Gamma Distribution

A continuous R.V. X is said to follow an Erlang distribution (or) General Gamma distribution with parameters $\lambda > 0$ and $a > 0$, if its pdf is given by

$$f(x) = \frac{a^{\lambda}}{\Gamma(\lambda)} e^{-ax} x^{\lambda - 1}, \quad x > 0$$

Note :-

A R.V. 'X' is said to follow Gamma distribution with parameter $\lambda > 0$, if its p.d.f is,

$$f(x) = \frac{e^{-x} x^{\lambda - 1}}{\Gamma(\lambda)}, \quad \lambda > 0, \quad x > 0$$

2. The function $f(x)$ defined above represents a probability function

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}\,dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-x} x^{\lambda-1}\,dx$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,dx$$

$$= \frac{1}{\Gamma(\lambda)} \underbrace{\hspace{3em}}_{\Gamma(\lambda)} = 1.$$

3. $M_X(t) = (1-t)^{-\lambda}, \ |t| < 1.$

4. Mean $= \dfrac{\lambda}{a}$

5. Variance $= \dfrac{\lambda}{a^2}$

$\Big\}$ Erlang distribution

6. When $a = 1$ Erlang distribution is called Gamma distribution

7. When $\lambda = 1$, the Erlang reduces exponential.

$$f(x) = a e^{-ax}, \quad x > 0$$

Q Mean and variance of simple gamma distribution is

Mean $= \lambda$

Variance $= \lambda$

Ex

The daily consumption of milk in a city, in excess of 20,000 litres is approximately distributed as a Gamma variate with parameters $a = \frac{1}{10000}$ and $\lambda = 2$. The city has a daily stock of 30,000 lt. What is the probability that the stock is insufficient on a particular day?

Sol :

'X' denotes daily consumption of milk

then r.v, $Y = X - 20000$ has a

Gamma distribution with pdf

$$f(y) = \frac{\left(\frac{1}{10000}\right)^2 e^{-\frac{y}{10000}} y^{2-1}}{\Gamma(2)}, \quad 0 < y < \infty.$$

$$= \frac{y\, e^{-y/10000}}{(10000)^2}$$

$$P(X > 30000) = P[Y > 10{,}000]$$

$$= \int_{1000}^{\infty} \frac{y\, e^{-y/10000}}{(1000)^2}\, dy$$

$t = y/10000$, $y = 10000t$

$dt = dy/10000$

Also,

$y = 10000, \quad t = 1$

$y = \infty, \quad t = \infty$

$$= \int_1^\infty \frac{\cancel{10000}\, t\, e^{-t}}{\cancel{(10000)^2}} \, \cancel{(10000)}\, dt$$

$$= \int_1^\infty t e^{-t}\, dt$$

$$= \left[\left(\frac{e^{-t}}{-1}\right) - \left(\frac{e^{-t}}{1}\right)\right]_{1}^{0}$$

$$= 0 - \left[\frac{e^{-1}}{-1} - e^{-1}\right]$$

$$\therefore \quad 2e^{-1} \approx 2/e = 0.7357$$

2) In a certain city, the daily consumption of electric power in millions of kilowatt hours can be treated as a random variable having an erlang distribution with parameters $\left(\frac{1}{2}, 3\right)$. If
(α, λ)

the power plant of this city has a daily capacity of 12 million kilowatt hours, what is the probability that this power supply will be inadequate on any given day?

Soln :-

Let 'x' represent the daily consumption of electric power in millions of kilowatt hours.

Then pdf f(x)

$$f(x) = \frac{\left(\frac{1}{2}\right)^3 e^{-\frac{1}{2}x} x^2}{\Gamma(3)}, \quad x > 0$$

$$P(X>12) = \int_{12}^{\infty} \frac{x^2 e^{-x/2}}{2^3 (2)} \, dx$$

$$\because \Gamma(3) = 2!$$

$$= \frac{1}{16} \int_{12}^{\infty} x^2 e^{-x/2} \, dx$$

$$= \frac{1}{16} \left\{ \frac{x^2 e^{-x/2}}{-1/2} - \frac{(2x) e^{-x/2}}{1/4} + \frac{(2) e^{-x/2}}{-1/8} \right\}_{12}^{\infty}$$

$$= \frac{1}{16} \left\{ 0 - \left[-2e^{-\frac{12}{2}} (12)^2 - 8(12) e^{-12/2} - 16 e^{-12/2} \right] \right\}$$

$$= \frac{1}{16} e^{-6} \{ 288 + 96 + 16 \}$$

$$= \frac{1}{16} e^{-6} [400]$$

$$= 25 \times 0.0025 = \underline{\underline{0.0619}}$$