UNIT-II: Electrodynamics

2.1 The Lorentz Force Law:

2.1.1 Magnetic Fields:

Magnetic fields are produced by electric currents, which can be macroscopic currents in wires, or microscopic currents associated with electrons in atomic orbits. If we hold a tiny compass in the vicinity of a current-carrying wire, we will find that the field neither points *towards* the wire, nor *away* from it, but rather it *circles around the wire*. Thus if we grab the wire in our right hand with the thumb pointing in the direction of the current, then our curling fingers round the wire give the direction of the magnetic field (Fig. 2.1). Such a field leads to a force of attraction on a nearby parallel current (Fig. 2.2).

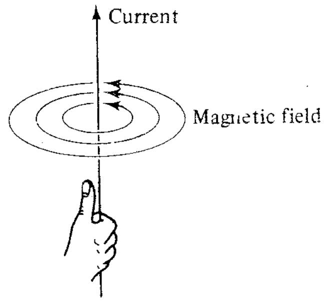

Fig. 2.1

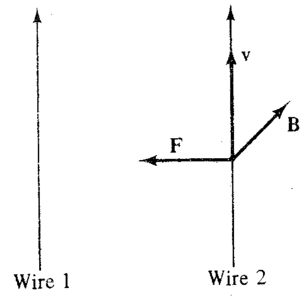

Fig. 2.2

2.1.2 Magnetic Fields:

The magnetic force in a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B}) \qquad \text{---- (i)}$$

This is known as **Lorentz force law**. In the presence of both electric and magnetic fields, the t**otal Lorentz force** on Q would be

$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})] \qquad \text{---- (ii)}$$

Note: *Since* $\vec{F}$ *and* $\vec{v}$ *are vectors,* $\vec{B}$ *is actually a pseudovector.*

2.2 Electric Currents:

The ***current*** in a wire is the *charge per unit time* passing a given point. The negative charges moving to the left count the same as positive ones to the right. Thus all phenomenon involving moving charges depend on the *product* of charge and velocity because if we change the sign of q and $\vec{v}$, we get the same answer.

Shri Mata Vaishno Devi University

Electric current is measured in coulombs-per-second, or ***amperes*** (A):

1 A = 1 C/s

2.2.1 Line Current Density:

A line charge λ travelling down a wire at a speed v (Fig. 2.3) constitutes a current

$$I = \lambda v \quad \text{---- (i)}$$

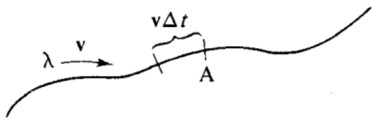

Fig. 2.3

because a segment of length $v\Delta t$, carrying charge $\lambda v\Delta t$, passes point P in a time interval Δt. Current is actually a vector given by

$$\vec{I} = \lambda\vec{v} \quad \text{---- (ii)}$$

Note: *The charge density in Eq. (i) refers only to the moving charges.*

When both types of charges move, we may write

$$\vec{I} = \lambda_+\vec{v}_+ + \lambda_-\vec{v}_- \quad \text{---- (iii)}$$

The magnetic force on a segment of current-carrying wire is, therefore,

$$\vec{F}_{mag} = \int(\vec{v}\times\vec{B})dq = \int(\vec{v}\times\vec{B})\lambda dl = \int(\vec{I}\times\vec{B})dl \quad \text{---- (iv)}$$

As $\vec{I}$ and $d\vec{l}$ both point in the same direction, we can just as well write it as

$$\vec{F}_{mag} = \int I(d\vec{l}\times\vec{B}) \quad \text{---- (v)}$$

For current of constant magnitude along the wire,

$$\vec{F}_{mag} = I\int(d\vec{l}\times\vec{B}) \quad \text{---- (vi)}$$

2.2.2 Surface Current Density:

When charge flows over a surface, we describe it by the ***surface current density, K*** which is defined as *the current per unit width-perpendicular-to-flow*. If we consider a "*ribbon*" of infinitesimal width $dl_\perp$, running parallel to the flow (Fig. 2.4), with line current density $d\vec{I}$, the surface current density is

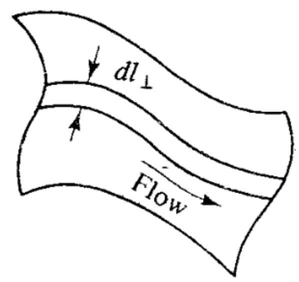

Fig. 2.4

$$\vec{K} = \frac{d\vec{I}}{dl_{\perp}} \quad \text{---- (vii)}$$

If the mobile surface charge density is σ and its velocity is $\vec{v}$, then

$$\vec{K} = \sigma\vec{v} \quad \text{---- (viii)}$$

Note: $\vec{K}$ *vary from point to point over the surface, reflecting variations in* σ *and/or* $\vec{v}$.

The magnetic force on the surface current is

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\sigma\, da = \int (\vec{K} \times \vec{B}) da \quad \text{---- (ix)}$$

2.2.3 Volume Current Density:

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the ***volume current density***, $\vec{J}$ which is defined as *current per unit area-perpendicular-to-flow.* If we consider a "*tube*" of infinitesimal cross-section $da_{\perp}$, running parallel to the flow (Fig. 2.5), with the line current density $d\vec{I}$, the volume current density is

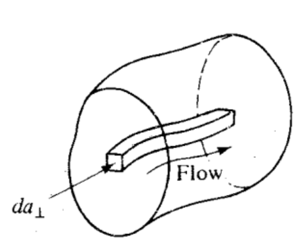

Fig. 2.5

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}} \quad \text{---- (x)}$$

If the *mobile* volume charge density is ρ and the velocity is $\vec{v}$, then

$$\vec{J} = \rho\vec{v} \quad \text{---- (xi)}$$

The magnetic force on a volume current is, therefore,

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\rho\, d\tau = \int (\vec{J} \times \vec{B}) d\tau \quad \text{---- (xi)}$$

2.3 Equation of Continuity:

Statement: *According to this equation,*

"The net outward flow of electric current per unit area at a point is equal to time rate of decrease of charge per unit volume at that point".

Proof: According to the definition of volume current density

Shri Mata Vaishno Devi University

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}}$$

$$\Rightarrow \qquad I = \oint_S J da_{\perp} = \oint_S \vec{J} \cdot d\vec{a} \qquad \text{---- (xii)}$$

Applying Gauss's divergence theorem to the integral on R.H.S. of above equation we have

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V \left(\vec{\nabla} \cdot \vec{J}\right) d\tau$$

Because *charge is conserved*, whatever flows out through the surface must come at the expense of that remaining inside:

$$\int_V \left(\vec{\nabla} \cdot \vec{J}\right) d\tau = -\frac{d}{dt}\int_V \rho\, d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

The *minus* sign reflects the fact that an *outward* flow *decreases* the charge left in V. Since this applies to *any* volume, we conclude that

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \qquad \text{---- (xiii)}$$

This is the precise mathematical statement of *local charge conservation.* It is called the ***Equation of Continuity.***

2.4 Biot-Savart Law:

This law is helpful in determining the magnitude and direction of magnetic field due to a current at any point.

2.4.1 Steady Currents:

Stationary charges produce electric fields that are constant in time; hence the term ***electrostatics***. *Steady currents* produce magnetic fields that are constant in time; the theory of steady currents is called ***magnetostatics.***

When a steady current flows in a wire, its magnitude I must be the same all along the line; otherwise the charge would be piling up somewhere, and it wouldn't be a steady current. On the same lines, $\frac{\partial \rho}{\partial t} = 0$ in magnetostatics, and hence the continuity equation

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

becomes

Shri Mata Vaishno Devi University

$$\vec{\nabla} \cdot \vec{J} = 0$$

2.4.2 The Magnetic Field of a Steady Current:

The magnetic field of a steady line current is given by the **Biot-Savart law:**

$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{I} \times \hat{r}'}{r'^2} dl' = \frac{\mu_o}{4\pi} I \int \frac{d\vec{l} \times \hat{r}'}{r'^2} \text{ ---- (i)}$$

The integration is along the current path, in the direction of the flow; $d\vec{l}$ is an element of length along the wire, and $\vec{r}'$ is the vector from the source to the point $\vec{r}$ (Fig. 2.6). The constant μ_o is called the ***permeability of free space.***

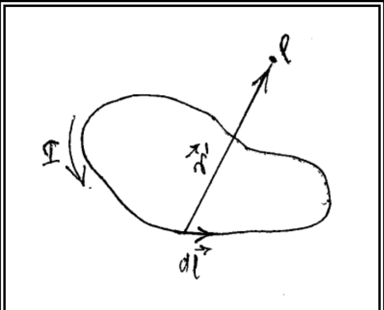

Fig. 2.6

$$\mu_o = 4\pi \times 10^{-7} \text{NA}^{-2} \text{ ---- (ii)}$$

The units of $\vec{B}$ will therefore, become ***tesla*** (**T**).

1 T = 1 N/(A . m) ---- (iii)

Note: *The Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. The 1/r² dependence is common to both laws.*

Direction of $\vec{B}$: The direction of $\vec{B}$ is given by the cross product of $d\vec{l}$ and $\vec{r}$ i.e. perpendicular to the plane of paper and directed inwards. This can also be determined by ***Right-Hand-Cork-Screw-Rule***.

2.4.3 Applications of Biot-Savart law:

(a) Magnetic Field due to a Long Straight Conductor of Infinite Length:

Consider a conductor of infinite length carrying current I steadily (Fig. 2.7) with an observation point P distant 'z' ⊥ar from it. Here $Id\vec{l}$ is the current element and $d\vec{l} \times \hat{r}$ points "*out of page*" and has a magnitude

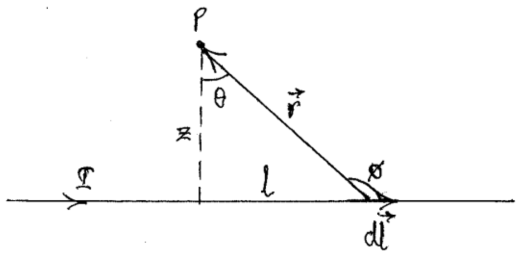

Fig. 2.7

$$dl\sin\phi = dl\sin\left(90^{\circ}+\theta\right) = dl\cos\theta$$

Also, $l = z\tan\theta$, so $dl = \dfrac{z}{\cos^2\theta}$

and $\dfrac{z}{r} = \cos\theta$ so $\dfrac{1}{r^2} = \dfrac{\cos^2\theta}{z^2}$

Thus, $$B = \frac{\mu_o I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{\left|d\vec{l}\times\hat{r}\right|}{r^2} = \frac{\mu_o I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{z^2}\right)\left(\frac{z}{\cos^2\theta}\right)\cos\theta\, d\theta$$

$$= \frac{\mu_o I}{4\pi}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_o I}{4\pi}\left(\sin\theta_2 - \sin\theta_1\right)$$

Where θ_1 and θ_2 are the initial and final angles. For an infinite conductor, $\theta_1 = -\dfrac{\pi}{2}$ and $\theta_2 = \dfrac{\pi}{2}$ so that

$$B = \frac{\mu_o I}{4\pi z}\left[\sin\frac{\pi}{2} - \left(-\sin\frac{\pi}{2}\right)\right] = \frac{\mu_o I}{4\pi z}\left[1-(-1)\right] = \frac{\mu_o I}{4\pi z}(1+1) = \frac{\mu_o I}{2\pi z}$$

$$B_{\infty} = \frac{\mu_o I}{2\pi z}$$

(b) <u>Magnetic field due to a Circular Current Carrying Loop at any point</u>:

Consider a circular loop of radius '*a*' carrying steady current *I*. Let P be any point distant '*z*' from centre O on the axis of the loop.

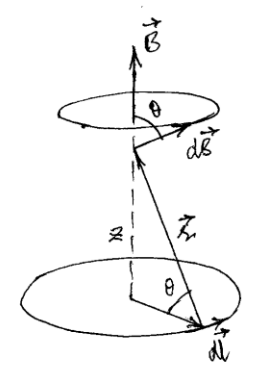

Fig. 2.8

The magnetic field $d\vec{B}$ associated with the current element $d\vec{l}$ is a shown (Fig. 2.8). As $d\vec{l}$ is integrated around the loop, $d\vec{B}$ sweeps out a cone. The horizontal components $dB\sin\theta$ cancel and the vertical components combine to give

$$\left|\vec{B}\right| = \frac{\mu_o I}{4\pi}\oint\frac{dl}{r^2}\cos\theta = \frac{\mu_o I}{4\pi}\frac{\cos\theta}{r^2}\oint dl$$

As $\cos\theta = \dfrac{a}{r}$

and $\oint dl$ = circumference, $2\pi a$

$$\therefore B = \frac{\mu_o I}{4\pi} \cdot \frac{a}{r^3}(2\pi a) = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}}$$

Special Case: At the centre of the loop, z = 0

and $$B_{centre} = \frac{\mu_o I a^2}{2a^3} = \frac{\mu_o I}{2a}$$

For n-turns of the loop, $B_{centre} = \dfrac{n\mu_o I}{2a}$

2.5 Ampere's Circuital Law: (Integral Form)

This law gives the mathematical means of calculating the curl of $\vec{B}$ from the total current threading the closed loop.

Statement: "The line integral of magnetic field around any closed curve C is equal to the product of μ_o and the current through the surface bounded by the curve."

i.e. $$\oint_C \vec{B}\cdot d\vec{l} = \mu_o I$$

Note: *Ampere's circuital law plays the same role in magnetostatics as Gauss's law does in electrostatics.*

Proof: Consider a closed circular loop (Fig. 2.9) threading the current I through it at its centre. Assume a small element of length $d\vec{l}$ such that

$$\vec{B}\cdot d\vec{l} = Bdl\cos 0^o = Bdl$$

Now, $$\oint_C \vec{B}\cdot d\vec{l} = \oint_C Bdl = B\oint_C dl$$

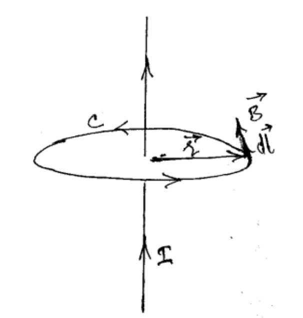

Fig. 2.9

Since for a straight conductor $B = \dfrac{\mu_o I}{2\pi r}$ and $\oint_C dl = 2\pi r$,

therefore,

$$= \frac{\mu_o I}{2\pi r} \times (2\pi r)$$

$\therefore$ $$\oint_C \vec{B}\cdot d\vec{l} = \mu_o I$$

This proves the law

Note: *Ampere's law is independent of the shape of the path. This can be shown as follows:*

Consider a current carrying wire ⊥ar to the plane of paper and the irregular loop in the XY plane.

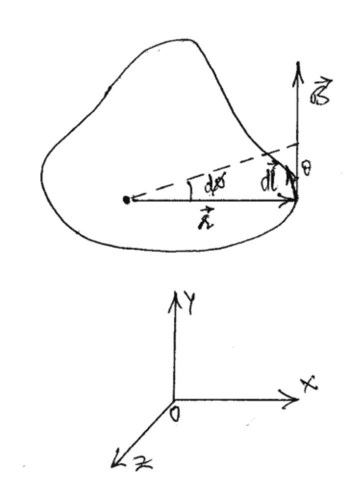

Fig. 2.10

$\therefore \qquad \vec{B}\cdot d\vec{l} = Bdl\cos\theta$ ----(i)

where $dl\cos\theta$ is the component of $d\vec{l}$ along the direction of $\vec{B}$, θ being the angle between $d\vec{l}$ and $\vec{B}$.

Thus $\qquad dl\cos\theta \perp \vec{r}$

$$\therefore \qquad d\phi = \frac{dl\cos\theta}{r} \qquad \Rightarrow dl\cos\theta = rd\phi$$

$$\therefore \qquad \vec{B}\cdot d\vec{l} = Bdl\cos\theta = Brd\phi = \frac{\mu_o I}{2\pi r}\cdot rd\phi = \frac{\mu_o I}{2\pi}d\phi$$

Hence, $\oint_C \vec{B}\cdot d\vec{l} = \frac{\mu_o I}{2\pi}\oint_C d\phi = \frac{\mu_o I}{2\pi}\cdot 2\pi = \mu_o I \qquad \left\{\because \oint_C d\phi = 2\pi\right\}$

Thus *Ampere's law is also valid for irregular path.*

As $\qquad I = \int_S \vec{J}\cdot d\vec{a}$

$$\therefore \qquad \oint_C \vec{B}\cdot d\vec{l} = \mu_o \int_S \vec{J}\cdot d\vec{a}$$

2.5.1 Limitations of Ampere's Law:

The Ampere's law holds true for steady fields only and not for time-varying fields.

Suppose we are in the process of charging up of a capacitor (Fig. 2.11). During the charging process, a current i_C flows through the connecting wires which changes with time and magnetic field $\vec{B}$ is setup around the wires. Such a current is called "*conduction current*". However, no current can flow through the space between two plates P and Q, but an electric field $\vec{E}$ exists in the space between the two plates.

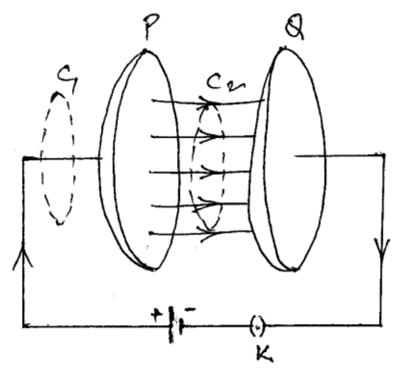

Fig. 2.11

If we consider two loops C_1 and C_2 parallel to the plates of the capacitor such that C_1 lies outside the region between plates enclosing the connecting wires and C_2

lies completely inside the region between the plates, then Ampere's law for two loops C_1 and C_2 separately give

$$\oint_{C_1} \vec{B}\cdot d\vec{l} = \mu_o i_c \qquad \text{---- (i)}$$

and $$\oint_{C_2} \vec{B}\cdot d\vec{l} = 0 \qquad \text{---- (ii)}$$

As the right hand sides of the two equations are not equal, it is followed that Ampere's law is not logically consistent.

2.5.2 Modified form of Ampere's Circuital Law:

To remove this difficulty, Maxwell modified the above equation by introducing the concept of *"displacement current"* as required below:

The differential form of Ampere's law is

$$\vec{\nabla}\times\vec{B} = \mu_o \vec{J} \qquad \text{---- (i)}$$

Taking the divergence of both sides of the equation, we have

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \mu_o\left(\vec{\nabla}\cdot\vec{J}\right)$$

But $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right) = 0$

$$\therefore \qquad \mu_o\left(\vec{\nabla}\cdot\vec{J}\right) = 0 \qquad \Rightarrow \qquad \vec{\nabla}\cdot\vec{J} = 0$$

which is the equation of continuity for steady fields and does not apply to time varying fields.

From Gauss's Law

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_o}$$

Differentiating, we get $\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{\varepsilon_o}\cdot\frac{\partial\rho}{\partial t}$

or $$\varepsilon_o\vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}\right) = \frac{\partial\rho}{\partial t}$$

Adding $\vec{\nabla}\cdot\vec{J}$ to either side, we have

$$\vec{\nabla}\cdot\vec{J} + \varepsilon_o\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}$$

विज्ञानं ब्रह्म
Shri Mata Vaishno Devi University

$$\Rightarrow \qquad \vec{\nabla}\cdot\vec{J} + \varepsilon_o \vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t} = 0 \qquad \left\{\because \vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0\right\}$$

or $$\vec{\nabla}\cdot\left(\vec{J} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right) = 0$$

$$\Rightarrow \qquad \vec{\nabla}\cdot\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0$$

where, $\vec{D} = \varepsilon_o \vec{E} =$ *displacement vector* in free space.

Replacing $\vec{J}$ by $\vec{J} + \frac{\partial \vec{D}}{\partial t}$ in Eq. (i), Ampere's Law gets modified to

$$\vec{\nabla}\times\vec{B} = \mu\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \qquad \text{---- (ii)}$$

The term $\frac{\partial \vec{D}}{\partial t}$ added to $\vec{J}$ is called *displacement current density*. It shall be effective only when $\vec{E}$ is a time varying field.

As $i_d = \int \vec{J}\cdot d\vec{a}$, therefore, in integral form Eq. (i) reads

$$\oint \vec{B}\cdot d\vec{l} = \mu_o i_c + \mu_o \varepsilon_o \int \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a} \qquad \text{---- (iii)}$$

2.5.3 <u>Significance of Displacement Current</u>:

The concept of displacement current rendered the relation $\vec{\nabla}\times\vec{B} = \mu_o \vec{J}$ between the current and magnetic field consistent with the continuity equation. The concept also led to a new induction effect according to which a *time-varying electric field should give rise to magnetic effect as a time-varying magnetic field gives rise to electric field*. Further, it helped to retain the notion that the flow of current in a circuit is continuous.

2.6 <u>Divergence of Magnetic Field</u>:

We have, form Biot-Savart's law for general case of volume current

$$\vec{B} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}\times\hat{r}}{r^2} d\tau' \qquad \text{---- (i)}$$

It may be noted that here magnetic field $\vec{B}$ is a function of (x, y, z) and current density $\vec{J}$ is a function of (x', y', z') (Fig. 2.12). .

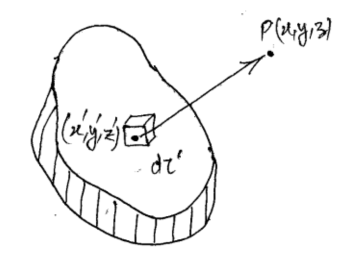

Fig. 2.12

Thus, $\vec{r} = (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}$

and $d\tau' = dx'dy'dz'$

Applying the divergence to Eq. (i), we have

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_o}{4\pi} \int \vec{\nabla} \cdot \left(\frac{\vec{J} \times \hat{r}}{r^2} \right) d\tau' \qquad \text{---- (ii)}$$

From product rule,

$$\vec{\nabla} \cdot \left(\vec{J} \times \frac{\hat{r}}{r^2} \right) = \frac{\hat{r}}{r^2} \cdot \left(\vec{\nabla} \times \vec{J} \right) - \vec{J} \cdot \left(\vec{\nabla} \times \frac{\hat{r}}{r^2} \right)$$

But $\vec{\nabla} \times \vec{J} = 0$, as $\vec{J}$ is independent of (x, y, z) and $\vec{\nabla} \times \frac{\hat{r}}{r^2} = 0$

In this light, Eq. (ii) becomes

$$\vec{\nabla} \cdot \vec{B} = 0$$

2.6.1 Significance of Divergence of $\vec{B}$:

Since magnetic field lines are continuous, therefore, as much lines enter into a closed surface as those leave. Thus net outflux of magnetic field i.e. div $\vec{B}$ is zero. The above statement also gives us the Gauss's law in magnetostatics as per which flux of magnetic field over any closed surface is always zero. This is due to the reason that there is no isolated monopole in magnetostatics which acts as a source of net outflux. Hence, $\vec{\nabla} \cdot \vec{B}$ is also an indication of the fact that "*no magnetic monopole exists*".

2.7 Magnetic Scalar and Vector Potentials:

2.7.1 Magnetic Scalar Potential:

We have, electric field $\vec{E} = -\vec{\nabla} V_E$

From this we may conclude that $\vec{E}$ is an *irrotational field* as its curl vanishes

i.e. $\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla} V_E) = 0 \qquad \{\because \vec{\nabla} \times \vec{\nabla} = 0\}$

विज्ञान श्रम
Shri Mata Vaishno Devi University

This gives us an idea that electric potential exists even in charge free space.

Now taking the curl of magnetic field $\vec{B}$ which is non-zero

i.e. $\vec{\nabla}\times\vec{B}=\mu_o\vec{J}$ (Differential form of Ampere's Law)

For $\vec{\nabla}\times\vec{B}$ to be zero, $\vec{J}$ should be zero i.e. the *region should be current free space.* Thus for charge/current free space, $\vec{\nabla}\times\vec{B}=0$. This will be the case only when $B=-\vec{\nabla}V_m$ = negative of gradient of same scalar magnetic potential.

Thus for magnetic scalar potential to exist, the region should be *current free* i.e. $\vec{J}=0$.

2.7.2 <u>Magnetic Vector Potential</u>:

The vector whose curl at a point gives the magnetic field induction at that point is called *magnetic vector potential*. It is denoted by $\vec{A}$.

Thus $\vec{B}=\vec{\nabla}\times\vec{A}$

In this sense, divergence of $\vec{B}$ should be zero as

$$\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=(\vec{\nabla}\times\vec{\nabla})\cdot\vec{A}=0$$

<u>Expression for $\vec{A}$</u>:

We have from Ampere's law,

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2A=\mu_o\vec{J} \text{ ---- (i)}$$

Now, we can add to the magnetic vector potential any function whose curl vanishes (e.g. gradient of a scalar λ), without affecting $\vec{B}$. Thus divergence of $\vec{A}$ can be determined.

Suppose original vector potential be $\vec{A}'$ whose $\vec{\nabla}\cdot\vec{A}'\neq 0$. We add to it the gradient of λ, so that

$$\vec{A}=\vec{A}'+\vec{\nabla}\lambda \quad \text{---- (ii)}$$

and the new divergence is

$$\vec{\nabla}\cdot\vec{A}=\vec{\nabla}\cdot\vec{A}'+\nabla^2\lambda \quad \text{---- (iii)}$$

Now, $\vec{\nabla}\cdot\vec{A}$ will vanish provided there exists a function λ that satisfies

$$\nabla^2\lambda = -(\vec{\nabla}\cdot\vec{A}') \qquad \text{---- (iv)}$$

Thus $\vec{\nabla}\cdot\vec{A} = 0 \Leftrightarrow \nabla^2\lambda = -(\vec{\nabla}\cdot\vec{A}')$ ---- (v)

The condition (iv) is equivalent to Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o} \qquad \text{---- (vi)}$$

If ρ goes to zero at infinity, then

$$V = \frac{1}{4\pi\varepsilon_o}\int\frac{\rho}{r}d\tau \qquad \text{---- (vii)}$$

Using (v) in (i), we have

$$\nabla^2\vec{A} = -\mu_o\vec{J}$$

By analogy, when $\vec{J}$ goes to zero at infinity,

$$\vec{A} = \frac{\mu_o}{4\pi}\int\frac{\vec{J}}{r}d\tau \qquad \text{---- (viii)}$$

This gives the required expression.

For line and surface currents,

$$\vec{A} = \frac{\mu_o}{4\pi}\int\frac{\vec{I}}{r}dl = \frac{\mu_o I}{4\pi}\int\frac{1}{r}d\vec{l}; \quad \frac{\mu_o}{4\pi}\int\frac{\vec{K}}{r}da$$

2.7.3 Derivation of Biot-Savart's law from Vector Potential:

Consider a current element of vector length $Id\vec{l}$ having an area of cross-section 'a' such that vector $d\vec{l}$ points in the direction of positive current (Fig 2.13).

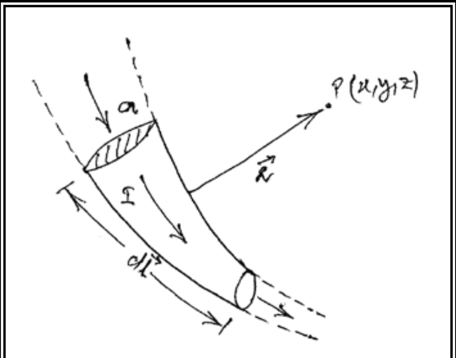

Fig. 2.13

Then volume of the current element

$$d\tau = \vec{a}\cdot d\vec{l} = a(dl)$$

If $\vec{J}$ is the current density, then its magnitude

$$J = \frac{I}{a}$$

or $$Jd\tau = \frac{I}{a} \times d\tau = \frac{I}{a} \times a(dl) = I(dl)$$

Since $\vec{J}$ and $d\vec{l}$ have same direction

$$\vec{J}d\tau = Id\vec{l}$$

Now vector potential, $\vec{A} = \frac{\mu_o}{4\pi}\int_V \frac{\vec{J}d\tau}{r}$

or $$d\vec{A} = \frac{\mu_o}{4\pi} \cdot \frac{\vec{J}d\tau}{r} = \frac{\mu_o}{4\pi} \cdot \frac{Id\vec{l}}{r} \quad \text{---- (i)}$$

Further, $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\therefore \quad d\vec{B} = \vec{\nabla} \times d\vec{A} = \vec{\nabla} \times \frac{\mu_o}{4\pi} \cdot \frac{Id\vec{l}}{r} = \frac{\mu_o I}{4\pi}(\vec{\nabla} \times d\vec{l}) \cdot \frac{1}{r}$$

$$= -\frac{\mu_o I}{4\pi}(d\vec{l} \times \vec{\nabla})\frac{1}{r} = -\frac{\mu_o I}{4\pi} d\vec{l} \times \vec{\nabla}\left(\frac{1}{r}\right)$$

$$= -\frac{\mu_o I}{4\pi} d\vec{l} \times \left(\frac{-\vec{r}}{r^3}\right) = \frac{\mu_o I}{4\pi}\frac{d\vec{l} \times \vec{r}}{r^3} \qquad \left\{\because \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}\right\}$$

or $$d\vec{B} = \frac{\mu_o I}{4\pi}\frac{(d\vec{l} \times \vec{r})}{r^3} \quad \text{---- (ii)}$$

This is *Biot-Savart's Law*.

Note: *The direction of $d\vec{B}$ is ⊥ar to the plane containing $d\vec{l}$ and $\vec{r}$ as given by right hand screw rule for cross product of two vectors.*

2.8 Electromagnetic Induction:

Whenever there is a change in magnetic flux linked with a circuit, an e.m.f. is induced in the circuit. If the circuit is closed, a current is also induced in it. ***The e.m.f. and current so produced last so long as the change in flux linked with the circuit lasts***. The phenomenon is called **Electromagnetic Induction**. To show the phenomenon of electromagnetic induction, Faraday gave a series of three experiments:

Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 2.14 (a)). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop (Fig. 2.14 (b)).

Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 2.14 (c)), the changed the *strength* of the field (by using and electromagnet and varying the current in the coil). Once again current flowed in the loop.

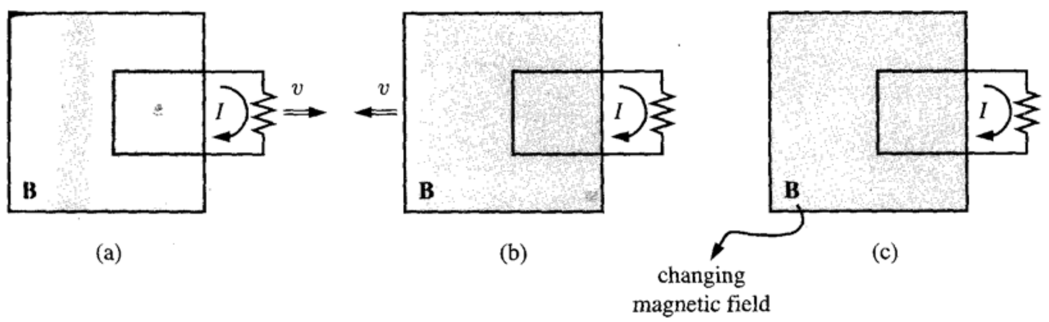

Fig. 2.14

2.8.1 Laws of Electromagnetic Induction:

There are two laws governing the phenomenon of electromagnetic induction:

(i) ***Faraday's laws***, that give the ***magnitude*** of induced e.m.f. and (ii) ***Lenz's law***, which tells us the ***direction*** of induced e.m.f.

2.8.2 Faraday's Laws Of Electromagnetic Induction:

I[st] law: "***Whenever there is a change in the magnetic flux linked with a circuit, an e.m.f. is induced in the circuit***". **The induced e.m.f. lasts only for the time for which the magnetic flux is actually changing**.

II[nd] law: ***"The magnitude of the induced e.m.f. is equal to the rate of change of magnetic flux passing through the circuit"***.

If '$d\Phi$' is the change in magnetic flux linked with the circuit in time 'dt' at any instant, then the induced e.m.f. in the circuit is given by

$$\varepsilon = -\frac{d\Phi}{dt} \qquad \text{----- (i)}$$

Here –ve sign indicates that induced e.m.f. always opposes any change in magnetic flux associated with the circuit. Also ε is in volt and $\frac{d\Phi}{dt}$ is in Wbs^{-1}.

If the resistance of the circuit is R, the current is

$$i = \frac{\varepsilon}{R} = \frac{1}{R} \cdot \frac{d\Phi}{dt} \quad \text{----- (ii)}$$

The e.m.f. developed by a changing flux is called ***induced e.m.f.*** and the current produced by this e.m.f. is called ***induced current***.

Note: *For e.m.f. to be induced in a coil, the magnetic flux linking the coil should change continuously.*

2.8.3 Lenz's Law:

This law is useful in determining the ***direction*** of the induced current in the circuit. This law states that:

"***The direction of induced e.m.f. (or induced current if the circuit is closed) is such that it opposes the change in flux that has induced it***".

If a current is induced by an ***increasing flux***, it will ***weaken*** the original flux. If a current is induced by a ***decreasing flux***, it will ***strengthen*** the original flux.

2.8.4 Faraday's Law of Electromagnetic Induction (Integral and Differential Forms):

According to Faraday's law of electromagnetic induction, the emf induced in the circuit is directly proportional to time rate of change of magnetic flux linked with it.

i.e. $\varepsilon \propto -\frac{d\Phi}{dt}$

or $\varepsilon = -k\frac{d\Phi}{dt}$

In S.I., k = 1 $\therefore \varepsilon = -\frac{d\Phi}{dt}$ ---- (i)

As $\Phi = \int_S \vec{B} \cdot d\vec{a}$ $\therefore$ $\frac{d\Phi}{dt} = \int_S \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{a}$ $\{\because B = B(x,t)\}$

Thus, $\varepsilon = \int_S \left(-\frac{\partial B}{\partial t}\right) \cdot d\vec{a}$ ---- (ii)

Eq. (ii) gives the *Integral Form of Faraday's Law.*

Further, $\varepsilon = \oint_C \vec{E} \cdot d\vec{l}$ ---- (iii)

Comparing (ii) and (iii), we get

$$\oint_C \vec{E} \cdot d\vec{l} = \int_S \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{a} \quad \text{---- (iv)}$$

Shri Mata Vaishno Devi University

From Stoke's law, $\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a}$ ---- (v)

Using (v) in (iv), we get

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = \int_S \left(-\frac{\partial B}{\partial t}\right)\cdot d\vec{a}$$

or $$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{---- (vi)}$$

This gives *Faraday's Law in Differential Form.*

2.9 Maxwell's Equations:

These are the four vector differential relations based on the phenomenon of electric and magnetic fields which help in developing electromagnetic field theory. The two relations are independent of time and are known as *steady state* equations .the other two equations depend upon time and are therefore known as *time varying* equations.

***Steady State Equations*:**

1) *Maxwell's First* ***Steady State*** *Equation:*

	Differential Form	***Integral Form***
In Vacuum,	$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_o}$	$\oint_S \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_o}$
In a Medium, (Linear Isotropic)	$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$	$\oint_S \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon}$

or $\vec{\nabla}\cdot(\varepsilon\vec{E}) = \rho \quad \Rightarrow \vec{\nabla}\cdot\vec{D} = \rho$

2) Maxwell's Second Steady State Equation:

	Differential Form	***Integral Form***
In Vacuum,	$\vec{\nabla}\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{a} = 0$
In a medium,	$\vec{\nabla}\cdot(\mu\vec{H}) = 0 \quad \Rightarrow \vec{\nabla}\cdot\vec{H} = 0$	$\oint_S \vec{H}\cdot d\vec{a} = 0$

3) Maxwell's First Time Varying Equation:

In Vacuum, $Curl\,\vec{E} = \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\quad\oint_C \vec{E}\cdot d\vec{l} = \int_S\left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{a}$

विज्ञानं ब्रह्म Shri Mata Vaishno Devi University

In a Medium, $\vec{\nabla}\times\vec{E} = -\dfrac{\partial}{\partial t}(\mu\vec{H}) = -\mu\dfrac{\partial\vec{H}}{\partial t}$ $\oint_C \vec{E}\cdot d\vec{l} = -\mu\int_S\left(\dfrac{\partial\vec{H}}{\partial t}\right)\cdot d\vec{a}$

4) Maxwell's Second Time Varying Equation:

In Vacuum, $\vec{\nabla}\times\vec{B} = \mu_o\left[\vec{J}+\varepsilon_o\dfrac{\partial\vec{E}}{\partial t}\right] = \mu_o\left[\vec{J}+\dfrac{\partial\vec{D}}{\partial t}\right]$ $\oint_C \vec{B}\cdot d\vec{l} = \mu_o(I_C + I_D)$

or $\vec{\nabla}\times\vec{H} = \vec{J}+\varepsilon_o\dfrac{\partial\vec{E}}{\partial t}$ or $\oint_C \vec{H}\cdot d\vec{l} = I_C + I_D$

In a Medium, $\vec{\nabla}\times\vec{B} = \mu\left[\vec{J}+\dfrac{\partial\vec{D}}{\partial t}\right]$ $\oint_C \vec{B}\cdot d\vec{l} = \mu(I_C + I_D)$

or $\vec{\nabla}\times\vec{H} = \vec{J}+\dfrac{\partial\vec{D}}{\partial t}$ $\left(\because H = \dfrac{B}{\mu}\right)$ $\oint_C \vec{H}\cdot d\vec{l} = I_C + I_D$

For vacuum, in a charge and current free region,

$\rho = 0$ and J = 0

(i) $\vec{\nabla}\cdot\vec{E} = 0$

(ii) $\vec{\nabla}\cdot\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} = -\mu_o\dfrac{\partial\vec{H}}{\partial t}$

(iv) $\vec{\nabla}\times\vec{H} = \varepsilon_o\dfrac{\partial\vec{E}}{\partial t} = \dfrac{\partial\vec{D}}{\partial t}$

2.9.1 Physical Interpretation of Maxwell's Equation:

1) $\vec{\nabla}\cdot\vec{D} = \rho$: It is a steady state equation independent of time. As $\vec{\nabla}\cdot\vec{D}$ = a scalar, therefore, charge density ρ is a scalar quantity. The positive charge acts as a "*source*" and negative charge as a "*sink*" for electric lines of force.

2) $\vec{\nabla}\cdot\vec{B} = 0$: It is also a steady state time independent equation. This equation signifies that *isolated magnetic monopoles cannot exist*. As $\vec{\nabla}\cdot\vec{B} = 0$, this means that equal No. of lines of magnetic force enter and leave a given volume. Also there is *no source or sink for lines of magnetic force*.

3) $\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$ **:** It is a time dependent equation varying with time which relates space variation of $\vec{E}$ with time variation of $\vec{B}$. It is a statement of *Faraday's and Law's* of electromagnetic induction.

4) $\vec{\nabla}\times\vec{B}=\mu_o\left[\vec{J}+\varepsilon_o\dfrac{\partial\vec{E}}{\partial t}\right]=\mu_o\left[\vec{J}+\dfrac{\partial\vec{D}}{\partial t}\right]$ **:** It is also a time dependent equation and relates the space variation of $\vec{B}$ with time variations of $\vec{D}$ or $\vec{E}$. It is a statement of Ampere's Law. This equation gives us an idea that magnetic field vector $\vec{B}$ can be generated by current density vector $\vec{J}$.

विज्ञानं ब्रह्म
Shri Mata Vaishno Devi
University