Moment Generating function

If 'X' is a Discrete or Continuous R.V. then the MGF is

$$M_X(t) = E[e^{tx}] = \sum_x e^{tx} p(x)$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Note

$$M_X(t) = E[e^{tx}]$$

$$= E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots + \frac{t^r x^r}{r!} + \cdots\right]$$

$$= E[1] + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \cdots \frac{t^r}{r!}E[X^r] + \cdots$$

$$= 1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E[X^2] + \cdots \frac{t^r}{r!}E[X^r] + \cdots$$

Coeff of $\frac{t}{1!} = E[X]$, Mean

" $\frac{t^2}{2!} = E(X^2)$, Second moment

Coefficient of $\frac{t}{1!}$ = $E(X)$, Mean

" " $\frac{t^2}{2!}$ = $E(X^2)$, Second Moment

$$V(x) = E(X^2) - [E(X)]^2$$

In General

The n^{th} moment is the coefficient of

$$\frac{t^n}{n!} = E[X^n].$$

(mean)

2. $\mu_1' = \left[M_x'(0) \right] = \underbrace{\left[\frac{d}{dt} [M_x(t)] \right]_{t=0}}$

3. $\mu_2' = \left[M_x''(0) \right] = \underbrace{\left[\frac{d^2}{dt^2} M_x(t) \right]_{t=0}}$

(second moment)

$\therefore$ Variance, $\underline{\underline{\mu_2}} = \mu_2' - (\mu_1')^2$

Ex :-

Find the moment generating function for $p(x) = \begin{cases} \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases}$

Soln :-

$$M_X(t) = E[e^{tx}] = \sum_x e^{tx} p(x)$$

$$= \sum_{x=1,2} e^{tx} p(x)$$

$$= e^{t} \cdot \frac{1}{2} + e^{2t} \cdot \frac{1}{4}$$

$$= \frac{e^{t}}{2} \left[1 + e^{t}/2 \right]$$

$$= \frac{e^{t}}{2} \left[\frac{2 + e^{t}}{2} \right]$$

$$\therefore M_X(t) = \frac{e^{t}}{4} (2 + e^{t})$$

Ex 2 :-

Find the MGF for the given function $p[X=x] = \frac{1}{2^x}$, $x = 1, 2, 3 \dots$

Sol :-

$$M_x(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$\because (1-x)^{-1} = 1 + x + x^2 + \cdots$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2 - e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{\cancel{2}} \times \frac{\cancel{2}}{2 - e^t}$$

$$M_x(t) = \frac{e^t}{2 - e^t} \quad //$$

3) Find $M_x(t)$ for $P[X = x] = 2\left(\frac{1}{3}\right)^x$, $x = 1, 2, 3, \ldots \ldots$ Also find Mean, μ_1' and Variance μ_2.

$$M_x(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} \cdot 2\left[\frac{1}{3}\right]^x$$

$$= 2 \sum_{x=1}^{\infty} \left(\frac{e^t}{3}\right)^x$$

$$= 2\left\{\frac{e^t}{3} + \left(\frac{e^t}{3}\right)^2 + \left(\frac{e^t}{3}\right)^3 + \cdots\right\}$$

$$= \frac{2e^t}{3}\left\{1 + \frac{e^t}{3} + \left(\frac{e^t}{3}\right)^2 + \cdots\right\}$$

$$= \frac{2}{3}e^t\left(1 - \frac{e^t}{3}\right)^{-1}$$

$$= \frac{2}{3} e^t \left[1 - \frac{e^t}{3} \right]^{-1}$$

$$= \frac{2e^t}{3} \left[\frac{3 - e^t}{3} \right]^{-1}$$

$$= \frac{2e^t}{\cancel{3}} \times \frac{\cancel{3}}{3 - e^t} = \frac{2e^t}{3 - e^t}$$

$$M_x(t) = \frac{2e^t}{3-e^t}$$

$$M_x'(t) = \frac{(3-e^t)2e^t - 2e^t[-e^t]}{(3-e^t)^2}$$

$$\mu_1' = M_x'(0) = \frac{(3-1)(2) - 2(-1)}{(3-1)^2} = \frac{2(2)+2}{2^2}$$

$$\mu_1' = \frac{6}{4}$$

$$\therefore \mu_1' = \frac{3}{2} \quad //$$

$$M_x'(t) = \frac{(3-e^t)\,2e^t - 2e^t(-e^t)}{(3-e^t)^2}$$

$$= \frac{6e^t - \cancel{2e^{2t}} + \cancel{2e^{2t}}}{(3-e^t)^2} = \frac{6e^t}{(3-e^t)^2}$$

$$M_X''(t) = \frac{(3-e^t)^2\, 6e^t - 6e^t \cdot 2(3-e^t)(-e^t)}{(3-e^t)^4}$$

$$\mu_2' = M_X''(0) = \frac{\left((3-1)^2 \cdot 6\right) - 6 \cdot 2(3-1)(-1)}{(3-1)^4}$$

$$= \frac{(4 \cdot 6) - 6 \cdot 2 \cdot 2 \cdot (-1)}{2^4} = \frac{24+24}{16}$$

$$= \frac{48}{16} = 3$$

$$\therefore \mu_2' = 3$$

Variance $\mu_2 = \mu_2' - (\mu_1')^2$

$$= 3 - \left(\frac{3}{2}\right)^2$$

$$= 3 - \frac{9}{4} = \frac{12-9}{4} = 3/4$$

4) The M.G.F of a random variable X is given by $M_X(t) = \dfrac{1}{1-t^2}$.

Find Mean and Variance of X.

<u>Sol</u> :-

$$M_X(t) = \frac{1}{1-t^2}$$

$$M_X'(t) = \frac{-1}{(1-t^2)^2}[-2t] = \frac{2t}{(1-t^2)^2}$$

$$\mu_1' = M_x'(t) = 0$$

$$M_x''(t) = \frac{(1-t^2)^2 \cdot 2 - 2t\left[2(1-t^2)(-2t)\right]}{(1-t^2)^4}$$

$$\mu_2' = M_x''(0) = \frac{(2) - 0}{1} = 2$$

Variance. $\mu_2 = \mu_2' - \mu_1'^2 = 2 - 0 = 2$ //

5) The pdf of a random variable X is $f(x) = e^{-x}$, $\underline{x \geq 0}$. Find $M_x(t)$

Find μ_1' and μ_2.

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} \cdot e^{-x} \, dx$$

$$= \int_0^\infty e^{-x + tx} \, dx$$

$$= \int_0^\infty e^{-x[1 - t]} \, dx$$

$$= \left[\frac{e^{-x(1-t)}}{-(1-t)} \right]_0^\infty$$

$$= \frac{-1}{(1-t)} \left\{ 0 - (1) \right\}$$

$$M_x(t) = \frac{1}{(1-t)}$$

$$M_x(t) = \frac{1}{1-t}$$

$$M_x'(t) = \frac{-1}{(1-t)^2}(-1)$$

$$= \frac{1}{(1-t)^2}$$

$$\mu_1' = M_x'(0) = 1$$

$$M_x''(t) = \frac{-2}{(1-t)^3}(-1) = \frac{2}{(1-t)^3}$$

$$\mu_2' = M_x''(0) = 2$$

Variance, $\mu_2 = \mu_2' - \mu_1'^2$

$$= 2 - (1)^2$$

$$= 2-1 = 1$$

Theoretical Results

1. $M_{ax}(t) = M_x(at)$

2. Let X_1 and X_2 be two Independent Random Variables, then

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t).$$

3. Effect of change of origin and scale on MGF:

$$M_{\frac{X-a}{c}}(t) = e^{-at/c} M_X(t/c),$$

where a and c are constants

4. MGF of a random variable, if exists, is unique

5. The MGF if it exists, uniquely determines the distribution function.