Finite amplitude waves in viscoelastic medium

ME723A - (Wave Propagation in Solids)
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Presentation I

Problem Definition

The aim of this project is to generalize the idea of wave propagation in the solid domain in two ways.

1. Allowing finite amplitude oscillation of particles of the medium.
2. Incorporating viscosity of the medium to include damping effects.

Defining terms in the problem statement

Finite Amplitude

In the most basic analysis of waves, the major assumptions include that the amplitude of wave oscillation is much smaller than the wavelength.
This assumption breaks when we have finite amplitude oscillation like in the case of **Tacoma Bridge** failure.

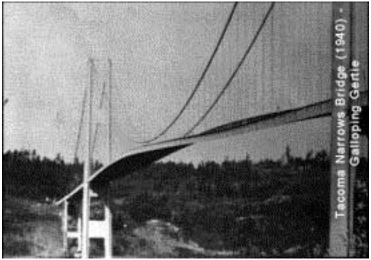

Figure 1: Tacoma Bridge Failure. courtesy -ASCE

Viscoelasticity

It is a generalization of elastic modeling of material to consider their dissipative nature. As we know by the second law of thermodynamics, when particles of any material oscillate, they convert mechanical energy to heat. This makes it important to consider damping for realistic results from our model.

Caution: Viscoelasticity is different from plasticity.

Viscoelasticity vs Plasticity

Viscoelasticity is reversible with respect to the state of materials. **Plasticity** is irreversivle even with respect to the state of the material.

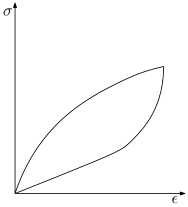

Figure 2: Viscoelastic Response

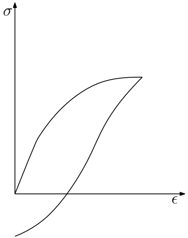

Figure 3: Plastic Response

Path Plan

Path Plan

We will go as follows:

1. Start by stating the assumption of the linear solution of wave propagation
2. Modifying the model to leave some assumptions.
3. Rederive the governing equations in terms of parameters of the new model.

Underlying Assumptions of linear elastic model

We will go as follows:

1. **Geometric Assumption** The displacement of all particles in the domain of interest is small as compared to the wavelength of the wave
2. **Material Assumption**
   - The stress responses of the material are instantaneous and depend only on the present state of the material.
   - The stress responses vary linearly with the strain of the material

Generalising Geometry of the model

Geometry in linear elasticity

Geometry in linear elasticity consists of a strain-displacement relationship.

$$\underline{\underline{\epsilon}} = \frac{1}{2}(\nabla \underline{\boldsymbol{u}} + (\nabla \underline{\boldsymbol{u}})^T)$$

Finite Strain Geometry

Consider the space $\mathbb{R}^3$. This space is a normed space. Now define the coordinate system of this space.
Let the coordinates of the initial material is $\underline{\boldsymbol{X}}$ and it is transformed by a function $\underline{\chi}$ to coordinates of current configuration $\underline{\boldsymbol{x}}$

$$\underline{\chi} : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{x}}$$

Now let $\underline{\boldsymbol{u}}$ denote the displacement of the material points.

$$\underline{\boldsymbol{u}} = \underline{\boldsymbol{x}} - \underline{\boldsymbol{X}}$$

Finite Strain Geometry

Taking gradient on both sides we get

$$\nabla \underline{\boldsymbol{u}} = \nabla \underline{\chi}(\underline{\boldsymbol{X}}) - \nabla \underline{\boldsymbol{X}}$$

Note that the gradient is with respect to the initial coordinate i.e., $\underline{\boldsymbol{X}}$
Lets denote $\nabla \underline{\chi}$ as $\underline{\underline{\boldsymbol{F}}}$. It is called as **Deformation Gradient**.

Finite Strain Geometry

We define the new strain measure as

$$\underline{\underline{\epsilon}} = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^T \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\boldsymbol{I}}})$$

Upon further simplification, it becomes

$$\underline{\underline{\epsilon}} = \frac{1}{2}(\nabla\underline{\boldsymbol{u}} + (\nabla\underline{\boldsymbol{u}})^T + \nabla\underline{\boldsymbol{u}}(\nabla\underline{\boldsymbol{u}})^T)$$

Presentation II

Notations

There are three quantities of our interest.

1. **Scalars:** It is represented by a non-bold mathematical symbol. For example: a,A
2. **Vectors:** It is represented by a boldface mathematical symbol in lowercase. For example: $\boldsymbol{a}$
3. **Tensors:** We deal mainly with two-point tensors, so by default, it is a linear map between two 3-D vector spaces unless it is explicitly mentioned otherwise. They are represented by boldface mathematical symbols in the upper case. For example: $\boldsymbol{A}$

Stress Responses

We are following the work of Destrade and Saccomandi, "Finite amplitude elastic waves propagating in compressible solids" We start by introducing our Cauchy-stress tensor. It is defined as usual in Chadwick, *Continuum Mechanics: Concise Theory and Problems* (Chapter-III). We are denoting it by $\boldsymbol{T}$. In this model, it is decomposed into two parts, namely elastic and viscous parts.

$$\boldsymbol{T} = \boldsymbol{T}^E + \boldsymbol{T}^D$$

For modeling the elastic response in the stress tensor, the hyperelastic model is used, and for modeling the viscous part, the viscous-like model in Landau, *EM Lifshitz Theory of elasticity Pergamon Press* (Chapter-V) is used.

Notations Again

We are using the following notation to denote the following quantities.

- Strain energy:Σ
- Cauchy stress tensor: $\boldsymbol{T}$
- Deformation gradient: $\boldsymbol{F}$
- Right Cauchy Green strain tensor: $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$
- Left Cauchy Green strain tensor: $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$
- Standard three invariants of $\boldsymbol{B}$: I_1, I_2, I_3
- Identity tensor: $\boldsymbol{I}$
- Green Lagrange strain tensor: $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$
- Second Piola-Kirchhoff stress tensor: $\boldsymbol{S} = det(\boldsymbol{F})\boldsymbol{F}^{-1}\boldsymbol{T}\boldsymbol{F}^{-T}$

Material Responses

Material Responses

We now proceed to make a framework to include material responses. There are two representation theorems. One for scalars and another for tensors.

Theorem 1

Consider a scalar-valued function f of a symmetric tensor $\boldsymbol{B}$. If $\forall \boldsymbol{R} \in SO(3)$

$$f(\boldsymbol{RBR}^T) = f(\boldsymbol{B})$$

then

$$f(\boldsymbol{B}) = \tilde{f}(I_1(\boldsymbol{B}), I_2(\boldsymbol{B}), I_3(\boldsymbol{B}))$$

Material Responses

Theorem 2

Consider a Tensor-valued function $\boldsymbol{A}$ *of a symmetric tensor* $\boldsymbol{B}$ *such that* $\boldsymbol{A}$ *is also symmetric. If* $\forall \boldsymbol{R} \in SO(3)$

$$\boldsymbol{A}(\boldsymbol{R}\boldsymbol{B}\boldsymbol{R}^T) = \boldsymbol{R}\boldsymbol{A}(\boldsymbol{B})\boldsymbol{R}^T$$

then

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{B}) =& \phi_1(I_1(\boldsymbol{B}), I_2(\boldsymbol{B}), I_3(\boldsymbol{B}))\boldsymbol{I} \\ &+ \phi_2(I_1(\boldsymbol{B}), I_2(\boldsymbol{B}), I_3(\boldsymbol{B}))\boldsymbol{B} \\ &+ \phi_3(I_1(\boldsymbol{B}), I_2(\boldsymbol{B}), I_3(\boldsymbol{B}))\boldsymbol{B}^2\end{aligned}$$

Material Responses

Using Theorem 1, Theorem 2, Cayley-Hamilton Theorem, and observer agreement, one can prove that the final form of $\boldsymbol{T}^E$ can be evaluated as

$$\boldsymbol{T}^E = 2\left(\frac{I_2}{\sqrt{I_3}}\frac{\partial \Sigma}{\partial I_2} + \sqrt{I_3}\frac{\partial \Sigma}{\partial I_3}\right)\boldsymbol{I} + \frac{2}{\sqrt{I_3}}\frac{\partial \Sigma}{\partial I_1}\boldsymbol{B} - 2\sqrt{I_3}\frac{\partial \Sigma}{\partial I_2}\boldsymbol{B}^{-1}$$

Look at Beatty, "Topics in finite elasticity: hyperelasticity of rubber, elastomers, and biological tissues—with examples" for more details, and it's proof.

Material Responses

Using Theorem 1, Theorem 2, Cayley-Hamilton Theorem, and observer invariance, one can prove that the final form of $\boldsymbol{T}^E$ for the homogeneous isotropic hyperelastic medium can be evaluated as

$$\boldsymbol{T}^D = 2\eta(\dot{\boldsymbol{E}} - \frac{1}{3}tr(\dot{\boldsymbol{E}})\boldsymbol{I}) + (\zeta + \chi)tr(\dot{\boldsymbol{E}})\boldsymbol{I}$$

where η, ζ and χ are dissipation coefficients of materials in response to various factors like thermal, viscous effects, and so on. Look at Landau, *EM Lifshitz Theory of elasticity Pergamon Press* (Chapter-V) for more details

Remarks on Constraint

Constraints are the extra equations that need to be satisfied together with the balance laws. It can be seen as if this equation creates an additional force that we sometimes call constraint force. The mechanical energy balance in localized form can be given by this

$$\left(2Sym(\partial_{\boldsymbol{C}}\hat{\Sigma}) - \boldsymbol{S}\right) \cdot \dot{\boldsymbol{C}} = 0$$

, where $\partial_{\boldsymbol{A}}(a)$ is defined as

$$da = \partial_{\boldsymbol{A}}(a) \cdot d\boldsymbol{A}$$

and for two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \cdot \boldsymbol{B} = tr(\boldsymbol{A}\boldsymbol{B}^T)$ and $\hat{\Sigma}$ is the extension of Σ over whole tensor space from the symmetric tensor space.

Remarks on Constraint (Continued)

$$\left(2Sym(\partial_{\boldsymbol{C}}\hat{\Sigma}) - \boldsymbol{S}\right) \cdot \dot{\boldsymbol{C}} = 0$$

If there are no extra constraints, $\dot{\boldsymbol{C}}$ can be any arbitrary tensor from the space of symmetric tensors.

$$\left(2Sym(\partial_{\boldsymbol{C}}\hat{\Sigma}) - \boldsymbol{S}\right) = 0$$

But if we have n constraint equations of the form

$$H_i(\boldsymbol{C}) = 0 \;\; i \in \{1, 2, ..., n\}$$

Then we can write using the Lagrange multiplier method to get

$$\left(2Sym(\partial_{\boldsymbol{C}}\hat{\Sigma}) - \boldsymbol{S}\right) = -\sum_{i=1}^{n} p_i \partial_{\boldsymbol{C}} H_i$$

, where p is the Lagrange multiplier.

$$\left(2Sym(\partial_{\boldsymbol{C}}\hat{\Sigma}) - \boldsymbol{S}\right) = -\sum_{i=1}^{n} p_i \partial_{\boldsymbol{C}} H_i$$

If we write the expression for Cauchy stress tensor from it using the following relation,

$$\boldsymbol{T} = \frac{1}{det(\boldsymbol{F})}\boldsymbol{FSF}^T$$

we will get an extra term

$$\boldsymbol{T}^C = \frac{1}{det(\boldsymbol{F})}\boldsymbol{F}\left(\sum_{i=1}^{n} p_i \partial_{\boldsymbol{C}} H_i\right)\boldsymbol{F}^T$$

where $\boldsymbol{T}^C$ is to include the constraint term. Note that $n \leq 6$. For examples of this, please refer Destrade and Saccomandi, "Finite amplitude elastic waves propagating in compressible solids".

Balance Laws

Balance Laws

We now move ahead by stating the balance laws. Let us consider $\boldsymbol{x_{ref}} = X\hat{\boldsymbol{i}} + Y\hat{\boldsymbol{j}} + Z\hat{\boldsymbol{k}}$ denotes the referential position coordinates and let $\boldsymbol{x} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ denotes the current position coordinates.

Let the divergence of $(\cdot)$ with respect to $\boldsymbol{x_{ref}}$ is denoted by $Div(\cdot)$ and the divergence of $(\cdot)$ with respect to $\boldsymbol{x}$ is denoted by $div(\cdot)$. Let $\boldsymbol{b}$ denotes the body force per unit volume and let ρ_o and ρ denote the referential and current densities. Then the angular and linear momentum balance will together give.

$$div(\boldsymbol{T}) + \boldsymbol{b} = \rho\ddot{\boldsymbol{x}}$$

This is equivalent to

$$Div(det(\boldsymbol{F})\boldsymbol{T}(\boldsymbol{F}^{-1})^T) + \boldsymbol{b} = \rho_o\ddot{\boldsymbol{x}}$$

Reduction to plane waves

Reduction to plane waves

We are now moving ahead to reduce the formulation to some simple cases. The first simplification is to the planar wave.

Let us consider $\boldsymbol{x_{ref}} = X\hat{\boldsymbol{i}} + Y\hat{\boldsymbol{j}} + Z\hat{\boldsymbol{k}}$ denotes the coordinates in the reference configuration with the canonical basis of $\mathbb{R}^3$. Following the previous assumption of the planar wave, we can assign that our dependent coordinate for displacement as X. Let the displacement be

$$\boldsymbol{u} = u(X,t)\hat{\boldsymbol{i}} + v(X,t)\hat{\boldsymbol{j}} + w(X,t)\hat{\boldsymbol{k}}$$

Let the final position be denoted by $\boldsymbol{x}$.

$$\boldsymbol{x} = (X + u(X,t))\hat{\boldsymbol{i}} + (Y + v(X,t))\hat{\boldsymbol{j}} + (z + w(X,t))\hat{\boldsymbol{k}}$$

Reduction to plane waves

With these simplifications,

$$[\boldsymbol{F}] = \begin{bmatrix} (1+u_X) & 0 & 0 \\ v_X & 1 & 0 \\ w_X & 0 & 1 \end{bmatrix}, \ [\boldsymbol{F}^{-1}] = \frac{1}{(1+u_X)} \begin{bmatrix} 1 & 0 & 0 \\ -v_X & (1+u_X) & 0 \\ -w_X & 0 & (1+u_X) \end{bmatrix}$$

, where $(\cdot)_v = \frac{\partial(\cdot)}{\partial v}$

Reduction to plane waves

$$[\boldsymbol{B}] = \begin{bmatrix} (1+u_X)^2 & v_X(1+u_X) & w_X(1+u_X) \\ (1+u_X)v_X & (v_X^2+1) & v_X w_X \\ (1+u_X)w_X & w_X v_X & (w_X^2+1) \end{bmatrix},$$

$$[\boldsymbol{C}] = \begin{bmatrix} (1+u_X)^2 + v_X^2 + w_X^2 & v_X & w_X \\ v_X & 1 & 0 \\ w_X & 0 & 1 \end{bmatrix}$$

From here,

$$[\dot{\boldsymbol{E}}] = \frac{1}{2}\begin{bmatrix} ((1+u_X)^2 + v_X^2 + w_X^2)_t & v_{Xt} & w_{Xt} \\ v_{Xt} & 0 & 0 \\ w_{Xt} & 0 & 0 \end{bmatrix}$$

Reduction to plane waves

We can evaluate

$$
\begin{aligned}
I_1 &= 2 + (1 + u_X)^2 + v_X^2 + w_X^2 \\
I_2 &= 1 + 2(1 + u_X)^2 + v_X^2 + w_X^2 \\
I_3 &= (1 + u_X)^2
\end{aligned}
$$

and our equilibrium equation will become,

$$
\begin{aligned}
\rho_o u_{tt} &= \left[2\left(\frac{\partial \Sigma}{\partial I_1} + 2\frac{\partial \Sigma}{\partial I_2} + \frac{\partial \Sigma}{\partial I_3}\right)(1 + u_X) + \frac{1}{2}\kappa((1 + u_X)^2 + v_X^2 + w_X^2)_t\right]_X \\
\rho_o v_{tt} &= \left[2\left(\frac{\partial \Sigma}{\partial I_1} + \frac{\partial \Sigma}{\partial I_2}\right)v_X + \eta v_{Xt}\right]_X \\
\rho_o w_{tt} &= \left[2\left(\frac{\partial \Sigma}{\partial I_1} + \frac{\partial \Sigma}{\partial I_2}\right)w_X + \eta w_{Xt}\right]_X
\end{aligned}
$$

, where $\kappa = \zeta + \chi + \frac{4\eta}{3}$

Some Comments on Voigt Notation

Voigt Notation is in the following form for a general case.

$$[\boldsymbol{\sigma}] = [\boldsymbol{C}][\epsilon]$$

where $[\boldsymbol{\sigma}]$ is the corresponding stress vector, $[\epsilon]$ is the corresponding strain vector and the $[\boldsymbol{C}]$ is the stiffness matrix.

Once writing this, it is clear that stress components are linearly dependent on the strain components, and hence the stress depends linearly on the strain. But for our problem, we are working with the most general form of the hyperelastic model, which may or may not be linear. Hence, we can not convert our relation in the Voigt notation at this stage.

Presentation III

Specific Materials

Saint Venant-Kirchhoff Model

This model is the extension of the isotropic linear-elastic constitutive model from geometrically linear to non-linear cases. contributors", *"Hyperelastic material — Wikipedia, The Free Encyclopedia"*

$$\Sigma(\boldsymbol{B}) = \tilde{\Sigma}(\boldsymbol{E}) = \frac{\lambda}{2}(tr(\boldsymbol{E}))^2 + \mu tr(\boldsymbol{E}^2)$$

where λ and μ are material constants. The constitutive equation is in terms of invariants of $\boldsymbol{E}$ or equivalently in terms of invariants of $\boldsymbol{C}$ but we want it in terms of invariants of $\boldsymbol{B}$.

Saint Venant-Kirchhoff Model

$$
\begin{aligned}
\Sigma &= \frac{\lambda}{2}(tr(\boldsymbol{E}))^2 + \mu tr(\boldsymbol{E}) \\
&= \frac{\lambda}{2}\left(tr\left(\frac{\boldsymbol{C}-\boldsymbol{I}}{2}\right)\right)^2 + \mu tr\left(\left(\frac{\boldsymbol{C}-\boldsymbol{I}}{2}\right)^2\right)
\end{aligned}
$$

We know that any invertible tensor can be decomposed into a unique product of an orthogonal and a symmetric tensor.

$$
\therefore \boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}
$$

where $\boldsymbol{R}$ is an orthogonal and $\boldsymbol{U}$ is a symmetric tensor.

Saint Venant-Kirchhoff Model

$$
\begin{aligned}
\boldsymbol{B} &= \boldsymbol{F}\boldsymbol{F}^T \\
&= \boldsymbol{R}\boldsymbol{U}\boldsymbol{U}^T\boldsymbol{R}^T \\
&= \boldsymbol{R}\boldsymbol{U}\boldsymbol{U}\boldsymbol{R}^T \text{ Since } \boldsymbol{U} \text{ is symmetric}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{C} &= \boldsymbol{F}^T\boldsymbol{F} \\
&= \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U} \\
&= \boldsymbol{U}\boldsymbol{U} \text{ Since } \boldsymbol{U} \text{ is symmetric}
\end{aligned}
$$

From the above calculation, it is clear that $\boldsymbol{B}$ is the rotation of $\boldsymbol{C}$. Therefore their eigenvalues are the same. Hence they have the same invariants.

Saint Venant-Kirchhoff Model

Using the expressions for I_1, I_2, and I_3, we get the following.

$$\frac{\partial \Sigma}{\partial I_1} = \frac{\lambda}{4}(I_1 - 3) + \frac{\mu}{2}(I_1 - 1)$$

$$\frac{\partial \Sigma}{\partial I_2} = -\frac{\mu}{2}$$

$$\frac{\partial \Sigma}{\partial I_3} = 0$$

Saint Venant-Kirchhoff Model

Using the expressions for I_1, I_2, and I_3, we get the following.

$$\rho_o u_{tt} = \Bigg[2\Bigg(\left(\frac{\lambda}{4}+\frac{\mu}{2}\right)((1+u_X)^2+v_X^2+w_X^2-1)\Bigg)(1+u_X) + \frac{1}{2}\kappa((1+u_X)^2+v_X^2+w_X^2)_t\Bigg]_X$$

$$\rho_o v_{tt} = \Bigg[2\Bigg(\left(\frac{\lambda}{4}+\frac{\mu}{2}\right)((1+u_X)^2+v_X^2+w_X^2)-\frac{\lambda}{4}\Bigg)v_X+\eta v_{Xt}\Bigg]_X$$

$$\rho_o w_{tt} = \Bigg[2\Bigg(\left(\frac{\lambda}{4}+\frac{\mu}{2}\right)((1+u_X)^2+v_X^2+w_X^2)-\frac{\lambda}{4}\Bigg)w_X+\eta w_{Xt}\Bigg]_X$$

,

Mooney-Rivlin Model

This model is generally used for the elastic response of rubber-like materials. contributors", *"Mooney-Rivlin solid — Wikipedia, The Free Encyclopedia"* It is given by

$$\Sigma = C_1\left(\bar{I}_1 - 3\right) + C_2\left(\bar{I}_2 - 3\right)$$

where C_1 and C_2 are the material constants and $\bar{I}_1 = I_1 det(\boldsymbol{F})^{-\frac{2}{3}}$ and $\bar{I}_2 = I_2 det(\boldsymbol{F})^{-\frac{4}{3}}$.
Using the relation

$$det(\boldsymbol{B}) = det(\boldsymbol{F})^2$$

we get,

$$\Sigma(\boldsymbol{B}) = C_1\left(\frac{I_1}{I_3^{\frac{1}{3}}} - 3\right) + C_2\left(\frac{I_2}{I_3^{\frac{2}{3}}} - 3\right)$$

Mooney-Rivlin Model

Evaluating the derivatives of Σ, we get

$$\frac{\partial \Sigma}{\partial I_1} = \frac{C_1}{(1+u_X)^{\frac{2}{3}}}$$
$$\frac{\partial \Sigma}{\partial I_2} = \frac{C_2}{(1+u_X)^{\frac{4}{3}}}$$
$$\frac{\partial \Sigma}{\partial I_3} = -\frac{1}{3(1+u_X)^{\frac{11}{3}}}\Big((1+u_X)^3 C_1 + 4(1+u_X)^{\frac{7}{3}} C_2 + (1+u_X)v_X^2 C_1$$
$$+ (1+u_X)w_X^2 C_1 + 2v_X^2(1+u_X)^{\frac{1}{3}} C_2 + 2w_X^2(1+u_X)^{\frac{1}{3}} C_2$$
$$+ 2(1+u_X)C_1 + 2C_2(1+u_X)^{\frac{1}{3}}\Big)$$

We can now use these expressions to write down the governing equation for displacement.

Hadamard Model

This model is taken from Destrade and Saccomandi, "Finite amplitude elastic waves propagating in compressible solids" It is given by

$$\Sigma = \frac{C}{2}(I_1 - 3) + \frac{D}{2}(I_2 - 3) + G(I_3)$$

where C and D are the material constants and $G(I_3)$ is a function of I_3. In other words, $G(I_3)$ takes care of compressibility.

Hadamard Model

Evaluating the partial derivatives of strain energy with respect to the invariants, we get the following expressions.

$$\frac{\partial \Sigma}{\partial I_1} = \frac{C}{2}$$
$$\frac{\partial \Sigma}{\partial I_2} = \frac{D}{2}$$
$$\frac{\partial \Sigma}{\partial I_3} = G'(I_3)$$

Hadamard Model

The equilibrium equations will become,

$$\rho_o u_{tt} = \left[\left(C + 2D + 2G'(I_3)\right)(1 + u_X) + \frac{1}{2}\kappa((1 + u_X)^2 + v_X^2 + w_X^2)_t\right]_X$$

$$\rho_o v_{tt} = \left[\left(C + D\right)v_X + \eta v_{Xt}\right]_X$$

$$\rho_o w_{tt} = \left[\left(C + D\right)w_X + \eta w_{Xt}\right]_X$$

,

Hadamard Model

Some important points to note here are:

1. The shear wave components are governed by linear partial differential equations.
2. We can seek the ansatz $v = v_o \exp(ikX - i\omega t)$. We get the following dispersion relation:

$$\omega^2 + i\eta k^2 \omega - (C + D)k^2 = 0$$

Poynting's Theorem

Poynting's Theorem

We will closely follow Auld, *Acoustic fields and waves in solids* (Chapter-V) The rate of change in the stored energy in a volume equals the sum of the total energy influx and the total work done on the volume.
Here we have

$$\Sigma = \text{Strain energy density } = U_s$$

$$\frac{1}{2}\rho|\dot{\boldsymbol{u}}|^2 = \text{Kinetic energy density } = U_v$$

In order to derive Poynting's Theorem in this case, we start with the force balance equation.

$$Div(\boldsymbol{T}) = \frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{b}$$

Poynting's Theorem

Previous equation is a vector equation, and we can take a dot product with velocity $\boldsymbol{v}$ on both sides. We get,

$$\boldsymbol{v} \cdot Div(\boldsymbol{T}) = \boldsymbol{v} \cdot \frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{v} \cdot \boldsymbol{b}$$

We can now write the strain displacement relation as

$$\nabla \boldsymbol{x} = \boldsymbol{F}$$

Taking time derivative on both sides with respect to time, we get

$$\nabla \boldsymbol{v} = \dot{\boldsymbol{F}}$$

This is a tensorial equation, so we can take the tensor dot product on both sides with $\boldsymbol{T}$. and we get

$$\boldsymbol{T} \cdot \nabla \boldsymbol{v} = \boldsymbol{T} \cdot \dot{\boldsymbol{F}}$$

Poynting's Theorem

Now adding the previous two equations, we get,

$$\boldsymbol{v} \cdot Div(\boldsymbol{T}) + \boldsymbol{T} \cdot \nabla \boldsymbol{v} = \boldsymbol{v} \cdot \frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{v} \cdot \boldsymbol{b} + \boldsymbol{T} \cdot \dot{\boldsymbol{F}}$$

Clearly, R.H.S. is reducible to $Div(\boldsymbol{T}^T \boldsymbol{v})$
Integrating both sides over a control volume V and using the divergence theorem, we get

$$\oint_{\partial V} \boldsymbol{T}^T \boldsymbol{v}.\hat{\boldsymbol{n}} dS = \int_V \boldsymbol{v} \cdot \frac{\partial \boldsymbol{p}}{\partial t} dV - \int_V \boldsymbol{v} \cdot \boldsymbol{b} dV + \int_V \boldsymbol{T} \cdot \dot{\boldsymbol{F}} dV$$

Observing the right-hand side of the equation, the first term is the rate of change in the kinetic energy. The second term is the power due to the body force, and the last term is the internal energy change of rate.

Poynting's Theorem

If we remove the power due to the body forces, our equation will look like

$$\oint_{surface} (-\boldsymbol{T}^T \boldsymbol{v} \cdot \hat{\boldsymbol{n}}) dS + \frac{\partial \int_{volume} (U_s + U_v) dV}{\partial t} = 0$$

Clearly, here dissipation is not involved. In order to add dissipation, we need to add a model for power loss due to dissipation. Since our dissipative stress is linearly dependent on the rate of change of the Green-Lagrange strain tensor, We can use power loss due to dissipation P_d,

$$P_d = \int_V \boldsymbol{T_D} \cdot \dot{\boldsymbol{E}} dV$$

In this case,

$$\boldsymbol{T_D} = 2\eta\dot{\boldsymbol{E}} + \left(\zeta - \frac{2\eta}{3} + \chi\right) tr(\dot{\boldsymbol{E}})\boldsymbol{I}$$

$$\begin{aligned}
P_d &= \int_V \boldsymbol{T_D} \cdot \dot{\boldsymbol{E}} dV \\
&= \int_V \left(2\eta\dot{\boldsymbol{E}} + \left(\zeta - \frac{2\eta}{3} + \chi\right) tr(\dot{\boldsymbol{E}})\boldsymbol{I}\right) \cdot \dot{\boldsymbol{E}} dV \\
&= \int_V \left(2\eta tr((\dot{\boldsymbol{E}})^2) + \left(\zeta - \frac{2\eta}{3} + \chi\right)(tr(\dot{\boldsymbol{E}}))^2\right) dV
\end{aligned}$$

Poynting's Theorem

With dissipation, Poynting's theorem will be

$$\oint_{\partial V} (-\boldsymbol{T}_{\boldsymbol{E}}^{T} \boldsymbol{v} \cdot \hat{\boldsymbol{n}}) dS + \frac{\partial \int_V (U_s + U_v) dV}{\partial t} + P_d = 0$$

Here Poynting's vector is

$$\boldsymbol{P} = -\boldsymbol{T}_{\boldsymbol{E}}^{T} \boldsymbol{v}$$

Poynting's Theorem for Mooney-Rivlin Model

Now we can look at some examples of Poynting's vector for a material following the given constitutive relation. The first one is Mooney-Rivlin Model. We start with the general form of the elastic part of the stress tensor in terms of the constitutive parameters and the strain.

$$\boldsymbol{T}^E = 2\left(\frac{I_2}{\sqrt{I_3}}\frac{\partial\Sigma}{\partial I_2} + \sqrt{I_3}\frac{\partial\Sigma}{\partial I_3}\right)\boldsymbol{I} + \frac{2}{\sqrt{I_3}}\frac{\partial\Sigma}{\partial I_1}\boldsymbol{B} - 2\sqrt{I_3}\frac{\partial\Sigma}{\partial I_2}\boldsymbol{B}^{-1}$$

$$\boldsymbol{P} = -\left(2\left(\frac{I_2}{\sqrt{I_3}}\frac{C_1}{I_3^{\frac{2}{3}}} + \sqrt{I_3}\left(-\frac{C_1 I_1}{3 I_3^{\frac{4}{3}}} - \frac{2 C_2 I_2}{3 I_3^{\frac{5}{3}}}\right)\right)\boldsymbol{I} + \frac{2}{\sqrt{I_3}}\frac{C_2}{I_3^{\frac{1}{3}}}\boldsymbol{B} - 2\sqrt{I_3}\frac{C_2}{I_3^{\frac{4}{3}}}\boldsymbol{B}^{-1}\right)\boldsymbol{v}$$

Poynting's Theorem for Hadamard Model

We can use the constitutive relation for the Hadamard material to write Poynting's vector as

$$\boldsymbol{P} = \left(2\left(\frac{I_2}{\sqrt{I_3}}\frac{D}{2} + \sqrt{I_3}\frac{\partial G(I_3)}{\partial I_3} \right)\boldsymbol{I} + \frac{2}{\sqrt{I_3}}\frac{C}{2}\boldsymbol{B} - 2\sqrt{I_3}\frac{D}{2}\boldsymbol{B}^{-1} \right)\boldsymbol{v}$$

Presentation IV

Poynting's Theorem for Pre-stressed Media

Pre-stressed Media

In this section, we study the propagation of waves in a pre-stressed media. Let us consider that the body is initially in the state of stress with some distribution

$$\boldsymbol{T} = \boldsymbol{T_{ini}}(\boldsymbol{X})$$

Due to the equilibrium condition, this stress distribution must satisfy

$$div(\boldsymbol{T_{ini}}) + \boldsymbol{b} = \rho\ddot{\boldsymbol{x}}$$

Pre-stressed Media

There are two ways to solve this problem.

- The first way is to use the finite wave formulation. In this case, we don't have to consider pre-stress explicitly.
- The second and more useful way is when the wave amplitude is small. In this case, even if the pre-stress is large, we can still use the small deformation theory by considering

$$\boldsymbol{T} = \boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}$$

Poynting's Theorem

We will closely follow Auld, *Acoustic fields and waves in solids* (Chapter-V) The rate of change in the stored energy in a volume equals the sum of the total energy influx and the total work done on the volume.
Here we have

$$\Sigma = \text{Strain energy density } = U_s$$

$$\frac{1}{2}\rho|\dot{\boldsymbol{u}}|^2 = \text{Kinetic energy density } = U_v$$

In order to derive Poynting's Theorem in this case, we start with the force balance equation.

$$Div(\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) = \frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{b}$$

Poynting's Theorem

Previous equation is a vector equation, and we can take a dot product with velocity $\boldsymbol{v}$ on both sides. We get,

$$\boldsymbol{v} \cdot Div(\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) = \boldsymbol{v} \cdot \frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{v} \cdot \boldsymbol{b}$$

We can now write the strain displacement relation as

$$\nabla \boldsymbol{x} = \boldsymbol{F}$$

Taking time derivative on both sides with respect to time, we get

$$\nabla \boldsymbol{v} = \dot{\boldsymbol{F}}$$

This is a tensorial equation, so we can take the tensor dot product on both sides with $\boldsymbol{T}$. and we get

$$(\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) \cdot \nabla \boldsymbol{v} = (\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) \cdot \dot{\boldsymbol{F}}$$

Poynting's Theorem

Now adding the previous two equations, we get,

$$\boldsymbol{v} \cdot Div(\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) + (\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) \cdot \nabla \boldsymbol{v} = \boldsymbol{v} \cdot \frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{v} \cdot \boldsymbol{b} + (\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) \cdot \dot{\boldsymbol{F}}$$

Clearly, R.H.S. is reducible to $Div((\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}})\boldsymbol{v})$
Integrating both sides over a control volume V and using the divergence theorem, we get

$$\oint_{\partial V} (\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) \boldsymbol{v}.\hat{\boldsymbol{n}} dS = \int_V \boldsymbol{v} \cdot \frac{\partial \boldsymbol{p}}{\partial t} dV - \int_V \boldsymbol{v} \cdot \boldsymbol{b} dV + \int_V (\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}}) \cdot \dot{\boldsymbol{F}} dV$$

Observing the right-hand side of the equation, the first term is the rate of change in the kinetic energy. The second term is the power due to the body force, and the last term is the internal energy change of rate.

Poynting's Theorem

If we remove the power due to the body forces, our equation will look like

$$\oint_{surface} (-(\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}})\boldsymbol{v} \cdot \hat{\boldsymbol{n}})dS + \frac{\partial \int_{volume}(U_s + U_v)dV}{\partial t} = 0$$

Clearly, here dissipation is not involved. In order to add dissipation, we need to add a model for power loss due to dissipation. Since our dissipative stress is linearly dependent on the rate of change of the Green-Lagrange strain tensor, We can use power loss due to dissipation P_d,

$$P_d = \int_V \boldsymbol{T_D} \cdot \dot{\boldsymbol{E}} dV$$

Poynting's Theorem

In this case,

$$\boldsymbol{T_D} = 2\eta\dot{\boldsymbol{E}} + \left(\zeta - \frac{2\eta}{3} + \chi\right) tr(\dot{\boldsymbol{E}})\boldsymbol{I}$$

$$\begin{aligned} P_d &= \int_V \boldsymbol{T_D} \cdot \dot{\boldsymbol{E}} dV \\ &= \int_V \left(2\eta\dot{\boldsymbol{E}} + \left(\zeta - \frac{2\eta}{3} + \chi\right) tr(\dot{\boldsymbol{E}})\boldsymbol{I}\right) \cdot \dot{\boldsymbol{E}} dV \\ &= \int_V \left(2\eta tr((\dot{\boldsymbol{E}})^2) + \left(\zeta - \frac{2\eta}{3} + \chi\right)(tr(\dot{\boldsymbol{E}}))^2\right) dV \end{aligned}$$

Poynting's Theorem

With dissipation, Poynting's theorem will be

$$\oint_{\partial V} (-(\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}})^T \boldsymbol{v} \cdot \hat{\boldsymbol{n}}) dS + \frac{\partial \int_V (U_s + U_v) dV}{\partial t} + P_d = 0$$

Here Poynting's vector is

$$\boldsymbol{P} = -(\boldsymbol{T_{ini}} + \boldsymbol{T_{wave}})\boldsymbol{v}$$

Computational Model

Computational Models

We are now in a position to solve the above problems numerically. We have the governing equation of plane wave propagation in many material models. We will use two models and solve the finite amplitude wave problem in two of them by considering the values of material parameters.

- Saint Venant-Kirchhoff Model
- Hadamard Model

Saint-Venant Kirchhoff Model

We assume a plane strain condition. A longitudinal wave is propagating in a material which is having a finite length in the direction of propagation.
With this assumption, the governing partial differential equation will become,

$$\rho_o u_{tt} = \left[2\left(\left(\frac{\lambda}{4}+\frac{\mu}{2}\right)((1+u_X)^2-1)\right)(1+u_X)+\frac{1}{2}\kappa((1+u_X)^2)_t\right]_X$$

Simplifying this equation further, we get

$$\rho_o u_{tt} = \left(\frac{\lambda}{2}+\mu\right)\left(6u_x u_{xx}+2u_{xx}+3u_x^2 u_{xx}\right)+\kappa(u_x u_{xxt}+u_{xxt}+u_{xx}u_{xt})$$

Discretization

We use the explicit finite difference method. For that, we discretize the space and time domain. We denote u_i^n as the displacement at the i^{th} node at n^{th} time step. We use the following expressions for derivatives.

$$
\begin{aligned}
u_x^n &= \frac{u_{i+1} - u_i}{\Delta x} \\
u_{xx}^n &= \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \\
u_{tt}^n &= \frac{u^n - 2u^{n-1} + u^{n-2}}{(\Delta t)^2} \\
u_{xt}^n &= \frac{u_x^{n-1} - u_x^{n-2}}{\Delta t} \\
u_{xxt}^n &= \frac{u_{xx}^{n-1} - u_{xx}^{n-2}}{\Delta x}
\end{aligned}
$$

Values

We use the values of the material parameters for steel with

$$\begin{aligned}
\rho_o &= 7900 kg/m^3 \\
\lambda &= 1.05 \times 10^{11} Pa \\
\mu &= 8.25 \times 10^{10} Pa \\
\kappa &= 10^2 kg/m/sec
\end{aligned}$$

The order for value for κ is evaluated using non-dimensional analysis by comparing it with the order of other values present.

Hadamard Model

We assume the same geometric condition for this case also. With this assumption, the governing partial differential equation will become,

$$\rho_o u_{tt} = \left[\left(C + 2D + 2G'(I_3) \right) (1 + u_X) + \frac{1}{2} \kappa ((1 + u_X)^2)_t \right]_X$$

We have to first select the function $G(I_3)$ before proceeding ahead. Destrade and Saccomandi, "Finite amplitude elastic waves propagating in compressible solids" gives one example of a function for rubber-like material.

$$G(I_3) = (\lambda + \mu)(I_3 - 1) + 2(\lambda + 2\mu)(\sqrt{I_3} - 1)$$

Hadamard Model

We need $G'(I_3)$ for our purpose.

$$G'(I_3) = (\lambda + \mu) - (\lambda + 2\mu)\frac{1}{\sqrt{I_3}}$$

Also, we want our model to tend to a linear model for small strains. For that

$$C = 2\mu + G'(1)$$
$$D = -\mu - G'(1)$$

Destrade and Saccomandi, "Finite amplitude elastic waves propagating in compressible solids"

Hadamard Model

The equilibrium equation will simplify as

$$\rho_o u_{tt} = \left[\left(3\mu + 2\lambda - \frac{2(\lambda + 2\mu)}{(1 + u_x)} \right) (1 + u_X) + \frac{1}{2} \kappa ((1 + u_X)^2)_t \right]_X$$

Further simplifying this equation, we will get

$$\rho_o u_{tt} = (3\mu + 2\lambda) u_{xx} + \kappa (u_{xx} u_{xt} + u_x uxxt + uxxt)$$

Values

We use the values of the material parameters for rubber with

$$\rho_o = 916 kg/m^3$$
$$\text{Young's Modulus, } E = 10^7 Pa$$
$$\text{Possion's Ratio, } \nu = 0.4$$
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 1.42 \times 10^7 Pa$$
$$\mu = \frac{E}{2(1+\nu)} = 3.57 \times 10^6 Pa$$
$$\kappa = 10^5 kg/m/sec$$

Values for Young's modulus is taken from Toolbox", *"Young's Modulus, Tensile Strength and Yield Strength Values for some Materials"*

Computation of Poynting's Vector

Once we are done with the computation of field variable $\boldsymbol{u}$ as a function of X and t (position and time), we can start to compute stresses and Poynting's vector. We start by writing Left Cauchy Green Tensor for the previous case

$$[\boldsymbol{B}] = \begin{bmatrix} (1+u_X)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

, and

$$[\boldsymbol{B}^{-1}] = \begin{bmatrix} (1+u_X)^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Computation of Poynting's Vector

Clearly, displacement has only one component, i.e., $\boldsymbol{u} = u\hat{\boldsymbol{i}}$ and all the non-diagonal terms in $[\boldsymbol{B}]$ and $[\boldsymbol{B}^{-1}]$ are zero. The Poynting vector will contains only one component

$$\boldsymbol{P} = P\hat{\boldsymbol{i}}$$

$$P = \left(2\left(\frac{I_2}{\sqrt{I_3}}\frac{\partial\Sigma}{\partial I_2} + \sqrt{I_3}\frac{\partial\Sigma}{\partial I_3}\right) + \frac{2}{\sqrt{I_3}}\frac{\partial\Sigma}{\partial I_1}(1+u_X)^2 - 2\sqrt{I_3}\frac{\partial\Sigma}{\partial I_2}\frac{1}{(1+u_X)^2}\right)\frac{\partial u}{\partial t}$$

where, $\frac{\partial\Sigma}{\partial I_i}$ is calculated earlier using the constitutive equation and

$$I_1 = 2 + (1 + u_X)^2$$
$$I_2 = 1 + 2(1 + u_X)^2$$
$$I_3 = (1 + u_X)^2$$

Results

We will now move to Matlab to see the results. The program and the result (in .gif format) can be found here.

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