Tutorial Problem Set
(Based on Magnetostatics)

Q. 1 A current I is uniformly distributed over a wire of circular cross-section, with radius R. Find the surface and volume current densities.

Sol. (i) Here, it is obvious that length ⊥ar to the current flow is $2\pi R$ and therefore,

$$dl_\perp = 2\pi R$$

and $K = I/dl_\perp = I/2\pi R$

(ii) Here also area ⊥ar to the current flow is πR^2 and therefore,

$$da_\perp = 2\pi R$$

and $K = I/da_\perp = I/\pi R^2$

Q. 2 (a) The volume current density in a wire of radius R is proportional to the distance from the axis is given by

$$\vec{J} = k\vec{r}$$

for some constant k. Find the total current in the wire.

(b) Find the total current if the volume current density is inversely proportional to the distance from the axis.

Sol. (a) We have, $I = \int_S \vec{J}\cdot d\vec{a} = \int_S (kr)(r\,dr\,d\phi) = k\left(\int_{r=0}^{R} r^2 dr\right)\left(\int_{\phi=0}^{2\pi} d\phi\right) = k\cdot\frac{R^3}{3}\cdot 2\pi = \frac{2\pi k R^3}{3}$

(b) Here, $J = \frac{k}{r}$ so $I = \int_S \vec{J}\cdot d\vec{a} = \int_S \left(\frac{k}{r}\right)(r\,dr\,d\phi) = k\left(\int_{r=0}^{R} dr\right)\left(\int_{\phi=0}^{2\pi} d\phi\right) = k\cdot R\cdot(2\pi) = 2\pi k R$

Q. 3 Suppose the magnetic field in some region has the form

$$\vec{B} = kz\hat{x} \qquad \textbf{(}k\textbf{ is some constant)}$$

Find the force on a square loop of side s, lying in the yz plane, centered at the origin, which carries a current I.

$\left[Ika^2\hat{z}\right]$

Sol. Let I flows in anti-clockwise direction.

The force on the left side (towards the left) cancels the force on the right side (towards the right).

The force on the top is $IsB = Isk(s/2) = Iks^2/2$, (pointing upwards), and the force on the bottom is $IsB = -Iks^2/2$ (also points upwards).

∴ the net force is $\vec{F} = (Iks^2/2 + Iks^2/2) = Iks^2\hat{z}$.

Q. 4 For a configuration of charges and currents confined within a volume V, show that

$$\int_V \vec{J}\,d\tau = \frac{d\vec{p}}{dt},$$

where $\vec{p}$ is the total dipole moment. [*Hint*: Evaluate $\int_V \vec{\nabla}\cdot(x\vec{J})\,d\tau$]

Sol. Here, $\dfrac{d\vec{p}}{dt} = \dfrac{d}{dt}\int_V \rho\vec{r}\,d\tau = \int_V \left(\dfrac{\partial\rho}{\partial t}\right)\vec{r}\,d\tau = -\int_V \left(\vec{\nabla}\cdot\vec{J}\right)\vec{r}\,d\tau$ --- (i)

(using continuity equation)

Using the product rule,

$$\vec{\nabla}\cdot(x\vec{J}) = x(\vec{\nabla}.\vec{J}) + \vec{J}\cdot(\vec{\nabla}x) = x(\vec{\nabla}.\vec{J}) + J_x$$

Thus $$\int_V \left(\vec{\nabla}\cdot\vec{J}\right)x\,d\tau = \int_V \vec{\nabla}\cdot(x\vec{J})\,d\tau - \int_V J_x\,d\tau$$

The first term $\int_V \vec{\nabla}\cdot(x\vec{J})\,d\tau = \int_S x\vec{J}\cdot d\vec{a}$, by divergence theorem.

Since, $\vec{J}$ is entirely inside V, it is zero on the surface *S*.

$$\therefore \quad \int_V \left(\vec{\nabla}\cdot\vec{J}\right)x\,d\tau = -\int_V J_x\,d\tau \qquad \text{--- (ii)}$$

$$\text{Similarly,} \quad \int_V \left(\vec{\nabla}\cdot\vec{J}\right)y\,d\tau = -\int_V J_y\,d\tau \qquad \text{--- (iii)}$$

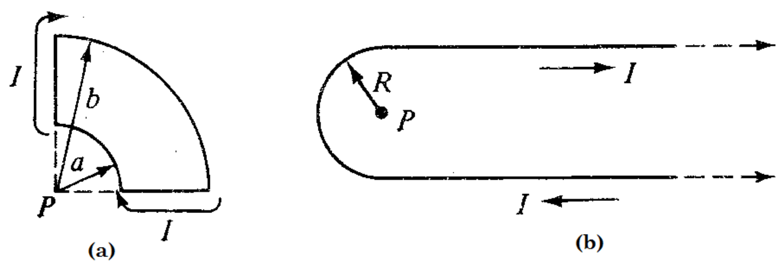
$$\text{and} \quad \int_V \left(\vec{\nabla}\cdot\vec{J}\right)z\,d\tau = -\int_V J_z\,d\tau \qquad \text{--- (iv)}$$

Combining (ii), (iii) and (iv), we get

$$\int_V \left(\vec{\nabla}\cdot\vec{J}\right)\vec{r}\,d\tau = -\int_V \vec{J}\,d\tau \qquad \text{--- (v)}$$

From (i) and (v), $$\frac{d\vec{p}}{dt} = \int_V \vec{J}\,d\tau$$

Q. 5 Find the magnetic field at point *P* for each of the steady current



configurations shown in Fig.

Sol. (a) The straight line segments produce no field at *P*. The two quarter-circles give

$$\vec{B} = \frac{\mu_o I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)$$

out of the page.

(b) The two half-lines are the same as one infinite line which has magnetic field as

$$B = \frac{\mu_o I}{2\pi R}$$

and the half-circle contribution is

$$B_{half} = \frac{\mu_o I}{8R}$$

so $\quad B = \frac{\mu_o I}{4R}\left(1+\frac{2}{\pi}\right)$ directed into the page perpendicular to it.

Q. 6 A steady current I flows down a long cylindrical wire of radius R. Find the magnetic field, both inside and outside the wire, if
(a) The current is uniformly distributed over the outside surface of the wire;
(b) The current is distributed in such a way that J is proportional to r, the distance from the axis.

Sol. (a) $\oint \vec{B}\cdot d\vec{l} = B(2\pi r) = \mu_o I_{enc} \qquad \Rightarrow \vec{\mathbf{B}} = \begin{cases} 0, & for\ r < R; \\ \frac{\mu_o I}{2\pi r}\hat{\phi} & for\ r > R. \end{cases}$

(b) $J = kr; \qquad I = \int_0^R J\,da = \int_0^R kr(2\pi r)\,dr = \frac{2\pi k R^3}{3} \qquad \Rightarrow k = \frac{3I}{2\pi R^3}$

Now, $I_{enc} = \int_0^r J\,da = \int_0^r kr'(2\pi r')\,dr' = \frac{2\pi k r^3}{3} = I\frac{r^3}{R^3}$, for $r < R$; $I_{enc} = I$, for $r > R$

So
$$\vec{\mathbf{B}} = \begin{cases} \frac{\mu_o I r^2}{2\pi R^3}\hat{\phi}, & for\ r < R; \\ \frac{\mu_o I}{2\pi r}\hat{\phi} & for\ r > R. \end{cases}$$

Q. 7 What current density would produce a constant azimuthal potential, $A_\phi = k$, in cylindrical co-ordinates?
Sol.

$$A_\phi = k \Rightarrow \mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\,\hat{\mathbf{z}} = \frac{k}{s}\hat{\mathbf{z}};\ \mathbf{J} = \frac{1}{\mu_0}(\nabla\times\mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \boxed{\frac{k}{\mu_0 s^2}\hat{\phi}.}$$

Q. 8 If $\vec{B}$ is uniform, show that $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$, where $\vec{r}$ is a vector from the origin to the point in question. That is, check that $\vec{\nabla}\cdot\vec{A} = 0$ and $\vec{\nabla}\times\vec{A} = \vec{B}$.
Sol.

$$\nabla\cdot\mathbf{A} = -\frac{1}{2}\nabla\cdot(\mathbf{r}\times\mathbf{B}) = -\frac{1}{2}[\mathbf{B}\cdot(\nabla\times\mathbf{r}) - \mathbf{r}\cdot(\nabla\times\mathbf{B})] = 0,$$

since $\nabla\times\mathbf{B} = 0$ ($\mathbf{B}$ is uniform) and $\nabla\times\mathbf{r} = 0$

$$\nabla\times\mathbf{A} = -\frac{1}{2}\nabla\times(\mathbf{r}\times\mathbf{B}) = -\frac{1}{2}[(\mathbf{B}\cdot\nabla)\mathbf{r} - (\mathbf{r}\cdot\nabla)\mathbf{B} + \mathbf{r}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{r})].$$

But $(\mathbf{r}\cdot\nabla)\mathbf{B} = 0$ and $\nabla\cdot\mathbf{B} = 0$ (since $\mathbf{B}$ is uniform), and

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

Finally, $(\mathbf{B}\cdot\nabla)\mathbf{r} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}} = \mathbf{B}.$

So $\nabla \times \mathbf{A} = -\frac{1}{2}(\mathbf{B} - 3\mathbf{B}) = \mathbf{B}.$

Q. 9 Find the vector potential above and below an infinite plane surface current $\vec{K} = K\hat{i}$ covering the *xy* plane.

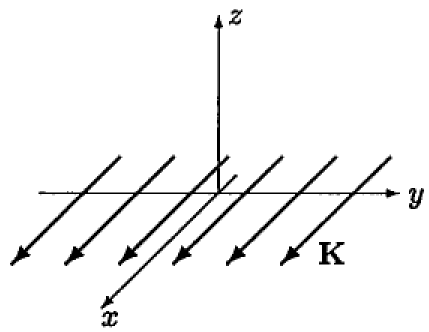
Sol.

$\mathbf{K} = K\,\hat{\mathbf{x}} \Rightarrow \mathbf{B} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}$ (plus for $z < 0$, minus for $z > 0$).

$\mathbf{A}$ is parallel to $\mathbf{K}$, and depends only on z, so $\mathbf{A} = A(z)\,\hat{\mathbf{x}}$.

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A(z) & 0 & 0 \end{vmatrix} = \frac{\partial A}{\partial z}\,\hat{\mathbf{y}} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}.$$

$$\boxed{\mathbf{A} = -\frac{\mu_0 K}{2}|z|\,\hat{\mathbf{x}}}$$



Q. 10 The magnetic vector potential at any point due to a dipole of magnetic moment $\vec{m}$ pointing in the *z*-direction is given by

$$\vec{A}_{dip} = \frac{\mu_o}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

Calculate the magnetic field $\vec{B}_{dip}$ at that point.

Sol. We have Curl $\vec{A} =$

$$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

Here, $A_\phi = \frac{\mu_o}{4\pi}\frac{m\sin\theta}{r^2}$, which on putting in above equation gives

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$