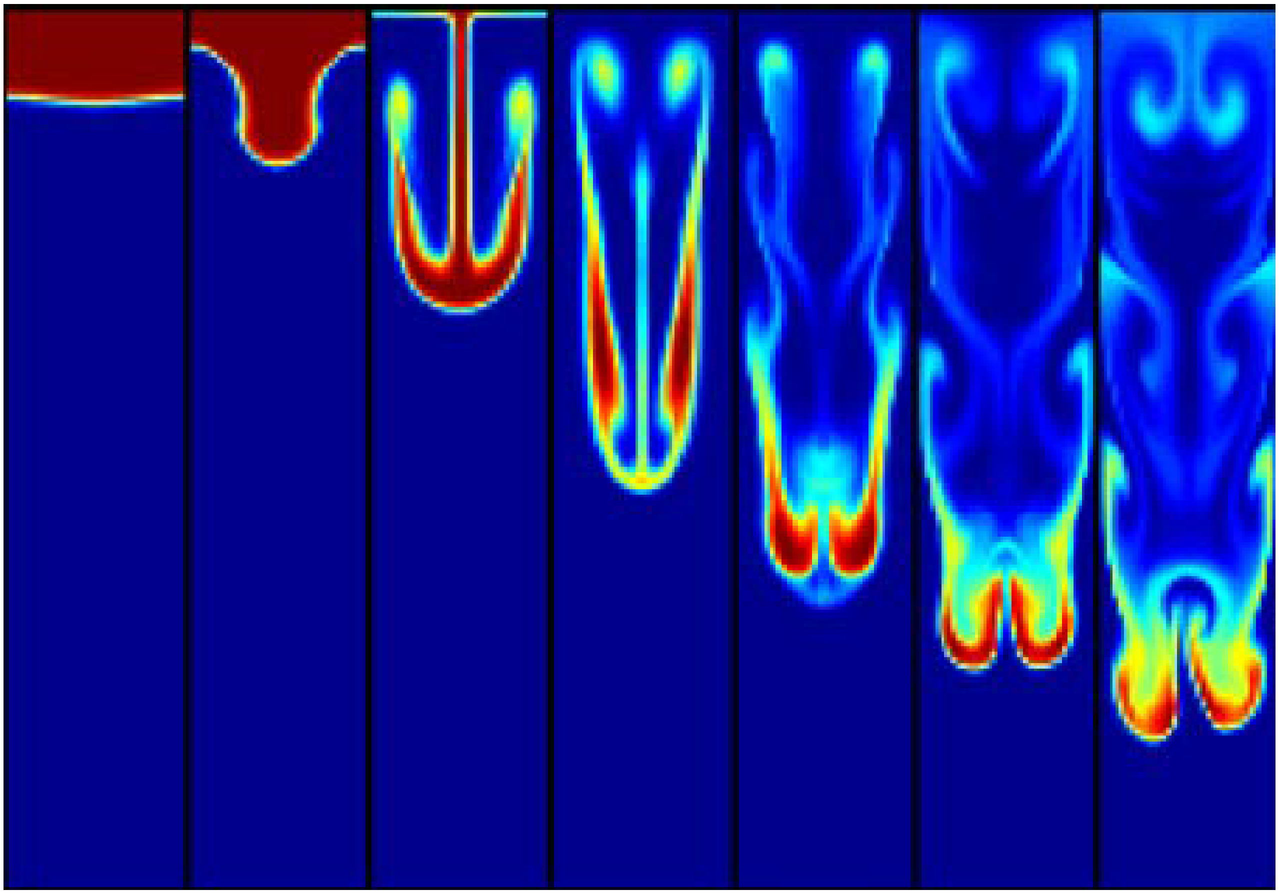
# Research Internship

*University Of Bordeaux, France*

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## Acknowledgments

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Furthermore, I am deeply appreciative of the resources and facilities made available to me, which played a crucial role in the successful execution of this research. The collaborative environment and academic atmosphere provided a stimulating space for innovation and critical thinking.

Lastly, I extend my heartfelt gratitude to my family and friends for their constant encouragement and support. Their belief in my abilities and their unwavering motivation have been a source of strength throughout this journey.

Being a part of this project has been an immensely rewarding experience, and I am truly grateful to everyone who contributed to making it a success.

# Contents

## Abstract

This study explores the intricate dynamics of interfacial instabilities in fluid systems, with a particular focus on the Rayleigh-Taylor (RT) instability. The RT instability occurs when a heavier fluid lies above a lighter fluid in a gravitational field, leading to the growth of perturbations at the interface. While the dispersion relation for binary fluids is well-established, this research extends the analysis to non-binary fluids, providing new insights into their behavior.

My work was guided by the diffuse interface method, a powerful approach for simulating two-phase flows of complex fluids. This method, based on energy-based variational formalism, allowed me to describe the interface through a mixing energy. The diffuse interface model, rooted in Van der Waals' work (1892), provides a strong foundation for phase field theory and reduces to the classical sharp-interface model as the capillary width approaches zero.

I derived the mixing energy density as a function of the phase-field variable and its gradient, which revealed the competition between phase separation ("phobic" effect) and mixing ("philic" effect) in determining the interface profile. Additionally, I derived expressions for interfacial tension and capillary width, demonstrating how the diffuse interface model transitions to the sharp-interface formulation. These derivations were crucial for understanding the stability and dynamics of the fluid interface.

To validate my theoretical findings, I conducted OpenFOAM simulations, comparing the evolution of the interface with literature values. While the simulations captured the overall dynamics of the RT instability, slight differences arose due to variations in initial conditions—specifically, the use of different interface descriptions (e.g., sine or tanh functions) in the literature. These discrepancies provided valuable insights into how initial conditions influence the growth of perturbations and the overall stability of the interface.

In conclusion, this research has deepened my understanding of dispersion relations for both binary and non-binary fluids, highlighting the critical interplay between surface tension, gravity, and fluid properties in determining interface stability.

# 1. Literature Review

Derivation of dispersion relation for binary fluids is mention. It is interesting to know how people have derived dispersion relation for non-binary fluids. A summary of the literature review done is presented below:

1.1. [Piriz et al.] The Rayleigh-Taylor instability is a fascinating phenomenon that occurs when a denser fluid sits on top of a less dense fluid in a gravitational field. It's like what happens when you try to balance a pencil upside down – it's inherently unstable. This instability isn't just a physics curiosity; it plays a crucial role in a wide range of natural and even industrial processes.

Think about turning a glass of water upside down. The water falls out because it's denser than the air above it, a classic example of the Rayleigh-Taylor instability in action. But this instability is also at work in much grander settings. It influences the formation of beautiful astronomical structures like the Eagle and Crab nebulas, remnants of exploded stars. It even affects deep-sea divers, contributing to the formation of air bubbles in their blood. Understanding this instability helps us to appreciate the intricate workings of our universe, from the everyday to the extraordinary.

1.2. [P. Yue et al. 2004] A diffuse interface method for simulating two phase flows of complex fluids

1.2.1. Energy based variational formalism (Phase field method) makes it possible to incorporate complex physics e.g. rheology easily, as long as it is due to the evolution of a microstructure desirable by a free energy.

1.2.2. The diffuse interface model can be viewed as a physically motivated level set method. Instead of choosing an artificial smoothing function for the interface, which affects the results in non-trivial ways if the radius of interfacial curvature approaches that of interfacial thickness, the diffuse interface model describes the interface by a mixing energy.

1.2.3. This idea can be traced to Van der Waals (1892) and is the foundation for the phase field theory for phase transition and critical phenomena. Thus, the structure of the interface is determined by molecular forces; the tendencies for mixing and demixing are balanced through non-local mixing energy.

1.2.4. When the capillary width approaches zero, the diffuse interface model becomes identical to a sharp-interface level set formulation. It also reduces properly to the classical sharp-interface model.

1.2.5. An additional advantage of the diffuse-interface method over other interface-regularizing methods is its energy conservation.

1.3. Theoretical and Numerical Models

1.3.1. The starting point is the Lagrangian, L = T - F, where T and F are the kinetic and potential energies of the system. The least action principle requires that the action integral2 I = ∫Ldt be stationary under variation of paths. This will lead to a momentum equation, with elastic stresses arising from the microstructural changes described by F and evolution equation3 for the field variable whose momenta are included in T.

1.3.2. Mixing energy:

In the diffuse-interface picture, the interface has a small but finite thickness, inside which the two components are mixed and store a mixing energy.

Let phase-field variable, $\phi$, represent the concentration of two components, $((1+\phi)/2)$ and $((1-\phi)/2)$.

Following Cahn and Hilliard, we express the mixing energy density as a function of $\phi$ and its gradient.

$f\_mix (\phi, \nabla \phi) = (1/2)\lambda|\nabla \phi|^2 + f\_0(\phi)$;

Where, $\lambda$ = Magnitude of mixing energy $\varepsilon$ = Thickness of the interface.

Double well potential: $f\_0 = (\lambda/4\varepsilon 2)(\phi^2-1)^2$

The bulk energy ($f_0$): prefers total separation of the phases into domains of pure components ($\phi = \pm 1$). This "phobic" effect produces classical sharp interface picture.

The gradient term ($(1/2)\lambda|\nabla\phi|^2$): represents weakly non-local interactions between the components that prefer complete mixing, "philic" effect.

The profile of $\phi$ across the interface is determined by the competition between the two effects.

### 1.3.3. Interfacial tension and Capillary width

Since the mixing energy (f_mix) represents the molecular interaction between the two phases, the classical concept of interfacial tension should be contained in it.

Consider a 1-dimensional surface (interface), we require that the diffuse mixing energy in the region be equal to traditional surface energy.

$$\sigma = \lambda \int_{-\infty}^{\infty} \{ (1/2) (d\phi/dx)^2 - f_0(\phi) \} dx$$

" λ" has been taken common.

Let us further assume that the diffuse interface is at equilibrium, and thus has zero chemical potential -

$$\mu = \delta f\_mix / \delta\phi = 0$$

$$\Rightarrow \partial(f\_mix)/\partial x - d/dx\, (\partial f/\partial x) = 0$$

$$\Rightarrow f_0'(\phi) - d/dx\, (d\phi/dx) = 0$$

$$\Rightarrow \lambda[-d^2\phi/dx^2 + f_0'(\phi)] = 0$$ (calculated using Euler-Lagrange equation)

Since $f_0(\pm\infty) = 0$ and $(d\phi/dx)|_{(x=\pm\infty)} = 0$, we can get (integration)

$$\mathbf{(1/2)(d\phi/dx)^2 = f_0(\phi)}$$

implies equal partition of free energy between two terms at equilibrium.

### 1.3.4. Important Derivations

#### 1.3.4.1. Equal partition of energy at equilibrium

The chemical potential is given as -

$$\mu = \lambda[d^2\phi/dx^2 + f_0'(\phi)] = 0 \text{ (at equilibrium)}$$

Multiplying both sides by $d\phi/dx$, we get:

$$\Rightarrow (-d^2\phi/dx^2)\, d\phi/dx + f_0'(\phi)\, d\phi/dx = 0$$

$$\Rightarrow f_0'(\phi)\, d\phi/dx = (d^2\phi/dx^2)\, d\phi/dx$$

Now integrating both sides w.r.t. 'x', we get:

$$\int_{-\infty}^{\infty} f_0'(\phi)\, d\phi/dx\, dx = \int_{-\infty}^{\infty} (d^2\phi/dx^2)\, d\phi/dx\, dx$$

$$\int_{-\infty}^{\infty} ((\partial f_0/\partial x)(\partial\phi/\partial x)\, dx = \int_{-\infty}^{\infty} (d/dx\, (1/2(d\phi/dx)^2))\, dx$$

$$\int_{-\infty}^{\infty} d/dx\, (f_0(\phi))\, dx = \int_{-\infty}^{\infty} d/dx\, (1/2(d\phi/dx)^2)\, dx$$

$$\mathbf{f_0(\phi) = (1/2)\, (d\phi/dx)^2}$$

### 1.3.4.2. Equilibrium profile for Phase field variable ($\phi$)

$$f_0(\phi) = (1/4\varepsilon^2)(\phi^2-1)^2 = (1/4\varepsilon^2)(1-\phi^2)^2$$

Now, using the above derived relation:-

$$(1/2)\,(d\phi/dx)^2 = f_0(\phi)$$

$$(1/2)\,(d\phi/dx)^2 = (1/4\varepsilon^2)(1-\phi^2)^2$$

$$d\phi/dx = (1/\sqrt{2}\,\varepsilon)\,(1-\phi^2)$$

$$d\phi \,/\, ((1-\phi)(1+\phi)) = dx \,/\, (\sqrt{2}\,\varepsilon)$$

Now, integrating from 0 to 'x', with the boundary condition $\phi(0) = 0$

$$\int_0^{\phi} d\phi/((1-\phi)(1+\phi)) = \int_0^{x} dx/(\sqrt{2}\,\varepsilon)$$

$$\int_0^{\phi} d\phi/((1-\phi)(1+\phi)) = x/(\sqrt{2}\,\varepsilon)$$

**Integration Steps:**

$$1/2 \int_0^{\phi} [\,(1+\phi) + (1-\phi)\,] \,/\, [\,(1-\phi)(1+\phi)\,]\, d\phi = x \,/\, (\sqrt{2}\,\varepsilon)$$

$$1/2 \int_0^{\phi} [\,1/(1-\phi) + 1/(1+\phi)\,]\, d\phi = x \,/\, (\sqrt{2}\,\varepsilon)$$

$$1/2\,[\,-\ln|1-\phi| + \ln|1+\phi|\,] = x \,/\, (\sqrt{2}\,\varepsilon)$$

$1/2 \ln | (1+\phi) / (1-\phi) | = x / (\sqrt{2} \varepsilon)$ —----- (1)

$\tanh x = \sinh x / \cosh x$

$= (e^x - e^{-x}) / (e^x + e^{-x})$

$\tanh x = (e^{2x} - 1) / (e^{2x} + 1)$

If $y = \tanh^{-1} x$

$\Rightarrow x = \tanh(y)$

$x = (e^{2y} - 1) / (e^{2y} + 1)$

$e^{2y} - 1 = e^{2y} * x + x$

$\Rightarrow e^{2y} (1-x) = 1+x$

$e^{(2y)} = (1+x) / (1-x)$

$2y = \ln((1+x) / (1-x))$

$y = (1/2) \ln((1+x) / (1-x)) = \tanh^{-1}x$ —--------- (2)

Using (2) in (1):

$\tanh^{-1}(\varnothing) = x / (\sqrt{2} \varepsilon)$

$\beta(x) = \tanh(x / (\sqrt{2} \varepsilon))$

1.3.4.3. Derivation of surface tension

We already know that:

$$(1/2)(d\phi/dx)^2 = f_0(\phi) \text{ and } \phi = \tanh(x/(\sqrt{2}\,\varepsilon))$$

Now, using,

$$\sigma = \lambda \int_{-\infty}^{\infty} \{ (1/2)(d\phi/dx)^2 + f_0(\phi) \}\, dx$$

$$\sigma = \lambda \int_{-\infty}^{\infty} (d\phi/dx)^2\, dx$$

$$\sigma = \lambda \int_{-\infty}^{\infty} [\, d/dx \tanh(x/(\sqrt{2}\,\varepsilon)) \,]^2\, dx$$

$$\sigma = \lambda \int_{-\infty}^{\infty} \mathrm{sech}^2(x/(\sqrt{2}\,\varepsilon))\, (1/(\sqrt{2}\,\varepsilon))^2\, dx$$

$$\sigma = \lambda/(2\varepsilon^2) \int_{-\infty}^{\infty} \mathrm{sech}^4(x/(\sqrt{2}\,\varepsilon))\, dx$$

$$\sigma = \lambda/(2\varepsilon^2) \int_{-\infty}^{\infty} (1 - \tanh^2(x/(\sqrt{2}\,\varepsilon)))\, \mathrm{sech}^2(x/(\sqrt{2}\,\varepsilon))\, dx$$

Let $a = \tanh x/(\sqrt{2}\,\varepsilon)$, $da = \mathrm{sech}^2\, x/(\sqrt{2}\,\varepsilon) \,.\, 1/(\sqrt{2}\,\varepsilon)\, dx$

$$\sigma = \lambda/(2\varepsilon^2) \int_{-\infty}^{\infty} (1 - a^2)\, da\, (\sqrt{2}\,\varepsilon)$$

$$\sigma = \lambda/(\sqrt{2}\,\varepsilon)\, [\, a - a^3/3 \,]\big|_{-1}^{1} \Rightarrow \sigma = \lambda/(\sqrt{2}\,\varepsilon)\, [\, 2 - 2/3 \,]$$

$$\boldsymbol{\sigma = (4\lambda)/(3\sqrt{2}\,\varepsilon)}$$

As the interfacial thickness, ε, shrinks towards 0, so should the energy density parameter, λ. The ratio gives interfacial tension.

### 1.3.5. Calculation of two Phase NS flows using Phase field

1.3.5.1. [David Jacqmin] Streaming flows can thin or thicken an interface and this must be resisted by high enough diffusion. On the other hand, too large a diffusion will overly damp the flows. These two constraints lead to an upper and lower bound for diffusivities.

1.3.5.2. Creation of interface energy by convection is always balanced by an equal decrease in kinetic energy caused by surface tension forcing.

1.3.5.3. Phase field methods are based on models of fluid free energy. The surface tension forces on the fluid is derived variationally from its energy density field.

1.3.5.4. It is easy to generate phase-field numerical implementations that are dissipative of energy and that therefore are free of parasitic flows.

1.3.5.5. Phase-field interface structure is important in determining interface energy and thus surface tension. Because of the need to calculate this structure, numerical phase field interfaces have usually been made wide, typically four to eight cells. -> Wider interfaces require stronger anti-diffusion in order to keep them from being distorted by advective straining.

1.3.5.6. **The derivation of diffuse-interface fluid dynamical equations is fairly simple, especially for compressible flow.** The key ideas are:

Convection can change the amount of free energy by either lengthening or thickening\thinning interfaces.

There must be a diffuse-interface force exerted by the fluid such that change in kinetic energy is always opposite to the change in free energy.

This must be true for arbitrary interface configuration and velocity fields.

1.3.5.7. The rate of change of free energy due to convection is -

$$\frac{d(T_{conv})}{dt} = \int \frac{\partial F_{conv}}{\partial t}\, d\Omega$$

$$= \int \left(\frac{\partial F}{\partial \Phi} + \frac{\partial \Phi}{\partial t}\right)\Big|_{conv}\, d\Omega$$

$$= \int \mu \left(\frac{\partial \Phi}{\partial t}\right)\Big|_{conv}\, d\Omega$$

$$= \int \mu(-\nabla \cdot (u\Phi))\, d\Omega$$

$= - \int \mu\ \nabla(u\Phi)\, d\Omega$ ; for incompressible flows $\nabla(u\Phi) = \Phi(\nabla.(u)) + u.\nabla(\Phi)$

$$= - \int \mu\, u \nabla \phi\, d\Omega$$

$$= - \int u.(\mu \nabla \phi)\, d\Omega$$

The rate of change of kinetic energy due to surface forces is

$$d(T)/dt\ (\text{kinetic}) = \int u \cdot F_s\, d\Omega$$

For change in free energy and kinetic energy to be always equal and opposite for any arbitrary velocity.

$[F_s] = \mu \nabla \phi$ ; $\phi$= Phase field order parameter, $\mu$= chemical potential

Forcing occurs only at interfaces.

Note:

Interface curvature changes phase-field surface tension. The incurred error in both surface tension and pressure jump is a quadratic function of interface thickness times curvature.

The chemical potential is the phase-field analogue to surface tension times curvature and as such it is a very important variable. It can sometimes provide the key to understanding a particular physical or numerical issue.

Solutions of purely diffusive Cahn-Hilliard equation have potential fields that are generally smooth. The Cahn-Hilliard $eq^n$ has curvature dependent solubility, which is what makes it useful for modelling nucleation, evaporation and coarsening. Regions of high curvature are generally also regions of high potential and high solubility - material from this region fluxes into the surrounding - lower potential medium. The extent of this solubility depends on potential ψ.

Where fluid convection is introduced, the chemical potential is no longer necessarily smooth. This has two causes. The first is that convective straining can lead to thicken or thin an interface. The second cause is related to curvature dependent solubility. Oscillation of interface changes its curvature and thus local solubility.

The desired asymptotic behaviour of the interface chemical potential is that it be constant across an interface.

### 1.3.5.8. Derivation of Alternate Surface Tension forcing

The change in energy due to convection $= \int ((\partial F/\partial \phi)(\partial \phi/\partial t)\, d\Omega \,|\_(\text{convection})$

$= \int \mu\, (\partial \phi/\partial t)\, d\Omega \,|\_(\text{convection})$

$= \int \mu\, (-\nabla \cdot (v\phi))\, d\Omega$   [As the phase field parameter should be conserved in the entire domain]

$= - \int \mu \nabla \cdot (v\phi) \, d\Omega$

$= - \int [ \nabla \cdot (\mu v\phi) - v\phi \nabla \mu ] \, d\Omega$

$= - \int_V \nabla \cdot (\mu v\phi) \, d\Omega + \int_V v\phi \nabla \mu \, d\Omega$

$= - \int_S \mu\phi \, v \cdot n \, dS + \int_V \phi v \nabla \mu \, d\Omega$

$v \cdot n = 0$ (at all boundaries :: No Flux)

$= \int_V \phi v \nabla \mu \, d\Omega$

The change in kinetic energy due to surface tension forcing

$= \int V F_s \, d\Omega$

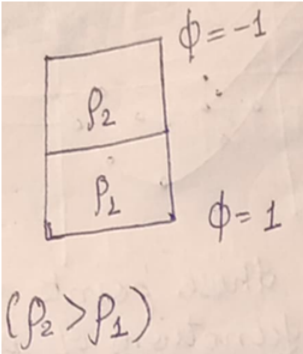
{change in energy due to convection} = - {change in K.E due to surface tension}

**$F_s = - \phi \nabla \mu$** → Applicable for both compressible and incompressible.

**Also**

**$F_\sigma = \mu \nabla \phi$** → For incompressible flow.

## 2. Dispersion Relation for immiscible fluids




*two fluids with densities $\rho_1$ and $\rho_2$ separated by an interface. $\rho_2 > \rho_1$. $\phi = -1$ in the lower fluid and $\phi = 1$ in the upper fluid*

$\phi$: phase field variable

$\phi = - r^{(1-q)/2} \tanh((y-h)/(\sqrt{2}\varepsilon) * r^{(1-p)/2})$

$\mathcal{F}(\phi) = \int \frac{1}{2}\lambda|\nabla\phi|^2 + (\lambda/4\varepsilon^2)(\phi^2-1)^2 \, dx$

$\nabla\phi = \phi_x\hat{i} + \phi_y\hat{j}$

$|\nabla\phi| = \sqrt{(\phi_x^2 + \phi_y^2)}$

$|\nabla\phi|^2 = \phi_x^2 + \phi_y^2$

$\mathcal{F}(\phi) = \int [\, (\lambda/4\varepsilon^2)(\phi^2-1)^2 + \frac{1}{2}\lambda\phi_x^2 + \frac{1}{2}\lambda\phi_y^2 \,] \, dx$

$\mathcal{F}(\phi) = F(x, y, \phi, \phi')$

$\mu = \delta\mathcal{F}(\phi)/\delta\phi \Rightarrow \partial F/\partial\phi - \partial/\partial x\, (\partial F/\partial\phi_x) - \partial/\partial y\, (\partial F/\partial\phi_y)$

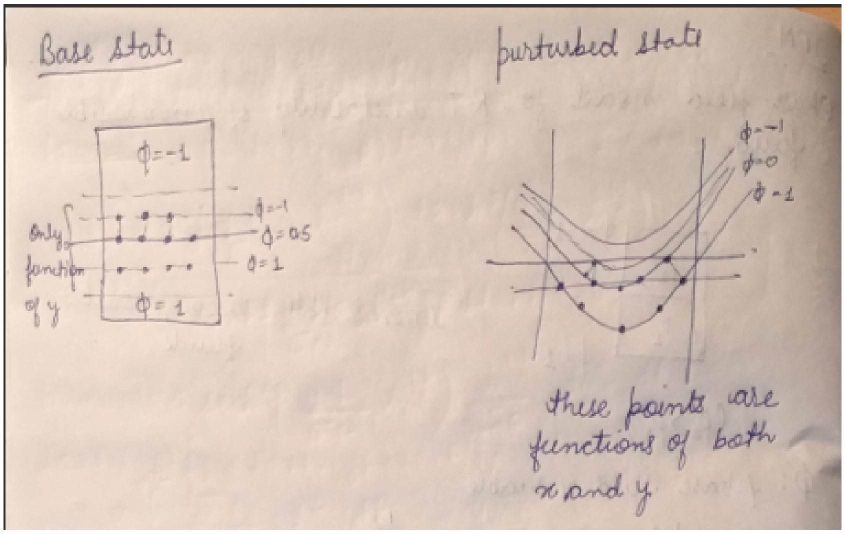
$\mu = \lambda/\varepsilon^2\, (\phi^3-\phi) - \lambda\, \partial^2\phi_x/\partial x^2 - \lambda\, \partial^2\phi_y/\partial y^2$

$\Rightarrow \lambda/\varepsilon^2\, (\phi^3-\phi) - \lambda\, \partial^2\phi/\partial x^2 - \lambda\, \partial^2\phi/\partial y^2$ --- (i)

$\mu_{base\_state} = \lambda/\varepsilon^2\, (\phi^3-\phi) - \lambda\, \partial^2\phi/\partial y^2$

From (i) and (ii)

$\mu = -\lambda\, \partial^2\phi/\partial x^2$




Now,

$\mu = -\lambda\, \partial^2\phi/\partial x^2$

$\phi = \phi = - r^{(1-q)/2} \tanh\left( ((y-h(x,t)) / (\sqrt{2}\,\varepsilon)) * r^{(1-p)/2} \right)$ {**For binary fluid**}

$\partial\phi/\partial x = r^{(1-q)/2}\, \mathrm{sech}^2 (((y-h(x,t)) / (\sqrt{2}\,\varepsilon)) * r^{(1-p)/2}) (-h' / (\sqrt{2}\,\varepsilon) * r^{(1-p)/2})$

$\partial^2\phi/\partial x^2 = r^{(1-q)/2}\, \mathrm{sech}^2 (((y-h(x,t)) / (\sqrt{2}\,\varepsilon)) * r^{(1-p)/2}) \tanh ( ((y-h(x,t)) / (\sqrt{2}\,\varepsilon)) * r^{(1-p)/2} ) (-h' / (\sqrt{2}\,\varepsilon) * r^{(1-p)/2}) + r^{(1-q)/2}\, \mathrm{sech}^2 (((y-h(x,t)) / (\sqrt{2}\,\varepsilon)) * r^{(1-p)/2}) (-h_{xx} / (\sqrt{2}\,\varepsilon) * r^{(1-p)/2})$

$\partial^2\Phi / \partial x^2 = -((h')^2/\varepsilon^2 * r^{1-p+(1-q)/2} )\, \mathrm{sech}^2( (y-h) / (\sqrt{2}\varepsilon) * r^{(1-p)/2} ) \tanh( (y-h) / (\sqrt{2}\varepsilon)\, r^{(1-p)/2} ) - r^{(2-p+q)/2} h'' / (\sqrt{2}\varepsilon)\, \mathrm{sech}^2( (y-h) / (\sqrt{2}\varepsilon)\, r^{(1-p)/2} )$

$\mu = -\lambda\, [ \partial^2\phi/\partial x^2 ]$

$\partial\mu/\partial y = \lambda((h')^2/\varepsilon^2) * r^{(3-2p-q)/2} * 2 * \mathrm{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) ( r^{(1-p)/2}/(\sqrt{2}$

$\varepsilon)) + \lambda((h')^2/\varepsilon^2) * r^{(3-2p-q)/2} * \mathrm{sech}^4 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) * ( r^{(1-p)/2}/(\sqrt{2}\,\varepsilon)) + \lambda(h''/\sqrt{2}\varepsilon^2) * 2 * \mathrm{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) ( r^{(1-p)/2}/(\sqrt{2}\,\varepsilon))$

$\phi\partial\mu/\partial y$ = terms + term2 + term3

$\text{term1} = - \lambda((h')^2/\varepsilon^2) * r^{(4-2p-2q)/2} * 2 * \mathrm{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh^3 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) ( r^{(1-p)/2}/(\sqrt{2}\,\varepsilon))$

$= a\, \mathrm{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh^3 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} )$

$\text{term2} = b \times \mathrm{sech}^4 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} )$

$\text{term3} = - \lambda(h''/\sqrt{2}\varepsilon^2) * r^{(3-2p-q)/2} * r^{(1-q)/2} * 2 * \mathrm{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} )$

Integrating $\phi\partial\mu/\partial y$ in vertical direction

Term1:

$$a \int_{-\infty}^{\infty} a\, \mathrm{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh^3 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} )\, dy$$

$\tanh ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) = t$

$\mathrm{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2}) ( r^{(1-p)/2}/(\sqrt{2}\,\varepsilon))\, dy = dt$

$$a (\sqrt{2}\,\varepsilon) \int_{-1}^{1} t^3\, dt \Rightarrow a (\sqrt{2}\,\varepsilon) [ t^4/4 ]|_{-1}^{1} = 0$$

Term2:

$$b \int_{-\infty}^{\infty} \mathrm{sech}^4 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} )\, dy$$

$\tanh\left( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} \right) = t$

$b * ((\sqrt{2}\,\varepsilon) / r^{(1-p)/2}) \operatorname{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) )\, dy = dt$

$b ((\sqrt{2}\,\varepsilon) / r^{(1-p)/2}) \int (1-t^2)\, dt$

$\int t^2\, dt \Rightarrow [\, t - t^3/3\, ]|_{-1}^{1} \Rightarrow 0$

Term3:

$- \lambda(h''/\sqrt{2}\varepsilon^2) * r^{(3-2p-q)/2} * r^{(1-q)/2} * 2 * \operatorname{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} ) \tanh^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} )$

$\tanh ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2}) = t$

$\operatorname{sech}^2 ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2}) \times ( r^{(1-q)/2}/(\sqrt{2}\,\varepsilon))\, dy = dt$

$\Rightarrow -\lambda/\varepsilon^2\, h''\, r^{(3-2p-q)/2}\, r^{(1-q)/2} \int \sqrt{2}\,\varepsilon/t^2 * (1 / r^{(1-q)/2} )\, dt$

$=> -\lambda/\varepsilon^2\, h''\, r^{(3-2p-q)/2}\, \sqrt{2}\varepsilon/3\, [t^3]^{1}_{1}$

$=> -\lambda/\varepsilon^2\, h''\, 2\sqrt{2}/3\, \gamma\, r^{(3-2p-q)/2}$

$=> -h''\, (2\sqrt{2}\, \lambda/3\, \varepsilon)\, \gamma\, r^{(3-2p-q)/2}$

$=> -\sigma_0\, h''\, r^{((3-2p-q)/2)}$

$$\int_{-\infty}^{\infty} \phi\, dy$$

$$\int_{-\infty}^{\infty} r^{(1-q)/2} \tanh ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} )\, dy$$

$$- r^{(1-q)/2} \left[ \int_{0}^{\infty} \tanh ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2}\, dy + \int_{-\infty}^{0} \tanh ( (y-h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2}\, dy \right]$$

$y = -t \quad dy = -dt$

$\int \tanh\left( (-t)/(\sqrt{2}\,\varepsilon) \right) r^{1/2} (-dt)$

$\int \tanh\left( (t)/(\sqrt{2}\,\varepsilon) \right) r^{1/2} dt$

$\int \tanh\left( (y+h)/(\sqrt{2}\,\varepsilon) \right) r^{1/2} dt$

$$- r^{(1-q)/2} \left[ \int_0^{\infty} \tanh\left( (y-h)/(\sqrt{2}\,\varepsilon) \right) r^{(1-p)/2} \, dy - \int_{-\infty}^{0} \tanh\left( (y+h)/(\sqrt{2}\,\varepsilon) * r^{(1-p)/2} \right) dy \right]$$

say $\beta = (1/(\sqrt{2}\,\varepsilon))\, r^{(1-p)/2}$

$$- r^{(1-q)/2} \left[ \int_0^{\infty} \tanh\left( (y-h)\beta \right) - \tanh\left( (y+h)\beta \right) dy \right]$$

$$- r^{(1-q)/2} \int_0^{\infty} \left[ \tanh(y\beta) - \tanh(h\beta) / (1 - \tanh(y\beta)\tanh(h\beta)) - (\tanh(y\beta) + \tanh(h\beta) / (1 + \tanh(y\beta)\tanh(h\beta)) \right] dy$$

$$- r^{(1-q)/2} \int_0^{\infty} \left[ (\text{Numerator}) / (1 - \tanh^2(y\beta)\tanh^2(h\beta)) \right] dy$$

Numerator: $\tanh(y\beta)\{1 + \tanh(y\beta)\tanh(h\beta)\} - \tanh(h\beta)\{1 + \tanh(y\beta)\tanh(h\beta)\}$

$- \tanh(y\beta)\{1 - \tanh(y\beta)\tanh(h\beta)\} - \tanh(h\beta)\{1 - \tanh(y\beta)\tanh(h\beta)\}$

$\Rightarrow \tanh(y\beta) + \tanh^2(y\beta)\tanh(h\beta) - \tanh(h\beta) - \tanh(h\beta)\tanh^2(y\beta) - \tanh(y\beta) + \tanh^2(y\beta)\tanh(h\beta) - \tanh(h\beta) + \tanh^2(h\beta)\tanh(y\beta)$

$\Rightarrow -2\tanh(h\beta) + 2\tanh^2(y\beta)\tanh(h\beta)$

$\Rightarrow -2\tanh(h\beta)\{1 - \tanh^2(y\beta)\}$

$\alpha = \tanh(h\beta)$

$\Rightarrow -2\alpha \operatorname{sech}^2(y\beta) = \text{Numerator}$

$- \; r^{(1-q)/2} \int_0^\infty [ (-2\alpha \operatorname{sech}^2(y\beta)) / (1 - \tanh^2(y\beta)\tanh^2(h\beta)) ] \, dy$

$\tanh(y\beta) = t; \;\; \operatorname{sech}^2(y\beta) \, \beta \, dy = dt$

$-2\alpha/\beta * r^{(1-q)/2} \int_0^1 dt / (1 - t^2\alpha^2)$

$1 / (1 - t^2\alpha^2) = 1 / ((1+t\alpha)(1-t\alpha)) = a / (1+t\alpha) + b / (1-t\alpha)$

$\Rightarrow 2\alpha/\beta \; r^{1/2} \int_0^1 dt / (1 - t^2\alpha^2)$

$1 / ((1+t\alpha)(1-t\alpha)) = 1/2 \, [ 1/(1+t\alpha) + 1/(1-t\alpha) ]$

$a = 1/2 \; b = 1/2$

$r^{(1-q)/2} \,) / \beta \, [ \ln(1+t\alpha)/\alpha + \ln(1-t\alpha)/-\alpha ] \,|_0^1$

$r^{(1-q)/2} \; / \beta \times \ln [ (1+\alpha) / (1-\alpha) ] \,|_0^1$

$r^{(1-q)/2} \; / \beta \times \ln [ (1+\alpha) / (1-\alpha) ]$

$r^{(1-q)/2} / \beta \times \ln [ (1+\tanh(h\beta)) / (1-\tanh(h\beta)) ]$

$r^{(1-q)/2} \; / \beta \times \ln [ (1 + (e^{h\beta} - e^{-h\beta}) / (e^{h\beta} + e^{-h\beta})) / (1 - (e^{h\beta} - e^{-h\beta}) / (e^{h\beta} + e^{-h\beta})) ]$

$r^{(1-q)/2} \; / \beta \times \ln [ (2e^{h\beta}) / (2e^{-h\beta}) ]$

$r^{(1-q)/2}$ / β × 2hβ

⇒ 2h $r^{(1-q)/2}$

N.S. eqn: $\rho\ [\ \partial V_i/\partial t + v\cdot(\nabla\cdot V)\ ] = -\partial p + \mu(\partial^2 V_i) - \rho g_i + \Phi\partial\mu/\partial xi$

$\rho_0\ (\partial V/\partial t) = -\partial/\partial y\ (p - \rho_0 g y) - \Phi\ \partial\mu/\partial y + (\rho - \rho_0)g_i$

$\rho = \rho_1\ (1+\Phi)/2 + \rho_2\ (1-\Phi)/2$

$\rho_0 = (\rho_1+\rho_2)/2$

$\rho_0\ (\partial V/\partial t) = -\partial P/\partial y - \Phi\partial\mu/\partial y - A\Phi\rho_0 g$

integrating along y:

$$\rho_0\ (\partial q\_y/\partial t) = 0 + \int_{-\infty}^{\infty} -\Phi\ (\partial\mu/\partial y) - A\rho_0 g \int_{-\infty}^{\infty} \Phi\ dy$$

$\rho_0\ (\partial q\_y/\partial t) = \sigma_0\ h''\ \lambda^{(3-h-p)/2} - Ag\rho_0\ (2h)\ r^{(1-q)/2}$

$\psi = \int e^{-ky}\ e^{ikx}\ \hat{y}(k,t)\ dk$ [ $\hat{y}$ → psi hat]

$V\_y = \int e^{-ky}\ e^{ikx}\ \hat{y}(k,t)\ dk$ ; $\bar{V} = \int -k\ e^{ikx}\ \hat{y}(k,t)\ dk$

$$q\_y = \int_{-\infty}^{\infty} V\ dy \Rightarrow \int_{0}^{\infty} V\ dy + \int_{-\infty}^{0} V\ dy \Rightarrow \int V\ dy + \int V\ (-dy) \Rightarrow 2^* \int_{0}^{\infty} V\ dq$$

$$q\_y = 2\int_{0}^{\infty} -k\ [\ \int_{0}^{\infty} e^{-ky}\ dy\ ]\ e^{ikx}\ \hat{y}(k,t)\ dk \Rightarrow -2\int_{0}^{\infty} e^{-ikx}\ \hat{y}(k,t)\ dk$$

**V*y = k q_y* / 2**

$h = \hat{h}(k,t)\, e^{ikx}$

$\rho_0\, (\partial/\partial t)\, (2\bar{V}/k) = \sigma_0\, h''\, r^{(3-2p-q)/2} - Ag\rho_0\ \ (2h)\, r^{(1-q)/2}$

$(2\rho_0/k)\, (\partial^2\hat{h}/\partial t^2) = -k^2\, \sigma_0\, r^{(3-2q-p)/2} + Ag\rho_0\ \ (2\hat{h})\, r^{(1-q)/2}$

$\partial^2\hat{h}/\partial t^2 = -k/(2\rho_0)\, (k^2\sigma_0 r^{(3-2q-p)/2} + 2A\rho\bar{g}_0 r^{(1-q)/2})\, \hat{h}$

$\omega^2 = k/(2\rho_0)\, (k^2\sigma_0 r^{(3-2q-p)/2} + 2A\rho\bar{g}_0 r^{(1-q)/2)})$

$\bar{g} = -g\hat{j}$

$\omega^2 = k^3/2\rho_0\, \gamma^{(3-2q-p)/2} - 2Ag\rho_0/2\rho_0\, k\, r^{(1-q)/2}$

$= k^3\sigma_0/2\rho_0\, \gamma^{(3-2q-p)/2} - kAg\, r^{(1-q)/2}$

$\partial^2\hat{h}/\partial t^2 = -\omega^2\hat{h}$

if $\omega^2 > 0$ stable $\hat{h} = A \sin(\omega t) + B \cos(\omega t)$

if $\omega^2 < 0$ unstable $\hat{h} = A\, e\hat{}(\omega t) + B\, e\hat{}(-\omega t)$

Considering unstable condition

$\omega(k) = \sqrt{}(\, -Agkr^{1-q}\, ) + k^2\sigma_0/2\rho_0\, r^{(3-2q-p)/2} = i\alpha(k)$

limit of instability

$\sigma_0 k^3/2\rho_0\, r^{(3-2q-p)/2} - kAg\, \gamma^{(1-q)/2} < 0$

$r^{(1-q)/2}\, [\, (\sigma_0 k^3/2\rho_0)\, r^{(2-p-q)/2} - kAg\, ] < 0$

$r^{(2-p-q)/2} \leq (kAg) / (\sigma_0 k^3/2\rho_0)$

$r^{(2-p-q)/2} \leq (2\rho_0 A g) / (\sigma_0 k^2)$

For $(\rho_1, \rho_2) = (1,3) \Rightarrow A = 0.5 \;\; \rho_0 = 2 \;\; \sigma_0 = 0.1$

$k = 2\pi$

$r^{1/2} \leq (20) / (2\pi)^2$

$r \leq 0.256 \;\; \Rightarrow$ theoretically unstable

## 3. Derivation for Non-binary fluid

$\mathscr{F}(\phi) = \int \tfrac{1}{2}\lambda|\nabla\phi|^2 + (\lambda/4\varepsilon^2)(\phi^2-1)^2 \, dx$

$\nabla\phi = \phi_x\hat{\imath} + \phi_y\hat{\jmath}$

$|\nabla\phi| = \sqrt{(\phi_x^2 + \phi_y^2)}$

$|\nabla\phi|^2 = \phi_x^2 + \phi_y^2$

$\mathscr{F}(\phi) = \int [\, (\lambda/4\varepsilon^2)(\phi^2-1)^2 + \tfrac{1}{2}\lambda\phi_x^2 + \tfrac{1}{2}\lambda\phi_y^2 \,] \, dx$

$\mu = \delta\mathscr{F}(\phi)/\delta\phi \Rightarrow \lambda/\varepsilon^2 \, (\phi^3-\phi) - \lambda \, \partial^2\phi/\partial x^2 - \lambda \, \partial^2\phi/\partial y^2$ --- (a)

$\mu_{\text{(base state)}} = \lambda/\varepsilon^2 \, (\phi^3-\phi) - \lambda \, \partial^2\phi/\partial y^2 = 0$ --- (b)

from a & b:

$\mu = -\lambda \, \partial^2\phi/\partial x^2 \, , \; \phi = f(\, (y-h)/\varepsilon \,)$

$\phi = -\tanh(\, (y-h) / (\sqrt{2}\,\varepsilon) \,)$

NS: $\rho_0 \, Dv_i/Dt = -\partial_i p + \mu\nabla^2 V_i - \phi\partial_i\mu + \rho g$

$\circlearrowleft(\epsilon)$

$\rho_0 (\partial V_i/\partial t) = -\partial_i p - \phi\partial_i\mu + (\rho-\rho_0)g + \rho_0 g$

in y-direction:

$\rho_0 (\partial V/\partial t) = -\partial P/\partial y - \phi\, \partial\mu/\partial y + (\rho-\rho_0)g$

$P = p - \rho_0 g y$

$\mu = \lambda/\varepsilon\ [\ f'\ \partial^2 h/\partial x^2 - f''\ (\partial h/\partial x)^2/\varepsilon\ ]$

$\rho_0 (\partial V/\partial t) = -\partial P/\partial y - \lambda/\varepsilon\ [\ f'f\ \ \partial^2 h/\partial x^2 - f''f\ (\partial h/\partial x)^2/\varepsilon\ ] + (\rho-\rho_0)g$

integrating in vertical direction

$$q\_y = \lim\ (L\rightarrow\infty) \int_{-L}^{L} V\ dy$$

$$\rho_0 (\partial q\_y/\partial t) = 0 - \lambda/\varepsilon\ [\ \partial^2 h/\partial x^2 \int_{-\infty}^{\infty} f''\ f\ \ dy - (\partial h/\partial x)^2 * 1/\varepsilon \int_{-\infty}^{\infty} f'''\ f\ dy] + (\rho-\rho_0)g$$

$\rho = \rho_1\ (1+\phi)/2 + \rho_2\ (1-\phi)/2$

$\rho_0 = (\rho_1+\rho_2)/2$

$$\rho_0 (\partial q\_y/\partial t) = -\ \lambda/\varepsilon\ [\ \partial^2 h/\partial x^2 \int_{-\infty}^{\infty} f''\ f\ \ dy - (\partial h/\partial x)^2 * 1/\varepsilon \int_{-\infty}^{\infty} f'''\ f\ dy] + \rho_0 g * \phi(\rho_1-\rho_2)/2\rho_0$$

$$\rho_0 (\partial q\_y/\partial t) = -\ \lambda/\varepsilon\ [\ \partial^2 h/\partial x^2 \int_{-\infty}^{\infty} f''\ f\ \ dy - (\partial h/\partial x)^2 * 1/\varepsilon \int_{-\infty}^{\infty} f'''\ f\ dy] - A\rho_0 g \int_{-\infty}^{\infty} \phi\ dy$$

$\Phi = f(y-h)/\varepsilon) = -\ \tanh(y-h)/(\sqrt{2}\varepsilon))$

$\int f'' f\, dy = -1/\varepsilon^2 \int \text{sech}^2(y-h)/(\sqrt{2}\varepsilon))\, \tanh^2(y-h)/(\sqrt{2}\varepsilon))\, dy$

$-1/\varepsilon^2 \int t^2 \sqrt{2}\, \varepsilon\, dt$

$-\sqrt{2}/\varepsilon\, [\, t^3/3\, ]|_{-1}^{1}$

$= -2\sqrt{2} / 3\varepsilon$

$$\int_{-\infty}^{\infty} \phi\, dy = -\int_{-\infty}^{\infty} \tanh(\,(y-h) / (\sqrt{2}\,\varepsilon)\,)\, dy$$

$$\int_{0}^{\infty} -\tanh(\,(y-h) / (\sqrt{2}\,\varepsilon)\,)\, dy + \int_{-\infty}^{0} -\tanh(\,(y-h) / (\sqrt{2}\,\varepsilon)\,)\, dy$$

$$\int_{-\infty}^{\infty} \tanh(\,(y) / (\sqrt{2}\,\varepsilon)\,) - \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,) / (1 - \tanh(\,(y) / (\sqrt{2}\,\varepsilon)\,) \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,)\,) - \int_{-\infty}^{\infty}$$

$$\tanh(\,(-y-h) / (\sqrt{2}\,\varepsilon)\,)\,(-dy)$$

$$\Rightarrow \int_{-\infty}^{\infty} \tanh(\,(y-h) / (\sqrt{2}\,\varepsilon)\,) - \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,) / (1 - \tanh(\,(y-h) / (\sqrt{2}\,\varepsilon)\,) \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,)\,) +$$

$$\tanh(\,(y+h) / (\sqrt{2}\,\varepsilon)\,)\, dy$$

$$\int_{-\infty}^{\infty} \tanh(\,(y-h) / (\sqrt{2}\,\varepsilon)\,) - \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,) / (1 - \tanh(\,(y-h) / (\sqrt{2}\,\varepsilon)\,) \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,)\,) + ($$

$$\tanh(\,(y+h) / (\sqrt{2}\,\varepsilon)\,) - \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,) / (1 + \tanh(\,(y+h) / (\sqrt{2}\,\varepsilon)\,) \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,)\,)\,)\, dy$$

$$\Rightarrow \int_{0}^{\infty} -2 \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,) / (1 - \tanh^2(\,(y) / (\sqrt{2}\,\varepsilon)\,) \tanh^2(\,(h) / (\sqrt{2}\,\varepsilon)\,)\,)\, dy$$

$$\Rightarrow \int_{0}^{\infty} 2 \tanh(\,(h) / (\sqrt{2}\,\varepsilon)\,) / (1 - \tanh^2(\,(y) / (\sqrt{2}\,\varepsilon)\,) \tanh^2(\,(h) / (\sqrt{2}\,\varepsilon)\,)\,)\, dy$$

$\{ 1 - \tanh^2 ( (y) / (\sqrt{2} \varepsilon) ) \tanh^2 ( (h) / (\sqrt{2} \varepsilon) ) \}$

$\Rightarrow 2 \tanh ( (h) / (\sqrt{2} \varepsilon) ) \int_0^\infty dt / (1 - a^2t^2) \{ a = \tanh ( (h) / (\sqrt{2} \varepsilon) ) \}$

$(2\sqrt{2} \varepsilon) / (2a) [ 1/(1-at) + 1/(1+at) ] = 1/(2a) \ln [ (1+at) / (1-at) ]_0^1$

$\sqrt{2} \varepsilon \ln ( (1+a) / (1-a) )$

$\sqrt{2} \varepsilon \times 2h$

$= 2h\sqrt{2} \varepsilon$

$\Rightarrow 2h$

$\rho_0 \partial q\_y/\partial t = - \partial/\partial y (-2\sqrt{2}/3) + A\rho_0 g (2h)$

$\rho_0 \partial q\_y/\partial t = \partial/\partial x^2 \sigma - 2A\rho_0 gh$

$\psi(x,y,t) = \int_0^\infty e^{-ky} e^{ikx} \hat{\psi}(k,t) dk$

$V(x,y,t) = -\int k e^{-ky} e^{ikx} \hat{Q}(k,t) dk$

$q\_y = \int_{-\infty}^\infty V dy \Rightarrow 2 \int_0^\infty V dy$

$\Rightarrow 2 \int_0^\infty ( \int -k e^{-ky} dy) e^{ikx} \hat{\psi}(k,t) dk$

$q\_y (V\_int) = -2 \int_0^\infty ik e^{-ky} \hat{\psi}(k,t) dk$

$$V\_int = \int_{0}^{\infty} \hat{\psi}(k,t)\, dk$$

$$V\_int = \hat{k}\, q\_y$$

$$\rho_0\, \partial q\_y/\partial t = \sigma\, \partial^2\psi/\partial x^2 - 2A\bar{\rho}g\, \Phi$$

$$\rho_0\, (\partial/\partial t)\, (2\bar{V}/k) = \sigma\, (\partial^2\psi/\partial x^2) - 2A\bar{\rho}g\, h$$

$$\rho_0/k\, (\partial^2\xi/\partial t^2) = -\sigma\, k^2\, \hat{\psi} - 2A\bar{\rho}g\, \hat{h}$$

$$h = \hat{h}(k,t)\, e^{ikx}$$

$$2\rho_0/k\, \partial^2\hat{h}/\partial t^2 = -(\sigma k^2 + 2A\bar{\rho}g)\, \hat{h}$$

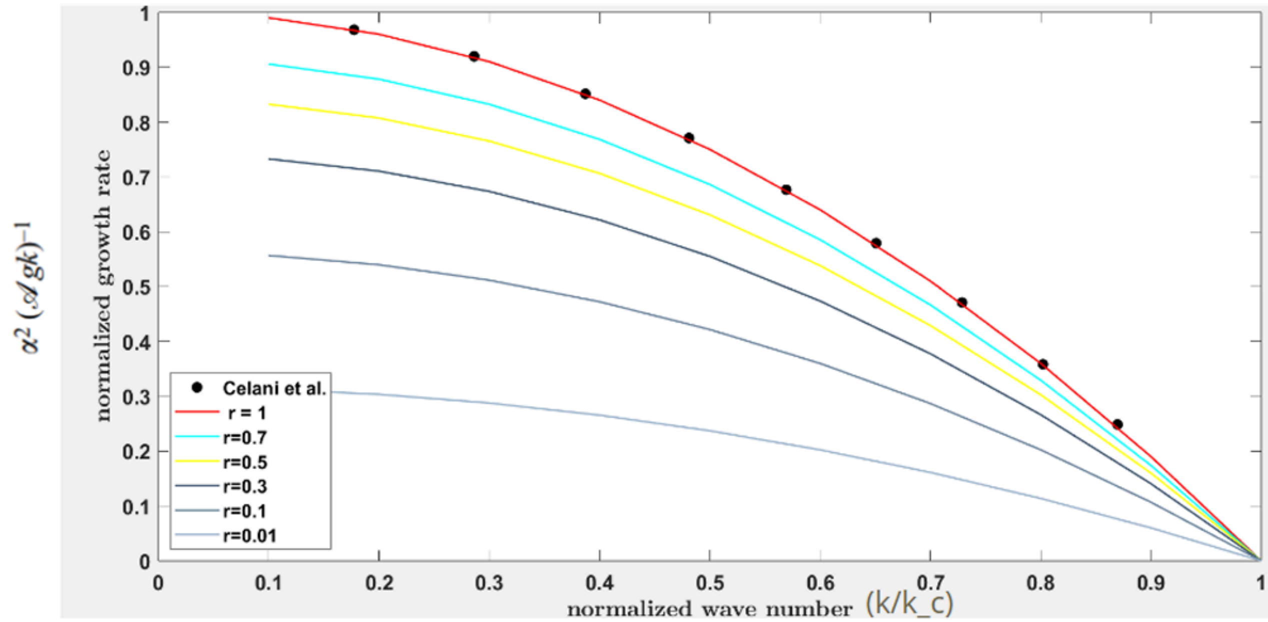
$$\partial^2\hat{h}/\partial t^2 = -k/2\rho_0\, (\sigma k^2 + 2A\bar{\rho}g)\, \hat{h}$$

$$\boldsymbol{\omega^2 = \sigma k^3/2\rho_0 + kAg}$$

## 4. Plots

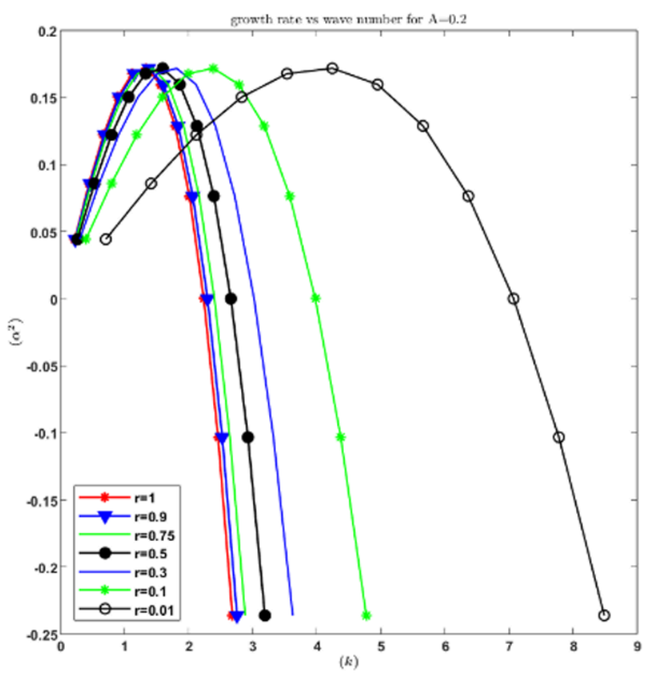
### 4.1. Plot of Non-dimensional growth-rate Vs Non-dimensional wave number

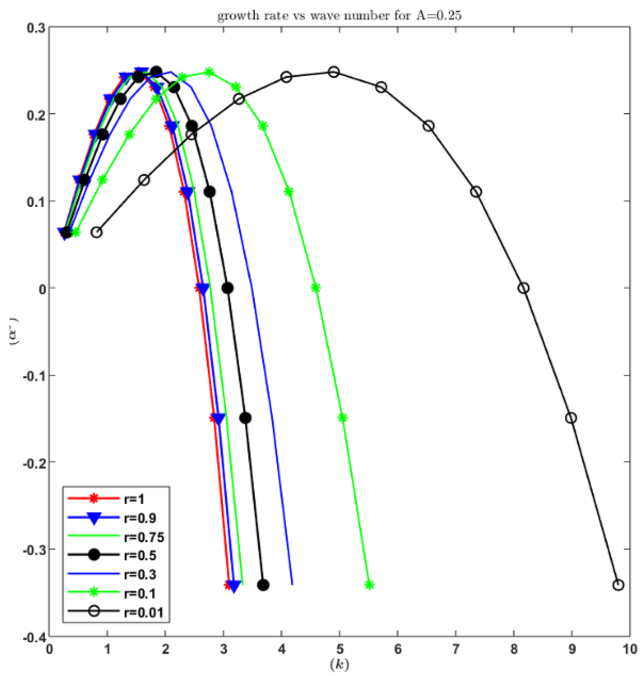
$$\frac{\alpha^2}{Agk} = -r^{\frac{1-q}{2}} \cdot \left[\left(\frac{k}{kc}\right)^2 + 1\right]$$

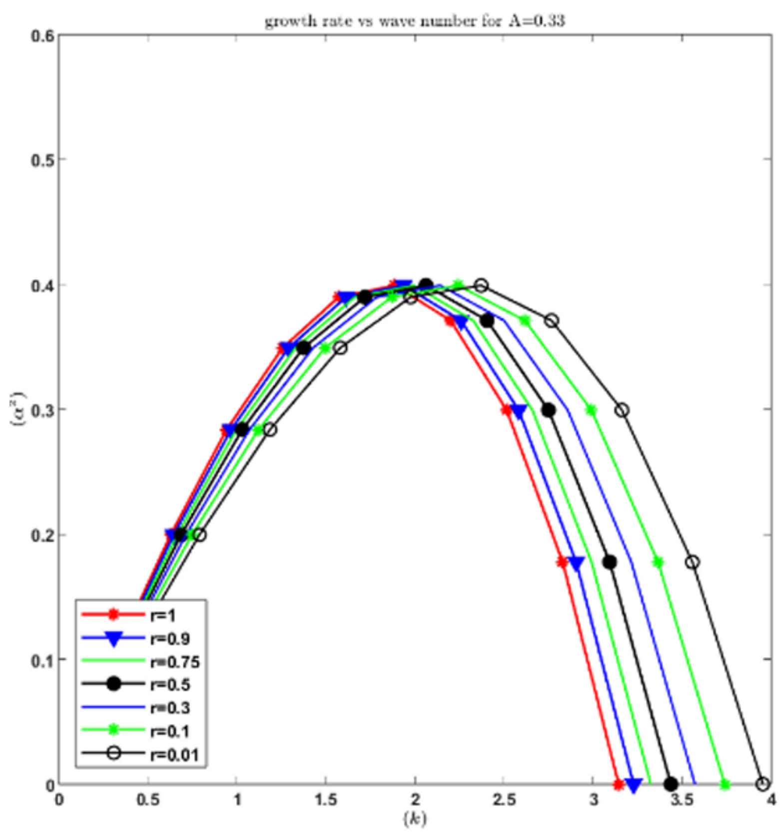


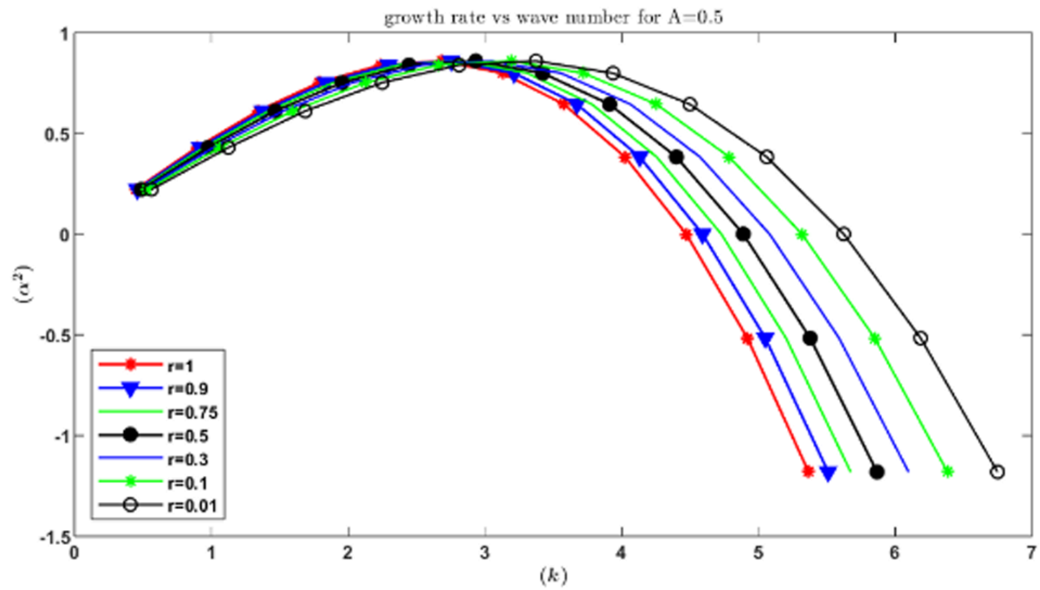

4.2. Plot of Dimensional growth-rate Vs Dimensional wave number

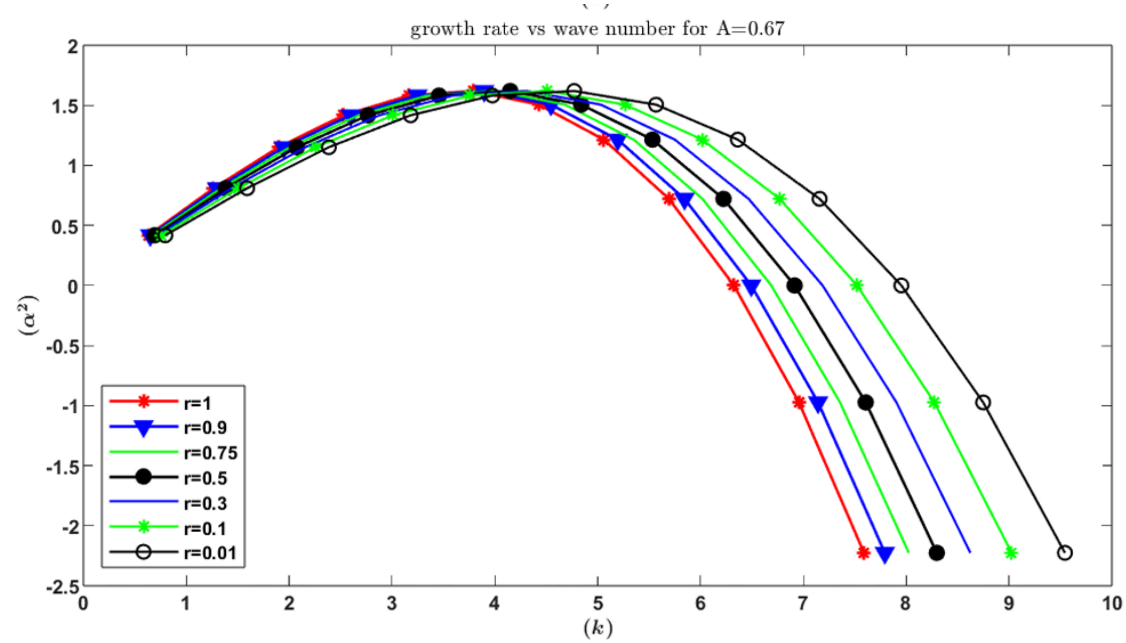
Surface Tension =0.1

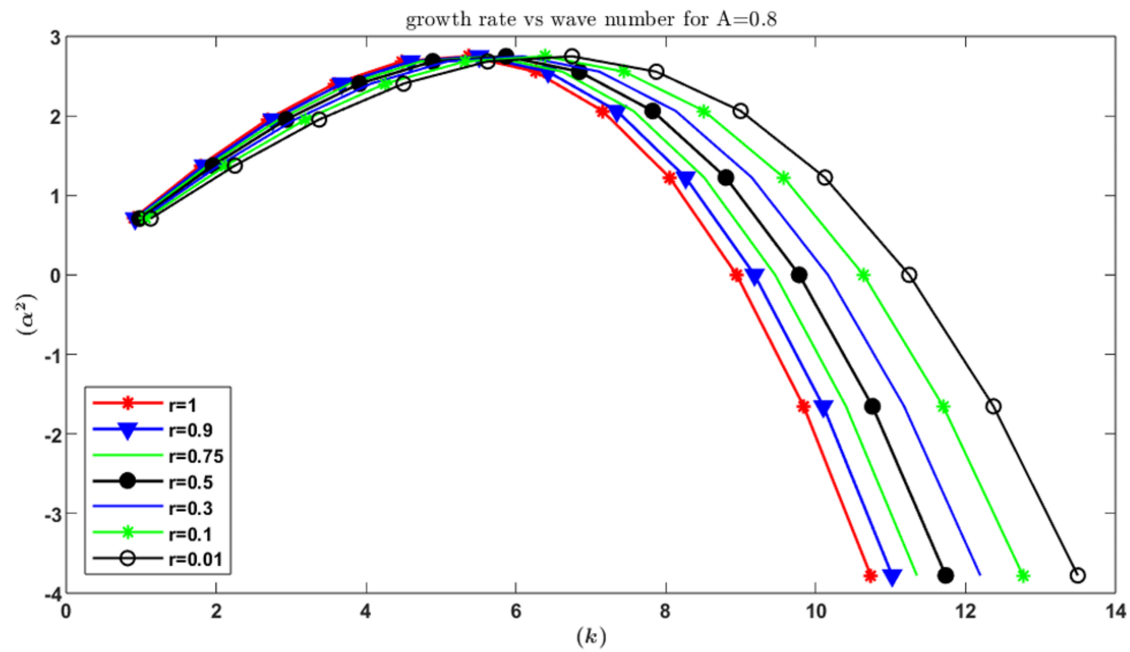


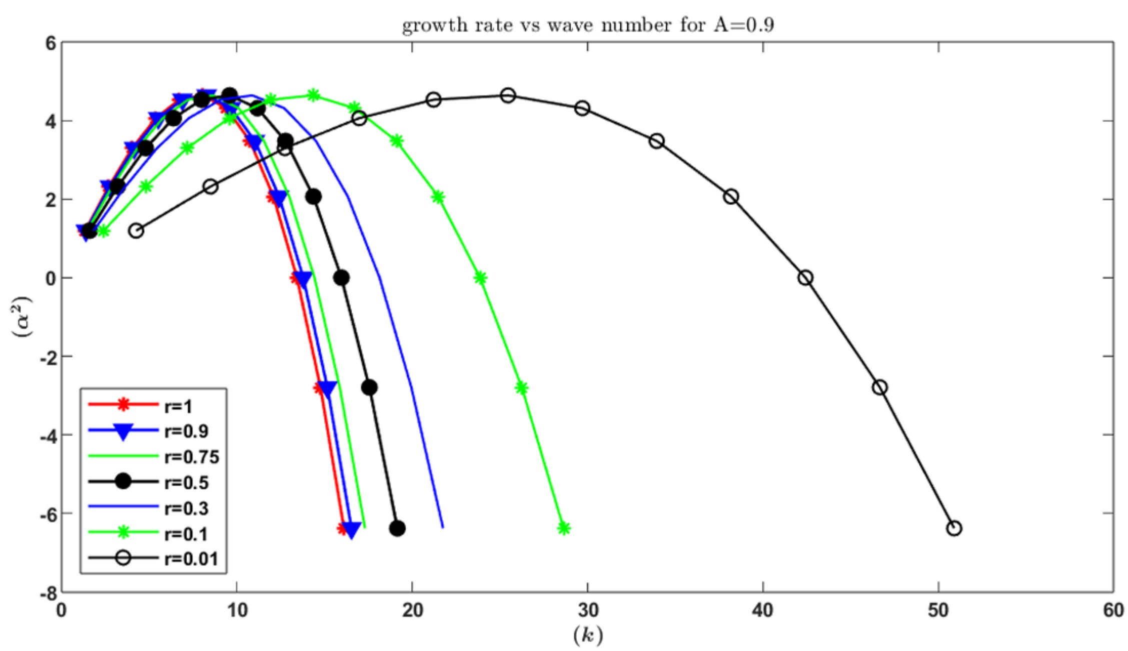

growth rate vs wave number for A=0.25
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01
(k)


growth rate vs wave number for A=0.33
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01
(k)

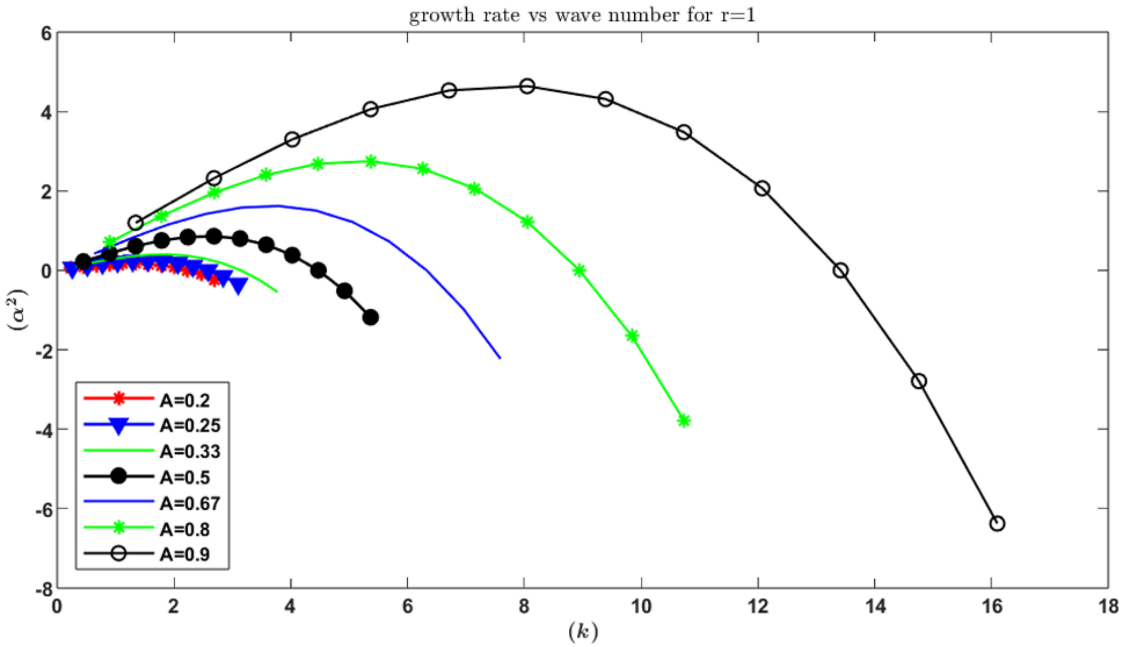

growth rate vs wave number for A=0.5
$(\alpha^2)$
$(k)$
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01

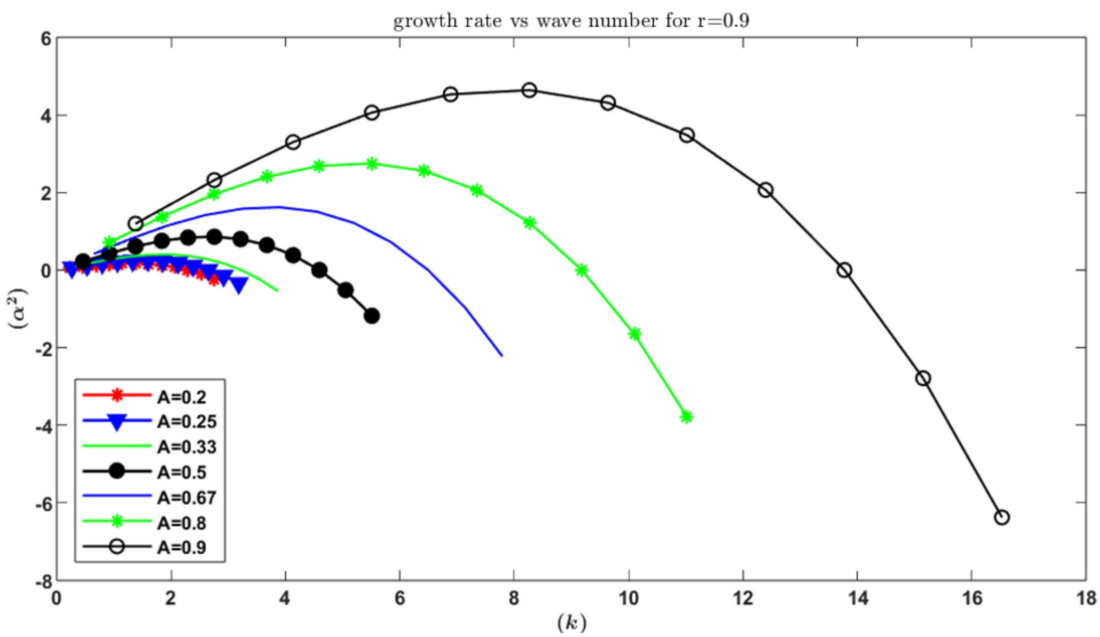

growth rate vs wave number for A=0.67
$(\alpha^2)$
$(k)$
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01

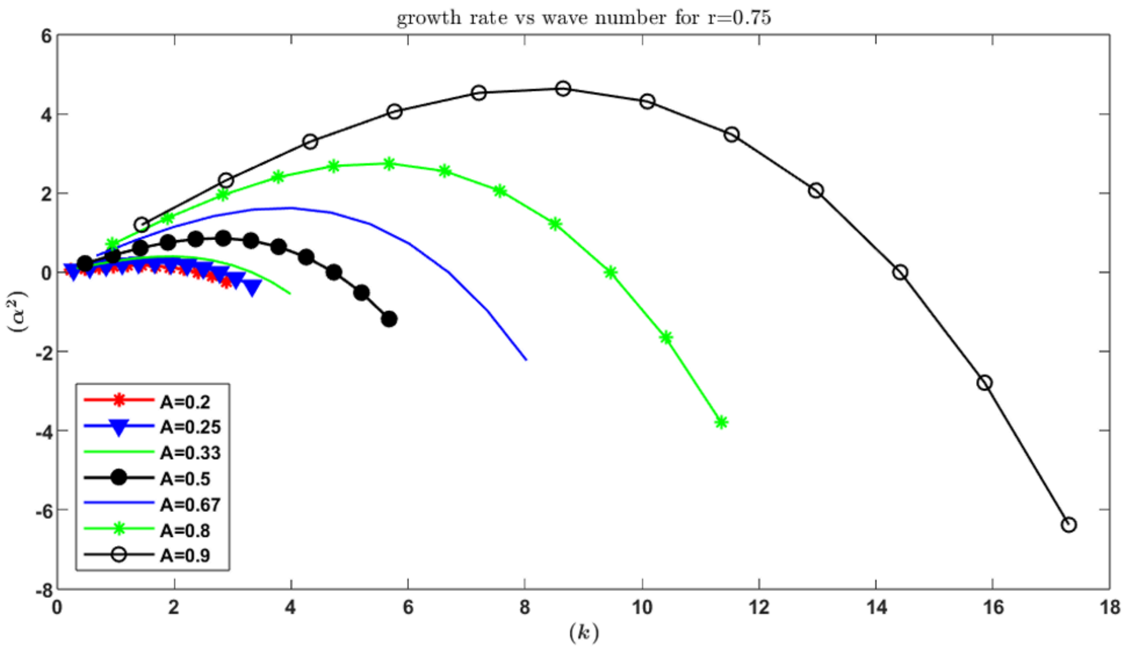


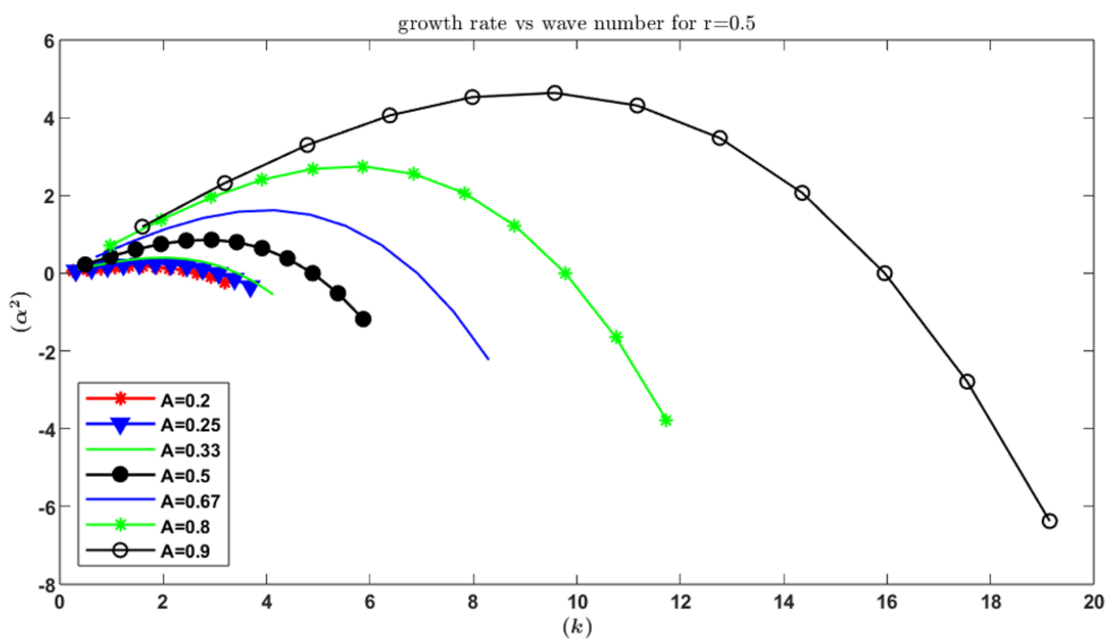

## 4.3. Different Atwood number for each r


growth rate vs wave number for r=1
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9
$(\alpha^2)$
$(k)$

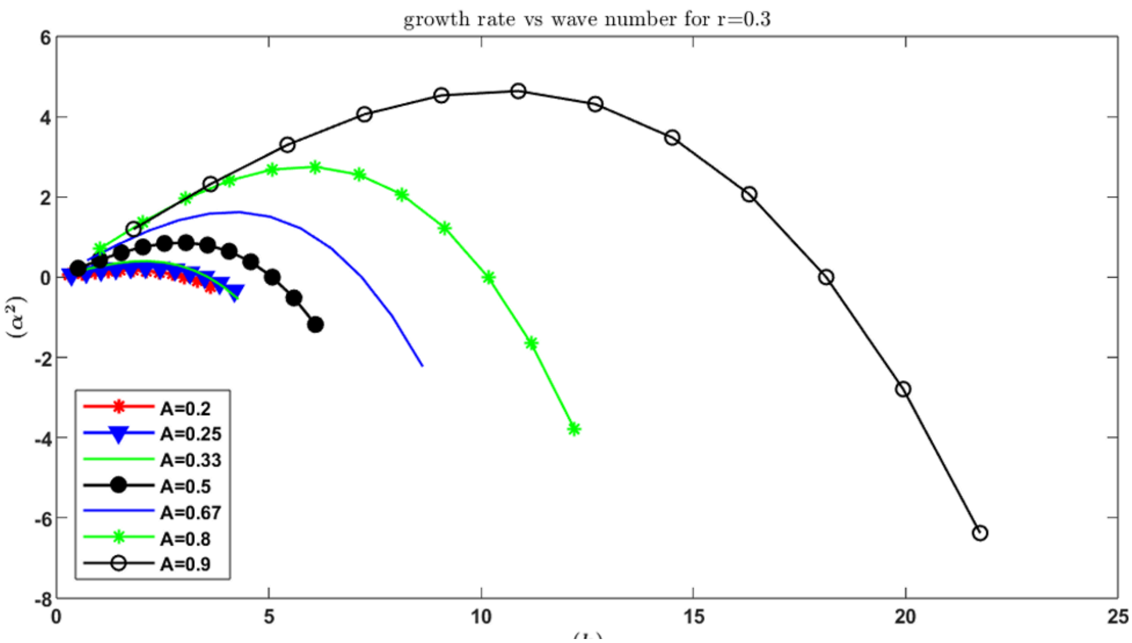

growth rate vs wave number for r=0.9
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9
$(\alpha^2)$
$(k)$



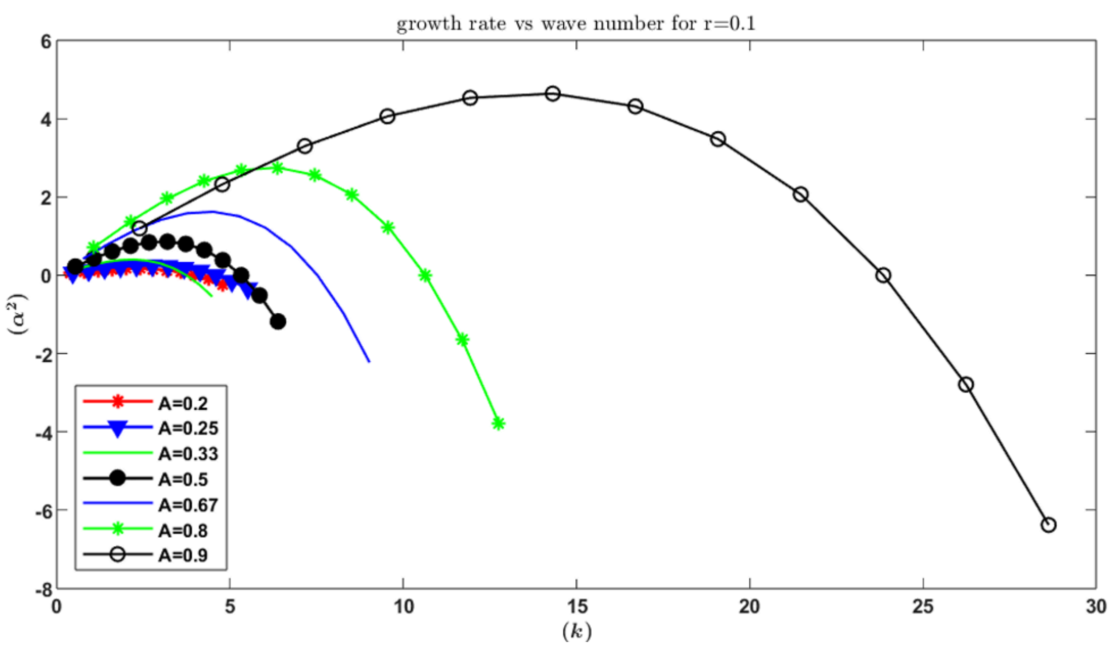
growth rate vs wave number for r=0.75
$(\alpha^2)$
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9

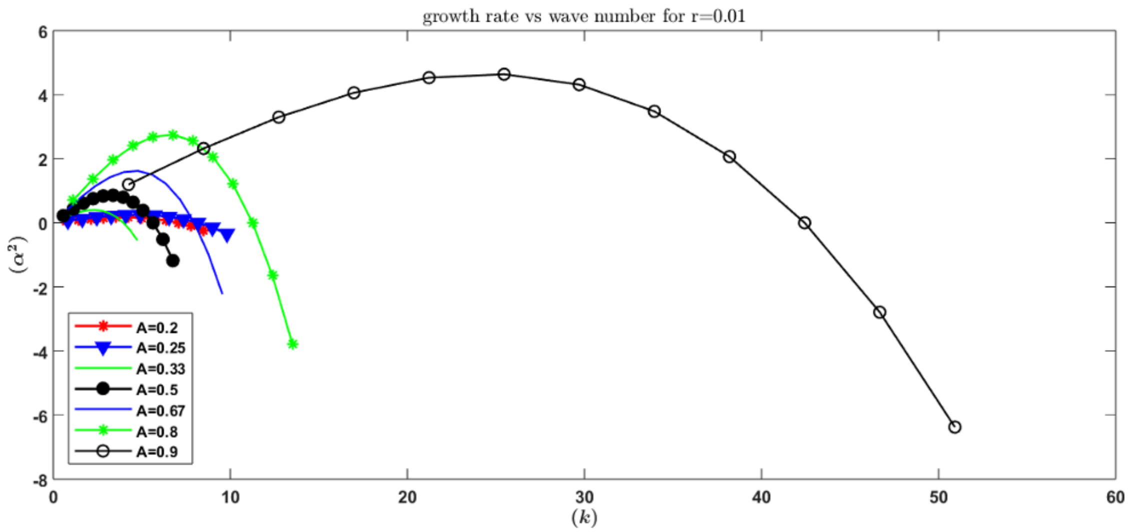
growth rate vs wave number for r=0.5
$(\alpha^2)$
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9



growth rate vs wave number for r=0.3
$(\alpha^2)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9

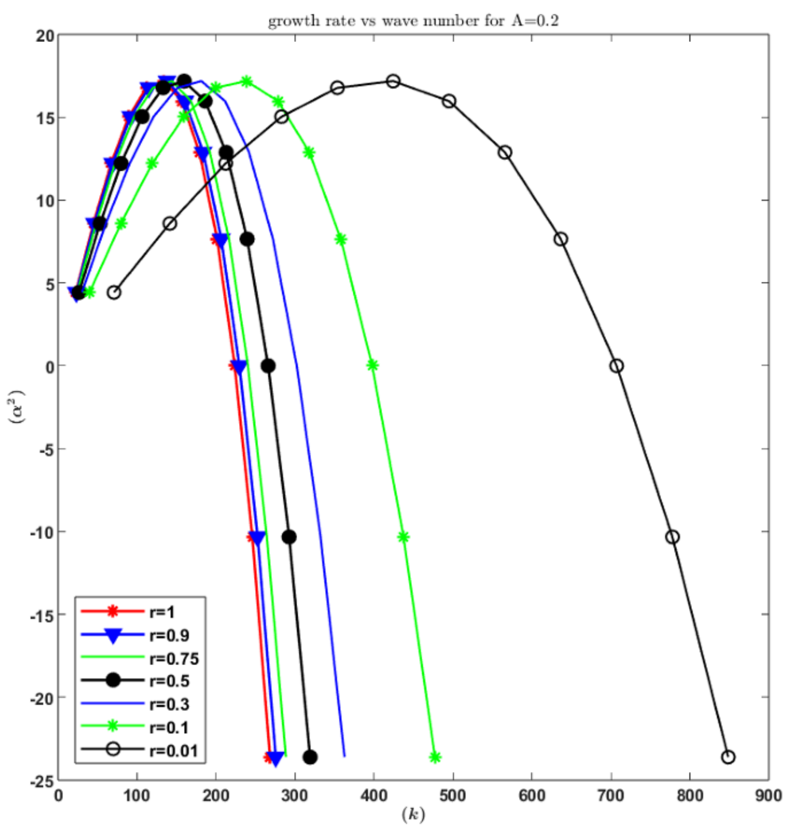



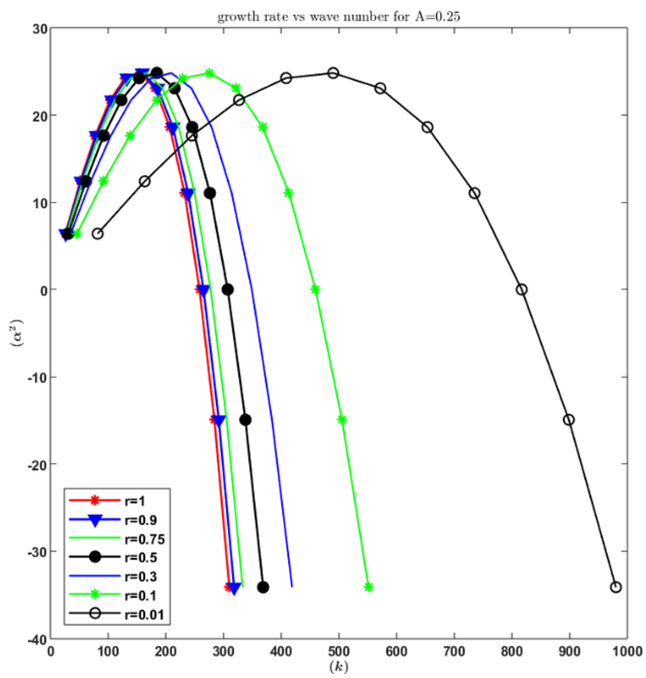
growth rate vs wave number for r=0.1
$(\alpha^2)$
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9

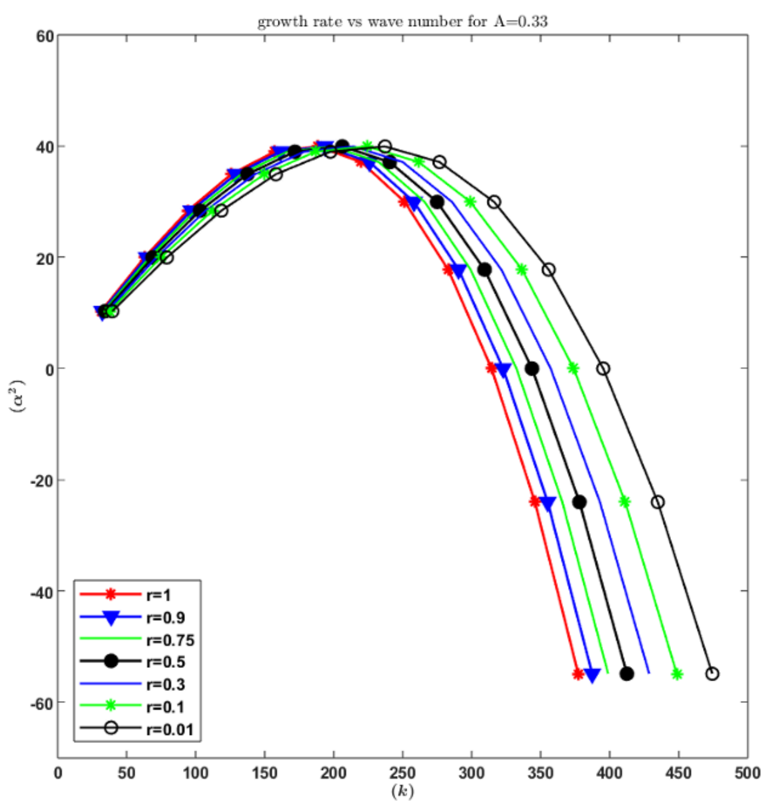


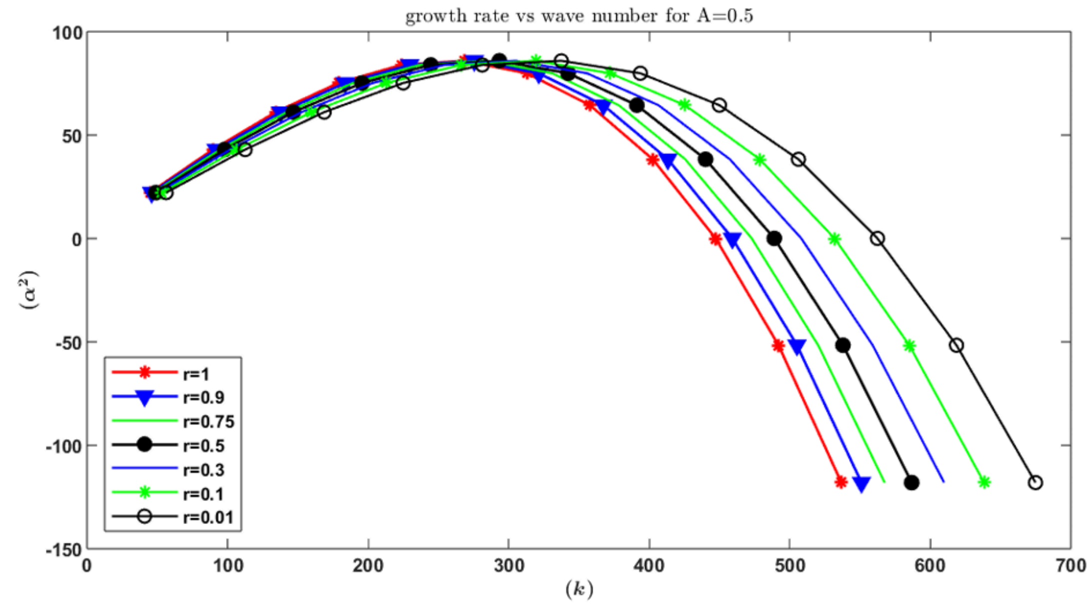

4.4.  Plot of Dimensional growth-rate Vs Dimensional wave number

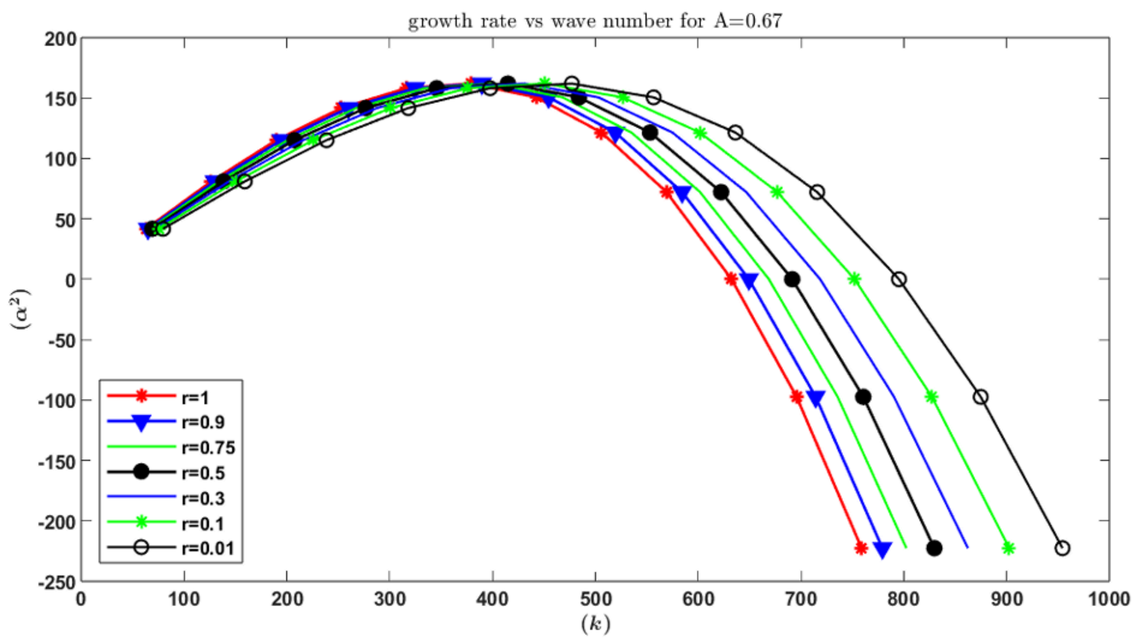
Surface Tension = 0.00001

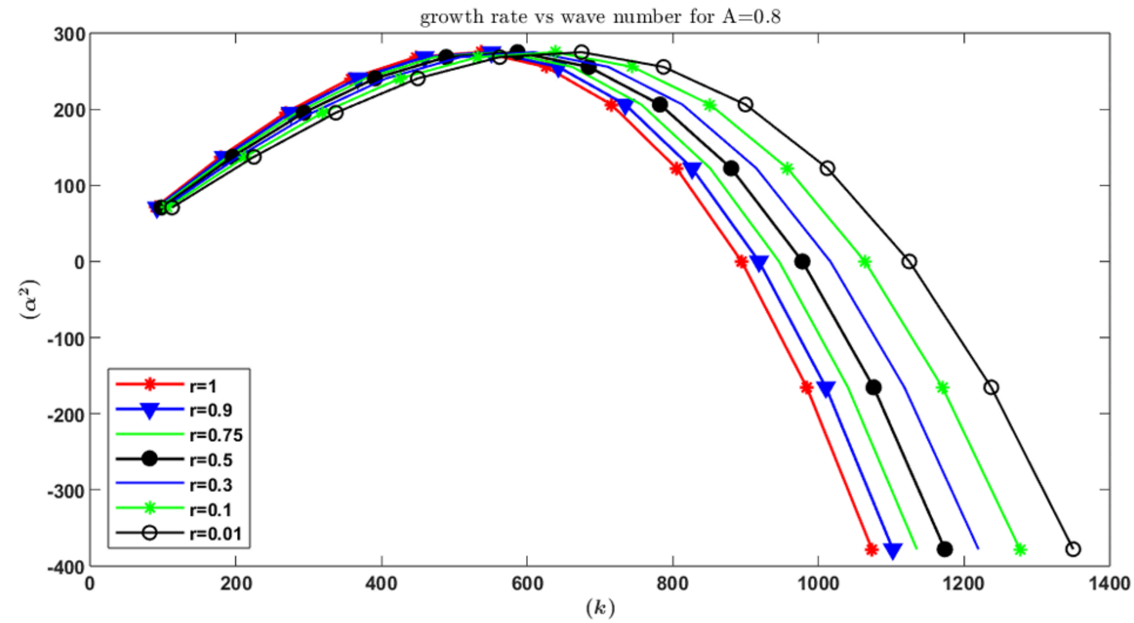


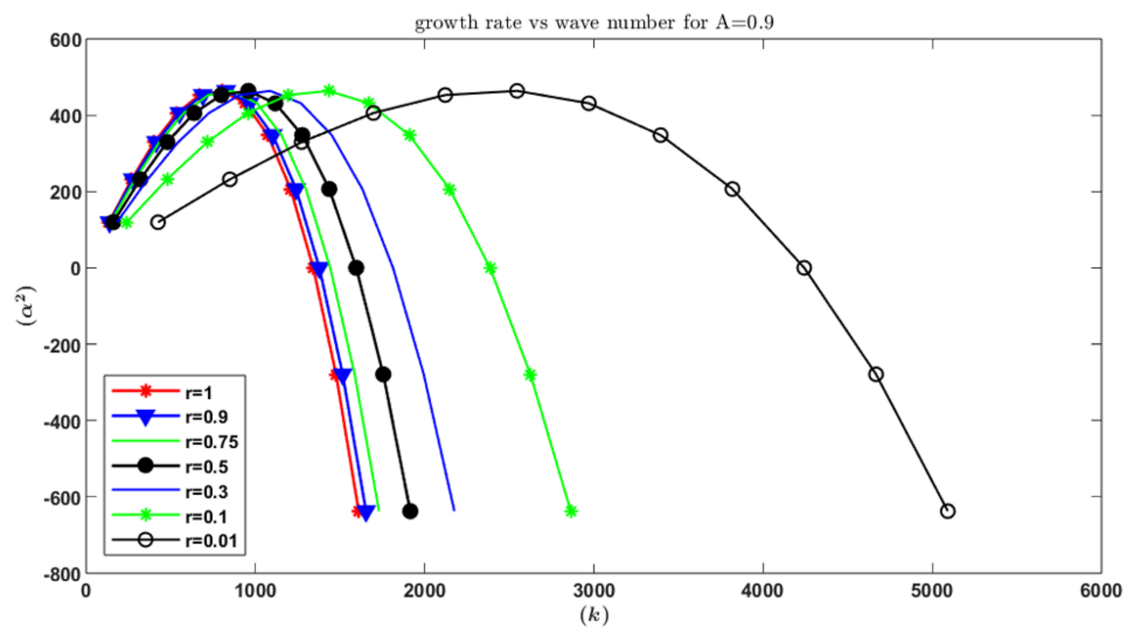

growth rate vs wave number for A=0.25
$(\alpha^2)$
$(k)$
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01


growth rate vs wave number for A=0.33
$(\alpha^2)$
$(k)$
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01

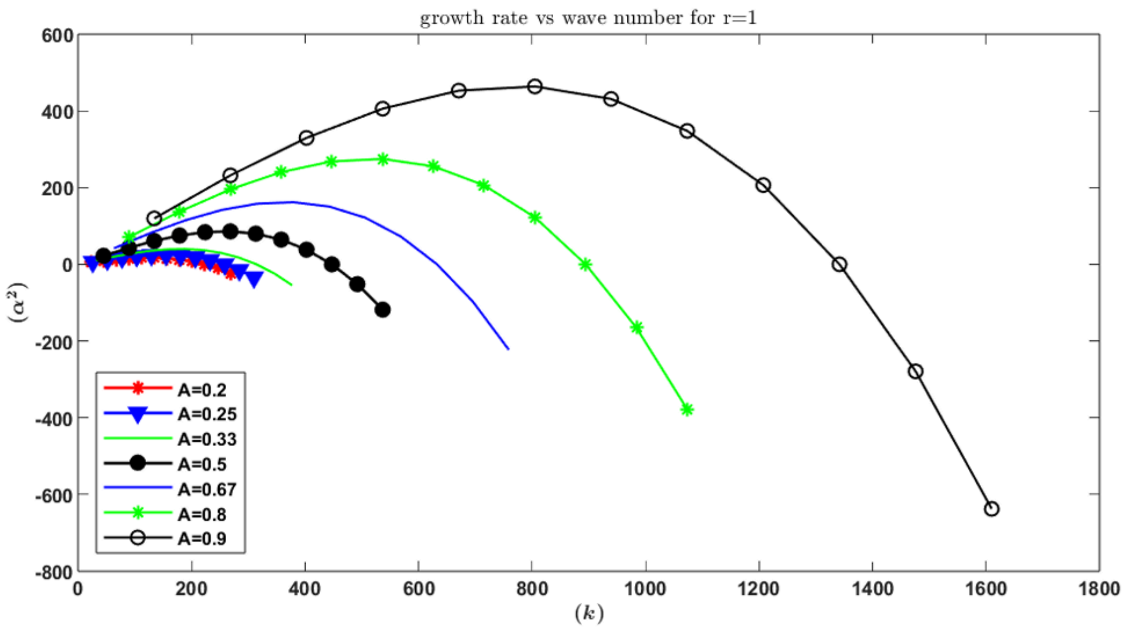

growth rate vs wave number for A=0.5
$(\alpha^2)$
$(k)$
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01

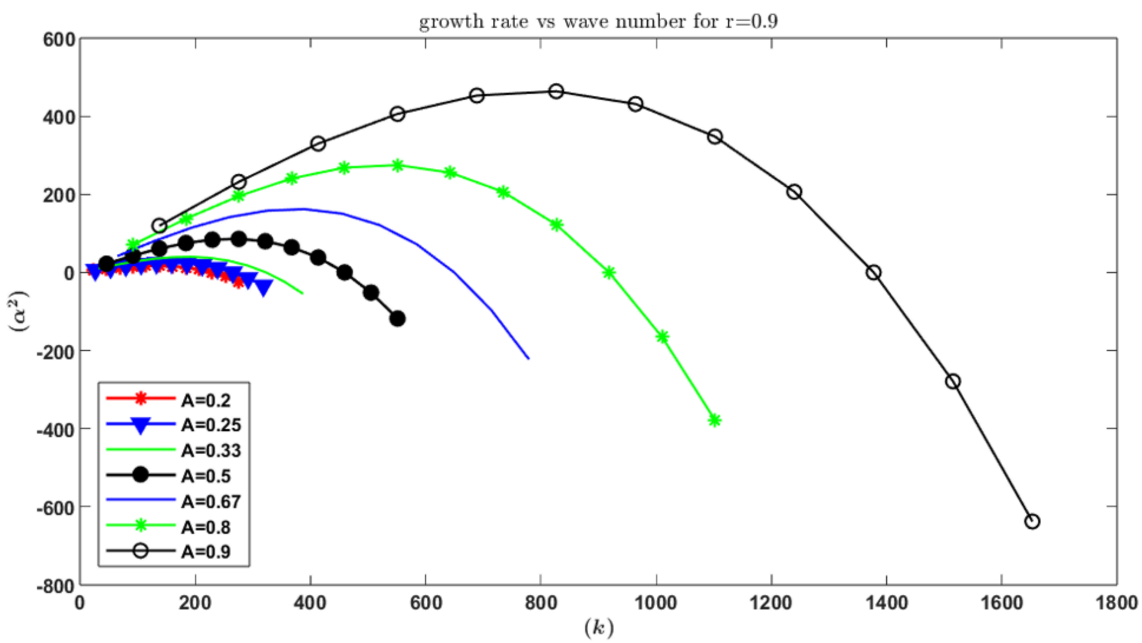

growth rate vs wave number for A=0.67
$(\alpha^2)$
$(k)$
r=1
r=0.9
r=0.75
r=0.5
r=0.3
r=0.1
r=0.01

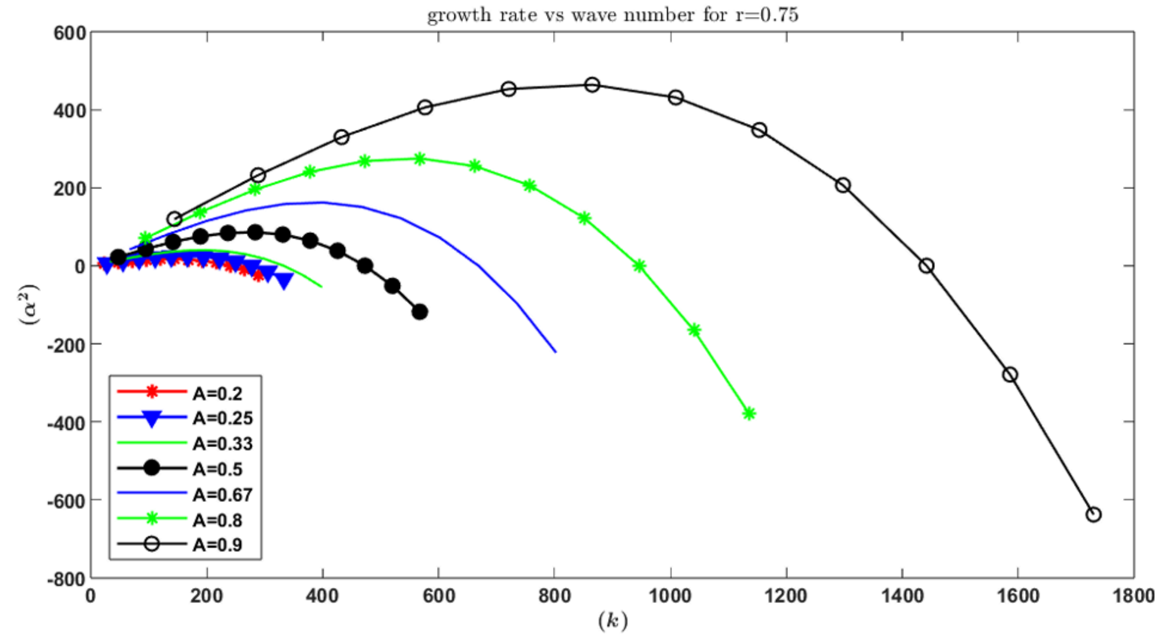


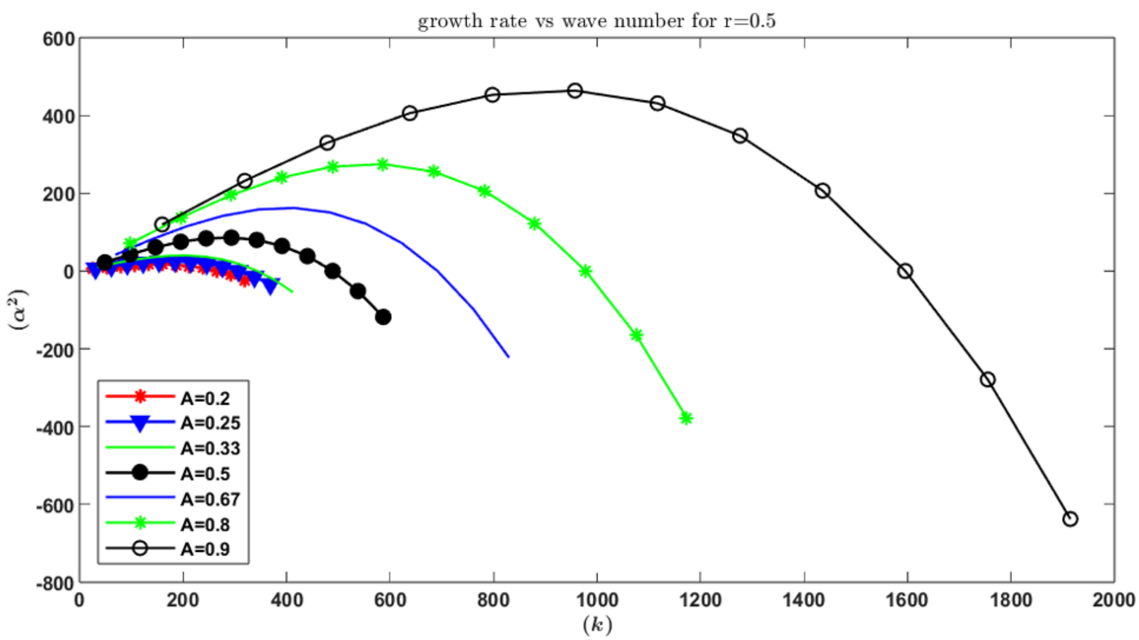



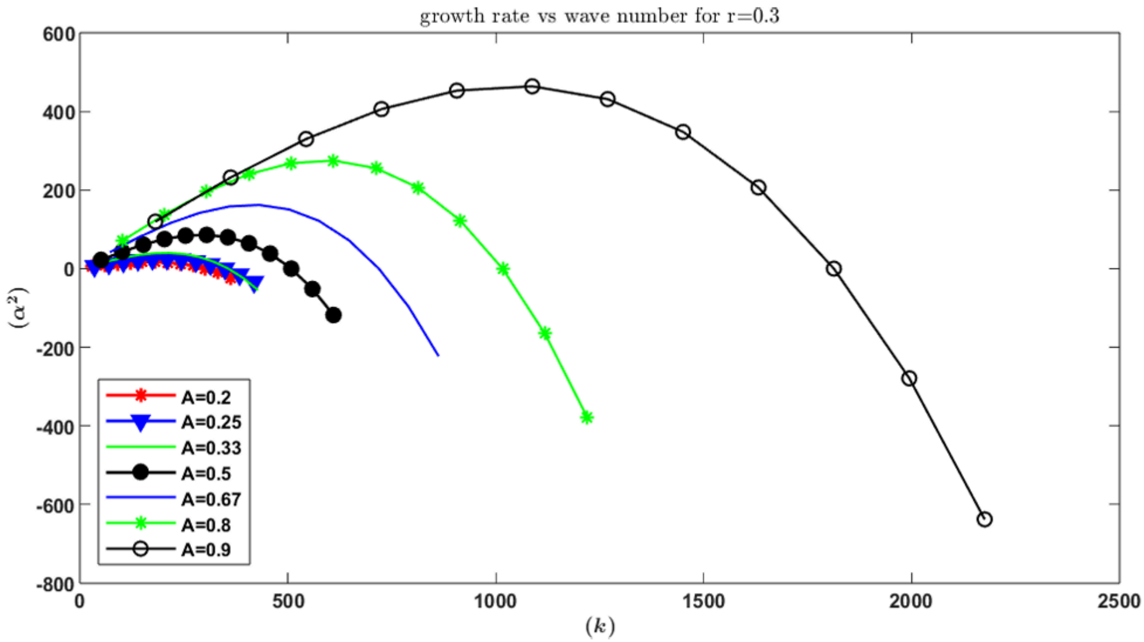

## 4.5. Different Atwood number for each r (Surface Tension = 0.00001)

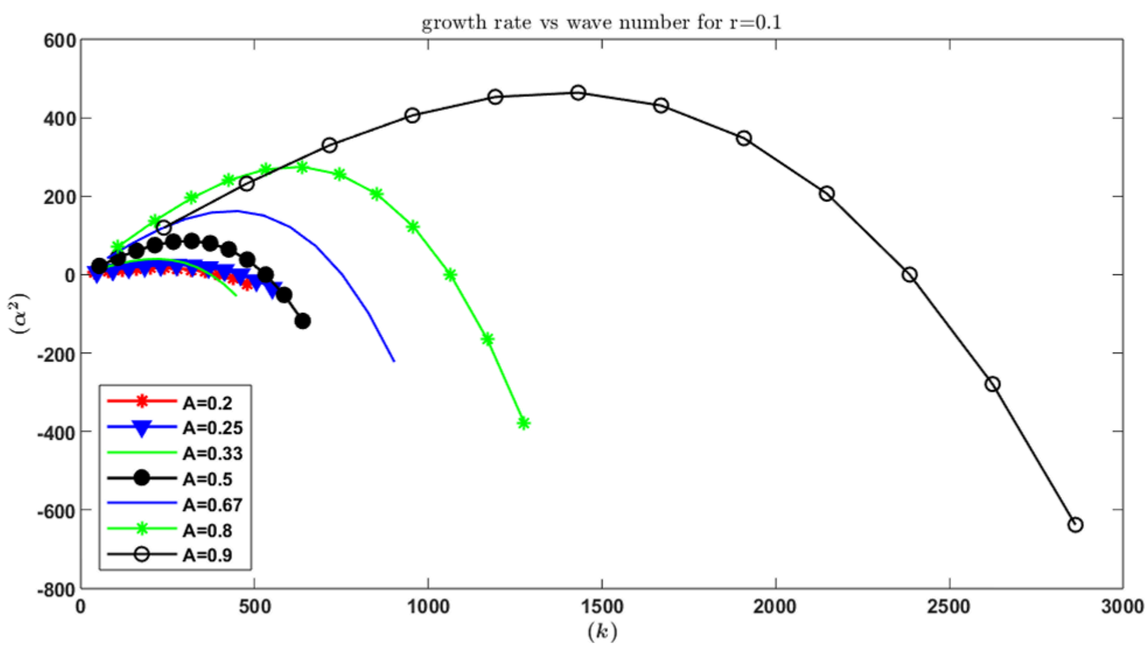

growth rate vs wave number for r=1
$(\alpha^2)$
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9

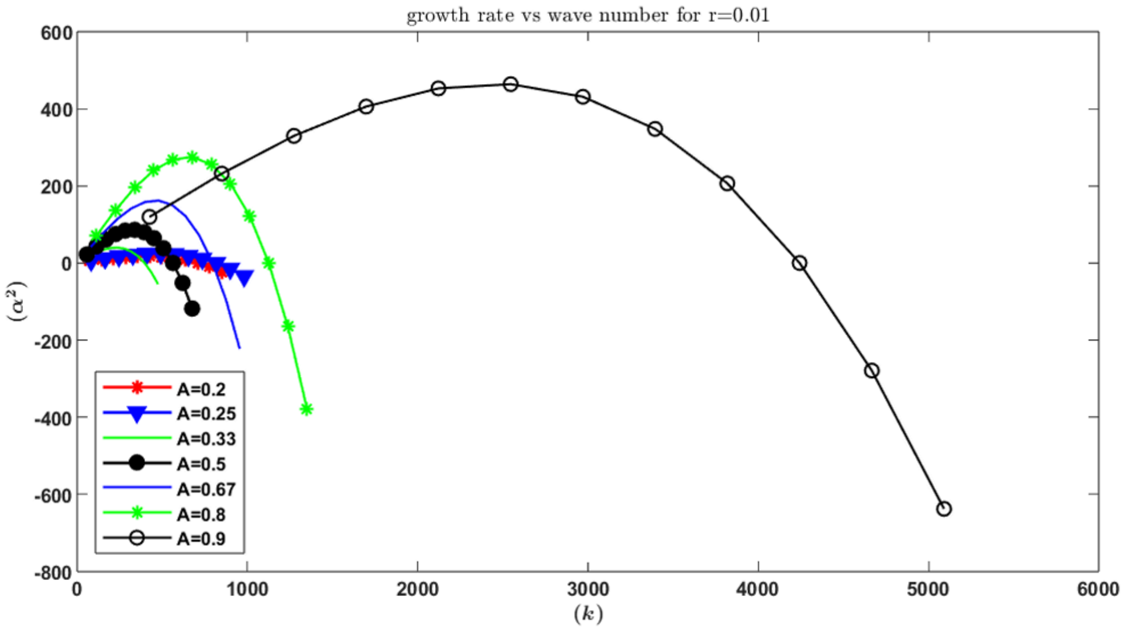

growth rate vs wave number for r=0.9
$(\alpha^2)$
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9


growth rate vs wave number for r=0.75
600
400
200
0
-200
-400
-600
-800
$(\alpha^2)$
0
200
400
600
800
1000
1200
1400
1600
1800
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9


growth rate vs wave number for r=0.5
600
400
200
0
-200
-400
-600
-800
$(\alpha^2)$
0
200
400
600
800
1000
1200
1400
1600
1800
2000
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9

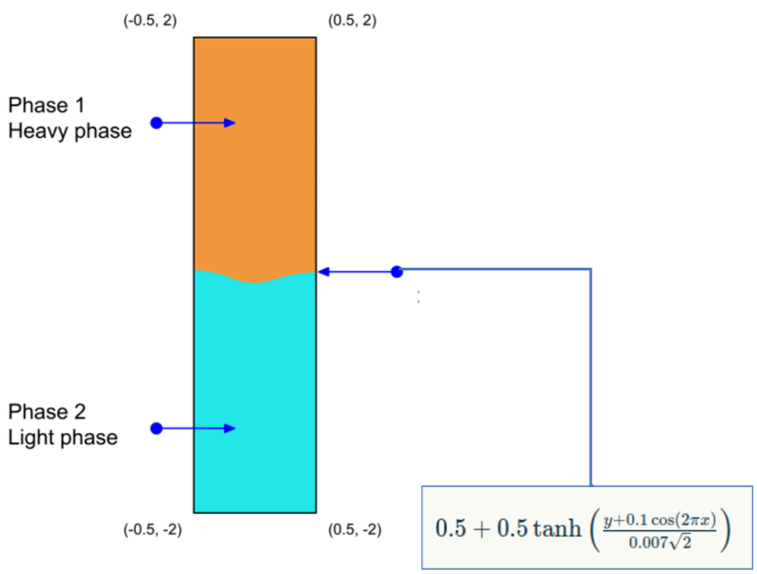

growth rate vs wave number for r=0.3
$(\alpha^2)$
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9


growth rate vs wave number for r=0.1
$(\alpha^2)$
$(k)$
A=0.2
A=0.25
A=0.33
A=0.5
A=0.67
A=0.8
A=0.9




## 5. Openfoam Simulation

### 5.1. Geometry:




### 5.2. Literature values

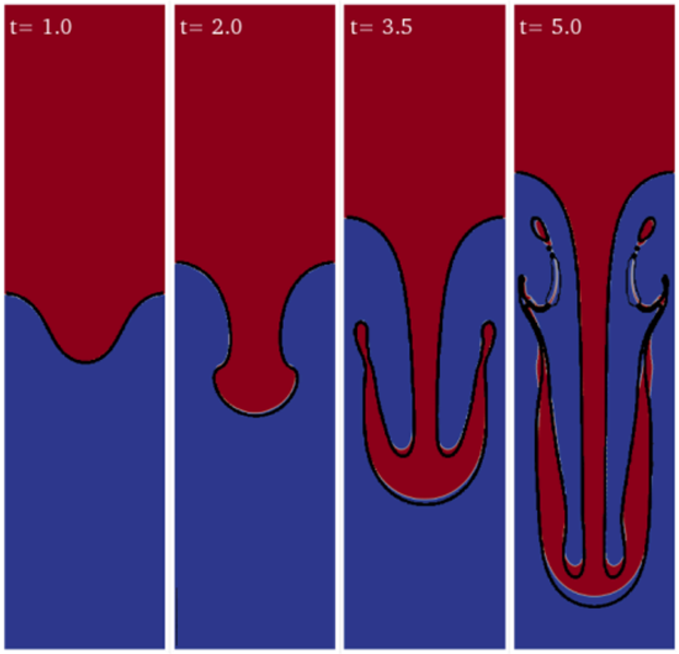



**Figure 4.** Rayleigh–Taylor instability ($A = 0.5, Re = 256$) computed with the OpenFOAM solver `interFoam`. The density field (color-coded) is compared with the density contours in He et al. (1999) (black lines).

5.3. Value Of transport properties for Re=256 & A=0.5

Re = (√gw )w/ $\nu$

Where: W is wavelength of the perturbation ( =1 in this case)

g is acceleration due to gravity

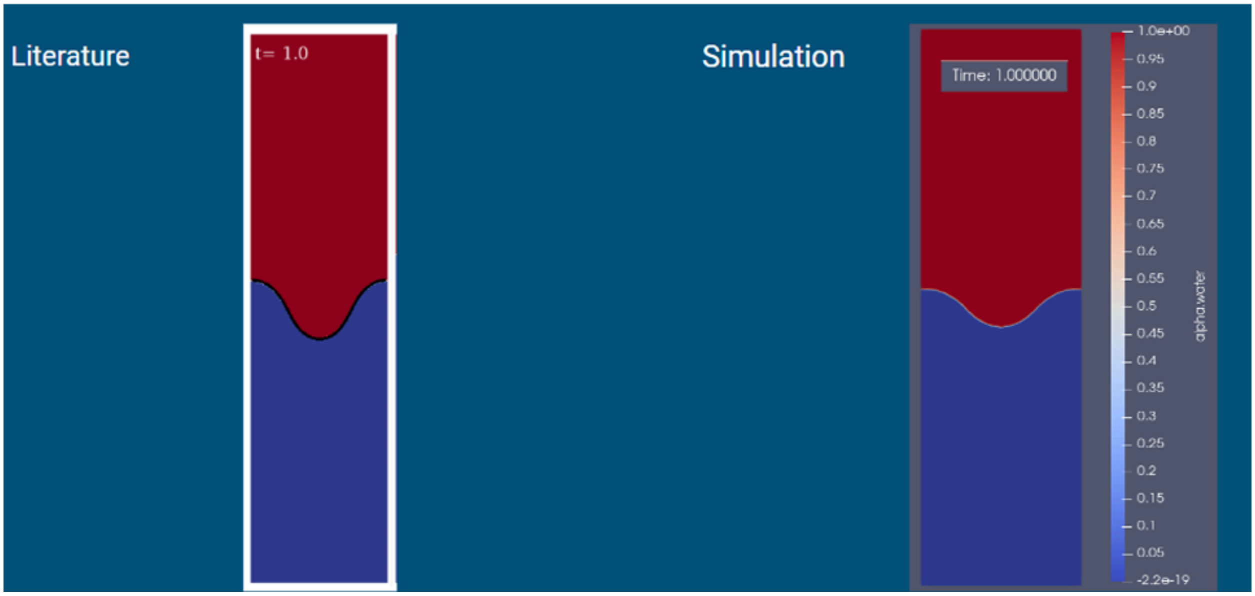
$\nu$ is kinematic viscosity ($\nu$_water = $\nu$_air )

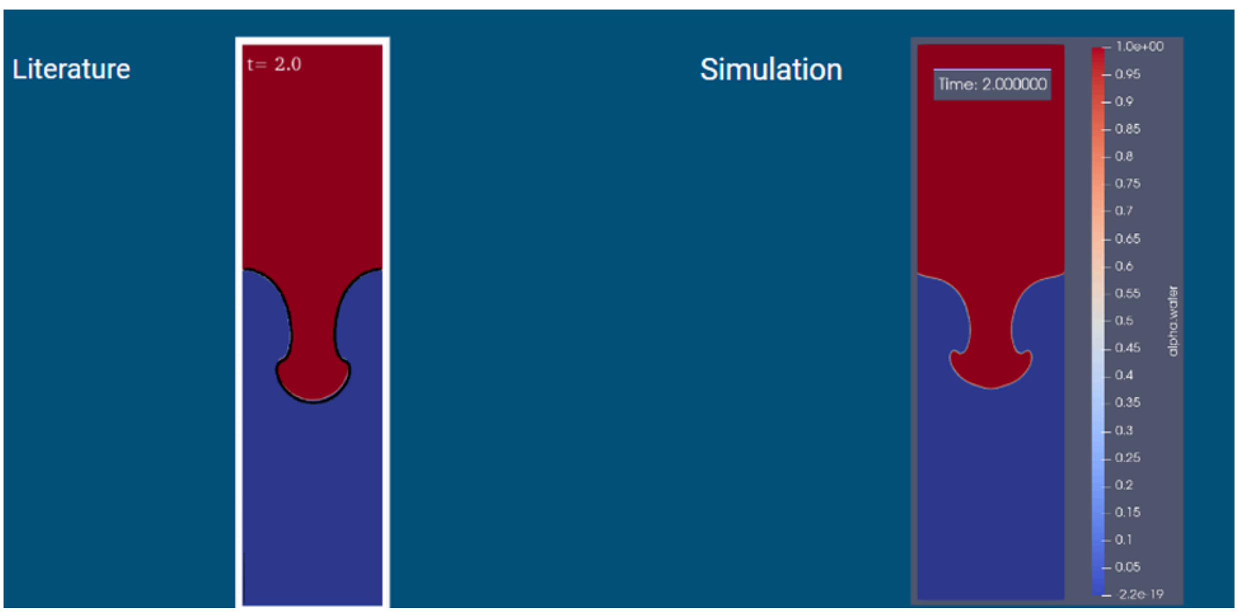
Density of air= 1 kg/m^3

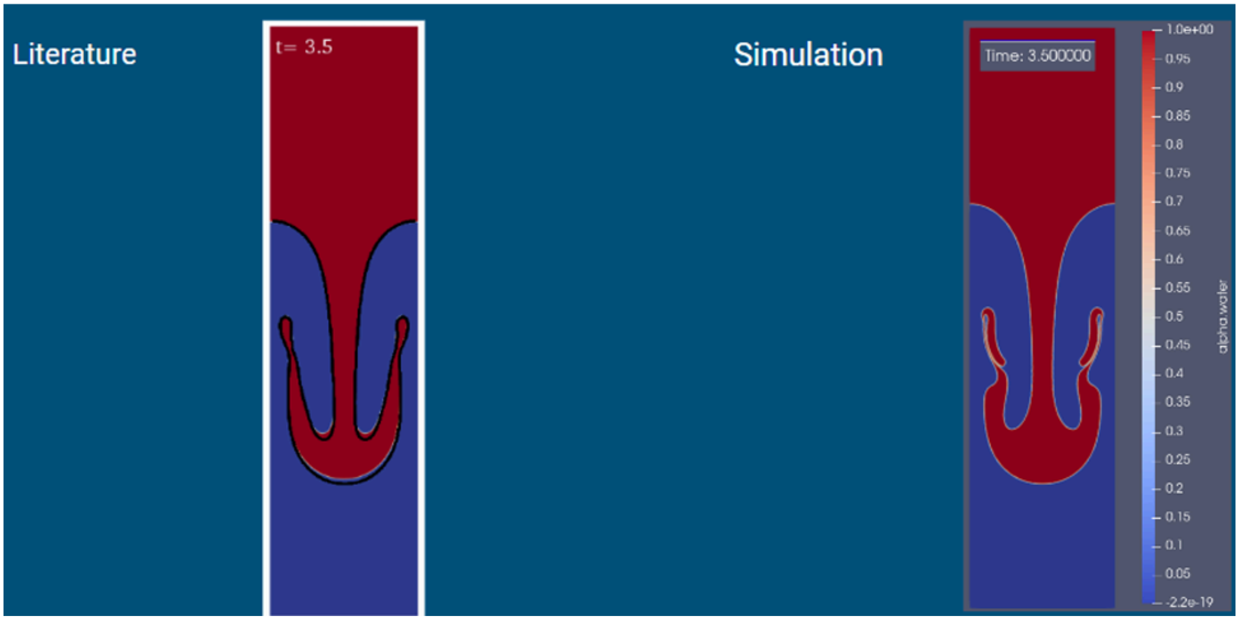
Density of water=3 kg/m^3

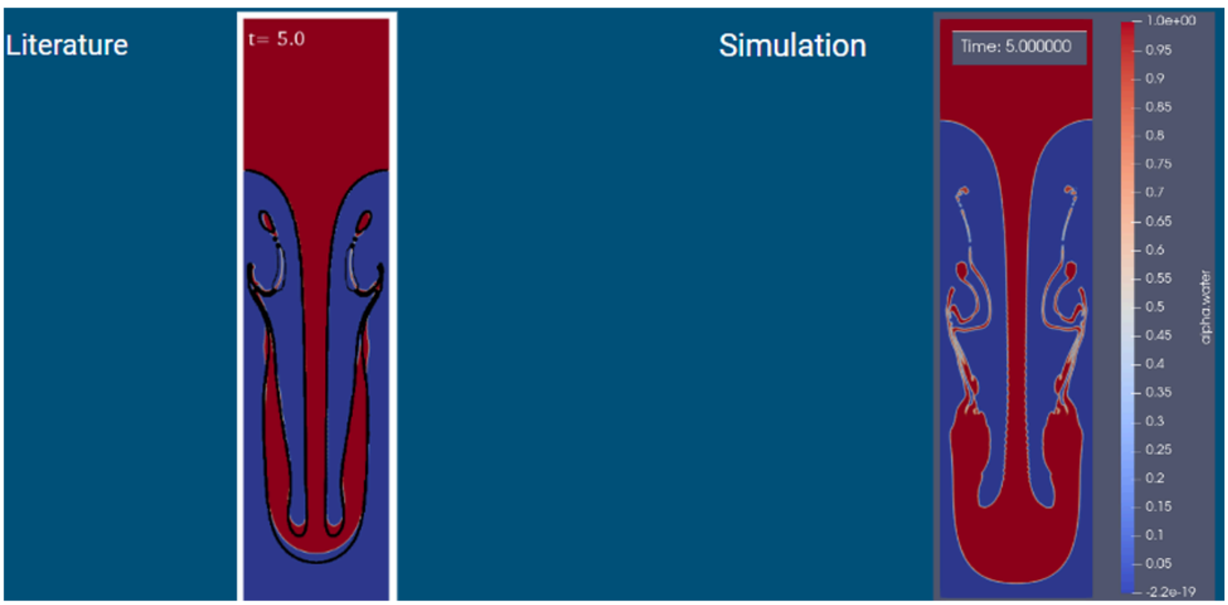
dynamic viscosity of air (nu) = 1.223e-02 N·s/m2 dynamic viscosity of water(nu)= 3.67e-02 N·s/m2

## 5.4. Simulation Results and comparison

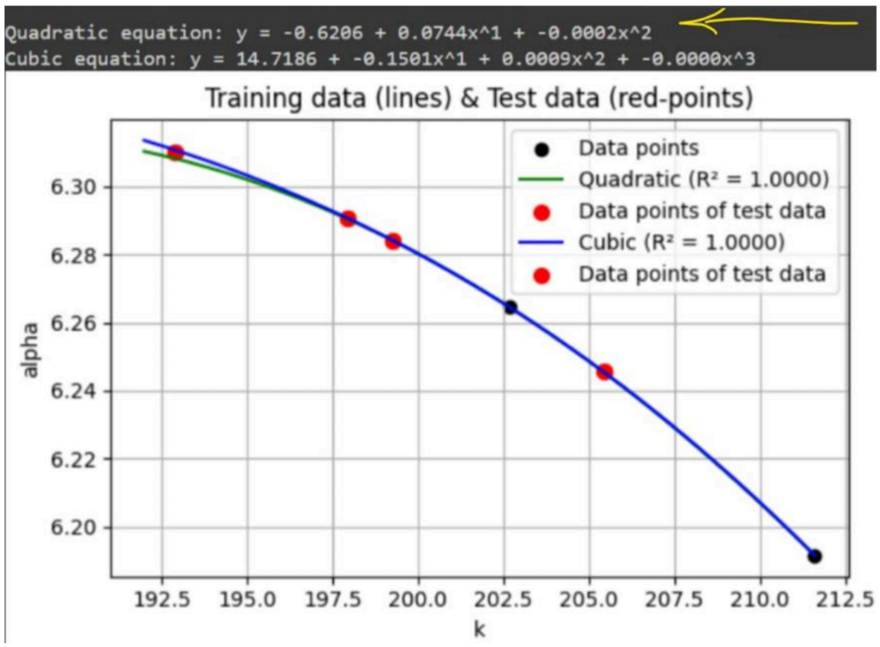





## 6. Overlap Curve for Miscible and Immiscible Cases




Point of intersection of r= 0.7, 0.63, 0.5, 0.3, 0.1, 0.001 with r=1 are as follows:

**(190.5489, 6.3161)**

**(192.9016, 6.3099)**

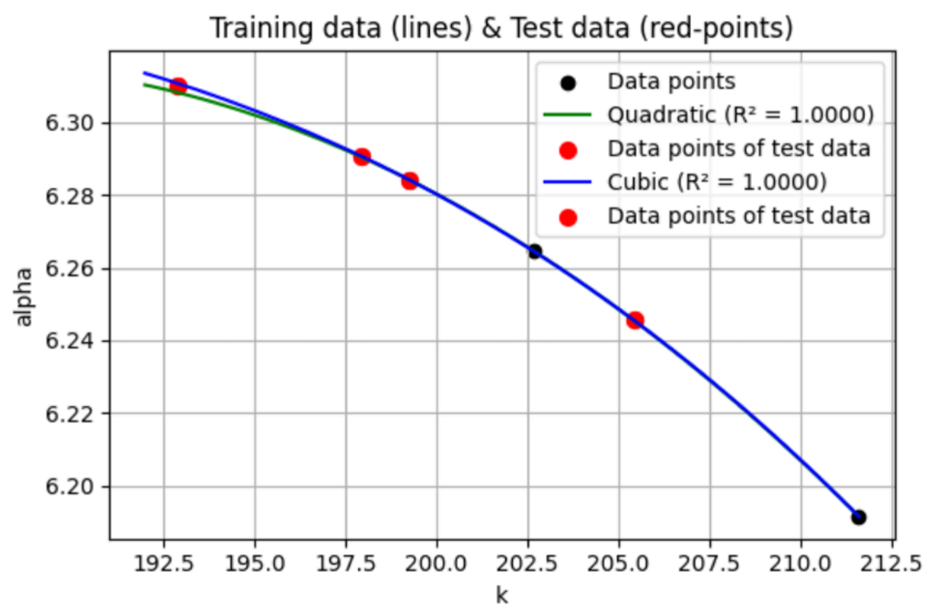
**(197.9417, 6.2906)**

**(208.5273, 6.2203)**

**(228.7088, 5.9506)**

**(285.2929, 4.0388)**

Now, finding equation of curve passing through these points: The equation is: y = -2.9334 + 0.0969x^1 + -0.0003x^2 Then, checking whether this curve satisfies different values of ‘r’ other than above used value. Checking for r=0.256, 0.35, 0.47, 0.4 with r=1




*Red dots are the data points which are used to check whether equation of curve is general or not*

## 7. Conclusion

For **Non-binary fluid**:

$\omega^2(k) = -Agk + (\sigma / 2\rho_0)\, k^3$

For **binary fluids**:

$\Phi = -r^{(1-q)/2} * \tanh((y-h)/(\sqrt{2}\varepsilon) * r^{(1-q)/2})$

$\omega^2 = (\sigma_0 k^3 / 2\rho_0) * r^{(3-2q-p)/2} - gkr\text{^}((1-q)/2)\ A$

$\alpha^2 = -(\sigma_0 k^3 / 2\rho_0) * r^{(3-2q-p)/2} + gkr^{(1-q)/2}\ A$

In both non-binary and binary fluid systems, the Rayleigh-Taylor instability exhibits distinct mathematical characteristics. For non-binary fluids, the growth rate of the instability is influenced by gravity and surface tension, as seen in the dispersion relation **$\omega^2(k) = -Agk + (\sigma / 2\rho_0)\, k^3$**. In binary fluids, additional complexity arises due to compositional effects, leading to modified growth rates and interface dynamics governed by **Φ, $\omega^2$, and $\alpha^2$**. These equations highlight how density stratification, gravity, and interfacial tension collectively shape the instability.

## 8. References


- [Piriz et al.] The Rayleigh-Taylor instability:

  https://www.researchgate.net/publication/260542299_The_Rayleigh-Taylor_instability

- [Hirt et al.] Volume of fluid (VOF) method for the dynamics of free boundaries

  https://www.sciencedirect.com/science/article/pii/0021999181901455?via%3Dihub

- [Jacqmin et al.] Calculation of Two-Phase Navier–Stokes Flows Using Phase-Field Modeling

  https://www.sciencedirect.com/science/article/pii/S0021999199963325

- [CELANI et al.] Phase-field model for the Rayleigh--Taylor instability of immiscible fluids

  https://www.researchgate.net/publication/1734748_Phase-field_model_for_the_Rayleigh--Taylor_instability_of_immiscible_fluids

- [brackbill et al.] A Continuum Method for Modeling Surface Tension

  https://www.sciencedirect.com/science/article/pii/002199919290240Y

- [Youngs et al.] Time-Dependent Multi-material Flow with Large Fluid Distortion

  https://www.researchgate.net/publication/249970655_Time-Dependent_Multi-material_Flow_with_Large_Fluid_Distortion?enrichId=rgreq-bbce311bed74192e65aa6b44192081a5-XXX&enrichSource=Y292ZXJQYWdlOzI0OTk3MDY1NTtBUzoxMzkxOTQwNjQxODMyOTZAMTQxMDE5Nzg1OTUxMw%3D%3D&el=1_x_2

- [Andrews et al.] A simple experiment to investigate twoMdimensional mixing by Rayleigh-Taylor Instability

  https://pubs.aip.org/aip/pof/article/2/6/922/402019/A-simple-experiment-to-investigate-two-dimensional

- [Ansary et al.]  Effects of surface tension and rotation on the Rayleigh–Taylor instability

  https://pubs.rsc.org/en/content/articlelanding/2002/cp/b106242p

- [COOK et al.] Transition stages of Rayleigh{Taylor instability between miscible fluids

  https://www.researchgate.net/publication/298933318_Transition_stages_of_Rayleigh-Taylor_instability_between_miscible_fluids

- [Elgowainy et al.] The Rayleigh±Taylor instability of viscous ¯uid layers

  https://pubs.aip.org/aip/pof/article/9/6/1635/260445/The-Rayleigh-Taylor-instability-of-viscous-fluid

- [Glimm et al.] A numerical study of bubble interactions in Rayleigh-Taylor instability for compressible fluids

  https://pubs.aip.org/aip/pof/article/2/11/2046/442044/A-numerical-study-of-bubble-interactions-in

- [NEWHOUSE et al.] The Rayleigh-Taylor instability of a viscous liquid layer resting on a plane wall

  https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/rayleightaylor-instability-of-a-viscous-liquid-layer-resting-on-a-plane-wall/842AAD3EBA549931E69C03C49234B740