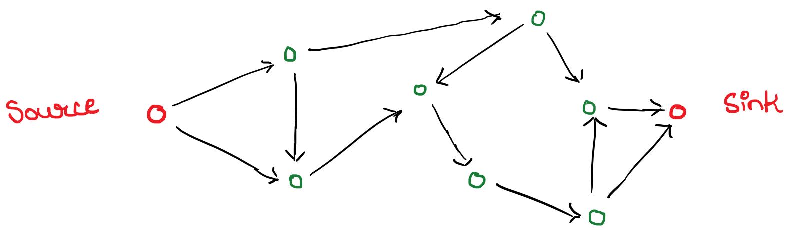
Flow Networks

Suppose that a Network consists of a Source, a Sink, and these two nodes are connected via switches. The bandwidth of each connection isn't the same. At the same time, each of the switch does cut-through switching without any Queue capacity.



This system can be modeled using a directed graph with two special nodes, the source and the sink.

Let the graph be $G(V, E)$, source be s and sink be t.

- By definition, no edge terminates at s and no edge begins at t.

Each edge has a capacity $c(e)$, which is the maximum possible capacity of that edge. We define $f(e)$, as the capacity of the edge in use currently.

$$\forall e \in E, \quad 0 \leq f(e) \leq c(e)$$ Capacity Constraint

* Flow

Following the example, each bridge has a data stream flowing through it. As no data can be accumulated, Inward flow = Outward flow

(Similar to Kirchoff's law from electricity) flow constraint

- The outwards flow for a node is represented by $f^{\rightarrow}(v)$, and is given by:-

$$f^{\rightarrow}(v) = \sum_{\substack{u \in V \\ (u,v) \in E}} f((u,v))$$

The value of inwards flow is similarly defined, and is represented as $f^{\leftarrow}(v)$.

- Value of the flow $|f|$

The total amount of data "flowing" through the network. It is given by $|f|$, and is calculated as shown.

Lemma

$$|f| = \sum_{\substack{v \in V \\ (s,v) \in E}} f(s,v) = f^{\rightarrow}(s) = \sum_{\substack{v \in V \\ (v,t) \in E}} f(v,t) = f^{\leftarrow}(t)$$

Proof

From above, we can see that

$$|f| = f^{\rightarrow}(s) + 0$$

$$= f^{\rightarrow}(s) + \sum_{v \in V \setminus \{s,t\}} \left(f^{\rightarrow}(v) - f^{\leftarrow}(v) \right) \rightarrow 0 \text{ by flow constraint}$$

$$= \sum_{v \in V \setminus \{t\}} \left(f^{\rightarrow}(v) - f^{\leftarrow}(v) \right) \qquad f^{\leftarrow}(s) = 0 \text{ by definition}$$

$$= \sum_{v \in V \setminus \{t\}} f^{\rightarrow}(v) - \sum_{v \in V \setminus \{t\}} f^{\leftarrow}(v)$$

$$= \underset{①}{f^{\rightarrow}(V \setminus \{t\})} - \underset{②}{f^{\leftarrow}(V \setminus \{t\})}$$

New notation!

Notice that all edges appear in ① as t has no flow leaving it. However, the edges "supplying" t would not be present in ②. These edges are left after subtraction.

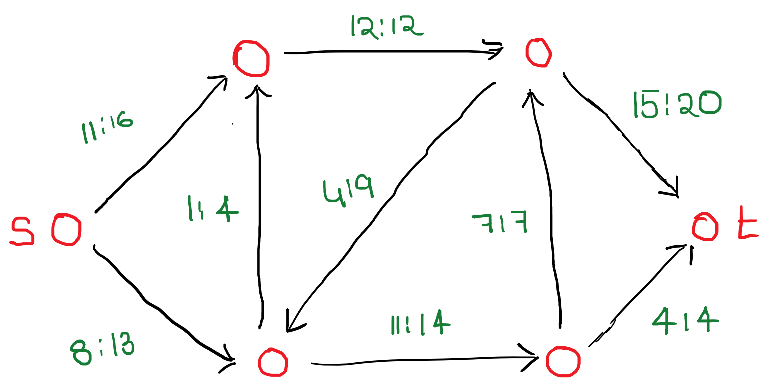
$$\rightarrow \quad |f| = f^{\rightarrow}(V \setminus \{t\}) - f^{\leftarrow}(V \setminus \{t\}) = \underline{f^{\leftarrow}(t)}$$

QED

* Flow Network notation

The network is represented as a directed acyclic graph, with designated source and sink nodes. Each edge is labelled as $f(e) : c(e)$.

Capacity constraints and flow constraints have to be satisfied.



We would like to know what the maximum possible flow in this network is. The following concept is introduced for this.

* An (s,t)-cut :-

Let the flow network be given by $G(V,E)$ with s as the source and t is the sink. An (s,t)-cut is given by partioning V into two sets, S and T which are mutully exclusive and exhaustive.

$$s \in S \qquad S \cup T = V$$
$$t \in T \quad , \quad S \cap T = \phi$$

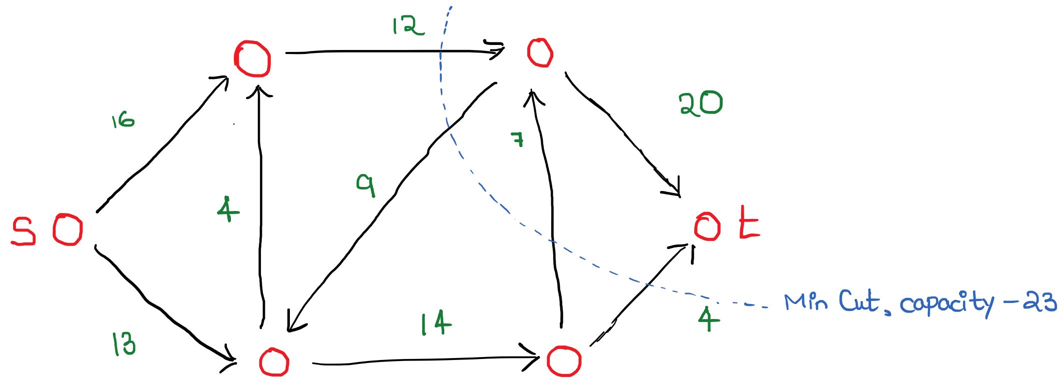
The capacity for an (s,t)-Cut (S,T) is given by:-

$$cap(S,T) = \sum_{\substack{u \in S,\, v \in T \\ (u,v) \in E}} c(u,v)$$

Notice how only one direction is considered, S→T.

- Mincut problem

Given a flow network, we wish to find the cut of the graph which has the least possible capacity. For example,



There is an inherent relationship between max-flow and min-cut classes of problems. The below lemma hints at what this could be.

Lemma
Weak duality

Consider a flow network G. For any (s,t)-Cut (S,T) over this G, the value of $|f| \leq$ capacity (S,T). The equality is acheived when the flow through edges S→T is saturated and through edges T→S is empty.