Moments & moment generating function

The obvious purpose of the mgf is in determining moments of random variables. However, the most important contribution is to establish distributions of functions of random variables.

If $g(x) = X^r$, $r = 0, 1, 2, \ldots$, then we get rth moment about the origin of the random variable X, denoted by μ_r'.

$E\,g(x) = \sum g(x) f(x)$

The rth moment about the origin of the r.v X is given by

$$\mu_r' = E(X^r) = \sum_x x^r f(x) \quad \text{if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} x^r f(x)\,dx \quad \text{if } X \text{ is cts.}$$

First moment about origin, $\mu_1' = E(X)$

& second " " , $\mu_2' = E(X^2)$

$$\mu = \mu_1' \quad \& \quad \sigma^2 = \mu_2' - \mu^2$$

Although the moments can be determined directly from above, an alternative procedure exists. This procedures to utilize a moment generating function.

The mgf of a r.v X is given by $E(e^{tX})$, denoted by $M_X(t)$,

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & \text{if } X \text{ is cts.} \end{cases}$$

mgf will exist only if the sum / integral above cgs.

If mgf of a r.v. X does exist, it can be used to generate all the moments of that variable.

Theorem Let $M_X(t)$ be the mgf for a r.v. X. Then

$$\boxed{\frac{d^r M_X(t)}{dt^r} = \mu_r'}$$

Proof Maclaurin's series expansion for e^{tX} is

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{(tX)^n}{n!} + \cdots$$

$$E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{(tX)^n}{n!} + \cdots\right]$$

$$M_X(t) = 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \cdots \quad \text{—} (1)$$

Diff. this series term by term w.r.t. 't'

$$\frac{d}{dt} M_X(t) = E(X) + t\,E(X^2) + \cdots + \text{terms containg } 't'$$

$$\left.\frac{d}{dt} M_X(t)\right|_{t=0} = E(X)$$

$$\frac{d^2}{dt^2} M_X(t) = E(X^2) + \cdots \text{ term contaig } t$$

$$\left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0} = E(X^2) \quad \text{\& so on.}$$

OR

from (1), $M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

$\therefore$ coeff of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r' (about origin)

$$\left.\frac{d^r}{dt^r} M_X(t)\right|_{t=0} = \left.\frac{\mu_r'\, r!}{r!} + \mu_{r+1}'\, t + \mu_{r+2}'\frac{t^2}{2!} + \cdots\right|_{t=0}$$

$$\mu_r' = \left.\frac{d^r}{dt^r} M_X(t)\right|_{t=0}$$

$M_X(t)$ (about $X=a$) $= E\left(e^{t(X-a)}\right)$

$$= E\left[1+t(X-a)+\frac{t^2}{2!}(X-a)^2+\ldots+\frac{t^r}{r!}(X-a)^r+\ldots\right]$$

$$= 1+t\mu_1'+\frac{t^2}{2!}\mu_2'+\ldots+\frac{t^r}{r!}\mu_r'+\ldots$$

where $\mu_r' = E\left[(X-a)^r\right]$, is the rth moment about $X=a$.

Properties of mgf

1. $M_{aX}(t) = M_X(at)$, a is constant.

LHS $M_{aX}(t) = E\left(e^{taX}\right)$

RHS $M_X(at) = E\left(e^{atX}\right) =$ LHS.

2. $M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$

LHS $= E\left[e^{t(X_1+X_2+\ldots+X_n)}\right] = E\left[e^{tX_1}.e^{tX_2}\ldots e^{tX_n}\right]$

$= E\left(e^{tX_1}\right).E\left(e^{tX_2}\right)\ldots E\left(e^{tX_n}\right)$

$= M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$

3. $M_{X+a}(t) = e^{at} M_X(t)$

LHS $E\left[e^{t(X+a)}\right] = e^{at} E\left(e^{tX}\right) = e^{at} M_X(t)$.

* mgf uniquely determines the distribution, i.e. $\exists$ 1-1 correspondence b/w the mgf & the dist. fn of a r.v.

Ex Let the distribution for X be given by

$$f_X(x) = ce^{-x} \quad x = 1, 2, 3 \ldots$$

a) find the value of c that make $f(x)$ a density fn.

$$\sum_x f(x) = 1 \quad \text{or} \quad \sum_{x=1}^{\infty} ce^{-x} = 1$$

$$c[e^{-1} + e^{-2} + \ldots] = 1 \quad \Rightarrow \quad c\frac{e^{-1}}{1-e^{-1}} = 1$$

$$\text{or} \quad c = \frac{1 - 1/e}{1/e} = \frac{e-1}{1}.$$

b) find the mgf for X

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} f(x) = c\sum_1^{\infty} e^{tx} e^{-x} = c\sum e^{(t-1)x}$$

$$= c(e^{t-1} + e^{2(t-1)} + \ldots) = c\frac{e^{t-1}}{1-e^{t-1}}$$

$$= \frac{(e-1)e^{t-1}}{1-e^{t-1}}$$

c) find $E(X)$ using $M_X(t)$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = E(X)$$

$$E(X) = \frac{d}{dt}\left[\frac{(e-1)e^{t-1}}{1-e^{t-1}}\right]_{t=0}$$

$$= (e-1)\left[\frac{(1-e^{t-1})e^{t-1} - e^{t-1}(-e^{t-1})}{(1-e^{t-1})^2}\right]_{t=0}$$

$$= (e-1)\left[\frac{(1-e^{-1})e^{-1} + e^{-1}e^{-1}}{(1-e^{-1})^2}\right]$$

$$= (e-1)\left(\frac{e^{-1} - \cancel{e^{-2}} + \cancel{e^{-2}}}{(e-1)^2}\right)e^2$$

$$= \frac{(e-1)e}{(e-1)^2} = \frac{e}{e-1}$$

Ex Let X denote the length in mins, of a long distance telephone conversation. The density for X is given by

$$f(x) = \frac{1}{10} e^{-x/10}, \quad x > 0$$

a) find mgf $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \frac{1}{10}\int_0^\infty e^{tx} e^{-x/10}\, dx$$

$$= \frac{1}{10}\int_0^\infty e^{\frac{(10t-1)x}{10}}\, dx$$

$$= \frac{1}{10} \left. \frac{e^{\frac{(10t-1)x}{10}}}{\frac{10t-1}{10}} \right|_0^\infty = \frac{1}{10}\left[-\frac{1}{\frac{10t-1}{10}}\right] = -\frac{1}{10t-1} = \frac{1}{1-10t}$$

$$t < 1/10$$

b) Use $M_X(t)$ to find the average length of such a call.

$$\left.\frac{d}{dt} M_X(t)\right|_{t=0} = \frac{-(-10)}{(1-10t)^2} = \left.\frac{10}{(1-10t)^2}\right|_{t=0} = 10 \text{ min}$$

c) find the variance & standard deviation for X.

$$E(X^2) = 10(1-10t)^{-2} = \frac{10 \times -10 \times -2}{(1-10t)^3} = 200$$

$$V(X) = E(X^2) - E(X)^2$$

$$= 200 - 100 = 100 \text{ min}.$$

Ex Let X be a discrete r.v with pmf.

$$f(x) = \begin{cases} 1/3 & x = 1 \\ 2/3 & x = 2 \end{cases}$$ find mgf.

$$M_X(t) = E(e^{tX}) = \sum_x f(x) e^{tx} = \frac{1}{3}e^t + \frac{2}{3}e^{2t}$$

$\forall t.$

$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{1}{3}e^t + \frac{4}{3}e^{2t} \Big|_{t=0} = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$

$E(X^2) = \frac{1}{3}e^t + \frac{8}{3}e^{2t} \Big|_{t=0} = \frac{9}{3} = 3$

$V(X) = E(X^2) - E(X)^2 = 3 - (5/3)^2 = 2/9$

<u>Ex</u> Let X & Y be 2 independent r.v with respective mgf:

$$M_X(t) = \frac{1}{1-5t}, \quad t < 1/5$$

$$M_Y(t) = \frac{1}{(1-5t)^2}, \quad t < 1/5$$

find $E(X+Y)^2$.

Let $W = X+Y$

$$M_W(t) = M_{X+Y}(t) = M_X(t)\, M_Y(t) = \frac{1}{1-5t} \cdot \frac{1}{(1-5t)^2}$$

$$= \frac{1}{(1-5t)^3}$$

$$E(W^2) = \frac{d^2}{dt^2} M_W(t) \Big|_{t=0}$$

$$\frac{15}{(1-5t)^4}\Big|_{t=0} = 15 = E(W)$$

$$= \frac{15(-4)(-5)}{(1-5t)^5}\Big|_{t=0}$$

$$= 300$$

Ex The value of a piece of factory equipment after 3 yrs of use is $100(0.5)^X$ where X is a r.v having mgf

$$M_X(t) = \frac{1}{1-2t}, \quad t < 1/2$$

Calculate the expected value of this piece of equipment after 3 yrs of use.

Let $Y = 100(0.5)^X$

$E(Y) = E(100(0.5)^X)$

$= 100\,E(0.5)^X = 100\,E(e^{\ln 0.5^X})$

$= 100\,E(e^{X\ln 0.5})$

$M_X(t) = E(e^{tX})$

$= 100\,M_X(\ln 0.5) = \dfrac{100}{1-2\ln 0.5} = 41.9060$