UNIT-1: "Electrostatics"

1.1 Coulomb's Law and Electric Field:

This law states that the force of interaction between two point charges at rest is

(a) directly proportional to the product of magnitude of charges i.e. $F \propto qQ$

and (b) inversely proportional to the square of distance of their separation

i.e. $F \propto \frac{1}{r^2}$

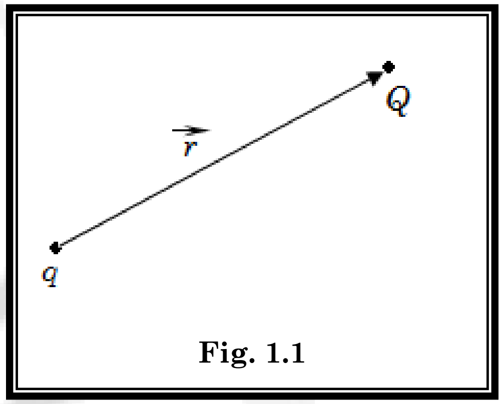


Fig. 1.1

which on combining gives

$$F \propto \frac{qQ}{r^2}$$

$$\text{or} \qquad F = \frac{1}{4\pi\varepsilon_o} \cdot \frac{qQ}{r^2} \qquad \text{----- (i)}$$

The constant ε_o is called the ***permittivity of free space.*** In SI units, $\varepsilon_o = 8.85 \times 10^{-12}\ C^2N^{-1}m^{-2}$. The charge q exerting force is called the *"source charge"* while the charge Q experiencing force is called the "*test charge*".

In vector form, $\vec{F} = \frac{1}{4\pi\varepsilon_o} \cdot \frac{qQ}{|\vec{r}|^2}\hat{r}$ ----- (ii)

Here $\vec{r}$ is position vector of Q w.r.t. q which is given as $\vec{r} = \vec{r}_Q - r_q$, so that Eq. (ii) takes the form

$$\vec{F} = \frac{1}{4\pi\varepsilon_o} \cdot \frac{qQ}{|\vec{r}_Q - \vec{r}_q|^2}\hat{r} \qquad \text{----- (iii)}$$

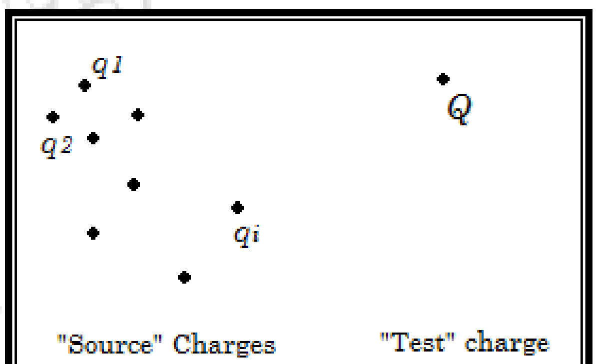


Fig. 1.2

NOTE: *The same amount of force is also experienced by the source charge because the Coulomb's forces obey Newton's Third law of motion.*

If we have a *Discrete Distribution of* several point charges $q_1, q_2, \ldots, q_i, \ldots, q_n$ *at distances* $r_1, r_2, \ldots, r_i, \ldots, r_n$ respectively from Q, the total force on Q is given by the superposition principle in electrostatics as

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots\ldots\ldots + \vec{F}_n = \frac{1}{4\pi\varepsilon_o}\cdot\left(\frac{q_1 Q}{r_1^2}\hat{r}_1 + \frac{q_2 Q}{r_2^2}\hat{r}_2 + \ldots\ldots + \frac{q_n Q}{r_n^2}\hat{r}_n\right)$$

$$= \frac{Q}{4\pi\varepsilon_o}\cdot\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \ldots\ldots + \frac{q_n}{r_n^2}\hat{r}_n\right)$$

or $$\vec{F} = Q\vec{E}$$

where $$\vec{E}(r_Q) = \frac{1}{4\pi\varepsilon_o}\sum_{i-1}^{n}\frac{q_i}{r_i^2}\hat{r}_i \quad \text{----- (iv)}$$

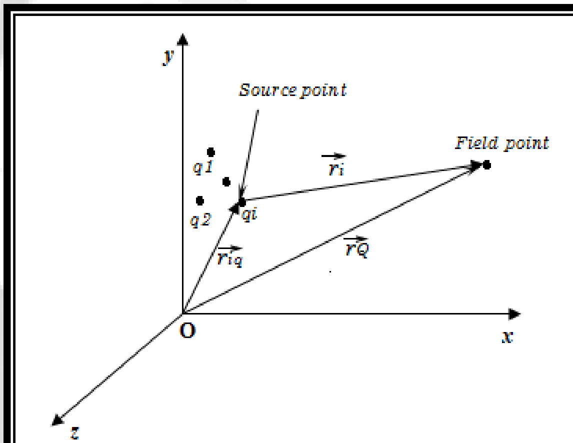


Fig. 1.3

Here $\vec{E}$ is the electric field of the source charges and is a function of position (r_Q).

<u>NOTE</u>: *Electric field $\vec{E}(r)$ is the force per unit charge that would be exerted on a test charge, if we were to place one at P.*

1.2 <u>Continuous Charge Distributions</u>:

If the charge is distributed continuously over some region, the sum in Eq. (iv) becomes an integral as

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int\frac{1}{r^2}\hat{r}dq \quad \text{----- (v)}$$

Depending upon whether the *region* is line, surface or volume, the charge distributions can be classified as

(a) Line Charge Distribution

(b) Surface Charge Distribution

(c) Volume Charge Distribution

For *Line Charge Distribution,* the charge is spread along a *line* with charge per unit length λ such that $dq = \lambda dl$, where dl is the element of length along the line. The electric field is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int_L \frac{\lambda}{r^2}\hat{r}\,dl \quad \text{----- (vi)}$$

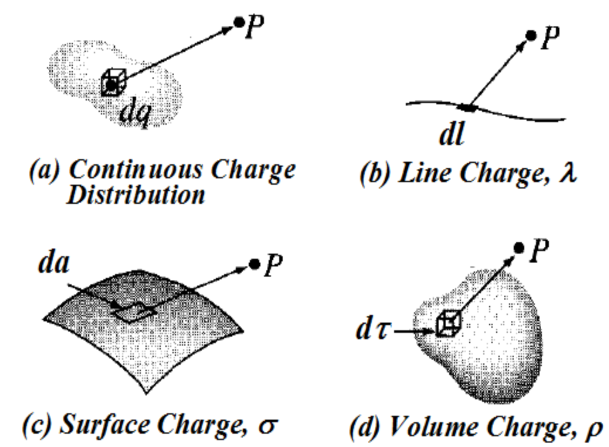


Fig. 1.4

For *Surface Charge Distribution*, the charge is smeared out over a *surface* with charge per unit area σ such that $dq = \sigma da$, where da is the element of area on the surface. The electric field is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma}{r^2}\hat{r}\,da \quad \text{----- (vii)}$$

For *Volume Charge Distribution*, the charge fills a *volume* with charge per unit volume ρ such that $dq = \rho d\tau$, where $d\tau$ is the element of volume. The electric field is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int_V \frac{\rho}{r^2}\hat{r}\,d\tau \quad \text{----- (viii)}$$

<u>NOTE</u>: *In all of the above cases, to calculate total force on a test charge Q due a given charge distribution, we simply multiply Q by* $\vec{E}$ *i.e.* $\vec{F} = Q\vec{E}$.

Ex. 1.1 ***Find the electric field at a distance z above the midpoint of a straight line segment of length 2L, which carries a uniform line charge λ (Fig. 1.5).***

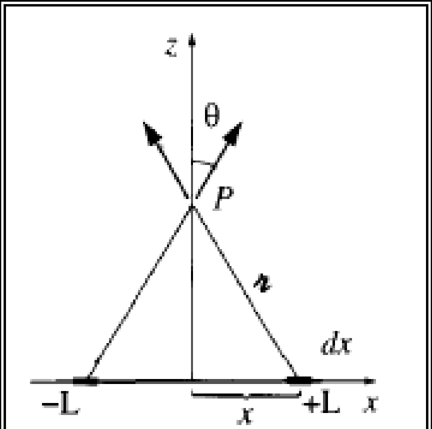


Fig. 1.5

Sol. We may divide the line into symmetrical placed pairs (at ±x), for then the horizontal components of the two fields cancel, and the net field of the pair is

$$d\vec{E} = 2\frac{1}{4\pi\varepsilon_o}\left(\frac{\lambda dx}{r^2}\right)\cos\theta\,\hat{z}$$

Here $\cos\theta = \dfrac{z}{r}$, $r = \sqrt{z^2 + x^2}$, and x runs from 0 to L.

$$\therefore \qquad E = \frac{1}{4\pi\varepsilon_o}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}dx$$

$$= \frac{2\lambda z}{4\pi\varepsilon_o}\left|\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\right|_0^L$$

$$= \frac{1}{4\pi\varepsilon_o}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$ and it acts in the z-direction.

For z >> L, this result becomes

$$E = \frac{1}{4\pi\varepsilon_o}\frac{2\lambda L}{z^2}$$

In the limiting case when L $\to\infty$, we obtain the field of an infinite straight wire:

$$E = \frac{1}{4\pi\varepsilon_o}\frac{2\lambda}{z}$$

Ex. 1.2 ***Find the electric field at a distance z above the center of a circular loop of radius a (Fig. 1.6), which carries a uniform line charge λ.***

Sol. We chop the circle into symmetrically placed pairs in opposition, for then the horizontal components of the two fields cancel, and the net field of the pair is

$$d\vec{E} = \frac{1}{4\pi\varepsilon_o}\left(\frac{\lambda dl}{r^2}\right)\cos\theta\,\hat{z}$$

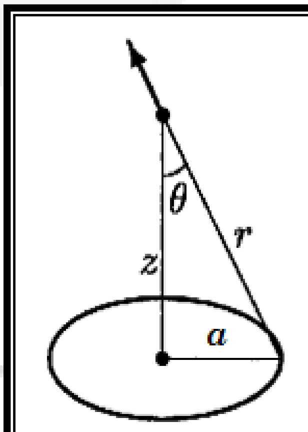


Fig. 1.6

Here $\cos\theta = \frac{z}{r}$, $r = \sqrt{z^2+a^2}$ (both constants), while $\int dl$ = $2\pi a$

Thus $$\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{\lambda(2\pi a)z}{(a^2+z^2)^{3/2}}\hat{z}$$

1.3 <u>The Operator</u> $\vec{\nabla}$:

$\vec{\nabla}$ is a vector operator which performs differentiation what follows it. It acts upon both vectors and scalars. It is read as "*del*" or "*nebla*". Mathematically,

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

When $\vec{\nabla}$ acts

(i) on a scalar ϕ, i.e. $\vec{\nabla}\phi$ gives *gradient* of ϕ which is a vector.

(ii) on a vector $\vec{v}$, via the dot product i.e. $\vec{\nabla}\cdot\vec{v}$ gives *divergence* of $\vec{v}$ which is a scalar.

(iii) on a vector $\vec{v}$, via the cross product i.e. $\vec{\nabla}\times\vec{v}$ gives *curl* of $\vec{v}$ which is a vector.

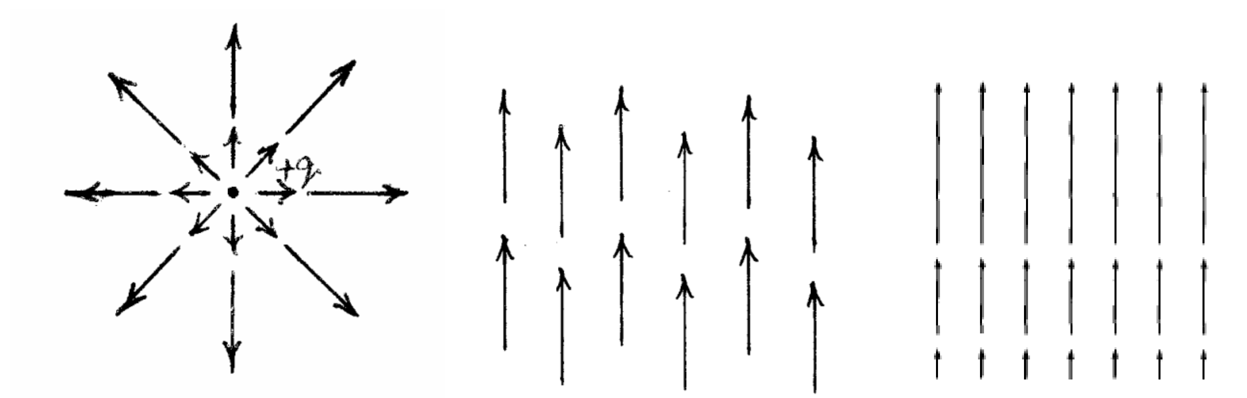
1.4 Divergence of a Vector Function:

Divergence of a vector function $\vec{v}$ is a "*scalar*" which *measures the degree of spreading of vector $\vec{v}$ from a point.* Mathematically, divergence of $\vec{v}$ is written as

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Fig. 1.7(a) shows a vector function with large *positive* divergence; Fig. 1.7 (b)



(a) **(b)** **(c)**

Fig. 1.7

shows the function with *zero* divergence while Fig. 1.7 (c) represents the function with *negative* divergence.

Note: *A point of positive divergence is a "source"; a point of negative divergence is a "sink".*

1.4.1 The Fundamental Theorem for Divergences:

***Statement*:** According to this theorem,

"The v*olume integral of divergence of a vector function is equal to the flux of the function over the surface bounding the volume.*"

If $\vec{v}$ is the vector function whose divergence within volume V bounded by the closed surface S is $\vec{\nabla}\cdot\vec{v}$, then

$$\int_V \vec{\nabla}\cdot\vec{v}\,d\tau = \oint_S \vec{v}\cdot d\vec{a}$$

i.e. $\int$ (divergence within the volume) = $\oint$ (net outflow through the surface)

Note: *This theorem gives the way to transform from volume integral to the surface integral for a vector function.*

1.4.2 The Divergence of $\hat{r}/r^2$:

Let us consider a vector function

$$\vec{v} = \frac{1}{r^2}\hat{r} \qquad \text{----(i)}$$

which has positive divergence such that $\vec{v}$ is directed radially outward (Fig. 1.8). Now divergence of $\vec{v}$ is given by

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \qquad \text{----(ii)}$$

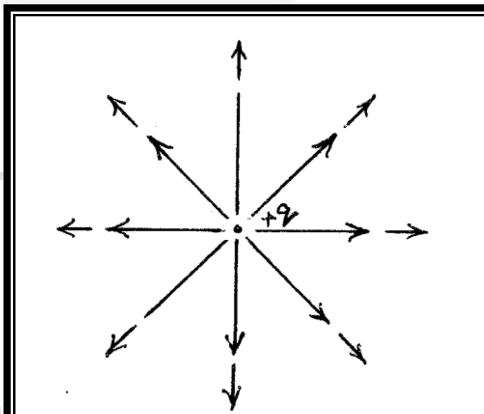


Fig. 1.8

This yields the volume integral $\int_V \vec{\nabla}\cdot\vec{v} = 0$ ----(iii)

Now if we integrate over a sphere of radius R, centered at the origin, the surface integral is

$$\oint_S \vec{v}\cdot d\vec{a} = \int\left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\sin\theta\, d\theta\, d\phi\,\hat{r}\right) = \left(\int_0^{\pi}\sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi \quad \text{----(iv)}$$

This is a paradox which invalidates the *Gauss's Divergence Theorem*.

Actually, here at point r = 0, the vector function $\vec{v}$ blows up. For this reason $\vec{\nabla}\cdot\vec{v}$ vanishes everywhere except at one point, and yet its *integral* over any volume containing that point is 4π.

1.5 <u>The Curl of a Vector Function</u>:

Curl of a vector function $\vec{v}$ is a "*vector*" which *measures the degree of curling of the vector* $\vec{v}$ *about a point*. Mathematically, curl of $\vec{v}$ is written as

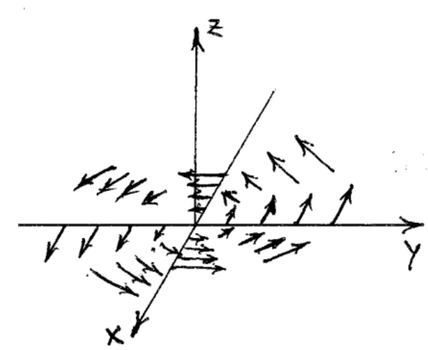


Fig. 1.9

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

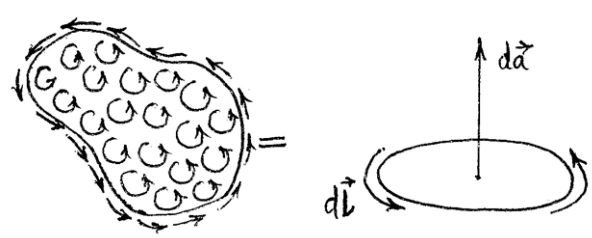
Fig. 1.9 shows a vector function which has non-zero curl, pointing in the z-direction.

1.5.1 <u>The Fundamental Theorem of Curls</u>:

<u>Statement</u>: According to this theorem,

"*The surface integral of curl of a vector function is equal to the line integral of the function around the closed path.*"

If $\vec{v}$ is the vector function whose curl over the closed curve bounding surface S (Fig. 1.10) is $\vec{\nabla}\times\vec{v}$, then



(a) **(b)**

Fig. 1.10

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_C \vec{v}\cdot d\vec{l}$$

1.6 <u>Electric Field Lines, Electric Flux and Gauss's Law</u>:

1.6.1 <u>Electric Field Lines</u>: If a small test charge $q_o \to 0$ is placed in an electric field $\vec{E}$, then it will move along a curve such that the tangent to the curve at every point gives the direction of electric field $\vec{E}$ at that point. Such a line (or curve) is known as *electric field line* or *line of electric force* (Fig. 1.11).

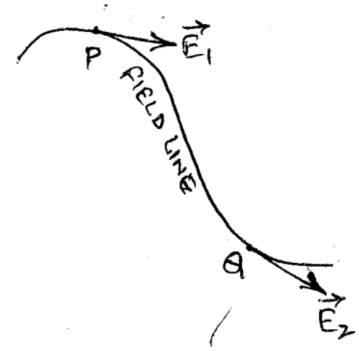


Fig. 1.11

Suppose we have a point charge q and we draw a sphere of radius r around the point, then magnitude of electric field strength at any point on the spherical surface is

$$E = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2} = \frac{q/\varepsilon_o}{4\pi r^2} = \frac{\textit{No. of electric field lines}}{\textit{Area of the sphere}}$$

$= \textit{No. of electric lines of force per unit area}$

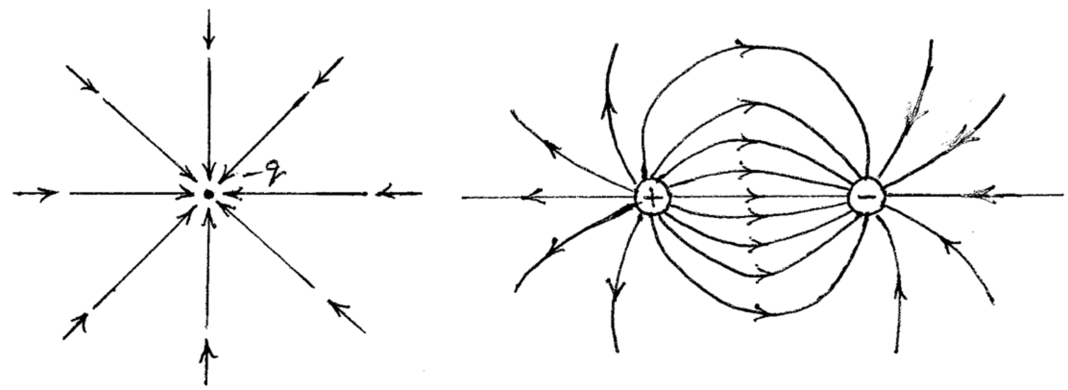
The strength of the electric field at a point is, therefore, taken to be proportional to *the number of lines crossing a unit area around that point.*

<u>**Note**</u>: ***The larger the field lines per unit area the greater the electric field strength.***

Following points are worth noting:

1. For a +ve point charge, the electric field lines act radially outwards and for a –ve point charge, these are directed radially inwards (Fig.1.12(a)).
2. Field lines are strong near the charge and grow weaker as we move farther out.
3. They originate from positive charge and terminate at the negative charge (Fig. 1.12(b)).

4. No two field lines can intersect because if it were so then at the point of intersection there will be two different directions of electric field which is impossible.



(a) **(b)**

Fig. 1.12

5. The field lines have longitudinal tension. This fact accounts for the attraction between unlike charges.
6. The field lines have lateral pressure. This fact accounts for the repulsion between like charges.
7. The tangent drawn to the field line at any point gives the direction of the electric field at that point.

1.6.2 <u>Electric Flux</u>: *This is defined as the number of electric field lines passing through a given area in a direction perpendicular to the surface. It is denoted by ϕ_E and is a scalar quantity.*

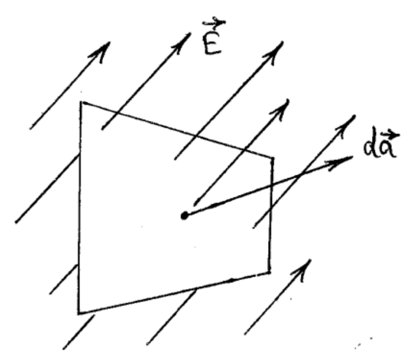


Fig. 1.13

Since $\boldsymbol{Electric\ Field} = \dfrac{\boldsymbol{No.\ of\ electric\ field\ lines}}{\boldsymbol{Area}}$

$\Rightarrow$ *No. of electric field lines = Electric Field × Area*

Mathematically, *electric flux ϕ_E through an area S is calculated as the product of the area and the component of electric field vector $\vec{E}$* normal to the area (Fig. 1.13).

i.e. $$\phi_E = \int_S \vec{E}\cdot d\vec{a}$$

ϕ_E is expressed in $\mathrm{Nm^2C^{-1}}$.

<u>**Note**</u>: $\vec{E}\cdot d\vec{a}$ ***is proportional to the number of lines passing through the infinitesimal area*** $d\vec{a}$***.***

1.6.3 <u>Gauss's Law</u>: This law states that

"The total electric flux over a closed surface S enclosing volume V in vacuum is equal to $\frac{1}{\varepsilon_o}$ *times the total charge enclosed by the surface."*

i.e. $$\phi_E = \oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_o}Q_{enc} \qquad \text{--- (i)}$$

<u>**Note**</u>: ***There is no contribution to the electric flux due to any charge lying outside the closed surface.***

The relation (i) can be proved as follows for a spherical surface (Fig.1.14):

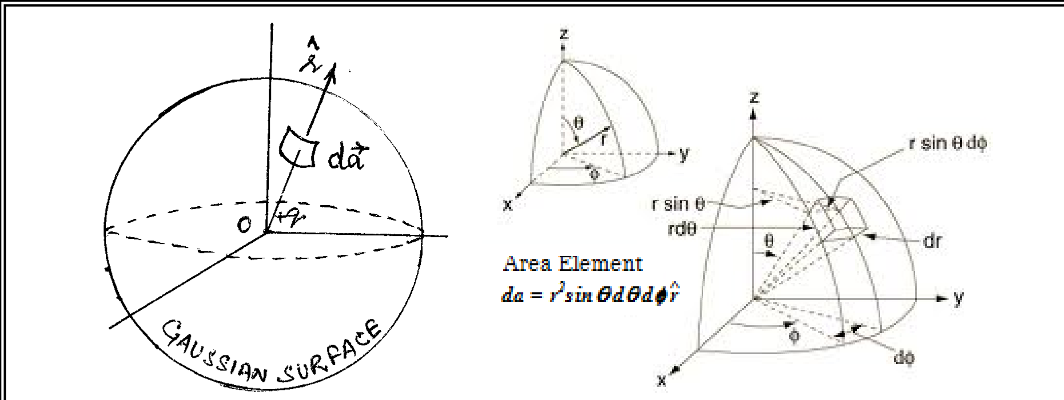


Fig. 1.14

$$\phi_E = \oint_S \vec{E}\cdot d\vec{a} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\hat{r}\right)\cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$= \frac{q}{4\pi\varepsilon_o}\left(\int_0^{\pi}\sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = \frac{q}{4\pi\varepsilon_o}\cdot 4\pi = \frac{q}{\varepsilon_o}.$$ Hence proved. --- (ii)

If there are a bunch of charges scattered about, then according to the principle of superposition, the total field is the vector sum of all the individual fields:

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i \qquad \text{--- (iii)}$$

The flux through the surface that encloses them all is, therefore,

$$\phi_E = \oint_S \vec{E} \cdot d\vec{a}$$

$$= \sum_{i=1}^{n} \left(\oint_S \vec{E}_i \cdot d\vec{a} \right) = \sum_{i=1}^{n} \left(\frac{1}{\varepsilon_o} q_i \right) \qquad \text{--- (iv)}$$

$\therefore$ for any closed surface,

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_o} Q_{enc} \qquad \text{---- (v)}$$

1.7 Divergence and Curl of Electrostatic Fields:

1.7.1 Divergence of $\vec{E}$:

Eq. (v) represents the Gauss's Law in *integral form*, but it can be turned into a *differential form*, by applying the *Gauss's Divergence Theorem* as follows:

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{E})\, d\tau \qquad \text{.... (vi)}$$

Rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{enc} = \int_V \rho\, d\tau \qquad \text{---- (vii)}$$

So that Gauss's Law becomes

$$\int (\vec{\nabla} \cdot \vec{E})\, d\tau = \int_V \left(\frac{\rho}{\varepsilon_o} \right) d\tau \qquad \text{---- (ix)}$$

Since the integrand is arbitrary and the integral holds for any volume

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_o} \cdot \rho \qquad \text{---- (x)}$$

This gives the ***Differential form of Gauss's Law.***

1.7.2 Curl of $\vec{E}$:

We have the electric field due to a charge q at a point is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_o} \cdot \frac{q}{r^2}\hat{r} \qquad \text{---- (i)}$$

The line integral of electric field from some point A to some other point B is given by

$$\int_A^B \vec{E} \cdot d\vec{l}$$

In spherical polar co-ordinates, $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$, so

$$\vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_o} \cdot \frac{q}{r^2} dr$$

Therefore,

$$\int_A^B \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_o} \cdot \int_A^B \frac{q}{r^2} dr = -\frac{1}{4\pi\varepsilon_o} \cdot \frac{q}{r}\Bigg|_{r_A}^{r_B} = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r_A} - \frac{q}{r_B}\right) \qquad \text{---- (i)}$$

where r_A is the distance from the origin to the point A and r_B is the distance to (B) The integral around a *closed* path is zero because for then $r_A = r_B$.

i.e. $$\oint \vec{E}.d\vec{l} = 0 \qquad \text{---- (ii)}$$

Applying the *Stoke's theorem*, we have

$$\vec{\nabla} \times \vec{E} = 0 \qquad \text{---- (iii)}$$

Thus Curl of $\vec{E}$ vanishes. This is because the electric field lines are discontinuous curves starting at positive charge and terminating at negative charge.

If we a discrete distribution of charges q_1, q_2, q_3, ……, q_n, the principle of superposition gives the total electric field as a vector sum of individual fields due to charges:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots\ldots + \vec{E}_n \qquad \text{---- (iv)}$$

so that

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{E}_1 + \vec{\nabla} \times \vec{E}_2 + \ldots\ldots\ldots + \vec{\nabla} \times \vec{E}_n = 0 \qquad \text{---- (v)}$$

1.8 Applications of Gauss's Law:

1.8.1 Electric Field due to a Uniformly Charged Solid Sphere of Radius R and Total Charge *q*:

As can be seen from the Fig. 1.15, the Gaussian surface is a sphere of radius r > R surrounding the charge distribution. Now according to Gauss's law

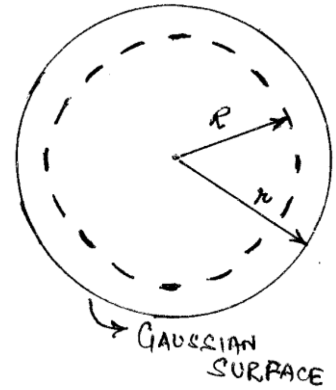


Fig. 1.15

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_o}\cdot Q_{enc} \quad \text{---- (i)}$$

and $Q_{enc} = q$. Here, $\vec{E}$ points radially outward as does $d\vec{a}$. So that the dot product $\vec{E}\cdot d\vec{a} = |\vec{E}|da$ and the integral becomes

$$\int_S \vec{E}\cdot d\vec{a} = \int_S |\vec{E}|da = |\vec{E}|\int_S da = |\vec{E}|4\pi r^2$$

Thus $$|\vec{E}|4\pi r^2 = \frac{1}{\varepsilon_o}q$$

or $$\vec{E} = \frac{1}{4\pi\varepsilon_o}\cdot\frac{q}{r^2}\hat{r} \quad \text{---- (ii)}$$

It can be seen that the field outside the sphere is exactly the *same as it would have been if all the charge had been concentrated at the center.*

Note: *Gaussian surface is assumed based on symmetry of charge distribution. For* ***spherical symmetry****, Gaussian surface is* ***concentric sphere****; for* ***cylindrical symmetry****, it is* ***coaxial cylinder****; and for* ***plane symmetry****, it is "****pillbox****" which pierces through the surface.*

1.8.2 Electric Field at Any Point due to an Infinite Sheet carrying a Uniform Surface Charge:

Let 'σ' be the surface charge density of the sheet. Draw the "Gaussian pillbox", extending equal distances above and below the plane as shown in Fig. 1.16. Apply Gauss's law to this surface, we have

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_o}\cdot Q_{enc} \qquad \text{---- (i)}$$

In this case, $Q_{enc} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, $\vec{E}$ points away from the plane (upward for points above, downward for points below). Thus, the top and bottom surfaces yield

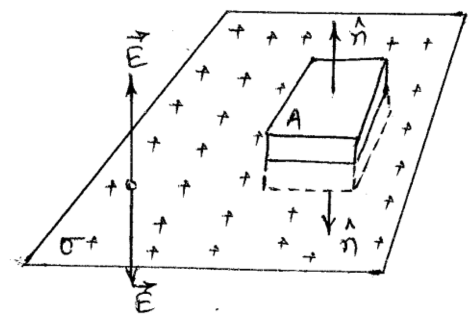


Fig. 1.16

$$\int \vec{E}\cdot d\vec{a} = 2A\left|\vec{E}\right|$$

whereas the sides contribute noting. Thus

$$2A\left|\vec{E}\right| = \frac{1}{\varepsilon_o}\sigma A$$

or
$$\vec{E} = \frac{\sigma}{2\varepsilon_o}\hat{n} \qquad \text{---- (ii)}$$

where $\hat{n}$ is a unit vector pointing away from the surface.

1.8.3 <u>Electric Field due to a Uniformly Charged Hollow Cylinder</u>:

For this case, we shall choose the Gaussian surface to be cylindrical coaxial with the charged hollow cylinder. Now the field at any outside point P at a distance r from the axis of the cylinder is then given by

$$E = \frac{\lambda}{2\pi\varepsilon_o r} \qquad \text{---- (i)}$$

where λ is the charge per unit length of the cylinder. Let P lie infinitely close to the surface of the cylinder so that r can be taken to be the radius of the cylinder. (Fig. 1.17). Let l be the length of the cylinder enclosed by the Gaussian surface. Then

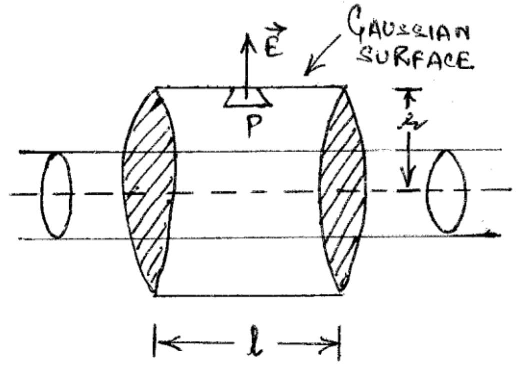


Fig. 1.17

surface area of the enclosed cylinder = $2\pi rl$

Let surface density of charge = σ

Charge on enclosed cylinder = $2\pi rl\sigma$

Charge per unit length of the cylinder = $\frac{2\pi rl\sigma}{l} = 2\pi r\sigma = \lambda$

$\therefore$ $$E = \frac{\lambda}{2\pi\varepsilon_o r} = \frac{2\pi r\sigma}{2\pi\varepsilon_o r} = \frac{\sigma}{\varepsilon_o}$$

or $$E = \frac{\sigma}{\varepsilon_o} \qquad \text{---- (ii)}$$

The field at an inside point will be zero as a Gaussian surface through this point will include no charges.

1.8.4 Electric Field due to a Uniformly Charged Solid Cylinder:

If the charge is uniformly distributed throughout the volume of an infinitely long solid cylinder, the electric field at any outside point is given by

$$E = \frac{\lambda}{2\pi\varepsilon_o r} \qquad \text{---- (i)}$$

Here λ is the charge per unit length of the cylinder and is given by

$$\lambda = \pi r^2 \rho \qquad \text{---- (ii)}$$

where ρ is the charge per unit volume of the cylinder

$\therefore$

$$E = \frac{\pi r^2 \rho}{2\pi\varepsilon_o r} = \frac{r\rho}{2\varepsilon_o} \qquad \text{---- (iii)}$$

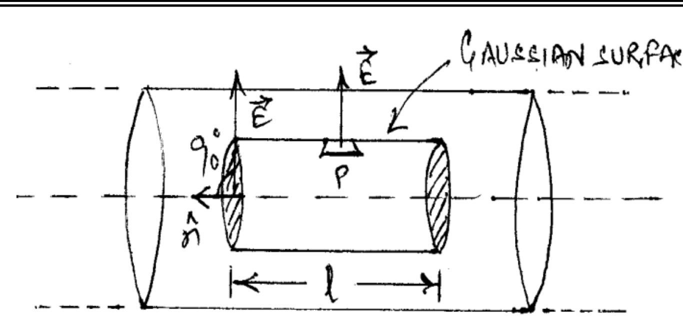


Fig. 1.18

The electric field at any point inside the cylinder can be determined as follows:

Imagine a Gaussian surface in the form of a cylinder of length l, co-axial with the charged cylinder and passing through P at which the electric field is to be determined (Fig. 1.18). The field will be normal to the curved surface and will have equal magnitude at all points lying on it.

Contribution of the curved surface towards electric field

$$\phi_E = \oint_S \vec{E} \cdot \hat{n} da = E \oint da = E \cdot 2\pi r l \quad \text{---- (iv)}$$

where $2\pi rl$ is the area of the curved surface of the cylinder. On the edges, $\vec{E}$ and $\hat{n}$ are at right angles to each other so that $\vec{E} \cdot \hat{n} = 0$. Hence the edges do not contribute to the electric flux.

$\therefore$ total flux over the entire closed surface is given by

$$\phi_E = E \cdot 2\pi r l \quad \text{---- (v)}$$

But from Gauss's law

$$\phi_E = \frac{q}{\varepsilon_o} = \frac{\pi r^2 l \rho}{\varepsilon_o}$$

where ρ is the charge per unit volume of the cylinder.

$\therefore$
$$E(2\pi r l) = \frac{\pi r^2 l \rho}{\varepsilon_o}$$

or
$$E = \frac{r\rho}{2\varepsilon_o}$$

<u>Note</u>: *It shows that the electric field inside a charged cylinder is directly proportional to the distance of the point from the axis of the cylinder.*

1.8.5 <u>Electric Field on the Surface of a Charged Conductor</u>:

Consider a surface S of a charged conductor having a surface charge density σ, placed in vacuum (Fig. 1.19). Let us find the electric field $\vec{E}$ at the surface of the conductor.

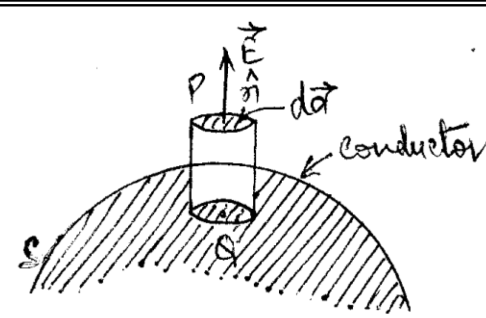


Fig. 1.19

Construct an imaginary cylinder as Gaussian surface, with the caps P and Q equidistant from but *infinitesimally* close to the surface. Let the cross-section area of the cylinder be *da*.

Electric flux over the cap P = $\vec{E}\cdot\hat{n}da = E\,da$ $(\because \vec{E}\ ||\ \text{to}\ \hat{n})$

Since the electric field inside the conductor is zero, the electric flux over the cap Q is zero. Again, as $\vec{E}$ is parallel to the curved cylindrical surface, the flux over the curved surface is also zero.

Hence the total flux ϕ over the entire cylindrical surface is given by $\phi = E\,da$ ---- (i)

From Gauss's law, $\phi = \dfrac{q}{\varepsilon_o} = \dfrac{\sigma\,da}{\varepsilon_o}$ ---- (ii)

From equations (i) and (ii),

$$E\,da = \frac{\sigma\,da}{\varepsilon_o}$$

or

$$E = \frac{\sigma}{\varepsilon_o} \qquad \text{---- (iii)}$$

Thus the *magnitude of electric field* $\vec{E}$ *at any point infinitesimally close to a charged conducting surface is equal to* $\dfrac{1}{\varepsilon_o}$ *times the surface density of charge.* This is called ***Coulomb's theorem***.

1.9 <u>Electric Potential</u>:

1.9.1 <u>Electric Potential and Electric Potential Difference</u>:

Electric potential at a point is defined as *the amount of work done in moving a unit positive test charge from infinity to that point, without acceleration, against electrical forces due to charge distribution.* Mathematically,

$$V_P = \frac{W_{\infty P}}{q_o} \qquad \text{---- (i)}$$

It is expressed in JC^{-1} or volt (V). It is a scalar quantity.

If V_A and V_B are the potentials of two points A and B in the electrostatic field of some charge distribution, then potential difference between two points is given by

$$V_A - V_B = \frac{W_{AB}}{q_o} \qquad \text{---- (i)}$$

It is also a scalar quantity and is expressed in JC^{-1} or volt (V).

1.9.2 <u>Electric Potential Difference as Line Integral of Electric Field</u>:

Let A and B be two points situated in an electric field set up by some far off stationary distribution of point charges. Let a positive charge q_o be moved without acceleration along the path AB in the field. (Fig. 1.20)

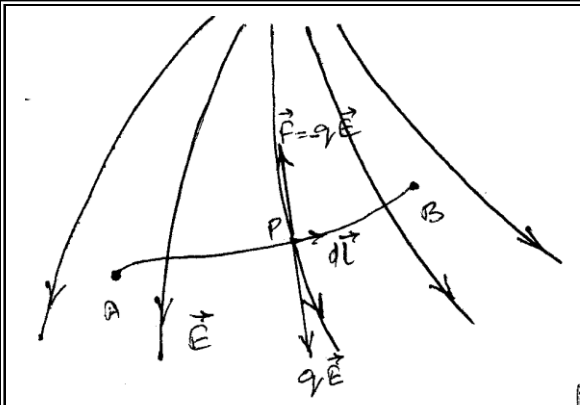


Fig. 1.20

If $\vec{E}$ is the electric field intensity at any point P, the test charge q_o placed at P shall experience a force $q_o\vec{E}$ in the direction of the field. In order that the test charge may not get accelerated along the direction of the field, an external agent must apply a force $\vec{F}$ exactly equal to $-q_o\vec{E}$ for all positions of the test charge. Let the external agent cause the test charge to move through a small displacement $d\vec{l}$ along the path AB. The elementary work done by the external agent is then given by

$$dW = \vec{F}\cdot d\vec{l} \qquad \text{---- (i)}$$

The total amount of work done in moving the test charge along the path AB is found by integrating Eq. (i) on both sides within the path length from A to B

$$\text{i.e.} \qquad W_{AB} = \int_A^B \vec{F}\cdot d\vec{l} \qquad \text{---- (ii)}$$

$$= -q_o\int_A^B \vec{E}\cdot d\vec{l} \qquad (\because\ \vec{F} = -q_o\vec{E})$$

Now, since $$V_B - V_A = \frac{W_{AB}}{q_o}$$

$\therefore$ potential difference, $$V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} \qquad \text{---- (ii)}$$

<u>Note</u>: *The potential difference between any two points A and B in an electric field is given by the negative of the line integral of the electric field along the path AB.*

If the point A were taken at infinity, V_A becomes zero so that the Eq. (ii) gives the potential V_B. Thus

$$V_B = -\int_\infty^B \vec{E}\cdot d\vec{l} \qquad \text{---- (iii)}$$

More generally, we can write

$$V(r) = -\int_\infty^r \vec{E}\cdot d\vec{l} \qquad \text{---- (iv)}$$

<u>Note</u>: *V depends only on the point* $\vec{r}$.

1.9.3 <u>Relation between Electric Field and Electric Potential</u>:

We know that the electric potential difference between two points A and B is

$$V_B - V_A = -\int_\infty^B \vec{E}\cdot d\vec{l} + \int_\infty^A \vec{E}\cdot d\vec{l} \qquad \text{---- (v)}$$

$$= -\int_\infty^B \vec{E}\cdot d\vec{l} - \int_\infty^A \vec{E}\cdot d\vec{l} = -\int_A^B \vec{E}\cdot d\vec{l} \qquad \text{---- (vi)}$$

Now, the fundamental theorem of gradients states that

$$V_B - V_A = \int_A^B (\vec{\nabla}V)\cdot d\vec{l} \qquad \text{---- (vii)}$$

so $$\int_A^B (\vec{\nabla}V)\cdot d\vec{l} = -\int_A^B \vec{E}\cdot d\vec{l}$$

Since this is true for any points A and B, the integrands must be equal:

$$\vec{E} = -\vec{\nabla} V \qquad \text{---- (viii)}$$

Note: *The electric field is negative of the gradient of the scalar potential.*

This suggests that if curl of $\vec{E}$ is taken then $\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla} V) = 0$. Such fields like $\vec{E}$ whose curl vanishes are called as *conservative fields*. Conservative fields have two important properties:

1. The work done on charges in moving them from one point to the other in the conservative field is independent of the path and depends only on the *initial* and *final* points.
2. The work done on the charges round a *closed trip* in the conservative field is always *zero*.

1.9.4 Poisson's Equation and Laplace's Equation:

We know that electric field can be written as the gradient of a scalar potential as

$$\vec{E} = -\vec{\nabla} V \qquad \text{---- (i)}$$

From the differential form of Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o} \qquad \text{---- (ii)}$$

Putting for $\vec{E}$ from Eq. (i) into Eq. (ii), we have

$$\vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{\rho}{\varepsilon_o}$$

or

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o} \qquad \text{---- (iii)}$$

This is known as ***Poisson's Equation***. It relates the charge density to the second derivative of potential.

In regions where there is no charge, so that $\rho = 0$, Poisson's equation reduces to ***Laplace's equation***.

i.e. $$\nabla^2 V = 0 \qquad \text{---- (iv)}$$

1.9.5 The Potential of a Localized Charge Distribution:

We know that the potential at any point $\vec{r}$ is given by

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} \quad \text{---- (i)}$$

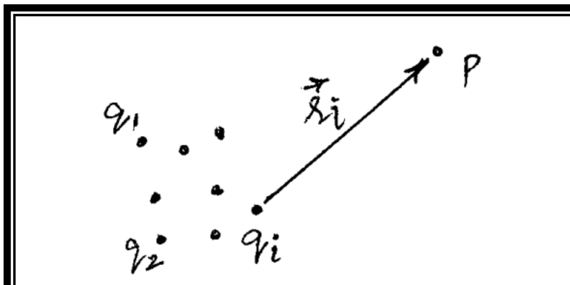


Fig. 1.21

Now electric field at any point $\vec{r}$ due to a point charge q at the origin is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\cdot\frac{q}{r'^2}\hat{r}' \quad \text{---- (ii)}$$

and $$d\vec{l} = dr'\hat{r}' + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \quad \text{---- (iii)}$$

so that $$V(r) = -\frac{1}{4\pi\varepsilon_o}\int_{\infty}^{r}\frac{q}{r'^2}dr' = \frac{1}{4\pi\varepsilon_o}\cdot\frac{q}{r}\Big|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_o}\cdot\frac{q}{r} \quad \text{---- (iv)}$$

To generalize for distribution of n charges (Fig.1.21), the potential is

$$V(r) = \frac{1}{4\pi\varepsilon_o}\cdot\sum_{i=1}^{n}\frac{q_i}{r_i} \quad \text{---- (v)}$$

and for a continuous distribution,

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int\frac{1}{r}dq \quad \text{---- (vi)}$$

In particular, for a line charge, it is

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int\frac{\lambda(r')}{r}dl'$$

for a surface charge

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int\frac{\sigma(r')}{r}da' \quad \text{---- (vii)}$$

and for volume charge

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(r')}{r}d\tau' \quad \text{---- (viii)}$$

1.9.6 Work Done in Assembling the Charges:

Consider a stationary configuration of source charges (Fig.1.22) and we want to move a test charge q_o from point A to point B in its electric field. Then as discussed above, the work done carrying the charge through an arbitrary path AB is given by

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{l} = -q_o\int_A^B \vec{E}\cdot d\vec{l} \qquad \text{---- (i)}$$

or

$$W_{AB} = q_o(V_B - V_A) \qquad \text{---- (ii)}$$

In particular, if charge q_o is brought from ∞ to a point $\vec{r}$, the work done will be

$$W = q_o[V(\vec{r}) - V(\infty)] \qquad \text{---- (iii)}$$

and if the reference point is set at infinity

$$W = q_oV(\vec{r}) \qquad \text{---- (iv)}$$

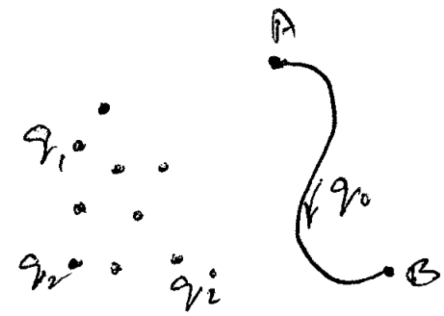


Fig. 1.22

<u>Note</u>: *This potential is potential energy (or work it takes to create the system) per unit charge.*

1.9.7 <u>Energy of a Point Charge Distribution</u>:

Consider a discrete distribution of n point charges $q_1, q_2, \ldots\ldots, q_n$ at positions $\vec{r}_1, \vec{r}_2, \ldots\ldots, \vec{r}_n$ respectively (Fig.1.23). Let us bring the charges one by one from infinity as follows:

It takes *no work* to carry charge q_1 since there is no field yet to fight against.

$$\therefore \qquad W_1 = 0 \qquad \text{---- (i)}$$

When charge q_2 is carried from infinity to point $\vec{r}_2$, then amount of work done will be $q_2V_1(\vec{r}_2)$, where V_1 is the potential due to q_1 at position $\vec{r}_2$.

Thus

$$W_2 = \frac{1}{4\pi\varepsilon_o}\cdot q_2\left(\frac{q_1}{r_{12}}\right) \qquad \text{---- (ii)}$$

where r_{12} is the distance between q_1 and q_2 once they are in position.

When charge q_3 is brought in the field of q_1 and q_2, the work that is required to be done will be $q_3 V_{12}(\vec{r}_3)$, where V_{12} is the potential due to charges q_1 and q_2 and is equal to $(1/4\pi\varepsilon_o)(q_1/r_{13} + q_2/r_{23})$.

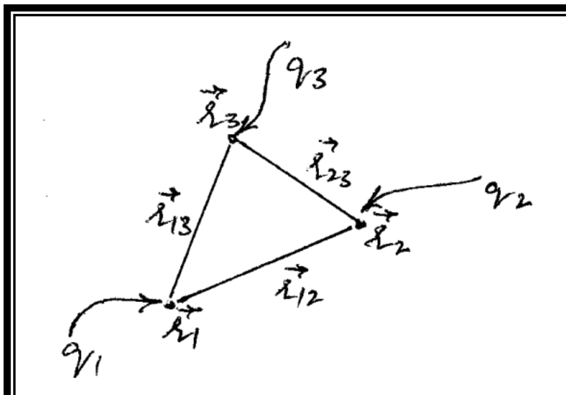


Fig. 1.23

Thus, $W_3 = \dfrac{1}{4\pi\varepsilon_o} \cdot q_3 \left(\dfrac{q_1}{r_{13}} + \dfrac{q_2}{r_{23}} \right)$ ---- (iii)

Similarly, the extra work to bring q_4 will be

$$W_4 = \frac{1}{4\pi\varepsilon_o} \cdot q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right) \quad \text{---- (iv)}$$

The *total* work necessary to assemble the first four charges, then, is

$$W = \frac{1}{4\pi\varepsilon_o} \cdot \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right)$$

To generalize, for n point charges,

$$W = \frac{1}{4\pi\varepsilon_o} \cdot \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>1}}^{n} \frac{q_i q_j}{r_{ij}} \quad \text{---- (v)}$$

A better way to accomplish this is to count each pair twice and then divide by 2,

$$\text{i.e.} \quad W = \frac{1}{8\pi\varepsilon_o} \cdot \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{r_{ij}} \quad \text{---- (vi)}$$

or
$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{4\pi\varepsilon_o} \cdot \frac{q_j}{r_{ij}} \right) \quad \text{---- (vii)}$$

The term in the parenthesis is the potential $V(\vec{r}_i)$ at point $\vec{r}_i$, which is the position of q_i, due to all the other charges.

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i) \quad \text{---- (viii)}$$

This gives the total work done in assembling a configuration of point charges. This work done will be stored into the system of charges as its *electric potential energy*.

1.9.8 Energy of a Continuous Charge Distribution:

For a volume charge distribution of volume density 'ρ', the Eq. (viii) becomes

$$W = \frac{1}{2}\int \rho V\, d\tau \qquad \text{---- (ix)}$$

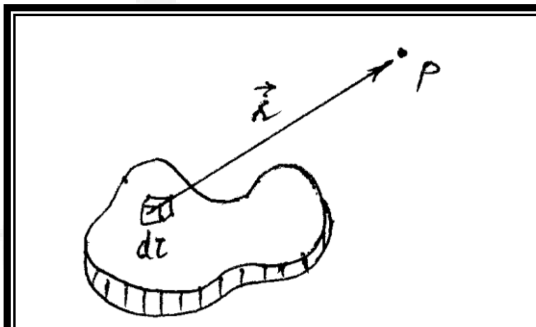


Fig. 1.24

From Gauss's law, we have

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_o} \Rightarrow \rho = \varepsilon_o \vec{\nabla}\cdot\vec{E} \qquad \text{--- (x)}$$

so that
$$W = \frac{\varepsilon_o}{2}\int (\vec{\nabla}\cdot\vec{E})V\, d\tau \qquad \text{---- (xi)}$$

Now, we have

$$\vec{\nabla}\cdot(V\vec{E}) = V(\vec{\nabla}\cdot\vec{E}) + \vec{E}\cdot(\vec{\nabla}V)$$

Integrating over volume and applying the divergence theorem, we get

$$\int_V \vec{\nabla}\cdot(V\vec{E})d\tau = \int_V V(\vec{\nabla}\cdot\vec{E})d\tau + \int_V \vec{E}\cdot(\vec{\nabla}V)d\tau = \oint_S V\vec{E}\cdot d\vec{a}$$

or
$$\int_V V(\vec{\nabla}\cdot\vec{E})\, d\tau = -\oint_V \vec{E}\cdot(\vec{\nabla}V)d\tau + \oint_S V\vec{E}\cdot d\vec{a} \qquad \text{---- (xii)}$$

Using Eq. (xii) in Eq. (xi), we get

$$W = \frac{\varepsilon_o}{2}\left[-\int_V \vec{E}\cdot(\vec{\nabla}V)d\tau + \oint_S V\vec{E}\cdot d\vec{a}\right]$$

But $\vec{\nabla}V = -\vec{E}$, so

$$W = \frac{\varepsilon_o}{2}\left[\int_V E^2\, d\tau + \oint_S V\vec{E}\cdot d\vec{a}\right] \qquad \text{---- (xiii)}$$

Now if the volume integral on R.H.S. of equation (xiii) is taken over *all space*, then the *surface integral* vanishes since it goes like $\frac{1}{r}$. Thus we may say

$$\text{As } V \rightarrow \infty, \oint_S V\vec{E} \cdot d\vec{a} \rightarrow 0$$

In this light, $$W = \frac{\varepsilon_o}{2} \int_{all\,space} E^2 \, d\tau \qquad \text{---- (xiv)}$$

<u>Note</u>: *Work done on the charges in assembling them in a distribution is stored as electric potential energy which is dependent on* $\vec{E} \cdot \vec{E} = E^2$.