Quicksort

Sorting algorithms

- Insertion, selection and bubble sort have quadratic worst-case performance
- The faster comparison based algorithm ?

  O(nlogn)

- Mergesort and Quicksort

Merge Sort

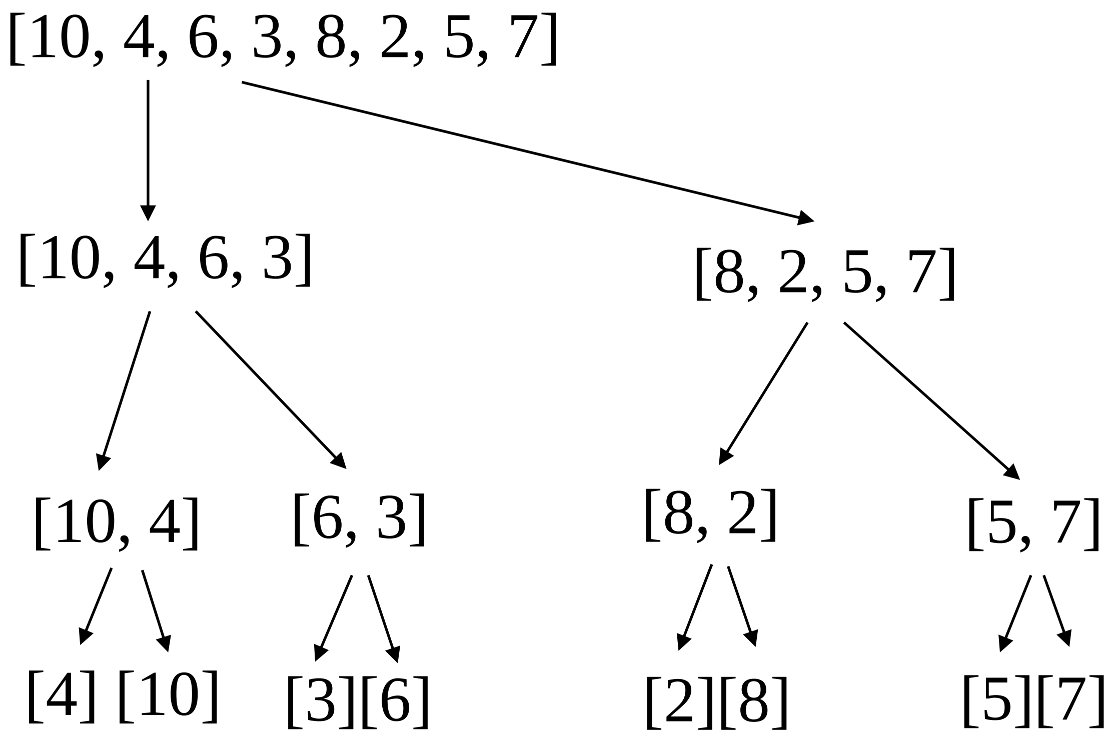
- Apply divide-and-conquer to sorting problem
- Problem: Given *n* elements, sort elements into non-decreasing order
- Divide-and-Conquer:
  - If n=1 terminate (every one-element list is already sorted)
  - If n>1, partition elements into two or more sub-collections; sort each; combine into a single sorted list
- How do we partition?

Partitioning -

- A gets *n/2* elements, B gets rest half
- Sort A and B recursively
- Combine sorted A and B using a process called *merge*, which combines two sorted lists into one
  - How? We will see soon

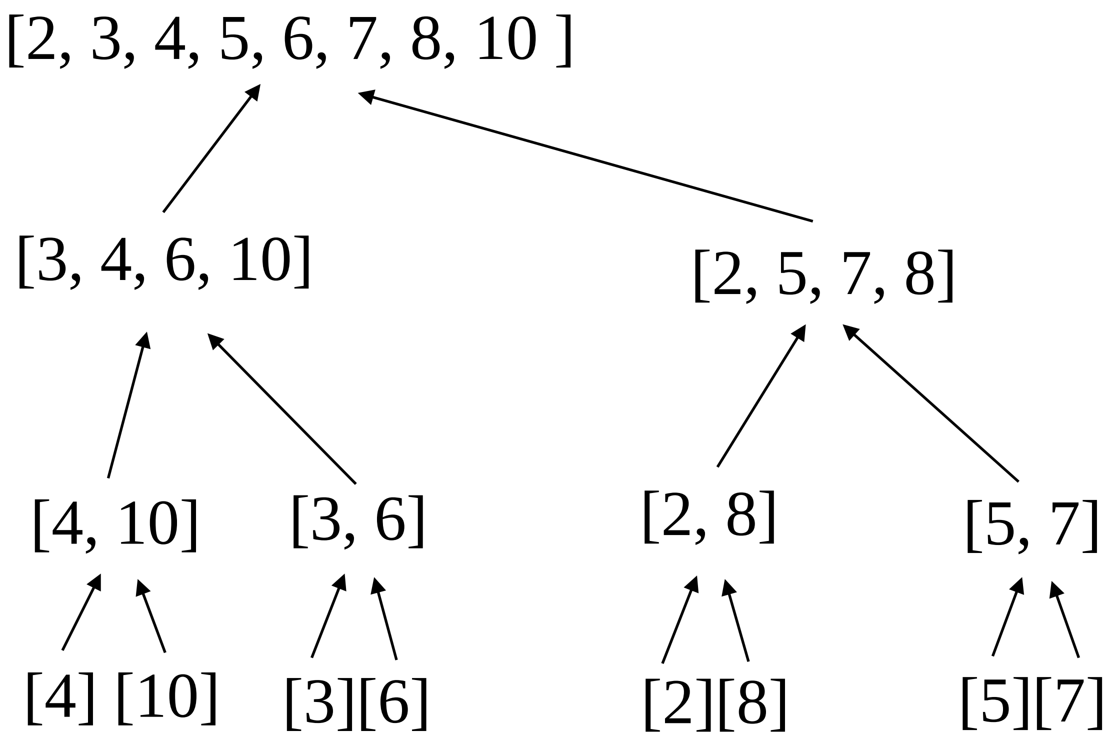
Example

- Partition into lists of size n/2



Example Cont’d

- Merge



Evaluation

- Recurrence equation:
- Assume n is a power of 2

$$T(n) = \begin{cases} c_1 & \text{if } n=1 \\ 2T(n/2) + c_2 n & \text{if } n>1,\ n=2^k \end{cases}$$

Solution

By Substitution:

$T(n) = 2T(n/2) + c_2n$

$T(n/2) = 2T(n/4) + c_2n/2$

$T(n) = 4T(n/4) + 2\ c_2n$

$T(n) = 8T(n/8) + 3\ c_2n$

$T(n) = 2^iT(n/2^i) + ic_2n$

Assuming $n = 2^k$, expansion halts when we get $T(1)$ on right side; this happens when $i=k$ $T(n) = 2^kT(1) + kc_2n$

Since $2^k=n$, we know $k=\log n$; since $T(1) = c_1$, we get

$T(n) = c_1n + c_2n\log n$;

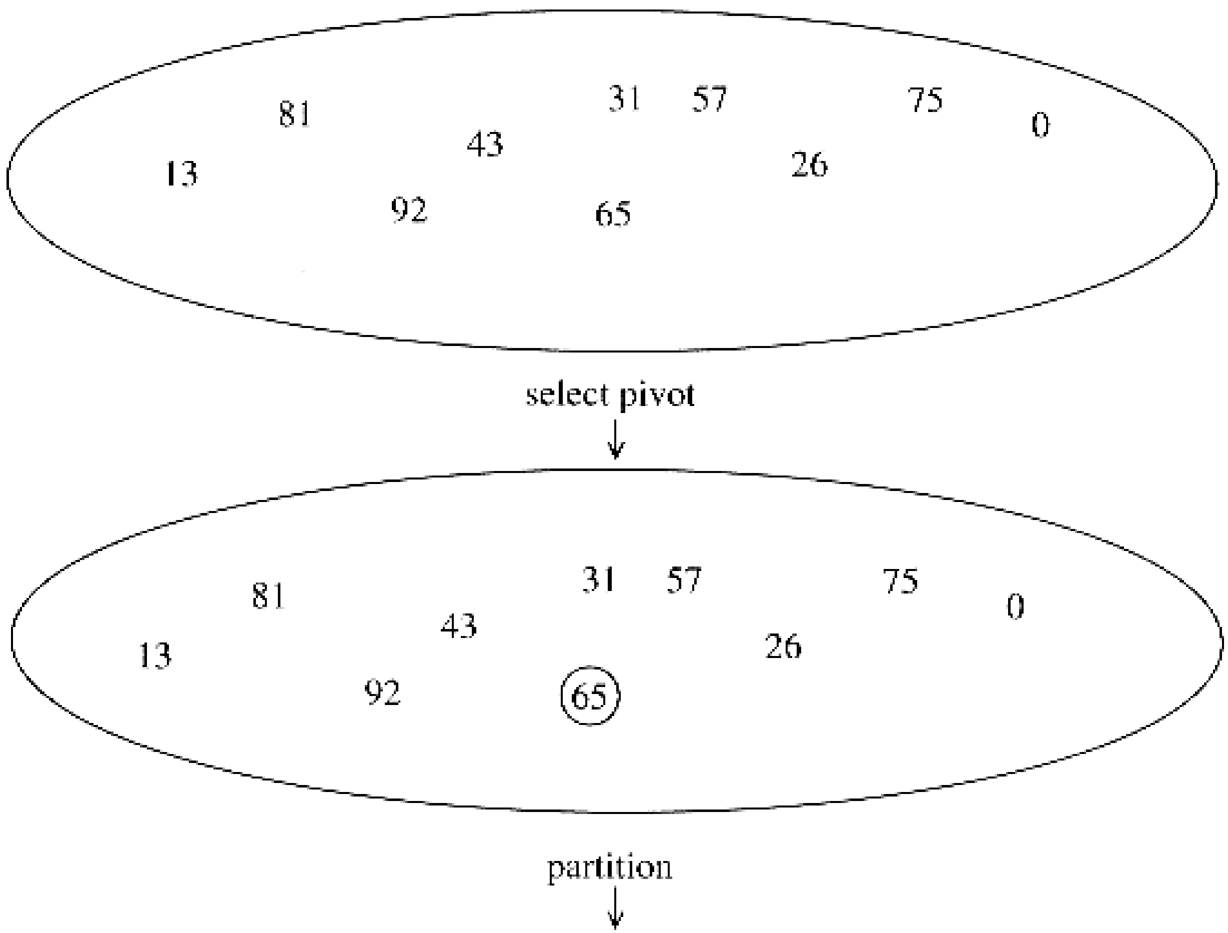
thus an upper bound for $T_{mergeSort}(n)$ is $O(n\log n)$

Quick Sort

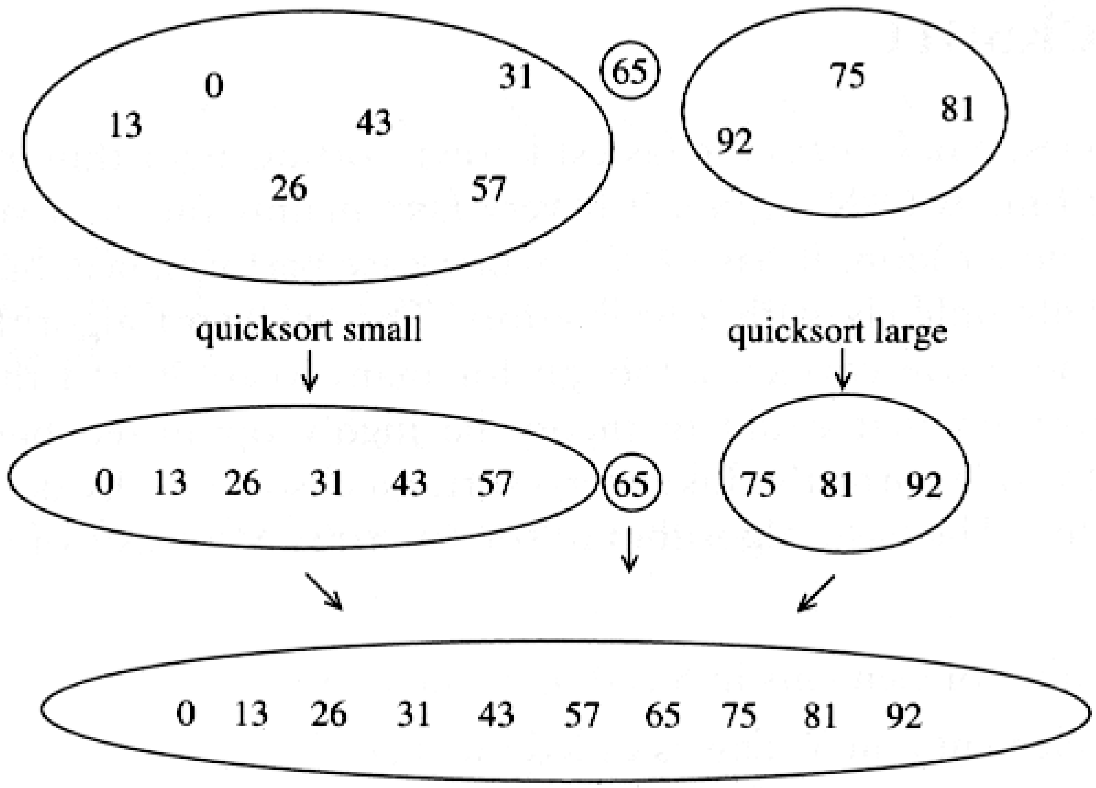
Quick Sort

- **Fastest** known sorting algorithm in practice
- Average case: O(N log N)
- Worst case: $O(N^2)$
  - But the worst case can be made exponentially unlikely.
- Another divide-and-conquer recursive algorithm, like merge sort.

Quick Sort: Example



Example of Quick Sort...



Issues To Consider

- How to pick the pivot?
  - Many methods (discussed later)
- How to partition?
  - Several methods exist.
  - The one we consider is known to give good results and to be easy and efficient.
  - We discuss the partition strategy first.

Quicksort Algorithm

Given an array of *n* elements (e.g., integers):

- If array only contains one element, return
- Else
  - pick one element to use as *pivot.*
  - Partition elements into two sub-arrays:
    - Elements less than or equal to pivot
    - Elements greater than pivot
  - Quicksort two sub-arrays
  - Return results

Example

We are given array of n integers to sort:

40	20	10	80	60	50	7	30	100

Pick Pivot Element

There are a number of ways to pick the pivot element. In this example, we will use the first element in the array:

40	20	10	80	60	50	7	30	100

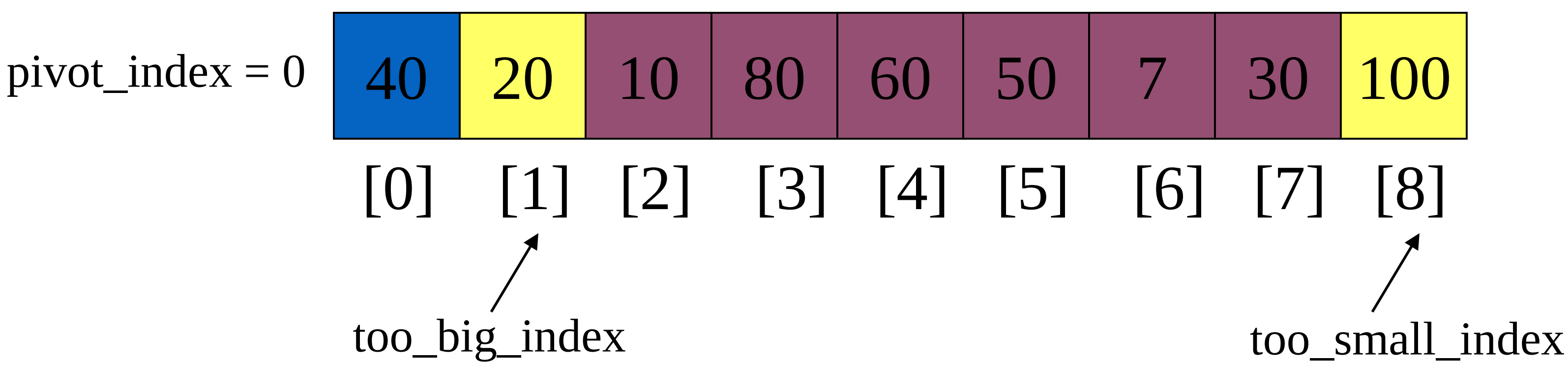
Partitioning Array

Given a pivot, partition the elements of the array such that the resulting array consists of:

1. One sub-array that contains elements >= pivot
2. Another sub-array that contains elements < pivot

The sub-arrays are stored in the original data array.

Partitioning loops through, swapping elements below/above pivot.


pivot_index = 0
40
20
10
80
60
50
7
30
100
[0]
[1]
[2]
[3]
[4]
[5]
[6]
[7]
[8]
too_big_index
too_small_index

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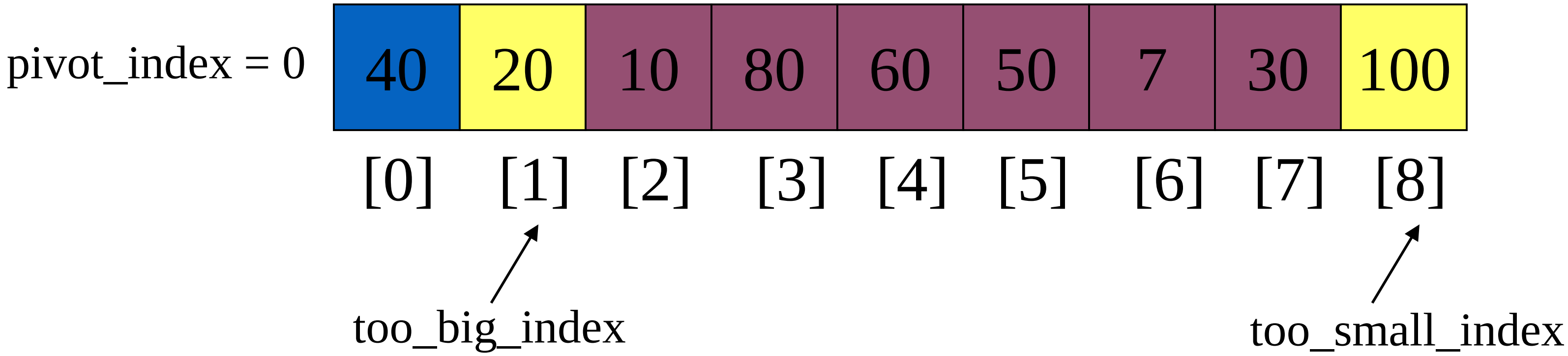

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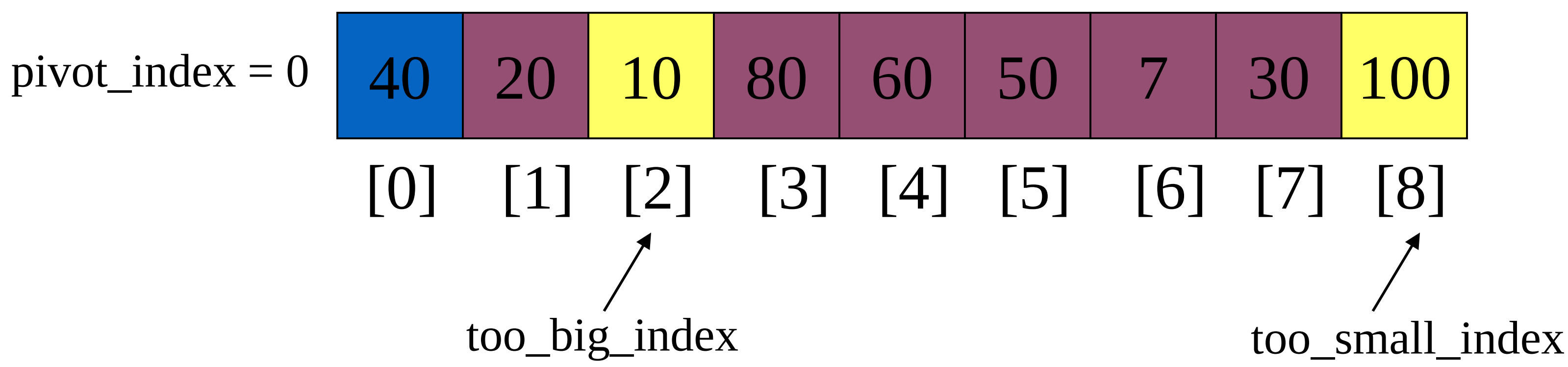

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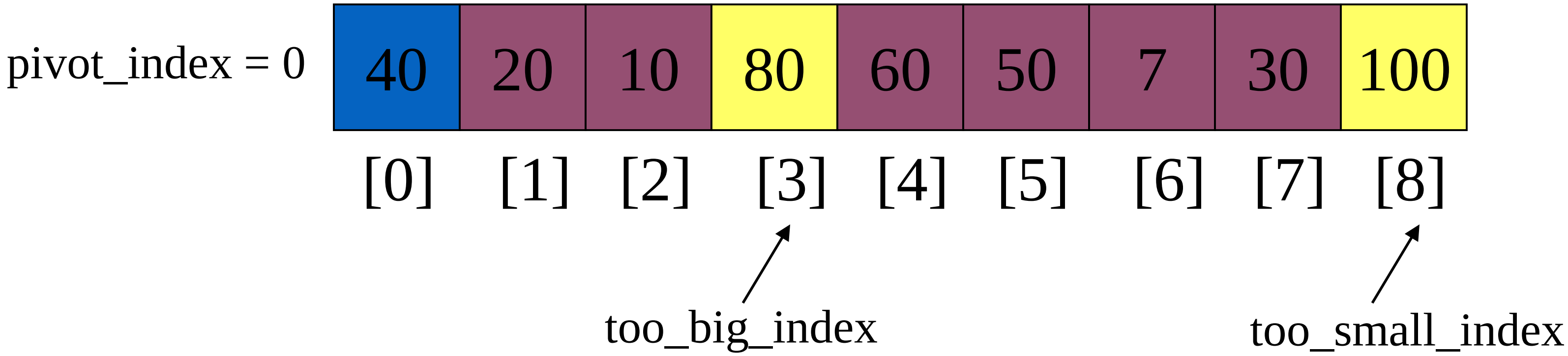

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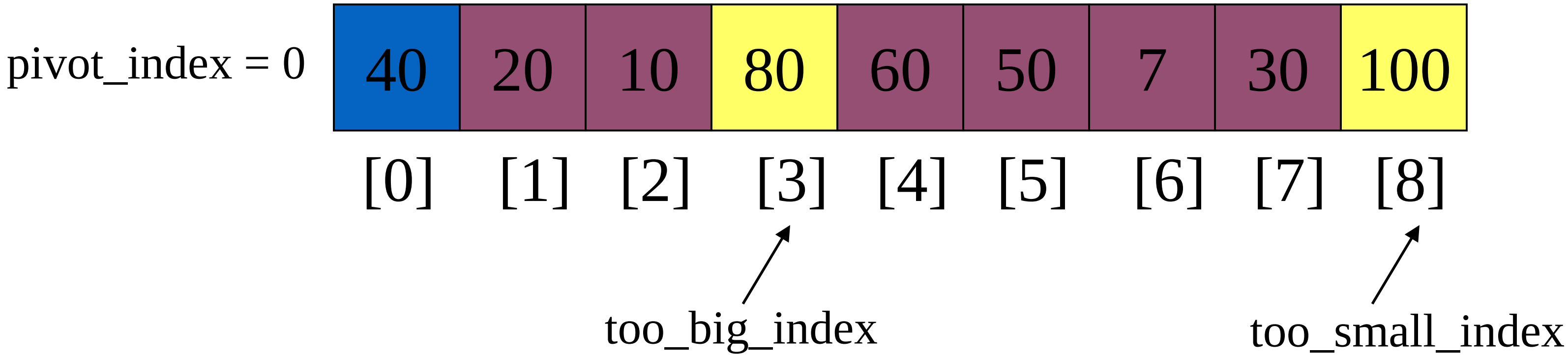

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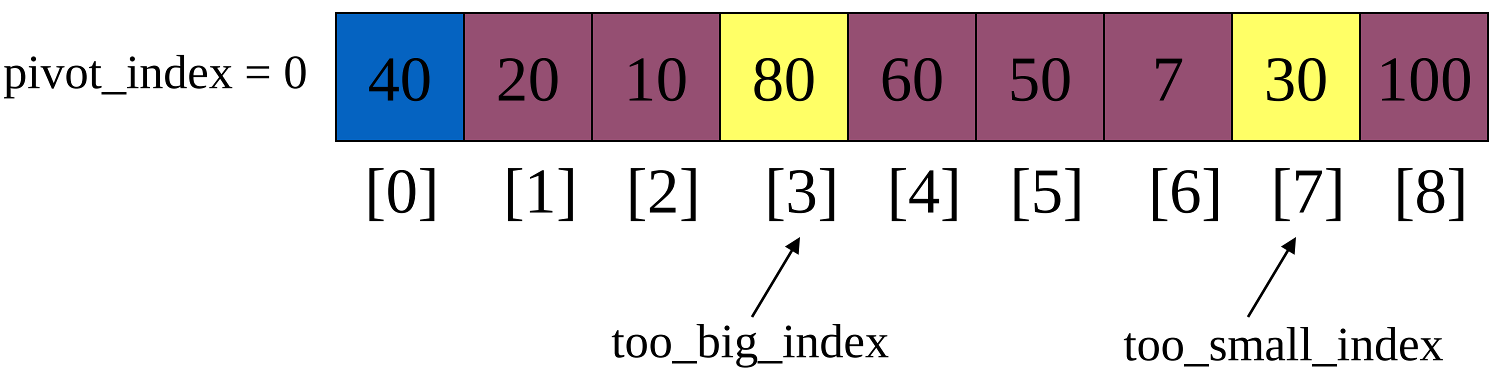

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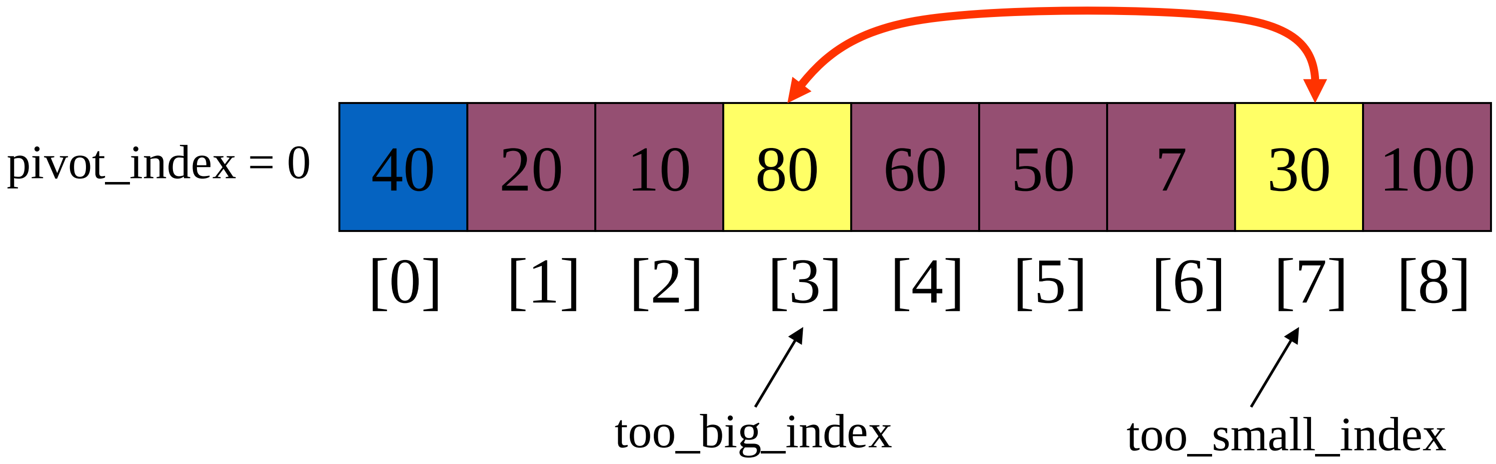

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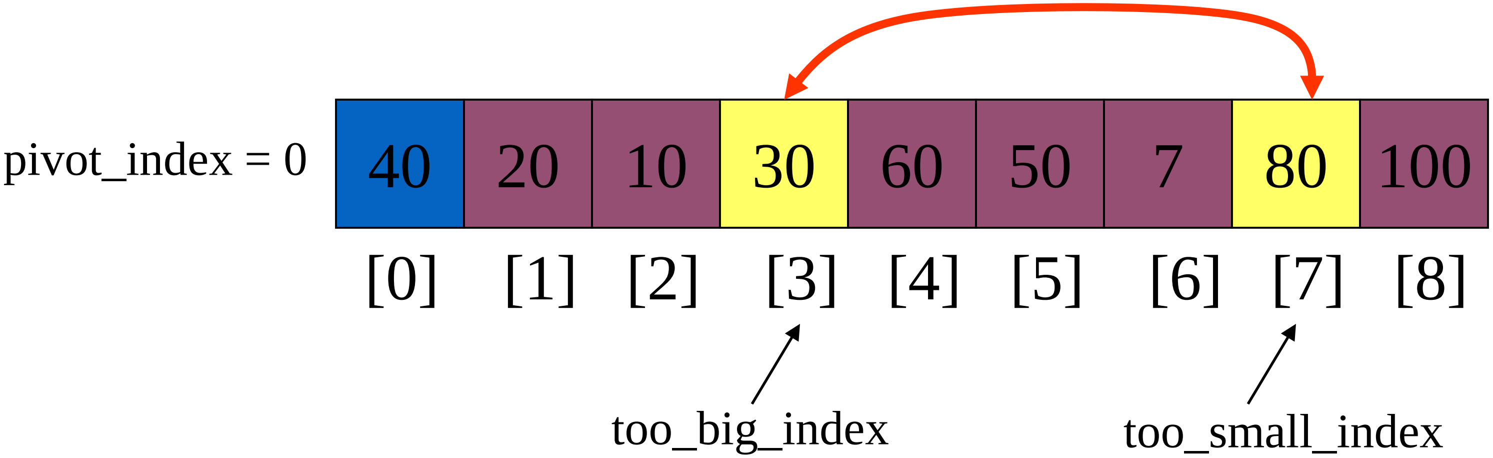

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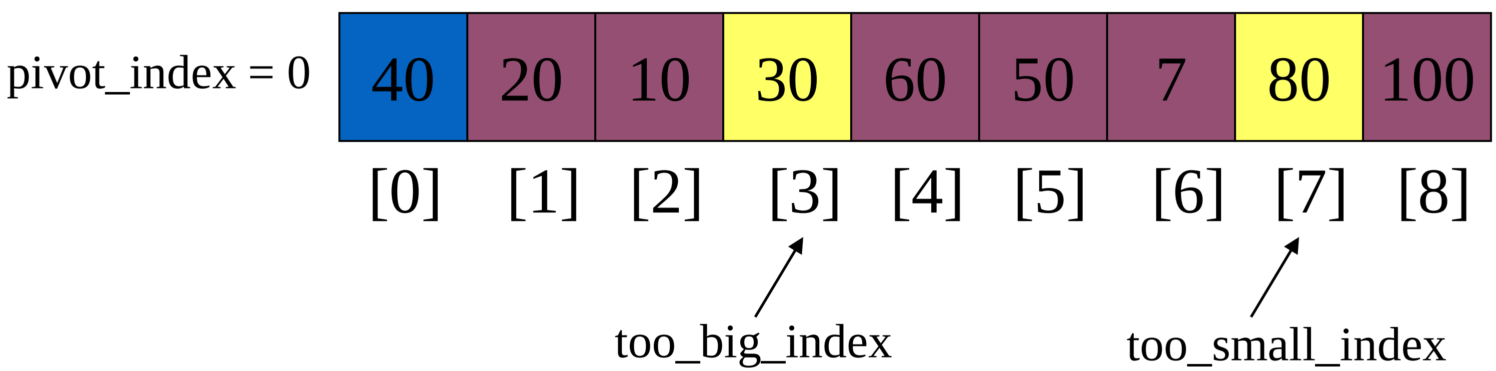

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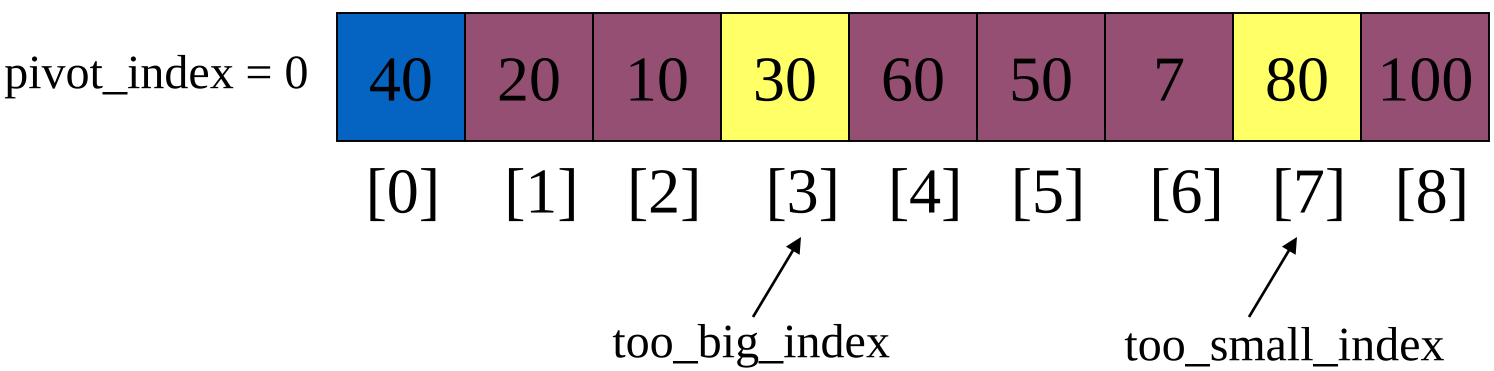

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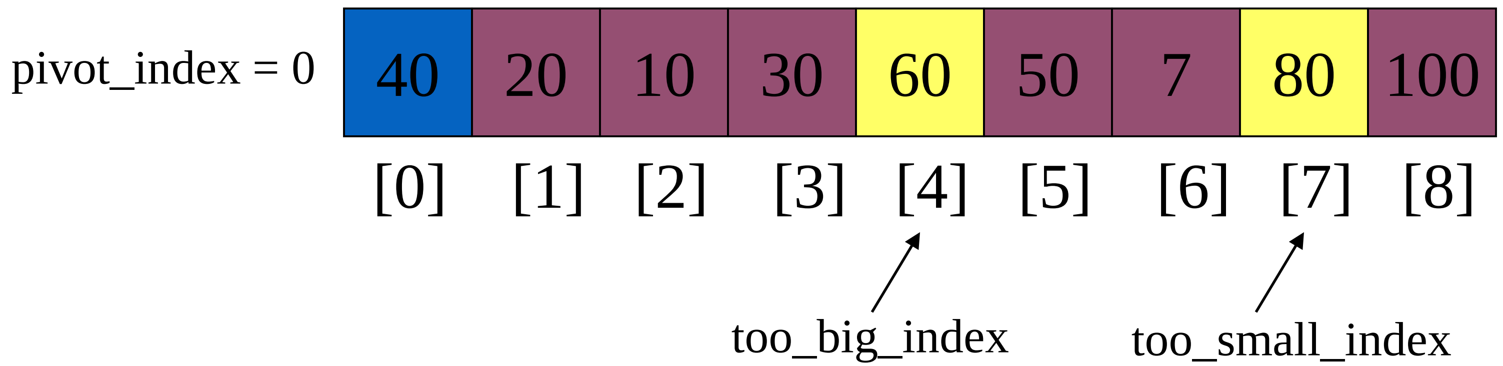

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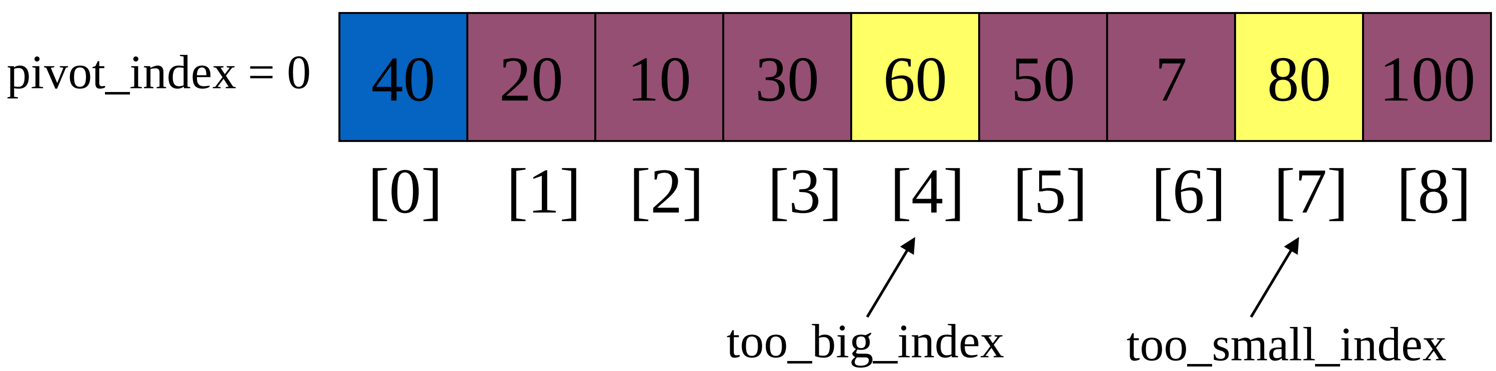

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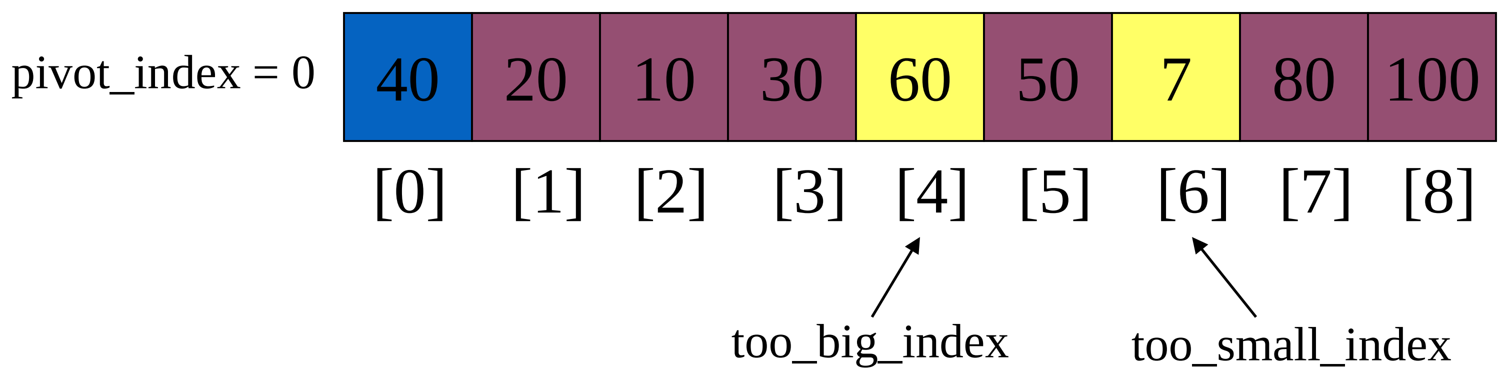

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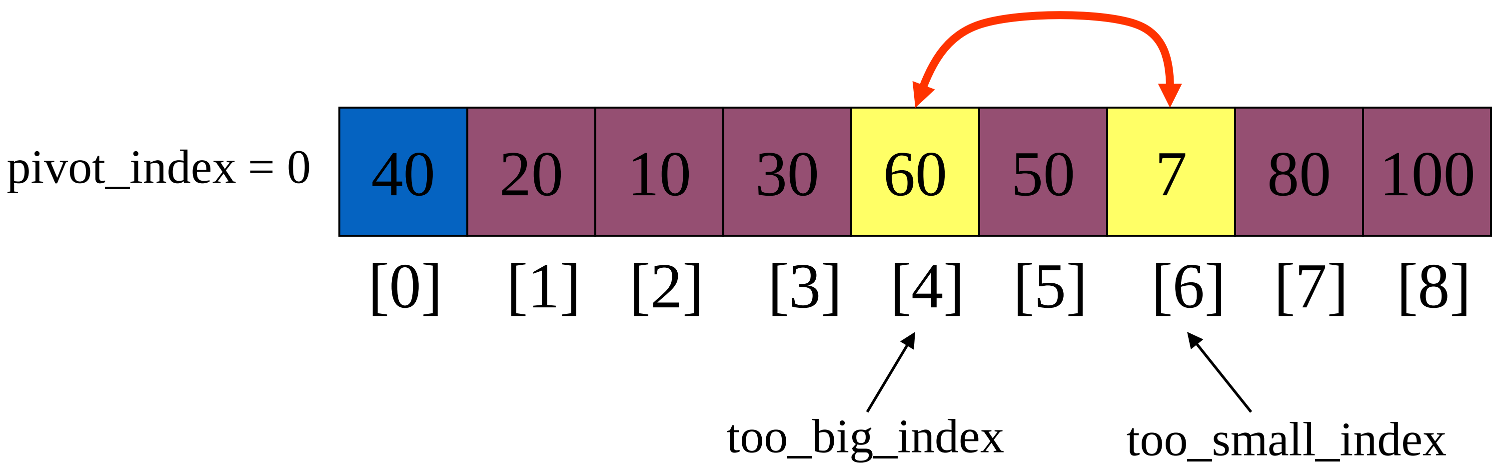

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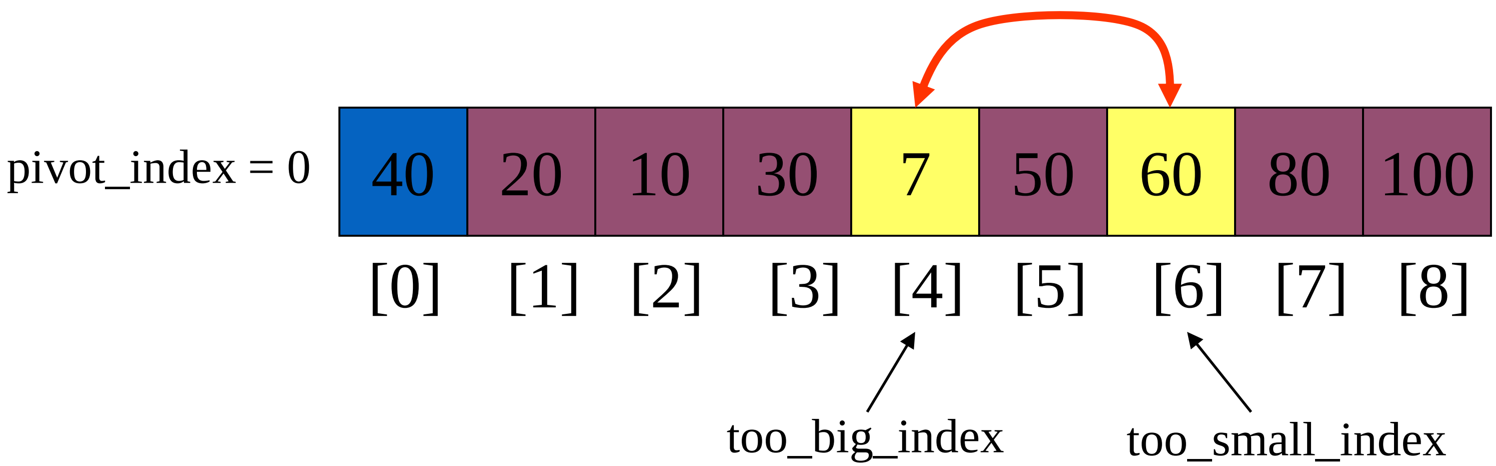

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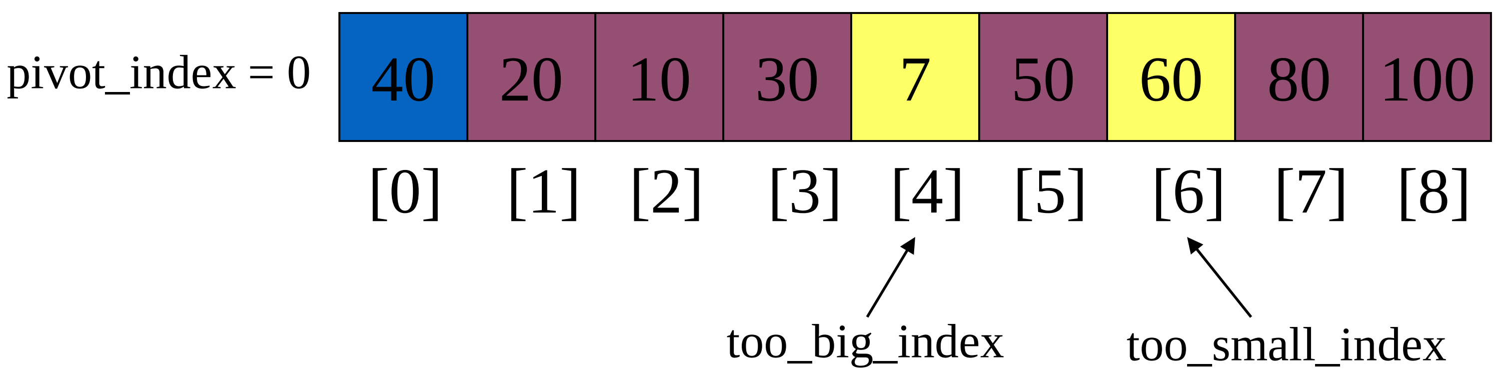

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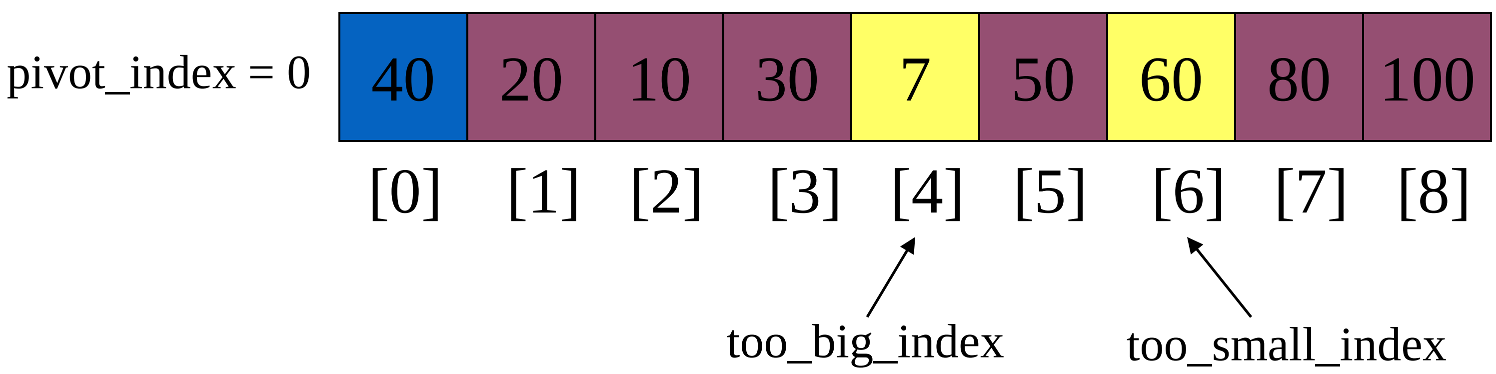

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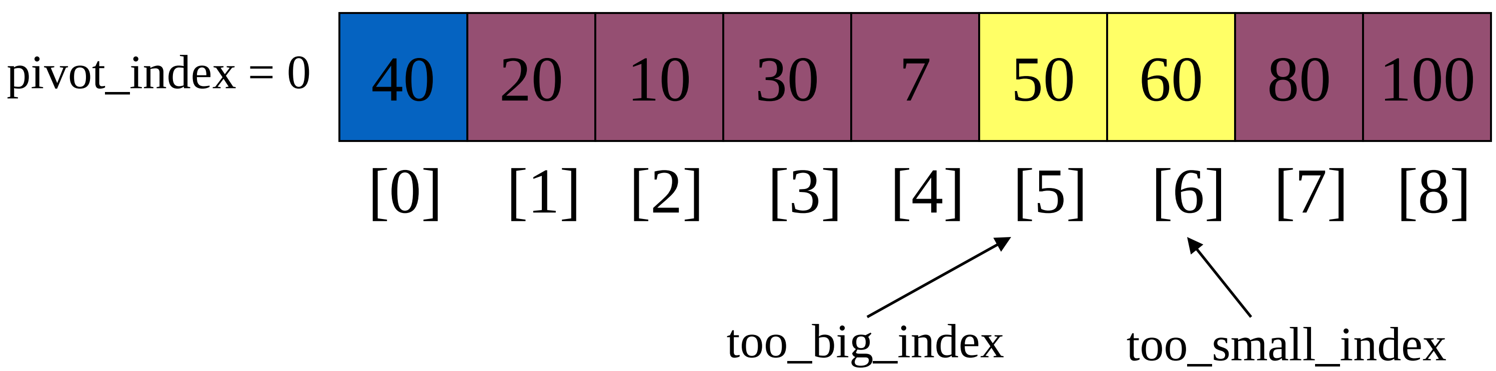

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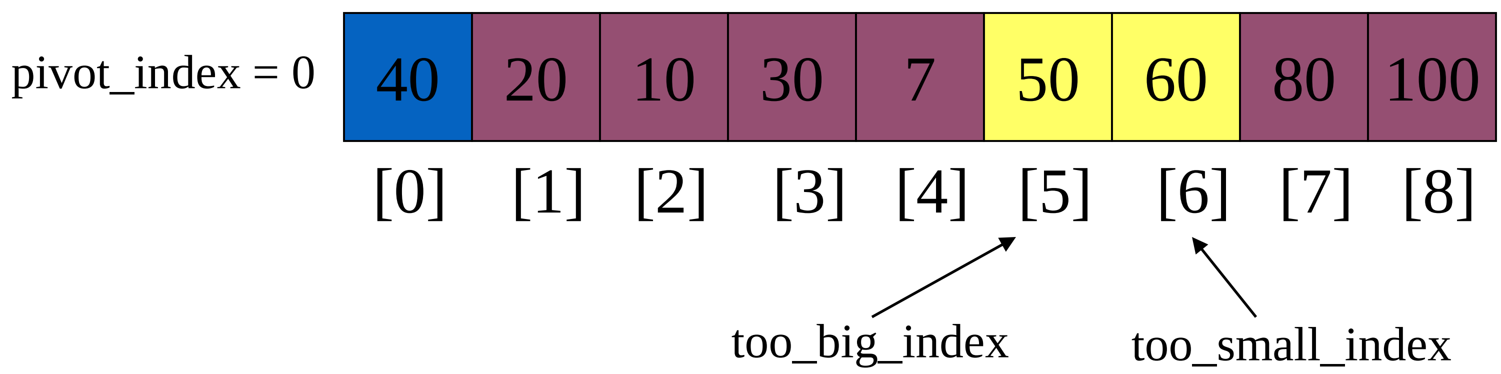

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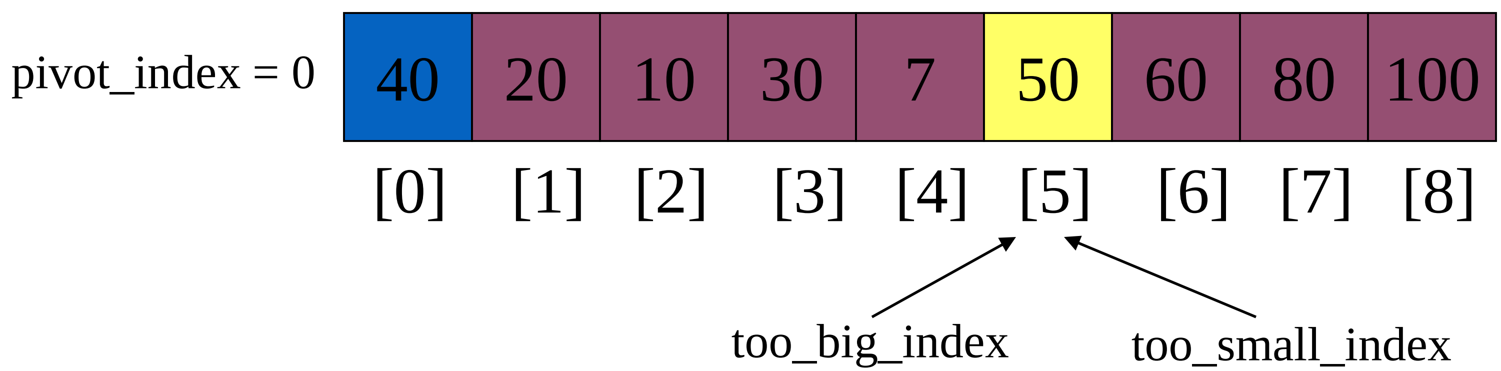

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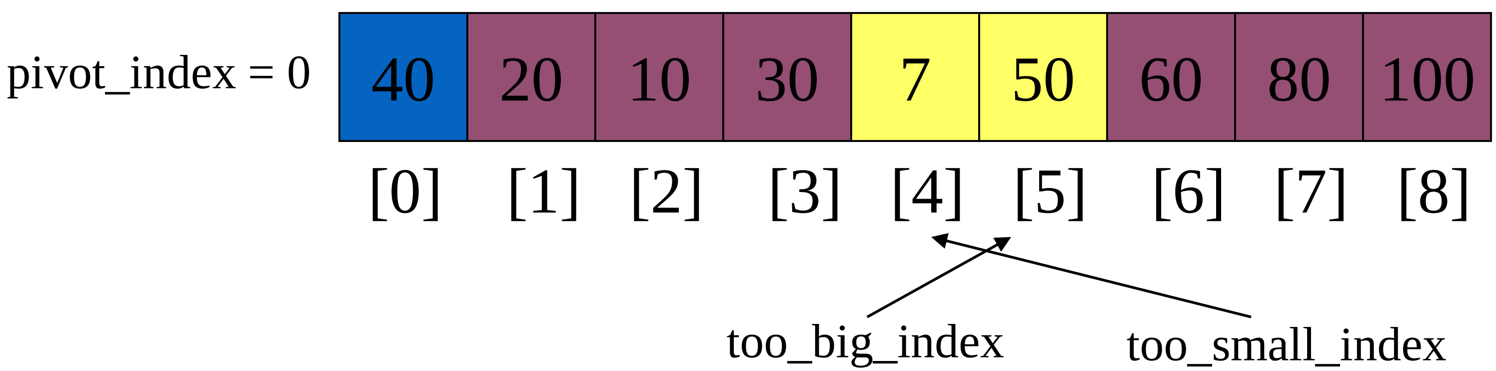

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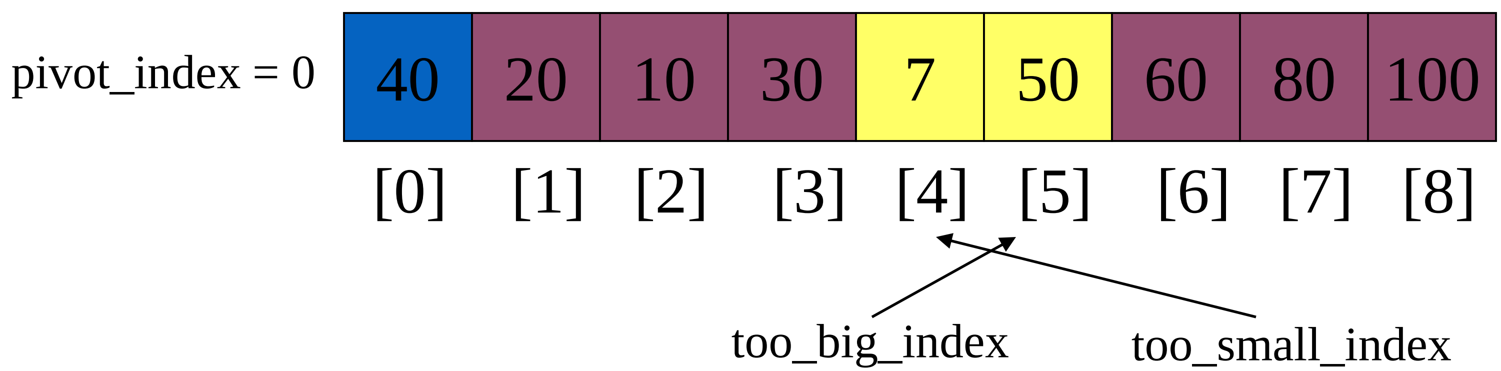

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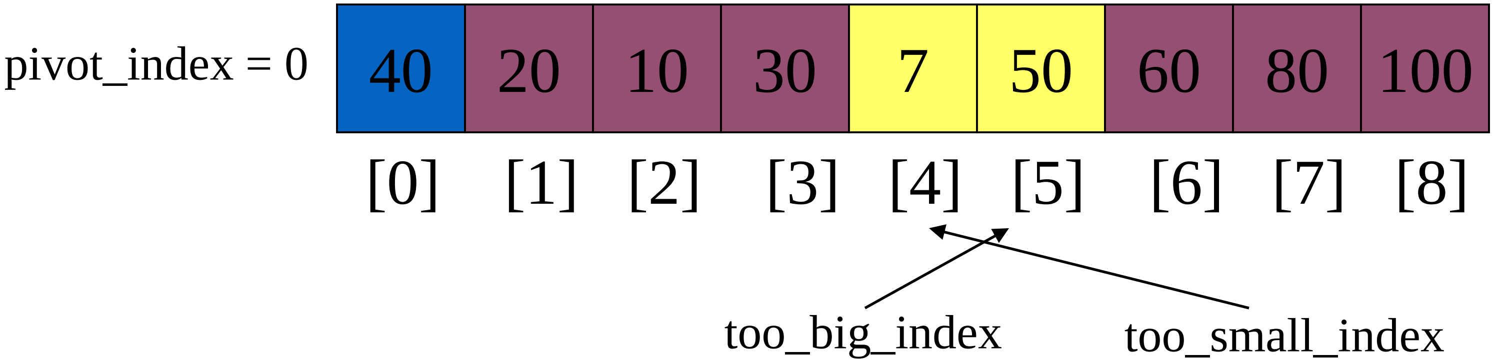

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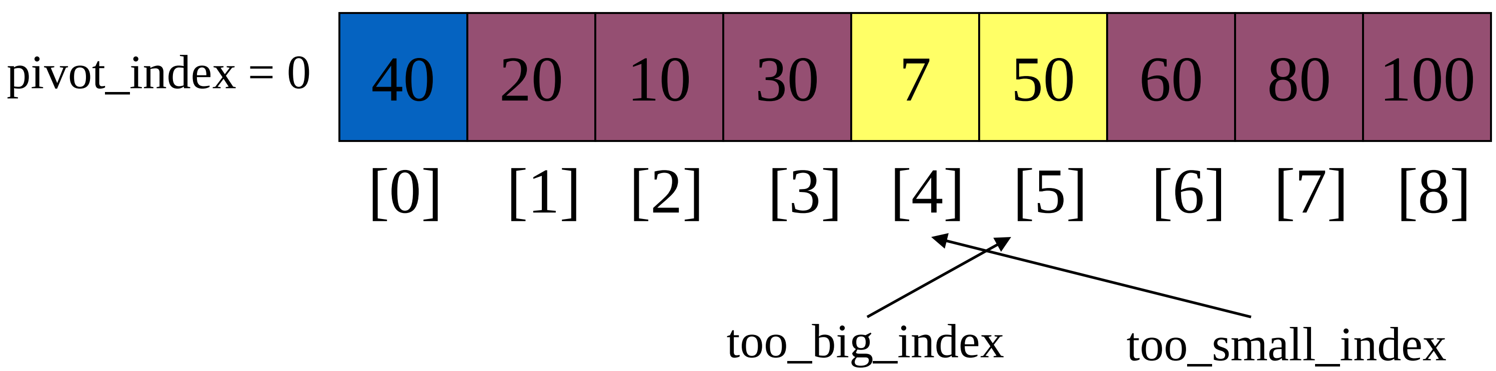

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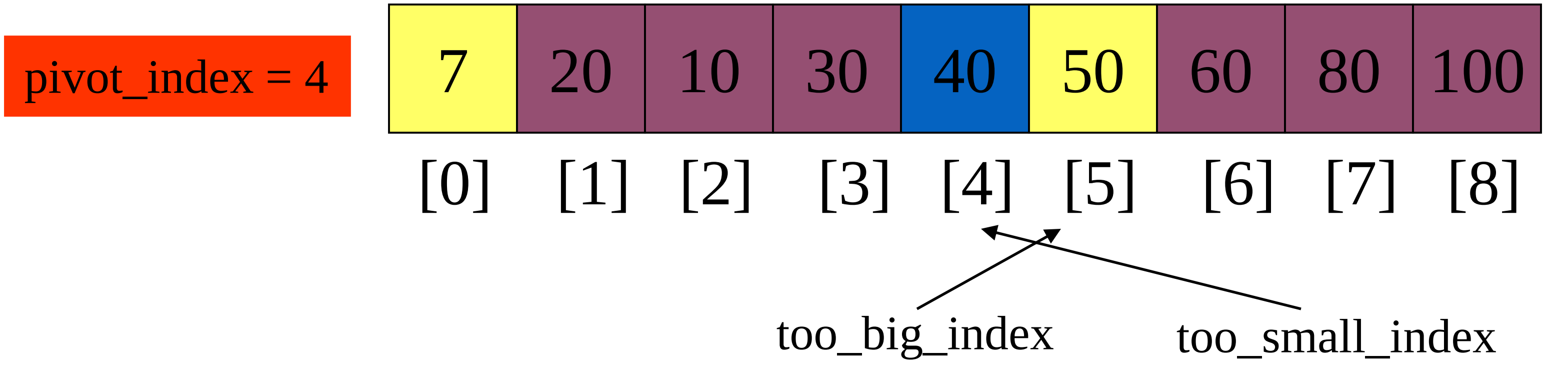
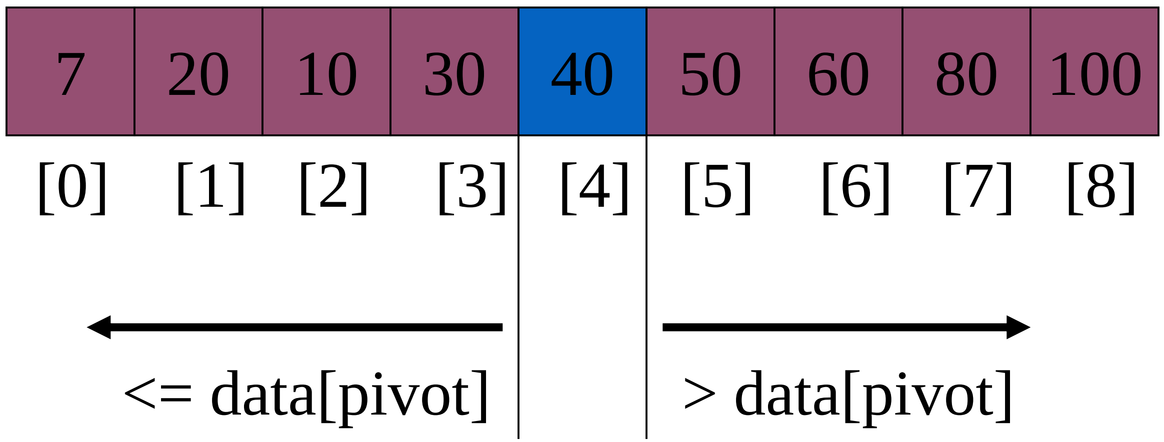
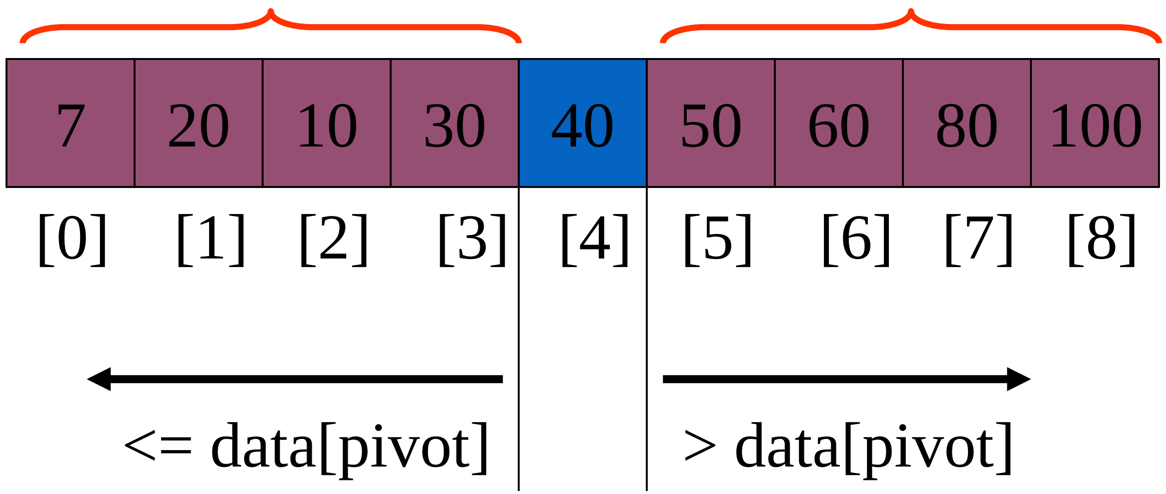


Partition Result



Recursion: Quicksort Sub-arrays



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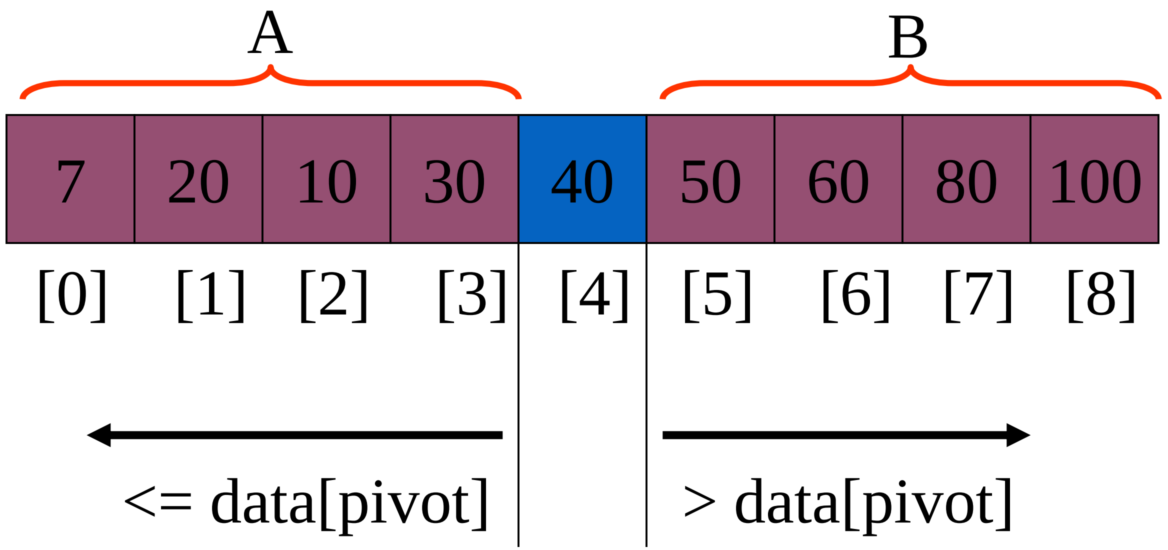

Picking the Pivot

- There are several ways to pick a pivot.

- Objective: Choose a pivot so that we will get 2 partitions of (almost) equal size.

Picking the Pivot (2)

- Use the first element as pivot
  - if the input is random, ok.
  - if the input is presorted (or in reverse order)
    - all the elements go into S_2 (or S_1).
    - this happens consistently throughout the recursive calls.
    - results in $O(N^2)$ behavior (we analyze this case later).
- Choose the pivot randomly
  - generally safe,
  - but random number generation can be expensive and does not reduce the running time of the algorithm.

Picking the Pivot (3)

- Use the median of the array (ideal pivot)
  - The $\lceil N/2 \rceil$ *th* largest element
  - Partitioning always cuts the array into roughly half
  - An optimal quick sort (O(N log N))
  - However, hard to find the exact median

- Median-of-three partitioning
  - eliminates the bad case for sorted input.
  - reduces the number of comparisons by 14%.

Quicksort Analysis

- Assume that keys are random, uniformly distributed.
- What is best case running time?

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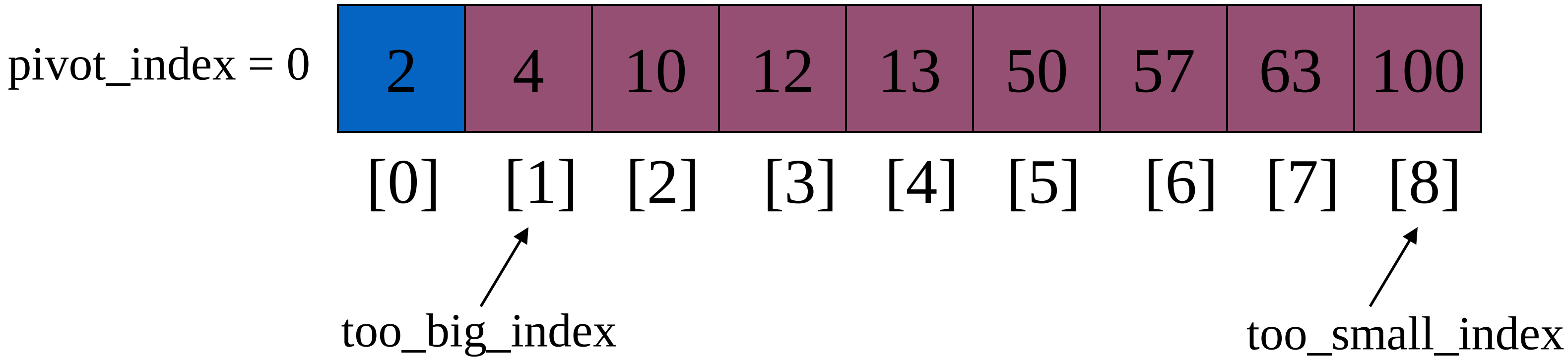
- Worst case running time?

Quicksort: Worst Case

- Assume first element is chosen as pivot.
- Assume we get array that is already in order:



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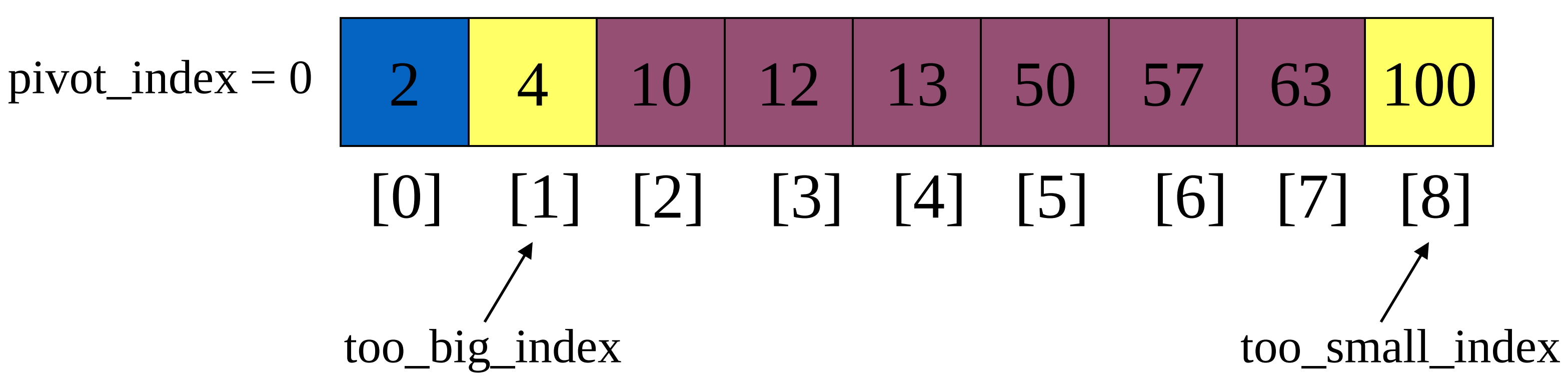

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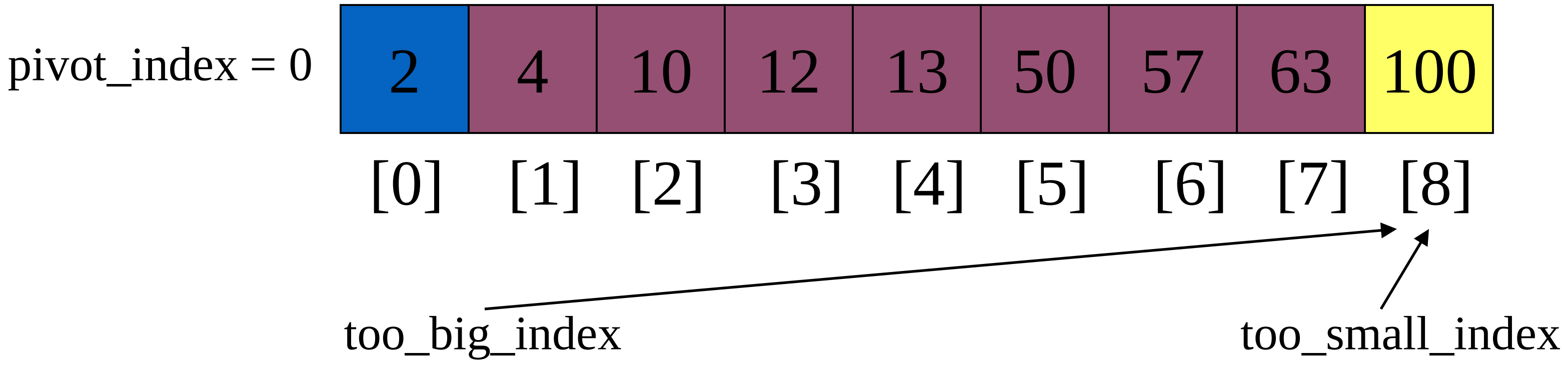

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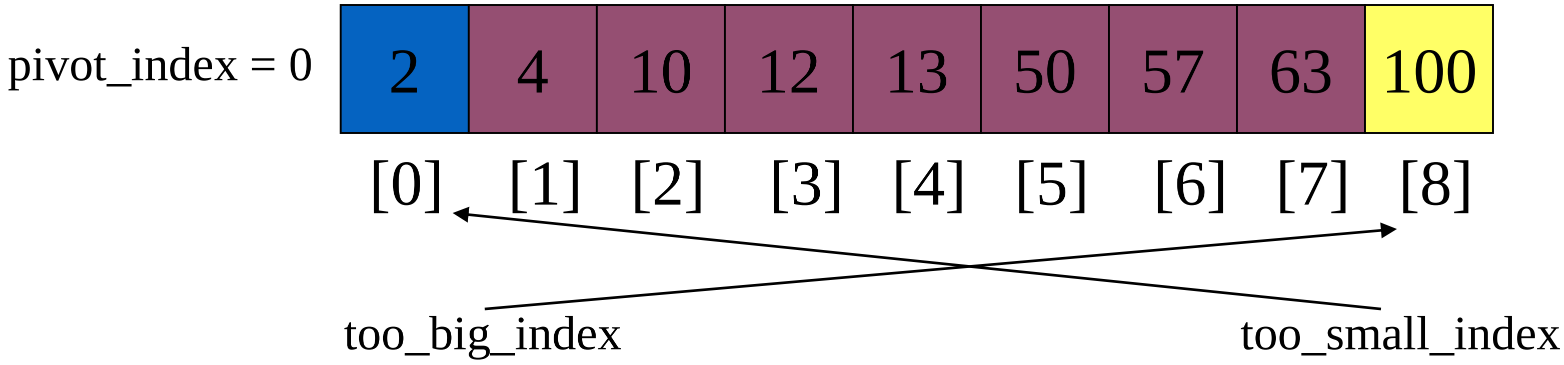

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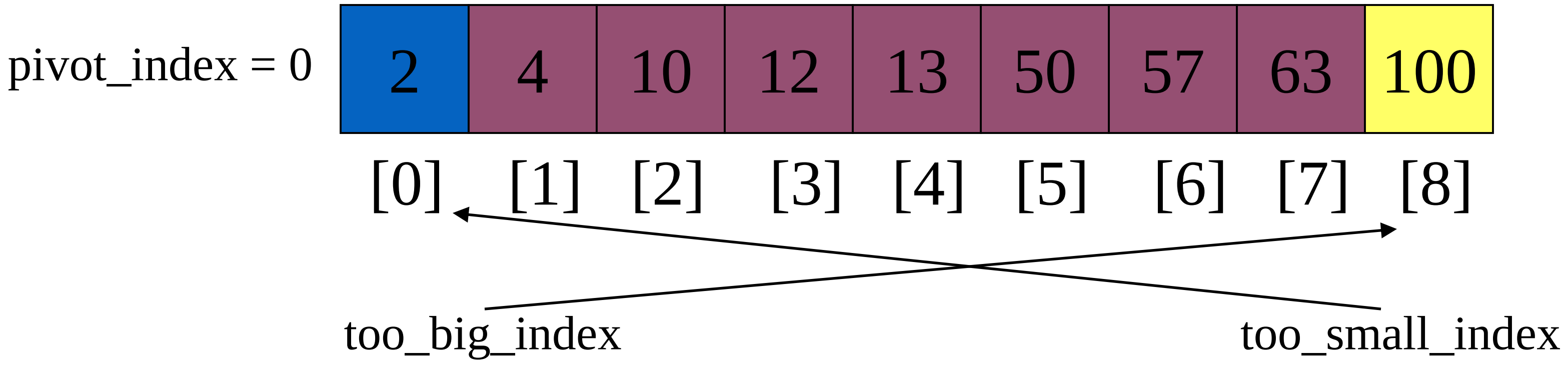

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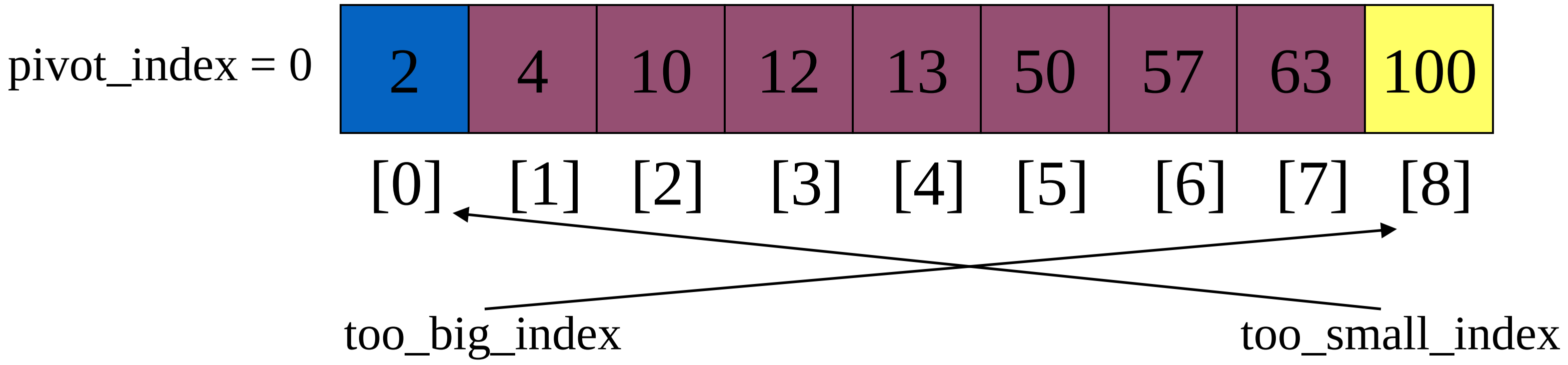

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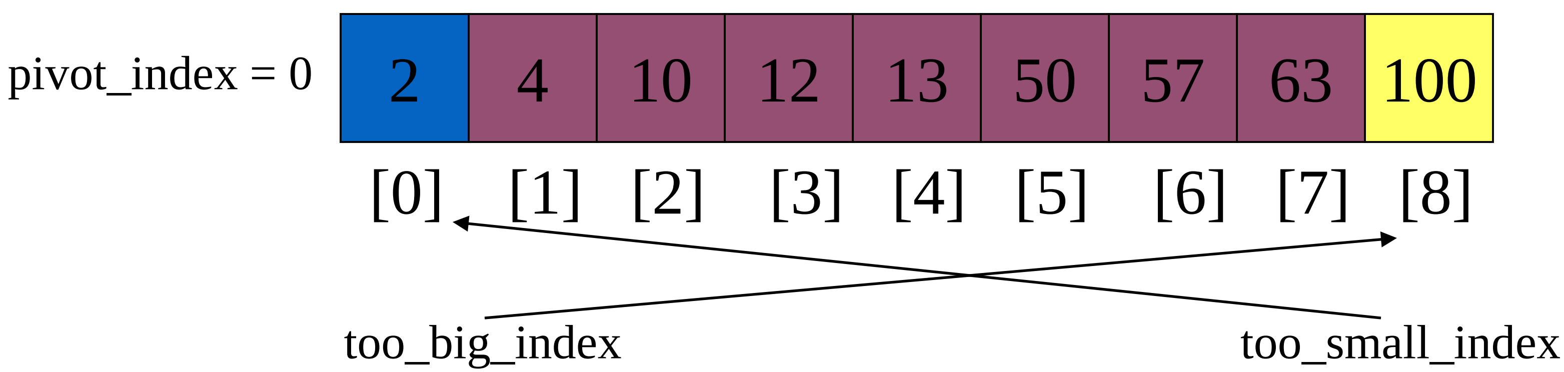

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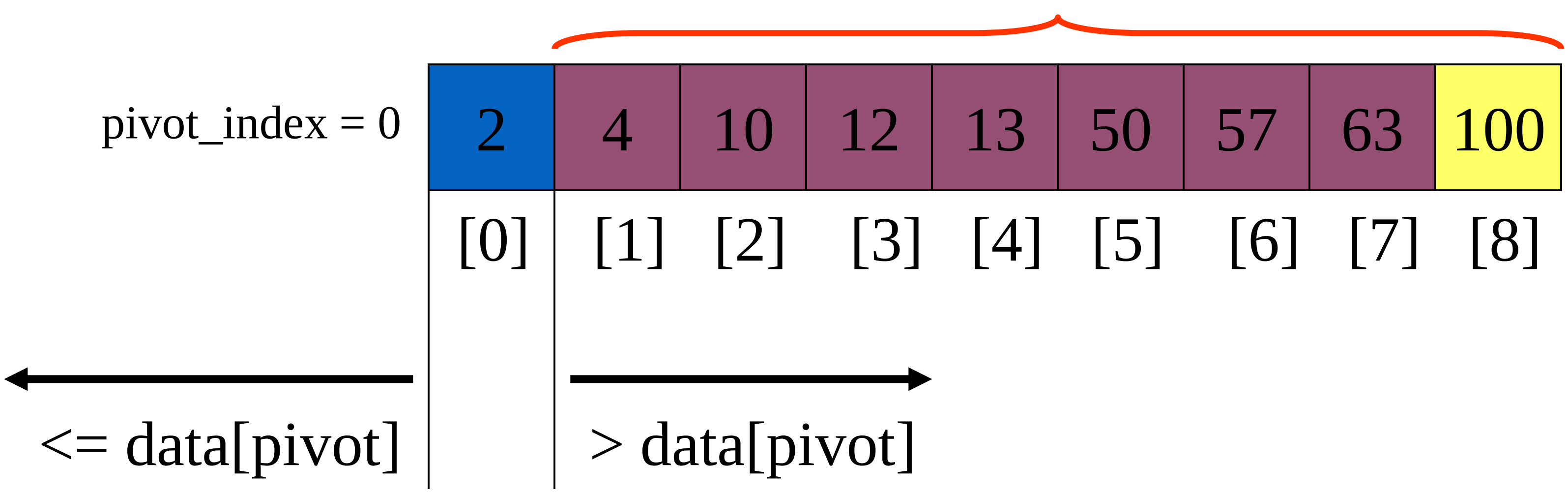

Quicksort Analysis

- Assume that keys are random, uniformly distributed.
- Best case running time: $O(n \log_2 n)$
- Worst case running time?
  - Recursion:
    1. Partition splits array in two sub-arrays:
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    2. Quicksort each sub-array
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- What can we do to avoid worst case?

Improved Pivot Selection

Pick median value of three elements from data array:

data[0], data[n/2], and data[n-1].

Use this median value as pivot.

Improving Performance of Quicksort

- Improved selection of pivot.
- For sub-arrays of size 3 or less, apply brute force search:
  - Sub-array of size 1: trivial
  - Sub-array of size 2:
    - if(data[first] > data[second]) swap them
  - Sub-array of size 3: left as an exercise.