Module-1
Probability and its Axioms

EXPERIMENT

Deterministic Experiment

(Genetic determination)

Random Experiment

(Hitting the target)

Definition 12.1

An **experiment** is defined as a process for which its result is well defined.

Definition 12.2

Deterministic experiment is an experiment whose outcomes can be predicted with certain, under ideal conditions.

Definition 12.3

A random experiment (or non-deterministic) is an experiment

(i) whose all possible outcomes are known in advance,

(ii) whose each outcome is not possible to predict in advance, and

(iii) can be repeated under identical conditions.

A die is 'rolled', a fair coin is 'tossed' are examples for random experiments.

Definition 12.4

A **simple event** (or elementary event or sample point) is the most basic possible outcome of a random experiment and it cannot be decomposed further.

Definition 12.5

A **sample space** is the set of all possible outcomes of a random experiment. Each point in sample space is an elementary event.

Illustration 12.1

(1) (i) If a die is rolled, then the sample space $S=\{1, 2, 3, 4, 5, 6\}$

(ii) A coin is tossed, then the sample space $S=\{H, T\}$

(2) (i) Suppose we toss a coin until a head is obtained. One cannot say in advance how many tosses will be required, and so the sample space.

$S=\{H, TH, TTH, TTTH, \dots\}$ is an infinite set.

(ii) The sample space associated with the number of passengers waiting to buy train tickets in counters is $S=\{0,1,2,\dots\}$.

(3) (i) If the experiment consists of choosing a number randomly between 0 and 1, then the sample space is $S = \{ x: 0< x <1\}$.

(ii) The sample space for the life length (t in hours) of a tube light is $S = \{ t: 0 < t<1000\}$.

Finite Space

In this section we restrict our sample spaces that have at most a finite number of points.

Types of events

Let us now define some of the important types of events, which are used frequently in this chapter.

- Sure event or certain event
- Complementary event
- Mutually inclusive event
- Equally likely events
- Impossible event
- Mutually exclusive events
- Exhaustive events
- Independent events (defined after learning the concepts of probability)

Definition 12.6

When the sample space is finite, any subset of the sample space is an **event**. That is, all elements of the power set $\mathscr{P}(S)$ of the sample space are defined as **events**. An event is a collection of sample points or elementary events.

The sample space S is called **sure event or certain event.** The null set $\varnothing$ in S is called an **impossible event.**

Definition 12.7

For every event A, there corresponds another event $\overline{A}$ is called the **complementary event** to A. It is also called the event 'not A'.

Definition 12.8

Two events cannot occur simultaneously are mutually exclusive events. $A_1, A_2, A_3, \ldots, A_k$ are **mutually exclusive or disjoint events** means that, $A_i \cap A_j = \varnothing$, for $i \neq j$.

Definition 12.9

Two events are **mutually inclusive** when they can both occur simultaneously.

$A_1, A_2, A_3, \ldots, A_k$ are mutually inclusive means that, $A_i \cap A_j \neq \varnothing$, for $i \neq j$

Illustration 12.3

When we roll a die, the sample space $S = \{1,2,3,4,5,6\}$.

(i) Since $\{1, 3\} \cap \{2, 4, 5, 6\} = \varnothing$, the events $\{1,3\}$ and $\{2, 4,5,6\}$ are mutually exclusive events.

(ii) The events $\{1,6,\}, \{2,3,5\}$ are mutually exclusive.

(iii) The events $\{2,3,5\}, \{5,6\}$ are mutually inclusive, since $\{2, 3, 5\} \cap \{5, 6\} = \{5\} \neq \varnothing$

Definition 12.10

$A_1, A_2, A_3, \ldots, A_k$ are called **exhaustive events** if, $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Definition 12.11

$A_1, A_2, A_3, \ldots, A_k$ are called **mutually exclusive and exhaustive events** if,
(i) $A_i \cap A_j \neq \varnothing$, for $i \neq j$ (ii) $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Illustration 12.4

When a die is rolled, sample space $S = \{1,2,3,4,5,6\}$.

Some of the events are $\{2,3\},\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$.

(i) Since $\{2, 3\} \cup \{1, 3, 5\} \cup \{4, 6\} = \{1, 2, 3, 4, 5, 6\} = S$ (sample space), the events $\{2,3\},\{1,3,5\},\{4,6\}$ are exhaustive events.

(ii) Similarly $\{2,3\},\{4,6\}$ and $\{1,5\}$ are also exhaustive events.

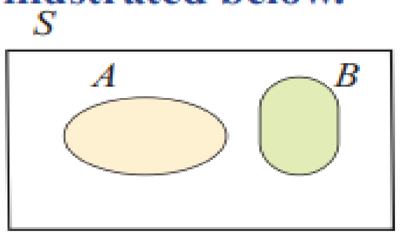
(iii) $\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$ are not exhaustive events.

(Since $\{1, 3, 5\} \cup \{4, 6\} \cup \{6\} \cup \{1, 5\} \neq S$)

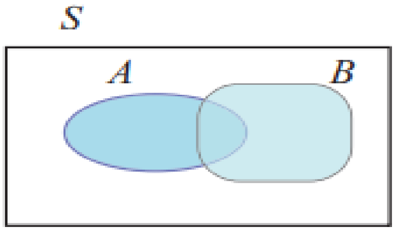
(iv) $\{2,3\},\{4,6\}$, and $\{1,5\}$ are mutually exclusive and exhaustive events, since

$$\{2,3\} \cap \{4,6\} = \varnothing, \{2,3\} \cap \{1,5\} = \varnothing, \{4,6\} \cap \{1,5\} = \varnothing \quad \text{and} \quad \{2, 3\} \cup \{4, 6\} \cup \{1, 5\} = S$$

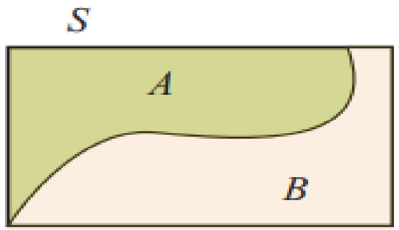
Types of events associated with sample space are easy to visualize in terms of Venn diagrams, as illustrated below.



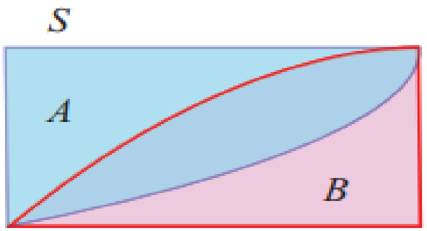
A and B are Mutually exclusive



A and B are Mutually inclusive



A and B are Mutually exclusive and exhaustive



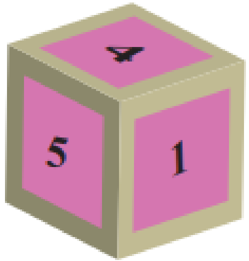
A and B are Mutually inclusive and exhaustive

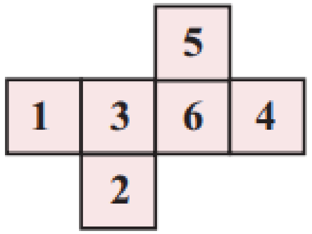
Definition 12.12

The events having the same chance of occurrences are called **equally likely** events.

Example for equally likely events: Suppose a fair die is rolled.

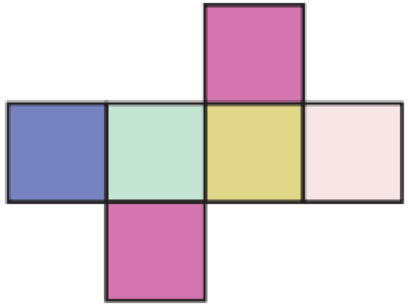
Number on the face	1	2	3	4	5	6
Chance of occurrence	1	1	1	1	1	1

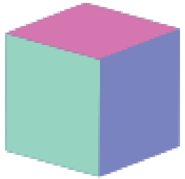




Example for not equally likely events: A colour die is shown in figure is rolled.

Colour on the face	(blue)	(light green)	(yellow)	(pink)	(light pink)
Chance of occurrence	1	1	1	2	1





Similarly, suppose if we toss a coin, the events of getting a head or a tail are equally likely.

Methods to find sample space

Illustration 12.5

Two coins are tossed, the sample space is

(i) $S = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$ or $\{HH, HT, TH, TT\}$

(ii) If a coin is tossed and a die is rolled simultaneously, then the sample space is

$$S = \{H, T\} \times \{1, 2, 3, 4, 5, 6\} = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\} \text{ or}$$

$$S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}.$$

Also one can interchange the order of outcomes of coin and die. The following table gives the sample spaces for some random experiments.

Random Experiment	**Total Number of Outcomes**	**Sample space**
Tossing a fair coin	$2^1 = 2$	$\{H, T\}$
Tossing two coins	$2^2 = 4$	$\{HH, HT, TH, TT\}$
Tossing three coins	$2^3 = 8$	$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
Rolling fair die	$6^1 = 6$	{1, 2,3, 4, 5,6}
Rolling Two dice or single die two times.	$6^2 = 36$	{(1,1) ,(1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2),(6,3), (6,4), (6,5), (6,6)}
Drawing a card from a pack of 52 playing cards	$52^1 = 52$	Heart ♥ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Diamond ♦ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Spade ♠ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour Club ♣ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour

12.4 Probability

12.4.1 Classical definition (A priori) of probability (Bernoulli's principle of equally likely)

Earlier classes we have studied the frequency (A posteriori) definition of probability and the problems were solved. Now let us learn the fundamentals of the axiomatic approach to probability theory.

Posteriori : Knowledge which precedes from experience or observation.

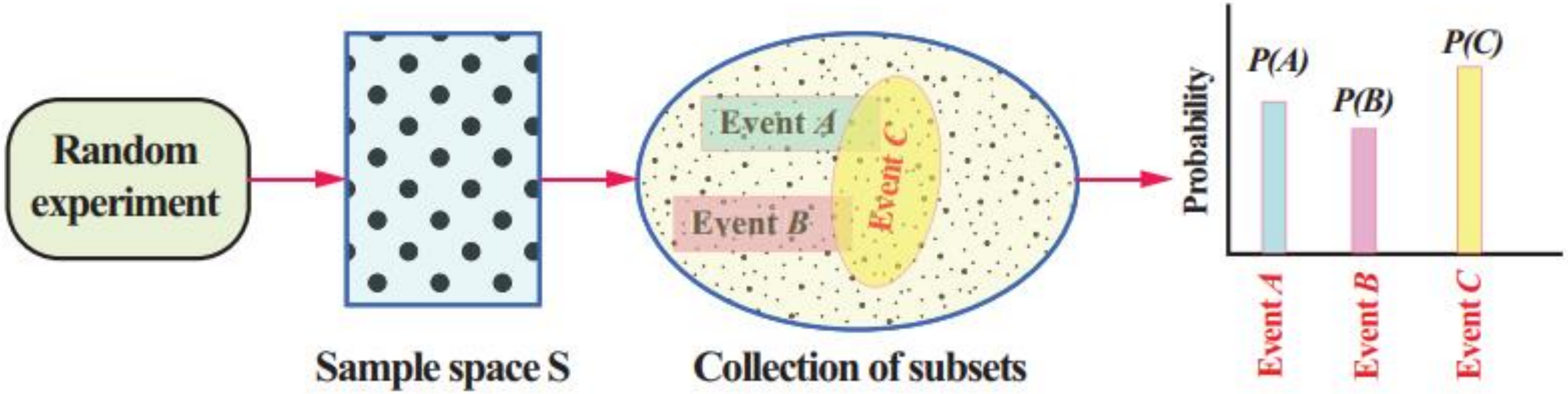
The basic assumption of underlying the classical theory is that the outcomes of a random experiment are equally likely. If there are n exhaustive, mutually exclusive and equally likely outcomes of an experiment and m of them are favorable to an event A, then the mathematical probability of A is defined as the ratio $\frac{m}{n}$. In other words, $P(A) = \frac{m}{n}$.

Definition 12.13

Let S be the sample space associated with a random experiment and A be an event. Let $n(S)$ and $n(A)$ be the number of elements of S and A respectively. Then the probability of the event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

Every probabilistic model involves an underlying process is shown in the following figure.



The classical definition of probability is limited in its application only to situations where there are a finite number of possible outcomes. It mainly considered discrete events and its methods were mainly combinatorial. This renders it inapplicable to some important random experiments, such as 'tossing a coin until a head appears' which give rise to the possibility of infinite set of outcomes. Another limitation of the classical definition was the condition that each possible outcome is 'equally likely'.

Axioms of probability

Let S be a finite sample space, let $\mathscr{P}(S)$ be the class of events, and let P be a real valued function defined on $\mathscr{P}(S)$ Then $P(A)$ is called probability function of the event A, when the following axioms are hold:

[P_1] For any event A, $P(A) \geq 0$ (Non-negativity axiom)

[P_2] For any two mutually exclusive events

$$P(A \cup B) = P(A) + P(B) \quad \text{(Additivity axiom)}$$

[P_3] For the certain event $P(S) = 1$ (Normalization axiom)

Note 12.1

(i) $0 \leq P(A) \leq 1$

(ii) If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events in a sample space S, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$$

Note 12.1

(i) $0 \le P(A) \le 1$

(ii) If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events in a sample space S, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$$

Theorems on finite probability spaces (without proof)

When the outcomes are equally likely Theorem 12.1 is applicable, else Theorem 12.2 is applicable.

Theorem 12.1

Let S be a sample space and, for any subset A of S, let $P(A) = \dfrac{n(A)}{n(S)}$. Then $P(A)$ satisfies axioms of probability $[P_1], [P_2]$, and $[P_3]$.

Theorem12.2

Let S be a finite sample space say $S = \{a_1, a_2, a_3, \ldots, a_n\}$. A finite probability space is obtained by assigning to each point a_i in S a real number p_i, is called the probability of a_i, satisfying the following properties:

(i) Each $p_i \ge 0$. (ii) The sum of the p_i is 1, that is, $\sum p_i = p_1 + p_2 + p_3 + \cdots + p_1 = 1$.

If the probability $P(A)$, of an event $[P_1]$ is defined as the sum of the probabilities of the points in A, then the function $P(A)$ satisfies the axioms of probability $[P_1], [P_2]$, and $[P_3]$.

Note: Sometimes the points in a finite sample space and their assigned probabilities are given in the form of a table as follows:

Outcome	a_1	a_2	a_3	$\ldots$	a_n
Probability	P_1	P_2	P_3	$\ldots$	P_n

If an experiment has exactly the three possible mutually exclusive outcomes A, B, and C, check in each case whether the assignment of probability is permissible.

(i) $P(A)=\frac{4}{7}$, $P(B)=\frac{1}{7}$, $P(C)=\frac{2}{7}$.

(ii) $P(A)=\frac{2}{5}$, $P(B)=\frac{1}{5}$, $P(C)=\frac{3}{5}$.

(iii) $P(A)=0.3$, $P(B)=0.9$, $P(C)=-0.2$.

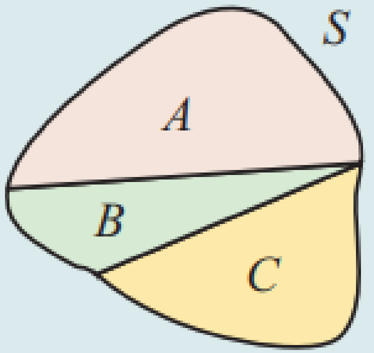
(iv) $P(A)=\frac{1}{\sqrt{3}}$, $P(B)=1-\frac{1}{\sqrt{3}}$, $P(C)=0$.

(v) $P(A)=0.421$, $P(B)=0.527$ $P(C)=0.042$.

Solution

Since the experiment has exactly the three possible mutually exclusive outcomes A, B and C, they must be exhaustive events.

$$\Rightarrow S = A \cup B \cup C$$



Therefore, by axioms of probability

$P(A) \geq 0, \quad P(B) \geq 0, \quad P(C) \geq 0$ and

$P(A \cup B \cup C) = P(A) + P(B) + P(C) = P(S) = 1$

(i) Given that $P(A) = \frac{4}{7} \geq 0$, $\quad P(B) = \frac{1}{7} \geq 0$, and $\quad P(C) = \frac{2}{7} \geq 0$

Also $P(S) = P(A) + P(B) + P(C) = \frac{4}{7} + \frac{1}{7} + \frac{2}{7} = 1$

Therefore the assignment of probability is permissible.

(ii) Given that $P(A) = \frac{2}{5} \geq 0$, $\quad P(B) = \frac{1}{5} \geq 0$, and $\quad P(C) = \frac{3}{5} \geq 0$

But $P(S) = P(A) + P(B) + P(C) = \frac{2}{5} + \frac{1}{5} + \frac{3}{5} = \frac{6}{5} > 1$

Therefore the assignment is not permissible.

(iii) Since $P(C) = -0.2$ is negative, the assignment is not permissible.

(iv) The assignment is permissible because

$$P(A) = \frac{1}{\sqrt{3}} \geq 0, \quad P(B) = 1 - \frac{1}{\sqrt{3}} \geq 0, \quad \text{and} \quad P(C) = 0 \geq 0$$

$$P(S) = P(A) + P(B) + P(C) = \frac{1}{\sqrt{3}} + 1 - \frac{1}{\sqrt{3}} + 0 = 1.$$

(v) Even though $P(A) = 0.421 \geq 0$, $P(B) = 0.527 \geq 0$, and $P(C) = 0.042 \geq 0$, the sum of the probability

$$P(S) = P(A) + P(B) + P(C) = 0.421 + 0.527 + 0.042 = 0.990 < 1.$$

Therefore, the assignment is not permissible.

An integer is chosen at random from the first ten positive integers. Find the probability that it is (i) an even number (ii) multiple of three.

Solution

The sample space is

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \quad n(S) = 10$$

Let A be the event of choosing an even number and

B be the event of choosing an integer multiple of three.

$$A = \{2, 4, 6, 8, 10\}, \quad n(A) = 5,$$

$$B = \{3, 6, 9\}, \quad n(B) = 3$$

$$P\text{ (choosing an even integer)} = P(A) = \frac{n(A)}{n(S)} = \frac{5}{10} = \frac{1}{2}.$$

$$P\text{ (choosing an integer multiple of three)} = P(B) = \frac{n(B)}{n(S)} = \frac{3}{10}.$$

Three coins are tossed simultaneously, what is the probability of getting (i) exactly one head (ii) at least one head (iii) at most one head?

Solution:

Notice that three coins are tossed simultaneously = one coin is tossed three times.

The sample space $S = \{H,T\}\times\{H,T\}\times\{H,T\}$

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}, \quad n(S) = 8$$

Let A be the event of getting one head, B be the event of getting at least one head and C be the event of getting at most one head.

$$A = \{HTT, THT, TTH\}; \; n(A) = 3$$

$$B = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}; \quad n(B) = 7$$

$$C = \{TTT, HTT, THT, TTH\}; \quad n(C) = 4.$$

Therefore the required probabilities are

(i) $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{8}$

(ii) $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{7}{8}$

(iii) $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}.$

Suppose ten coins are tossed. Find the probability to get (i) exactly two heads (ii) at most two heads (iii) at least two heads

Suppose a fair die is rolled. Find the probability of getting

(i) an even number (ii) multiple of three.

When a pair of fair dice is rolled, what are the probabilities of getting the sum

(i) 7 (ii) 7 or 9 (iii) 7 or 12?

Three candidates X, Y, and Z are going to play in a chess competition to win FIDE (World Chess Federation) cup this year. X is thrice as likely to win as Y and Y is twice as likely as to win Z. Find the respective probability of X, Y and Z to win the cup.

Thank you