Concentration Inequalities Intro

Markov's Inequality: For any nonnegative random variable X and $t > 0$,

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Chebyshev's Inequality: For any random variable X and $c > 0$,

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq c] \leq \frac{\text{Var}(X)}{c^2}.$$

Law of Large Numbers: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. We have the following:

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \mu$$

$$\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}.$$

Applying Chebyshev's inequality on the sample mean $\frac{1}{n}\sum_{i=1}^{n} X_i$, we have that

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right] \leq \frac{\sigma^2}{n\varepsilon^2}$$

which means that as $n \to \infty$, the probability of the sample mean deviating from the true mean by any $\varepsilon > 0$ approaches zero.

1 Probabilistic Bounds

Note 17

A random variable X has variance $\mathrm{Var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of X is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

(a) $\mathbb{E}[X^2] = 13$.

(b) $\mathbb{P}[X = 2] > 0$.

(c) $\mathbb{P}[X \geq 2] = \mathbb{P}[X \leq 2]$. (Feel free to skip the variance computations for this subpart.)

For the below parts, you should use Markov's and Chebyshev's inequalities to provide bounds on the probabilities. Remember that Markov's inequality requires a nonnegative random variable Y, and Chebyshev's inequality provides a bound on the absolute deviation from the mean $|X - \mu|$.

(d) $\mathbb{P}[X \leq 1] \leq 8/9$.

(e) $\mathbb{P}[X \geq 6] \leq 9/16$.

2 Vegas

Note 17

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip their coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Let X be the proportion of coin flips which are heads. Find $\mathbb{E}[X]$.

(b) Given the results of your experiment, how should you estimate p? (*Hint:* Construct an unbiased estimator for p using part (a). Recall that $\hat{p}$ is an unbiased estimator if $\mathbb{E}[\hat{p}] = p$.)

(c) How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

3 Working with the Law of Large Numbers

Note 17

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.