Linear Algebra

* **Vectors in R^n** :- Let R be the set of all real numbers and n is a fixed +ve integer ≥ 2, then

$$R^n = \{(x_1, x_2, \ldots x_n) \mid x_i \in R\}$$ is the set of all ordered n-tuples of real numbers.

If $x = (x_1, x_2 \ldots, x_n) \in R^n$, then x is called a point or a row-vector in R^n-space, which is also denoted by $x = (x_1\ x_2 \ldots x_n)$ or $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and in later case, it is called column-vector.

Each component in a vector $x = (x_1, x_2 \ldots x_n)$ is a real number and is called a scalar. Two vectors $x = (x_1, x_2 \ldots x_n)$ and $y = (y_1, y_2 \ldots y_n)$ in R^n are said to be equal, written as $x = y$, if $x_i = y_i$, $\forall\ i = 1, 2, \ldots n$.

$(0, 0 \ldots 0)$ is a vector in R^n and is called the zero vector, which is denoted by $\bar{0}$ or 0.

The sum of two vectors, $x = (x_1, x_2 \ldots x_n)$ and $y = (y_1, y_2 \ldots y_n)$ in R^n is denoted and defined as

$x + y = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$ which is also a unique vector in R^n. So, R^n is closed under vector addition.

The product of a scalar K in R and a vector $x = (x_1, x_2 \ldots x_n)$ in R^n is denoted and defined as

$Kx = (Kx_1, Kx_2, \ldots Kx_n)$, which is also a unique vector in R^n. R^n is closed under scalar multiplication.

For any vectors $x, y, z \in R^n$ and scalars α, β, in R, it is easy to see that -

(i) $(x+y)+z = x+(y+z)$.

(ii) $x+0 = x = 0+x$.

(iii) $-x+x=0=x+(-x)$

(iv) $x+y=y+x$.

(v) $\alpha(x+y)=\alpha x+\alpha y$.

(vi) $(\alpha+\beta)x=\alpha x+\beta x$.

(vii) $(\alpha\beta)x=\alpha(\beta x)$.

(viii) $1x=x$.

The set $\mathbb{R}^n$ in which all these (viii) conditions are satisfied is called a vector-space or linear-space of dimension n.

If W is a non-empty subset of $\mathbb{R}^n$ and W is also a vector-space under the two operations as defined above restricted to the vectors in W, then W is called a linear subspace in $\mathbb{R}^n$.

Note:- i) $-y=(-1)y=(-y_1,-y_2,\ldots,-y_n)$

ii) $x-y=x+(-y)=(x_1-y_1,x_2-y_2,\ldots x_n-y_n)$

iii) $0x=(0x_1,0x_2,\ldots 0x_n)$
$=(0,0,\ldots 0)$
$=0$.

* Linear combinations of vectors and linear span of a set :-

If $v_1,v_2,\ldots,v_m$ are m vectors in $\mathbb{R}^n$ and $\alpha_1,\alpha_2,\ldots,\alpha_m$ are m scalars in $\mathbb{R}$, then a vector of the form $\alpha_1v_1+\alpha_2v_2+\ldots+\alpha_mv_m$, is called a linear combination of vectors $v_1,v_2\ldots v_m$.

If $S=\{v_1,v_2\ldots v_m\}\subseteq\mathbb{R}^n$, then the set of all linear combinations of vectors in S is denoted by $L(S)$ and it is called the linear span of S, and this $L(S)$ is a linear subspace of $\mathbb{R}^n$, called sub-space generated by S.

If $L(S)=\mathbb{R}^n$, then S is called a generating system of $\mathbb{R}^n$.

* Linearly Dependent (L.D.) and Linearly Independent (L.I.) vectors:-

A finite no. of vectors $V_1, V_2 \dots V_m$ in $\mathbb{R}^n$ are said to be L.D. if there exists scalars $\alpha_1, \alpha_2 \dots \alpha_m$ in $\mathbb{R}$, not all zero, i.e., atleast one $\alpha_i \neq 0$, such that $\alpha_1 V_1 + \alpha_2 V_2 + \dots \alpha_m V_m = 0$.

Also, $V_1, V_2 \dots V_m$ in $\mathbb{R}^n$ are said to be L.I. if they are not L.D., i.e., if $\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_m V_m = 0$ where $\alpha_i \in \mathbb{R}$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 \dots = \alpha_m = 0.$$

If $S = \{V_1, V_2, \dots, V_m\} \subseteq \mathbb{R}^n$, then S is said to be L.D. or L.I. according as $V_1, V_2 \dots V_m$ are L.D. or L.I., respectively.

E.g. → (1) in $\mathbb{R}^3$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ are L.I.

(2) in $\mathbb{R}^3$, ~~eg~~ $u = (1,0,-2)$, $v = (2,1,0)$, $w = (-1,-1,-2)$ are L.D.

Justification:-

(1) Let $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$

$$\Rightarrow \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1) = (0,0,0)$$

$$\Rightarrow (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$, So, e_1, e_2, e_3 are L.I.

(2) $u - v = (-1,-1,-2)$
$= w$

$u - v - w = 0$, where coefficients are non-zero.

So, u, v, w are L.D.

Example:-7. Prove that the four vectors $(1,0,0), (0,1,0), (0,0,1)$ and $(1,1,1)$ are L.D., but any three of them are L.I.

→ Let $V_1(1,0;0)$, $V_2 = (0,1,0)$, $V_3 = (0,0,1)$, $V_4 = (1,1,1)$

$$\therefore V_1 + V_2 + V_3 = (1,1,1)$$
$$= V_4 .$$

$$\therefore V_1 + V_2 + V_3 + (-1) V_4 = 0 .$$

So, V_1, V_2, V_3, V_4 are L.D.

(i) $aV_1 + bV_2 + cV_3 = 0$

$(a,b,c) = (0,0,0)$

$a = b = c = 0.$

$\therefore V_1, V_2, V_3$ are L.I.

(ii) $aV_1 + bV_2 + cV_4 = 0$

$a(1,0,0) + b(0,1,0) + c(1,1,1) = (0,0,0)$

$\Rightarrow (a+c,\ b+c,\ c) = (0,0,0)$

$a+c=0,\quad b+c=0,\quad c=0$

$a=0,\quad b=0,\quad c=0.$

So, V_1, V_2 and V_4 are L.I.

(iii) $aV_1 + bV_3 + cV_4 = 0.$

$a(1,0,0) + b(0,0,1) + c(1,1,1) = 0$

$\Rightarrow (a+c, c; b+c) = 0$

$a+c=0,\quad c=0,\quad b+c=0$

$a = b = c = 0.$

So, V_1, V_3 and V_4 are L.I.

(iv) $aV_2 + bV_3 + cV_4 = 0$

$a(0,1,0) + b(0,0,1) + c(1,1,1) = 0$

$\Rightarrow (c, a+c, b+c) = 0$

$c = 0, \quad a+c = 0, \quad b+c = 0$

$a = b = c = 0.$

So, V_2, V_3 and V_4 are L.I.

2.* Linear Span :-

Let $S = \{V_1, V_2, \ldots, V_n\} \subseteq \mathbb{R}^n$, then the set of all linear combinations of vectors in S is denoted by $L(S)$ and is called the linear span of S.

If $L(S) = W$, then W is called linear subspace of $\mathbb{R}^n$ and is called subspace generated by S or S is said to be a generating system of W.

If $L(S) = \mathbb{R}^n$, then S is called generating system of $\mathbb{R}^n$.

If $L(S) = \mathbb{R}^n$ and S is L.I., then S is called a basis of $\mathbb{R}^n$; i.e., a linearly independent generating system of $\mathbb{R}^n$ is called a basis of $\mathbb{R}^n$.

e.g. $S = \{e_1 = (1,0,0, \ldots, 0), e_2 = (0,1,0, \ldots, 0), \ldots, e_n = (0,0,0, \ldots, 1)\}$ is a basis of $\mathbb{R}^n$, called standard or canonical basis of $\mathbb{R}^n$.

The number of vectors in any basis of $\mathbb{R}^n$ is fixed and it is n and is called dimension of $\mathbb{R}^n$.

Note:- 1. Maximum number L.I. vectors in $\mathbb{R}^n$ is n.

2. any set containing zero vectors is always L.D.

e.g. → if $S = \{0, V_1, V_2, \ldots, V_m\} \subseteq \mathbb{R}^n$, then

$\alpha 0 + 0V_1 + 0V_2 + \ldots + 0V_m = 0$, where $0 \neq \alpha \in \mathbb{R}$

$\therefore S$ is L.D.

* Rank of a matrix :- Rank of a non-zero matrix A is defined as the order of the largest non-singular square sub-matrix of A, and it is denoted by rank(A) or $\rho(A)$.

The rank of a zero-matrix (null-matrix) is defined to be 0.

e.g. (i) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$ is a square sub-matrix of A

Also, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

$\therefore$ rank(A) = 2.

(ii) If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}$, then there are ${}^4C_2 = 6$ no. of square sub-matrices of A. & clearly all are singular

$\therefore$ rank(A) = 1.

Note :- 1. Rank of a matrix is a non-negative integer and rank of a non-zero matrix is positive.

2. Rank of a non-singular square matrix of order n is n, and in particular, the rank of a unit matrix of order n is n.

1. Find the rank of the matrix :-

(a)
$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{pmatrix}$$

The are ${}^4C_3 = 4$ nos of sub-matrices (square) of A of order 3.

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ -1 & 0 & -2 \end{vmatrix} = -8 + 12 \cancel{+} - 4 = 0. \qquad \begin{vmatrix} 1 & -1 & 3 \\ 3 & 0 & -1 \\ -1 & -2 & 7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \\ -1 & 0 & 7 \end{vmatrix} = 28 - 4\cancel{2}4 + 12 = 0. \qquad \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 0 & -2 & 7 \end{vmatrix} = 0$$

$\therefore$ All the 3X3 square sub-matrices of A are singular

Now, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a sq. sub-matrix of A of order 2 and $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \neq 0$

$\therefore$ rank$(A) = 2$.

* **Elementary operations or Elementary transformations:-**

The following operations when applied to a matrix are called elementary row operations:-

1. Interchange of any two row. $\therefore R_i \leftrightarrow R_j$
2. Multiplication of every element of a row by a scalar K (non-zero)

   $\therefore R_i \rightarrow KR_i$.
3. Addition to the elements of a row, a constant multiple of corresponding elements of another row.

   $R_i \rightarrow R_i + KR_j$.

Similarly, elementary column operations are defined which are denoted as:-

$C_i \leftrightarrow C_j$; $C_i \rightarrow KC_i$; $C_i = C_i + KC_j$.

Theorem:- Elementary operations do not alter the rank of a matrix.

* **Echelon matrix or Echelon form of a matrix:-**

A matrix is said to be in Echelon form if the no. of zeroes preceding the first non-zero entries of any row increases row by row, until only zero rows remain (if zero row exist).

The first non-zero entries of the rows of an Echelon matrix are called distinguished elements and if each of them is unity and also it is the only non-zero element in the corresponding column, then the matrix is said to be a row-reduced Echelon matrix.

e.g.— (i) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 5 & -1 & 2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is an Echelon matrix.

(ii) $\begin{pmatrix} 0 & 2 & -3 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an Echelon matrix.

(iii) $\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a ~~redu~~ row-reduced Echelon matrix.

(iv) $\begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ is not an Echelon matrix.

* **Row space and column space of a matrix and rank of a matrix:-**

Let A be a $m \times n$ matrix over real numbers and let $R_1, R_2, \ldots, R_m$ be the m rows of A and $C_1, C_2, \ldots, C_n$ be the n-columns of A. Each row of A can be viewed as a vector in $\mathbb{R}^n$ and each column as a vector in $\mathbb{R}^m$.

The linear subspace of $\mathbb{R}^n$ generated by the rows $R_1, R_2, \ldots, R_m$ is called the row space of A and the linear subspace of $\mathbb{R}^m$ generated by the columns of $C_1, C_2, \ldots, C_n$ is called column space of A.

The dimension of the row-space or column ~~of~~-space of a matrix A are same, i.e., maximum number of L.I. rows and L.I. columns are same, and this number is called the rank of matrix A.

* **Theorem** :- The non-zero rows of a matrix in Echelon form are L.I. (and they constitute maximum no. of L.I. rows).

Note :- 1. To find the rank of a matrix (non-zero), we reduce it to Echelon form using elementary operations and then the rank of the matrix is equal to the no of non-zero rows in its Echelon form.

2. Every diagonal matrix, unit matrix, scalar matrix and upper triangular matrix are clearly in Echelon form.

* **Null space and nullity of a matrix** :-

Let A be a $m \times n$ matrix. The null space of A is the set of all column vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ such that $Ax = 0$. and is denoted by $N(A)$, i.e., $N(A)$ is the set of all solutions of homogenous linear equations, $Ax = 0$ and it is a linear subspace of $\mathbb{R}^n$.

The number of L.I. solutions of $Ax = 0$ is $(n-r)$, where r is the rank of A. and this $(n-r)$ is called the nullity of A.

1. Find the rank and nullity of the matrix:-

(i) $A = \begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$;

$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$ $\quad R_1 \longleftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$ $\quad R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

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$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix} \qquad \begin{array}{l} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 9R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 \to R_4 - 2R_3 ,$$

which is in Echelon form with 3 non-zero rows.

$\therefore$ rank $(A) = 3$.

no. of column is $n = 4$.

$\therefore$ nullity $(A) = 4 - 3 \quad (n - r)$

$= 1$.

<u>Note</u> :-
$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 \times \frac{1}{33} .$$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_2 \to R_2 + 6R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -5/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in row-reduced echelon form.

(ii) $A = \begin{bmatrix} -3 & 1 & 4 & -5 \\ 1 & 1 & 1 & 2 \\ -2 & 0 & 1 & -3 \\ 1 & 1 & -2 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ -3 & 1 & 4 & -5 \\ -2 & 0 & 1 & -3 \\ 1 & 1 & -2 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_2$.

$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 7 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 7 & 1 \\ 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & -3 & 3 \end{bmatrix}$ $R_3 \to R_3 - \frac{1}{2}R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 7 & 1 \\ 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_4 \to R_4 - 6R_3$

which is in Echelon form with 3-non-zero rows.

$\therefore$ rank$(A) = 3$.

nullity$(A) = 4 - 3 = 1$.

4

* System of Linear Equations :-

Consider the following m linear equation in n unknowns (variables) $x_1, x_2, \ldots, x_n$:

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m$$

, where a_{ij} and ~~b_{ij}~~ b_i are constants. If atleast one ~~b_i~~ $b_i \neq 0$, then the system is said to be a non-homogenous system of linear equation.

If $b_i = 0, \forall i = 1, 2, \ldots, m$, then the system is said to be a homogenous system of linear equation.

In matrix form, the above system can be written as :-

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix},$$

or

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

or ~~AXB~~ $AX = B$. (say).

where $A = (a_{ij})_{m\times n}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

A is called the coefficient matrix or associated matrix and

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

is called Augmented matrix of the system which is also denoted as $[AB]$ or $[A|B]$.

A system of equations $AX=B$ is said to be consistent or solvable if it has a solution, ~~i.e., if it has a solution,~~ i.e., if there exists values of the unknowns which satisfy all the equations of the system simultaneously otherwise the system is said to be inconsistent.

The homogenous system $AX=0$ is always consistent because $X=0$, i.e, $x_1=0$, $x_2=0$, ... is clearly a solution called zero solution or trivial solution and any other solution is called non-trivial solⁿ.

* **Theorem :-** The necessary and sufficient solution for a non-homogenous system of linear eqⁿ, $AX=B$ (in matrix form) to be consistent, i.e., to have a solution is that $\text{rank}(A) = \text{rank}([A:B])$.

Nature of solution:- Consider a system of m linear eqⁿ in n unknown given by $AX=B$ (in matrix form), then,

(i) If $\text{rank}(A) \neq \text{rank}([A:B])$, then the system is inconsistent, i.e., has no solⁿ.

(ii) If $\text{rank}(A) = \text{rank}([A:B]) = n$, = no. of unknowns, then the ~~sy~~ system is consistent and has a unique solution.

(iii) If $\text{rank}(A) = \text{rank}([A:B]) = r < n$, no. of unknowns, then the system is consistent and has an infinite no. of solutions but only $n-r+1$ of them are linearly independent, when $B \neq 0$ and if $B=0$, i.e., $AX=0$, then in this case, it has $(n-r)$ linearly independent solutions.

* ~~system of eq~~

* Solution of a system of n-linear eqⁿ in n-unknowns by matrix inversion method & Cramer's rule :-

Consider the following n linear equation in n-unknown in matrix form $AX=B$ ..... (i), where $A=(a_{ij})_{n\times n}$.

$$B=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \text{ and } X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Let $|A| \neq 0$ so that $A^{-1} = \frac{1}{A}\,\text{adj}(A)$ exists with the property

$$AA^{-1} = I = A^{-1}A$$

then, $A^{-1}AX = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow \boxed{X = A^{-1}B}$, which is the unique soln by matrix inversion method.

$X = A^{-1}B$

$X = \left(\frac{1}{A}\,adj(A)\right) B$.

Cramer's Rule:-

$x = \frac{\Delta x}{\Delta}$; $y = \frac{\Delta y}{\Delta}$; $z = \frac{\Delta z}{\Delta}$.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

* Gaussian Elimination Method :-

In this case, the given system of linear equations is first written in matrix form, and then applying elementary row operations ONLY, the coefficient matrix is reduced to echelon form, and the ranks of the coefficient matrix and augmented matrix are obtained and if they are equal, i.e., if the system is consistent, then the system is again written in individual form and then the unknowns are obtained by back substitutions. This method is general and can be applied when the numbers of equations and no. of unknowns are equal or unequal and also can be applied when matrix inversion method or Cramers rule fails.

Example:- A student wrongly copied a consistent system of linear equations as below:-

$x - 2y + z = 1$

$2x + y + z = 1$

$5y - z = 3$.

On checking it is found that the constant term in the 3rd eqn was wrongly copied while other parts are correct for the system. Determine the correct form of the system and solve it.

→ In matrix form, the given system is $AX = B$.

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

converting into Echelon form,

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \quad \ldots (i)$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_2$

A & [A:B] are in Echelon form, but the eqn now is not consistent.

∴ wrongly copied constant term of the third eqn must be -1.

so that rank(A) = rank([A:B]) = 2, and the system is consistent

∴ correct form of the system is,

$x - 2y + z = 1$

$2x + y + z = 1.$

$5y - z = -1.$

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∴ Eqn (i) becomes.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

∴ reduced eqn solution is

$x - 2y + z = 1$ – (ii)

$5y - z = -1$

$\Rightarrow z = 5y + 1$ .... (iii)

∴ $x = -3k$

$y = k.$

$z = 5k + 1$, where k is any real number.

2. Investigate for what values of λ and μ, the system of

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$, has (i) no solution, (ii) a unique soln.

(iii) infinite no. of solns.

$\rightarrow$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$ In matrix form, $AX = B$.

$$[A:B] = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{pmatrix}$$ $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$

which is in Echelon form.

(i) If $\lambda = 3$ and $\mu \neq 10$, then

$\text{rank}(A) = 2$ and $\text{rank}([A:B]) = 3$.

$\because$ $\text{rank}(A) \neq \text{rank}([A:B])$

$\therefore$ the system is inconsistent, i.e. has no soln.

(ii) If $\lambda \neq 3$ then for any value of μ.

$\text{rank}(A) = 3$ and $\text{ran}([A:B]) = 3$.

$\because$ $\text{rank}(A) = \text{rank}([A:B]) = 3 =$ no. of unknowns

$\therefore$ the system has a unique solution.

(iii) If $\lambda = 3$ and $\mu = 10$, then

$\text{rank}(A) = \text{rank}([A:B]) = 2 <$ no. of unknowns

$\therefore$ in this case, the system is consistent and has infinite no. of soln.

3. For what values of λ, the following system is consistent and solve its system:-

$$(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$$
$$(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$$
$$2x + (3\lambda+1)y + 3(\lambda-1)z = 0.$$

$\rightarrow$ In matrix form,

$AX = O$. $\begin{pmatrix} (\lambda-1) & (3\lambda+1) & 2\lambda \\ (\lambda-1) & (4\lambda-2) & (\lambda+3) \\ 2 & (3\lambda+1) & (3\lambda-3) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Since, the system is homogenous, so $X=0$, i.e., $x=0, y=0, z=0$ is clearly a solution for any value of λ.

$\therefore$ The system is consistent for all real values of λ.

$$|A| = \begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2\lambda & 3\lambda+1 & 3\lambda-3 \end{vmatrix}$$

$$= \begin{vmatrix} 6\lambda & 3\lambda+1 & 2\lambda \\ 6\lambda & 4\lambda-2 & \lambda+3 \\ 6\lambda & 3\lambda+1 & 3\lambda-3 \end{vmatrix} \qquad C_1 \to C_1+C_2+C_3$$

$$= 6\lambda \begin{vmatrix} 1 & 3\lambda+1 & 2\lambda \\ 1 & 4\lambda-2 & \lambda+3 \\ 1 & 3\lambda+1 & 3\lambda-3 \end{vmatrix}$$

$$= 6\lambda \begin{vmatrix} 1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & -\lambda+3 \\ 0 & 0 & \lambda-3 \end{vmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= 6\lambda \{(\lambda-3)^2\}$$

$$= 6\lambda(\lambda-3)^2$$

The system will have only trivial solution (zero sol^n) if $|A| \neq 0$, i.e., if $\lambda \neq 0$ and $\lambda \neq 3$.

When $\lambda = 0$, the system becomes

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$R_3 \to R_3 + R_2$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ The reduced equivalent system is

$-x+y=0$

$\Rightarrow x=y \quad -(1).$

$-3y+3z=0$

$\Rightarrow y=z \quad -(2).$

From (1) and (2)

$\Rightarrow x=y=z.$

$\therefore$ solⁿ is given by

$x=k,\ y=k,\ z=k$, where k is any real no.

(iii) When $\lambda=3$, the system becomes.

$$\begin{pmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 10 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{array}{l} R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \end{array}$$

$\therefore$ reduced equivalent system is

$2x+10y+6z=0.$

$x=-5y-3z.$

$\therefore$ solⁿ is given by

$x=-5k_1-3k_2,\ y=k_1,\ z=k_2,$

where k_1 and k_2 are real nos.

4.(i) Test consistency and solve if consistent, the system of eqⁿ :-

$x+2y-z=3$

$3x-y+2z=1$

$2x-2y+3z=2$

$x-y+z=-1.$

$\rightarrow$ In matrix form, the system is $AX=B$.

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ -4 \\ -4 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ \cancel{-8}\,-4 \\ 4 \\ -4 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_3 \\ R_3 \to R_3 - 2R_4 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 4 \\ 0 \end{pmatrix} \qquad R_4 \to R_4 - 3R_2 - 2R_3$$

, the system is in Echelon form.

$\therefore$ rank(A) = rank$([A:B])$ = no. of unknowns

$\therefore$ The system is consistent and has unique sol^n.

$\therefore$ The reduced equivalent system is.

$x + 2y - z = 3 \quad -(1)$

$-y = -4$

$\Rightarrow y = 4 \quad -(2)$

$z = 4 \quad -(3).$

From (1) $\Rightarrow x = -1.$

sol^n is $x = -1,\ y = 4,\ z = 4.$

(ii)

$x_1 - 2x_2 + x_3 + x_4 = 1.$

$x_1 + 2x_2 - x_3 + x_4 = 2$

$x_1 + 7x_2 - 5x_3 - x_4 = 3.$

$\rightarrow$ In matrix form, the system is $AX = B$.

$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 4 & -2 & 0 \\ 0 & 9 & -6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & -\frac{3}{2} & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1/4 \end{pmatrix}$$

$R_3 \to R_3 - \frac{7}{4} R_2$

, which is in Echelon form.

$\therefore$ rank(A) = rank([A:B]) = 3 < no. of unknowns.

$\therefore$ The system has infinite solⁿ (consistent).

The reduced equivalent system is

$$x_1 - 2x_2 + x_3 + x_4 = 1 \quad -(1)$$

$$4x_2 - 2x_3 = 1 \quad -(2)$$

$$-3/2\, x_3 - 2x_4 = -1/4$$

$$\Rightarrow 3x_3 + 4x_4 = 1/2 \quad -(3).$$

$$\Rightarrow x_4 = \left(\tfrac{1}{2} - 3K\right)/4 = \frac{1-6K}{8} \; ; \quad x_3 = K.$$

$$x_2 = \frac{1+2K}{4}$$

$$x_1 - \frac{1+2K}{2} + K + \frac{1-6K}{8} = 1$$

$$\Rightarrow 8x_1 - 4 - \cancel{8K} + \cancel{8K} + 1 - 6K = 8$$

$$\Rightarrow x_1 = \frac{11+6K}{8}.$$

$$\therefore x_1 = \frac{11+6K}{8} \; ; \; x_2 = \frac{1+2K}{4} \; ; \; x_3 = K \; ; \; x_4 = \frac{1-6K}{8}.$$

(5.) Prove that the system has no solⁿ unless $a+b+c=0$

$\rightarrow$ In matrix form, the system is $AX = B$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\therefore [A:B] = \begin{bmatrix} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}.$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \\ 1 & 1 & -2 & c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 3 & -3 & c-b \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 0 & 0 & a+b+c \end{bmatrix} \quad R_3 \to R_3 + R_2$$

, which is in Echelon form.

with rank$(A) = 2$

The system is consistent if rank$[A:B]$ = rank $(A) = 2$.

which is possible if $a+b+c=0$ only; other system is inconsistent, i.e., has no solution.

$\therefore$ Given system has no solution unless $a+b+c=0$

(6) Consider the system:-

$x+2y+z=3$; $ay+5z=10$, $2x+7y+az=b$.

(i) For what value of 'a', the system has a unique solⁿ?

(ii) Find those values of a and b for which the system has more than one solⁿ.

→ In matrix form, the system is

$AX = B$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & a & 5 \\ 2 & 7 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ b \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & a & 5 \\ 2 & 7 & a \end{vmatrix}$$

$= (a^2-35) + 2(10-a)$

$= a^2 - 2a - 15$

$= a^2 - 5a + 3a - 15$

$= (a-5)(a+3)$

(i) $\therefore$ The system will have a unique solution if $|A| \neq 0$, i.e., $a \neq -3$ & $a \neq 5$.

∴ The system has a unique soln for all real values of 'a' except -3 and 5.

(ii)(a) When 'a' $= -3$, the augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 2 & 7 & -3 & b \end{bmatrix}.$$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 0 & 3 & -5 & b-6 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 0 & 0 & 0 & b+4 \end{bmatrix}$$, which is in echelon form.

$\therefore$ rank$(A) = 2$.

The system will be consistent and will have more than one soln if rank(A) = rank$([A:B]) = 2$, i.e., if $b+4=0$

$\therefore b = -4$.

(b) When $a = 5$, the augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 2 & 7 & 5 & b \end{bmatrix}.$$

$\underline{b=12}$

$R_3 \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 0 & 3 & 3 & b-6 \end{bmatrix}$$

$R_3 \to R_3 - \frac{3}{5} R_2$

$$= \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 0 & b-12 \end{bmatrix}$$, which is in echelon form.

rank$(A) = 2$.

The system will be consistent and will have more than one soln if rank(A) = rank$([A:B]) = 2$, i.e., if $b-12=0$

$\therefore b = 12$.

Solve using Gaussian Elimination Method:-

$3x+2y+z=3$.
$2x+y+z=0$.
$6x+2y+4z=6$.

→ In matrix form, the system is $AX=B$.

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

$R_1 \to R_1 - R_2$
$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix}$$ • which is in echelon form.

The system is inconsistent since, $\text{rank}(A) = 2 \neq 3 = \text{rank}([A:B])$.

∴ The system has no solution.

* Gauss-Jordan Method of solving system of linear equations:-

In this method of solving a system of n linear equations in n unknowns, the system is first written in matrix form, and then using elementary row operations only, the augmented matrix is first converted to echelon form and if consistent, then the coefficient matrix A is converted to a diagonal matrix/ unit matrix and the unknowns are readily calculated.

1. Solve by Gauss-Jordan method:

$x+y+z=9$; $2x+3y+4z=13$, $3x+4y+5z=40$.

→ In matrix form, the system is

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{bmatrix}$$

$$\cancel{\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \end{bmatrix}} \qquad \cancel{R_3 \to R_3 + \tfrac{1}{5} R_2}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & -5 & 2 & -5 \end{bmatrix} \qquad R_3 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{bmatrix} \qquad R_3 \to R_3 + 5R_2 .$$

, which is in echelon form,

$\text{rank}(A) = \text{rank}([A:B]) = 3$, so the system has unique solution:

~~Ans~~ $$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad R_3 \to \frac{1}{12} R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad R_2 \to R_2 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad R_1 \to R_1 - R_2 - R_3 .$$

$\therefore$ solution is

$$x = 1,\ y = 3,\ z = 5.$$

* Elementary Matrix or E-matrix :-

A matrix obtained from a unit matrix by an application of a single elementary operation is called an elementary matrix or E-matrix.

e.g., applying $R_2 \to R_2 - 3R_1$ to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we get $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

which is an E-matrix.

(ii) applying $C_2 \leftrightarrow C_3$ to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ which is an E-matrix.

(iii). applying $R_2 \to R_2 - 3R_1$ to $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix}$ we get $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 5 \end{pmatrix}$ is not E-matrix.

Now,
$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 5 \end{pmatrix}$$

Also,
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}$$ which is nothing but matrix with $C_2 \leftrightarrow C_3$.

• **Theorem** :- Every elementary row (column) operation of a matrix can be obtained by pre (post) multiplication of the given matrix by corresponding E-matrix.

• **Matrix inversion by Gauss-Jordan method** :-

Let A is an invertible matrix, i.e., a non-singular matrix and let I is the unit matrix of same order as that A, then

$$A = IA$$

Applying elementary row operations only, we convert A of LHS to I and accordingly I of RHS will be converted to a matrix P (say), i.e., $I = PA \Rightarrow A^{-1} = P$.

1. Find A^{-1} using Gauss-Jordan method given:-

(a) $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (b) $A = \begin{pmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 6 & 8 \end{pmatrix}$

→ (a). $A = IA$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + 2R_1$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -9 \\ 0 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} A$$

$R_3 \to R_3 + 2R_2$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -9 \\ 0 & 0 & -16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix} A$$

$R_1 \to R_1 - R_2$
$R_3 \to -\frac{1}{16} R_3$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ \frac{1}{8} & \frac{-1}{8} & \frac{-1}{16} \end{pmatrix} A$$

$R_1 \to R_1 - 12 R_3$
$R_2 \to R_2 + 9R_3$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3/2 & 1/2 & \frac{3}{4} \\ -\frac{7}{8} & -1/8 & -\frac{9}{16} \\ 1/8 & -1/8 & -1/16 \end{pmatrix} A$$

$I = PA$

$$\therefore A^{-1} = \begin{pmatrix} 3/2 & 1/2 & 3/4 \\ -7/8 & -1/8 & -9/16 \\ 1/8 & -1/8 & -1/16 \end{pmatrix}$$

* Normal form of matrix:- Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformations, to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called normal form of A.

Note:- rank(A) = r iff it can be reduced to normal form.

To find two P and Q such that PAQ is in normal form, put

$A = IAI \Rightarrow \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$.

(row change)

(b) $A = IA$

$$\Rightarrow \begin{pmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 4 \\ 7 & 6 & 2 \\ 3 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \qquad R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 4 \\ 0 & 20 & 30 \\ 0 & 12 & 20 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 7 & 0 \\ 0 & 3 & 1 \end{pmatrix} A \qquad \begin{matrix} R_2 \to R_2 + 7R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 4 \\ 0 & 2 & 3 \\ 0 & \cancel{6}3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1/10 & 7/10 & 0 \\ 0 & 3/4 & 1/4 \end{pmatrix} A \qquad \begin{matrix} R_2 \to \frac{1}{10} R_2 \\ R_3 \to \frac{1}{\cancel{2}4} R_3 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 3/4 & 1/4 \\ 1/10 & 7/10 & 0 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1/10 & 1/20 & 1/4 \\ \frac{1}{10} & 7/10 & 0 \end{pmatrix} A \qquad R_2 \to R_2 - R_3$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1/10 & 1/20 & 1/4 \\ 3/10 & 6/10 & -1/2 \end{pmatrix} A \qquad R_3 \to R_3 - 2R_2.$$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/10 & -1/10 & -1/2 \\ -1/10 & 1/20 & 1/4 \\ 3/10 & 6/10 & -1/2 \end{pmatrix} A \qquad R_1 \to R_1 - 2R_2$$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/10 & -1/10 & -1/2 \\ -7/10 & -23/20 & 5/4 \\ 3/10 & 6/10 & -1/2 \end{pmatrix} A \qquad R_2 \to R_2 - 2R_3$$

$$A^{-1} = \begin{pmatrix} -1/5 & 1/10 & 1/2 \\ -7/10 & \frac{-23}{20} & 5/4 \\ 3/10 & 3/5 & -1/2 \end{pmatrix}$$

* Eigen value or Eigen vector of a square matrix :-

Let A be a square matrix of order n, then a scalar λ is called an Eigen value of A, if there exists a non-zero column vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ such that $AX = \lambda X$. — (1)

In this case, X is called an Eigen vector of A corresponding to the Eigen value λ.

• **Theorem :-** λ is an Eigen value of a square matrix A iff

$$|A - \lambda I| = 0.$$

Proof :- λ is an eigen value of a square matrix A of order n

$\Leftrightarrow AX = \lambda X$, for some non-zero column vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\Leftrightarrow AX - \lambda X = O$, where $O = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$.

$\Leftrightarrow AX - \lambda IX = 0$, where I is unit matrix of order n.

$\Leftrightarrow (A - \lambda I)X = 0$, where $X \neq 0$

$\Leftrightarrow A - \lambda I$ is not invertible, otherwise $X = 0$, a contradiction

$\Leftrightarrow A - \lambda I$ is singular

$\Leftrightarrow |A - \lambda I| = 0$.

Note :- 1. Eigen values of A are roots of the equation $|A - \lambda I| = 0$, and eigen vectors are non-zero solution vectors of $(A - \lambda I)X = 0$

Definitions :-

Let A is a square matrix of order n and I is the unit matrix of order n and also λ is a scalar ~~int~~ ~~indete~~ indeterminate, then

(i) $A - \lambda I$ is called characteristic matrix of A.

(ii) $|A - \lambda I|$ is called characteristic function of A. and when expanded it becomes a polynomial of degree n in λ and is called characteristic polynomial.

(iii) $|A - \lambda I| = 0$ is called characteristic equation of A and the roots of these equation are called characteristic roots or latent roots of A.

and a column vector $X = 0$, satisfying $AX = \lambda X$ or $(A-\lambda I)X = 0$,
is called characteristic vector of A corresponding to λ.

Note:- characteristic roots (vectors) are same as eigen ~~roots~~ values (vectors)

Q. Find eigen value and eigen vectors of the matrix :-

(i) $A = \begin{pmatrix} 5 & -2 \\ 9 & -6 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$ (iii) $A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}$

(iv) $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

(i) Eigen values of A are roots of

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)(-6-\lambda)+18 = 0$

$\Rightarrow 18 = 30+5\lambda-6\lambda-\lambda^2$

$\Rightarrow \lambda^2+\lambda-12 = 0$

$\Rightarrow \lambda^2+4\lambda-3\lambda-12 = 0$

$\Rightarrow (\lambda+4)(\lambda-3) = 0$.

$\therefore \lambda = 3$ or -4, which are eigen values.

Eigen vectors are non-zero solution vectors of

$$(A-\lambda I)X = 0$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{— (1)}$$

when $\lambda = 3$,

$$(1) \Rightarrow \begin{pmatrix} 2 & -2 \\ 9 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

~~$R_1 \to R_1$~~
$R_1 \to \frac{1}{2} R_1$
$R_2 \to \frac{1}{9} R_2$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad R_2 \to R_2 - R_1$$

$\therefore$ reduced equivalent system is

$$x - y = 0$$
$$\Rightarrow x = y.$$
$$\Rightarrow \frac{x}{1} = \frac{y}{1}$$

$\therefore X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = 3$.

When $\lambda = -4$. (1) $\Rightarrow$

$$\begin{pmatrix} 9 & -2 \\ 9 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 9 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad R_2 \to R_2 - R_1$$

$\therefore$ reduced equivalent system is

$$9x - 2y = 0$$
$$\Rightarrow 9x = 2y.$$
$$\Rightarrow \frac{x}{2} = \frac{y}{9}.$$

$\therefore X = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = -4$.

* Observation:-

(i) Let $P = \begin{pmatrix} 1 & 2 \\ 1 & 9 \end{pmatrix}$

$$\therefore |P| = \begin{vmatrix} 1 & 2 \\ 1 & 9 \end{vmatrix} = 7.$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P.$$

$$P^{-1} = \frac{1}{7} \begin{pmatrix} 9 & -2 \\ -1 & 1 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{7} \begin{pmatrix} 9 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 9 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 9 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ 3 & 36 \end{pmatrix}$$

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$$P^{-1}AP = \frac{1}{7}\begin{pmatrix} 21 & 0 \\ 0 & -28 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}$$

$$(P^{-1}AP)^2 = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}$$

$$P^{-1}AP\,P^{-1}AP = \begin{pmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{pmatrix}$$

$$\Rightarrow P^{-1}A^2P = \begin{pmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{pmatrix}$$

$$\Rightarrow A^2 = P\begin{pmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{pmatrix}P^{-1}.$$

(ii) Similarity of matrices:- A square matrix $\hat{A}$ of order n is said to be similar to another square matrix of order n, A, if $\exists$ a non-singular matrix P of order n such that $\hat{A} = P^{-1}AP$.

This transformation of A by a non-singular matrix P to $\hat{A}$ is ~~set~~ called a similarity transformation.

Theorem:- Similar matrices have the same eigen value or same characteristic roots.

Proof:- Let $\hat{A}$ is a square matrix which is similar to another sq. matrix A, then $\exists$ a non-singular matrix P such that $\hat{A} = P^{-1}AP$.

characteristic polynomial of $\hat{A}$ is $|\hat{A} - \lambda I| = |P^{-1}AP - \lambda P^{-1}P|$

$$= |P^{-1}AP - P^{-1}\lambda IP|$$
$$= |P^{-1}(A - \lambda I)P|$$
$$= |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= \frac{1}{|P|}|A - \lambda I|\,|P|$$
$$= |A - \lambda I|.$$

= characteristic polynomial of A.

$$\therefore |\hat{A} - \lambda I| = 0$$
$$\Leftrightarrow |A - \lambda I| = 0$$

$\therefore$ $\hat{A}$ and A have characteristic equation and hence they have same characteristic roots or eigen values.

Note:- If X is an eigen vector of A corresponding to the eigen value λ, then $AX = \lambda X$ and $X \neq 0$. Also, if $\hat{A}$ is similar to A, then $\hat{A} = P^{-1}AP$, where P is a non-singular matrix.

$$\therefore P^{-1}AX = P^{-1}\lambda X$$

$$\Rightarrow P^{-1}AIX = \lambda P^{-1}X$$

$$\Rightarrow P^{-1}APP^{-1}X = \lambda P^{-1}X$$

$$\Rightarrow \hat{A}Y = \lambda Y,$$ where $Y = P^{-1}X \neq 0$, otherwise

$$P^{-1}X = 0$$

$$\Rightarrow X = P0 = 0,$$

a contradiction.

$Y = P^{-1}X$ is an eigen vector of $\hat{A}$ corresponding to eigen value λ.

*** Diagonalization of a square matrix by similarity transformation:-**

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix, whose diagonal elements are nothing but eigen values of A and P is the matrix whose columns are eigen vectors of A.

Note:- 1. The matrix P which diagonalize A is called the modal matrix of A and the resulting diagonal matrix $P^{-1}AP = D$ (say) is called spectral matrix of A.

2. $$D = P^{-1}AP$$

$$D^2 = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}AIAP$$

$$= P^{-1}A^2P$$

similarly,

$$D^3 = P^{-1}A^3P$$

and in general, $D^n = P^{-1}A^nP$.

$$\therefore A^n = PD^nP^{-1}.$$

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1) (iii) $A = \begin{pmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{pmatrix}$

Eigen values of A are roots of the eqn

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & -2 & 3 \\ -2 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -2 & 3 \\ 5-\lambda & 1-\lambda & 6 \\ 5-\lambda & 2 & 2-\lambda \end{vmatrix} = 0 \qquad C_1 \to C_1 + C_2 + C_3$$

$$\Rightarrow (5-\lambda) \begin{vmatrix} 1 & -2 & 3 \\ 1 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\Rightarrow (5-\lambda) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 3-\lambda & 3 \\ 0 & 4 & -(\lambda+1) \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)\{(\lambda-3)(\lambda+1) - 12\} = 0$$

$$\Rightarrow (5-\lambda)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (5-\lambda)(\lambda-5)(\lambda+3) = 0$$

$$\Rightarrow \lambda = 5, 5, -3 .$$

$\therefore$ eigen values of A are -3 and 5.

Eigen vectors are non-zero solution vectors of

$$(A - I\lambda)X = 0$$

$$\Rightarrow \begin{pmatrix} 4-\lambda & -2 & 3 \\ -2 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{— (i)}$$

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When $\lambda = -3$

$$\begin{pmatrix} 7 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_3$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ -2 & 4 & 6 \\ 7 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - 7R_1$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & 8 & 16 \\ 0 & -16 & -32 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to \frac{1}{8} R_2$
$R_3 \to \frac{-1}{16} R_3$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system has $(n-r) = 3-2 = 1$, L.I. sol^n. vector.

The reduced equivalent system is

$$x + 2y + 5z = 0$$

$$\cancel{x+2y=0} \quad y + 2z = 0$$

$$\cancel{z=} \quad y = -2z.$$

$$\therefore x = -5z + 4z$$
$$= -z.$$

$\therefore x = -k,\ y = -2k,\ z = k$, where k is any real no.

$\therefore$ The eigen vector corresponding to $\lambda = -3$ is $X_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

When $\lambda = 5$, (1) $\Rightarrow$

$$\begin{pmatrix} -1 & -2 & 3 \\ -2 & -4 & 6 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & -2 & 3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 \to \frac{-1}{2} R_2$$

$$\Rightarrow \begin{pmatrix} -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 \to R_3 + R_1$
$\cancel{R_3 \to R_3 - R_2}$
$R_2 \to R_2 + R_1$

$\therefore$ The system has $n-r = 3-1 = 2$, L.I. soln vectors.

$\therefore$ The reduced system is

$$-x-2y+3z = 0$$

Two L.I. eigen vectors are

$$X_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \quad \text{and } X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

* Note:- Let $P = (X_1 \ X_2 \ X_3)$

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$|P| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$|P| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -3 & 0 \\ -2 & -2 & 0 \end{vmatrix}$$

$$|P| = -1(6+2)$$

$$|P| = -8.$$

~~cofactors of R.A (A)~~ Matrix of co-factors is $M = \begin{pmatrix} -1 & -3 & -1 \\ -2 & 2 & -2 \\ 3 & 1 & -5 \end{pmatrix}$

$$\therefore P^{-1} = \frac{1}{|P|} \operatorname{adj} P.$$

$$P^{-1} = \frac{1}{-8} M'$$

$$P^{-1} = \frac{-1}{8} \begin{pmatrix} -1 & -2 & 3 \\ -3 & 2 & 1 \\ -1 & -2 & -5 \end{pmatrix}$$

$$P^{-1} = \frac{1}{8} \begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & -1 \\ 1 & 2 & 5 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{8} \begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & -1 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{8} \begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & -1 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} -3 & 10 & 5 \\ -6 & -5 & 5 \\ 3 & 0 & 5 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{8}\begin{pmatrix} -24 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = D \text{ (say)}.$$

$$\therefore D^3 = P^{-1}A^3P$$

$$A^3 = PD^3P^{-1}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -27 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 125 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & -1 \\ 1 & 2 & 5 \end{pmatrix}$$

Q. Diagonalise A by similarity transformation and find A^5, where

(i) $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

(ii) Eigen value of A are roots of

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda)(2-\lambda) = 0$$

$\Rightarrow \lambda = 1, 2, 3.$, which are eigen values.

Eigen vectors of A are non-zero soln vectors of

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{--- (1)}$$

when $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 \to \tfrac{1}{2}R_2, \quad R_3 \to R_3 - R_1$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_1 \leftrightarrow R_2 .$$

The system has only one L.I. soln.

$\therefore$ The reduced equivalents system is

$$y + z = 0$$
$$z = 0$$

$\therefore y = 0,\ z = 0$, if we take $x = 1$, then it is a non-zero soln.

$\therefore$ e-vector corresponding to $\lambda = 1$ is

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 2$, (1) becomes

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The reduced system is

$$-x + z = 0$$
$$y + 2z = 0$$

$\therefore x = z \quad y = -2z.$

$\therefore x = 1,\ y = -2,\ z = 1.$

$\therefore$ e-vector corresponding to $\lambda = 2$ is

$$X_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 \to R_3 + \tfrac{1}{2} R_2$$

The reduced equivalent system is

$$-2x + z = 0$$
$$2z = 0$$

$\Rightarrow x = 0,\ z = 0$, if we take $y = 1$, then it is a non-zero soln.

$$X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Let $P = (X_1 \; X_2 \; X_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$\therefore |P| = 1(0-1) = -1.$

Matrix of co-factor is, $M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & -2 \end{bmatrix}$

$\therefore P^{-1} = \frac{1}{|P|} \operatorname{adj}(P) = \frac{1}{|P|} M' = (-1) \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$\therefore P^{-1}AP = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, which is required diagonalization of A.

$= D$ (say).

$P^{-1}AP = D$

$A = PDP^{-1}$

$A^5 = PD^5P^{-1}$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 32 & 0 \\ 0 & -64 & 243 \\ 0 & 32 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 31 \\ 0 & 243 & 422 \\ 0 & 0 & 32 \end{pmatrix}.$$

$\frac{486}{64}$

* **Cayley-Hamilton Theorem:-**

Every square matrix satisfies its own characteristic equation.

or Every square matrix is a root of (zero of) its characteristic equation (polynomial).

• Note:- Ch. polynomial of a sq. matrix A of order n is

$$\phi(\lambda) = |A - \lambda I|$$
$$= a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \ldots\ldots + a_n\lambda^n, \text{ (say)}.$$

C.H. Theorem says that,

$$a_0 I + a_1 A + a_2 A^2 + a_3 A^3 + \ldots\ldots + a_n A^n = 0.$$

Also, $\phi(0) = |A - 0I| = |A|$

$$\Rightarrow a_0 = |A|$$

$\therefore |A| \neq 0 \Leftrightarrow a_0 \neq 0$ and in this case A has its inverse A^{-1}.

$$\therefore A^{-1}\left(a_0 I + a_1 A + a_2 A^2 + \ldots\ldots + a_n A^n\right) = A^{-1} 0$$

$$\Rightarrow A^{-1} a_0 I + a_1 A^{-1} A + a_2 A^{-1} A^2 + \ldots\ldots + a_n A^{-1} A^n = 0$$

$$\Rightarrow a_0 A^{-1} + a_1 + a_2 A + a_3 A^2 + \ldots\ldots + a_n A^{n-1} = 0$$

$$\therefore A^{-1} = -\frac{1}{a_0}\left(a_1 I + a_2 A + a_3 A^2 + \ldots\ldots + a_n A^{n-1}\right).$$

Q. Verify Cayley-Hamilton Theorem for:-

(i) $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$

(ii) ch. equation of A is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 0 & 1-\lambda & 1-\lambda \end{vmatrix} = 0 \qquad R_3 \to R_3 - R_2$$

$$\Rightarrow (2-\lambda)\left\{(2-\lambda)(1-\lambda) + (1-\lambda)\right\} + 1\left(-2(1-\lambda)\right) = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda)(3-\lambda) - 2(1-\lambda) = 0.$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 4\lambda + 3) + 2\lambda - 2 = 0$$

$$\Rightarrow (-\lambda^3 + 6\lambda^2 - 11\lambda + 6) + 2\lambda - 2 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify:

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$A^2 = AA = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - \begin{pmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{pmatrix} + \begin{pmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

1. Using Cayley-Hamilton Theorem, find A^{-1} and A^{100} if :-

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Sol^n → (i) ch. eq^n of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

By Cayley-Hamilton Theorem,

$$A^2 - 2A + I = 0$$

$$\Rightarrow A^{-1}A^2 - 2A^{-1}A + A^{-1}I = 0$$

$$\Rightarrow A - 2I + A^{-1} = 0$$

$$\Rightarrow A^{-1} = 2I - A$$

$$A^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Now, $A^2 - 2A + I = 0$

$$A^2 = 2A - I \quad \text{— (i)}$$

$$AA^2 = A(2A - I)$$

$$A^3 = 2A^2 - AI$$

$$A^3 = 2(2A - I) - A \quad \text{— (using (i))}$$

$$A^3 = 3A - 2I \quad \text{— (ii)}$$

$$AA^3 = 3A^2 - 2A$$

$$A^4 = 3(2A - I) - 2A \quad \text{[using (i)]}$$

$$\Rightarrow A^4 = 4A - 3I.$$

$$\therefore A^{100} = 100A - 99I$$

$$= \begin{pmatrix} 100 & 100 \\ 0 & 100 \end{pmatrix} - \begin{pmatrix} 99 & 0 \\ 0 & 99 \end{pmatrix} = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}.$$

(b) ch. eqⁿ of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-1) = 0.$$

$$\Rightarrow (\lambda^2 - 1 - \lambda^3 + \lambda) = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

By Cayley - Hamilton Theorem :-

$$A^3 - A^2 - A + I = 0$$

$$A^{-1}A^3 - A^{-1}A^2 - A^{-1}A + A^{-1}I = 0$$

$$\Rightarrow A^2 - A - I + A^{-1} = 0$$

$$\Rightarrow A^{-1} = -A^2 + A + I.$$

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$\therefore A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Now, $A^3 = A^2 + A - I$ — (i)

$A^4 = A^3 + A^2 - A.$

$A^4 = 2A^2 - I$ — (ii)

$A^5 = 2A^3 - A$

$A^5 = 2A^2 + 2A - 2I - A = 2A^2 + A - 2I$ — (iii)

$$A^6 = 2A^3 + A^2 - 2A$$
$$= 2(A^2 + A - I) + A^2 - 2A$$
$$= 3A^2 - 2I \quad \text{— (iv).}$$

$$\therefore A^{100} = 50A^2 - 49I$$

$$= 50\begin{pmatrix} 1 & 0 & 0 \\ \cancel{1} & \cancel{1} & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{pmatrix}.$$

* **Note:-** In $\mathbb{R}^n$- space, any $n+1$ or more vectors are always L.D.

Q. Check whether the following vectors are L.D. or. L.I in $\mathbb{R}^3$ or $\mathbb{R}^4$ as the case may be.

(i) $(3,4,7), (2,0,3), (5,5,6)$. (ii) $(3,-2,0,4), (5,0,1,1), (-6,1,0,1), (2,0,0,3)$

(ii) → Let $v_1 = (3,-2,0,4)$, $v_3 = (-6,1,0,1)$
$v_2 = (5,0,1,1)$, $v_4 = (2,0,0,3)$

Also let, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$, where $\alpha_i \in \mathbb{R}$.

$$\alpha_1(3,-2,0,4) + \alpha_2(5,0,1,1) + \alpha_3(-6,1,0,1) + \alpha_4(2,0,0,3) = (0,0,0,0)$$

$$\Rightarrow (3\alpha_1 + 5\alpha_2 - 6\alpha_3 + 2\alpha_4, -2\alpha_1 + \alpha_3, \alpha_2, 4\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4) = (0,0,0,0)$$

$3\alpha_1 + 5\alpha_2 - 6\alpha_3 + 2\alpha_4 = 0$; $\alpha_2 = 0$.

$4\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 = 0$ $\qquad -2\alpha_1 + \alpha_3 = 0$

$\alpha_3 = 2\alpha_1$

→ $3\alpha_1 - 6\alpha_3 + 2\alpha_4 = 0$;

$4\alpha_1 + \alpha_3 + 3\alpha_4 = 0$;

~~$\Rightarrow 3\alpha_1 - 12\alpha_1 + 2\alpha_4 = 0$~~

In matrix form,

$$AX = 0$$

$$\begin{pmatrix} 3 & 5 & -6 & 2 \\ -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\therefore |A| = \begin{vmatrix} 3 & 5 & -6 & 2 \\ -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 1 & 1 & 3 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 3 & -6 & 2 \\ -2 & 1 & 0 \\ 4 & 1 & 3 \end{vmatrix}$$

$$= (-1) \left\{ 2(-2-4) + 3(3-12) \right\}$$

$$= -1 (-12-27)$$

$$= 39. \neq 0.$$

$\therefore |A| \neq 0$

$\therefore A^{-1}$ exists.

$\therefore X = A^{-1}0 = 0$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.$

$\therefore$ The vectors are L.I.

* Note :- If $|A| = 0$, then the system has infinite no. of sol^n.
So, has a non-zero solution, i.e, $\exists$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero, such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 = 0.$$

$\therefore V_1, V_2, V_3, V_4$ are L.D. //

3. Find all real values of λ for which the rank of matrix A is 2.

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \lambda \\ 1 & 1 & 6 & \lambda+1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{bmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & \lambda-1 \\ 0 & 2 & -4 & \lambda^2-5 \end{pmatrix}$ $\quad R_2 \to R_2 - R_1$, $R_3 \to R_3 - 5R_1$

$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-2\lambda-3 \end{pmatrix}$ $\quad R_3 \to R_3 - 2R_2$, which is echelon form.

Rank $(A) = 2$ if $\lambda^2 - 2\lambda - 3 = 0$

$$\lambda = \frac{2 \pm \sqrt{4+12}}{2}$$

$$\lambda = \frac{2 \pm 4}{2}$$

$$\lambda = 3, -1.$$