Science Coffee House, IIT Kanpur

Group Theory and Quantum Error Correction
Assignment 1

INSTRUCTIONS

- *Questions are made to build up your understanding; referring is fine, but copying is discouraged*
- *You are strongly encouraged to do this task alone, but if you do collaborate with someone else, both should mention their collaborator's name in the submitted document*
- *You may resort to any (classical) computation when needed*
- *Everything not forbidden is allowed!* [1]

Question 1: Prove that the set of all rational numbers with odd denominators is a group with respect to addition.

Question 2: Prove that if G is an Abelian group, then $\forall a, b \in G$ and $\forall n \in \mathbb{Z}$, $(ab)^n = a^n b^n$.

The symmetric group S_n is the set of all bijections from an n element set to itself.

Question 3: In S_3 give an example of two elements x, y such that $(xy)^2 \neq x^2 y^2$.

Question 4: If G is a group of even order, prove that it has an element $a \neq e$ such that $a^2 = e$.

Question 5: Prove that every subgroup of a cyclic group is cyclic.

Question 6: $\forall a, b \in \mathbb{Z}$, let $\sim$ be an equivalence relation such that $a \sim b$ if $a - b$ divides 4 (show this). Find all the distinct equivalence classes of $\sim$.

[1] A certain physics professor (based) in IITK

Question 7: Find the subgroups generated by each of the elements of S_3.

The center of a group G, $Z(G)$ is defined as the set of elements in the group that commute will all elements of the group; in other words, $Z(G) = \{g \in G \mid ag = ga \; \forall a \in G\}$.

Question 8: Show that $Z(G)$ is a subgroup of G.

Question 9: For a finite group G, show that for all $a \in G$, there exists some integer n such that $a^n = e$.

The generators of a group are the smallest set of elements that 'generate' the entire group in the sense that repeated multiplication of these elements creates the entire group. Ex: The generators of D_3 are $R_{120^\circ}, \mathrm{Flip}_A$.

Note: The generators of a group need not be unique; for instance $R_{240^\circ}, \mathrm{Flip}_B$ are also valid generators.

Question 10: Check explicitly that both the above generators generate D_3.

Question 11.1: Let there be k generators of a group G such that each of them squares to identity. Show that G is Abelian.

Question 11.2: Show that $|G| = 2^k$. Using this conclude that for a general group G, $|G| \geq 2^k$.