1. Verify whether the given is a distribution function?

$$F(x) = \begin{cases} 0, & x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right), & -a < x < a \\ 1, & x > a. \end{cases}$$

Soln:-

$$\frac{dF(x)}{dx} = f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & x > a \end{cases}$$

$$\int_{-a}^{a} \frac{1}{2a} \, dx$$

$$= \frac{1}{2a} (x)_{-a}^{a}$$

$$= \frac{1}{2a} (a+a)$$

$$= 1$$

$\Rightarrow f(x)$ is a c.d.f.

Two Dimensional Random variable

Let 'S' be the sample space. Let $X = X(s)$, $Y = Y(s)$ be two functions each assigning a real number to each outcome $s \in S$. The (X, Y) is called as a Two dimensional R.V.

Joint probability mass function [JPmf]

If (X, Y) is a two Dimensional R.V. such that $\underline{P[X = x_i, Y = y_j]} = p_{ij}$ is called JPmf of (X, Y) if it satisfies the following condition.

i) $p_{ij} \geq 0 \ \forall \ i, j$

ii) $\sum_j \sum_i p_{ij} = 1$.

Joint probability density function (Jpdf)

If (X, Y) is a two dimensional continuous r.v. such that

$$P\left[x - \frac{dx}{2} \le X \le x + \frac{dx}{2},\ y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right] = f(x, y)$$

is called Jpdf if it satisfies the conditions that,

$$\text{(i)}\ f(x, y) \ge 0 \quad \forall (x, y) \in R$$

$$\text{(ii)}\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$$

Cumulative Distribution function

If (X, Y) is a two dimensional Random Variable (Discrete or continuous), then

$$F(x, y) = P(X \leq x \text{ and } Y \leq y)$$ is

called CDF of (X, Y) and is defined

$$F(X, Y) = \sum_j \sum_i p_{ij} \quad - \text{ Discrete}$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\, dx\, dy \quad - \text{ Cont}$$

Marginal probability function of X and Y

(X, Y) – Discrete R.V

The marginal probability function of X is given by:

$$P[X=x_i] = P[X=x_i, Y=y_1] + P[X=x_i, Y=y_2] + \cdots$$

$$= \sum_j p_{ij} = p_{i\cdot}$$

The marginal probability function of Y given by,

$$P[Y = y_j] = P[X = x_1, Y = y_j] + P[X = x_2, Y = y_j] + \dots$$

$$= \sum_i p_{ij}$$

$$= p_{\cdot j}$$

Conditional probability function of X and Y

(i) The conditional probability function of X given $Y = y_j$ is given by,

$$P[X = x_i / Y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]} = \frac{P_{ij}}{P_{\cdot j}}$$

(ii) The conditional probability of Y given $X = x_i$ is given by

$$P[Y = y_j / X = x_i] = \frac{P[X = x_i, Y = y_j]}{P(X = x_i)}$$

$$= \frac{p_{ij}}{p_{i.}}$$

Note :-

The two Rv's are Independent if

$$P[X = x_i, Y = y_j] = P(X = x_i) * P[Y = y_j]$$

(or)

$$p_{ij} = p_{i\cdot} * p_{\cdot j} \quad \forall \; i, j.$$

Marginal density function of X and Y

(i) The marginal density function of X,

$$f_X(x) \text{ (or) } f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

(ii) The marginal density function of Y,

$$f_Y(y) \text{ (or) } f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Conditional density functions of X and Y

$$f_{X/Y}(x/y) \text{ (or) } f(x/y) = \frac{f(x,y)}{f(y)}$$

$$f_{Y/X}(y/x) \text{ (or) } f(y/x) = \frac{f(x,y)}{f(x)}$$

Note :-

1. If x and y are Independent then

$$f(x,y) = f(x) * f(y) \quad \forall\, x, y \in R$$

2. The Relationship between $f(x,y)$ and $F(x,y)$ is

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) \quad (\text{or}) \quad \frac{\partial^2}{\partial y \partial x} F(x,y)$$

Example :-

1. Given is the probability function

$p(x,y) = \frac{x+y}{21}$, $x = 1, 2, 3$, $y = 1, 2$. Find

Marginal probability functions of X and Y and Conditional probability of X given Y and Y given X.

$y \backslash x$	1	2	3
1	$P(1,1) = \frac{2}{21}$	$P(2,1)$ $\frac{3}{21}$	$P(3,1)$ $\frac{4}{21}$
2	$P(1,2)$ $\frac{3}{21}$	$P(2,2)$ $\frac{4}{21}$	$P(3,2)$ $\frac{5}{21}$

$$P(X=1) = P[1,1] + P(1,2)$$

$$= \frac{2}{21} + \frac{3}{21} = \frac{5}{21}$$

$$P(X=2) = P(2,1) + P(2,2)$$

$$= \frac{2}{21} + \frac{4}{21} = \frac{7}{21}$$

$$P(X=3) = P(3,1) + P(3,2)$$

$$= \frac{4}{21} + \frac{5}{21} = \frac{9}{21}$$

X :	1	2	3
$P(X)$:	$\frac{5}{21}$	$7/21$	$9/21$

$$P(Y=1) = P(1,1) + P(2,1) + P(3,1)$$

$$= \frac{2}{21} + \frac{3}{21} + \frac{4}{21} = 9/21$$

$$P(Y=2) = P(1,2) + P(2,2) + P(3,2)$$

$$= \frac{3}{21} + \frac{4}{21} + \frac{5}{21} = \frac{12}{21}$$

X :	1	2
$P(Y)$:	$9/21$	$\frac{12}{21}$

$$P[X=1/Y=1] = \frac{P(1,1)}{P(Y=1)} = \frac{2/21}{9/21} = \frac{2}{9}$$

$$P(X=1/Y=2] = \frac{P(1,2)}{P(2)} = \frac{3/21}{12/21} = \frac{3}{12} = 1/4$$

$$P[X=2/Y=1] = \frac{P(2,1)}{P(Y=1)} = \frac{3/21}{9/21} = \frac{3}{9} = \frac{1}{3}$$

$$P[X=2/Y=2] = \frac{P(2,2)}{P(Y=2)} = \frac{4/21}{12/21} = \frac{4}{12} = \frac{1}{3}$$

$$P(X=3/Y=1) = \frac{P(3,1)}{P(Y=1)} = \frac{4/21}{9/21} = \frac{4}{9}$$

$$P(X=3/Y=2) = \frac{P(3,2)}{P(Y=2)} = \frac{5/21}{12/21} = \frac{5}{12}$$

$$P[Y=1/X=1] = \frac{P(1,1)}{P(X=1)} = \frac{2/21}{5/21} = \frac{2}{5}$$

$$P[Y=1/X=2] = \frac{P(2,1)}{P(X=2)} = \frac{3/21}{7/21} = 3/7$$

$$P[Y=1 / X=3] = \frac{P(3,1)}{P(X=3)} = \frac{4/21}{9/21} = \frac{4}{9}$$

$$P[Y=2 / X=1] = \frac{P(1,2)}{P(X=1)} = \frac{3/21}{5/21} = 3/5$$

$$P[Y=2 / X=2] = \frac{P(2,2)}{P(X=2)} = \frac{4/21}{7/21} = \frac{4}{7}$$

$$P[Y=3 / X=3] = \frac{P(3,2)}{P(X=3)} = \frac{5/21}{9/21} = 5/9$$