Ex

1. If a Random Variable takes the values 1, 2, 3 and 4 such that

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4).$$

Find the probability distribution and the CDF

Soln :-

Given:

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K$$

$\therefore$ $2P(X=1) = K \Rightarrow P(X=1) = K/2$

$3P(X=2) = K \Rightarrow P(X=2) = K/3$

$P(X=3) = K$

$5P(X=4) = K \Rightarrow P(X=4) = K/5$

$X: \quad 1 \quad 2 \quad 3 \quad 4$

$P(x): \quad \frac{k}{2} \quad \frac{k}{3} \quad k \quad \frac{k}{5}$

$\Rightarrow \frac{61k}{20} = 1$

$\sum p(x) = 1$

$\boxed{k = \frac{30}{61}}$

$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$\frac{15k + 10k + 6k + 30}{30} = 1$

X :	1	2	3	4
$P(x)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$F(x)$:	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1 .

$F(x) = P(X \leq x)$

$F(1) = P(X \leq 1) = \frac{15}{61}$

$F(2) = P(X \leq 2) = \frac{25}{61}$

2) Is the function defined as follows a density function?

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Soln :-

$$\int_0^\infty e^{-x}\,dx = \left(-e^{-x}\right)_0^\infty$$

$$= -\left[e^{-\infty} - e^{0}\right] = -1\left[0 - 1\right]$$

$$= 1.$$

Yes, f(x) is a pdf

3) A continuous R.V. 'X' has a pdf,

$f(x) = 3x^2$, $\underline{0 \le x \le 1}$. Find 'a' such that

$P(X \le a) = P(X > a)$

<u>Sol'n</u> :-

Given: $P(X \le a) = P(X > a)$

$$\int_0^a 3x^2\,dx = \int_a^1 3x^2\,dx$$

$$= \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_0^a = \left(\frac{\cancel{3}x^3}{\cancel{3}}\right)_a^1$$

$$a^3 = 1 - a^3$$

$$2a^3 = 1$$

$$a^3 = 1/2$$

$$a = \sqrt[3]{\frac{1}{2}}$$

$$\underline{\underline{a = 0.794}}$$

4) If the density function of a continuous R.V. 'X' is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & x > 3. \end{cases}$$

Find the value of 'a' and find the cdf of X.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{3} f(x)\,dx = 1$$

$$\int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (3a - ax)\,dx = 1$$

$$a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^3 = 1$$

$$a\left(\frac{1}{2}\right) + a + \left[9a - \frac{9a}{2}\right] - \left[6a - \frac{\cancel{4}^{2}a}{\cancel{2}}\right] = 1$$

$$\frac{a}{2} + a - 4a + \frac{9a}{2} = 1$$

$$\frac{\cancel{10}^{5}a}{\cancel{2}} - 3a = 1$$

$$5a - 3a = 1$$

$$2a = 1$$

$$\boxed{a = 1/2}$$

$$f(x) = \begin{cases} \frac{x}{2}, & 0 < x \leq 1 \\ \frac{1}{2}, & 1 \leq x \leq 2 \\ \left(\frac{3}{2} - \frac{x}{2}\right), & 2 \leq x \leq 3 \\ 0, & x > 3. \end{cases}$$

$F(x) = 0$, when $x < 0$

$$F(x) = \int_0^x \frac{x}{2}\,dx, \quad 0 \leq x \leq 1$$

$$= \left(\frac{x^2}{4}\right)_0^x = \frac{x^2}{4} \;//$$

$$F(x) = \int_0^1 \frac{x}{2}\,dx + \int_1^x \frac{1}{2}\,dx \quad , \qquad 1 \leq x \leq 2$$

$$= \left(\frac{x^2}{4}\right)_0^1 + \left(\frac{x}{2}\right)_1^x$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \frac{x}{2} - \frac{1}{4} \quad //$$

$$F(x) = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x \left(\frac{3-x}{2}\right) dx$$

$$= \left(\frac{x^2}{4}\right)\Bigg|_0^1 + \left(\frac{x}{2}\right)\Bigg|_1^2 + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)\Bigg|_2^x$$

$$= \frac{1}{4} + \frac{1}{2}(2-1) + \frac{1}{2}\left[3x - \frac{x^2}{2}\right] - \frac{1}{2}\left[6 - \frac{4}{2}\right]$$

$$= \left(\frac{1}{4} + \frac{1}{2} \text{[redacted]} 2\right) + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)$$

$$= \left(\frac{2+4-16}{8}\right) + \frac{3x}{2} - \frac{x^2}{4}$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} \quad //$$

$f(x) = 1$, when $x > 3$

5) A random variable 'x' has a density function

$$f(x) = \begin{cases} K \cdot \dfrac{1}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of K and F(x).

Soln:-

$$\int_{-\infty}^{\infty} K \cdot \frac{1}{1+x^2} \, dx = 1$$

$$k\left[\tan^{-1}x\right]_{-\infty}^{\infty} = 1$$

$$k\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = 1$$

$$k\left[\frac{\pi}{2} - \left[-\frac{\pi}{2}\right]\right] = 1$$

$$k\left[\frac{\cancel{2}\pi}{\cancel{2}}\right] = 1$$

$$k = \frac{1}{\pi}$$

$$\therefore f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

$$F(x) = P[X \le x] = \int_{-\infty}^{x} \frac{1}{\pi} \cdot \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right] //$$

b) Find the value of 'K' and hence find Mean and Variance of the distribution:

$$dF = Kx^2 e^{-x} \underline{\underline{dx}}, \quad 0 < x < \infty$$

<u>Sol</u>

Sol:

$$\because \frac{dF(x)}{dx} = \underline{\underline{f(x)}}$$

$$\frac{dF}{dx} = Kx^2 e^{-x}$$

$$f(x) = Kx^2 e^{-x}, \quad 0 < x < \infty \quad //$$

$$\int_0^\infty k x^2 e^{-x}\, dx = 1$$

$$k \underbrace{\int_0^\infty x^2 e^{-x}\, dx}_{} = 1$$

$$k\,[2!] = 1$$

$$k = \frac{1}{2} \quad //$$

Result:

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

$$\forall\, n \in \mathbb{Z}^+$$

$$f(x) = \frac{1}{2} x^2 e^{-x}, \quad 0 < x < \infty .$$

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x \cdot \frac{1}{2} x^2 e^{-x} dx$$

$$= \frac{1}{2} \int_{0}^{\infty} x^3 e^{-x} dx$$

$$= \frac{1}{2} [3!] = \frac{1}{\cancel{2}} \times \cancel{6}^{\,3} = 3$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x - \text{mean})^2 f(x)\, dx$$

$$= \int_{0}^{\infty} (x-3)^2 \frac{x^2}{2} e^{-x}\, dx$$

$$= \frac{1}{2}\left\{ \int_{0}^{\infty} (x^2 - 6x + 9) \cdot x^2 e^{-x}\, dx \right\}$$

$$= \frac{1}{2}\left\{ \int_{0}^{\infty} x^4 e^{-x}\, dx - 6\int_{0}^{\infty} x^3 e^{-x}\, dx + 9\int_{0}^{\infty} x^2 e^{-x}\, dx \right\}$$

$$= \frac{1}{2}\left\{ 4! - 6(3!) + 9(2!) \right\}$$

$$= \frac{1}{2}\left\{24 - 36 + 18\right\}$$

$$= \frac{1}{2}\left\{24 - 18\right\}$$

$$= \frac{6}{2} = 3$$

$\therefore$ Variance $= 3$.

7. If a R.V. 'X' has a pdf

$$f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

find Mean and Variance.

<u>Soln</u> :

$$\text{Mean} = \int_{-1}^{1} x \cdot \left(\frac{x+1}{2}\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2+x)\, dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}+\frac{1}{2}\right)-\left(-\frac{1}{3}+\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}+\cancel{\frac{1}{2}}+\frac{1}{3}-\cancel{\frac{1}{2}}\right]$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right)$$

$$= \frac{1}{3}$$

$$\text{Variance} = \int_{-1}^{1} \left(x - \frac{1}{3}\right)^2 \left(\frac{x+1}{2}\right) dx$$

$$= \frac{1}{2 \times 9} \int_{-1}^{1} (3x-1)^2 (x+1)\, dx$$

$$= \frac{1}{18} \int_{-1}^{1} (9x^2 - 6x + 1)(x+1)\, dx$$

$$= \frac{1}{18} \int_{-1}^{1} \left[9x^3 + 9x^2 - 6x^2 - 6x + x + 1\right] dx$$

$$= \frac{1}{18}\int_{-1}^{1}\left(9x^3 + 3x^2 - 5x + 1\right)dx$$

$$= \frac{1}{18}\left[\frac{9x^4}{4} + \frac{3x^3}{3} - \frac{5x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{1}{18}\left[\left(\cancel{\frac{9}{4}} + 1 - \cancel{\frac{5}{2}} + 1\right) - \left(\cancel{\frac{9}{4}} - 1 - \cancel{\frac{5}{2}} - 1\right)\right]$$

$$= \frac{1}{18}\left[1 + 1 + 1 + 1\right] = \frac{4}{18} = \frac{2}{9} \;//$$