MA105:-

/ •proof by contradictions.

• completeness property:-

If a set in R is bounded above; then it has a supremum.

→ 'R' has completeness property

'Q' dosen't have completeness property

'Z' has completeness property.

supremum: sup (or) lub

infremum: inf (or) glb.

sequence → ∞ terms

• Archemedian principle:-

if x is a real no., then there exists $n \in \mathbb{N}$; such that $n > x$.

+AM > GM proof

f: domain → codomain.

toward convergence of series:

- if a series dosent converge, then it diverges.

" there exist $a \in \mathbb{R}$; such that

$|a_n - a| < \varepsilon \quad \forall \varepsilon \in \mathbb{R}^+$

$\forall n \geq n_0$ & there exists such n_0 "

$\varepsilon - n_0$ definition.

we need

$\frac{1}{n_0} < \varepsilon$

$\therefore \boxed{n_0 > \frac{1}{\varepsilon}}$

but n_0 is integer;

so write $n_0 = [1/\varepsilon] + 1$

if we need $\left|\frac{2}{n_0^2+1}\right| < \varepsilon$

then take $\frac{2}{n_0^2+1} < \frac{2}{n_0^2} < \varepsilon$

$\therefore \boxed{n_0 \geq \sqrt{\frac{2}{\varepsilon}}}$

nonconvergence:

proof of $(-1)^n$ by taking, it is

say n_0; so that $|(-1)^n - a| < \varepsilon$.

take $\varepsilon = 1/3$.

$|(-1)^{2n_0} - a| < \frac{1}{2}$

$|(-1)^{2n_0+1} - a| < 1/2$

$\big| |1-a| + |a+1| \big| < 1$

$|1-a+1+a| \leq$

$\therefore 2 < 1$ contradiction.

→ every convergent sequence is bounded.

every bounded; is not convergent. $(-1)^n$

every unbounded is divergent.

* if $a_n \le b_n$ for all large n; then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

- sandwich theorem.

proof.

n_0 for $|a_n - a'| \le \varepsilon$ | m_0 for $|b_n - b'| < \varepsilon$

* if $a_n = 2^n$! prove that it is not convergent.

2^n is unbounded. So divergent.

→ increasing: } monotonic.

$a_1 \le a_2 \le a_3 \cdots a_n$

∴ constant function is both increasing & decreasing.

convergence theorems.

1) An increasing sequence $\{a_n\}$, that is bounded above is convergent to its supremum.

Show proof;

Eg: $a_1 = \frac{3}{2}$. $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ is this convergent?

lower bounded by $a_n \ge 0$! God!.

$$a_n - a_{n+1} = \frac{a_n - \frac{2}{a_n}}{2} = \frac{a_n^2 - 2}{2a_n}$$

$$\& \quad a_{n+1}^2 - 2 = \frac{(a_n - 2)^2}{2a_n} = +ve$$

∴ $a_n - a_{n+1} = +ve$.

∴ decreasing series

& lower bounded.

∴ convergent.

∴ $a_{n_0} \to a$

$a_{n_0+1} \to a$.

∴ $a = \frac{1}{2}\left(a + \frac{2}{a}\right)$

Theorem: every bounded seq. has a convergent sub sequence.

volzaro weilstrauss theorem

part sequence; made out of a sequence.

not discussed.

1) sub-sequency

2) cauchy-sequency

sequency; where $|a_n - a_m| < \varepsilon$ $\forall n, m \ge$ $\exists n_0$.

for all $\varepsilon \in \mathbb{R}^+$

→ continity of a function of real variable:-

can do Tut-1

1) by sequence definition

2) for every $\epsilon > 0$; there exist $\delta > 0$

$|x-c| < \delta$ & $|f(x) - f(c)| < \epsilon$.

for limit at c.

→ intermediate value property:-

(a,b).

= $\sin(1/x)$ has IVP.

= 0, x = 0

∴ function need not be continuous.

* Intermediate value theorem:-

" if 'f' is continuous, it has IVP."

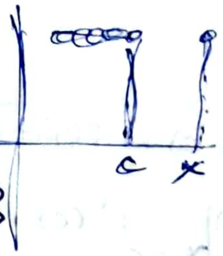
~~sleepy~~ " By intermediate value theorem".

fk!

inverse is false...

→ extreme value theorem:-

if 'f' is continuous on $[a,b]$; then $f(x)$ is bounded & attains its bounds on $[a,b]$!



∴ $\exists\, c_1, c_2$ s.t. $f(c_1) = \max\{f(x) : x \in [a,b]\}$

$f(c_2) = \min\{f(x) : x \in [a,b]\}$

→ Limits of a function:-

Satwik Howla.

in $(c-r, c+r) - \{c\}$; $f(x_n) \to l$.

$x_n \to c$

so, $\forall \{x_n\}$ sequences; it is true.

$f(x) = x \cdot \sin(\frac{1}{x})$

$|x \sin(1/x)| < \epsilon$

tools:

sandwich theorem

inequality

no-ϵ_0

$\exists\, \delta\ \forall \epsilon$ s.t. $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

an

* Limit point, in domain.

→ equivalent condition for differentiation:-

Lemma:-

if $f(x)$ is diff. at c ⟺, there exists $f_1(x)$ satisfying.

$f(x) - f(c) = (x-c)\, f_1(x)$

& $f_1(x)$ → 1) is continuous at c

2) $f'(c) = f_1(c)$.

$\boxed{f_1 \to \text{increment function}}$.

connected with 'f' and "c"!!

Oldest trick in book! add and subtract.

proof for $(f \cdot g)'(c) = f \cdot g'(c) + f' \cdot g(c)$!

* $g(f(x))$

continuous fun has IVP. so range is also an interval.

* $(f^{-1})' f(c) = \frac{1}{f'(c)}$ — no proof. inverse fun. derivative theorem.

we prove $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

C-lemma!

$\boxed{f_1(x)\,(x-c) = f(x) - f(c)}$, $f_1(x)$ cont. at c

& $f_1(x)\big|_{x=c} = f'(c)$

Rolle's theorem:-

sandwich:-

$-|x| \le \left| x \sin\left(\frac{1}{x}\right) \right| \le |x| \to 0$

$\to 0$

$\{ \quad c \to$

$\epsilon - \epsilon < a_n - c$

$c = 10$

0.1

10^{-3}

$\vdots$

Doubt-1:-

1) IVP for $\sin(1/x)$.
↳ in definition, it is mentioned interval.

'wat about $f(x)=1; x>0$
$f(x)=-1, x<0$.

→ Roll's theorem:-

→ Lagrangean mean value theorem:-

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

* convexity & concavity:- (differentiable; not needed)

↳ hence;

(i) $|x|$ is convex; (from first principles).

(ii) $y=\alpha x+\beta$ is both convex and concave. (but not strictly).

convex (downwards) concave (downwards...)

if st-line segment; above curve.

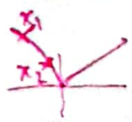
convex says:- ~~downwards~~

$$f((1-\lambda)x_1 + \lambda(x_2)) \le (1-\lambda)f(x_1) + \lambda f(x_2) \quad \forall \lambda \in [0,1]$$

similarly for concave... downwards.

* to say; on an interval $[a,b]$ take any $x_1, x_2 \in [a,b]$ & check defn.

∴ on $[-1,1]$ $|x|$ is not strictly convex.



conditions:- * if $f: I \to \mathbb{R}$ is once differentiable.
f' is increasing; $\Leftrightarrow$ f is convex on I

* if $f: I \to \mathbb{R}$ is double differentiable... (so; no need to check much!). if f'' exists.

$f'' \ge 0 \Longleftrightarrow f$ is convex on I.

$f'' > 0 \Longrightarrow f$ is strictly convex on I.
but not converse... (u know na!).

Critical points:- → Not needed "f is continuous there!"

an interior point of domain; where f' is zero or not defined.

Check for global extremum

for $f:[a,b]\to\mathbb{R}$; f is continuous.

at endpoints & critical points.

→ Local extremum & first derivative test :-

$D\in\mathbb{R}$, & c is an interior point & f is continuous at c.

necessary for local extremum

but need not cont.

(i) f is continuous at c.

(ii) there is $\delta>0$ such that $f'\leq 0$ on $(c-\delta,c)$ & $f'\geq 0$ on $(c,c+\delta)$

Then f: has local maxima at c.

2nd der. test:-

if $f'(x_0)=0$ & $f''(x_0)\neq 0$; then extremum.

3 ways to find POI.

* Point of inflection:-

if $f''(a)=0$, $f'''(a)\neq 0$; $x=a$ is inflection. (when differentiable twice & thrice); if not; use def^n!

An interior point c of I is ____ ; if there is a $\delta>0$; s.t.

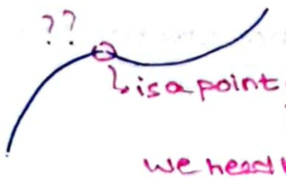
f is concave on $(c-\delta,c)$ & convex on $(c,c+\delta)$

or

reverse

Eg: $x^{1/3}\to$ at $(0,0)$ here; at $x=0$; f is not differentiable.

nowhere in def.^{n}

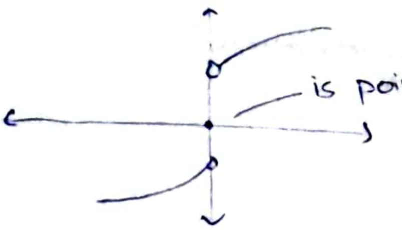


?? ↳ is a point of inflection..?

we need not have f', f'' here.

hitesh remembers; point of inflect^n must have ~~differ~~ tangent...

$\boxed{\text{doubt}}$



is point of inflection.

3 ways to show POI:

1) $f'(x)$ decreasing on $(c-\delta,c)$ increasing on $(c,c+\delta)$

2) convex on $(c-\delta,c)$, concave on $(c,c+\delta)$

3) $f''(c)=0$; $f'''(c)\neq 0$ (if twice differentiable).

* if a sequence is unbounded; then; is it necessary $a_n \to \infty$ or $a_n \to -\infty$.
$\{a_n\}$

no!

$$a_n = (-1)^n \cdot n$$

unbounded.

Asymptode:-
horizontal
vertical.
oblique.

* we say $a_n \to \infty$
if for every $\alpha \in \mathbb{R}$;
$\exists\, n_0$ s.t. $a_n > \alpha \;\; \forall n \geq n_0$

* avoid $\lim_{n\to\infty} a_n = \infty$
concept.

1) horizontal:-

if $\lim_{x\to\infty \text{ or } x\to-\infty} f(x) = b$; $y = b$ is h.asymptode.

2) oblique:-

line $y = ax+b$ is asymptode;
if
$$\lim_{x\to\infty \text{ or } x\to-\infty} (f(x) - ax - b) = 0.$$

check for graph; if $x\to\infty$ or $x\to-\infty$.

$\frac{x^2+1}{x-2} \longrightarrow$ remainder is linear....

3) verticle:-

$$\lim_{x\to a^+ \text{ or } x\to a^-} f(x) = +\infty \text{ or } -\infty.$$

check; when $f(x) \to \infty$.

Example!!....

$e^{-x} \cdot \sin x$.

here x-axis is "ASYMPTODE". Sweet

→ extended mean value theorem:-

mean value theorem:- $f(x)$ is cont. on $[a,b]$, diff. on (a,b).
closed open

$$f(b) = f(a) + f'(c)\cdot(b-a).$$

extended one:- $f(x)$, $f'(x)$ continuous on $[a,b]$; $f'(x)$ diff. on (a,b).
exists and

Proof: slides

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}\cdot(b-a)^2$$

↓
extend many times.

Taylor's theorem:-

$$f(b) = f(a) + \frac{f'(a)(b-a)}{1!} + \frac{f''(a)}{2!}(b-a)^2 + \cdots \frac{f^{n}(a)}{n!}(b-a)^n + \frac{f^{n+1}(c)}{(n+1)!}(b-a)^{n+1}$$

there exists such 'c'!

Taylor's formula:-

> Tut·problem
> $$\lim_{n\to\infty} \frac{a^n}{n!} = 0.$$

$$f(x) = P(x) + \boxed{R(x)}$$

any fu^n can be approximated as — polyn^ml function

reminder. "error in approx^n"

$$P(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots \frac{f^n(a)(x-a)^n}{n!}.$$

$$R(x) = \frac{f^{n+1}(c)\cdot(x-a)^{n+1}}{(n+1)!}.$$

one integration class.
completely slept ☹

$L(P,f) \le L(f) \le U(f) \le U(P,f)$ useful in solving problems.

$\sup\{L(P,f)\} = L(f)$

$\inf\{U(P,f)\} = U(f)$

Tut, tip.

Taylor's theorem:-

- $f : [a,b] \to R$.
- $f', f'', \dots f^n$ are ~~continuous~~ existing on $[a,b]$ closed!
- f^n is continuous on $[a,b]$
  differentiable on (a,b)

then c exists $c \in (a,b)$ s.t.

$$\Rightarrow f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)(b-a)^2}{2!} + \dots \frac{f^n(a)(b-a)^n}{n!} + \underbrace{\frac{f^{n+1}(c)(b-a)^{n+1}}{(n+1)!}}_{R_n}$$

Proof:

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots \frac{f^n(a)(x-a)^n}{n!}.$$

(note the $(P(b) + R_n)$ is RHS of taylor's theorem).

$$F(x) = f(x) - p(x) - \frac{f(b) - P(b)}{(b-a)^{n+1}}(x-a)^{n+1}$$

take $F(x)$; such that

$F(a) = F(b) = 0 \to F'(c_1) = 0$

$F'(a) = 0 \to F''(c_2) = 0$

$F''(a) = 0$

$\vdots$

$F^n(a) = 0 \to F^n(c_n) = 0$

$F^{n+1}(c_n) = 0 \to$ gives taylor's R_n term.

$P(x) = P_n(x)$ → n^{th} order approximation of $f(x)$.

matured taylor's theorem:- nth talyor polynomial around a.

$\boxed{f(x) = P_n(x) + R_n(x)}$ nth lagrangean reminder around a.

$P_n(x)$ is polynomial of degree $\leq n$.

$P_n(x) \rightarrow n^{th}$ order approximation of $f(x)$.

→ Apply for sinx:-

taylor series Around zero!

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - x \cdots + \frac{(-1)^{n/2} x^{n+1}}{(n+1)!} \cdot \cos cx$ if n is even.

$\frac{(-1)^{\frac{n+1}{2}} x^{n+1}}{(n+1)!} \cdot \sin cx$ if n is odd.

$\boxed{|R_n| \leq \frac{x^{n+1}}{(n+1)!}}$

↳ & we can show $\boxed{\lim_{n\to\infty} \frac{a^n}{n!} = 0}$ $\frac{a}{1} \cdot \frac{a}{2} \cdots \frac{a}{a} \Big| \frac{a}{a+1} \cdot \frac{a}{a+2} \cdots$

$\therefore \boxed{\sin x = \lim_{n\to\infty} P_n(x) \text{ around zero}}$ what a fat proof.

Reimann Integration:-

* Partion of $[a,b]$ is a finite ordered set $\{x_0, x_1, \cdots x_n\}$ such that

$a = x_0 < x_1 < \cdots x_n = b$.

$[x_{i-1}, x_i] \rightarrow i^{th}$ subinterval.

$L(P,f) = \sum_{i=1}^{n} m_i(f) \cdot (x_i - x_{i-1})$ $U(P,f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$

* Refinement of P: P^* is a partition with all the elements of P and more.

$L(P,f) \leq L(P^*,f)$
$U(P,f) \geq U(P^*,f)$

* Any lower sum is less than any upper sum. (prove by P_1, P_2

$P^* = P_1 \cup P_2$.

$L(P_1,f) \leq L(P^*,f) \leq U(P^*,f) \leq U(P_2,f)$

① Idk why; but it is here!

* A bounded function is said to be integrable, if $\boxed{L(f) = U(f)}$

$\sup(L(P,f))$ $\inf(U(P,f))$

*for dirichlet fun;
$m_i(f) = 0$, $M_i(f) = 1$
$\therefore L(f) = 0$, $U(f) = b - a$ } not integrable.

* Riemann condition :-

② $f:[a,b]$ be a bounded function; is integrable; if and only if; for every ϵ; there exists a partition P_ϵ of $[a,b]$ such that $U(P_\epsilon,f)-L(P_\epsilon,f)<\epsilon$.

Riemann condition.

→ in proofs: we use.

$U(f) \leq U(P_\epsilon,f)$

$L(f) \geq L(P_\epsilon,f)$.

P_ϵ = refinment of P_1,P_2.

part of proof ($\Rightarrow$).

let $f(a,b)$ be a "BOUNDED".

③ Colloary: $f[a,b]$ be BOUNDED; is integrable, if and only if there exist a sequence P_n such than $U(P_n,f)-L(P_n,f) \longrightarrow 0$ as $n\to\infty$.

we can show, $U(f)-L(f)\to 0$.

proof: ($\Rightarrow$)

it tends to 0.

let $\epsilon = 1/n$, $n\in\mathbb{N}$.

now by Riemann condition;

there exist P_ϵ

s.t. $U(P_n,f)-L(P_n,f)<1/n$

$n\to\infty$. we have a sequence of P_ϵ!

($\Leftarrow$) we find a n_0 s.t. $U-L<\epsilon$.

- so by $\epsilon = n_0$; we have Reimann condition.

$\therefore f$ is integrable.

→ Integrable functions:

1) domain additivity, → $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$c\in(a,b)$

* if $f:[a,b]\to\mathbb{R}$ is monotonic (never said continuous); f is integrable ✓

* if $f:[a,b]\to\mathbb{R}$ is bounded; & has almost, finite no of discontinuities in $[a,b]$, then f is integrable.

unlike dirchelt.

this $[a,b]$ is important.

other wise; we would be saying

$f(x) = 1/x \to (0,1]$ is integrable.

domain.

Eg: $f:[0,1]\to\mathbb{R}$ by $f(0)=0$.

$f(x) = \sin(\frac{1}{x})$.

→ bounded

→ finite discontinuity (at $x=0$).

$\therefore$ integrable.

Fundamental theorem of calculus (FTC):-

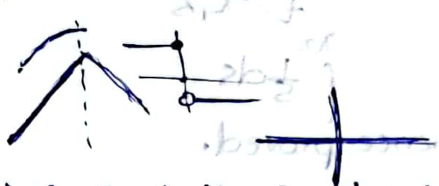
$f: [a,b] \to \mathbb{R}$; f is Rheeman integrabale.

say $\boxed{F(x) = \int_a^x f(x)\,dx}$; $x \in (a,b)$.

Then; F is continuous on (a,b) & if some $c \in (a,b)$; if $f(x)$ is continuous at $x=c$; then $F(x)$ is diffrentiable at c; and $\boxed{F'(c) = f(c)}$.

Outcoms: if $f(x)$ is continuous on $[a,b]$; then $F(x)$ is continuosly diffrentiable on (a,b).

* Antiderivative of f is $F(x)$; $F'(x) = f(x)$.



Part-II: FTC:

$f(x)[a,b]$ be a diffrentiable function; such that $f'(x)$ is integrable;

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

* integration & diffrentiation are inverse of eachother.

→ ANTIDERIVATIVE:-

$f: I \to \mathbb{R}$; we say f has an antiderivative on I; if there exist $F: I \to \mathbb{R}$ such that $F' = f$. F is called antiderivative, primitive, an indefinite integral of f.

* Integration by parts.

$f(a,b)$ is diffrentiable, f' is integrable

$g(a,b)$ is integrable & G be antiderivative of

f(x) is continuosly diffrentiable

⇓

$f'(x)$ exists & is continuous.

∴ $x^2 \sin(1/x)$ is NOT continuosly diffrentiable.

* Integration by substitution:-

$f: [a,b] \to \mathbb{R}$ be continuous

$\phi[\alpha,\beta] \to [a,b]$ be a "continuosly diffrentiable" function.

is subset of

then $$\boxed{\int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx = \int_\alpha^\beta f(\phi(x))\cdot\phi'(x)\,dx}$$

Darboux theorem.

there is $\delta > 0$; ϵ sq $U(P,f) - L(P,f) < \epsilon$, $\mu(P) < \delta$.

↓

Stronger than Rheemann

defining new functions; using Riemann integr^n.

* logarithmic & exponential fu^n:-

1) $$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt \;;\; x \in (0,\infty)}$$ definition!

Properties:-

- $\ln(x_1 x_2) = \ln x_1 + \ln x_2$

Proof: $$\int_1^{x_1 x_2} \frac{1}{t}\,dt = \int_1^{x_1} \frac{1}{t}\,dt + \underbrace{\int_{x_1}^{x_1 x_2} \frac{1}{t}\,dt}_{t = x_1 s}$$ revise ind^n by substitution.

$$\int_1^{x_2} \frac{1}{s}\,ds$$

hence proved.

2) exponential; $R \to (0,\infty)$

inverse of log function. ($\because$ log is bijection $(0,\infty) \to R$).

$\exp(x) = y \Rightarrow \log y = x.$

i) $\exp(0) = 1$

ii) $\exp'(x) = \exp(x)$

} show using fact. $\exp(x)$ is inverse f^n of $\ln(x)$

& say $\boxed{\exp(1) = e}$

$2 < e < 4.$

→ real powers of a +ve real number:-

note that for any $r \in \mathbb{Q}$

$$\boxed{\ln a^r = r \ln a}$$

Proof: $f(x) = \ln a^x - x \ln a$

$f'(x) = 0. \;\forall x$

$\therefore$ as $\ln(a^r) = r \ln a$

$$\boxed{a^r = \exp(x \ln a)}$$ definition

$\therefore e^x = \exp(x \ln e)$

$= \exp(x)$

$\therefore \boxed{\exp(x) = e^x}$ show!

two functions:- {see slides}.

$f(x) = a^x$

$g(x) = x^b$

trignometric & inverse trignometric:-

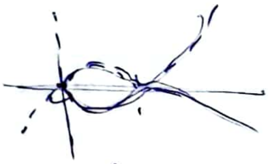
$$\boxed{\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt}$$ fking definition!

& Define tan $\downarrow$ sin, cos.

defining $\pi = 2\cdot \sup\{\arctan x : x\in(0,\infty)\}$

• **Areas:**

$y^2 = x(1-x)^2$



Area definition:- $\int_a^b f(x)\,dx$? signed area.

normal; area between curves: $\int_a^b |f_1(x)-f_2(x)|\,dx$

Polar co ordinates:-

$r=\sqrt{x^2+y^2}$

* (r,θ) is polar coordinate of $P(x,y)$.

$\theta\in(-\pi,\pi]$

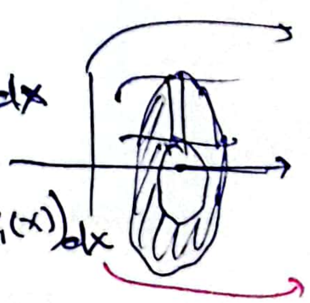
$$\theta = \begin{cases} \cos^{-1}\left(\frac{x}{r}\right), & \text{if } y\geq 0 \\ -\cos^{-1}\left(\frac{x}{r}\right), & \text{if } y<0 \end{cases}$$

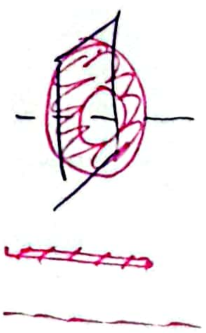
$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right)$

Area & volume.

* $A=\int_{\theta_1}^{\theta_2}\frac{1}{2}\left(r_2^2(\theta)-r_1^2(\theta)\right)d\theta$

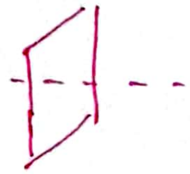
washer: $V=\pi\cdot\int_a^b\left((f_2(x))^2-(f_1(x))^2\right)dx$





Shell method: $2\pi\cdot\int_a^b x\cdot\left(f_2(x)-f_1(x)\right)dx$

Slice method: $\int_a^b A(x)\cdot dx.$



* Parametric curve :-> need not be function : $x = a\cos(\theta)$
$y = a\sin(\theta)$

* if $f(x)$ is continuously differentiable; then $f'(x)$ is continuous.

$$\boxed{\text{Arc length} = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2}\, dt}$$ definition.

where $x(t), y(t) : [\alpha, \beta] \to \mathbb{R}$ are continuous functions.

* if $y = f(x)$:

$$\text{length} = \int_0^x \sqrt{1 + (f'(x))^2}\, dx$$

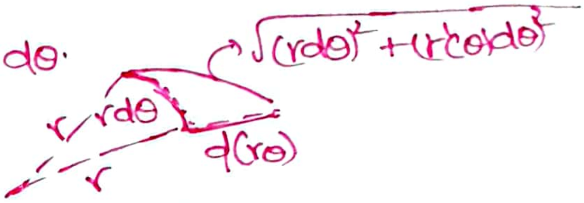
if $x = f(y)$, $\text{length} = \int_0^{y_0} \sqrt{(f'(y))^2 + 1}\, dy$

in polar coordinates :-

$x = r(\theta)\cos\theta$

$y = r(\theta).\sin\theta$

$$\text{Arc length} \neq \int_{\theta_1}^{\theta_2} r(\theta)\, d\theta.$$



$$= \sqrt{(r(\theta)\sin\theta(-1) + r'(\theta)\cdot\cos\theta)^2 + (r'(\theta)\sin\theta + r(\cos\theta))^2}$$

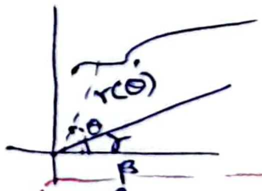
$$\boxed{\text{Arc} = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta.}$$

$\therefore$ in 3d; $\text{Arc} = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$

* Area of rotation:-

$\pi(d_1 + d_2)\cdot l.$

*



$$\boxed{2\pi \int_\alpha^\beta r(\theta)|\sin(\theta - \gamma)| \cdot \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta}$$

$$2\pi \int_{t_1}^{t_2} \rho(t) \cdot \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2}\, dt}_{\text{length of elt part.}}$$

$$\rho(t) = \frac{|a x(t) + b y(t) + c|}{\sqrt{a^2 + b^2}}.$$

Eg:

(i) $y = \frac{x^3}{3} + \frac{1}{4x}$. rotate about $y = -1$

$$= 2\pi \int_1^x (y - (-1)) \sqrt{1 + (y')^2}\, dx$$

2-variable calculus:-

$\mathbb{R}^m$: $x = \{x_1, x_2, \dots x_m\}$.

* norm of x = $\|x\| = (x_1^2 + x_2^2 + \dots x_m^2)^{1/2}$

* schwaz inequality or triangle inequality:-

$x = \{x_1, x_2, \dots x_m\}$
$y = \{y_1, y_2, \dots y_m\}$

we define $x \cdot y = x_1 y_1 + x_2 y_2 + \dots x_m y_m$
↳ scalar product; dot product.

$$|x \cdot y| \le \|x\| \cdot \|y\| \quad \& \quad \|x+y\| \le \|x\| + \|y\|$$

* orthogonal def^n: $x \cdot y = 0$.

angle definition: $\boxed{\cos\theta = \dfrac{x \cdot y}{\|x\| \|y\|}}$ $\theta \in [0, \pi)$

• neighbour of a point x_0:-
$\{x \in \mathbb{R} \mid \|x - x_0\| < r\}$

Functions:- $f(x,y)$.

now; the plot will be a 3D curve.

∴ $f(x,y,z)$ cannot be visualized.

- contour lines & level curve
(x,y,c) (x,y)
$f(x,y) = c$.

proving continuity of $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$

⋮

$\le \dfrac{|x|}{2}$

as $(x_n, y_n) \to (0,0)$

$f(x,y) \le 0 \cdot \to 0$ ✓

continuous $f(x,y)$:

1) as $(x_n, y_n) \to (x_0, y_0) \Rightarrow f(x_n, y_n) \to f(x_0, y_0)$ for every (x_n, y_n).

2) if for every $\epsilon > 0$; there exist a $\delta > 0$; s.t.

$$\|(x,y) - (x_0,y_0)\| < \delta \Rightarrow |f(x,y) - f(x_0,y_0)| < \epsilon.$$

Limit for $f(x,y)$:-

* suppose $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$

$f(0,0) = 0$.

is $f(x,y)$ continuous at $(0,0)$.

to show continuous;
→ dont use sequences;
→ use $\epsilon - \delta$ method.

Sol^n)

$x_n = 1/n$
$y_n = 1/n$.

$\therefore \quad \|(x_n, y_n) - (0,0)\| \to 0$

$f(x_n, y_n) \to \frac{1}{2}$

doubt!
if $x_n = 1/n$
$y_n = 2/n$.
then $f(x_n, y_n) \to \frac{2}{5}$.
now;
how is $f(0,0)$ continuous?

not continuous

Eg: $\frac{x^2 y}{x^4+y^2} = f(x,y)$; $f(0,0) = (0,0)$

Here $x_n = 1/n$, $y_n = 1/n$.

then we have $f(x_n, y_n) \to 0$

but $x_n = \frac{1}{n}$, $y_n = \frac{1}{n^2}$;

then we have $f(x_n, y_n) \to 1/2$.

$\therefore$ not continuous at $(0,0)$

one eg. is sufficent; to say.
(one sequence)

Limits:-

we say $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$ exists; if there is l such that

for every sequence (x_0, y_0) in Domain with $(x_n, y_n) \to (x_0, y_0)$

i.e $(1/n, 1/n)$
or
$(1/n, 1/n^2)$
etc.

& $(x_n, y_n) \neq (x_0, y_0)$
$\Downarrow$
$f(x_n, y_n) = l$

* Hence, $f(x,y) = \frac{xy}{x^2+y^2}$ is 'Limit-doesn't exist' at $(0,0)$.

* if we find; atleast one sequence; making $f(x_n, y_n) \not\to l$;
then we say: Limit don't equal l.

* prove continuity at (0,0) for $f(x,y) = \frac{x^2 y}{x^2+y^2}$.

Alternate ϵ, δ definition:-

now let $\|(x,y)-(0,0)\| < \delta \to |f(x,y)-0|$

$$= \left|\frac{x^2y}{x^2+y^2}\right| < \frac{|x|}{2} \le \frac{\|(x,y)\|}{2} < \frac{\delta}{2} < \delta$$

$$\boxed{\delta = 2\epsilon}$$

(or). $|f(x,y)| \le \frac{|x|}{2}$. so for any $(x_n,y_n) \to (0,0)$; $f(x,y) \to 0$.

Thumbrules for Limits:-

$f(x,y) = \frac{p(x,y)}{q(x,y)}$ where $p(x,y)$ is m^{th} order homog. poly. $q(x,y)$ n^{th} order homog. poly.

if $m>n$; $f(0^+,0^+) \to 0$ if $\frac{p(0,0)}{q(0,0)} \to \frac{0}{0}$

if $m \le n$; $f(0^+,0^+)$ not defined

$m=n$; we approach along different lines & get different.

Partial Diff.:

$f(x,y): D \to R$; is said to hav partial derivative wrt x; if

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} \text{ exists.}$$

on a horizontal line.

$= f_x(x_0,y_0)$ or $\frac{\partial f}{\partial x}(x_0,y_0)$

* Gradiant:-

if $f_x(x_0,y_0), f_y(x_0,y_0)$ exist. then

$\nabla f(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$ is gradient of $f(x,y)$ at (x_0,y_0).

* $f = \sqrt{x^2+y^2}$.

$$f_x(0,0) = \lim_{h\to 0} \frac{\sqrt{h^2+(0)^2}}{h}$$

zero! writ zero.

$= N.A.$

* $f_{xx}(x_0,y_0)$ or $\frac{\partial^2 f}{\partial x^2}(x_0,y_0)$

* $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}(x_0,y_0)$

take care.

existance of Partial derivative doesnt imply continuity.

Eg:

$f(x,y) = \frac{xy}{x^2+y^2}$

$$f_x(0,0) = \lim_{h\to 0} \frac{\frac{0}{0+h^2}}{h} = 0.$$

$\frac{f(h,0)-f(0,0)}{h}$

$\therefore f_y(0,0) = 0$

But $f(x,y)$ not continuous at $(0,0)$.

Mixed partials:-

in general $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ but; equal in most cases.

Eg: $f(x,y) = xy \cdot \frac{x^2-y^2}{x^2+y^2}$ . . .

Mixed partials theorem:- (if continuous; then equal).

$f: D \to R$: let (x_0, y_0) be a point. & there exist r; such that

$S = \{(x,y) \in R^2 / |x-x_0| < r$ & $|y-y_0| < r\} \in D$ & f_x, f_y exists on S.

Square region; centered on (x_0,y_0)

If one of f_{xy} / f_{yx} (x_0,y_0) exists & is continuous, then the other mixed partial exists at (x_0,y_0) & $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$

"proof omitted"

Directional derivatives:-

$u = (u_1, u_2)$ is a unit vector; $\|u\| = 1$

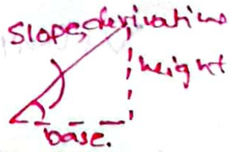
then

$$\lim_{t \to 0} \frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t} \text{ exists; implies}$$

$\boxed{D_u f(x_0,y_0) \text{ exists}}$ nice!

* $\boxed{D_{-u} f(x_0,y_0) = -D_u f(x_0,y_0)}$

c lemma: (for directional derivative).



in one-D; $\overbrace{f(x)-f(a)}^{\text{height}} = \overbrace{(x-a)}^{\text{base}} \cdot f_1(x)$; $x \to a$.

Here; direction is restricted to a line; so we write $x \to a$.

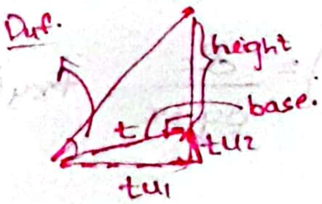
But now;

$D_u f(x_0,y_0)$ exists $\Leftrightarrow$ there is a function; f_1 which

- is continuous at (x_0,y_0)
- satisfies $\overbrace{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}^{\text{height}} = \underbrace{t}_{\text{base}} \cdot f_1(x_0+tu_1, y_0+tu_2)$. (variable is t!)

& $(D_u f)(x_0,y_0) = f_1(x_0,y_0)$

both are 1 variable limit.



ii) $f(x,y) = x^2+y^2$;

$$D_u(f(x,y)) = \lim_{t \to 0} \frac{(x+tu_1)^2+(y+tu_2)^2-x^2-y^2}{t}$$

$$= 2x_0u_1+2y_0u_2.$$

$\therefore$ $\boxed{D_u f(x,y) = u_1(D_x f(x,y)) + u_2(D_y f(x,y))}$

$\therefore$ $\boxed{D_u f(x,y) = \nabla f(x,y) \cdot u}$ (Here).

Scalar dot product.
(u = unit vector)

iii) $f(x) = \sqrt{x^2+y^2}$.

now; perform $D_u f(0,0)$.

Basics! "will get right answer; Like this only".

$$= \frac{f(tu_1, tu_2) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt{t^2}}{t} \text{ - Not defined.}$$

but $D_u f(x_0,y_0) = \dfrac{x_0u_1+y_0u_2}{\sqrt{x_0^2+y_0^2}} = \nabla f(x_0,y_0) \cdot u$
$(x_0,y_0) \neq (0,0)$ except $(x_0,y_0) \neq (0,0)$

think; on what topic; you have bad mindset in MA105.

* Diffrentiability & total derivative:-

we say f is diff. at (x_0,y_0) if there is vector (α,β) s.t.

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0+\alpha h, y_0+\beta k) - f(x_0,y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0$$

& (α,β) is called total derivative of f at (x_0,y_0).

here $f_x(x_0,y_0) = \alpha$.
$f_y(x_0,y_0) = \beta$.

don't confuse differentiability with Gradient's existence.

* if $f(x_0, y)$ is diffrentiable;

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

* Linear map:- f, derivatives:

a function $L: R^m \to R$ is called a linear map; if:

$L(rx+sy) = r\,L(x) + s.L(y) \to$ Linear function.

* f(x) is diffrentiable; if linear map $L(x) = dx$ exists; s.t.

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - L(h)}{h} = 0$$

for $f(x,y)$

linear map $L(h,k) = \alpha h + \beta k$ gotta exist

s.t. $\lim_{\|h,k\| \to 0} \frac{f(x+h, y+k) - f(x,y) - L(h,k)}{\|(h,k)\|} = 0.$

* if a $f(x,y)$ is total derivable; then (at (x_0, y_0))

~~Necessary conditions:~~ used to tell non-differentiable.

1) ~~$D_u f(x,y)$ is defined at (x_0, y_0) for all u.~~

2) ~~$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$.~~

Increment Lemma:- (two variable limit; unlike cathadory lemma).

$f: D \to R$ is differentiable at (x_0, y_0) if and only if; there exist functions f_1, f_2.

such that f_1, f_2 are cont. at (x_0, y_0) &

$\forall (x,y) \in D$.

not; unique pair.

$$f(x,y) - f(x_0, y_0) = (x - x_0) f_1(x,y) + (y - y_0) f_2(x,y)$$

here; $\nabla f(x_0, y_0) = (f_1(x_0, y_0), f_2(x_0, y_0))$

Proof: *didnt read nicely*.

* consequently; if $f(x,y)$ is differentiable at (x_0, y_0): then it is continuous at (x_0, y_0).

* Diff^n: necessary condt's: (means; even if all these are valid; even then; function might not be differentiable?).

i) both $f_x(x_0, y_0)$ & $f_y(x_0, y_0)$ exist.

ii) $D_u f(x_0, y_0)$ exist $\forall u \in \mathbb{R}^2$ & valid.

iii) $\boxed{D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u}$ $\to$ imp. condition; to dismiss differentiability.

iv) f is continuous.

* Sufficient condⁿ for diffrentiability:-

(using Increment lemma)

S = 

$|x-x_0|<r$

$|y-y_0|<r$

square region.

one of the partial derivatives exist on S & is continuous at (x_0,y_0)

another one atleast exists at (x_0,y_0)

then f is diffrentiable at (x_0,y_0).

proof: we find two increment functions.

& then, show diffrentiability....

(didnt read perfectly)

* checking diffrentiability:-

1) if any necessary condition is violated.

2) if sufficient condition is satisfied. ($f_x(x,y)$, $f_y(x,y)$.....)

3) check definition.

* $f(x,y)=\frac{x^2y}{x^2+y^2}$; $=0$ for $(0,0)$

cont. ✓

$D_u f(0,0) = u_1^2 u_2$

∴ we see $\nabla f(0,0)=(0,0)$

& thus

$D_u f(0,0) \neq \nabla f(0,0)\cdot u$

∴ not diffrentiable at (0,0)

∴ first calculate

$\boxed{D_u f(0,0)} \cong \lim_{t\to 0} \frac{f(tu_1,tu_2)}{t}$ ✓

dont say diffrentiable; just because all $D_u f(x_0,y_0)$ exist (you're weakness!)

(iv) $f(x,y)=x^2+y^2$.

show f(x,y) is diffrentiable !

Method!

$f_x(x,y)=2x$ } exists on S & cont. at (x_0,y_0)

$f_y(x,y)=2y$; exists at (x_0,y_0)

Sufficient condition!

24th Sept

tut notes:

* 1) f_x and f_y exists on (x_0, y_0)

one of f_x / f_y ~~exists~~ is continuous at (x_0, y_0)

$\Rightarrow$ f is differentiable.

we can say; f is not 'differentiable' when {we use these practically}.

1) f_x or f_y donot exist.
2) $D_u f$ is not defined for some u.
3) $D_u f(x_0, y_0) \neq \nabla f(x_0, y_0) \cdot u$ fore some u.
4) f is not continuous.

their negation statements are necessary conditions of differentiability.

Geometric interpretation of Gradient :-

Results from Increment Lemma :- (most - "get-by-thinking-proof-shit")

product "rules"!

for 1 varbl. → $f'(x)$

for 2 variables → $\nabla f(x,y)$ as $(\alpha,\beta) = \nabla f(\alpha,\beta)$

* if f, g are differentiable at (α,β).

- $\underbrace{\nabla fg(\alpha,\beta)}_{(x,x)} = \underbrace{g(\alpha,\beta)}_{\text{scalar}} \cdot \underbrace{(\nabla f(\alpha,\beta))}_{(x,x)} + \underbrace{f(\alpha,\beta)}_{\text{scalar}} \cdot \underbrace{(\nabla g(\alpha,\beta))}_{(x,x)}$

- $$\nabla\left(\frac{f}{g}\right)(x_0,y_0) = \frac{g(x_0,y_0)\nabla f(x_0,y_0) - f(x_0,y_0)\cdot\nabla g(x_0,y_0)}{(g(x_0,y_0))^2}$$

chain rules:-

*1) $f(x,y)$ is diff. at (x_0,y_0) $\qquad \frac{\partial h}{\partial x} = \frac{\partial g}{\partial z}\cdot\frac{\partial z}{\partial x}$. where $z = f(x,y)$

$z_0 = f(x_0,y_0)$ $\qquad$ $g(z)$ is a fun.

& $g(x)$ is diff. at z_0 ; $h(x,y) = g(f(x,y))$

$\therefore$ $h_x(x_0,y_0) = g'(z_0)\cdot f_x(x_0,y_0)$; $h_y(x_0,y_0) = g'(z_0)\cdot f_y(x_0,y_0)$

the most obvious things

2) $f(x(t),y(t))$ is there.

$x(t_0) = x_0,\ y(t_0) = y_0$. $\qquad z = f(x,y)$. , $x = x(t)$.

$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = f_x(x_0,y_0)x'(t) + f_y(x_0,y_0)\cdot y'(t)$.

$\phi(t) = f(x(t), y(t))$

$\phi'(t) = f_x(x(t_0),y(t_0))\cdot x'(t_0) \boxed{+} f_y(x(t_0),y(t_0))\, y'(t_0)$

true; if you try to think of proof.

3) $F(u,v) = f(x(u,v), y(u,v))$ $\qquad$ almost same

$\left.\begin{aligned} F_u(u,v) &= f_x(x_0,y_0)\cdot x_u(u,v) + f_y(x_0,y_0)\cdot y_u(u,v) \\ F_v(u,v) &= f_x(x_0,y_0)\cdot x_v(u,v) + f_y(x_0,y_0)\cdot y_v(u,v) \end{aligned}\right\}$ overaction

Geometrical interpret^n:-

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$$
$$= \|\nabla f(x_0, y_0)\| \cos\theta.$$

1) $\cos\theta = 1$;

$$u = \frac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}$$

2) $\cos\theta = -1$

$D_u f(x_0, y_0)$ is minimum.

dont think $= 0$ is minimum.

$= -1$ is minimum $\times \|\nabla f(x_0, y_0)\|$

* direction of "steepest ascent"

$$u = \boxed{\frac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}}$$

even if $\nabla f(x_0, y_0) = (-1, -1)$

this -ve implies direction on the xy plane.

direct^n of steepest descent

$$u = -\frac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}$$

* direct^n of no change in height:-

let $\nabla f(x_0, y_0) = 1, 1$

$\therefore$ $u_1 + u_2 = 0$

and we know $u_1^2 + u_2^2 = 1$

$$\therefore \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \;\&\; \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

* Tangent:- a curve;

$F(x, y) : D \to R$.

when $F(x, y) = 0$ is a curve C. (in xy plane; $z = 0$)

then line tangent at $P_0(x_0, y_0)$ is.

$$F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) = 0$$

tangent plane:- (PTO)

$\rightarrow$ to $(x_0, y_0, f(x_0, y_0))$ to curve $y = f(x_0, y_0)$;

* $z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$\rightarrow$ or. * $f(x, y, z) = 0 \rightarrow$ 3D surface described....

tangent $\Sigma f_x(x_0, y_0, z_0)(x - x_0) = 0$

* Tangent planes :-

case i: at $(x_0, y_0, f(x_0, y_0))$

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

case ii: $f: R^3 \to R$

$$f(x, y, z) = 0$$

$\rightarrow$ describes a surface.

tang. plane (x_0, y_0, z_0) is

$$\boxed{\sum \underset{(x_0, y_0, z_0)}{f_x}(x - x_0) = 0}$$

* Normal Line:

1) $F(x, y, z) = 0$

Line eqn:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

2) $z = f(x_0, y_0)$

$$-z + f(x, y) = 0$$

$$\frac{z - f(x_0, y_0)}{-1} = \frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)}$$

---

* Tangent vector:- $\rightarrow$ tangent to curve $(x(t), y(t), z(t))$ (not surface).

on a 3-D surface 'S'; let there be curve $C = (x(t), y(t), z(t))$ $t \in [\alpha, \beta]$.

the vector $(x'(t_0), y'(t_0), z'(t_0))$ $\rightarrow$ is called the tangent vector at $C(t_0)$.

* now; let S be $F(x, y, z) = 0$.

$\therefore$ curve is $\phi(t) = (x(t), y(t), z(t))$; some $F(\phi(t)) = 0$.

diffn $(F(\phi(t)) = 0)$

$\sum F_x(P_0) \cdot x'(t) = 0$ orthogonal!

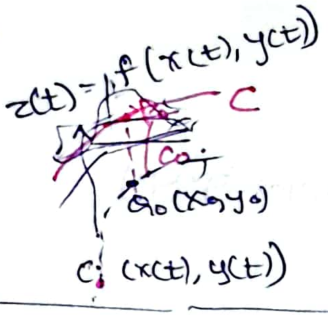
$$\underbrace{\qquad\qquad}_{\text{using this;}}$$

$\therefore$ normal line $\perp$ to tangent vector.
(F_x, F_y, F_z) $(x'(t), y'(t), z'(t))$

"normal is $\perp^{lar}$ to contour line at that point."

(F_x, F_y, F_z) point: $(x(t), y(t), c_0)$

* $z = f(x,y)$

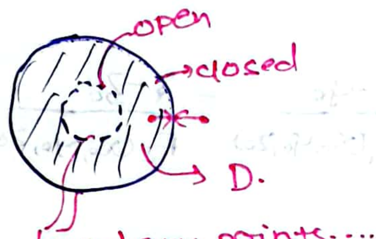


30th sept. 2019

* what does closed ~~set~~ interval $\{(x,y)\}$ mean?

- Boundary point:-

A point $(x_0, y_0) \in \mathbb{R}^2$ is boundary point of D if there exists sequences; both in D & R−D; such that these sequences go to (x_0, y_0)



• closed interval for 2 variables:-

def^n 1: interval $D(\in \mathbb{R}^2)$ is said to be closed; if every sequence in D; ~~that~~ converges in $\mathbb{R}^2$, converges to a point of D itself.

def^n 2:

$\partial D \rightarrow$ boundary of set D. ≡ set of all boundary points.

- D is closed $\Leftrightarrow \partial D \in D$.

* bounded set: $\|(x,y)\| \leq \alpha \quad \forall (x,y) \in D$

*

$\rightarrow$ whole region $D: [a,b] \times [c,d]$

$\rightarrow$ boundary $\partial D = \{(x,y) \in D : x \in \{a,b\} \text{ or } y \in \{c,d\}\}$

only, 4 sides.

Finding global extremum:- extreme value theorem:-

D is a closed, nonempty, bounded subset of R &
$f: D \to R$ is continuous on D. then f is bounded
& attains its bounds on D.

* critical point:

$\nabla f(x_0, y_0) = (0,0)$ or $\nabla f(x_0, y_0)$ is not defined.

* for a continuous f, on D; global min. & max. is attained either at a critical point of f or boundary point of D.

· orthogonal gradient theorem:- (tangent vector on xy plane $\perp$ gradient vector)

curve through (x_0, y_0) . . . .

let curve C $(x(t), y(t))$ pass through (x_0, y_0). & $(x(t_0); y(t_0)) = (x_0, y_0)$ &
if f; restricted on C has extremum at (x_0, y_0) then.

$\frac{d}{dt} f(x(t), y(t)) = 0$

$f_x(x_0, y_0) \cdot x'(t_0) + f_y(x_0, y_0) \cdot y'(t_0) = 0.$

$\therefore$ $\boxed{\nabla f(x, y)\big|_{x_0, y_0} \cdot \text{Tangent vector} = 0}$

$\boxed{D_{u = \hat{t}} f(x,y) = 0}$ → we are finding the directional derivative; which is zero.

(we can also find curve C's tangent vector so that (x_0, y_0) is an extremum point.)

* constrained extrema:-

(x,y) is constrained; like lie on $g(x,y) = 0$ or so. $(y = \eta(x))$

now; find extrema of $f(x,y)$ for such x, y . . . $g(x, \eta(x)) = 0 \;\; \forall x.$

* $f: D \to R$ has an extrema at (x_0, y_0); subjected to $g(x,y) = 0$.

result: lagrange multiplier theorem.:

$\nabla f(x_0, y_0) = \lambda_0 \nabla g(x_0, y_0)$

(x_0, y_0) → required extremum.

λ_0 → lagrange multiplier.

Result: $\nabla g(x_0, y_0)$; $\nabla f(x_0, y_0)$ are parallel.

feel! now; we have surface $z = f(x,y)$.
& (x,y) should satisfy $g(x,y) = 0$.

now pull $g(x,y) = 0$; then; at extrema;
the $z = f(x,y)$ & $g(x,y) = 0$ both would be "same".....

so; $\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0)$

just check (ig.)

$\boxed{f_x \cdot g_y = g_x \cdot f_y}$

even if $\nabla g(x,y) = 0$;
this will work.

* Global extremum can be found with such $\nabla f = \lambda \nabla g$, $\nabla g \neq (0,0)$ & $\nabla g = (0,0)$ & $g = 0$ points.

more eqns. courant & Fret Jhon.

3D! (after 2 pages)

trivariate mean value theorem:-

$D \subseteq \mathbb{R}^3$

let (x_0,y_0,z_0) & $(x_1,y_1,z_1) \in D$.

then there is $(c,d,e) \in D$ on open line segment;

s.t. $f(x_1,y_1,z_1) - f(x_0,y_0,z_0) = \sum f_x(c,d,e)(x_1-x_0)$

st. line

* closed & unbounded.

Proof: $\phi(t) = f(x_0+t(x_1-x_0), y_0+t(y_1-y_0), z_0+t(z_1-z_0))$.

* proposition:- (for local extremum). necessary cond.

(x_0,y_0) — is extremum; if $D_u f(x_0,y_0)$ exists;

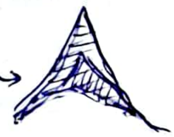
then $D_u f(x_0,y_0) = 0$.

(or) if $\nabla f(x_0,y_0)$ exists; $\nabla f(x_0,y_0) = (0,0)$

converse is not true.

* ex: $f(x,y) = xy$ → all the above is true; but no point of extremum.

* Saddle point:-

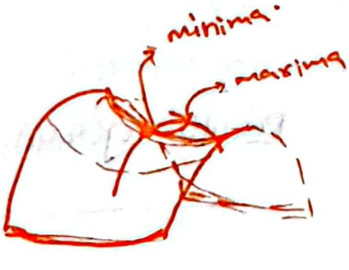


def^n!

Point (x_0,y_0); s.t. the tangent plane is horizontal.

(i.e. $\nabla f(x_0,y_0) = (0,0)$).

& it is not extremum point; then it is saddle point.



we dont have order relation on vectors.

* Before considering "sufficent conditions" for local extremum or saddle point; let us look into mean value theorem (bivariete).

* Make sure continuous partial derivative condition.

Bivariate mean value theorem:-

"$f(x,y)$ has continuous partial derivatives & is, thus differentiable!!

(x_0,y_0) (c,d) (x_1,y_1) $f(x,y)$ has continuous partial derivatives.

$$\underline{f(x_1,y_1) = f(x_0,y_0) + (x_1-x_0)\cdot f_x(c,d) + (y_1-y_0)\cdot f_y(c,d)}$$

* Partial differential operator:-

$$D_{h,k} = h\cdot\frac{\partial}{\partial x} + k\cdot\frac{\partial}{\partial y}$$

* $D^2_{h,k} = h^2\cdot\frac{\partial^2}{\partial x^2} + 2hk\cdot\frac{\partial^2}{\partial x\partial y} + k^2\frac{\partial^2}{\partial y^2}$ (also applying mixed partials theorem).

~~Extended~~ Extended BMVT:

f has continuous partial derivatives of order $\leq n+1$; then there exist (c,d) on the open line segment (x_0,y_0) & (x_1,y_1) s.t. - [polynomial in x,y]

$$\begin{bmatrix} h = x_1 - x_0 \\ k = y_1 - y_0 \end{bmatrix}$$

$$f(x_1,y_1) = f(x_0,y_0) + \frac{D_{h,k}f(x_0,y_0)}{1!} + \frac{D^2_{h,k}f(x_0,y_0)}{2!} + \cdots \;\Big|\; + \underbrace{\frac{D^{n+1}_{h,k}\cdot f(c,d)}{n+1!}}_{\text{Reminder}}$$

$$\underline{P_n(x,y) = \sum_{i=0}^{n} \frac{1}{i!}\left(D^i_{x-x_0,y-y_0}f\right)(x_0,y_0)}$$

n^{th} bivariate taylor polynomial of f around (x_0,y_0)

$$R_n^{(x,y)} = \frac{1}{n+1!}\left(D^{n+1}_{x-x_0,y-y_0}f\right)(c,d)$$

n^{th} lagrange reminder of f, around (x_0,y_0)

* Discriminant test: (for saying local max/min or saddle point).

$\nabla f(x_0,y_0) = 0$; then, we will check (as it is necessary cond^n) further..

$$\boxed{\Delta f_{(x_0,y_0)} = f_{xx}(x_0,y_0)\cdot f_{yy} - f_{xy}^2(x_0,y_0)}$$

(i) if $\Delta f(x_0,y_0) > 0$, $f_{xx}(x_0,y_0) > 0$; local minima at (x_0,y_0)

$f_{xx}(x_0,y_0) < 0$, local maxima at (x_0,y_0).

(ii) if $\Delta f(x_0,y_0) < 0$; saddle point.

(iii) if $\Delta f(x_0,y_0) = 0$; cant decide.

→ **Double integrals:-**

$f(x,y) = x\sin y$
$= x^2+y^2$ } continuous partial derivatives; & differentiable everywhere....

$R: [a,b]\times[c,d]$

P is a partition of R.; given by

$P = [x_i, y_j]$; $i = 0,1,2,\dots n$
$j = 0,1,2,\dots m$

R is divided into $\{nm\}$ nonoverlapping rectangles;

$[x_{i-1}, x_i]\times[y_{j-1}, y_j]$ $\quad i,j = (1,1),\dots\dots$

Lower double sum: $\sum m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = L(P,f)$

Upper — — : $\sum M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = U(P,f)$

$\underbrace{L(f)}_{\text{lower double integral}} = \sup(\underbrace{L(P,f)}_{\text{lower double sum.}})$

* let $f(x,y)$ be positive on R;
the $\iint_R f(x,y)\,d(x,y)$ = volume of solid $\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in R;\ 0\le z\le f(x,y)\}$

* **evaluation of double integrals:-**
$\boxed{\text{I - repeated reimann integration}}$ — even in variable transformat."
Later; we will see Green/Stokes theorem.

* $\boxed{\text{Fubini}}$ **theorem on rectangle:**

$R = [a,b]\times[c,d]$

$f: R\to\mathbb{R}$, $\boxed{\text{Let } I = \iint_R f(x,y)\,d(x,y)}$

postulate i. for every fixed $x\in(a,b)$;
if $\int_c^d f(x,y)\,dy$ exists; then $\int_a^b\int_c^d f(x,y)\,dy\,dx$ exists & $= I$

postulate ii: ↳ for every fixed $y\in(c,d)$ if $\int_a^b f(x,y)\,dx$ exists; then $\int_c^d\int_a^b f(x,y)\,dx\,dy = I$.

then

$$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

fubini theorem.

Integrable functions:-

- if f is monotonic in each of two variables then f is integrable on R.
- if f is bounded on R; & at most, finite no. of discontinuities; then integrable.

* if $f(x,y) = \phi(x)\cdot\varphi(y)$.

then $\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b \phi(x)\,dx \int_c^d \varphi(y)\,dy$

OK.✓

Fichtenn & lichtenstein theorem:-

$f(x,y)$ is bounded

Theorem: if $f(x,y)$ is bounded on R & if $\int_c^d f(x,y)\,dy$ exist for each x & $\int_a^b f(x,y)\,dx$ exist for each y; then the iterated integrals exist & are equal.

Stronger than fubini theorem; ∵ assumes only boundedness, not integrability.

* Mesh:

$$P(x_i,y_i) = \left\{(x_i,y_i);\ \begin{matrix} i=0,1,2\cdots n \\ j=0,1,2,\cdots k \end{matrix}\right\}$$

$$\mu(p) = \max\{x_1-x_0, x_2-x_1, \cdots x_n-x_{n-1}, y_1-y_0, y_2-y_1, \cdots y_n-y_{n-1}\}$$

& not, by some $\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}$ ✓.

(ε, δ definition for integrability)

Rieemann sum:- (the approximate sum thing...)

$$S_n = \frac{1}{n^4}\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)^2 \text{ for } n\in\mathbb{N}.$$

$$= \frac{1}{n^2}\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{i}{n}+\frac{j}{n}\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{\left(\left(\frac{i}{n}\right)+\left(\frac{j}{n}\right)\right)^2}_{M_{ij}(f)}\cdot\left(\frac{i}{n}-\frac{i-1}{n}\right)\left(\frac{j}{n}-\frac{j-1}{n}\right)$$

$$\lim_{n\to\infty} S_n = \int_0^1\int_0^1 (x+y)^2\,\boxed{dx}\,dy$$ partial integration only!...

$x^2+2xy+y^2$

$\frac{x^3}{3}+x^2y+y^2x.$

$\frac{1}{3}+y+y^2$

$\frac{1}{3}+\frac{1}{2}+\frac{1}{3}$ $\frac{2}{3}+\frac{1}{2}=\frac{7}{6}$ ✓

→ Double integral over a bounded set: need not be rectangle.

take $R:[a,b]\times[c,d]$ such that domain $D\subseteq R$. → can be a line.

& define $f^*: R\to\mathbb{R}$ by

$$f^* = \begin{cases} f(x,y) & \text{if } (x,y)\in D \\ 0 & \text{if } (x,y)\in R-D. \end{cases}$$

$$\underbrace{\iint_D f\,d(x,y) = \iint_R f^*\,d(x,y)}_{\text{definition.}}$$

we take a new function f^* such that its domain is Rectangle. we know, how to do on rectangle.

* Now, seek conditions under which a bounded function defined on a bounded interval is integrable. content zero.

subset of $\mathbb{R}^2$ can be point, line, shape or perimeter of shape.

* A bounded subset $D \in \mathbb{R}^2$ is said to be of "content zero", ...

if for every $\epsilon > 0$; there are finitely many rectangles, whose union containing D & sum of area $< \epsilon$.

* 2D content zero set → thin subset of $\mathbb{R}$.

Eg: 1) finite subset of $\mathbb{R}^2$.

2) $(\frac{1}{n}, \frac{1}{k})$ subset $n, k \in \mathbb{N}$.

3) $(x, \psi(x))$ or $(\phi(y), y)$ are content zero.

4) $(x(t), y(t))$ $t \in [\alpha, \beta]$ is content zero... } proof...

Theorem: sufficient condition.

$D \rightarrow$ bounded subset of $\mathbb{R}^2$.

$f: D \rightarrow \mathbb{R}$ be bounded function.

if ∂D is of content zero & the set of points of discontinuity is also content zero; } none of these is "necessary" for integrability.

then, f is double integrable.

subsets with boundary-not content zero:-

$(x, y) \in [0,1] \times [0,1]$ & $x \in \mathbb{Q}$ $y \in \mathbb{Q}$.... Sheker told!

Dirichlet!

→ Elementary regions:-

type 1:

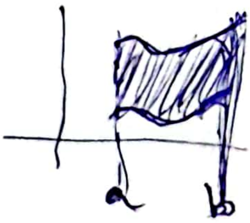
$a \le x \le b$

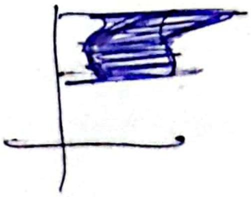
$\phi_1(x) \le y \le \phi_2(x)$

type 2:

$c \le y \le d$

$\phi_1(y) \le x \le \phi_2(y)$





boundary is content zero;

Hence a continuous fu^n; defined on such a boundary is integrable.

2^nd condition;

Points of discontinuity is content zero.

Stuff of; "before 2 pages."

R^3 functions:-

1) Trivariet mean value theorem. there exist P; s.t.

$$f(P_1)-f(P_0) = f_x(P)(x_1-x_0) + f_y(P)(y_1-y_0) + f_z(P)(z_1-z_0)$$

2) Orthogonal gradient theorem:-

$f(x,y,z)$ & the curve $(x(t), y(t), z(t))$.

then if f has a extremum at $P_0(x(t_0)\ldots)$ then.

$$\boxed{\nabla f(P_0)\cdot(x'(t_0), y'(t_0), z'(t_0)) = 0}$$

put $x,y,z(t_0)$ in $f(x,y,z)$ & proceed....

(~~derive~~ remember from $f(x,y)$ result)

3) Lagrange multiplier:-

1) global extremum of f on $\{(x,y,z): g(x,y,z)=0\}$ can be found by comparing the values at sol^ns of:

$\underbrace{\nabla f(x,y,z) = \lambda\nabla g(x,y,z)}_{3\text{ equ}^n}$, $\nabla g \neq 0$ & $\underbrace{\boxed{g=0}}_{1\text{ equ}^n}$ dont forget this.

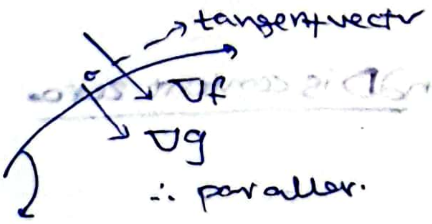
★ where $\boxed{g=0 \text{ \& } \nabla g = 0}$. dont forget this!!.

2) Two constraints:- $g=0, h=0$.

$\underbrace{\nabla f = \mu\nabla g + \lambda\nabla h}_{3\text{ equ}^n}$, $\underbrace{g=0, h=0}_{2\text{ equ}^n}$ & here $\nabla g \neq 0$, $\nabla h \neq 0$. & $\nabla g \not\parallel \nabla h$.

★ Also gotta consider points where $\nabla g = 0$ or $\nabla h = 0$ or $\nabla g \parallel \nabla h$.

Proof: in 1^st case.



∴ paraller.

$g(x,y)=0$.

equ^n: $x(t), y(t) = \Phi(t)$

so; $g(\Phi(t)) = 0$ identity

$$\therefore g_x(\Phi(t))\cdot\frac{\partial x}{\partial t} + g_y(\Phi(t))\cdot\frac{\partial y}{\partial t} = 0.$$

fine.

& max on f. so; orthogradient....

in 2^nd case.

∇f ($\perp$ to tangent vector, by ortho-gradient)

∴ a 2-D result,

$$\boxed{\nabla f = \mu\nabla g + \lambda\nabla h}$$

Fubini; over elementary regions:-

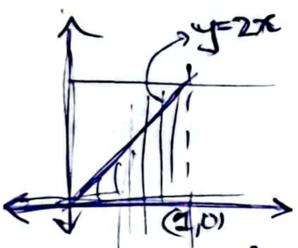
i) $f: D \to \mathbb{R}$

a) if D is elem. region-type-I;

then the iterated integral $\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$ exists & equals $\iint_D f(x,y)\,d(x,y)$

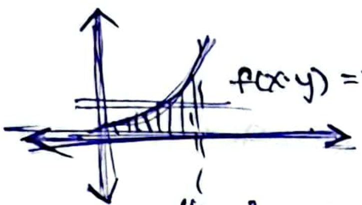
b) if D is type-II elem. region;

then $\iint_D f(x,y)\,d(x,y) = \int_c^d \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx\,dy$



$f(x,y) = e^{x^2}$.

$\int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx$



$f(x,y) = x+y$.

$= \int_0^{1/2} \int_0^{x^2} x+y\,dy\,dx$

$xy + \frac{y^2}{2}$

$\left[x^3 + \frac{x^4}{2}\right]_0^{1/2}$

$\frac{x^4}{4} + \frac{x^5}{10}$.

$\frac{1}{4} \cdot \frac{1}{16} + \frac{1}{320}$

$\frac{6}{320} = \frac{3}{160}$ ✓.

or $\sqrt{y}$ to $1/2$

$\int_0^{1/4} \int_{\sqrt{y}}^{1/2} (x+y)\,dx\,dy$.

$\left[\frac{x^2}{2} + yx\right]_{\sqrt{y}}^{1/2}$.

$\frac{1}{8} - \frac{y}{2} + y\left(\frac{1}{2} - \sqrt{y}\right)$

* Constant function over bounded subset of $\mathbb{R}^2$:-

* if ∂D is content zero; $\underset{\overset{\shortparallel}{1}}{1_D(x,y)}$ is integrable.

converse holds:

if $1_D(x,y)$ is integrable; then ∂D is content zero.

But; hence; say

$(x,y) \in [0,1] \times [0,1]$

$(x)(y) \in \mathbb{Q}$; dirchlet.

then $\iint 1_D(x,y)\,d(x,y)$ dosent exist...

make irregular domain into rectangle domain. $\star \to$ (we do; by writing $1_D^* = 1$ when $(x,y) \in D$
$= 0$ when $(x,y) \in [0,1] \times [0,1] - D$.

& definition of double integral.

* Area of a bounded subset:- (definition.)

$$\text{Area}(D) = \iint_D I_D(x,y)\,d(x,y)$$

Area exists $\Leftrightarrow$ ∂D is content zero.

* Domain additivity (non rectangular too):-

$D \equiv D_1 \cup D_2$

where $D_1 \cap D_2$ is content zero

then $\iint_D f = \iint_{D_1} f + \iint_{D_2} f$.

$\longrightarrow$ Variable change in double integral:-

let $\left.\begin{array}{l} x = x_0 + a_1 u + b_1 v \\ y = y_0 + a_2 u + b_2 v \end{array}\right\}$ linear! have a unique solution; in (u,v)

$\therefore a_1 b_2 \neq a_2 b_1$

$x = x_0 + \varphi_1(u,v);\ y = y_0 + \varphi_2(u,v)$

$\therefore \varphi = (\varphi_1, \varphi_2)$ } this is called affine transformation.

* Here (u,v): $[0,1] \times [0,1]$ square is transformed into $|a_1 b_2 - a_2 b_1|$ unit area parallelogram. in (u,v) in (x,y).

$\therefore \boxed{dx \cdot dy = |a_1 b_2 - a_2 b_1| \cdot du\,dv}$.

Now; in general for some $\phi = (\phi_1, \phi_2): E \to \mathbb{R}^2$. need not be linear (i.e. affine)

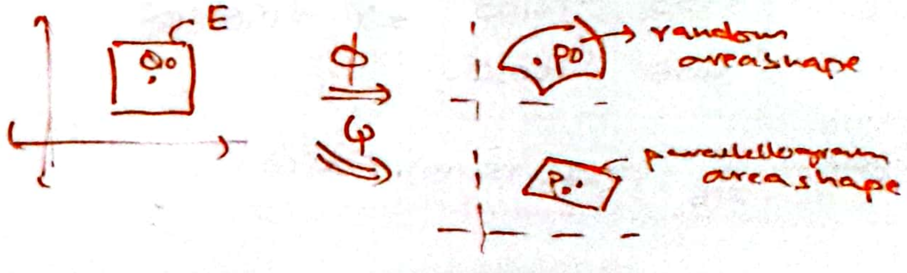
let $Q(u_0,v_0)$ be point in E;

let $x = \phi_1(u_0,v_0)$, $y = \phi_2(u_0,v_0)$

(then we approximate as instantaneously $\varphi(u,v)$ $\left\{\begin{array}{l} x = x_0 + \dfrac{\partial \phi_1(u_0,v_0)}{\partial u} \cdot u + \dfrac{\partial \phi_1}{\partial v}(u_0,v_0) \cdot v \\ y = y_0 + \dfrac{\partial \phi_2(u_0,v_0)}{\partial u} u + \dfrac{\partial \phi_2(u_0,v_0)}{\partial v} \cdot v \end{array}\right.$)

* we define Jacobian as $\begin{vmatrix} \dfrac{\partial \phi_1}{\partial u} & \dfrac{\partial \phi_2}{\partial u} \\ \dfrac{\partial \phi_1}{\partial v} & \dfrac{\partial \phi_2}{\partial v} \end{vmatrix}$.

$$J(\phi)(u_0,v_0) = \frac{\partial(x,y)}{\partial(u_0,v_0)}(u_0,v_0)$$



Theorem (change of variables):-

blah! (too many conditions) can be a fu! of u,v.

$$\iint_D f(x\cdot y)\, d(x\cdot y) = \iint_E f\circ\phi(u,v)\, d(u\cdot v)\, |J(\phi)(u,v)|$$

gotta be of same sign.

$\phi(u,v) = (\phi_1(u,v), \phi_2(u,v))$

condts:

let D; be a closed, bounded subset of $\mathbb{R}^2$; which has area

& $f: D \to \mathbb{R}$ is continuous.

suppose Ω is an open subset of $\mathbb{R}^2$ & $\phi: \Omega \to \mathbb{R}^2$ is a one-one transformation

such that $\phi = (\phi_1, \phi_2)$;

where ϕ_1, ϕ_2 have continuous partial durivatives in Ω.

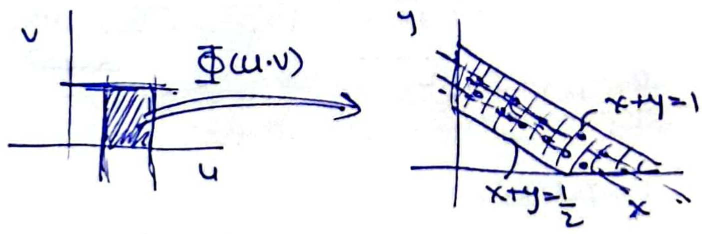
$J(\phi)(u,v) \neq 0 \;\forall\; (u,v) \in \Omega$.

let $E \in \Omega$ s.t. $\Phi(E) = D$; then E is closed, bounded subset of Ω & E has an area.

Moreover $f\circ\phi: E \to \mathbb{R}$ is continuous, and

$$\iint_D f(x\cdot y)\, d(x\cdot y) = \iint_E f\circ\phi(u\cdot v)\, d(u\cdot v).$$

Eg:



let $x+y=u$

$\frac{y}{x+y} = v$

then we have $u: [1/2, 1]$

$v: [0, 1]$

* polar transformation:-

$\Phi: \mathbb{R}^2 \to \mathbb{R}^2$; $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$

$$J(\phi)(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r \;\forall (r,\theta) \in \mathbb{R}^2$$

* $$\boxed{\iint_D f(x\cdot y)\, d(x\cdot y) = \iint_E f(r\cos\theta, r\sin\theta)\cdot r\, d(r\cdot\theta)}$$

Triple Integrals:-

domain.

* $K: [a,b]\times[c,d]\times[e,f]$ cuboid. ;

  $f: K \to R$

* if $f=1$; then; if f is integrable on K.; its triple integrable = volume.

  * if f is monotonic in each of the three variable or if f is continuous; then it is Integrable.

· Fubinis; on K.,

for every x; if $\iint f(x,y,z)\,d(y,z)$ exists; integral of f exists & = I.

then $\iiint_K f \cdots$ exists & = I.

* non cuboid domains:-

D is bounded subset of R^3 & let $f: D \to R$ be continuous fun.. take $K: [a,b]\times[c,d]\times[e,f]$.

$D \subset K$.

define $f^* = 1 \;\forall\; (x,y,z) \in D$.

$\qquad\quad\; 0 \;\forall\; (x,y,z) \in K-D$.

$$\iiint_D f = \iiint_K f^*$$

→ 3-D content zero:-

E is subset of R^3; for every ϵ; there are finite no. of cuboids such that union of these cuboids contain E & their sum of volume $< \epsilon$.

Eg: D is subset of $\mathbb{R}^2$; f is function; integrable over D;

then $(x,y,f(x,y)) \,/\, (x,y) \in D$ is a content zero.

(u,v) $(u\cos v, u\sin v)$ Triple integrals...

x, y

grad.

divergence } easily extended to $\mathbb{R}^4, \mathbb{R}^5 \ldots$

curl : } only for $\mathbb{R}^3$

$e^{x^2+y^2}$

$e^{x^2} y\, dx\, dy$

e^{x^2}

(ii) let $x(u,v)$: $[a,b]\times[c,d]\to R$
$y(u,v)$
$z(u,v)$

then the surface $(x(u,v), y(u,v), z(u,v))$ is 3-D content zero.

→ theorem for integrability :-

D be a bounded subset of R^3;

i) points of discontinuities of f is 3-D content zero.

ii) ∂D is 3D-content zero.

then 'f' is integrable over D.

→ Cavaleiri principle :-

D is bounded ....

$(x,y)\in D_0$; $\varphi_1(x,y)\le z\le\varphi_2(x,y)$;

then

$$\iint\left(\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz\right)d(x,y) \text{ exists.}$$

if D_0 is elementary region;
(of type 1)..

$$\int_a^b\int_{f_1(x)}^{f_2(x)}\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz\,dy\,dx \text{ exists.}$$

→ Constant function over $\mathbb{R}^3$ subset, bounded :-

if $1_D(x,y,z)$ is integrable on D; then ∂D is 3D-content zero.

Here integrability on D $\Leftrightarrow$ ∂D is 3D-content zero....

* if $|f|\le\alpha$

$$\left|\iiint f\right|\le\iiint|f|\le\alpha\cdot(\text{volume}\cdots)$$

* $$\iiint_D=\iiint_{D_1}+\iiint_{D_2}$$

when $D=D_1\cup D_2$

$D_1\cap D_2\equiv$ 3D-content zero.

→ Transformation of variables:-

$$J(\phi)\underbrace{(u_0, v_0, w_0)}_{Q_0} = \frac{\partial x \cdot y \cdot z}{\partial u \cdot v \cdot w}(Q_0) = \begin{vmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} & \frac{\partial \phi_1}{\partial w} \\ (\ ) & (\ ) & (\ ) \\ (\ ) & (\ ) & (\ ) \end{vmatrix}$$

* $J(\phi)\, Q_0 \neq 0.$

then

→ gotta be same sign.

$$\iiint_D f \, d(x \cdot y \cdot z) = \iiint_D f \circ \phi(Q) \, |J(\phi) Q| \, d(u \cdot v \cdot w)$$

$J(\phi)Q$ can be zero for a 3-D content zero space.

→ Important cases:-

1) cylindrical:-

$$\phi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

$$J(\phi)(r, \theta, z) = \begin{vmatrix} \cos\theta & r(-\sin\theta) & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

2) spherical:-

$$\Phi(r, \theta, \phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

$$J(\Phi)(r, \theta, \phi) = r^2\sin\theta.$$

→ Vector Algebra:-

An element of $\mathbb{R}^1$ is called scalar.

An element of $\mathbb{R}^m$; $m>1$; is called vector.

* $x\cdot(\bar{y}\times\bar{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

* $x\times(y\times z) = (x\cdot z)y - (x\cdot y)z$.

→ Scalar fields & vector feilds:-

Scalar fields	Vector fields
association of a scalar to each point in space (temp. on earth)	association of a vector to each point in space (velocity profile)

* let $m\in\mathbb{N}$; $D\subset\mathbb{R}^m$

vector field; is function from D to $\mathbb{R}^m$.

~~if m=2, vector~~

* Smooth scalar & vector fields:-

D is an open subset of $\mathbb{R}^m \Rightarrow$ every point is interior point.

* Scalar field $f: D\to\mathbb{R}$ is smooth if

$\frac{\partial f}{\partial x_i}$ exists & is continuous on D;

for $i=1,2,\dots m$.

* the set of all smooth scalar fields on D is $C^1(D)$.

the set; with continuous 1^{st} & 2^{nd} order is $C^2(D)$.

* vector field $f: D\to\mathbb{R}^m$ is smooth if.

$\frac{\partial f_i}{\partial x_j}$ is continuous & existing on D.

$\forall\ i=1,2,\dots m$
$j=1,2,\dots m$

→ Gradient; Divergence, curl:-

1) Grad:

if f is a smooth scalar field on D;

$$\text{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

this "vector field" is called the grad field.

2) Divergence;

if $F = (P,Q,R)$ is a smooth vector field on D;

then the scalar field;

$$\text{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

3) curl:

curl of the vector $F(P,Q,R)$ is the vector field;

$$\text{curl} F = \nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

* curl can be definied; only for an $\mathbb{R}^3$ element.

→ GCD sequence:-

$$\left.\begin{aligned} &\text{curl}(\text{grad} f) = (0,0,0) \\ &\text{div}(\text{curl} F(P,Q,R)) = 0. \end{aligned}\right\} \text{use these; to verify something; or find variable.}$$

* following questions arise;

1) if G is a smooth vector field; & curl $(G) = (0,0,0)$

then must G be a gradient field?

2) if H is a smooth vector field; such that div.$(H) = 0$;

must H be a curl field?

Like; should a continuous function $g(x)$; be a derivative of "some" function $f(x)$??

FTC-part-1 answers Yes! $f(x) = \int_a^x g(x)dx$

$f'(x) = g(x)$.

to answer ①,②; we turn to suitable theorem of Integrations.

→ Laplacian field:-

f: → scalar field.

$$\text{div}(\text{grad } f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian is a scalar field.; acting on scalar field.

→ Paths:

$\alpha < \beta$; A path or parameterised curve; is a continuous fu^n;

$\gamma [\alpha, \beta] \to \mathbb{R}^m$

i.e. if $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_m)$;

then $\gamma_j : [\alpha, \beta] \to \mathbb{R}$; $\forall j = 1, 2, 3, \ldots m$.

1) Closed path: $\gamma(\alpha) = \gamma(\beta)$

2) Simple path:- $\gamma(t_1) \neq \gamma(t_2)$ unless $(t_1 = \alpha$ & $t_2 = \beta)$ or $(t_1 = \beta$ & $t_2 = \alpha)$ $\forall\ t_1 \neq t_2$

* $\frac{\partial \gamma}{\partial t} = \left(\frac{\partial \gamma_1}{\partial t}, \ldots\right)$.

tangent vector. to γ; at t.

3) Smooth path:-

$\gamma = (\gamma_1, \gamma_2, \ldots \gamma_m)$

$\forall j = 1, 2, \ldots m$ γ_j is continuously differentiable (i.e. γ_j' exists continuously).

& also; if $\gamma(\alpha) = \gamma(\beta)$; then $\gamma'(\alpha) = \gamma'(\beta)$ too.

smooth path is C^1-path.

4) Regular: A smooth path is called regular if $\gamma'(t) \neq (0,0)$ $\forall\ t \in [\alpha, \beta]$

i.e. unit tangent vector exists $\forall\ t \in [\alpha, \beta]$.

- piecewise smooth: at some middle points; not smooth.

piecewise regular: at some middle points; not regular.

→ Path connected Subsets & convex subsets:-

- A subset D of $\mathbb{R}^m$ is called path connected; if for all $u, v \in D$; there is a path $\gamma : [\alpha, \beta] \to \mathbb{R}^m$; such that

$$\gamma[\alpha] = u$$
$$\gamma(\beta) = v$$
$$\gamma(t) \in D \ \forall \ t \in (\alpha, \beta).$$

not p. connected.

- subset is convex; if the line connecting any two points lies in the subset.

convex ✓ path conn. ✓

convex ✗ path conn. ✓

→ Path length:-

$\gamma = (\gamma_1, \gamma_2, \dots \gamma_m) : [\alpha, \beta] \to \mathbb{R}^m$. be piesewise smooth, parametis curve.

$$\text{arc Length} = \int_\alpha^\beta \|\gamma'(t)\| \, dt = \int_\alpha^\beta \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2 + \dots} \, dt.$$

* line int. of scalar field.

$f(\gamma(t))$

$$= \int_\alpha^\beta f(\gamma(t)) \cdot \overbrace{\|\gamma'(t)\| \, dt}^{dl}.$$

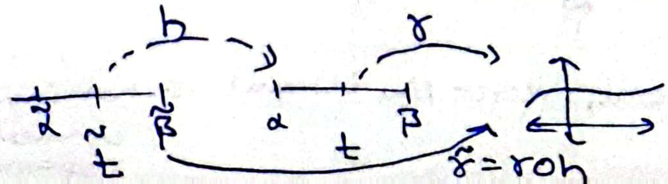
fine. ✓

→ Reparameterisation:-

$\gamma = (\gamma_1, \gamma_2, \dots \gamma_m) : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth parametrized curve...

let $h : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ be a continuosly differentiable function. & $h'(\tilde{t}) \neq 0 \ \forall \ \tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$ & $h([\tilde{\alpha}, \tilde{\beta}]) = (\alpha, \beta)$ then

$\tilde{\gamma} \equiv \gamma \circ h$ is called reparametrisat^n.



$\therefore \ \gamma(t) = \tilde{\gamma}(\tilde{t})$

* the line integral is invariant under reparameterisat^n!

rough: $\int_\alpha^\beta f(\gamma(t)) \cdot \|\gamma'(t)\| dt = \int_{\tilde\alpha}^{\tilde\beta} f(\tilde t) \cdot \|\gamma(h(\tilde t))\| \cdot d\tilde t$

$t = h(\tilde t)$

$\int_{\tilde\alpha}^{\tilde\beta} f(h(\tilde t)) \cdot \|\gamma(h(\tilde t))\| \cdot h'(\tilde t) \cdot d\tilde t$

$$\int_\alpha^\beta f(t)\, \|\gamma'(t)\| dt = \int_{\tilde\alpha}^{\tilde\beta} f(\gamma(h(\tilde t))) \cdot \|\gamma'(h(\tilde t))\| \cdot h'(\tilde t) d\tilde t$$

* arc length parameterisation:-

if $s(t) = \int_\alpha^t \|\gamma'(t)\| dt$

then; $t = h(s)$

where h is the inverse function.

$\hat t = \dfrac{\frac{\partial\gamma}{\partial t}}{\|\frac{\partial\gamma}{\partial t}\|}$ & as $ds = \|\frac{\partial r}{\partial t}\| \partial t$

$\boxed{\hat t = \dfrac{\partial\gamma}{\partial s}}$!!

unit!

$\therefore$; now; $\tilde\gamma = \gamma(h(s))$ is the new parameterisation.

- vector field; line integrals:-

vector field means Bold $\mathbf{F} : C \to \mathbb{R}^m$;

$C = (\gamma[\alpha, \beta])$

$= \int \mathbf{F} . d\mathbf{s}$

$= \int_\alpha^\beta \vec{\mathbf{F}}(\gamma(t)) \cdot \vec{\gamma'(t)}\, dt$

don't write $\mathbf{F}(t)$.. take care.

$\gamma(t)$ is a point.; not t.

$\therefore ds = \|\gamma'(t)\| dt$

$d\mathbf{s} = \gamma'(t) dt$.

* on reparameterisation;

vector integral can change; but only in sign.

i.e.

if $\int_{\tilde\gamma} \vec F . \vec{ds} \neq \int_\gamma \vec F . \vec{ds}$

then $\int_{\tilde\gamma} \vec F . \vec{ds} = - \int_\gamma \vec F . \vec{ds}$

$\therefore$ in general; vector line integral is not invariant under reparameterisation

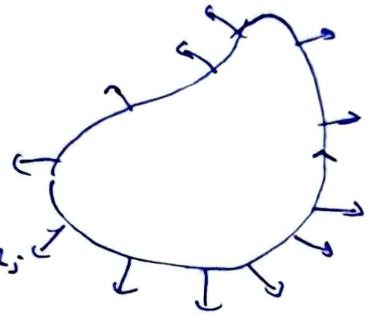
→ Flux integral:-

$r : [\alpha, \beta] \to \mathbb{R}^2$

let $\hat{n}$ denote the continuous outward unit normal to r;

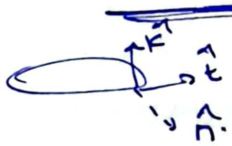
then $\int_\gamma \mathbf{F} \cdot \hat{n}\, ds$ is flux integral.

$\gamma \to$ simple closed path.



→ Suppose $\gamma(t)$ → anticlockwise; as seen from above;

then $\hat{n} = \hat{t} \times \hat{k} = \frac{\partial r}{\partial s} \times \hat{k} = \left(\frac{\partial x}{\partial s}\hat{i} + \frac{\partial y}{\partial s}\hat{j} \right) \times \hat{k}$



$$\boxed{\hat{n} = \frac{\partial y}{\partial s}\hat{i} - \frac{\partial x}{\partial s}\hat{j}}$$

→ $\int_\gamma F \cdot \hat{n}\, ds = \int_\gamma \left(P\frac{dy}{ds} - Q\frac{\partial x}{\partial s} \right) ds = \int_\gamma (P\,\partial y - Q\,\partial x)$

flux - 2D.

rough:- $F = (x - y, x)$.

γ: circle $x^2 + y^2 = 1$

$(\cos t, \sin t)$.

$F(r(t)) = (\cos t - \sin t, \cos t)$

$d \quad dx\hat{i} + dy\hat{j}$ circulation $\int_0^{2\pi} -\cos t + \sin t \ldots\, dt$

$(-\sin t\, dt, \cos t\, dt)$ $\quad 2\pi$.

flux: $\cdot \left(\frac{dy}{dt}\hat{i} - \frac{\partial x}{\partial t}\hat{j} \right) dt$.

$(\cos t\,\hat{i} + \sin t\,\hat{j}) \cdot (\cos t - \sin \ldots$

$\int_{-\pi}^{\pi} \cos^2 t - \sin t \cos t \ldots\, dt$.

$\frac{1 + \cos 2t}{2}$.

$\frac{1}{2}(2\pi) + 0 = \pi$

* path independence of f: in $D \subset \mathbb{R}^3$ dosent mean that $\int_\lambda f\,ds = \int_{\tilde{\lambda}} f \cdot ds$ for any $\lambda, \tilde{\lambda}$. it is true; only for same start & endpoints of $\lambda, \tilde{\lambda}$.

proposition:-

let F be a continuous vector field in $D \subset \mathbb{R}^m$.

then line integrals of F are path independent

$\Updownarrow$

the line integral, along every closed loop path, that lies in D, is zero.

path independence;
Specially for vector fields.

Not intrested, in discussing for scalar fields.

Theorem: (FTC-1 for line integrals):-

in an open subset $D \subset \mathbb{R}^m$; if G be a continuous ~~function~~ field! Scalar field (~~Scalar~~ Vector field)

& let D be path connected. ; if G is path independent line integral.

Define

$$f(x) = \int_{\gamma_x} G \cdot dS \; ; \; \gamma_x \text{ is a path from a fixed } x_0 \text{ to } x.$$

then $\nabla f = G$.

consequently; G is a gradient field.

when D is path connected;

$\int_\gamma G \cdot d\vec{s}$ is path independent.

FTC-2:-

$D \to$ open bounded subset on $\mathbb{R}^m$.

& let 'f' be a smooth scalar field.

$$\int_\gamma (\nabla f) \cdot ds = f(\beta) - f(\alpha)$$

consequently; Line integral of a continuous gradient field are path independent.

* we need methods to assert that a vector field G is gradient or not....

Necessary conditions:-

1) path independence

2) $\oint_c$ along every closed loop is zero.

3) Cross-derivative test:-

$F = (F_1, F_2, \cdots F_m) : D \to \mathbb{R}^m$.

if F is a gradient field;

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall i,j = 1,2,\cdots m.$$

not sufficient (in a light sense).

Eg: $D: \mathbb{R}^2 - \{0,0\}$.

$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

not gradient; but $P_y = Q_x$.

$\Rightarrow$ if $m=3$;

then necessary condition is; $curl(F) = (0,0,0)$

sufficient on simply connected domains.

* Cross derivative test becomes sufficient; when domain is $\mathbb{R}^m$.

$D \to$ simply connected; (domain shouldn't have holes.

every closed path in D; encloses; only points of D.

$D: \mathbb{R}^2 - \{(0,0)\}$ is not simply connected).

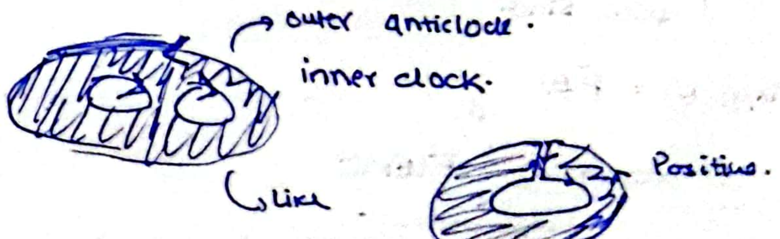
* when a function (vector field) satisfies cross derivative test on a open simply connected domain;

then the Greens theorem & stokes theorem will show; $\oint F \cdot ds = 0$.

over a closed loop

& thus implies path independence; & therefore is a Gradient field.

* Boundary, positively oriented:-



→ Greens Theorem:-

- relates line integrals; along boundary of a domain to double integral over the domain.

- 2D analogue of Part II FTC

$$\int_{\boxed{\partial D}} P\,dx + Q\,dy = \iint_D \underbrace{(Q_x - P_y)}_{\text{curl.}}\,d(x,y)$$

should be positive orientation.

→ Computational advantages:- (consequence of - curl).

1) evaluation of a double integral:-

$$\text{Area}(D) = \iint 1 \cdot d(x,y) \quad \text{(area of subset of } \mathbb{R}^2\text{)}$$

$$Q_x - P_y = \frac{1}{2} - \left(\frac{-1}{2}\right)$$

$$Q = \frac{x}{2},\ P = \frac{-y}{2}.$$

$$\therefore = \int_{\partial D} \frac{-y}{2}\,dx + \frac{x}{2}\,dy$$

now; use parametric

Note! Area; also $= \int_{\partial D} x\,dy$ or $-\int_{\partial D} y\,dx$

$\therefore$ if Boundary $x(t), y(t)$

$$\text{Area} = \frac{1}{2} \cdot \int_\alpha^\beta (x(t) \cdot y'(t) - y(t)\,x'(t))\,dt.$$

$$= \boxed{\frac{1}{2} \int_\alpha^\beta W(x,y)(t)\,dt}$$

$$W(x,y) = \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} \rightarrow \text{wronskian.}$$

for finding area.

Like; if we want $\int_\alpha^\beta f'(x)\,dx$

we do $f(\beta) - f(\alpha)$. (this is 1 der. to 0 der.)

Now we doing 2. der to 1. der.

Same; again in 3D - we will get. for volume finding ✓.

- if ∂D is given in polar

$r = p(\theta)$

then $x = p(\theta)\cos\theta$

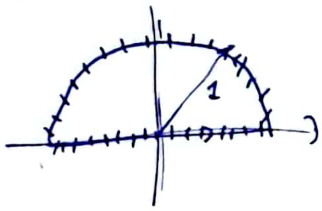
$y = p(\theta)\sin\theta$.

$$W(x,y)(\theta) = p^2(\theta)$$

$$\therefore \text{area} = \frac{1}{2} \int_\alpha^\beta p^2(\theta)\,d\theta$$

(we did this earlier).

(ii) calculation of line integral; along oriented boundary:-



$F = (y^2, 3xy)$

calculate $\int_{\partial D} \vec{F}\cdot d\vec{s}$.

positive orientation

(we don't do $\int_a^b f'(x)dx$ for $f(b)-f(a)$

but we do $\iint$ curl F dxy for $\int \vec{F}\cdot d\vec{s}$).

sometimes; this is easy.

$= \iint_D (3y-2y)\,dx\,dy$

$= \iint_D y\cdot dx\,dy$

$Q_x - P_y$.

let $Q_x = 0$; $P_y = -y$

$P = -\frac{y^2}{2}$

$\therefore$ new $F' = \left(-\frac{y^2}{2}, 0\right)$

successfully changed into others.

$\therefore = \int_{\partial D} -\frac{y^2}{2}\,dx$.

$= 0 + \int_{-1}^{1} -\frac{1}{2}\cdot(1-x^2)\,dx$ → orientation is wrong.

$= -\frac{1}{2}\left(x-\frac{x^3}{3}\right]_{-1}^{1}$

$= -\frac{1}{2}(2) + \frac{1}{6}(2)$

$= -1+\frac{1}{3} = -\frac{2}{3}$.

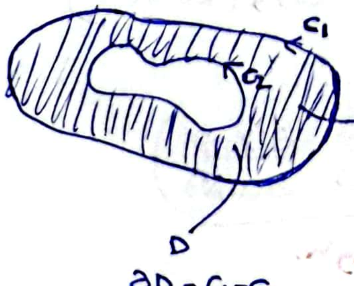
→ conseq. of greens theorem:-

if $Q_x = P_y$ on a closed, bounded interval D,

$\int_{\partial D} \vec{F}\cdot d\vec{s} = 0$.

* invariance of some line integrals:- Deformation principle.

* important.



→ $Q_x = P_y$ in this region;

then

$\int_{\partial C_1} \vec{F}\cdot d\vec{s} = \int_{\partial C_2} \vec{F}\cdot d\vec{s}$

both are same oriented.

$\partial D = C_1 - C_2$.

• Formulations of greens theorem:- Awesome Interpretation.

1) circulation-curl form:-

$$\int_{\partial D} \vec{F}\cdot\hat{t}\,ds = \iint_D (\text{curl}F)\cdot k\; d(x,y) \quad [\text{let } F=(P,Q,0)].$$

Stokes theorem is its 3D analogue.

2) flux-divergence form:-

$$\int_{\partial D} \vec{F}\cdot\hat{n}\,ds = \iint_D (\text{div}\,F)\,d(x,y) \quad [\text{let } F=(Q,P,0)].$$

Gauss theorem is its 3D analogue.

→ We now; move one dimension up; & go from line integrals to surface integrals.

* $F(x,y,z)=0$; – implicitly defined

or $(x(u,v), y(u,v), z(u,v))$; $(u,v)\in E$. are surfaces.

parametrically defined

independent parameters.

→ 3D - parametrized surfaces:-

let a point $P_0 = \Phi(u_0,v_0)$; $(u,v)\in E$

$$\left(\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)\right)$$

normal vector at P_0; to the surface $\Phi(u,v)$ is,

$(\Phi_u \times \Phi_v)(u_0,v_0)$. ($\because$ this is $\perp$ to all tangent vectors).

↳ the course of proof.

also called **fundamental product** for Φ at $(u_0,v_0)\in E$.

* $\Phi = (x,y,z)$

$$\Phi_u \times \Phi_v = \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}, \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right)$$

$$= \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)$$

fair enough.

* parameterized surface is regular; if

$$\|\Phi_u \times \Phi_v\| \neq 0. \ \forall (u,v)\in E.$$

$$\therefore \hat{n}(u,v) = \frac{(\Phi_u \times \Phi_v)(u_0,v_0)}{\|\Phi_u \times \Phi_v(u,v)\|}$$

unit normal vector.

& so; tangent plane eqn. is $(x-x_0, y-y_0, z-z_0)\cdot(\Phi_u \times \Phi_v)(u,v) = 0$

graph of a function!

→ Special case: $(x, y, f(x,y))$

∟ $\Phi_u \times \Phi_v(x,y) = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1)$

∟ $\|\Phi_x \times \Phi_y(x,y)\| = \sqrt{1+f_x^2+f_y^2}$

∟ always a regular surface.

→ Surface Area:-

$$\text{Area}(\phi) = \iint \|(\Phi_u \times \Phi_v)(u,v)\| \, d(u,v)$$

$$dS = \|\phi_u \times \phi_v(u,v)\| \, d(u,v)$$

* Area cosine principle ∟

$$\boxed{dS = \frac{dE}{\cos\theta}}$$

→ Surface integral of scalar field:-

$$\iint_\phi f(\phi(u,v)) \cdot \|\phi_u \times \phi_v(u,v)\| \, d(u,v)$$

→ Reparameterisation of a surface:-

$h(\tilde{E}) = E$ & $J(u,v) \neq 0 \ \forall (\tilde{u}, \tilde{v}) \in \tilde{E}$

Jacobian is never zero.

$\tilde{\phi} = \phi \circ h(\tilde{u},\tilde{v})$

called as reparameterisation.

- Surface integral of scalar; is invariant under reparameterisation.

→ Surface integral of vector field:- (aka flux integral)

$$\iint_\phi \mathbf{F} \cdot d\mathbf{S} = \iint_\phi F(\phi(u,v)) \cdot (\phi_u \times \phi_v(u,v)) \, d(u,v)$$

bold! means vector.

$$= \iint_\phi \left(P \cdot \frac{\partial(y,z)}{\partial(u,v)} + Q \cdot \frac{\partial(z,x)}{\partial(u,v)} + R \cdot \frac{\partial(x,y)}{\partial(u,v)} \right) \partial(u,v)$$

written as:

$$= \iint_\phi P \, dy \wedge dz + Q \cdot dz \wedge dx + R \cdot dx \wedge dy.$$

differential notation.

* surface ∫ of a vector field is in general; not invariant under reparameterisation, but only upto its sign.

reparametrisation:-

$$\tilde{\Phi}(\tilde{u},\tilde{v}) = \Phi \circ h(\tilde{u},\tilde{v})$$

$$\boxed{(\tilde{\Phi}_{\tilde{u}} \times \tilde{\Phi}_{\tilde{v}})(\tilde{u},\tilde{v}) = (\Phi_u \times \Phi_v)(h(\tilde{u},\tilde{v}))\, J(h)(\tilde{u},\tilde{v})} = \frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}$$

(whole integral is surface area)

factor is fine!

area factor!

Σ is invariant.

we have;

if $J(h)(\tilde{u},\tilde{v}) > 0 \ \forall (\tilde{u},\tilde{v}) \in \tilde{E}$

then $\iint_{\tilde{\Phi}} \vec{F}\cdot d\vec{S}$ is invariant.

if $J(h)(\tilde{u},\tilde{v}) < 0 \ \forall (\tilde{u},\tilde{v}) \in \tilde{E}$ then $\iint_{\tilde{\Phi}} \vec{F}\cdot d\vec{S}$ is sign reversed.

Eg: 1) $\tilde{E} = (v,u) / (u,v) \in E$; $h(\tilde{u},\tilde{v}) = (\tilde{v},\tilde{u})$

$\therefore \tilde{\Phi}(\tilde{u},\tilde{v}) = \Phi(\tilde{v},\tilde{u})$

$$\& \ J(h)(\tilde{u},\tilde{v}) = \begin{vmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$$

$$\therefore \iint_{\tilde{\Phi}} F\, dS = (-1)\cdot \iint_{\Phi} F\, dS$$

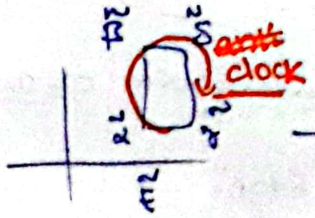
2) $\tilde{E} = \left(\frac{u}{2}, \frac{v}{2}\right) / (u,v) \in E$ $\quad h(\tilde{u},\tilde{v}) = (2\tilde{u}, 2\tilde{v})$

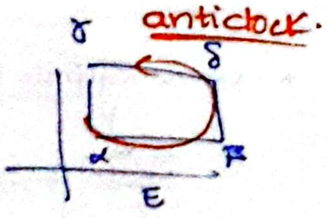
$\therefore h(\tilde{E}) = E$.

$$\text{jacobian} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

$$\therefore \iint_{\tilde{\Phi}} F\cdot dS = \iint_{\Phi} F\cdot dS \ ✓$$

· opposite of a parametrised surface:-





let $\Phi(u,v)$; $(u,v) \in E$.

now; define $\tilde{E} : (v,u) / (u,v) \in E$

$h(\tilde{u},\tilde{v}) = (\tilde{v},\tilde{u})$

$\tilde{\Phi}(\tilde{u},\tilde{v}) = \Phi(\tilde{v},\tilde{u})$

$$\begin{vmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{vmatrix}$$

$\tilde{\Phi}$ is opposite parametrisation of Φ

$$\iint_{\Phi^{op}} F\, dS = -1 \cdot \iint_{\Phi} F\, dS$$

Orientable surfaces :-

$\Phi_{(u,v)} = (u^3+v^2, uv^4)$

$\tilde{\Phi}(\vec{u},\vec{v}) = (\vec{v}^3+\vec{u}^2, \vec{v}\vec{u}^4) = \Phi^{op}$

S → geometrical surface in $\mathbb{R}^3$. → means pure x, y. no parameter.

there exist a continuous function $P \to n(P)$ point.

from S to $\mathbb{R}^3$

s.t.

$n(p)$ is unit normal vector at P.

then S is orientable.

$n(p)$ is called orientation of S.

* Say S is oriented:

a parametrisation Φ of S; if it gives normal vectors, same as specified by orientation $n(p)$ (ie. same direction; not opposite direction).

then Φ is orientation-preserving parametr-

and Φ^{op} is orient^n reversing parametrisation.

all because we choose $\hat{n}$ be $\Phi_u \times \Phi_v$

* Geometric surface: $(x, y, f(x,y))$

$P: (x_0, y_0, f(x_0, y_0))$

$n(p): (-f_x, -f_y, 1) / \|(-f_x, -f_y, 1)\|$ → pure upwards normals. hence orientation. ✓

pure x,y terms.

also: $\Phi(u,v) = (u, v, f(u,v))$ is orientation-preserving.

Intrinsic Boundary :-

different from Boundary of a surface!

for a surface: everything is boundary in (in 3D).

- no 2-D disk like thing in space → intrin. boundary point.

* set of intrinsic boundary point of surface S is ∂S.

also called "Edge".

if $\partial S = \phi$; closed surface or surface without edges.

a vector field can be both gradient & curl field.

Eg: (yz, zx, xy)

grad. of xyz

→ $\operatorname{div} F = 0$ curl field.

→ Intrinsic boundary orientation:- (predetermines).

Orientation of surface ($P \to n(P)$) induces an orientation to its intrinsic boundary.

→ Head on $n(P)$; walk, such as surface is towards left.

→ the rotation (mostly ~~do~~ anti clockwise)

is the induced orientation. on ∂S.

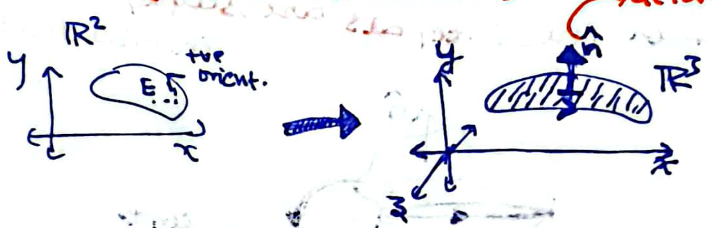
NOTE:

E is subset of $\mathbb{R}^2$ & $\Phi: E \to \mathbb{R}^3$ is a smooth, orientation preserving parameterisation of a surface S.

then $\boxed{\Phi(\partial E) = \partial S}$ and the induced orientation on ∂S corresponds to the positive orientation of ∂E defined; before.

(head on $+\hat{k}$ direction)

→ determined by $\Phi_u \times \Phi_v$ too...



"so both are linked".

"can roughly show why this is so."

this positive orientation is

is useful indetermining; whether we should integrate from left to right or right to left. (on line integrals)

theorems say the orientation.

Stokes theorem:- • 3D Greens Th. • analogue of FTC-II.

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \text{curl}(F) \cdot d\vec{S}$$

normal.

connection - both (+) or both (-).

* Also positively oriented...

$$\int_{\partial S} P\,dx + Q\,dy + R\,dz = \iint_S (R_y - Q_z)\,dy \wedge dz + (\ ) + (\ ).$$

INDUCED ORIENTATION:- By right hand thumbs rule! along $\hat{n}$

Mathematicians find right hand thumb rule informal; so found out induced orientation;

* if for a surface $\partial S = 0$ (i.e closed surface.

$$\iint_S (\text{curl}F) \cdot ds = 0 \checkmark$$

Greens theorem ↔.

$$\iint_S (Q_x - P_y)\,ds = \int_{\partial S} \vec{F} \cdot d\vec{s}$$

which sense; we got to integrate !! positive orientat.

* Surface - Independence:-

if
$\iint_{\Phi} F\cdot dS = \iint_{\tilde{\Phi}} F\cdot ds$

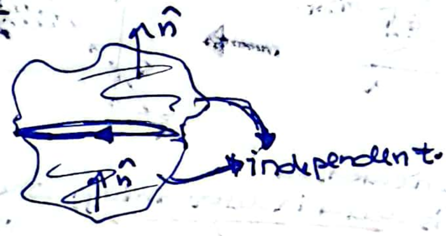
where $\partial\Phi$; $\partial\tilde{\Phi}$ is same & have same positive orientation.

" then F is surface integral are independent.

* Theorem 1:-

let $D \in \mathbb{R}^3$ / F be a continuous curl field.

then flux integrals are surface independent.



How to confirm whether curl field or not!?

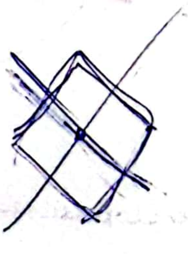
necessary condition:- div(F) = 0 ; if F is a curl field. (almost sufficient too; some technical problem).

div(F) — scalar

if D is such that; every closed Surface in D, encloses space; which is subset of D;
domain $\subset \mathbb{R}^3$
then
if div(F) = 0; F is curl.

$\mathbb{R}^3 - \{(0,0,0)\}$ is not one such

2D mei closed loop;
3D mei closed surface.



* say $F = \text{curl } G$
$F = \text{curl } H$
$\therefore 0 = \text{Curl } (H - G)$
i.e. $H - G$ is grad field
$\therefore \boxed{H = G + \nabla(f)}$! f gotta be scalar field.
not unique solution...

Stokes theorem:-

D computational uses:-

1) calculation of flux integral of a curl field:-

- by using surface independent flux integrals
- By making into single integral over the ly oriented ∂S.

2) Calculating line integral over an oriented boundary:-

- by changing into flux integral.

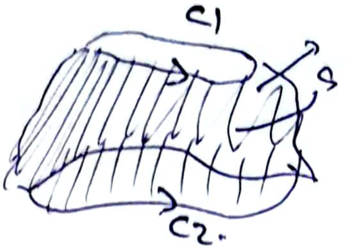
(where boundary isn't so easy; such as

this boundary.)

Eg: $(-y^3, x^3, z^3)$ ... Pg. 6, Lecture: 26.

2) Consequences:-

1)



case F is a grad field on S, or curl F ≠ (0,0,0)

$\partial S = C_2 \cup (-C_1)$

$\therefore \oint_{C_1} \vec{F} \cdot d\vec{s} = \oint_{C_2} \vec{F} \cdot d\vec{s}$ - if curl F = 0

* D : closed & bounded subset of $\mathbb{R}^3$

∂D (usual boundary) is consists of orientable, closed geometrical surfaces.

&

their unit normal vectors all point out!

like; domain is solid objects!.

then ∂D is positively oriented!

(till now; we bought surface to line positive orient^n; now volume to surface positive orient^n;)

Eg:

this is positive orientation of ∂D.

{ Gauss divergence theorem }

→ Gauss-Divergence Theorem :- (not derived by some FCQ, -ve) in stokes.)

D is subset of $\mathbb{R}^3$ such that ∂D is of closed geometrical surfaces (i.e. $\partial s = \emptyset$

& the ∂D is positively oriented. F is vector field;

(surface ka tue orent^n reading 1st time).

$$\boxed{\iint_{\partial D} \vec{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}(F)\cdot dV}$$

flux = divergence.

also

$$* \quad \iint_{\partial D} P\, dy\wedge dz + Q\, dz\wedge dx + R\, dx\wedge dy = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz.$$

→ computational uses:-

1) evaluate 3-integral:-

volume;

$$\therefore \iiint_D 1\cdot d(x,y,z)$$

$\operatorname{Div}(F) = 1$

let $F = \frac{x}{3}, \frac{y}{3}, \frac{z}{3}$

∴ by Gauss theorem:-

$$= \iint_{\partial D} \left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)\cdot \underbrace{(dy\wedge dz, dz\wedge dx, dx\wedge dy)}_{\text{surface area; general.}}$$

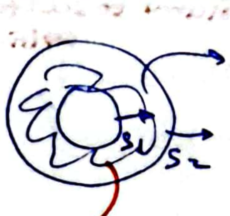
$$= \frac{1}{3}\iint_{\partial D} W\cdot d(u,v)$$

DONT FORGET

where

Wronskian $W_{(x,y,z)} = \begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$

→ ★ Deformation principle; for surface integrals:



→ $\operatorname{div}(F) = 0$ here. ∴ $\iint_{\partial D} \vec{F}\cdot ds = 0$ $\quad \partial D = S_2 \cup (-S_1)$

the orientation.

then $\iint_{S_1} \vec{F}\cdot dS = \iint_{S_2} \vec{F}\cdot dS$

if div = 0

i.e. F is a curl field.

Greens:

$$\underbrace{\int_{\partial D} \vec{F}\, ds}_{\text{vector } \int} = \underbrace{\iint_D (Q_x - P_y)\, dx\, dy}_{\text{scalar } \int}$$

↳ Hence, write wronskian & find 2D area (∂x ∂y) planar.

Stokes:

$$\underbrace{\int_{\partial S} \vec{F}\cdot d\mathbf{s}}_{\text{vector}} = \underbrace{\iint_S \mathrm{curl}(\vec{F})\cdot d\mathbf{S}}_{\text{vector}}$$

can do; if $\vec{F}$ is some $\hat{n}$.

↳ (absent)

(3D surface area is $(dy\wedge dz, dz\wedge dx, dx\wedge dy)$)

Gauss:

$$\underbrace{\iint_{\partial D} \vec{F}\cdot d\mathbf{S}}_{\text{vector}} = \underbrace{\iiint_D \mathrm{div}(F)\cdot d(x,y,z)}_{\text{scalar.}}$$

↳ Hence, write wronskian & find 3D volume (∂x·∂y·∂z)

The surface integral of vector field can be evaluated:

1) direct (u,v).
2) if it aint closed, by stokes.
3) if it is closed, by Gauss.

* $\frac{\mathbf{r}}{r^3} = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ is both a gradient & ~~curl~~ (leaving out (0,0,0)).

electrostatic.

div(·) = 0

---

scalar integral can be written as vector integral.

⊥ar distance = $\vec{r}\cdot\hat{n}$.

normal area mei '1' = $\hat{n}\cdot\hat{n}$.

curl $\vec{F}(x,y,z) = \hat{n}$ (if $\hat{n}$ is constant [illegible]