Complex Variable.

* **Limit of a function**:- A function $f(z)$ is said to have a limit l as z approaches 'a' if for any $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(z)| < \varepsilon$ for $|z| < \delta$.

* **Continuity**:- A function $f(z)$ is said to be continuous if
$$\lim_{z \to a} f(z) = f(a).$$

* **Differentiability**:- $f(z)$ is said to be differentiable if
$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$
exists, if this limit exists and has the same value for all the ways in which δz approaches 0.

Analytic function:- A function $f(z)$ which is single valued and possesses a unique derivative w.r.t. z at all points of a region R, then $f(z)$ is said to be analytic in that region R.

Theorem:- The necessary and sufficient condition for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are:-

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions in R.

(ii) $\left.\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array}\right\}$ C.R. equations or Cauchy-Riemann equations.

1. Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at origin but C.R. equations are satisfied at origin.

→ Here, $u = \sqrt{|xy|}$, $v = 0$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} \qquad ; \qquad \frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{0-0}{x}$$
$$= \frac{0-0}{x} = 0 \qquad\qquad = 0.$$

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 0.$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

So, C.R. equations are satisfied at origin.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\sqrt{|xy|}-0}{z}$$

Let $z\to 0$ along the path $y=mx$

$z\to 0 \Rightarrow x\to 0$; $z = x+iy$

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|xy|}-0}{x+iy}$$

$$= \lim_{x\to 0} \frac{\sqrt{x(mx)}}{x(1+im)}$$

$$= \lim_{x\to 0} \frac{\sqrt{m}}{1+im}$$, which depends on m,

i.e. $f'(0)$ does not exists, so, $f(z)$ is not analytic

so, limit is not unique.

** 2. $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$

$= 0$, $z = 0$

Prove that $f(z)$ is continuous and C.R. equations are satisfied at origin, yet $f'(0)$ does not exist.

$\rightarrow$ $\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} = \lim_{y\to 0} [-y(1-i)] = 0$

$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} f(z) = \lim_{x\to 0} x(1+i) = 0$

$f(0) = 0$.

Let $x\to 0$ and $y\to 0$ simultaneously, along the path $y=mx$.

$$\therefore \lim_{\substack{x\to 0\\ y\to mx}} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} = \lim_{x\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{x^2(1+m^2)}$$

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \cdot \frac{x\{(1+i) - m^3(1-i)\}}{(1+m^2)} = 0$$

$\therefore f(z)$ is continuous at origin. $u = \frac{x^3+y^3}{x^2+y^2}$, $v = \frac{x^3+y^3}{x^2+y^2}$.

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} \qquad \frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{x\to 0} \frac{x-0}{x} = 1 \qquad = -1.$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{x-0}{x} = 1. \qquad \frac{\partial v}{\partial y} = 1.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the C.R. equations are satisfied at origin

Now, $$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

Let $z \to 0$ along the path $y = mx$.

$z \to 0 \Rightarrow x \to 0$ $\quad z = x+iy$.

$$\therefore f'(0) = \lim_{x\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{(x+imx)(x^2+m^2x^2)}$$

$$= \lim_{x\to 0} \frac{(1+i) - m^3 + im^3}{(1+im)(1+m^2)}$$

$$= \lim_{x\to 0} \frac{(1-m^3) + i(1+m^3)}{(1+im)(1+m^2)}$$ which depends on m

$\therefore f'(0)$ does not exist. limit is not unique.

So, $f(z)$ is not analytic.

1. Derive Cauchy-Reimen' Equation in cartesian form, i.e., if $f(z) = u+iv$ is analytic in a region R. Show that

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in R.

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$.

→ Let $f(z) = u+iv$ be analytic at $P(x,y,z)$ in the region R.

$\therefore f(z)$ has a unique derivative at P.

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - (u(x,y) + iv(x,y))}{\delta x + i\delta y}$$

$\because \delta z \to 0$, along any path to P, first we assume δz to be wholly real and then we assume δz to be wholly imaginary.

Firstly, when δz is wholly real, δy is 0 and $\delta z = \delta x$.

So, $\delta z \to 0 \Rightarrow \delta x \to 0$.

$$\therefore f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - (u(x,y) + iv(x,y))}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad -(1)$$

When δz is wholly imaginary, δx is 0 and $\delta z = i\delta y$

as $\delta z \to 0 \Rightarrow i\delta y \to 0$

$$f'(z) = \lim_{i\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i\frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial u} \quad -(2)$$

n every $\varepsilon > 0$, there exists $|f(z) - l| < \varepsilon$ whenever

$f(z) = l = f(a)$

From (1) and (2),

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$ which are C.R. equations.

As, $f'(z)$ involves, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, so they are continuous in R.

* **Harmonic function:**– A function $u(x,y)$ is said to be harmonic if u satisfies Laplace equation, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

1. If $f(z) = u + iv$ is analytic, then show that both u and v are harmonic.

→ $f(z) = u + iv$ is analytics, so the C.R. equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Diff. w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2}$$

Diff. w.r.t. y

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

∴ Both u and v are harmonic.

* Polar form of Cauchy-Riemann (CR) equation:

Let (r,θ) be the polar coordinates of a point $P(x,y)$, and $f(z) = u+iv$, be analytic, then

$z = x+iy = r\cos\theta + ir\sin\theta$

$z = re^{i\theta}$.

$f(z) = f(re^{i\theta}) = u+iv$.

Diff. partially w.r.t. r and θ.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^{i\theta} f'(re^{i\theta}) \quad -(1)$$

Also, $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = re^{i\theta} . i f'(re^{i\theta}) \quad -(2).$$

(1) and (2) $\Rightarrow$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ri\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts,

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \quad ; \quad \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

1. If $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$, is analytic, find the value of p.

$\rightarrow$ $u = r^2\cos 2\theta$, $v = r^2\sin p\theta$.

$\therefore \frac{\partial u}{\partial \theta} = r^2\, 2(-\sin 2\theta)$, $\frac{\partial v}{\partial \theta} = pr^2\cos p\theta$.

$\frac{\partial u}{\partial r} = 2r\cos 2\theta$. $\frac{\partial v}{\partial r} = 2r\sin p\theta$.

$\because f(z)$ is analytic,

$\therefore -2r^2\sin 2\theta = -r.2r\sin p\theta$ | $2r\cos 2\theta = \frac{1}{r}pr^2\cos p\theta$

$\Rightarrow p = 2.$ | $\Rightarrow \cos p\theta = \cos 2\theta.$

$\therefore p = 2.$

2. Derive $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$, from C.R. equations.

→ From C.R. equation we have,

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta\,\partial r} \quad \text{and} \quad \frac{\partial^2 u}{\partial r^2} = \frac{-1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{-1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r} \qquad \frac{\partial^2 u}{\partial r^2} = \frac{-1}{r}\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r}$$

$$\therefore \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

* Harmonic function (in terms of polar coordinates) :-

$v(r,\theta)$ is said to be harmonic, if v satisfies the Laplace equation in polar form, i.e.

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

1. Show that :-

(a) $u = e^{-x}(x\sin y - y\cos y)$ is harmonic

(b) $v = r^2\cos 2\theta - r\cos\theta + 2$ is harmonic.

(a) $$\frac{\partial u}{\partial x} = e^{-x}(\sin y - 0) - e^{-x}(x\sin y - y\cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(-\sin y) + (-e^{-x})(\sin y - x\sin y + y\cos y)$$
$$= -e^{-x}(2\sin y - x\sin y + y\cos y).$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$$
$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\cos y + \sin y + y\cos y + \sin y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}(-2\sin y + x\sin y - y\cos y$$
$$+2\sin y + y\cos y - x\sin y)$$

$$= 0$$

$\therefore$ u is harmonic.

(ii) $v = r^2\cos 2\theta - r\cos\theta + 2.$

$\rightarrow \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta$.:, $\frac{\partial v}{\partial \theta} = 2r^2(-\sin 2\theta) + r\sin\theta.$

$$\frac{\partial^2 v}{\partial \theta^2} = -4r^2\cos 2\theta + r\cos\theta.$$

$$\frac{\partial^2 v}{\partial r^2} = 2\cos 2\theta$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta - \cos\theta)$$
$$+\frac{1}{r^2}(-4r^2\cos 2\theta + r\cos\theta)$$

$$= 2\cos 2\theta + 2\cos 2\theta - \frac{1}{r}\cos\theta - 4\cos$$
$$+\frac{1}{r}\cos\theta$$

$$= 0.$$

$\therefore$ v is harmonic.

- **Milne Thomson Method for obtaining $f(z)$; when real part or imaginary part is given.**

Case I :- Real part (u) is given.

Let $\dfrac{\partial u}{\partial x} = \phi_1(x,y)$ and $\dfrac{\partial u}{\partial y} = \phi_2(x,y)$

Find $\phi_1(z,0)$ and $\phi_2(z,0)$

$$\therefore f(z) = \int \left[\phi_1(z,0) - i\phi_2(z,0)\right] dz .$$

Case II :- Imaginary part (v) is given.

Let $\dfrac{\partial v}{\partial x} = \psi_1(x,y)$ and $\dfrac{\partial v}{\partial y} = \psi_2(x,y)$

Find $\psi_1(z,0)$ and $\psi_2(z,0)$

$$\therefore f(z) = \int \left[\psi_2(z,0) - i\psi_1(z,0)\right] dz .$$

1. Find $f(z) = u + iv$, where $u = e^{-x}(x\sin y - y\cos y)$.
2. Find $f(z) = u + iv$, where $v = e^{-x}(x\sin y - y\cos y)$.

Given $f(z)$ is analytic in both problems.

1.→ $$\frac{\partial u}{\partial x} = e^{-x}(\sin y - x\sin y + y\cos y) = \phi_1(x,y).$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y) = \phi_2(x,y).$$

$$\phi_1(z,0) = e^{-z}(0) = 0$$

$$\phi_2(z,0) = e^{-z}(z-1)$$

ze^{-z}
$-ze^{-z} - \int -e^{-z}$
$-ze^{-z} + e^{-z} \ldots$

$$\therefore f(z) = \int -i(e^{-z})(z-1)\, dz .$$

$$= -i\left[z(-e^{-z}) - e^{-z} + e^{-z}\right] = ize^{-z} + iC .$$

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Put $z = x+iy$.

$$f(z) = i(x+iy)\, e^{-x} e^{-iy} + iC.$$

$$= (ix-y)\, e^{-x}(\cos y - i\sin y) + iC.$$

$$= (ix\cos y + x\sin y - y\cos y + iy\sin y)\, e^{-x} + iC.$$

$$= (x\sin y - y\cos y) + i(x\cos y + y\sin y)e^{-x} + i$$

$$u+iv = e^{-x}(x\sin y - y\cos y) + i(x\cos y + y\sin y)$$

$$\therefore v = (x\cos y + y\sin y)\, e^{-x} + C.$$

2. $$\frac{\partial v}{\partial x} = e^{-x}(\sin y - x\sin y + y\cos y) = \phi_1(x,y).$$

$$\frac{\partial v}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y) = \phi_2(x,y).$$

$$\phi_1(z,0) = 0$$

$$\phi_2(z,0) = e^{-z}(z-1).$$

$$\therefore f(z) = \int [\phi_2(z,0) - i\phi_1(z,0)]\, dz$$

$$= \int e^{-z}(z-1)\, dz$$

$$= ze^{-z} + C.$$

Put $z = x+iy$.

$$f(z) = (x+iy)e^{-x}.e^{-iy} + C.$$

$$u+iv = (x+iy)(\cos y - i\sin y)\, e^{-x} + C$$

$$u+iv = (x\cos y - ix\sin y + iy\cos y + y\sin y)\, e^{-x} + C$$

$$u+iv = (x\cos y + y\sin y)\, e^{-x} + C + ie^{-x}(y\cos y - x\sin y)$$

$$u = (x\cos y + y\sin y)e^{-x} + C.$$

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Complex Variable

* Complex Integration :-

$f(z) = u+iv$ is a function and c be any closed curve

$$\oint_c f(z)\,dz = \oint_c (u+iv)(dx+idy)$$

$$= \int_c (udx - vdy) + i\int_c (udy+vdx)$$

1. Evaluate.

$\oint_c \frac{dz}{z-a}$, where C is circle, $|z-a| = r$.

$\Rightarrow z-a = re^{i\theta}$, $0 \leq \theta \leq 2\pi$

$dz = rie^{i\theta}\,d\theta$

$$= \int_0^{2\pi} \frac{rie^{i\theta}\,d\theta}{re^{i\theta}}$$

$$= i\int_0^{2\pi} d\theta$$

$$= i\,(2\pi - 0)$$

$$= 2\pi i.$$

2. $\int_0^{2+i} (\bar{z})^2\,dz$, ~~where c is~~ along $y = \frac{x}{2}$.

$\Rightarrow x = 2y.$

$dx = 2dy$

$dz = dx + idy$

$= (2+i)dy$

$$= \int_0^1 (x-iy)^2\,dz$$

$$= \int_0^1 (2y-iy)^2\,(2+i)\,dy$$

$$= \int_0^1 (2-i)^2 y^2 (2+i)\,dy$$

$$= 5\int_0^1 (2-i)\,y^2\,dy$$

$$= 5(2-i)\left|\frac{y^3}{3}\right|_0^1$$

$$= 5(2-i)\frac{1}{3}$$

$$= \frac{5}{3}(2-i)$$

Theorem:- Cauchy's Theorem:-

If $f(z)$ is analytic and $f'(z)$ is continuous at each point within and on a closed curve C, then

$$\int_C f(z)\,dz = 0$$

* **Cauchy's Integral Formula:-**

* **Note:-** 1. The line integral of an analytic function $f(z)$ in a region D is independent of the path joining any two points of D.

2. If $f(z)$ is analytic in the region D, bounded by two closed curve C_1 and C_1 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Theorem:-

If $f(z)$ is analytic within and on a closed curve C and a is any within the curve C, then ~~of C, then~~

$$\boxed{f(a) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-a}}$$

Extension of Cauchy's Integral Formula:-

1. $$f'(a) = \frac{\lfloor 1}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2}$$

$$f''(a) = \frac{\lfloor 2}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^3}.$$

$$f^n(a) = \frac{\lfloor n}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

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$$\int_C \frac{z^2-z+1}{z-1}\,dz, \quad \text{where } C \text{ is (i) } |z| = 1. \quad \text{(ii) } |z| = 1/2.$$

→ (i) Here,

$f(z) = z^2 - z + 1$ is analytic within and on C and $a = 1$

∴ By Cauchy's Integral Formula.

$$f(1) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-1}$$

$$\Rightarrow 2\pi i\,(1-1+1) = \int_C \frac{(z^2-z+1)\,dz}{z-1}$$

$$\Rightarrow \int_C \frac{z^2-z+1}{z-1}\,dz = 2\pi i.$$

(ii) Here,

$f(z) = z^2 - z + 1$ is analytic and $f'(z)$ is always continuous on C, since $a = 1$ lies outside C.

$$\therefore \int_C \frac{z^2-z+1}{z-1}\,dz = 0. \quad \text{[By Cauchy's Theorem].}$$

2. $$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz, \quad C \text{ is } |z| = 3.$$

3. $$\oint_C \frac{e^{tz}}{z^2+1}\,dz, \quad C \text{ is } |z| = 3.$$

$(\sin\pi z^2 + \cos\pi z^2)\, dz$ $z = a+ib$

2. → $\int_C \left(\frac{1}{z-2} - \frac{1}{z-1}\right)(\sin\pi z^2 + \cos\pi z^2)$

$\Rightarrow f(z) = \sin\pi z^2 + \cos\pi z^2$ is analytic within and on C (3)

and $a = 1, 2$.

$\sin 4\pi + \cos 4\pi$

$\oint_C \frac{e}{}$

$\therefore f(2) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-2}$

$4\pi i$

$\int \frac{f(z)dz}{z-2} = 2\pi i\, f(2)$

Also, $\int \frac{f(z)\,dz}{z-1} = 2\pi i\, f(1)$.

$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} = 2\pi i\{\sin 4\pi + \cos 4\pi - \sin\pi - \cos\pi\}$

$= 2\pi i\{0 + 1 - 0 - (-1)\}$

$= 4\pi i$. (4)

(3) $\oint_C \frac{e^{tz}\,dz}{z^2+1} = \int_C \frac{e^{tz}}{(z-i)(z+i)}\,dz$ [By . C.I.F.]

$= \frac{-1}{2i}\int\left(\frac{1}{(z+i)} - \frac{1}{(z-i)}\right)e^{tz}\,dz.$

$= \frac{-1}{2i}\left\{2\pi i\, f(-i) - 2\pi i\, f(i)\right\}$

$= \left(-i f(-i) + i f(i)\right)\pi.$

$= \pi i\left(-e^{-ti} + e^{it}\right)$

$= \pi i\left(e^{it} - e^{-it}\right)$

$\cancel{= \pi i(\sin t + i\cos t)}$

$= \pi i\left(\cos t + i\sin t - \cos t + i\sin t\right)$

$= 2\pi i \sin t.$

(4) $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$, C is $|z|=1$.

$\Rightarrow f(z) = \sin^2 z$ is analytic and $a = \pi/6$ lies inside C.

By extension of C.I.F.

$f''(\pi/6) = \frac{2}{2\pi i}\int \frac{f(z)\,dz}{(z-\pi/6)^3}$

$f'(z) = 2\sin z\cos z$
$= \sin 2z$

$f''(z) = 2\cos 2z.$

$f''(\pi/6) = 2\cos\pi/3$
$= 1.$

$\oint_C \frac{f(z)\,dz}{(z-\pi/6)^3} = \pi i\, f''(\pi/6).$

$= \pi i$

5. $\int_C \frac{e^{2z}}{(z+i)^4}dz$, C is $|z|=3$.

→ Here, $f(z)=e^{2z}$ is analytic and $a=-i$ lies inside C.

∴ By extension of C.I.F.

$$f'''(-i) = \frac{\cancel{2}\,3}{\cancel{2}\pi i}\int_C \frac{f(z)\,dz}{(z+i)^4}$$

$$\Rightarrow \int_C \frac{f(z)\,dz}{(z+i)^4} = \frac{\pi i}{3} f'''(-i)$$

$$= \frac{8\pi i}{3} e^{-2i}$$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f'''(-i) = 8e^{-2i}$

6. $\int_C \frac{5z-2}{z^2-z}dz$, where C is $|z|=2$.

7. $\int_C \frac{e^{2z}\,dz}{z^2+3z+2}$, where C is $|z|=3$.

(a) 6. $\int \frac{5z-2}{z^2-z}dz = \int \frac{5z-2}{z(z-1)}dz = \int \left(\frac{5z-2}{z-1} - \frac{5z-2}{z}\right)dz$

$f(z)=(5z-2)$ is analytic and $a=1,0$ lies inside C.

By C.I.F.:

$$\int_C \frac{f(z)\,dz}{z(z-1)} = 2\pi i f(1) - 2\pi i f(0)$$

$$= 2\pi i(5-2) - 2\pi i(0-2)$$

$$= 6\pi i + 4\pi i$$

$$= 10\pi i$$

7. $\int \frac{e^{2z}\,dz}{z^2-3z+2}$ ~~dz~~

$f(z) = e^{2z}$ is analytic and $a = 1, 2$ lies inside C.

$$\therefore \int \frac{e^{2z}\,dz}{z^2-3z+2} = \int \frac{e^{2z}\,dz}{(z-1)(z-2)}$$

$$= \int \left(\frac{1}{z-2} - \frac{1}{z-1}\right) e^{2z}\,dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i \left(e^4 - e^2\right).$$

8. Show that $\oint (z+1)\,dz = 0$, where C is boundary of a square whose vertices are at points $z=0, z=1, z=1+i, z=i$.

* Morera's Theorem:- If $f(z)$ is continuous in a region and $\oint f(z)dz = 0$ around every simple closed curve C in D, then $f(z)$ is analytic in D.

* Liouville's Theorem:- If $f(z)$ is analytic and bounded in entire complex plane, then $f(z)$ is a constant.

1. $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i|=2$.

2. $\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$ where C is $|z|=4$.

1. $\int_C \frac{(z-1)dz}{(z+1)^2(z-2)}$.

$$\frac{\cancel{(z-1)}1}{(z+1)^2(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{z-2}.$$

$$\Rightarrow \cancel{(z-1)}1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$$

For $z=-1$

$\cancel{-2}1 = -3B \Rightarrow B = \cancel{2/3} -1/3$

$z=2 \Rightarrow 1 = 9C \Rightarrow C = 1/9.$

$A+C=0$

$\Rightarrow A = -1/9.$

$\cancel{\int \frac{f}{z+1}}$ $\cancel{f(-1) = \frac{1}{2\pi i}\int \frac{f}{z+1}}$

$$\therefore \int_C \frac{(z-1)dz}{(z+1)^2(z-2)} = \int_C \left(\frac{-1}{9}\frac{1}{(z+1)} \cancel{+} -\frac{1}{3}\frac{1}{(z+1)^2} + \frac{1}{9}\frac{1}{z-2}\right) dz\,(z-1)$$

$\cancel{f(z_0)}$ $\cancel{= -\frac{1}{9}2\pi i(-2) + \frac{1}{9}2\pi i \cdot 1 + \frac{2}{3}2\pi i \cdot 0}$

$$= -\frac{1}{9}\cdot 2\pi i(-2) - \frac{1}{3}2\pi i + \frac{1}{9}2\pi i \cdot 1$$

$$= \frac{4\pi i}{9} + \frac{2\pi i}{9} - \frac{2\pi i}{3}$$

$$= 0.$$

$$\boxed{\text{Ans} = \frac{i}{\pi}}$$

2. $$\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$$

$$= \int_C \frac{e^z\,dz}{(z+i\pi)^2(z-i\pi)^2}.$$

$$\frac{1}{(z+i\pi)^2(z-i\pi)^2} = \frac{A}{z+i\pi} + \frac{B}{(z+i\pi)^2} + \frac{C}{(z-i\pi)} + \frac{D}{(z-i\pi)^2}$$

$$1 = A(z+i\pi)(z-i\pi)^2 + B(z-i\pi)^2 + C(z-i\pi)(z+i\pi)^2 + D(z+i\pi)^2$$

For $z = i\pi$ $\quad 1 = (2i\pi)^2 D \Rightarrow \cancel{\quad} D = \frac{-1}{4\pi^2}$

$z = -i\pi$, $\quad 1 = B(-2i\pi)^2$

$$B = \frac{-1}{4\pi^2}.$$

$A + C = 0.$

$\cancel{1 = A(z^3}$ $\quad 1 = \cancel{A(\quad} i\pi^3 A - B\pi^2 - Ci\pi^3 - D\pi^2$

$\Rightarrow (A-C)\,i\pi^3 = 1 + \pi^2(B+D)$

$\Rightarrow (A-C) = \left(1 + \pi^2\left(\frac{-1}{2\pi^2}\right)\right)\frac{1}{i\pi^3} = \frac{1}{i\pi^3}.$

$\therefore A = \frac{1}{2i\pi^3}$, $\quad C = \frac{-1}{2i\pi^3}$

$$\int \frac{f(z)\,dz}{(z^2+\pi^2)^2}$$

$$= \frac{1}{\cancel{2}\cancel{i}\pi^3}\cancel{2\pi i}\, f(-i\pi) + \left(\frac{-1}{4\pi^2}\right) 2\pi i\, f'(-i\pi) + \frac{-1}{\cancel{2}\cancel{i}\pi^3}\cancel{2\pi i}\, f(i\pi) - \frac{1}{4\pi^2} 2\pi i\, f'(i\pi)$$

$$= \frac{1}{\pi^2} f(-i\pi) - \frac{i}{2\pi} f'(-i\pi) - \frac{1}{\pi^2} f(i\pi) - \frac{i}{2\pi} f'(i\pi)$$

$$= \frac{1}{\pi^2}\left(e^{-i\pi} - e^{i\pi}\right) - \frac{i}{2\pi}\left(e^{-i\pi} + e^{i\pi}\right)$$

$$= \frac{1}{\pi^2}\left(\cancel{\cos\pi} - i\sin\pi - \cancel{\cos\pi} - i\sin\pi\right) - \frac{i}{2\pi}\left(\cos\pi - \cancel{i\sin\pi} + \cos\pi + \cancel{i\sin\pi}\right)$$

$$= \frac{-2i\sin\pi}{\pi^2} - \frac{i}{2\pi}2\cos\pi$$

$$= 0 - \frac{i}{2\pi}2\cdot(-1) = i/\pi.$$

* Singularities of Functions:- Poles, Residues, Cauchy's Residue Theorem and its Applica for finding Definite Integral.

* Singularity:-

A singular point of a function is that point at which the function ~~seizes~~ ceases to be analytic.

Laurent series of $f(z)$ about $z=a$ is

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$
$$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

Types of singularity:-

1. Isolated singularity → If $z=a$, is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood, then $z=a$ is called isolated singularity.

2. Removable Singularity:- If all the negative powers of $(z-a)$ in Laurent series is zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ and $z=a$ is called removable singularity.

$$\lim_{z\to a} f(z) = \text{finite}.$$

3. Poles:- If all the negative powers of $(z-a)$ in Laurent series of $f(z)$ after the n^{th} are missing, then $z=a$ is called a pole of order n.

A pole of first order is called a simple pole.

4. Essential singularity:- If the no. of negative powers of $(z-a)$ in Laurent series of $f(z)$ is infinite, then $z=a$ is called essential singularity

$$\lim_{z\to a} f(z) = \infty.$$

1. Find the nature of singularities of following f^n :-

1(a) $\dfrac{z-\sin z}{z^2}$

$$= \frac{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\cdots\right)}{z^2}$$

$$= \frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}-\cdots\cdots$$

$\because$ There are no negative powers of z in expansion,

$\therefore z=0$ is a removable singularity.

1(b) $(z+1)\sin\dfrac{1}{(z-2)}$

Let $z-2=t$

$z+1 = t+3$.

$(t+3)\sin\frac{1}{t}$.

$$\Rightarrow (t+3)\left\{\frac{1}{t}-\left(\frac{1}{t}\right)^3\frac{1}{3!}+\left(\frac{1}{t}\right)^5\frac{1}{5!}-\cdots\cdots\right)$$

$$\Rightarrow \left(1-\frac{1}{t^2 3!}+\frac{1}{t^4 5!}+\cdots\right)+3\left(\frac{1}{t}-\left(\frac{1}{t}\right)^3\frac{1}{3!}+\left(\frac{1}{t}\right)^5\frac{1}{5!}+\cdots\right)$$

$$= \left(1+\frac{3}{t}-\frac{1}{t^2\,3!}-\frac{3}{t^3\,3!}+\cdots\cdots\right)$$

$$= \left(1+\frac{3}{(z-2)}-\frac{1}{3!(z-2)^2}+\cdots\cdots\right)$$

$\because$ There are infinite number of negative powers of z.

$\therefore z=2$ is a essential singularity.

1(c) $e^{1/z}$

$$= \cancel{\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\cdots\right)} = \left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2\frac{1}{2!}+\cdots\cdots\right)$$

$\therefore z=0$ is an essential singularity

* **Residue**:- The coefficient of $(z-a)^{-1}$ is the expansion of Laurent series of $f(z)$ around an isolated singularity $z=a$ is called residue of $f(z)$ at $z=a$.

$$a_{-1} = \text{Residue of } f(z) \text{ at } z=a.$$

$$= \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

$$\Rightarrow \oint_C f(z)dz = a_{-1} 2\pi i \quad \cdot u$$

* **Cauchy's Residue Theorem**:- If $f(z)$ is analytic in a closed curve C except at a finite no. of singular points within C, then

$$\oint_C f(z)\, dz = 2\pi i\,(\text{sum of residues at singular points}).$$

* **Formula for finding residue**:-

1. If $f(z)$ has simple pole at $z=a$, then residue at $z=a$ is given by $\lim_{z\to a} (z-a)\, f(z)$

2. Residue at a pole $z=a$ of order n ~~is~~ is

$$\lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$$

Q. 1. Find all the residues of ~~(i)~~ $f(z) = \dfrac{\sin z}{z\cos z}$ at all its poles inside circle $|z| = 2$. ~~Hence~~

Q. 2. Find all the poles of $f(z) = \dfrac{z^2}{(z-1)^2\,(z+2)}$ and also residues at its poles. Hence, find ~~$f(z)dz$~~ integral at c of $|z|=$

2. → $f(z)$ has simple pole at $z = \cancel{a} = -2$ and a pole of order 2 at $z=1$.

Residue at $z=-2$ is $\lim_{z\to -2} (z+2)\, f(z)$

$$\Rightarrow \lim \, (z+2) \, \frac{z^2}{\ldots}$$

$= \frac{4}{9}$.

Residue at $z=1$ is $\lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{z^2}{(z-1)^2(z+2)}\right]$

$= \lim_{z\to 1} \frac{d}{dz} \frac{z^2}{(z+2)}$ $\Rightarrow \cancel{\lim_{z\to 1} \frac{d}{dz} \frac{z^2}{(z+2)^2}}$

$= \lim_{z\to 1} \frac{(z+2)2z - z^2}{(z+2)^2}$ $= \cancel{\lim_{z\to 1} \frac{(z+2)^2 2z - z^2 2(z+2)}{(z+2)^4}}$

$\Rightarrow \lim_{z\to 1} \frac{z^2+4z}{(z+2)^2}$ $= \cancel{\lim \frac{18-6}{81}}$

$\Rightarrow 5/9$. $= \cancel{\frac{12}{81}}$

$f(z)$ is analytic with C, $|z|=2.5$ except at

$\because$ ~~Both~~ the poles $z=1$ and $z=-2$,

$\therefore$ By C.R.T,

$$\oint_C f(z)dz = 2\pi i\,(\text{sum of residues})$$
$$= 2\pi i\left(\frac{4}{9}+\frac{5}{9}\right)$$
$$= 2\pi i.$$

1. $f(z)$ has simple poles at $z=0, \pm\pi/2, \pm 3\pi/2, \ldots$

Only, the simple poles $z=0$ and $z=\pm\pi/2$ lies inside C.

$\therefore$ Residue at $z=0$ is $\lim_{z\to 0} z\frac{\sin z}{z\cos z}$

$$= \lim_{z\to 0} \tan z$$
$$= 0.$$

Residue at $z=\pi/2$ is $\lim_{z\to \pi/2} (z-\pi/2)\frac{\sin z}{z\cos z}$.

$\cancel{\Rightarrow \lim_{z\to\pi/2}(z-\pi/2)}$

$$= \lim_{z\to\pi/2}\left\{\frac{(z-\pi/2)\sin z}{z\cos z}\right\}$$
$$= \lim_{z\to\pi/2}\left\{\frac{(z-\pi/2)\cos z + \sin z}{\cos z - z\sin z}\right\}$$
$$= \frac{-2}{\pi}.$$

Residue at $z=-\pi/2$ is

$$\lim_{z\to-\pi/2} \frac{(z+\pi/2)\sin z}{z\cos z}.$$

$$= \lim_{z\to-\pi/2} \left\{\frac{(z+\pi/2)\cos z+\sin z}{\cos z - z\sin z}\right\}$$

$$= \frac{-1}{-\pi/2} = \frac{2}{\pi}.$$

$$\text{Residue} = 0 - \frac{2}{\pi} + \frac{2}{\pi}$$

$$= 0.$$

3. Find residues of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at all its poles and hence evaluate $\int_C f(z)dz$ and C is $|z|=2.5$.

4. Evaluate $\oint_C \dfrac{z-3}{(z^2+2z+5)}$ over C, where C is

(i) $|z|=1$

(ii) $|z-i+1|=2$

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* Integration around unit circle of the form :-

(i) $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$.

Put $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta$

$\Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$

$\sin\theta = \frac{z - \frac{1}{z}}{2i}$; $\cos\theta = \frac{z + \frac{1}{z}}{2}$.

$= \int_C f(z)\, dz$, where C is $|z| = 1$.

1. Evaluate :-

$dz = ie^{i\theta} d\theta \Rightarrow \frac{dz}{iz}$

(a) $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$. $a>0, b>0$

Put $z = e^{i\theta} \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$

$\cos\theta = \frac{z+\frac{1}{z}}{2}$

$$= \oint_C \frac{\frac{1}{i}\frac{dz}{z}}{a + b\left(\frac{z + 1/z}{2}\right)}$$

$$= \oint_C \frac{1}{i}\frac{2\,dz}{2az + b(z^2+1)}$$

$$= \frac{2}{i}\oint_C \frac{dz}{2az + b(z^2+1)}$$

$$= \frac{2}{ib}\oint_C \frac{dz}{z^2 + \frac{2a}{b}z + 1}$$

$$= \frac{2}{ib}\oint_C \frac{dz}{z^2 + \frac{2a}{b}z + 1}$$

$= \frac{2}{ib} \cdot 2\pi i \sum R_i$ [By Cauchy's Residue theorem]

$I = \frac{4\pi}{b}\sum R_i$ — (i) where $\sum R_i$ denotes the sum of residues of $f(z) = z^2 + \frac{2a}{b}z + 1$, at the poles within and on C, $|z| = 1$.

Poles of $f(z)$ are given by $z^2 + \frac{2a}{b}z + 1 = 0$.

$$z = \frac{-2a/b \pm \sqrt{\frac{4a^2}{b^2} - 4}}{2}$$

$$z = -a/b \pm \sqrt{\frac{a^2}{b^2} - 1}$$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$z_1 = \frac{-a \pm \sqrt{a^2-b^2}}{b} \quad ; \quad z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}.$$

Here, only simple pole z_1 lies within C.

Residue at $z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}$ is

$$\lim_{z \to z_1} (z - z_1)\, f(z)$$

$$\lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ z - \left(\cancel{\frac{-a+\sqrt{a^2-b^2}}{b}} \right) \right\} \frac{1}{\left(z - \cancel{\frac{-a+\sqrt{a^2-b^2}}{b}}\right)\left(z - \frac{-a-\sqrt{a^2-b^2}}{b}\right)}$$

$$\Rightarrow \lim_{z \to \frac{-a \pm \sqrt{a^2-b^2}}{b}} \frac{1}{z - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)}$$

$$\Rightarrow \frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} - \frac{-a-\sqrt{a^2-b^2}}{b}}$$

$$= \frac{b}{-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}}$$

$$= \frac{b}{2\sqrt{a^2-b^2}}.$$

From (i)

$$I = \frac{4\pi}{b} \cdot \frac{b}{2\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}.$$

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. (c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$, $a^2<1$.

(d) $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$. Ans. $\left(\frac{\pi}{12}\right)$ (e) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$, a^2 [

(d) $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$

Put $z=e^{i\theta}$

$d\theta = \frac{1}{i}\frac{dz}{z}$

$\cos\theta = \frac{z+1/z}{2}$

$\cancel{\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4\left(\frac{z+\frac{1}{z}}{2}\right)^3 - 3/2 \ldots = \frac{1}{2}\left\{\left(z+\frac{1}{z}\right)\right\}\left\{(z\ldots\right.}$

$$\cancel{= \oint_C \frac{\frac{1}{2}(z+1/z)\{(z+\frac{1}{z})^2-3\}\frac{dz}{iz}}{5-\frac{4}{2}\left(\frac{z+1/z}{z}\right)}}$$

$$\cancel{= \oint_C \frac{1}{2iz}\frac{(z+1/z)\{(z+1/z)^2-3\}dz}{5-2(z+1/z)}}$$

$$\cancel{= \oint_C \frac{}{2iz}}$$

$\cos 3\theta = \frac{e^{3i\theta}+e^{-3i\theta}}{2}$

$= \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)$

$$= \oint_C \frac{\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)\frac{dz}{iz}}{5-4\left(\frac{z+1/z}{2}\right)}$$

$$= \oint_C \frac{1}{2}\frac{(z^6+1)dz \cdot z}{iz^4\{5z-2z^2-2\}}$$

$$= \frac{-1}{2i}\oint_C \frac{(z^6+1)\,dz}{z^3(2z^2-5z+2)}$$

$$= \frac{-1}{2i}\oint_C \frac{(z^6+1)\,dz}{z^3(2z-1)(z-2)}$$

$= \frac{-1}{\cancel{2i}} \cdot \cancel{2}\pi i \sum R_i$ [By Cauchy's Integral Formula]

$I = -\pi \sum R_i$ — (i)

Poles of $f(z)$ are given by $z=0$ (order 3), $z=2, 1/2$ (ord

But only simple pole $z=1/2$ and $z=0$ (order 3) lies within c

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Residue at $z=0$, $R_1 = \lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2}\left\{\frac{(z^6+1)}{(z-2)(2z-1)}\right\}$

$$\lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2}\left\{\frac{z^6+1}{2z^2-5z+2}\right\}$$

$$\lim_{z\to 0} \frac{1}{2}\frac{d}{dz}\left\{\frac{6z^5(2z^2-5z+2)-(z^6+1)(4z-5)}{(2z^2-5z+2)^2}\right\}$$

$$\lim_{z\to 0} \frac{1}{2}\frac{d}{dz}\left\{\frac{12z^7-30z^6+12z^5-4z^7+5z^6-4z+5}{(2z^2-5z+2)^2}\right\}$$

$$\lim_{z\to 0} \frac{1}{2}\frac{(2z^2-5z+2)^2(56z^6-125z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)\,2(2z^2-5z+2)(4z-5)}{(2z^2-5z+2)^4}$$

$$=\frac{1}{2}\left\{\frac{4(-4)-10\cdot 2(-5)}{16}\right\}$$

$$=\frac{1}{2}\left\{\frac{-16+100}{16}\right\}$$

$$=\frac{21}{8}.$$

$R_2 =$ Residue at $z=1/2$

$$\lim_{z\to 1/2} \frac{z^6+1}{2(z-2)(z-0)^3}$$

$$=\frac{1/64+1}{2(1/2-2)\,1/2} = \frac{65/64}{2(-3/2)(1/2)^3} = \cancel{\ }\ .\ -\frac{65}{24}$$

$$I = -\pi\left(\frac{21}{8}-\frac{65}{24}\right)$$

$$= \pi\cdot 2/24 = \frac{\pi}{12}.$$

(e) $\oint_C \frac{\left(\frac{z^2+1/z^2}{2}\right)\frac{dz}{iz}}{1-2a\left(\frac{z+1/z}{2}\right)+a^2}$

$$=\oint_C \frac{(z^4+1)\,dz\cdot z}{2z^2 i\,(z-az^2-a+a^2z)}$$

$$=\frac{1}{2i}\oint_C \frac{(z^4+1)\,dz}{z^2\left(-az^2+(a^2+1)z-a\right)}$$

$$=\frac{1}{2i}\int_C \frac{(z^4+1)\,dz}{z^2(z-a)\cancel{(z-1/a)}(1-za)} = \frac{2\pi i}{2i}\sum R_i$$

$\because a^2<1 \Rightarrow \frac{1}{a^2}>1 \Rightarrow \frac{1}{a}>1$ lies outside C.

$$\operatorname{Res} f(a) = \lim_{z \to a} \frac{z^4+1}{z^2(z - 1/a)} = \frac{a^4+1}{a^2(a-1/a)} = \frac{a^4+1}{(a^2-1)a} = \frac{a^4+1}{(1-a^2)a^2}$$

$$\operatorname{Res} f(0) = \lim_{z \to 0} \frac{d}{dz} \left\{ \frac{z^4+1}{(-az^2+(a^2+1)z-a)} \right\}$$

$$= \lim_{z \to 0} \frac{(-az^2+(a^2+1)z-a)(4z^3) - (z^4+1)(-2az+a^2+1)}{(-az^2+(a^2+1)z-a)^2}$$

$$= \frac{-(a^2+1)}{a^2}$$

$$\therefore I = \frac{2\pi i}{2i}\left(-\frac{a^4+1}{a^2(a^2-1)} - \frac{a^2+1}{a^2} \right)$$

$$= \frac{\pi}{a^2}\left\{ \frac{-a^4-1-a^4+1}{a^2-1} \right\}$$

$$= \frac{\pi}{a^2}\left(\frac{-2a^4}{a^2-1} \right)$$

$$= \frac{2\pi a^2}{1-a^2}.$$

(c)

$$\int_C \frac{\frac{dz}{iz}}{1 + a\left(\frac{z+1/z}{2}\right)}$$

$$= \frac{2}{i}\int_C \frac{dz}{2z + az^2 + a}$$

$$= 2\pi i \cdot \frac{2}{i} \int f(z)\, dz$$

$$= 4\pi \sum R_i$$

$$\operatorname{Res} f\left(\frac{-1+\sqrt{1-a^2}}{2}\right) = \lim_{z \to \frac{-1+\sqrt{1-a^2}}{2}} \left(z - \frac{-1+\sqrt{1-a^2}}{2}\right) \frac{1}{\left(z - \frac{-1+\sqrt{1-a^2}}{2}\right)\left(z - \frac{-1-\sqrt{1-a^2}}{2}\right)}$$

$$= \lim \frac{1}{\frac{-1+\sqrt{1-a^2}}{2} - \frac{-1-\sqrt{1-a^2}}{2}}$$

$$= \frac{1}{2\sqrt{1-a^2}}$$

$$\therefore I = \frac{4\pi}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}.$$

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(ii) Integration around a small semicircle of the form:-

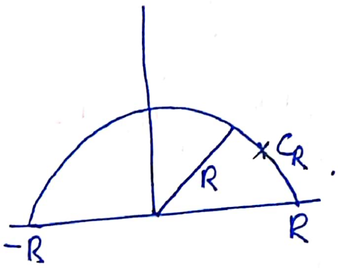
To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)\,dz$ where C is the contour consisting of semicircle $C_R : |z| = R$, together with the diameter that closes it.

Suppose $f(z)$ has no singular point on real axis, then by Cauchy's Residue theorem $\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$

$= 2\pi i \sum R_i$.

Finally, making $R \to \infty$, we get the value of

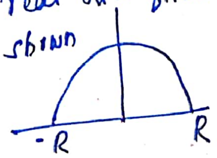
$\int_{-\infty}^{\infty} f(x)\,dx$, provided $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.



1. Evaluate:-

(a) $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Consider $\int_C \frac{z^2\,dz}{(z^2+1)(z^2+4)} = \int_C f(z)\,dz$, where C is the contour consisting of semicircle C_R of radius R, together with part of real axis from $-R$ to $+R$ as shown



Poles of $f(z)$ are given by

$(z^2+1)(z^2+4) = 0$

$\therefore z = \pm i,\ z = \pm 2i$.

Only the simple poles, $z = i$ and $z = 2i$ lies inside C (other's lies below x-axis).

$\therefore$ By C.I.T. $\int_C f(z)\,dz = 2\pi i \sum R_i$

Residue at $z = i$. $\lim_{z \to i} \cancel{(z-i)} \dfrac{z^2}{(z+i)\cancel{(z-i)}(z^2+4)}$

$$= \lim_{z \to i} \frac{z^2}{(z+i)3}$$

$$\Rightarrow \frac{-1}{2i} = \frac{-1}{6i} = \frac{i}{6}.$$

Residue at $z = 2i$

$$\lim_{z \to 2i} \cancel{(z-2i)} \frac{z^2}{(z^2+1)(z+2i)\cancel{(z-2i)}}$$

$$= \frac{-4}{-3.4i}$$

$$= \frac{-i}{3}.$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right) = \frac{\pi}{3}. \quad -(i)$$

Also $\displaystyle\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$ — (ii)

Making $R \to \infty$, we will now show that

$\displaystyle\int_{C_R} f(z)\,dz$ vanishes.

For any point on C_R, as $|z| \to \infty$

$f(z) = \dfrac{1}{z^2(1+z^{-2})(1+4z^{-2})}$, which decreases as $1/z^2$

and tends to 0, whereas the length of C_R increases with z.

consequently, $\displaystyle\lim_{|z| \to \infty} \int_{C_R} f(z)\,dz = 0.$

Hence, from (i) and (ii) we get (taking $R \to \infty$)

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}.$$

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(b) $\int_0^\infty \frac{\cos ax \, dx}{1+x^2}$ $\left[= \frac{\pi}{2} e^{-a}\right.$

$=$ Consider $\int_C \frac{\cos az}{z^2+1} dz = \int_C f(z)dz$, where C is contour consisting of semicircle C_R together with a part of real axis from (−R) to R.

$= \int \frac{e^{iaz}}{z^2+1} dz$

Poles of $f(z)$ are given by

$z^2+1=0$

$z = \pm i$

Only simple pole $z=i$ lies inside C.

$\therefore$ Residue at $z=i$, $\lim_{z\to i} \frac{\cos az}{(z+i)} = \frac{\cos ia}{2i}$

$\lim_{z\to i} \frac{e^{iaz}}{2i} = \frac{e^{-a}}{2i}$

$\therefore \int_C f(z)\,dz = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi e^{-a}$ — (i)

Also, $\int_C f(z)dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$ — (ii)

Making $R\to\infty$, we will show that

$\int_{C_R} f(z)dz$ vanishes.

For any point on C_R, as $|z|\to\infty$.

$|z| = R$ on C_R and $|z^2+1| \geq R^2-1$

$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax}.e^{-ay}| = e^{-ay} < 1$ $[\because y > 0]$

$\therefore \left|\frac{e^{iaz}}{z^2+1}\right| = |e^{iaz}| \frac{1}{|z^2+1|} < \frac{1}{R^2-1}$

$\therefore \int_{C_R} f(z)\,dz = \left|\int_{C_R} \frac{e^{iaz}}{z^2+1} dz\right| < \int_{C_R} \frac{1}{R^2-1} |dz| < \frac{\pi R}{R^2-1}$

which tends to 0 as $R\to\infty$.

$$\therefore \pi e^{-a} = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

Equating real and imaginary part.

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}.$$

$\because \cos ax$ is an even function

$$\therefore 2\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

$$\Rightarrow I = \frac{\pi e^{-a}}{2}$$