Ex

1. Given below is the pdf of a R.V. 'X'

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty.$$

Find $\phi_X(t)$ and Mean and Variance

Sol'n :-

$$\phi_X(t) = E\left[e^{itx}\right] = \int_{-\infty}^{\infty} e^{itx} \frac{1}{2} e^{-|x|} \, dx$$

$$= \int_{-\infty}^{0} e^{itx} \frac{1}{2} e^{-(-x)} dx + \int_{0}^{\infty} e^{itx} \frac{1}{2} e^{-(x)} dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{x+itx} dx + \frac{1}{2}\int_{0}^{\infty} e^{itx-x} dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{x(1+it)} dx + \frac{1}{2}\int_{0}^{\infty} e^{-x[1-it]} dx$$

$$= \frac{1}{2}\left[\frac{e^{x(1+it)}}{(1+it)}\right]_{-\infty}^{0} + \frac{1}{2}\left[\frac{e^{-x(1-it)}}{-(1-it)}\right]_{0}^{\infty}$$

$$= \frac{1}{2(1+it)}\left[1\right] - \frac{1}{2(1-it)}\left[0 - 1\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+it} + \frac{1}{1-it}\right] = \frac{1}{2}\left[\frac{1 \cancel{-it} + 1 \cancel{+it}}{1+t^2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1+t^2}\right]$$

$$\phi_x(t) = \frac{1}{1+t^2}$$

$$\phi_x'(t) = \frac{-1}{(1+t^2)^2}(2t)$$

$$\mu_1' = (-i)^1 \frac{d}{dt}\left[\phi_x(t)\right]_{t=0} = (-i)^n \left\{\frac{-2t}{(1+t^2)^2}\right\}_{t=0}$$

$$\Rightarrow k_1' = 0$$

$$\frac{d^2}{dt^2}\phi_x(t) = \frac{d}{dt}\left[\frac{-2t}{(1+t^2)^2}\right]$$

$$= \frac{(1+t^2)^2(-2) - 2(-2t)(1+t^2)(2t)}{(1+t^2)^4}$$

$$\left[\frac{d^2}{dt^2}\phi_x(t)\right]_{t=0} = \frac{(-2)}{1} = -2$$

$$\mu_2' = (-i)^2 \frac{d^2}{dt^2}\left[\phi_x(t)\right]_{t=0}$$

$$= (-i)^2 (-2)$$

$$= (-1)\cdot(-2)$$

$$\mu_2' = 2$$

$$\text{Variance} = \mu_2' - \mu_1'^2 = 2 - 0 = 2$$

$$\therefore \ \mu_2 = 2$$

Inversion theorem

Suppose $\phi(t)$ be the characteristic function of the distribution function $F(x)$.

For any two points, a and $a+h$, $(h>0)$ of continuity of $F(x)$ we have

$$F(a+h) - F(a) = \lim_{T \to \infty} \int_{-T}^{T} \frac{1 - e^{-ith}}{it} \phi(t)\, dt.$$

Remark 1 :-

If $f(x)$ is the pdf of x

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt$$

Ex

Find the pdf $f(x)$ whose characteristic function is $\phi(t) = e^{-|t|}$, $-\infty < t < \infty$.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} \cdot e^{-|t|} \, dt$$

$$= \frac{1}{2\pi}\left\{ \int_{-\infty}^{0} e^{-itx} \cdot e^{+t} \, dt + \int_{0}^{\infty} e^{-itx} \cdot e^{-t} \, dx \right.$$

$$= \frac{1}{2\pi} \left\{ \int_{-\infty}^{0} e^{t[1-ix]} \, dt + \int_{0}^{\infty} e^{-t[1+ix]} \, dt \right]$$

$$= \frac{1}{2\pi} \left\{ \left[\frac{e^{t(1-ix)}}{(1-ix)} \right]_{-\infty}^{0} + \left[\frac{e^{-t(1+ix)}}{-(1+ix)} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{1}{1-ix} \right\} + \left\{ 0 + \frac{1}{1+ix} \right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1+\cancel{jx}+1-\cancel{jx}}{1+x^2}\right\}$$

$$= \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{2}}{1+x^2}\right]$$

$$\therefore f(x) = \frac{1}{\pi}\cdot\left[\frac{1}{1+x^2}\right] \quad ; \quad -\infty < x < \infty.$$

Ex 2.

Find the pdf for the given

$$\phi_x(t) = e^{-t^2/2}, \quad -\infty < t < \infty.$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \cdot e^{-t^2/2} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - t^2/2} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[t^2 + 2itx\right]} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(t^2 + 2itx + (ix)^2] + (ix)^2\right]} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2} \cdot e^{+(ix)^2} \, dt$$

$$= \frac{1}{2\pi} e^{-x^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2} \, dt}$$