APMA E2000 Midterm Exam 2

Department of Applied Physics & Applied Mathematics

Fall 2020

This exam is scheduled for 75 minutes, though you will have 120 minutes to upload your exams as a single PDF file to Gradescope. Present your work neatly and legibly; The Short Answer section may be on a single sheet, but each Free Response question should be on its own page. Do not submit work written on the exam paper itself. Late submissions will not be accepted. No notes or other outside materials are permitted with the exception of a scientific calculator. It is understood that by submitting the exam you affirm you have not discussed its contents with any living being.

Good luck to you.

Short Answer

(5 points each) Write your answers neatly. No justification is required, but a brief, reasoned explanation can earn partial credit in the case of incorrect answers.

SA1 Consider $f(x,y) = x^{2020} + y^{2018}$. For each of the following statements, state whether true or false. (No justification required.)

(a) $(0,0)$ is a saddle point.

(b) 0 is the global minimum of f.

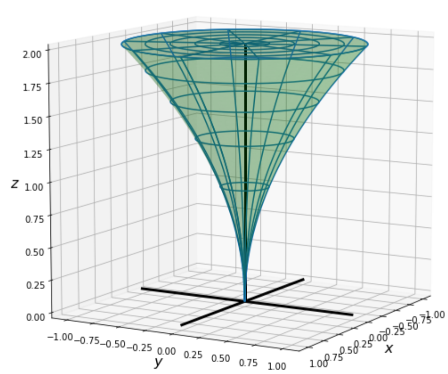
(c) $\nabla f(x,y)$ is always orthogonal to $f(x,y)$.

(d) The only critical point of f is $(0,0)$.

(e) f has no global maximum.

SA2 Consider $f(x,y,z) = x^2 + y^2$ constrained to the unit sphere $x^2 + y^2 + z^2 = 1$. At which point(s) does f obtain its maximum? No formal justification required.

SA3 Let E be the region graphed below. Without evaluating each integral, order the three integrals from least to greatest.

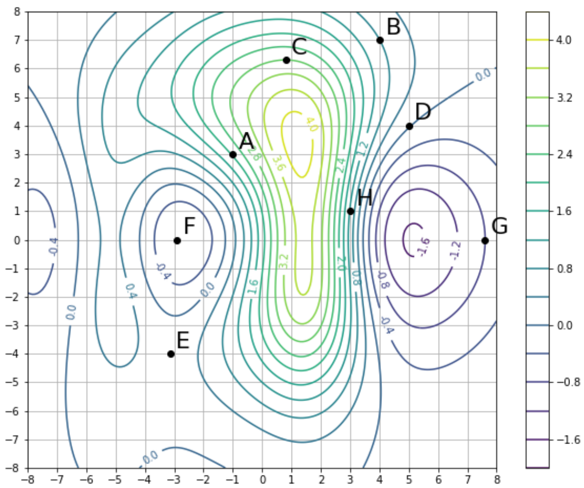


(a) $\iiint_E x\,dV$

(b) $\iiint_E z\,dV$

(c) $\iiint_E (2-z)\,dV$

SA4 Consider the following contour plot of a smooth function $f(x,y)$.



Match each description below with the appropriate marked point above. The same point can satisfy more than one description.

(a) A point where $|\nabla f|$ is as large as possible.

(b) A saddle point.

(c) A point where $f_x = 0$ but $f_y \neq 0$.

(d) A point where a local minimum occurs.

(e) A point where the gradient points "northeast" (i.e., is in the same direction as $\langle 1, 1\rangle$).

Free Response

Be sure to show all your work neatly and indicate your final answer where appropriate.

FR1 (20 points) Find the maximum and minimum values of $f(x,y) = 3x+4y$ on the closed annulus

$$1 \le x^2 + y^2 \le 4.$$

(A complete solution must show the whole process, including the method of Lagrange multipliers.)

FR2 (10 points each) Other coordinate systems

(a) Consider the integral

$$\int_{x=0}^{1} \int_{y=-\sqrt{4-x^2}}^{-\sqrt{1-x^2}} x\,dy\,dx.$$

- Sketch the domain of integration on a set of axes.
- Rewrite it as a sum of one or more iterated integrals in polar coordinates. (Be sure to translate the limits, the integrand, and the differentials correctly.)

(If needed, leave trigonometric expressions without evaluating them (i.e., $\sin(3), \arccos(1/3)$, etc.). You do not have to evaluate the integral.)

(b) Translate the following integral in spherical coordinates into an equivalent integral in rectangular (xyz) coordinates.

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{\frac{\pi}{2}} \int_{\rho=0}^{1} \rho^4 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

You need not evaluate it.

FR3 (20 points) Consider the surface (a hyperboloid of 1 sheet) given by

$$x^2 + y^2 = z^2 + 1.$$

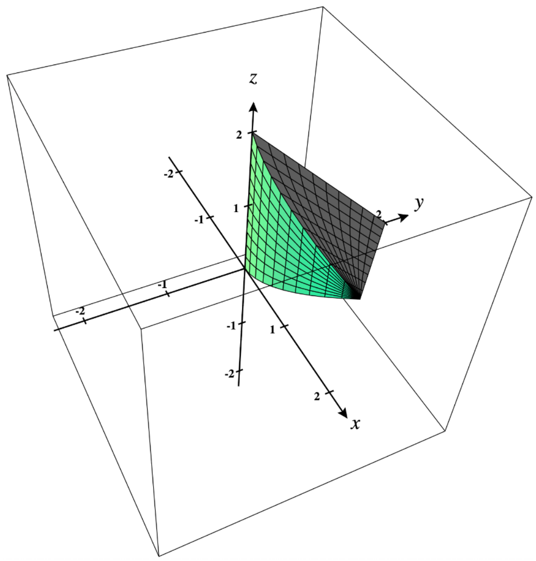
(a) Write an equation of the tangent plane to this surface at $(5,5,7)$.

(b) Find a unit vector $\mathbf{u}$ that satisfies **both** of the following:

- $\mathbf{u}$ is tangent to the surface at $(5,5,7)$, and
- the function $g(x,y,z) = x^2 + y^2 + z^2$ has directional derivative $D_{\mathbf{u}} g(5,5,7) = 0$.

FR4 (20 points) Consider the solid region E in the first octant described by

$$\begin{aligned} x+y+z &\leq 2 \\ x^2 &\leq y. \end{aligned}$$



Click here for a plot you can move around.

(a) Express the triple integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral 2 different ways:

i. in "$dx\,dy\,dz$" order, and

ii. in "$dy\,dz\,dx$" order.

(b) Find the volume of E.