Books

- Gupta (for Numerical)
- → ① N. K. Bajaj
- ② A. P. French

Syllabus :-

1) Waves and Oscillations
2) E. M. Theory
3) Quantum Mechanics
4) Solid State Physics

Absolute Marking

Distribution

End term – 50

mid term – 30

Minor Test – 10

Attendance
- 5 (Class)
- 5 (Tutorial)

100 – 91 →	AA	10
90 – 81 →	AB	9
80 – 71 →	BB	8
70 – 61 →	BC	7
60 – 51 →	CC	6

30 % Marks Numerical

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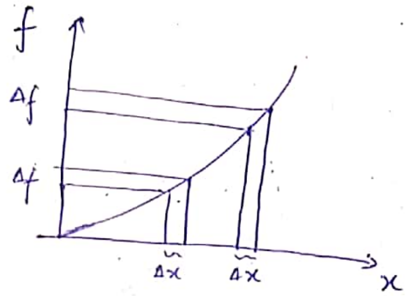
Introduction

· Tutorial-1

① Gradient

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\Delta f = \left(\frac{df}{dx}\right) \Delta x$$



change in function is directly proportional to independent variable

$\phi(x, y, z)$

$$\delta\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$$

$$\delta\phi = \left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right) \cdot \left(\hat{x}\, dx + \hat{y}\, dy + \hat{z}\, dz\right)$$

$$\boxed{\delta\phi = (\vec{\nabla}\phi) \cdot d\vec{r}}$$

very small displacement

gradient of ϕ

[Max slope at that particular point of funⁿ]

★ Gradient of function gives maxᵐ slope at particular pt.

$$\vec{\nabla}\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\phi$$

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$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

↳ del operator / vector operator

- Gradient is always taken on scalar funcn.
- Gradient of scalar funn is vector quantity.

(Q) $\phi = \frac{1}{r}$ $\quad r = \sqrt{x^2+y^2+z^2}$

(A) Find Gradient

$$\phi = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

$$= \hat{x}\left(-\frac{1}{2}\right)(2x)(x^2+y^2+z^2)^{-3/2}$$

$$+ \hat{y}\left(-\frac{1}{2}\right)(2y)(x^2+y^2+z^2)^{-3/2}$$

$$+ \hat{z}\left(-\frac{1}{2}\right)(2z)(x^2+y^2+z^2)^{-3/2}$$

$$\Rightarrow \left(-(x^2+y^2+z^2)^{-3/2}\right)[x\hat{x} + y\hat{y} + z\hat{z}]$$

OR

$$\vec{\nabla}\phi = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= \hat{x}(x^2+y^2+z^2)^{-3/2}\left(-\frac{1}{2}\right)(2x) + \text{---} \text{---}$$

$$= -\frac{[x\hat{x} + y\hat{y} + z\hat{z}]}{(x^2+y^2+z^2)^{3/2}} = -\frac{\vec{r}}{r^3}$$

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(5) $\phi = \ln r$ $\quad \vec{\nabla} . \phi = ?$

<u>Ans</u> $\vec{\nabla}\phi = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(\phi)$

$$= \frac{1}{2}\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\ln(x^2+y^2+z^2)\right)$$

$\Rightarrow$ $\cancel{x\hat{x} + y\hat{y} +}$

$$\Rightarrow \frac{1}{2}\left(\hat{x}(2x)\frac{1}{x^2+y^2+z^2} + \hat{y}(2y)\frac{1}{x^2+y^2+z^2} + \hat{z}(2z)\frac{1}{x^2+y^2+z^2}\right)$$

$$= (x\hat{x} + y\hat{y} + z\hat{z})\left(\frac{1}{x^2+y^2+z^2}\right)$$

$$= r\vec{r}$$

$$\delta\phi = (\vec{\nabla}\phi).\vec{dr} = |\vec{\nabla}\phi|\,|\vec{dr}|\cos\theta$$

$$|\vec{dr}| = \sqrt{dx^2+dy^2+dz^2}$$

when $\theta = 0$ $\quad \partial\phi$ is maximum

- directional derivative of ϕ along $\vec{A}$ is given as $(\vec{\nabla}\phi) . \frac{\vec{A}}{|\vec{A}|}$

A. (B+C)

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Q Find directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\hat{x} - \hat{y} - 2\hat{k}$.

$$(\vec{\nabla}\phi)_{(1,-2,-1)} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= \hat{x}\,2xyz + \hat{y}\,x^2z + \hat{z}x^2y$$
$$+ 4\hat{x}z^2 + 0 + \hat{z}(8xz)$$

$$= \cancel{x}(2xyz + 4z^2)\hat{x}$$
$$+ zx^2\hat{y} + (x^2y + 8xz)\hat{z}$$

$$= (+4+4)\hat{x} + (-\hat{y}) + (-2 \mp 8)\hat{z}$$

$$\Rightarrow 8\hat{x} - \hat{y} - 10\hat{z}$$

$$(\vec{\nabla}\phi)_{1,-2,-1} \cdot \frac{\vec{A}}{|\vec{A}|} = \frac{\cancel{37}}{3}(8\hat{x} - \hat{y} - 10\hat{z}) \cdot \frac{(2\hat{x} - \hat{y} - 2\hat{z})}{\sqrt{9}}$$

$$= \frac{37}{3}$$

$$\text{Max}^m \text{ slope} = \text{magnitude of gradient}$$

$$= \sqrt{100+64+1} = \sqrt{165}$$

End

$x_1 \rightarrow$ sol^n

$x_2 \rightarrow$ sol^n

$x_1 + x_2 \rightarrow$ will also be a sol^n

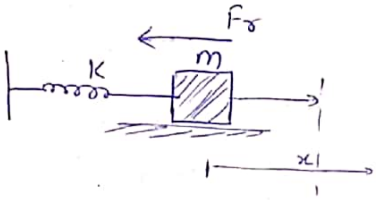
$\downarrow$

Linear eq^n

1) Waves and Oscillations

Books

1. Vibration and waves By A. P. French
2. The Physics of waves & oscillation N. K. Bajaj



$$F_r = -kx$$

$$m\frac{d^2x}{dt^2} = -kx \qquad \sqrt{\frac{k}{m}} = \omega_0$$

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$$

Linear Eq^n :-

If x_1 is a sol^n & x_2 is a sol^n then $c_1x_1 + c_2x_2$ should be a sol^n. Obeys Superposition principle.

$x_1 = a\sin\omega t$

$x_2 = b\cos\omega t$

general sol^n

$x = x_1 + x_2$

$= a\sin\omega t + b\cos\omega t$

$x = Ae^{\alpha t}$

$$\frac{dx}{dt} = A\alpha e^{\alpha t} \qquad \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

$$\frac{d^2x}{dt^2} = -\omega_0^2 x \Rightarrow A\alpha^2 e^{\alpha t} = -\omega_0^2 x$$

$$\alpha^2 = -\omega_0^2$$

$$\boxed{\alpha = \pm i\omega_0}$$

$\alpha = i\omega_0$, $\alpha = -i\omega_0$

$$x = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}$$

$$= A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t - i\sin\omega t)$$

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$$= (A_1 + A_2)\cos\omega t + i(A_1 - A_2)\sin\omega t$$

$$A_1 + A_2 = A\cos\delta \qquad i(A_1 - A_2) = A\sin\delta$$

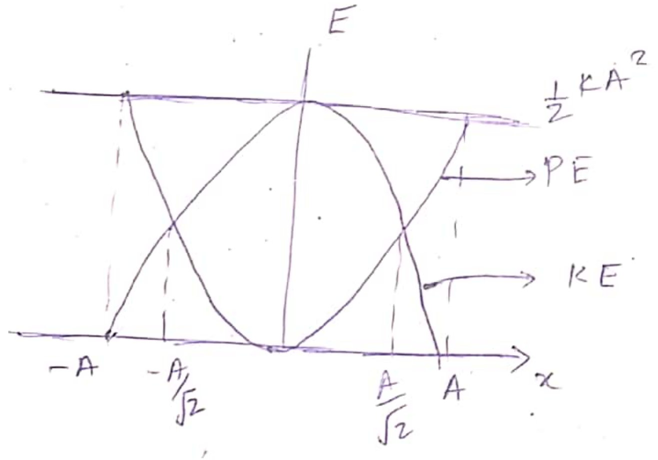
$$x = A\cos\delta \cdot \cos\omega t + A\sin\delta \cdot \sin\omega t$$

$$\boxed{x = A\cos(\omega t + \delta)}$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \delta)$$

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \delta)$$

Total Energy $= KE + PE = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0 A^2$



Damped Oscillation

$$F_d = -P\frac{dx}{dt} \qquad P = \text{damping coefficient}$$

eq^n of motion

$$m\frac{d^2x}{dt^2} = -kx - P\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{P}{m}\frac{dx}{dt}$$

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$\boxed{\frac{b}{m} = \gamma \text{ (Gamma)}}$ $\boxed{\frac{k}{m} = \omega_0^2}$ ω_0 = natural frequency

$$\boxed{\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0}$$

2nd order linear homogeneous

$x = Ae^{\alpha t}$ $\quad \frac{dx}{dt} = A\alpha e^{\alpha t}$ $\quad \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$

$$A\alpha^2 e^{\alpha t} + \gamma A\alpha e^{\alpha t} + \omega_0^2 x = 0$$

$$\alpha^2 x + \alpha \gamma x + \omega_0^2 x = 0$$

$\boxed{\alpha^2 + \gamma\alpha + \omega_0^2 = 0}$ Characteristics equation.

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

$$\alpha_1 = -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}, \quad \alpha_2 = -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

General Sol^n

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

(i) Under damping :- $(\gamma < 2\omega_0)$
(weak)

$$\left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2} = i\omega$$

$$\boxed{\omega = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}}$$

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$$x = A_1 \exp\left[-\frac{\gamma}{2} + i\omega\right]t + A_2 \exp\left(-\frac{\gamma}{2} - i\omega\right)t$$

$$x = e^{-\frac{\gamma}{2}t}\left[A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right]$$

$$\boxed{x = \underbrace{A e^{-\frac{\gamma t}{2}}}_{\text{Amplitude}} \cos(\omega t - \delta)}$$

$$\frac{dx}{dt} = A\left(-\frac{\gamma}{2}\right) e^{-\frac{\gamma}{2}t} \cos(\omega t - \delta) - A e^{-\frac{\gamma t}{2}} \omega \sin(\omega t - \delta)$$

at $t = 0$, $x = 0$, $\frac{dx}{dt} = V_0$

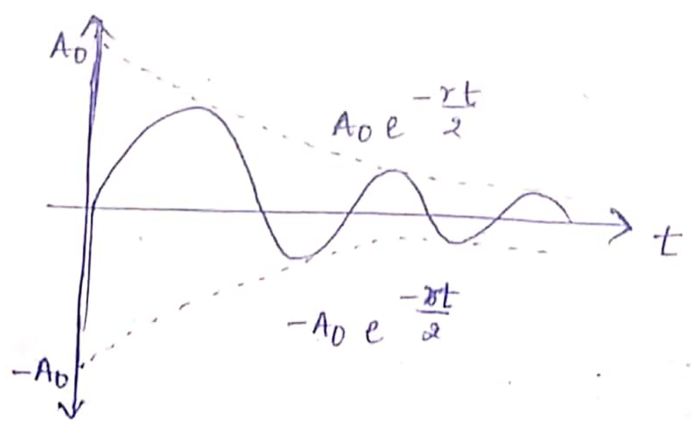
$$0 = A \cos \delta \qquad \boxed{\delta = \frac{\pi}{2}}$$

$\cancel{V_0 = A\left(-\frac{\gamma}{2}\right)}$ $\quad V_0 = 0 + A\omega$

$$A = \frac{V_0}{\omega}$$

$$x = \frac{V_0}{\omega} e^{-\frac{\gamma t}{2}} \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\boxed{x = \frac{V_0}{\omega} e^{-\frac{\gamma t}{2}} \sin(\omega t)} \Rightarrow x = A(t) \sin \omega t$$



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$$\omega = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2} \qquad T = \frac{2\pi}{\omega}$$

Not exactly periodic but Time period can be defined.

$$\boxed{t_{max} = \frac{1}{\omega} \tan^{-1}\left(\frac{2\gamma}{\omega}\right)}$$

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ii) Large - damping :- $(\gamma > 2\omega_0)$

$$\left(\frac{\gamma^2}{4} - \omega^2\right)^{1/2} = q$$

$$x = A_1 \exp\left[-\frac{\gamma}{2} + q\right]t + A_2 \exp\left[-\frac{\gamma}{2} - q\right]t$$

$$\frac{dx}{dt} = A_1\left(-\frac{\gamma}{2} + q\right)\exp\left[-\frac{\gamma}{2}+q\right]t + A_2\left(-\frac{\gamma}{2} - q\right)\exp\left[-\frac{\gamma}{2}-q\right]t$$

Condⁿ I

at $t = 0$, $x = 0$, $\frac{dx}{dt} = V_0$

$$\Rightarrow 0 = A_1 + A_2 \Rightarrow A_1 = -A_2$$

Condⁿ II

$$V_0 = A_1\left(-\frac{\gamma}{2} + q\right) + A_2\left(-\frac{\gamma}{2} - q\right)$$

$$= A_1\left(-\frac{\gamma}{2} + q\right) - A_1\left(-\frac{\gamma}{2} - q\right)$$

$$= A_1\left(-\cancel{\frac{\gamma}{2}} + q + \cancel{\frac{\gamma}{2}} + q\right)$$

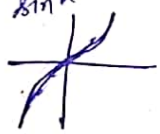
$$V_0 = 2q A_1$$

$$\boxed{A_1 = \frac{V_0}{2q}}$$

$$x = \frac{V_0}{2q}\left[e^{\left(-\frac{\gamma}{2}+q\right)t} - e^{\left(-\frac{\gamma}{2}-q\right)t}\right]$$

$$\boxed{x = \frac{V_0}{2q} e^{-\frac{\gamma t}{2}}\left(e^{qt} - e^{-qt}\right)}$$

sinhx



$$x = \frac{V_0}{2q} e^{-\frac{\gamma t}{2}} \cdot \sinh qt$$

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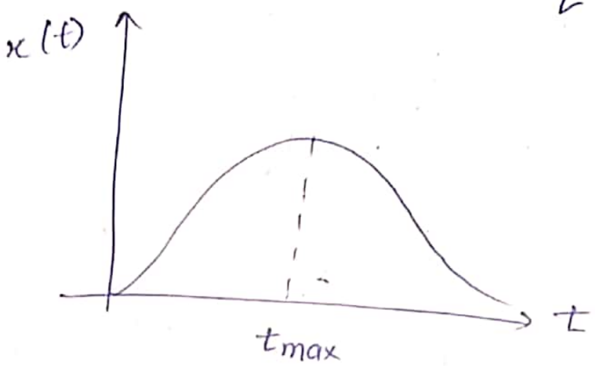
(ii) $x = \frac{v_0}{q} e^{-\frac{rt}{2}} \sinh qt$

$$t_{max} = \frac{1}{q} \tanh^{-1}\left(\frac{2q}{r}\right)$$



<u>For t_{max}</u>

$$\frac{dx}{dt} = \frac{v_0}{q}\left(-\frac{r}{2}\right) e^{-\frac{rt}{2}} \sinh qt + q \cdot \frac{v_0}{q} e^{-\frac{rt}{2}} \cosh qt$$

$$0 = \frac{v_0}{q}\left(-\frac{r}{2}\right) e^{-\frac{rt}{2}}\left(\frac{e^{qt} - e^{-qt}}{2}\right) + v_0 e^{-rt}\left(\frac{e^{qt} + e^{-qt}}{2}\right)$$

$$\cancel{-\frac{rv_0}{4q}\, e^{qt - \frac{rt}{2}}}$$

$$0 = e^{qt - \frac{rt}{2}}\left(-\frac{rv_0}{4q} + \frac{v_0}{2}\right) + e^{-qt - \frac{rt}{2}}\left(+\frac{rv_0}{4} + \frac{v_0}{2}\right)$$

III Critical damping :- $(r = 2\omega_0)$

$$\left(\frac{r^2}{4} - \omega_0^2\right)^{1/2} = 0$$

$$x = A_1 e^{-\frac{r}{2}t} + A_2 e^{-\frac{r}{2}t}$$

$$x = B e^{-\frac{rt}{2}} \qquad (A_1 + A_2 = B)$$

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at $t=0$, $x=0$, $\frac{dx}{dt} = v_0$

$0 = B$

$$x = ct\, e^{-\frac{rt}{2}}$$

$$x = (B+ct)\, e^{-rt/2}$$

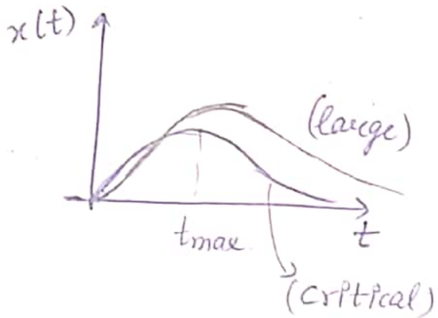
$$\frac{dx}{dt} = c\, e^{-rt/2} + (B+ct)\left(-\frac{r}{2}\right) e^{-rt/2}$$

at $t=0$, $x=0$, $\frac{dx}{dt} = v_0$

$\boxed{0 = B}$

$\cancel{v_0 = c}$ $\quad v_0 = c - \frac{rB}{2} = c - 0$

$\boxed{v_0 = c}$ $\quad \boxed{x = v_0 t\, e^{-\frac{rt}{2}}}$



over, critical, Large Damp?

$\ddot{x} = \frac{d^2x}{dt^2}$ $\quad \dot{x} = \frac{dx}{dt}$

① $\ddot{x} + \dot{x} + 3x = 0$

② $\ddot{x} + 4\dot{x} + 3x = 0$

③ $\frac{1}{4}\ddot{x} + \dot{x} + x = 0$

① $r = 1$ $\quad w_0^2 = 3$ $\quad \frac{r^2}{4} = \frac{1}{4}$ $\quad (2w_0 > r)$ Under Damping

$w_0 = \sqrt{3}$

$= 1.7$

② $r = 4$ $\quad w_0^2 = 3$ $\quad r > 2w_0$ $\quad$ Large Damping

③ $r = 4$ $\quad w_0^2 = 4$ $\quad r = 2w_0$ $\quad$ Critical Damping

$w_0 = 2$

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Logarithmic Decrement (λ)

$$x(t) = A_0 e^{-\frac{\gamma t}{2}} \sin \omega t$$

$$A_1(t) = A_0 e^{-\gamma t/2}$$

$$A_2(t+T) = A_0 e^{-\frac{\gamma}{2}(t+T)}$$

$$\frac{A_1}{A_2} = e^{\frac{\gamma T}{2}} = \frac{A_2}{A_3} = \frac{A_3}{A_4} = \frac{A_n}{A_{n+1}}$$

The logarithmic ratio of 2 successive amplitudes.

$$\Rightarrow \ln\left(\frac{A_n}{A_{n+1}}\right) = \frac{\gamma T}{2}$$

$$\Rightarrow \boxed{\lambda = \frac{\gamma T}{2}}$$

Quality factor (Q) :-

$$Q = 2\pi \times \frac{\text{energy stored in oscillation}}{\text{energy lost in one oscillation}}$$

$$\boxed{Q \simeq \frac{\omega_0}{\gamma}}$$

~~Tot~~

Total Energy →

$$E = KE + PE = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} k x^2$$

$E \propto (A)^2$

$A_0 e^{-\frac{\gamma t}{2}}$

$A_0 e^{-\frac{\gamma t}{2}}$

$A_0^2 e^{-\gamma t}$

$$\langle E \rangle = \frac{1}{2} m \omega_0^2 A_0^2 e^{-\gamma t}$$

$$= E_0 e^{-\gamma t}$$

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$$x = A_0 e^{-rt/2} \sin\omega t$$

$$\frac{dx}{dt} = A_0 \left(-\frac{r}{2}\right) e^{\frac{-rt}{2}} \sin\omega t + A_0 e^{-\frac{rt}{2}} (\omega) \cos\omega t$$

$$= A_0 e^{\frac{-rt}{2}} \left(-\frac{r}{2} \sin\omega t + \omega \cos\omega t\right)$$

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} kx^2$$

$$\frac{dE}{dt} = \frac{1}{2} m \, 2 \left(\frac{dx}{dt}\right) \cdot \frac{d^2x}{dt^2} + \frac{1}{2} k \, 2x \frac{dx}{dt}$$

$$= m \frac{dx}{dt} \left(\frac{d^2x}{dt^2} + \frac{k}{m} x\right)$$

$$= m \frac{dx}{dt} \left(\frac{d^2x}{dt^2} + \omega_0^2 x\right)$$

$$\frac{dE}{dt} = -mr \left(\frac{dx}{dt}\right)^2$$

$$\Delta E = \int_0^T mr \left(\frac{dx}{dt}\right)^2 dt$$

Energy lost in last cycle.

$$= mr \int_0^T A_0^2 e^{-rt} \left(-\frac{r}{2} \sin\omega t + \omega \cos\omega t\right)^2 dt$$

$$r << \omega_0 \,, \quad e^{-rt} \simeq 1$$

$$\Delta E = mr A_0^2 \int_0^T \left(\frac{r^2}{4} \sin^2\omega t + \omega^2 \cos^2\omega t - r\omega \sin\omega t \cos\omega t\right) dt$$

$$\omega t = \theta$$

$$\omega \, dt = d\theta \Rightarrow dt = \frac{d\theta}{\omega}$$

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$$AE = mrA_0^2 \int_0^{2\pi} \left(\frac{r^2}{4} \sin^2\theta + \omega^2 \cos^2\theta - r\omega \sin\theta \cos\theta \right) \frac{d\theta}{\omega}$$

$$= \frac{m r A_0^2}{\omega} \int_0^{2\pi} \left(\frac{r^2}{4} \left(\frac{1-\cos 2\theta}{2} \right) + \omega^2 \left(\frac{\cos 2\theta + 1}{2} \right) - \frac{r\omega}{2} \sin 2\theta \right) d\theta$$

$2c^2 - 1 = \cos 2\theta$

$1 - 2s^2 = \cos 2\theta$

$$= \frac{m r A_0^2}{\omega} \left(0 - \frac{r^2}{4 \times 2} \frac{\sin 2\theta}{2} + \frac{\omega^2}{2} \frac{\sin 2\theta}{2} + 0 + \frac{r\omega}{2 \times 2} \cos 2\theta \right)_0^{2\pi}$$

$$= \frac{m r A_0^2}{\omega} \left(0 - \frac{r^2}{16} (0) + \frac{\omega^2}{4} (0) + \frac{r\omega}{4} \right)$$

$$- \Big($$

$$= \frac{m r A_0^2}{\omega} \pi \left(\frac{r^2}{4} + \omega^2 \right)$$

$$\omega^2 = \omega_0^2 - \frac{r^2}{4} \Rightarrow \omega_0^2 = \omega^2 + \frac{r^2}{4}$$

$$AE = \frac{m r A_0^2 \pi}{\omega} \omega_0^2$$

$$Q = 2\pi \frac{E}{AE} = \frac{2\pi \, \frac{1}{2} \omega_0^2 A^2}{\frac{\pi m r A_0^2 \omega_0^2}{\omega}} = \frac{\omega}{r}$$

$$\omega \approx \omega_0 \quad , \quad r \ll \omega_0$$

$$Q \approx \frac{\omega_0}{r}$$

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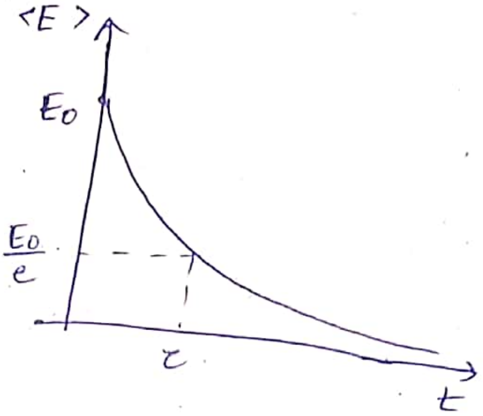
(3) Relaxation Time (τ)

$$\langle E \rangle = E_0 e^{-\gamma t}$$



$$E_0 e^{-\gamma \tau} = E_0 e^{-1}$$

$$\gamma \tau = 1$$

$$\boxed{\tau = \frac{1}{\gamma}}$$ More Relaxation if γ is less

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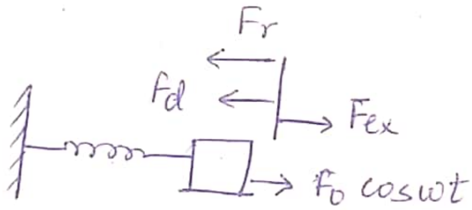
Forced Oscillation :-



$$m\frac{d^2x}{dt^2} = -kx - P\frac{dx}{dt} + F_0 \cos \omega t$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{P}{m}\frac{dx}{dt} + \frac{F_0}{m}\cos\omega t$$

$$\boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f_0 \cos\omega t} \qquad f_0 = \frac{F_0}{m}$$

2^{nd} Linear Non-Homogenous Eqn

x_1 Complementary function. ω = Driving force

$$x_1 = Ae^{-\frac{\gamma t}{2}}\cos(\omega' t - \delta)$$

$$\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \omega_0^2 x_2 = f_0\cos\omega t$$

x_2 = Particular integral

$x_2 = B\cos(\omega t - \phi)$

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General sol^n $= x_1 + x_2$

$$= A e^{-\frac{rt}{2}} \cos(\omega' t - s) + B\cos(\omega t - \phi)$$

Transient sol^n $\rightarrow$

$$\boxed{\omega' = \left(\omega_0^2 - \frac{r^2}{4}\right)^{1/2}}$$

natural frequency

$$x_s = B_s \cos(\omega t - \phi)$$

$$z = B_s e^{i(\omega t - \phi)}$$

$$\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \omega_0^2 x = f_0 \cos\omega t$$

$$Re\left(\frac{d^2z}{dt^2} + r\frac{dz}{dt} + \omega_0^2 z = f_0 e^{i\omega t}\right)$$

$$= f_0 e^{i(\omega t - \phi)} e^{i\phi}$$

$$\frac{dz}{dt} = B_s (i\omega) e^{i(\omega t - \phi)}$$

$$\frac{d^2z}{dt^2} = -\omega^2 B_s e^{i(\omega t - \phi)}$$

$$\Rightarrow -\omega^2 B_s e^{i(\omega t - \phi)} + r B_s (i\omega) e^{i(\omega t - \phi)}$$

$$+ \omega_0^2 B_s e^{i(\omega t - \phi)} = f_0 e^{i(\omega t - \phi)} \cdot e^{i\phi}$$

$$B_s e^{i(\omega t - \phi)} \left[-\omega^2 + i\omega r + \omega_0^2\right] - f_0 e^{i(\omega t - \phi)} e^{i\phi} = 0$$

$$\left[B_s (\omega_0^2 - \omega^2 + i\omega r) - f_0 e^{i\phi}\right] e^{i(\omega t - \phi)} = 0$$

$$B_s (\omega_0^2 - \omega^2 + i\omega r) - f_0 e^{i\phi} = 0$$

$$B_s e^{-i\phi} = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega r}$$

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$$B_s(\cos\phi - i\sin\phi) = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega\gamma} \quad -(1)$$

$$B_s(\cos\phi + i\sin\phi) = \frac{f_0}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad -(2)$$

Complex Conjugate.

$$B_s\cos\phi = \frac{f_0(\omega_0^2 - \omega^2)}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2} \quad -(3)$$

$$B_s\sin\phi = \frac{f_0\,\omega\gamma}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2} \quad -(4)$$

$$B_s^2\cos^2\phi + B_s^2\sin^2\phi = \frac{f_0^2(\omega_0^2-\omega^2)^2}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]} + \frac{f_0^2\omega^2\gamma^2}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]^2}$$

$$\Rightarrow B_s^2 = \frac{f_0^2\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]^2}$$

$$\boxed{B_s = \frac{f_0}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]^{1/2}}}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2-\omega^2}\right)}$$

At steady state the phase & Amplitude will not depend on initial condn

B_s & ϕ are independent of initial condition.

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Variation in B_s & ϕ with driving force frequency.

(1) Low frequency

$\omega << \omega_0$, $r << \omega_0$

$$B_s = \frac{f_0}{\omega_0^2} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

$$\phi = \tan^{-1}(0) = 0$$

Stiffness controlled motion

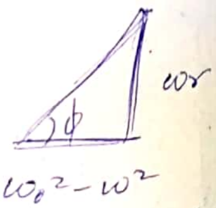
(2) High frequency ($\omega >> \omega_0$, $r << \omega_0$)

$$B_s = \frac{f_0}{\omega^2} = \frac{F_0}{m\omega^2}$$ (Mass control) $\boxed{f_0 = \frac{F_0}{m}}$

$$\phi = \tan^{-1}(0) = \pi\ (180^\circ)$$ out of phase

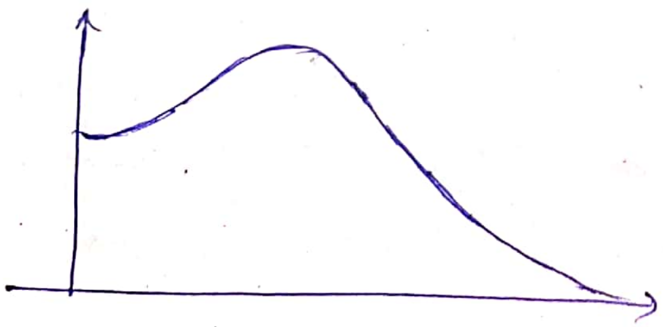
(3) $\omega = \omega_0$ (Mid frequency)

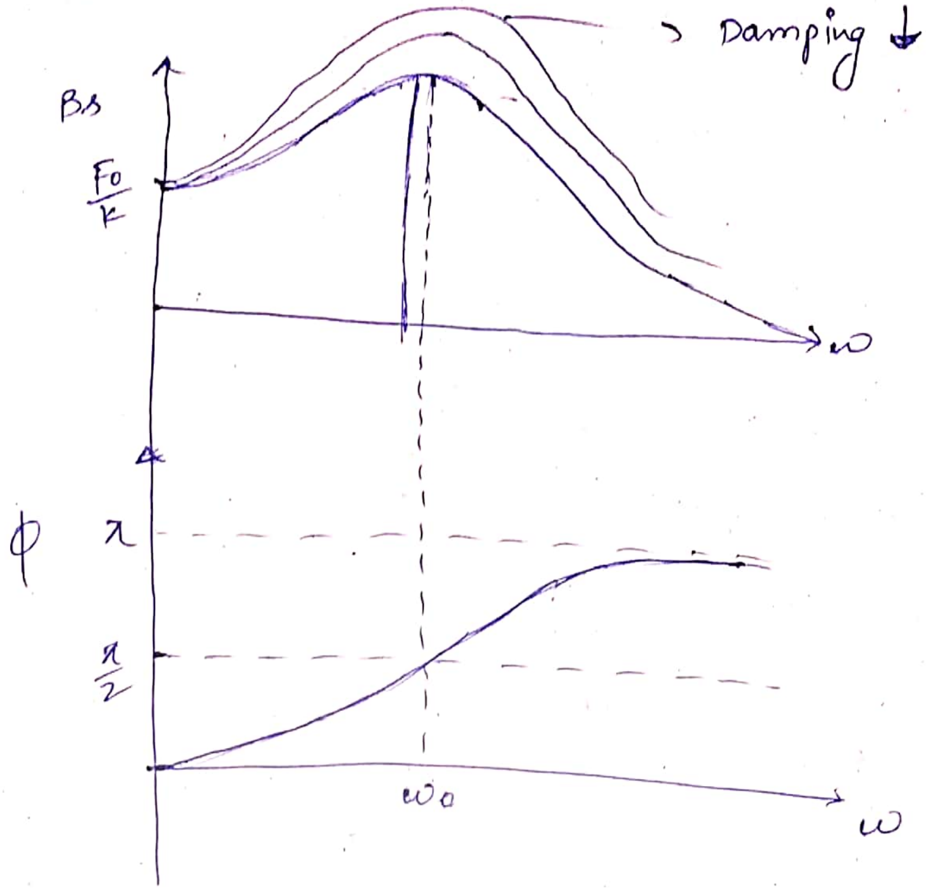
$$B_s = \frac{f_0}{\omega r} = \frac{F_0}{m\omega r} = \frac{F_0}{\omega p}$$



$$\phi = \tan^{-1}(0) = \frac{\pi}{2}$$

(Damping Control)





* More Damping, far is Resonance.

Resonance :- When you make a body oscillates at its natural frequency on which a driving force applied periodically act at particular frequency which is equal to that natural frequency.

$$x_s = B_s \cos(\omega t - \phi)$$

$$B_s = \frac{f_0}{\left[(\omega_0^2 - \omega^2) + \omega^2 r^2\right]^{1/2}}$$

$$D = (\omega_0^2 - \omega^2) + \omega^2 r^2$$

$$\frac{dD}{d\omega} = 0$$

$$\Rightarrow 2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega r^2 = 0$$

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$$\Rightarrow \quad -2(\omega_0^2-\omega^2)+r^2=0$$

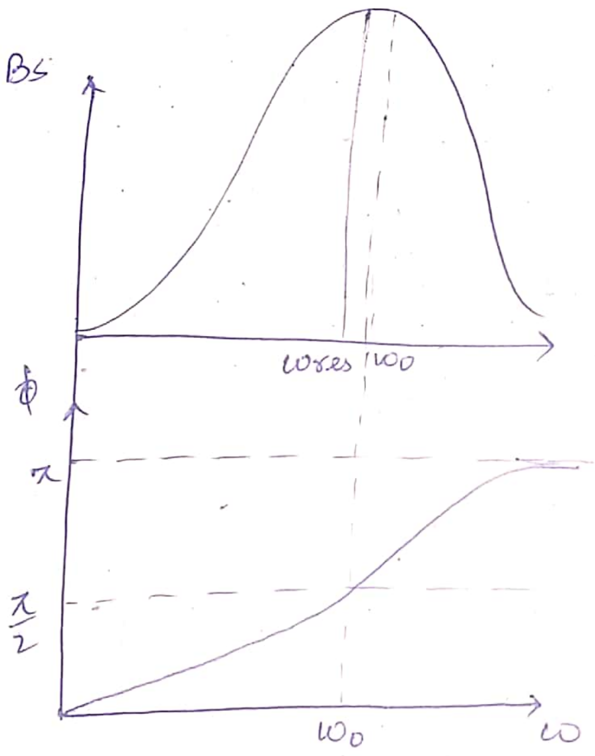
$$\omega_0^2-\omega^2=\frac{r^2}{2}$$

$$\omega^2=\omega_0^2-\frac{r^2}{2}$$

$$\boxed{\omega_{res}=\left(\omega_0^2-\frac{r^2}{2}\right)^{1/2}}$$

$\omega_{res} \rightarrow$ freq for amplitude resonance

- r very small $\quad \omega_{res} \simeq \omega_0$



Velocity Resonance

$$x_s=B_s\cos(\omega t-\phi)$$

$$\frac{dx_s}{dt}=-B_s\omega\sin(\omega t-\phi)$$

$$v_0=B_s\omega=\frac{f_0\,\omega}{\left[(\omega_0^2-\omega^2)^2+\omega^2r^2\right]^{1/2}}$$

$$=\frac{f_0}{\left[\frac{\omega_0^2-\omega^2}{\omega^2}+\frac{\omega^2r^2}{\omega^2}\right]^{1/2}}$$

$$V_0 = \frac{f_0}{\left[\omega^2\left(\frac{\omega_0^2}{\omega^2}-1\right)^2+\gamma^2\right]^{1/2}}$$

$$\boxed{\omega = \omega_0}$$

Power Supplied by driving force :-

$$\text{Instantaneous Power} = F_{ext}\frac{dx}{dt}$$

$$x_s = B_s \cos(\omega t - \phi)$$

$$= B_s \cos\phi \cos\omega t + B_s \sin\phi \sin\omega t$$

$$x_s = B_1 \cos\omega t + B_2 \sin\omega t$$

$$\frac{dx_s}{dt} = -B_1\omega \sin\omega t + B_2 \omega \cos\omega t$$

$$P = (F_0 \cos\omega t)(-B_1 \omega \sin\omega t + B_2 \omega \cos\omega t)$$

$$= -B_1 \omega F_0 \cos\omega t \cdot \sin\omega t + f_0 B_2 \omega \cos\omega t$$

$$\langle P \rangle = P_{avg} = -B_1 \omega F_0 \langle \cos\omega t \cdot \sin\omega t \rangle + f_0 B_2 \omega \langle \cos^2\omega t \rangle$$

$$P_{avg} = \frac{F_0 B_2 \omega}{2} = \frac{f_0 \omega}{2} \cdot \frac{\omega\gamma f_0}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

$$P_{avg} = \frac{F_0^2 \omega^2 \gamma^2}{2m\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]}$$

$$\frac{dP_{avg}}{d\omega} = 0 \Rightarrow \boxed{\omega = \omega_0}$$

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$$\left(P_{avg}\right)_{max} = \frac{F_0^2 \omega^2 \gamma}{2m\,\omega^2 \gamma^2} = \frac{F_0^2}{2m\gamma}$$

$$\boxed{P_{avg} = \left(P_{avg}\right)_{max} \frac{\omega^2 \gamma^2}{\left[(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2\right]}}$$

- Avg Power dissipated by the driven system

$$\text{Power} = F_d \frac{dx}{dt} = -P \frac{dx}{dt} \cdot \frac{dx}{dt}$$

$$= -P\left(\frac{dx}{dt}\right)^2$$

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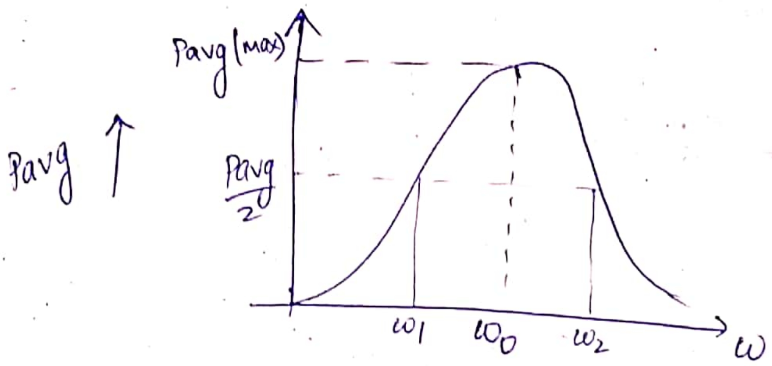

Full width at Half Maximum = FWHM

$\Rightarrow \Delta\omega = \omega_2 - \omega_1$

$$(P_{avg})_{max} = \frac{\omega^2\gamma^2}{\left[(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2\right]} = \frac{(P_{avg})_{max}}{2}$$

$$\Rightarrow (\omega_0^2-\omega^2)^2 + \omega^2\gamma^2 = 2\omega^2\gamma^2$$

$$\Rightarrow (\omega_0^2-\omega^2)^2 = \omega^2\gamma^2$$

$$\Rightarrow \omega_0^2 - \omega^2 = \pm\omega\gamma$$

① $\omega_0^2 - \omega^2 = \omega\gamma$

$-\omega_0^2 + \omega^2 + \omega\gamma = 0$

$$\omega = \frac{-\gamma}{2} \pm \left(\frac{\gamma^2}{4} + \omega_0^2\right)^{1/2}$$

$$\omega_1 = -\frac{\gamma}{2} + \left(\frac{\gamma^2}{4} + \omega_0^2\right)^{1/2}$$

$$\omega_1 = -\frac{\gamma}{2} + \omega_0$$

$$\boxed{\omega_1 = \omega_0 - \frac{\gamma}{2}}$$

② $\omega_0^2 - \omega^2 = -\omega\gamma$

$-\omega_0^2 + \omega^2 - \omega\gamma = 0$

$$\omega = \frac{\gamma}{2} \pm \left(\frac{\gamma^2}{4} + \omega_0^2\right)^{1/2}$$

$$\omega_2 = \frac{\gamma}{2} + \left(\frac{\gamma^2}{4} + \omega_0^2\right)^{1/2}$$

$$\boxed{\omega_2 = \frac{\gamma}{2} + \omega_0}$$

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$$\Delta\omega = \omega_2 - \omega_1$$

$$= \left(\omega_0 + \frac{\gamma}{2}\right) - \left(\omega_0 - \frac{\gamma}{2}\right)$$

$$\boxed{\Delta\omega = \gamma}$$

Sharpness of resonance $= \frac{1}{\Delta\omega} = \frac{1}{\gamma}$

$$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\Delta\omega}$$

$$Q = \frac{f_0}{\Delta f} \simeq 10^4$$

$$Q \simeq 10^5$$

1.(9) An object of mass 0.2 kg is hung from a spring with spring constant $K = 80\,N/m$. It is subjected to a resistive force given by $-PV$.

(1) Set up the eq^n of motion

(2) If damped freq is $\frac{\sqrt{3}}{2}$ of undamped frequency, what is the value of P.

(3) What is Q of the System.

(29) Find the steady state sol^n for a forced oscillator with a force as $F = F_0 \sin\omega t$.

$$x_s = B_s \sin(\omega t - \phi)$$

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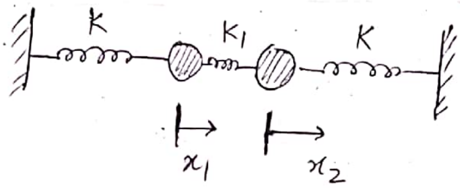
Coupled Oscillation



$$m\frac{d^2x_1}{dt^2} = -Kx_1 \qquad m\frac{d^2x_2}{dt^2} = -Kx_2$$

Now introduce K_1 spring

$$\Rightarrow m\frac{d^2x_1}{dt^2} = -Kx_1 - K_1x_1 + K_1x_2$$
$$= -Kx_1 - K_1(x_1 - x_2) \quad \text{---} (1)$$

K_1 get compressed by x_1 & elongated by x_2

$$\Rightarrow m\frac{d^2x_2}{dt^2} = -Kx_2 - K_1x_2 + K_1x_1$$
$$= -Kx_2 - K_1(x_2 - x_1) \quad \text{---} (2)$$

$$\frac{d^2x_1}{dt^2} + x_1\frac{(K+K_1)}{m} - \frac{K_1x_2}{m} = 0 \quad \text{---} (3)$$

$$\frac{d^2x_2}{dt^2} + x_2\frac{(K+K_1)}{m} - \frac{K_1x_1}{m} = 0 \quad \text{---} (4)$$

$\hookrightarrow$ Coupled differential equation.

Add eq^n (3) & (4)

$$\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} + \frac{K(x_1+x_2)}{m} = 0$$

$$\frac{d^2(x_1+x_2)}{dt^2} + \frac{K(x_1+x_2)}{m} = 0 \quad \text{---} (5)$$

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Subtract ④ from ③

$$\frac{d^2(x_1-x_2)}{dt^2} - \frac{2k_1(x_2-x_1)}{m} - \frac{k(x_2-x_1)}{m} = 0$$

$$\frac{d^2(x_1-x_2)}{dt^2} + \left(\frac{k+2k_1}{m}\right)(x_1-x_2) = 0$$

$x_1 + x_2 = q_1$, $x_1 - x_2 = q_2$ (q_1, q_2) are normal coordinates

$$x_1 = \frac{q_1+q_2}{2} , \quad x_2 = \frac{q_1-q_2}{2}$$

eq ③ $$\frac{d^2q_1}{dt^2} + \left(\frac{k}{m}\right) q_1 = 0$$

⑥ $$\frac{d^2q_2}{dt^2} + \left(\frac{k+2k_1}{m}\right) q_2 = 0$$

$$\sqrt{\frac{k}{m}} = \omega_1 \qquad \sqrt{\frac{k+2k_1}{m}} = \omega_2$$

$$\left.\begin{aligned} \frac{d^2q_1}{dt^2} + \omega_1^2 q_1 = 0 \\ \frac{d^2q_2}{dt^2} + \omega_2^2 q_2 = 0 \end{aligned}\right\} \text{SHM}$$

$q_1 = A_1 \cdot \cos \omega_1 t$

$q_2 = A_2 \cos \omega_2 t$

$$x_1 = \frac{q_1+q_2}{2} = \frac{1}{2}\left(A_1 \cos\omega_1 t + A_2 \cos\omega_2 t\right)$$

$$x_2 = \frac{q_1-q_2}{2} = \frac{1}{2}\left(A_1 \cos\omega_1 t - A_2 \cos\omega_2 t\right)$$

when the force sys oscillate with same freq → Normal Mode

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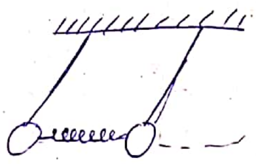
(1) At $t=0$, $x_1 = x_2 = A$. same dirn, same freq

COM moves

$$A = \frac{1}{2}(A_1 + A_2) \Rightarrow \boxed{A_2 = 0}$$

$$A = \frac{1}{2}(A_1 - A_2) \qquad \boxed{A_1 = 2A}$$



$$x_1 = \frac{1}{2}(2A\cos\omega_1 t + 0)$$

$$\boxed{x_1 = A\cos\omega_1 t} \qquad \boxed{x_2 = A\cos\omega_1 t}$$

---

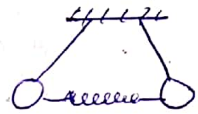
(2) at $t=0$, $x_1 = A$, $x_2 = -A$. In opp dirn, out of phase, same freq

$$A = \frac{1}{2}(A_1 + A_2) \Rightarrow \boxed{A_1 = 0} \quad \boxed{A_2 = 2A}$$

$$-A = \frac{1}{2}(A_1 - A_2)$$

COM doesn't move

$$\boxed{x_1 = A\cos\omega_2 t} \qquad \boxed{x_2 = -A\cos\omega_2 t}$$



(III) $x_1 = \frac{1}{2}(A\cos\omega_1 t + A\cos\omega_2 t)$

$$x_2 = \frac{1}{2}(A\cos\omega_1 t - A\cos\omega_2 t)$$

$$x_1 = \frac{1}{2}A \cdot 2\cos\left(\frac{\omega_1 - \omega_2}{2}\cdot t\right) \cdot \cos\left(\frac{\omega_1 + \omega_2}{2}\cdot t\right)$$

$$\boxed{x_1 = A\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t\,\cos\left(\frac{\omega_1 + \omega_2}{2}\right)t}$$

$$x_2 = -\frac{1}{2}A \cdot 2\sin\left(\frac{\omega_1 - \omega_2}{t}\right)t \cdot \sin\left(\frac{\omega_1 + \omega_2}{2}\right)t$$

$$= A\sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right)t\right]\sin\left(\frac{\omega_1 + \omega_2}{2}\right)t$$

$$\boxed{x_2 = A\cos\left[\frac{\pi}{2} + \left(\frac{\omega_1 - \omega_2}{2}\right)t\right]\sin\left(\frac{\omega_1 + \omega_2}{2}\right)t}$$

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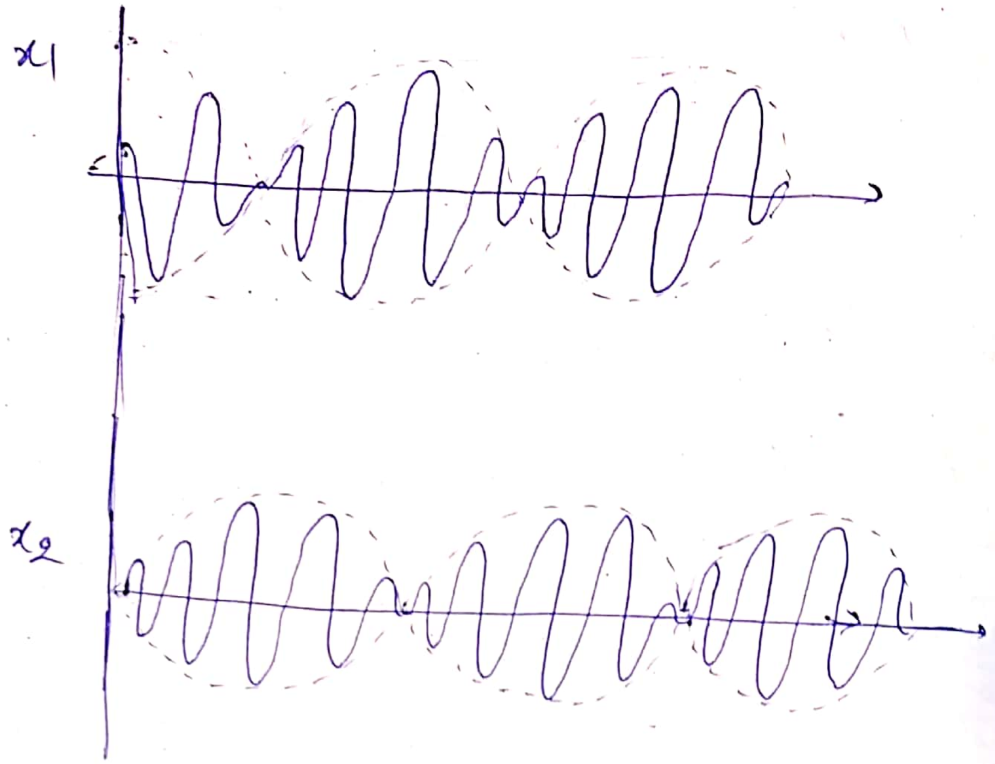

Analytical approach to find normal mode :-

$x_1 = A e^{i\omega t}$ $\qquad$ $x_2 = B e^{i\omega t}$

$\frac{d^2x_1}{dt^2} = -A\omega^2 e^{i\omega t}$ $\qquad$ $\frac{dx_2^2}{dt^2} = -B\omega^2 e^{i\omega t}$

$$-m\omega^2 A e^{i\omega t} + (k+k_1) A e^{i\omega t} - k_1 B e^{i\omega t} = 0$$

$$\Rightarrow (k+k_1 - m\omega^2) A e^{i\omega t} - k_1 B e^{i\omega t} = 0 \quad — (3)$$

from (2)

$$-m\omega^2 B e^{i\omega t} + (k+k_1) B e^{i\omega t} - k_1 A e^{i\omega t} = 0$$

$$\Rightarrow -k_1 A e^{i\omega t} + (k+k_1 - m\omega^2) B e^{i\omega t} = 0$$

(4)

$$\begin{pmatrix} k+k_1-m\omega^2 & -k_1 \\ -k_1 & k+k_1-m\omega^2 \end{pmatrix} \begin{pmatrix} A e^{i\omega t} \\ B e^{i\omega t} \end{pmatrix} = 0$$

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$$\begin{vmatrix} K+K_1-m\omega^2 & -K_1 \\ -K_1 & K+K_1-m\omega^2 \end{vmatrix} = 0$$

$$(K+K_1-m\omega^2)^2 - K_1^2 = 0$$

$\Rightarrow \quad K+K_1-m\omega^2 = +K_1 \quad ; \quad K+K_1-m\omega^2 = -K_1$

$\omega = \sqrt{\frac{K}{m}} \qquad \omega = \sqrt{\frac{K+2K_1}{m}}$

Put this in Eq^n (3)

$$e^{i\omega t}\left[(K+K_1-\cancel{K})A - K_1 B\right] = 0$$

$$e^{i\omega t} K_1 [A-B] = 0$$

$\Rightarrow \boxed{A = B}$

Put this in Eq^n (3) or (4)

$\Rightarrow \boxed{A = -B}$

Characteristic of normal modes (Pure & Stationary States)

1) When the system oscillate in normal mode then all its component oscillates with single frequency. ie called normal frequency.

2) Normal modes are independent of each other. So energy transfer does not occur from one to other normal mode

3) When one normal mode is excited then the other normal mode remain unexcited.

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