Linearity of Expectation

Functions of Two Variables

$(X, Y) \sim p(x, y)$ over $\mathbb{R} \times \mathbb{R}$

$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$g(X, Y)$ New random variable

$$\Pr(g(X, Y) = z) \quad = \Pr((X, Y) \in g^{-1}(z)) \quad = \sum_{(x,y) \in g^{-1}(z)} p(x, y)$$

Two Unconscious Statisticians

$$Eg(X) = \sum_z z \cdot P(g(x) = z)$$

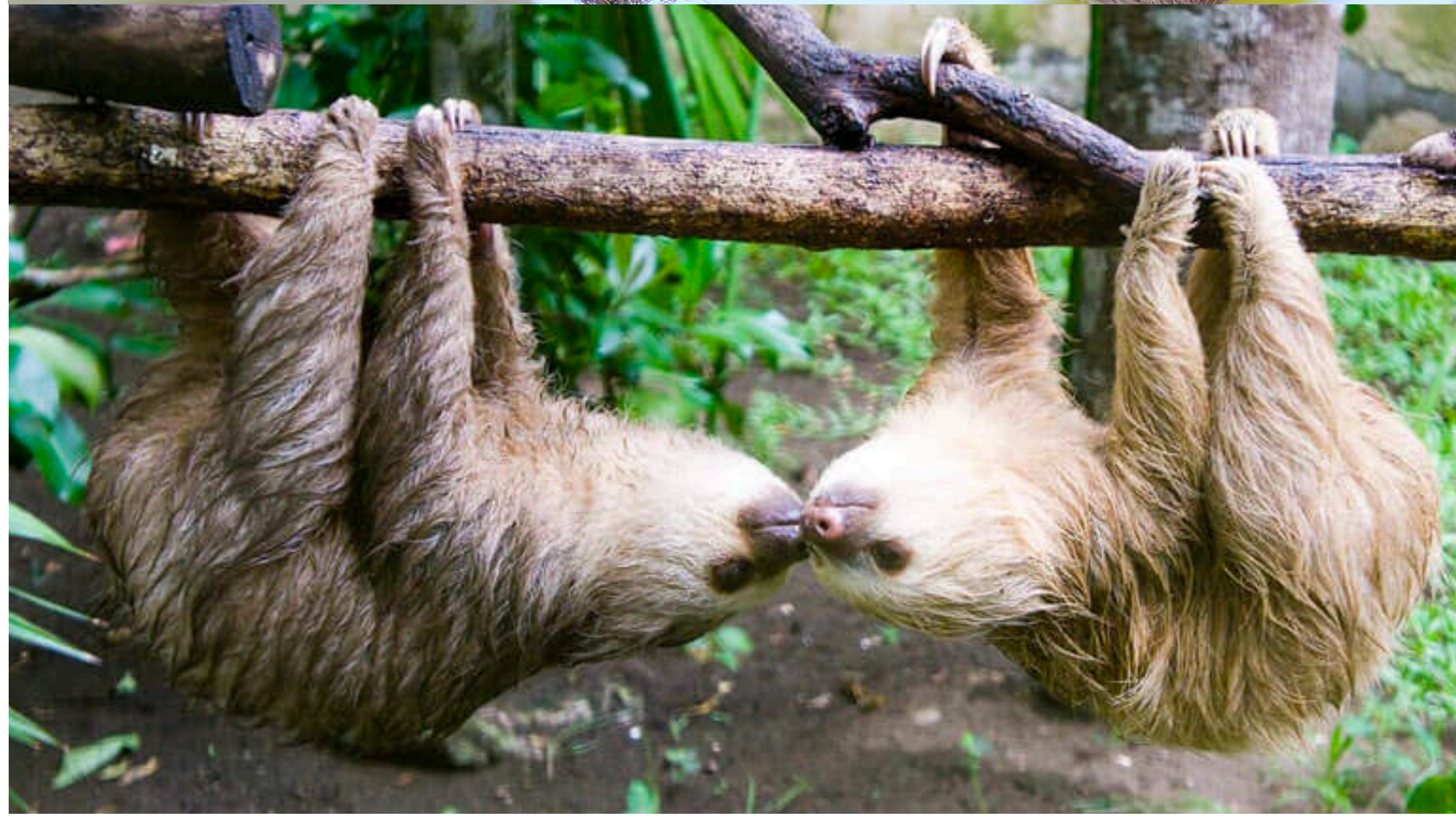
$$= \sum_z z \sum_{x \in g^{-1}(z)} p(x)$$

$$= \sum_z \sum_{x \in g^{-1}(z)} z \cdot p(x)$$

$$= \sum_z \sum_{x \in g^{-1}(z)} g(x)p(x)$$

$$= \sum_x g(x)p(x)$$

$$Eg(X,Y) =$$



$$X \to X, Y$$

$$x \to x, y$$

$$= \sum_{x,y} g(x,y)p(x,y)$$

Even proof is 

Linearity of Expectation

$$
\begin{aligned}
E(X+Y) &= \sum_x \sum_y (x+y) \cdot p(x,y) \\
&= \sum_x \sum_y x \cdot p(x,y) + \sum_x \sum_y y \cdot p(x,y) \\
&= \sum_x x \sum_y p(x,y) + \sum_y y \sum_x p(x,y) \\
&= \sum_x x \cdot p(x) + \sum_y y \cdot p(y) \\
&= EX + EY
\end{aligned}
$$

Expectation of sum =
sum of expectations

	0	1	3		y
1	.05	.15	.30		.5
3	.15	.25	.10		.5

x	.2	.4	.4

$E\,Y = .5 \cdot 1 + .5 \cdot 3 = 2$

$E\,X = .2 \cdot 0 + .4 \cdot 2 + .4 \cdot 4 =$

Variance

Expectations add $E(X+Y) = EX + EY$

Do variances? $V(X+Y) \stackrel{?}{=} V(X) + V(Y)$

$$\begin{aligned} V(X+Y) &= E(X+Y)^2 - (E(X+Y))^2 \\ &= E(X^2 + 2XY + Y^2) - (EX + EY)^2 \\ &= EX^2 + 2E(XY) + EY^2 - (E^2X + 2EX \cdot EY + E^2Y) \\ &= EX^2 - E^2X + EY^2 - E^2Y + 2(E(XY) - EX \cdot EY) \\ &= V(X) + V(Y) + 2(E(XY) - EX \cdot EY) \end{aligned}$$

$$E(XY) = EX \cdot EY?$$

Do expectations multiply?

Different Formula

$V(X)$ $= E(X - \mu)^2$

$= E(X^2 - 2\mu X + \mu^2)$

$= E(X^2) - E(2\mu X) + E(\mu^2)$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$

$= E(X^2) - (E\,X)^2$

$E(X)=\mu$

$2, \mu$ - constants

Bernoulli p Again

$X \sim B(p)$

Recall: $EX = p$ $V(X)=pq$

Re-derive using $V(X) = E\,X^2 - (EX)^2$

$E(X^2) = (1 - p) \cdot 0^2 + p \cdot 1^2 = p$

Even simpler $0^2=0,\ 1^2=1$ $\rightarrow$ $X^2=X$ $\rightarrow$ $EX^2 = EX = p$

$V(X) = E\,X^2 - (EX)^2 = p - p^2 = p(1 - p) = pq$ ✅

The Hat Problem

$\mathbb{1}_i$ - indicator function i^{th} student caught their own hat

H - # students who caught their own hat

$$H = \sum_{i=1}^{n} \mathbb{1}_i$$

$\mathbb{1}_i$ - Bernoulli

	H_1	H_2	H_3	H
1 2 3	1	1	1	3
1 3 2	1	0	0	1
2 1 3	0	0	1	1
2 3 1	0	0	0	0
3 1 2	0	0	0	0
3 2 1	0	1	0	1

$$P(\mathbb{1}_i = 1) = \frac{\#\text{ permutations of } (\sigma_1, \ldots, \sigma_n) \text{ when } \sigma_i = i}{\#\text{ permutations of } (\sigma_1, \ldots, \sigma_n)} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$E(\mathbb{1}_i) = P(\mathbb{1}_i = 1) = \frac{1}{n}$$

$$E(H) = E\left(\sum_{i=1}^{n} \mathbb{1}_i\right) = \sum_{i=1}^{n} E(\mathbb{1}_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Do variances add?

Expectations add   Do variances?

$V(X+Y) = V(X) + V(Y)$?

$Y = -X$ $V(X+Y) = V(X-X) = V(0) = 0$ $\neq V(X) + V(Y)$

$Y = X$ $V(X+Y) = V(X+X) = V(2X) = 4V(X)$

Next

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