CSE667: Decision Making for Multi Robot Systems

Assignment-3

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Question-1

Part a

To Show:

$H(X|Y) = H(X,Y) - H(Y)$

Solution:

We know that Joint Entropy is:

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y)$$

$$H(X,Y) = -\sum_{x \in X} \sum_{y \in Y} p(x,y) \log p(x,y)$$

Entropy is:

$$H(X) = -\sum_{x \in X} p(x) \log p(x)$$

Conditional Entropy is defined in the question :

$$H(X|Y) = \sum_{y \in Y} p(x) H(X|Y = y)$$

$$= -\sum_{y \in Y} p(y) \sum_{x \in X} p(x|y) \log p(x|y)$$

$$= -\sum_{x,y} p(x,y) \log p(x|y)$$

Proof:

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y)$$

$$= -\sum_{x,y} p(x,y) \log [p(x|y)p(y)]$$

$$= -\sum_{x,y} p(x,y) \log p(x|y) - \sum_{x,y} p(x,y) \log p(y)$$

So above equation becomes:

$$= -\sum_{x,y} p(x,y) \log p(x|y) - \sum_{y \in Y} p(y) \log p(y)$$

$$= H(X|Y) + H(Y)$$

Rearranging the equation to get:

$$H(X|Y) = H(X,Y) - H(Y)$$

Hence, proved.

Part b

To Prove:

$H(X|Y) = 0$ if and only if $X = g(Y)$ for some function g.

Given:

H(.) represents entropy, X and Y are the beliefs of two different robots.

Solution:

Let us assume that there exists an y_0 and two different values x_1, x_2 where $x_1 \neq x_2$ such that

$$p(x_1, y_0) > 0 \quad \text{and} \quad p(x_2, y_0) > 0.$$

Then clearly,

$$p(y_0) \geq p(x_1, y_0) + p(x_2, y_0) > 0,$$

and hence the conditional probabilities $p(x_1|y_0)$ and $p(x_2|y_0)$ are both strictly between 0 and 1

We know that conditional entropy is:

$$H(X|Y) = -\sum_y p(y) \sum_x p(x|y) \log p(x|y)$$

So for y_0, we have:

$$H(X|Y) \geq p(y_0) \left(-p(x_1|y_0) \log p(x_1|y_0) - p(x_2|y_0) \log p(x_2|y_0)\right)$$

According to our assumption $0 < p(x_1|y_0), p(x_2|y_0) < 1$, and the function $-t \log t$ is strictly positive for $0 < t < 1$ This would mean that

$$H(X|Y) > 0,$$

which contradicts our assumption that $H(X|Y) = 0$

Therefore, for all y with $p(y) > 0$, there must be exactly one value of x such that $p(x, y) > 0$, i.e., X is a function of Y.

Hence, $H(X|Y) = 0 \iff X = g(Y)$.

Question 2

Part a

The mathematical expression for $I(X;Y)$ in terms of uncertainty of X and conditional entropy between X and Y is:

$$I(X;Y) = H(X) - H(X|Y)$$

Proof:

$$\begin{aligned}
I(X;Y) &= \sum_{x,y} p(x,y)\log\left(\frac{p(x,y)}{p(x)p(y)}\right) \\
&= \sum_{x,y} p(x,y)\log\left(\frac{p(x/y)}{p(x)p(y)}\right) \\
&= -\sum_{x} p(x)\log p(x) + \sum_{x,y} p(x,y)\log p(x|y) \\
&= -\sum_{x} p(x)\log p(x) - \left(-\sum_{x,y} p(x,y)\log p(x|y)\right) \\
&= H(X) - H(X|Y)
\end{aligned}$$

Hence, Proved

Part b

To prove: $I(X;Y) = I(Y;X)$

$$I(X;Y) = \sum_{x,y} p(x,y)\log\left(\frac{p(x,y)}{p(x)p(y)}\right)$$

$$I(Y;X) = \sum_{y,x} p(y,x)\log\left(\frac{p(y,x)}{p(y)p(x)}\right)$$

We know that the joint probability distribution,

$$p(x,y) = p(y,x)$$

Thus,

$$I(X;Y) = I(Y;X)$$

Hence, Proved

Part c

Show $I(X;Y) = KL(p(x,y)||p(x)p(y))$

Solution:

We know by definition that:

$$I(X;Y) = \sum_{x,y} p(x,y)\log\left(\frac{p(x,y)}{p(x)p(y)}\right)$$

Kullback–Leibler distance is defined as

$$KL(p\|q) = \sum_{x\in X} p(x)\log\frac{p(x)}{q(x)}$$

$$= \mathbb{E}_p \log\frac{p(X)}{q(X)}$$

Mutual Information can we written as:

$$I(X;Y) = \sum_{x\in X}\sum_{y\in Y} p(x,y)\log\left(\frac{p(x,y)}{p(x)p(y)}\right)$$

This is equivalent to:

$$= KL(p(x,y)||p(x)p(y))$$

$$= \mathbb{E}_{p(x,y)} \log \frac{p(X,Y)}{p(X)p(Y)}$$

Hence, proved

$$I(X;Y) = KL(p(x,y)||p(x)p(y))$$

Question 3

Given:

- **Workspace:** $S = \{s_1, s_2, \ldots, s_n\}$ where each s_i represents a state in the environment.
- **Belief at Time t:** $P(X \mid \xi_t, a_t)$ where:
  - X: random variable representing the environment
  - ξ_t: history of states and actions up to time t
  - a_t: current action at time t.
- ξ_t is defined as $\xi_t = \langle s_1, a_1, \ldots, s_t \rangle$
- Current state s_t
- Next state Distribution S_{t+1}

Objective (Exploration Strategy):

Select an action that provides the maximum information about the environment. That is, the action should maximize the reduction in uncertainty (i.e., entropy) about the environment.

Part a

The action a_t should be chosen such that:

$$a_t^* = \arg\max_{a_t} I(X; S_{t+1}|\xi_t, a_t)$$

This maximizes the expected information gain. We want to reduce uncertainty about the environment X RV representing the environment, S_{t+1} is the state the robot will observe after taking action a_t, and ξ_t is the history of all past states and actions. The robot has belief over X and updates this belief based on exploration. The mutual information quantifies how much observing the next state S_{t+1} will tell the robot about X, conditioned on its past history and the action it chooses. By choosing the action a_t that maximizes this mutual information, the robot ensures that its next observation will be the most informative to update its belief about the environment. We take the action a_t^* for which the Mutual Information is maximum.

Part b

Solution:

$$\begin{aligned} I(X; S_{t+1}|\xi_t, a_t) &= \mathbb{E}_{S_{t+1}} \left[KL(P(X|\xi_t, a_t, S_{t+1}) \| P(X|\xi_t, a_t)) \right] \\ a_t^* &= \arg\max_{a_t} \mathbb{E}_{S_{t+1}} \left[KL(P(X|\xi_t, a_t, S_{t+1}) \| P(X|\xi_t, a_t)) \right] \end{aligned}$$

The next state S_{t+1} is a RV. When a robot takes an action a_t, it does not know exactly which state it will observe next, but it does have a probability distribution over possible outcomes based on its current belief and the environment. To evaluate the information of an action is, the robot must consider all possible observations it might make after taking that action and compute how much each possible observation would change its belief about the environment. The KL-divergence measures the information gained from a specific observation S_{t+1}, and taking the expectation over all such possible outcomes gives the total expected information gain.The expectation captures the average reduction in uncertainty about X by performing action a_t, for increasing information about the environment.

These equations are just the mathematical representation of equation in part a using KL divergence.

References