Lectures from John Watrous:-

L1:

lets talk about Quantum Information; a topic upon which quantum computatn is basedon.

physical device $X \to$ some states Σ.

meaning, classic states. later; by state we'll mean quantum superposition state of X.

Eg: $\Sigma = \{0, 1\}$

* we don't have info. about state of X. we'll represent our knowledge about state of X as probabilities.

$$Pr(\text{state } 0) = 1/4 \quad ; \quad Pr(\text{state } 1) = \frac{3}{4}.$$

* probability vector : colulm vector of non-negative real nos sum up to 1.

$$v = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$$ indexed by elements of Σ.

→ what happens when you look at device X?

→ we won't see the probability vector. we'll see some state $\in \Sigma$.

$\therefore$ the v is changed to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- operations on X:-

initialize to 0,1 $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; $[P_0 \; P_1] \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = v_0 P_0 + v_1 P_1$ probability vectors.

NOT operation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

do nothing $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

for $1/100^{th}$ of the times, do (NOT) operation $= \begin{pmatrix} \frac{99}{100} & \frac{1}{100} \\ \frac{1}{100} & \frac{99}{100} \end{pmatrix}$.

All these matrices have properties

i) all entries are non-negative real nos.

ii) Sum for each column is 1.

Stochastic matrices! every column is a probability vector

(qubits)

classical state $\longrightarrow$ human can look, touch, recognize without ambiguity.

$\Sigma = \{0, 1\}$
↳ classical states.

vectors represent superposition!

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4i/5 \end{pmatrix}$$

amplitudes! of classical states.

- can be negative or even imaginary.
- need not sum up to 1.

condition:-

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$$

vectors have eculedian length = 1.

Here after observing the qubit; we get 0 with $Pr. = |\alpha|^2$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

1 with $Pr = |\beta|^2$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

* Till now, this model seems eerly similar to before one.

but, lets see the operations on X.

Unitary matrices.

$$U \cdot U^{\dagger} = I = U^{\dagger} U$$
↳ transpose conjugate.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |c|^2 + |d|^2 \end{pmatrix}$$

1

∴ each column, row preserve euclidian norm & both rows are orthogonal.

Eg:

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$ Hadamard transform.

$$\text{Not} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Eg: $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. performing Hadamard transform;

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

// we dont know outcom of our measurment

again Hadamard trasfam

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

= we are certain about outcome of our measurment

* Two things we can do on a qubit :-

1. Perform a measurment: to collapse the probability.

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ becomes either $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ becomes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only!

2) Perform a $\underbrace{\text{unitary}}_{\text{since single qubit}}$ operation :-

- unitary matrix U.

superposition $v \longrightarrow$ superposition Uv.

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \xrightarrow{H} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{H} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

* Say I know x to be in state $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ or state $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$.

this complete is not again a new state.

dont get creative.

even qubit exists in a SINGLE quantum state/ superposition.

haha.

L2: multiqubit system & dirac notation for describing superpositions.

* 2-qubit system.

lets see... classical first. $\Sigma = \{0, 1\}$

we'll have 4 classical states.

(00, 01, 10, 11)

probability vector is

$$\begin{array}{c|c} \begin{pmatrix} 1/8 \\ 1/2 \\ 0 \\ 3/8 \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

sum = 1

operations are again stochastic matrices but 4×4.

column sum is 1

c_i column represents probabilities of state i to transit to all other states. $\sum_{j=1} c_{ij} = 1$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

if first bit = 0; do nothing.

if first bit = 1; random 2nd bit.

Quantum variant:-

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 1/2 \\ -1/2 \end{pmatrix}$$

if we measure just one qubit; then we'll see later lec; how the state will change.

hmm...

2-qubit unitary operation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

controlled-NOT operation.

2nd bit is NOTted if 1st bit == 1.

Tensor Product:-

say two uncorelated devices X, Y.

$$X = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \quad Y = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$$

$$\underbrace{X \otimes Y}_{\text{tensor product}} = \begin{pmatrix} \frac{1}{12} \\ \frac{3}{12} \\ \frac{2}{12} \\ \frac{6}{12} \end{pmatrix}$$

Kronecker's product.

* in general, for any 2 matrices $A_{n\times m}$ $B_{k\times l}$;

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & & a_{nm}B \end{pmatrix}_{nk \times ml}$$

crazy!

Properties:-

1) associative $\quad A \otimes (B \otimes C) = (A \otimes B) \otimes C$

2) $\underbrace{(A \otimes B)(C \otimes D)}_{\text{matrix product}} = (AC) \otimes (BD)$

since these are defined.

3) distributive law

$$A \otimes (B + C) = A \otimes B + A \otimes C.$$

4) for a scalar α;

$$(\alpha A) \otimes B = \alpha (A \otimes B)$$

5) Not commutative:-

in general $A \otimes B \neq B \otimes A$.

(position of terms are jumbled).

Nice Note:-

1) Not every probability vector of (X,Y) can be expressible as tensor product of X & Y.

Eg: $\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$ entangled qubits. correlated X,Y

- only when uncorelated; can we breakdown into tensor product XY

$ac = 1/2$
$bc = 0$
$bd = 1/2$

- we'll talk about entangelment a lot during the course.

2) Same for operations

$(2\times2) \otimes (2\times2) = (4\times4)$.

But not all operations can be broken into individuals.

* independent / uncorrelated operations

$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ = Not $\quad V = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$U \otimes V = \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{pmatrix}$$

unitary matrix!

this describes operations; even on entangled states.

$$\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

entanglement destroyed!

not individual particles, but an inseperable whole.

* entangled states means; if ~~state~~ qubit-1 is measured then ~~state~~ qubit-2 is also measured.

But doesn't mean that

if op. is carried out on qubit-1 then it is carried out on qubit-2 also....

operation on qubits; can destroy entanglment. can create entanglment.

Dirac notation:-

since vectors & matrices are growing exponentially with number of qubits; we'll have a tough time drawing them.

* column vectors are represented by "kets":

$$|0\rangle \overset{def}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |01\rangle \overset{def}{=} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

* $|\psi\rangle$ & $|\phi\rangle$ be arbitrary vectors.

$$|\psi\rangle|\phi\rangle \overset{def}{=} |\psi\rangle \otimes |\phi\rangle$$

tensor product

$\therefore$ $|1010\rangle$ is a 16-dimensional vector.

* any arbitrary vector, which shows the superposition of states of n-quibits can be written using base vectors

$$\{|x\rangle : x \in \{0,1\}^n\}$$

$$|\phi\rangle = \sum_{x \in \{0,1\}^n} a_x \cdot |x\rangle$$

* Now; say I have a state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

& gotta apply H-transformation to qubit-1 & nothing to qbit2.
can do using vectors & matrices.

$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

define the operation in this language. H is 1-qubit unary op.

& starting state $= \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$

transform qubit-1.

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)|1\rangle$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

done!

* we define bra $\langle\psi|$ for a ket $|\psi\rangle$ as

$$\langle\psi| = (|\psi\rangle)^{\dagger}$$

transpose conjugate.

$\therefore$ $\langle\psi|$ is a row vector.

$$|\psi\rangle = \begin{pmatrix} \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \langle\psi| = \begin{pmatrix} \frac{1-i}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ $\langle\psi|\psi\rangle$ = scalar = inner product of vectors.
matrix multiplication

$|\psi\rangle\langle\phi|$ is not scalar. its a matrix!

$$|\psi\rangle\langle\phi| \cdot \underbrace{|x\rangle}_{\text{col.}} = |\psi\rangle \underbrace{\langle\phi|x\rangle}_{\text{scalar}} = \underbrace{\langle\phi||x\rangle}_{\text{d.p}} \underbrace{|\psi\rangle}_{\text{col.}}$$

L3: superdense coding, quantum circuits, partial measurments

Superdense coding:-

* communicate 2 classical bits, using a single qubit.
a, b

(But its part of e-bit
or EPR-bit).
einstein Podolsky Rosen...

A has one entangled qbit
B has another.

A sends his qbit to B.

• initially; two qbits entangled

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

A has one, B has another.

Protocol:-

1. if $a=1$, Alice applies following operation on his qbit A.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

if $a=0$; no apply.

2. if $b=1$, Alice applies the unitary operation on A

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{NOT.}$$

If $b=0$; she doesnot.

3. Alicee sent A to bob. (this is the only qubit sent)

4. Bob applied Controlled NOT (a 2-qubit, unary op.) to AB.

A as control.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

5. Bob applies hadamard transform to A.

6) Bob measures A,B. Output will certainly be a,b.

Eg:

ab	State after step-2	after step 4	after step 5
0 1	$\frac{1}{\sqrt{2}}\|10\rangle+\frac{1}{\sqrt{2}}\|01\rangle$	$\underbrace{\left(\frac{1}{\sqrt{2}}\|1\rangle+\frac{1}{\sqrt{2}}\|0\rangle\right)}_{A}\underbrace{\|1\rangle}_{B}$	$\|0\rangle\|1\rangle$ same as ab.

for any ab; protocol give same AB.

Quantum Circuits:-

we need better ways to describe protocols. Not english.

As algorithms become complex; we need better techiques to show operations & measurments.

- Show using Quantum Circuit diagrams.

1) Time goes from left to right (in protocol)

2) Horizontal lines represent qubits.

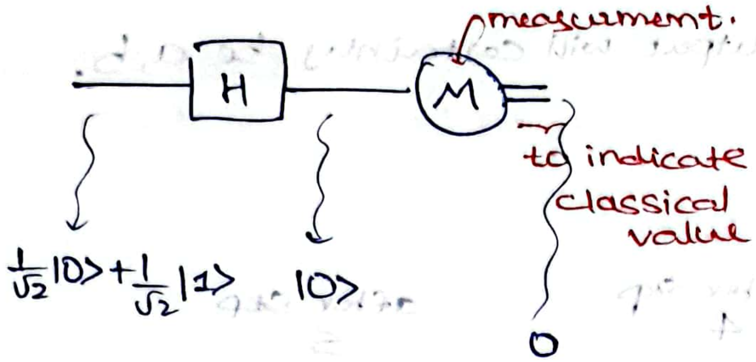
3) Operations & measurments shown by symbols.

Eg) 1)

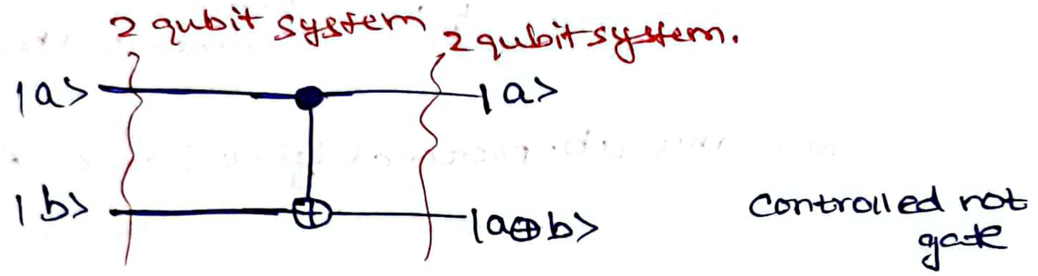
single qubit. Hadamard op.

$\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle$ $\qquad$ $|0\rangle$

Eg2:

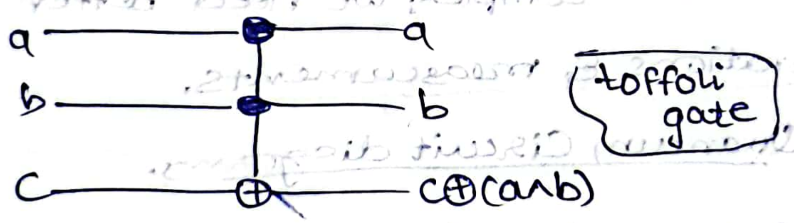


Eg3:

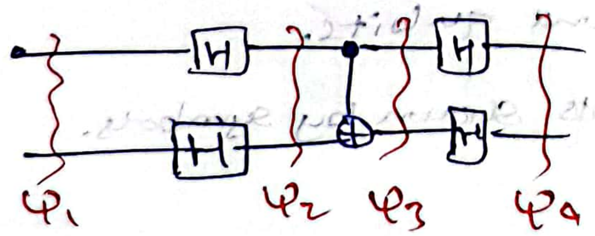


actually shown for classical inputs.

so,



Eg4:



$$\varphi_1 = |00\rangle$$

$$\varphi_2 = H|0\rangle \cdot H|0\rangle$$

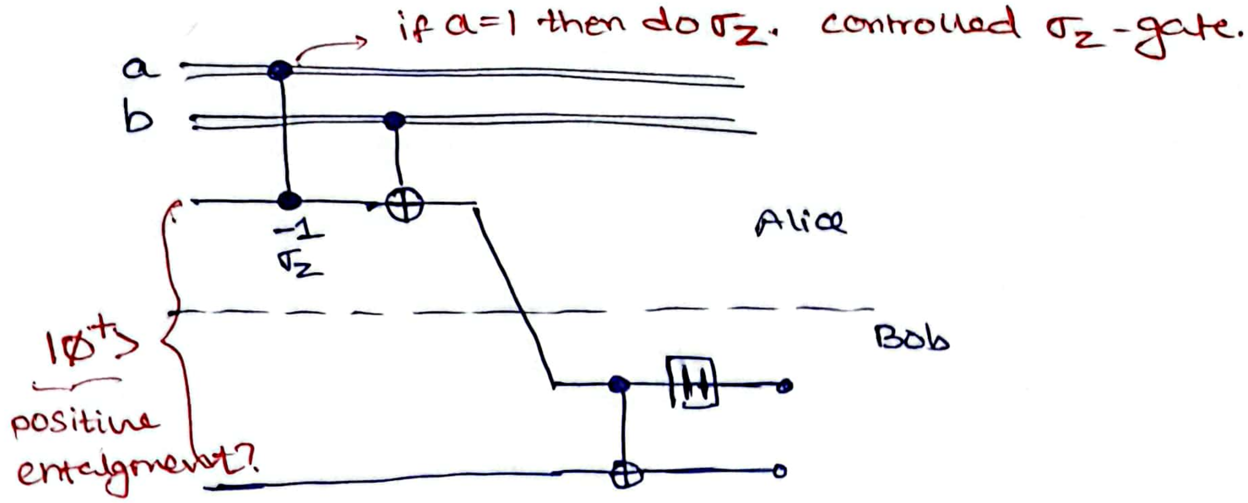
$$= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right).$$

$$\varphi_3 = \frac{1}{2}\left(|00\rangle + |01\rangle + |11\rangle + |10\rangle\right) = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\varphi_4 = (|0\rangle)(|0\rangle)$$

Egs: Superdense coding:-



Partial measurements:- easy!.

$$\alpha\langle 00| + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

when we measure both qbits; we'll finally be in a pure state

* Say we measured only 1st qbit.

∴ & say if $M_0 = 0$

then new State is $\alpha \underline{\langle 00| + \beta|01\rangle}$ typo

just renormalize this.

∴ new State

$$= \frac{\alpha|00\rangle + \beta|01\rangle}{\sqrt{|\alpha|^2+|\beta|^2}}$$] norm of this vector.

* Say we have a superposition $|\varphi\rangle = |0\rangle|\phi_0\rangle + |1\rangle|\phi_1\rangle$
& we measure the 1st qbit.

the new Superposition is

$$|\varphi_{new}\rangle = \frac{|\phi_0\rangle}{\|\phi_0\|} \text{ if 1st qbit is 0}$$

$$= \frac{|\phi_1\rangle}{\|\phi_1\|} \text{ if 1st qbit is 1.}$$

L4: Quantum teleportation, Deutsch's Algorithm:-

Quantum teleportation protocol:-

Alice has a qubit $= \alpha|0\rangle + \beta|1\rangle$; which is to be sent to bob.

How many classical bits communication needed?

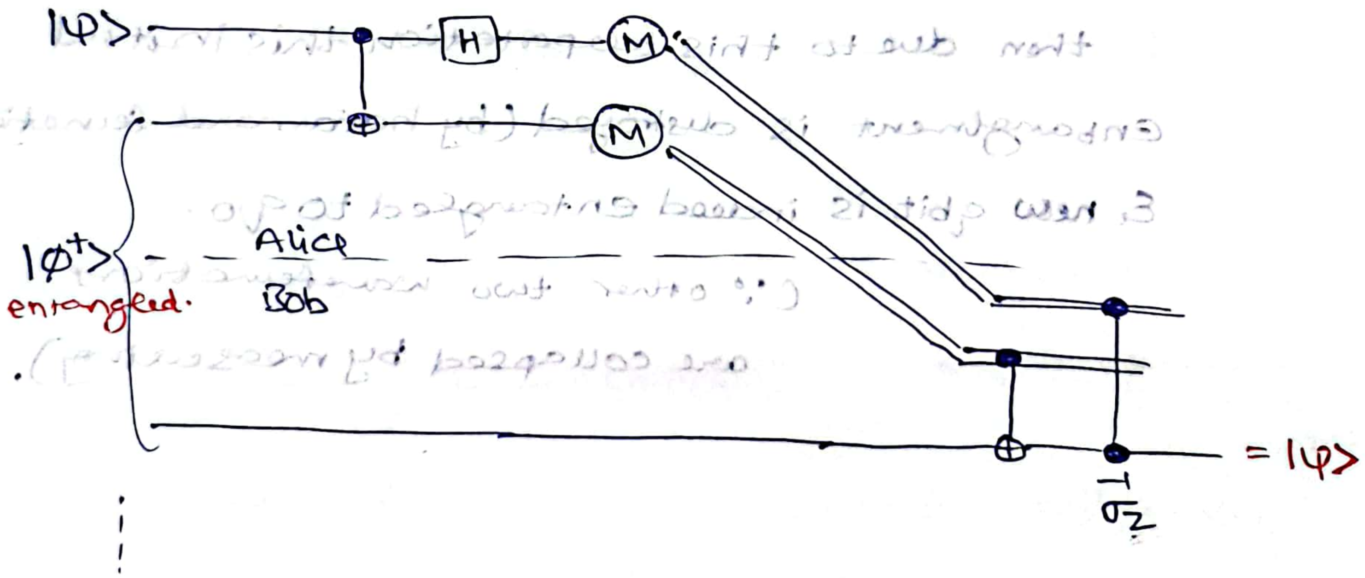
Ans 1:- as many as precision needed for α, β.

Ans 2:- dunno! α dunno! β dunno! can't perform experiment!

What to send!

* Now again; let Alice, bob share an e-bit.

Now 2 classical bits communicated.



initial state:-

$$= (\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

CNOT gate:- on qbit 2

$$= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|110\rangle + \frac{\beta}{\sqrt{2}}|101\rangle$$

H gate on qubit-1:-

$$= \frac{\alpha}{2}|000\rangle + \frac{\alpha}{2}|100\rangle + \frac{\alpha}{2}|011\rangle + \frac{\alpha}{2}|111\rangle + \frac{\beta}{2}|010\rangle - \frac{\beta}{2}|110\rangle + \frac{\beta}{2}|001\rangle - \frac{\beta}{2}|101\rangle$$

$$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$$

now measurment might give 00, 01, 10, 11.

but any case; our outcome finally is

$\alpha|0\rangle+\beta|1\rangle$

∴ Qubit teleported!

Something stronger is true.

* Here we wrote $\alpha|0\rangle+\beta|1\rangle$ as if the Alice's qubit is uncorrelated to anyother; but even if it is entangled,

the final transmitted qbit would be entangled with preserved.

∴ teleportation works like a perfect quantum channel.

Here ; if $|\psi\rangle$ was entangled with q_0;

then due to this teleportation this initial entanglment is destroyed (by hadamard function) & new qbit is indeed entangled to q_0.

(∵ other two wavefunctions are collapsed by measuring).

Deutsch's Algorithm:-

let $f:\{0,1\} \to \{0,1\}$

		f_0	f_1	f_2	f_3
Inp	0	0	0	1	1
	1	0	1	0	1

$f_0\ f_3 \to$ constant

$f_2 f_1 \to$ balanced.

In classic world;

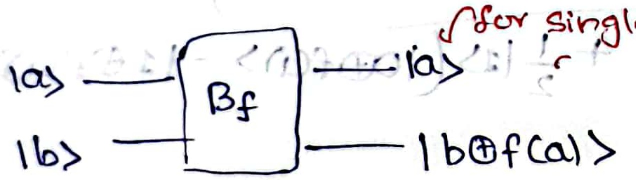
we need atleast two evaluations of f; to decide constant/balanced

but quantum information....

we won't have

Since $f_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not unitary.

lets have



✓for single classical state.

qubit 1 need not stay presented for quantum state.

now; for any f from $f_0 f_1 f_2 f_3$; we'll have unitary matrix as operation.

Claim: one evaluation of B_f is enough, to decide whether f is Constant or balanced.

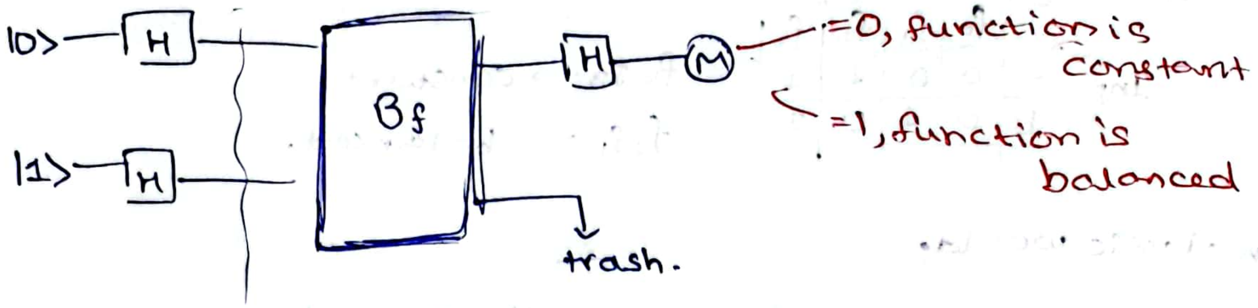
PTO

In general; for any $f:\{0,1\}^n \to \{0,1\}^m$ quantum transformation B_f is defined as

$B_f|x\rangle|y\rangle$

$= |x\rangle|y \oplus f(x)\rangle$

matrix is always a permutation & hence unitary.



$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

Since Bf's 2nd output is $|b \oplus f(a)\rangle$;

it will be ~~wiser~~ correct to choose a simple a.

$$|\psi\rangle = \frac{1}{2}|0\rangle(|0\rangle - |1\rangle) + \frac{1}{2}\cdot|1\rangle(|0\rangle - |1\rangle)$$

after Bf:

$$= (-1)^{f(0)}(|0\rangle - |1\rangle)$$

$$|\psi\rangle = \frac{1}{2}|0\rangle\left(|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle\right) + \frac{1}{2}|1\rangle\left(|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle\right)$$

$$= \frac{1}{2}(-1)^{f(0)}|0\rangle(|0\rangle - |1\rangle) + \frac{1}{2}(-1)^{f(1)}\cdot|1\rangle(|0\rangle - |1\rangle)$$

$$= \left(\frac{1}{2}(-1)^{f(0)}\cdot|0\rangle + \frac{1}{2}(-1)^{f(1)}|1\rangle\right)\underbrace{(|0\rangle - |1\rangle)}$$

Strange.... the 2nd qubit's state is preserved.

But; in our definition of Bf 1st qubit's state was supposed to be same!

$\therefore$ classically; 1st qubit preserved

quantumly; 2nd qubit preserved.

2. 1^{st} qubit after Hadamard

$$= \frac{1}{\sqrt{2}}(-1)^{f(0)} \cdot \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{2}(-1)^{f(1)}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{|0\rangle}{2}\left((-1)^{f(0)} + (-1)^{f(1)}\right) + \frac{|1\rangle}{2}\left((-1)^{f(0)} - (-1)^{f(1)}\right)$$

$\therefore$ if constant; $|0\rangle$ ✓

if ~~uniform~~ (balanced); $|1\rangle$ ✓

* There has been an important effect. The factors $(-1)^{f(0)}, (-1)^{f(1)}$ appeared in state of q_0 - Phase Kick-back

commonly used trick in Quantum algorithms.