Mathematical expectation

If you have a collection of numbers $x_1, x_2 \dots x_n$, their average is a single number that describes the whole collection.

Consider the R.V 'x', define its average, which is called expected value (or) mean in probability.

Expected value [mean (or) average]

Defn :-

If X is a D. RV then the expected value of X,

$$E(X) = \sum_{x} x\, p(x)$$

If X is a conti. R.V, then the expected value of X,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Remarks :-

1. Mathematical expectation of a R.V. 'X' gives the arithmetic mean $\bar{X}$

$E[X] = p_1 x_1 + p_2 x_2 + \dots + p_n x_n = \bar{X}$ [$\sum_i x_i p_i$

2. If $X \geq 0 \Rightarrow E[X] \geq 0$.

3. $E[x^r] = \begin{cases} \sum_x x^r p(x), & x \in \mathbb{R} \\ \int_{-\infty}^{\infty} x^r f(x)\,dx \end{cases}$ (moments) $r = 1, \dots, n$

Properties

1. $E[a] = a$, a is a constant

2. $E[aX \pm bY] = aE(X) \pm bE(Y)$, a & b are constants

3. Let X and Y be two random variable such that $X \le Y$ then $E(X) \le E(Y)$, provided that expected values exists

4. $|E(X)| \le E|X|$

5. Variance of X, $Var(X) = E[X^2] - \{E[X]\}^2$
or $[V(X)]$

6. Let X and Y be any two correlated Random Variables then the covariance

between X and Y is defined in terms of expectations as,

$$\text{cov}(X,Y) = E[XY] - E(X) \cdot E(Y)$$

7. If the R.Vs X and Y are independent then $\text{Cov}(X,Y) = 0$

i.e., $E(XY) = E(X) \cdot E(Y)$

8. The correlation coefficient of X and Y is denoted by r_{XY} and is

defined,

$$\underline{\underline{r_{XY}}} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \cdot \sigma_Y} , \quad \underline{-\infty \le \operatorname{cov}(X,Y) \le \infty}$$

$$\underline{-1 \le r_{XY} \le 1}$$

where $\operatorname{cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$

$$\sigma_X = \sqrt{E[X^2] - [E(X)]^2}$$

$$\sigma_Y = \sqrt{E[Y^2] - [E(Y)]^2}$$

Multiplication theorem $E(XY) = E(X) \cdot E(Y)$

where

$$E(XY) = \begin{cases} \sum_x \sum_y xy\, p(x,y), & x, y \in \mathbb{R} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy \end{cases}$$

Note :-

1. $V(aX \pm b) = a^2 \operatorname{Var}(X)$

$\operatorname{Var}(b) = 0$, b is a constant

2. $V(aX + bY) = a^2 \, var(X) + b^2 \, var(Y) + 2ab \, cov(X, Y)$.

3. $Cov(aX, bY) = ab \, cov(X, Y)$

Conditional expectation

Let (X, Y) be two dimensional R.V,

then the conditional expectation of X

given $Y = y_j$ is

$$E\left[X / Y = y_j\right] = \sum_{i=1}^{\infty} x_i \, p\left[X / Y = y_j\right]$$

Where $p\left[X / Y = y_j\right] = \dfrac{P(X, Y)}{P(Y = y_j)}$

$$E\left[X / _{Y=y_i}\right] = \int_{-\infty}^{\infty} x \, f(x/y) \, dx$$

Where $f(x/y) = \dfrac{f(x,y)}{f(y)}$

Ex

1. The following is a frequency distribution of demand for assembled computers in a shop

Demand :	10	11	12	13	14	15
No. of weeks :	5	15	30	20	16	14

If X stands for number of assembled

Computers demanded then find $E(x)$.

x:	10	11	12	13	14	15
$p(x)$:	$\frac{5}{100}$	$\frac{25}{100}$	$\frac{30}{100}$	$\frac{20}{100}$	$\frac{16}{100}$	$\frac{14}{100}$

$$E(x) = \sum_x x\, p(x)$$

$$= \left(10 \times \frac{5}{100}\right) + \left(11 \times \frac{15}{100}\right) + \left(12 \times \frac{30}{100}\right)$$

$$+ \left(13 \times \frac{20}{100}\right) + \left(14 \times \frac{16}{100}\right) + \left(15 \times \frac{14}{100}\right)$$

$$= \frac{50 + 165 + 360 + 260 + 224 + 210}{100}$$

$$= \frac{1269}{100}$$

$$E(x) = 12.69$$

2. Three coins are tossed. 'X' denotes number of heads secured. find mean and variance of X.

Sol :-

'X' = number of heads

$S = \{(HHH), (HTH), (THH), (HTT), (HHT), (TTH), (TTT), (THT)\}$

X :	0	1	2	3
$p(x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean $= \sum_x x\, p(x)$

$[E(X)]$

$= \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) +$

$\left(3 \times \frac{1}{8}\right)$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$

Variance $= E[X^2] - (E[X])^2$

Where

$$E(x^2) = \sum x^2 p(x)$$

$$= \left[0^2 * \frac{1}{8}\right] + \left[1^2 * \frac{3}{8}\right] + \left[2^2 * \frac{3}{8}\right]$$

$$+ \left[3^2 * \frac{1}{8}\right]$$

$$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8}$$

$$= 24/8 = 3$$

$\therefore \ Var(X) = E(X^2) - [E(X)]^2$

$= 3 - \left(\frac{3}{2}\right)^2$

$= 3 - \frac{9}{4}$

$= \frac{12-9}{4} = 3/4$ //