Chapter 1 – Linear Algebra

Q.1 Consider the matrix :

$$J_6 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

which is obtained by reversing the order of the columns of the identity matrix I_6.

Let $P = I_6 + \alpha J_6$, where α is a non-negative real number. The value of α for which det $(P) = 0$ is __________.

Q.2 Let A be an $m \times n$ matrix and B an $n \times m$ matrix. It is given that determinant $(I_m + AB) =$ determinant $(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(A) 2 (B) 5

(C) 8 (D) 16

Q.3 If the vectors $e_1 = (1, 0, 2)$, $e_2 = (0, 1, 0)$ and $e_3 = (-2, 0, 1)$ form an orthogonal basis of the three dimensional real space R^3, then the vector $u = (4, 3, -3) \in R^3$ can be expressed as

(A) $u = -\frac{2}{5}e_1 - 3e_2 + \frac{11}{5}e_3$

(B) $u = -\frac{2}{5}e_1 - 3e_2 - \frac{11}{5}e_3$

(C) $u = -\frac{2}{5}e_1 + 3e_2 + \frac{11}{5}e_3$

(D) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

Statement for Linked Answer Questions 4 & 5

Given that three vector as

$$P = \begin{bmatrix} -10 \\ 1 \\ 3 \end{bmatrix}^T, Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T, R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$$

Q.4 An orthogonal set of vectors having a span that contains P, Q, R is

(A) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

(B) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix}\begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(C) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$

(D) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix}\begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

Q.5 The following vector is linearly dependent upon the solution to the previous problem

(A) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ (B) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$

(C) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$ (D) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

Q.6 Choose the CORRECT set of functions, which are linearly dependent.

(A) $\sin x$, $\sin^2 x$ and $\cos^2 x$

(B) $\cos x$, $\sin x$ and $\tan x$

(C) $\cos 2x$, $\sin^2 x$ and $\cos^2 x$

(D) $\cos 2x$, $\sin x$ and $\cos x$

Q.7 Consider the matrices $X_{(4\times3)}, Y_{(4\times3)}$ and $P_{(2\times3)}$. The order of $[P(X^TY)^{-1}P^T]^T$ will be

(A) (2×2) (B) (3×3)

(C) (4×3) (D) (3×4)

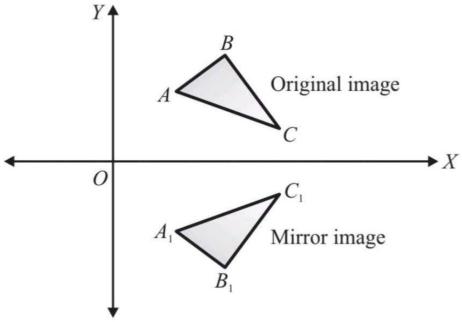
Q.8 Given that the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is –12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is

(A) – 96 (B) – 24

(C) 24 (D) 96

Q.9 For the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ if det stands for the determinant and A^T is the transpose of A then the value of $\det(A^TA)$ is ______.

Q.10 The figure shows a shape ABC and its mirror image $A_1B_1C_1$ across the horizontal axis (X-axis). The coordinate transformation matrix that maps ABC to $A_1B_1C_1$ is



(A) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Q.11 The rank of matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is____.

Q.12 A is a $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A^+ = (A^TA)^{-1}A^T$. Then which one of the following statements is FALSE?

(A) $AA^+A = A$ (B) $(AA^+)^2 = AA^+$

(C) $A^+A = I$ (D) $AA^+A = A^+$

Q.13 Given a system of equations

$$x + 2y + 2z = b_1$$
$$5x + y + 3z = b_2$$

which of the following is true regarding its solutions?

(A) The system has a unique solution for any given b_1 and b_2.

(B) The system will have infinitely many solutions for any given b_1 and b_2.

(C) Whether or not a solution exists depends on the given b_1 and b_2.

(D) The system would have no solution for any values of b_1 and b_2.

Q.14 The following system of equations

$$x + y + z = 3,\ x + 2y + 3z = 4$$
$$x + 4y + k = 6$$

will not have a unique solution for k equal to

(A) 0 (B) 5

(C) 6 (D) 7

Q.15 The Eigen values of a (2×2) matrix X are -2 and -3. The Eigen values of the matrix $(X+I)^{-1}(X+5I)$ are

(A) $-3, -4$ (B) $-1, -2$

(C) $-1, -3$ (D) $-2, -4$

Q.16 Consider the matrix $A=\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ whose Eigen values are $1, -1$ and 3. The trace of $(A^3 - 3A^2)$ is ________.

Q.17 The eigen values of a skew-symmetric matrix are

(A) always zero.

(B) always pure imaginary.

(C) either zero or pure imaginary.

(D) always real.

Q.18 The value of x for which all the eigen-values of the matrix given below are real is

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

(A) $5+j$ (B) $5-j$

(C) $1-j5$ (D) $1+j5$

Q.19 Consider the 5×5 matrix

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

It is given that A has only one real eigen value. Then the real eigen value of A is

(A) -2.5 (B) 0

(C) 15 (D) 25

Q.20 For a given 2×2 matrix A, it is observed that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Then matrix A is

(A) $A=\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

(B) $A=\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(C) $A=\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(D) $A=\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

Q.21 If $\{1, 0, -1\}^T$ is an Eigen vector of the following matrix,

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Then the corresponding Eigen value is

(A) 1 (B) 2

(C) 3 (D) 5

(1) $P = I_6 + \alpha J_6$, $\alpha = ?$

$P = 0$

$I_6 + \alpha J_6 = 0$

If I take I_2 , then J_2

$$\left|\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}\right| = 0$$

$$1 - \alpha^2 = 1$$

$$\alpha = \pm 1$$

So, $\boxed{\alpha = 1}$ Ans

(2) $|I_m + AB| = |I_n + BA|$

B

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + \left| \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \right|$$

$$= 1 + 4$$

$$= \boxed{5}$$ Ans

3 ⇒ $\vec{e_1} = (1, 0, 2)$, $\vec{e_2} = (0, 1, 0)$, $\vec{e_3} = (-2, 0, 1)$

$\boxed{D}$ $\vec{u} = (4, 3, -3)$

$$\begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$4 = A - 2C$

$3 = B$

$-3 = 2A + C$

$-11 = 3C$

$C = \frac{-11}{5}$

$A = 4 - \frac{22}{5} = \cancel{0}$

$A = \frac{-2}{5}$

$$u = \frac{-2}{5} e_1 + 3e_2 - \frac{11}{5} e_3$$ Ans

④

A

$\vec{P} = \begin{bmatrix} -10 \\ 1 \\ 3 \end{bmatrix}^T$, $\vec{Q} = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$, $\vec{R} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$

A -

$$\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -24 \\ 6 \\ 18 \end{bmatrix}$$

0

5

B

$C_1 = \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$, $C_2 = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, $C_3 = ?$

By option A

$$3C_1 + 4C_2 = C_3$$

$$C_3 = \begin{bmatrix} -18+16 \\ -9-8 \\ 18+12 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$$ Ans

6 $\Rightarrow$ Linearly –

c

A – $\sin x$, $\sin^2 x$, $\cos^2 x$

B – $\cos x$, $\sin x$, $\tan x = \left(\frac{\sin x}{\cos x}\right)$

C $\cos 2x = (\cos^2 x - \sin^2 x)$, $\cos^2 x$, $\sin^2 x$

7 $\Rightarrow$ $[P \cancel{(X^T Y)^{-1}} P^T]^T$

A

$= [P_{(2\times3)} \cdot P^T_{(3\times2)}]^T$

$= [(2\times3) \cdot (3\times2)]^T$

$= [2\times2]^T$

$= (2\times2)$ Ans

8 ⇒ $\left|\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}\right| = -12$

$\boxed{A}$

$$\left|\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}\right| = 2^3 \left|\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}\right|$$

$= 8 \times -12$

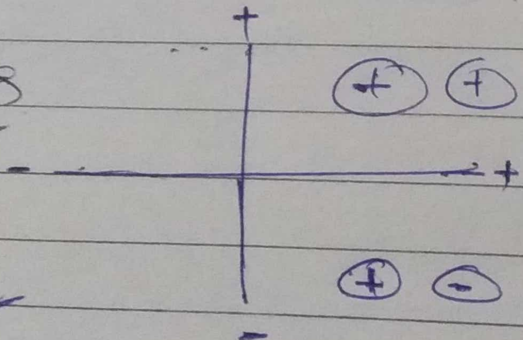
$= -96$ Ans

9 ⇒ $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$|A^T . A| = |I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = +1$ Ans

10 ⇒

$\boxed{D}$



$\Longrightarrow \begin{bmatrix} + & + \\ + & - \end{bmatrix}$

$\Downarrow$

D — $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Ans

(11) ⇒ Rank - $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}_{(3\times 4)}$

Rank = Min (3, 4) = 3 Ans

(12) ⇒ $m \times n \longrightarrow$ Full rank matrix, | $m > n$

D

$A^+ = (A^T \cdot A)^{-1} \cdot A^T$

$A^+ = A^{-1} \cdot (A^T)^{-1} \cdot A^T$ → Identity

$A^+ = A^{-1} \cdot I$

$A^+ = A^{-1}$

By option → False

A $(AA^{-1})A = A$ ✓
$I \Rightarrow I$

B $(AA^{-1})^2 = AA^{-1} \Rightarrow (I = I)$ ✓

C $A^{-1} \cdot A = I$ ✓

D $AA^+A = A^+$

$(AA^{-1})A = A^{-1}$

I $A = A^{-1}$ ✗

13 ⇒

B

$$x + 2y + 2z = b_1 \qquad \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$5x + y + 3z = b_2$$

$\rho = 2$, $n = 3$

The system will have infinitely many sol^n for any given b_1 & b_2

14 ⇒

D

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & K \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

not unique sol^n

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & K & 6 \end{array}\right]$$

not Unique — $|A| = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & K-1 \end{vmatrix} = 0$$

$$(K-1) - 6 = 0$$

$$K = 7 \quad \text{Ans}$$

15 ⇒ Eigen value — -2 & -3

C

$(X + I)^{-1} \cdot (X + 5I)$

$= (-2+1)^{-1} \times (-2+5)$

$= -1 \times 3$

$= -3$

$(-3+1)^{-1} \times (-3+5)$

$= \frac{-1}{2} \times 2$

$= -1$

-3 & -1 Ans

16 ⇒ Eigen value = $1, -1, 3,$

$trace(A^3 - 3A^2) \Longrightarrow$

$1 \rightarrow 1 - 3 = -2$

$-1 \rightarrow -1 - 3 = -4$

$3 \rightarrow 27 - 27 = 0$

$-2 - 4 + 0 = -6$ Ans

17 ⇒ eigen value of skew symmetric matrix

C ⟶ either zero or pure imaginary

18 ⇒ $x = ?$, eigen value real

B

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

⟶ real

$x = 5 - j$

19 ⇒

C

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 15-\lambda & 2 & 3 & 4 & 5 \\ 15-\lambda & 1-\lambda & 2 & 3 & 4 \\ 15-\lambda & 5 & 1-\lambda & 2 & 3 \\ 15-\lambda & 4 & 5 & 1-\lambda & 2 \\ 15-\lambda & 3 & 4 & 5 & 1-\lambda \end{bmatrix} = (15-\lambda)\begin{bmatrix} \quad \end{bmatrix}$$

$\lambda = 15$ Ans

20 ⇒ $A\begin{bmatrix}1\\-1\end{bmatrix} = -\begin{bmatrix}1\\-1\end{bmatrix}$ & $A\begin{bmatrix}1\\-2\end{bmatrix} = -2\begin{bmatrix}1\\-2\end{bmatrix}$

C

~~A = -1~~ ~~A = -2~~

$\lambda = -1$ $\lambda = -2$

$A = MDM^{-1}$

$$A = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$$ Ans.

21 ⇒ $\{1, 0, -1\}^T$ → Eigen vector

A

$$\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}_{3\times3}\begin{bmatrix}1\\0\\-1\end{bmatrix}_{3\times1} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

1, 0, -1 → Eigen value

1 is eigen value Ans