The joint pdf of R.V. X and Y is given by $f(x,y) = kxye^{-(x^2+y^2)},\ x>0,\ y>0$. Find the value of k and also prove that X and Y are independent.

Sol'n:-

$$\int_0^\infty\int_0^\infty kxy e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$\int_0^\infty kx\left(\int_0^\infty y e^{-y^2}\,dy\right) e^{-x^2}\,dx = 1$$

Let $t = y^2$, $y = 0$, $t = 0$

$dt = 2y\,dy$ $y = \infty$, $t = \infty$

$\frac{dt}{2} = y\,dy$

$$\int_0^\infty e^{-t} \frac{dt}{2} = \frac{1}{2}\left[-e^{-t}\right]_0^\infty = \frac{1}{2}.$$

$$\frac{k}{2}\int_0^\infty x e^{-x^2}\,dx = 1$$

$$\frac{k}{2}\left[\frac{1}{2}\right] = 1$$

$$\boxed{k = 4}$$

$$\therefore f(x,y) = 4xy\,e^{-(x^2+y^2)}, \quad x > 0,\ y > 0$$

To prove: $f(x,y) = f(x) \cdot f(y)$

$$f(x) = \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dy$$

$$= 4x\, e^{-x^2} \int_0^\infty y\, e^{-y^2}\, dy$$

$$= 4x e^{-x^2} \left[\frac{1}{2} \right] = 2x e^{-x^2}$$

$$f(y) = \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dx$$

$$= 4y\, e^{-y^2} \underbrace{\int_0^\infty x\, e^{-x^2}\, dx}$$

$$= 4y e^{-y^2} \cdot \frac{1}{2}$$

$$f(y) = 2y e^{-y^2}$$

$$f(x) \cdot f(y) = 2x e^{-x^2} \cdot 2y e^{-y^2}$$

$$= 4xy e^{-(x^2+y^2)}$$

$$= f(x, y)$$

$\Rightarrow$ X and Y are independent

- Given $f(x,y)=\begin{cases} Cx(x-y) & ,0<x<2,\ -x<y<x \\ 0 & otherwise \end{cases}$. (i) Find C (ii) Find $f(x)$ (iii) $f(y/x)$

Sol:-

$$\int_0^2 \int_{-x}^{x} c\,x(x-y)\,dy\,dx = 1$$

$$c\int_0^2 \left[\int_{-x}^{x} (x^2 - xy)\,dy \right] dx = 1$$

$$c\int_0^2 \left[x^2 y - \frac{x \cdot y^2}{2} \right]_{-x}^{x} dx = 1$$

$$c\int_0^2 \left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right) dx = 1$$

$$c\int_0^2 x^3 - \cancel{\frac{x^3}{2}} + x^3 + \cancel{\frac{x^3}{2}} \quad dx = 1$$

$$c\int_0^2 2x^3\,dx = 1$$

$$2c\left[\frac{x^4}{4}\right]_0^2 = 1$$

$$\frac{x}{\cancel{4}}\left[16\right]^{4} = 1$$

$$\boxed{c = \frac{1}{8}}$$

$$\therefore \cancel{f(x,y)} = \frac{1}{8}x(x-y), \quad 0<x<2$$

$$-x<y<x$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-x}^{x} \frac{1}{8} x (x-y)\,dy$$

$$= \frac{1}{8} x \left[xy - \frac{y^2}{2} \right]_{-x}^{x}$$

$$= \frac{1}{8} x \left[\left(x^2 - \frac{x^2}{2} \right) - \left(-x^2 - \frac{x^2}{2} \right) \right]$$

$$= \frac{1}{8}x\left[2x^2\right]$$

$$f(x) = \frac{1}{4}x^3$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{\cancel{8}_2}\cancel{x}(x-y)}{\frac{1}{\cancel{4}}x^{\cancel{3}\,2}}$$

$$= \frac{1}{2x^2}(x-y) \quad //$$

The joint pdf of a two-dimensional RV (X,Y) is given by $f(x,y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$. Compute $\underline{P(X > 1)}$, $\underline{P(Y < \frac{1}{2})}$, $\underline{P(X > 1 / Y < \frac{1}{2})}$.

<u>Sol:</u> -

$$f(x) = \int_0^1 xy^2 + \frac{x^2}{8} \, dy$$

$$= \left[\frac{xy^3}{3} + \frac{x^2 y}{8} \right]_0^1$$

$$f(x) = \left(\frac{x}{3} + \frac{x^2}{8} \right)$$

$$P(X>1) = \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8}\right) dx$$

$$= \left(\frac{x^2}{6} + \frac{x^3}{24}\right)_1^2$$

$$= \left\{\left(\frac{4}{6} + \frac{8}{24}\right) - \left(\frac{1}{6} + \frac{1}{24}\right)\right\}$$

$$= \frac{3}{6} + \frac{7}{24} = \frac{1}{2} + \frac{7}{24}$$

$$= \frac{24+14}{48} = \frac{38}{48} = \frac{17}{24}$$

$$f(y) = \int_0^2 xy^2 + \frac{x^2}{8}\, dx$$

$$= \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^2$$

$$= \frac{4y^2}{2} + \frac{8}{24} = 2y^2 + \frac{1}{3}$$

$$P[Y < \tfrac{1}{2}] = \int_0^{1/2} \left(2y^2 + \frac{1}{2}\right) dy$$

$$= \left[\frac{2y^3}{3} + \frac{y}{3}\right]_0^{1/2}$$

$$= \frac{\cancel{2}\left(\frac{1}{\cancel{8}}\right)_4}{3} + \frac{\left(\frac{1}{2}\right)}{3}$$

$$= \frac{1}{2}\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)$$

$$= \frac{1}{6}\left[\frac{1}{2}+1\right]$$

$$= \frac{1}{4} \cdot \frac{3}{2} \quad \frac{2}{2}$$

$$P[Y<\tfrac{1}{2}] = \frac{1}{4}$$

$$P[X>1 \mid Y<\tfrac{1}{2}] = \frac{P[(X>1) \cap Y<\tfrac{1}{2}]}{P[Y<\tfrac{1}{2}]}$$

$$P\left[X>1,\ Y<\frac{1}{2}\right] = \int_0^{\frac{1}{2}}\int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2 dy$$

$$= \int_0^{1/2} \left[\frac{4y^2}{2} + \frac{8}{24}\right] - \left[\frac{y^2}{2} + \frac{1}{24}\right] dy$$

$$= \int_0^{1/2} \left(\frac{3y^2}{2} + \frac{7}{24} \right) dy$$

$$= \left[\frac{3}{2} \left(\frac{y^3}{3} \right) + \frac{7y}{24} \right]_0^{1/2}$$

$$= \frac{1}{16} + \frac{7}{48}$$

$$= \frac{5}{24}$$

$$P\left[X > \tfrac{1}{4} \,\middle|\, Y < \tfrac{1}{2}\right] = \frac{5/24}{1/4} = \frac{5}{6}$$

Try this

If X and Y is a two dimensional R.V uniformly distributed over the triangular region R bounded by $y = 0, x = 3$ *and* $y = \frac{4x}{3}$. Find $f(x)$, $f(y)$, Mean of x, Var(X), mean of y and Var(Y)

The joint probability density function of a two-dimensional random variable (X,Y) is given by

$$f(x,y)=\begin{cases} 2; 0<x<1, 0<y<x; \\ 0, elswhere \end{cases}.$$

(i) Find the marginal density functions of x and y

(ii) Check for independency of X and Y.