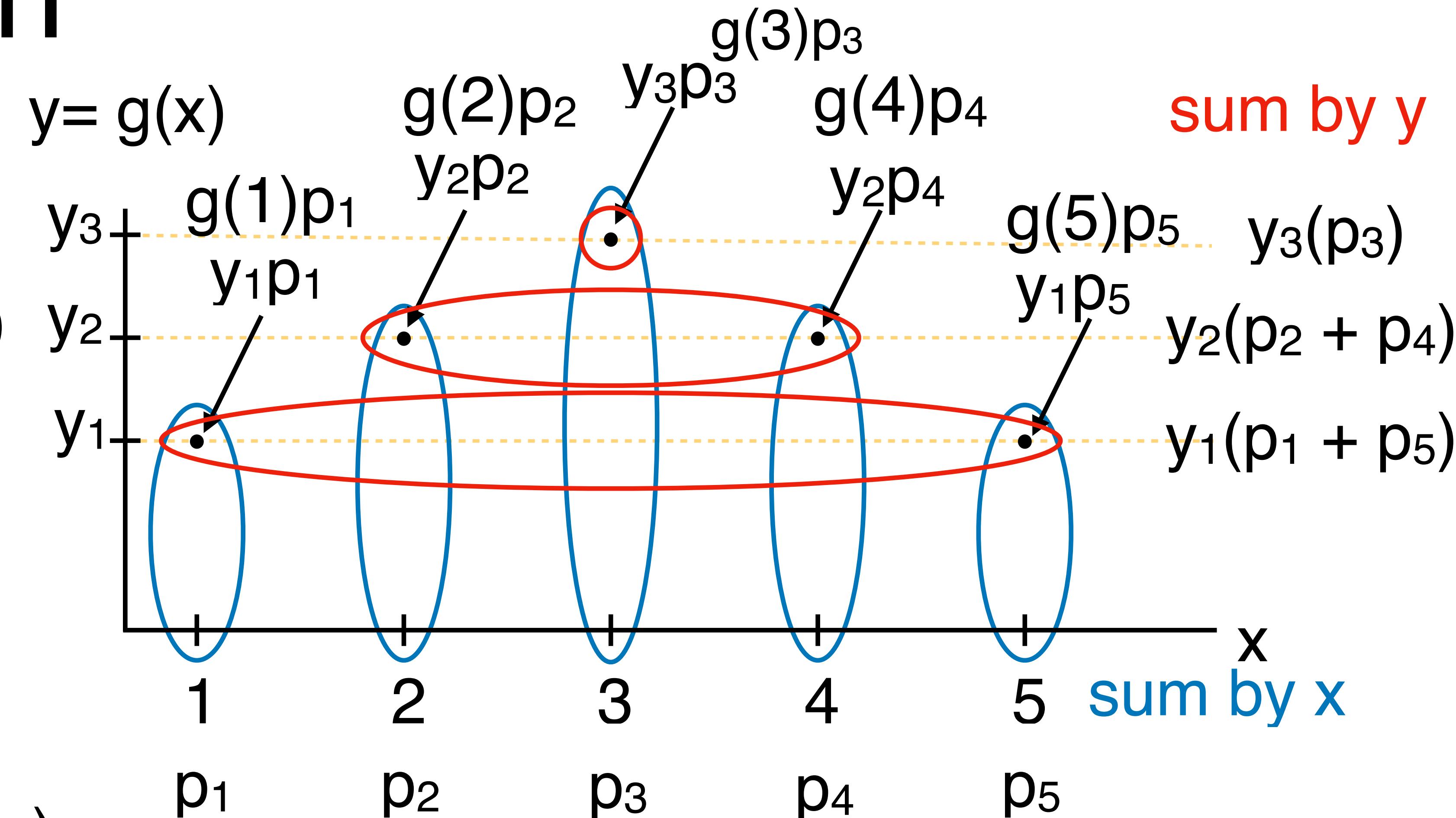
$$= \sum_y y \cdot P(X \in g^{-1}(y))$$

$$= \sum_y y \sum_{x \in g^{-1}(y)} p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} y \cdot p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} g(x) \cdot p(x)$$

$$= \sum_x g(x) \cdot p(x)$$



y : Fewer multiplications

x : Properties (next)

General Formulas

Constant Addition

$$E(X + b)$$

$$= \sum p(x) \cdot (x + b)$$

$$= \sum p(x) \cdot x + \sum p(x) \cdot b$$

$$= E(X) + b \cdot \sum p(x)$$

$$= E(X) + b$$

x	0	1
p(x)	1 - p	p

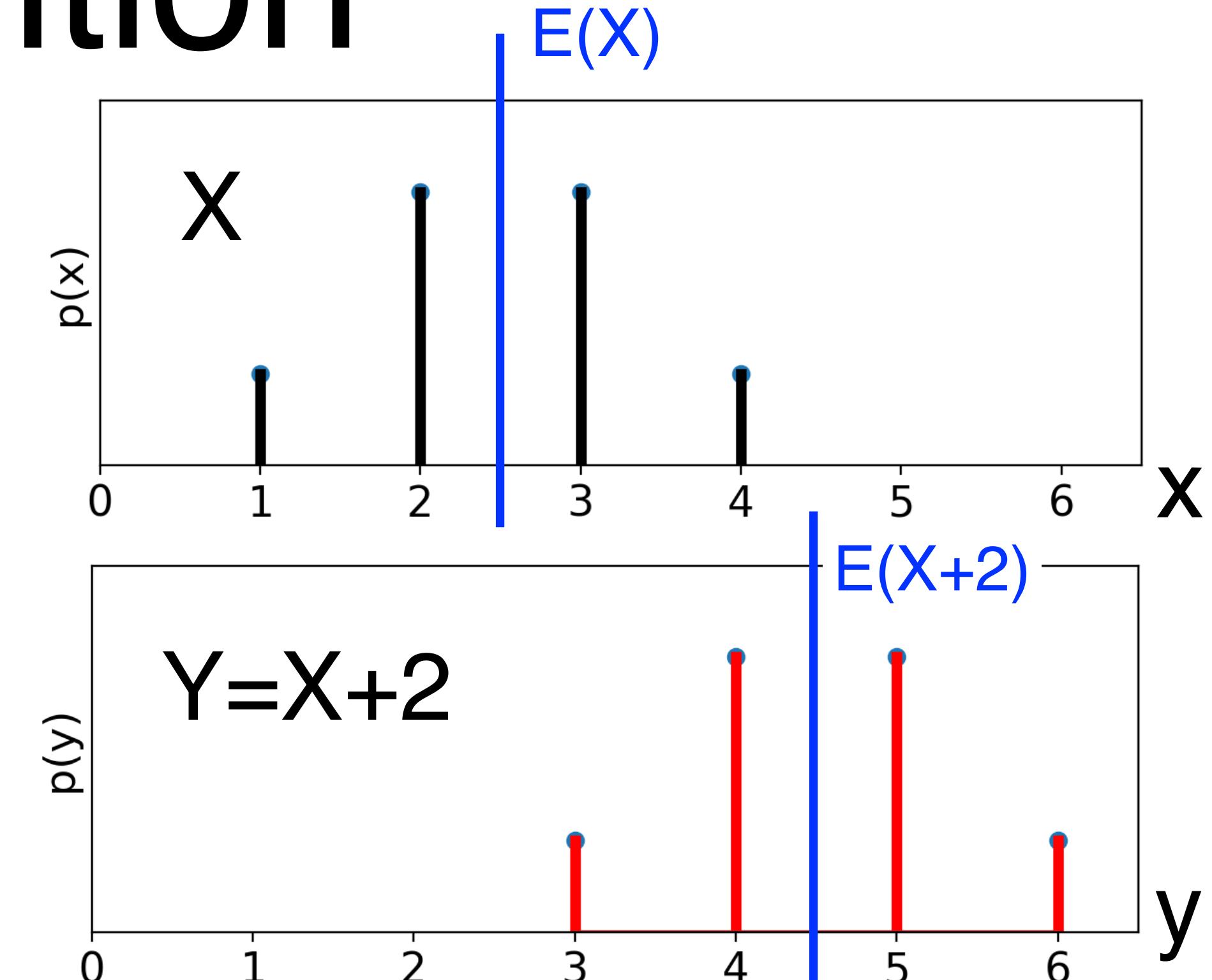
Bernoulli p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

$$E(X + 2) = (1 - p) \cdot (0 + 2) + p \cdot (1 + 2)$$

$$= 2 - 2p + 3p$$

$$= p + 2 = E(X) + 2$$



Constant Multiplication

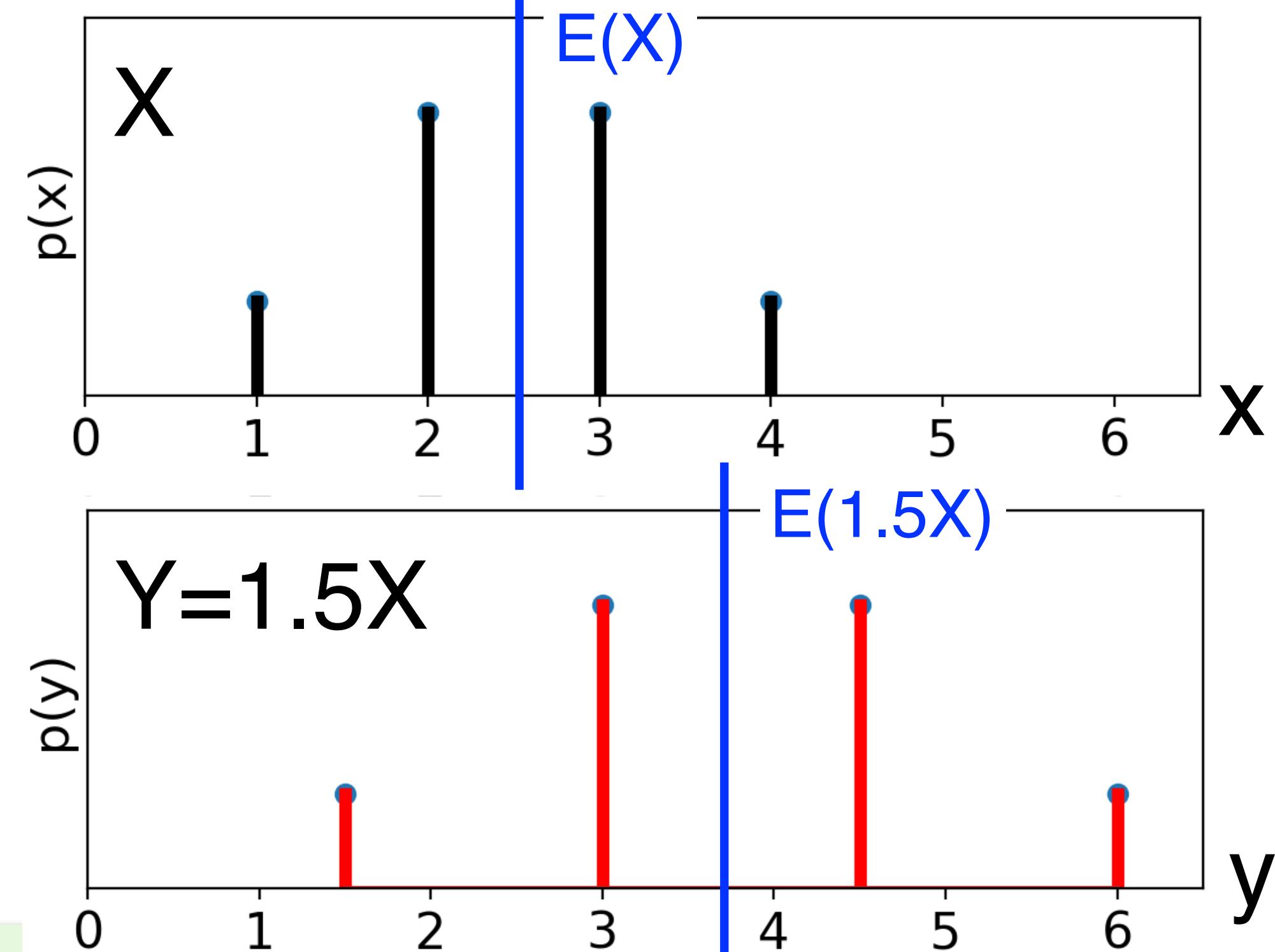
$$\begin{aligned} E(aX) &= \sum p(x) \cdot (ax) \\ &= a \sum p(x) \cdot x \\ &= aE(X) \end{aligned}$$

Bernoulli p

x	0	1
p(x)	1 - p	p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

$$\begin{aligned} E(3X) &= (1 - p) \cdot (3 \cdot 0) + p \cdot (3 \cdot 1) \\ &= 3p = 3E(X) \end{aligned}$$



Linearity of Expectation

$$E(aX + b) = E(aX) + b$$

$$= a E(X) + b$$

Bernoulli p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

x	0	1
p(x)	1 - p	p

$$E(2X + 3) = (1 - p)(2 \cdot 0 + 3) + p(2 \cdot 1 + 3)$$

$$= 3 - 3p + 5p$$

$$= 2p + 3 = 2E(X) + 3$$

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