JOINT DISTRIBUTION IN RESIDUE CLASSES OF FAMILIES OF MULTIPLICATIVE FUNCTIONS I

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Abstract. We study the joint distribution of families of multiplicative functions in residue classes, allowing the moduli to vary within a wide range and assuming some natural control on the average behavior of the functions at (some fixed power of) the primes. As an application, we obtain essentially best possible analogues of the Siegel–Walfisz theorem for families of multiplicative functions that can be controlled by polynomials at the first few powers of all primes. (This class includes several interesting arithmetic functions such as Euler's totient $\varphi(n)$, the sum-of-divisors $\sigma(n)$, and more generally, the sum-of-divisor-powers $\sigma_r(n) := \sum_{d|n} d^r$, and so on.) Our results extend (and give essentially optimal uniform analogues of) works of Narkiewicz, Rayner, Śliwa, Dobrowolski, Fomenko and others. One of the main ideas behind our arguments is the detection of a certain "mixing"/"quantitative ergodicity" phenomenon via methods from sieve theory and the anatomy of integers. Additionally, we need several ideas and machinery from classical analytic number theory, character sums, linear algebra over rings, as well as arithmetic and algebraic geometry.

1. Introduction

We say that a function $f : \mathbb{Z}^+ \to \mathbb{Z}$ is equidistributed modulo $q \in \mathbb{Z}^+$ if $\#\{n \le x : f(n) \equiv a \pmod q\} \sim x/q$ as $x \to \infty$. This notion has been studied for *additive* functions: See results of Delange [8, 9], which have been partially extended in [28, 29, 1, 35]. However, for *multiplicative* functions, it turns out that this notion is not the correct one to work with: For instance, using classical results (such as those implicit in [18]), it can be shown that Euler's totient function $\varphi(n)$ is almost always divisible by any fixed integer q, and hence is **not** equidistributed modulo any fixed $q > 1$. Motivated by this, Narkiewicz defines a function $f : \mathbb{Z}^+ \to \mathbb{Z}$ to be weakly equidistributed (or WUD) modulo $q \in \mathbb{Z}^+$ if there are infinitely many n for which $(f(n), q) = 1$, and if

$$\#\{n \le x : f(n) \equiv a \pmod q\} \sim \frac{1}{\varphi(q)} \#\{n \le x : (f(n), q) = 1\} \quad \text{as } x \to \infty,$$

for every coprime residue a mod q.[1] Hence, our sample space is $\{n : (f(n), q) = 1\}$, and every coprime residue mod q gets its fair share of the sample space. For example, if f is WUD mod 6, then the residues 1 and 5 mod 6 each asymptotically receive 50% of the sample space $\{n : (f(n), 6) = 1\}$. The notion of weak equidistribution extends naturally to a family of functions $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$: We say that $(f_1, \dots, f_K)$ are WUD mod q if there are infinitely many n for which $(f_1(n) \dots f_K(n), q) = 1$, and if for all coprime residues $a_1, \dots, a_K$ mod q, we have

(1.1)
$$\#\{n \le x : (\forall i)\ f_i(n) \equiv a_i \pmod q\} \sim \frac{1}{\varphi(q)^K} \#\{n \le x : (f_1(n) \dots f_K(n), q) = 1\} \quad \text{as } x \to \infty.$$

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[1]Here, (a, b) denotes the gcd of a and b, unless stated otherwise.

The notion of weak equidistribution has been studied by several authors. Narkiewicz gave a general criterion [21, Theorem 1] deciding weak equidistribution of a "polynomially–defined" multiplicative function; here, $f : \mathbb{Z}^+ \to \mathbb{Z}$ is polynomially–defined if there exist $V \in \mathbb{Z}^+$ and polynomials $\{F_v\}_{1 \le v \le V} \subset \mathbb{Z}[T]$ satisfying $f(p^v) = F_v(p)$ for all primes p and all $v \in [V]$. (Hence, F_v controls the behavior of f at the v-th powers of primes.) Using his criterion, he showed in [21] that $\varphi(n)$ is WUD mod q iff $(q, 6) = 1$. (Related are works of Dence–Pomerance [10] and Banks–Shparlinski [2, Theorem 3.1].) Śliwa [37] used Narkiewicz's criterion to show that the sum-of-divisors function $\sigma(n) = \sum_{d|n} d$ is WUD mod q iff $6 \nmid q$. This was extended to the functions $\sigma_r(n) = \sum_{d|n} d^r$ (which also appear as Fourier coefficients of Eisenstein series) by Narkiewicz, Rayner, Śliwa, Doborowolski and Fomenko [37, 13, 26, 24, 25, 30, 31]. Finally, the most general results were also obtained by Narkiewicz in [23, 22], where he gave an exact characterization deciding when an arbitrary *family* of polynomially–defined multiplicative functions is weakly equidistributed to a fixed modulus.

In all these works, the modulus q is always assumed to be *fixed*. It is then natural to ask what happens when q is allowed to *grow* with x, in analogy with the *Siegel–Walfisz theorem* from prime number theory. Analogues of Siegel–Walfisz are always sought both due to their independent interest and applications to other problems (and such analogues have been studied in other contexts, such as for smooth numbers and mean values of multiplicative functions). In our context, the explicit question is: Can we find analogues of the Siegel–Walfisz theorem for the joint distribution of a general family of multiplicative functions? To formalize this, given $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ and a set $\mathcal{Q} \subset \mathbb{Z}^+$, we say that $(f_1, \dots, f_K)$ are WUD mod q, uniformly for $q \in \mathcal{Q}$, if:

- For every $q \in \mathcal{Q}$, we have $(\prod_{i=1}^K f_i(n), q) = 1$ for infinitely many n, and
- (1.1) holds as $x \to \infty$, uniformly in $q \in \mathcal{Q}$ and in coprime residues $a_1, \dots, a_K$ mod q.

The only results in this direction seem to have been obtained in [19, 27, 29], which have made partial progress towards uniformizing (*highly special cases* of) Narkiewicz's criteria in [21, 23].
(i) In [19], Lebowitz-Lockard, Pollack and the author show that $\underline{\varphi(n)}$ is WUD uniformly mod $\underline{\text{prime}}$ $q \in [5, (\log x)^{K_0}]$ for any fixed K_0.

(ii) In [27], Pollack and the author consider multiplicative functions $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ for which there exist nonconstant polynomials $F_1, \dots, F_K \in \mathbb{Z}[T]$ with the product $\underline{F_1 \dots F_K \text{ separable}}$[2], such that $f_i(p) = F_i(p)$ for all primes p. We show that such $(f_1, \dots, f_K)$ are WUD mod q, uniformly in a <u>very small</u> range of q (far from optimal), such that q is also <u>"almost prime"</u> in a certain sense.

(iii) In [29], we partially improve the above result <u>for $K = 1$</u>, optimizing the range of q and replacing "almost prime" by the condition that q is supported on <u>sufficiently large primes</u>.

Although the underlined restrictions above are not necessary in Narkiewicz's criteria, they cannot be overcome using the arguments and methods available in the existing literature. For instance, several arguments in [29] do not generalize to families (i.e. to $K > 1$), and the separability of F_1 as well as the (roughness) condition on q are both needed for the arguments to work. *Further*, while Narkiewicz's criteria in [21, 23] allow a certain flexibility with prime powers (elaborated on below) and the possibility of sparse input sets – both of which are crucial in several applications – [19, 27, 29] cannot allow any such flexibility. Hence, these works cannot give satisfactory uniform extensions of many of the cited works of Rayner, Śliwa and others, either.

[2]i.e. having no repeated roots in $\mathbb{C}$

In this manuscript, we remove *all* these limitations and give essentially optimal uniform extensions of Narkiewicz's general criteria in [23, 22] (for a single growing modulus), hence also completely extending all the previously cited works. Our modulus q will vary either within optimal ranges *or* up to any fixed power of $\log x$, hence we also obtain essentially best possible qualitative analogues of the *Siegel–Walfisz theorem* for a general family of polynomially–defined multiplicative functions. We not only need to refine the methods in the previous works (which were restricted to the anatomy of integers), but also use several new ideas not considered in them (such as from linear algebra over rings, as well as from algebraic and arithmetic geometry). As mentioned in § 1.2, these ideas seem to have potential applications to other related problems as well.

1.1. **The Main Results.** Since the general version of Narkiewicz's criterion in [23] requires some technical set-up, we first give uniform analogues of a special case to illustrate the most important ideas. In section 7, we state our most general results and the additional ingredients they require.

Consider multiplicative functions $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ for which there exist polynomials $F_1, \dots, F_K \in \mathbb{Z}[T]$ satisfying $f_i(p) = F_i(p)$ for all primes p and all $i \in [K]$. For $q \in \mathbb{Z}^+$ we define

$$\boldsymbol{\alpha(q) := \varphi(q)^{-1} \#\{u \in U_q : F_1(u) \cdots F_K(u) \in U_q\}}, \text{ where } \boldsymbol{U_q := (\mathbb{Z}/q\mathbb{Z})^\times}.^{3}$$

Let $\boldsymbol{\mathcal{Q}(f_1, \dots, f_K)}$ denote the set of all $q \in \mathbb{Z}^+$ **with** $\boldsymbol{\alpha(q) \neq 0}$ that satisfy the following property: For all Dirichlet characters $(\chi_1, \dots, \chi_K) \neq (\chi_0, \dots, \chi_0) \bmod q$ satisfying $\prod_{i=1}^K \chi_i(F_i(u)) = 1$ on its "unit support" $\{u \in U_q : \prod_{i=1}^K F_i(u) \in U_q\}$, we have $\prod_{i=1}^K \chi_i(f_i(2^j)) = -1$ for all $j \geq 1$.

Here χ_0 is the trivial character mod q. Narkiewicz's criterion in this special setting is as follows.

Theorem 1.1. [23] *Assume $\alpha(q) \neq 0$. Then $(f_1 \dots, f_K)$ are WUD mod q iff $q \in \mathcal{Q}(f_1, \dots, f_K)$.*

To state our main results, we assume that $F_1, \dots, F_K$ are **multiplicatively independent**, i.e. that $\prod_{i=1}^K F_i^{c_i}$ is nonconstant in $\mathbb{Q}(T)$ for all integers $(c_1, \dots, c_K) \neq (0, \dots, 0)$.[4] Now factor each F_i as $r_i \prod_{j=1}^M G_j^{\mu_{i,j}}$ with $r_i \in \mathbb{Z}$, $\mu_{i,j} \in \mathbb{Z}_{\geq 0}$, and with $\boldsymbol{\{G_j\}_{j=1}^M} \subset \mathbb{Z}[T]$ being pairwise coprime primitive irreducibles, such that each G_j appears with a positive exponent $\mu_{i,j}$ in some F_i. With $\boldsymbol{E_0}$ denoting the $M \times K$ integer matrix $\left((\boldsymbol{\mu_{i,j}})_{\substack{\boldsymbol{1 \leq i \leq K} \\ \boldsymbol{1 \leq j \leq M}}}\right)^{\intercal}$, let $\boldsymbol{\beta := \beta(F_1, \dots, F_K)} \in \mathbb{Z}^+$ be the largest invariant factor of E_0. (Since $\{F_i\}_i$ are multiplicatively independent, the columns of E_0 are $\mathbb{Q}$–linearly independent, so β is the last diagonal entry in the Smith Normal Form of E_0.)

Fix any $B_0 > 0$. We say that q satisfies **Hypothesis** $\boldsymbol{IFH(F_1, \dots, F_K; B_0)}$ if all its prime divisors ℓ exceeding B_0 satisfy $\gcd(\ell - 1, \beta) = 1$. Note that if $\prod_{i=1}^K F_i$ is separable (as was assumed in previous works [27, 29]) then $\beta = 1$, so that any q satisfies $IFH(F_1, \dots, F_K; B_0)$ for any $B_0 > 0$.

Theorem 1.2. *Fix any $K_0, B_0 > 0$ and $\epsilon \in (0,1)$, and assume that $F_1, \dots, F_K$ are multiplicatively independent. Then $(f_1, \dots, f_K)$ are WUD mod $q \leq (\log x)^{K_0}$ lying in $\mathcal{Q}(f_1, \dots, f_K)$ and satisfying $IFH(F_1, \dots, F_K; B_0)$, provided at least one of the following two conditions holds:*

(i) $q \leq (\log x)^{(1-\epsilon)\alpha(q)(K-1/D_{\min})^{-1}}$, *where* $D_{\min} := \min\{\deg F_i : 1 \leq i \leq K\}$, ***or***

(ii) *q is squarefree and* $q^{K-1} D_{\min}^{\omega(q)} \leq (\log x)^{(1-\epsilon)\alpha(q)}$.

[3] In this paper, **bold** is used only for emphasis (for ease of reference), not as part of the notation itself.

[4] This hypothesis is easily satisfied in almost all applications: For instance, it is satisfied if $\prod_{i=1}^K F_i$ is separable, or (more generally), if each F_i has an irreducible factor that is not present in the other F_j (for $j \neq i$).

The conditions in (i) and (ii) are ignored if $K = D_{\min} = 1$ (i.e. a single multiplicative function controlled by a linear polynomial at the primes), so in this case, we have uniformity up to $(\log x)^{K_0}$ for any fixed K_0. Some applications of Theorem 1.2: $\varphi(n)$ is WUD mod $q \le (\log x)^{K_0}$ coprime to 6 (extending Narkiewicz [21, Corollary 2]). $\sigma(n)$ is WUD mod odd $q \le (\log x)^{K_0}$ (extending Śliwa [37]). Moreover, (φ, σ) are (jointly) WUD *uniformly* mod $q \le (\log x)^{(1-\epsilon)\alpha_0(q)}$ coprime to 6, where $\alpha_0(q) = \prod_{\ell | q \text{ prime}} (\ell - 3)/(\ell - 1)$ (extending [22, Theorem 1] for a single modulus).

In section 6, we show that for any K and $D_{\min}$ (except $K = D_{\min} = 1$), both ranges in (i) and (ii) are essentially optimal, in that "$1-\epsilon$" cannot be replaced by "$1+\epsilon$" in either. In fact, if $K > 1$ and $D_{\min} = 1$, then this optimality holds in a much stronger sense: The ranges in (i) and (ii) are optimal no matter how many of the F_i we assume to be linear and even if we take all the F_i to be essentially arbitrary linear polynomials. As a consequence, the range $q \le (\log x)^{(1-\epsilon)\alpha_0(q)}$ above is optimal even for the particular example of (φ, σ). Moreover, the multiplicative independence hypothesis and hypothesis $IFH(F_1, \dots, F_K; B_0)$ are also both necessary in Theorem 1.2 and in Theorem 1.3 below: We shall establish this in the sequel [36].

As our constructions in section 6 show, obstructions to uniformity come from prime inputs. We can modify those constructions to produce more obstructions coming from inputs n that have *too few* large prime factors. Hence to restore uniformity in the full Siegel-Walfisz range, we need to restrict our n to those with *sufficiently many* large prime factors. Let $\boldsymbol{P_R(n)}$ denote the R-th largest prime factor of n counted with multiplicity (with the convention $P_R(n) := 1$ if $\Omega(n) < R$).

Theorem 1.3. *Fix $K_0, B_0 > 0$ and assume that $F_1, \dots, F_K$ are multiplicatively independent. Then*

$$\sum_{\substack{n \le x:\ P_R(n) > q \\ (\forall i)\ f_i(n) \equiv a_i \pmod q}} 1 \sim \frac{1}{\varphi(q)^K} \sum_{\substack{n \le x:\ P_R(n) > q \\ (\prod_i f_i(n), q) = 1}} 1 \sim \frac{1}{\varphi(q)^K} \sum_{\substack{n \le x \\ (\prod_i f_i(n), q) = 1}} 1 \quad \text{as } x \to \infty, \tag{1.2}$$

uniformly in $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(f_1, \dots, f_K)$ and satisfying $IFH(F_1, \dots, F_K; B_0)$, as well as uniformly in coprime residues $a_1, \dots, a_K$ modulo q. Here

$$R = \begin{cases} KD + 1, & \text{in general, where } D := \sum_{i=1}^K \deg F_i, \\ 2K + 1, & \text{if } q \text{ is squarefree,} \\ 2, & \text{if } q \text{ squarefree}, K = 1,\ F_1 \text{ is } \textbf{not squarefull} \text{ (i.e. } F_1 \text{ has a simple root in } \mathbb{C}). \end{cases}$$

As an application, (φ, σ) are WUD uniformly modulo $q \le (\log x)^{K_0}$ coprime to 6 if we restrict to inputs n with $P_5(n) > q$. (Without this restriction, we only have the aforementioned optimal range $q \le (\log x)^{(1-\epsilon)\alpha_0(q)}$.) Another application: By Narkiewicz [22, 23], $(\varphi, \sigma, \sigma_2)$ are WUD modulo any fixed q having $P^-(q) > 23$. By Theorem 1.2, this weak equidistribution holds uniformly modulo any such $q \le (\log x)^{(1/2-\epsilon)c_q}$ where $c_q := \prod_{\ell | q:\ \ell \equiv -1 \pmod 4} (\ell - 3)/(\ell - 1) \cdot \prod_{\ell | q:\ \ell \equiv 1 \pmod 4} (\ell - 5)/(\ell - 1)$. Uniformity is restored modulo *all* $q \le (\log x)^{K_0}$ satisfying $P^-(q) > 23$, provided we restrict to n having $P_{13}(n) > q$. This last restriction can be weakened to $P_7(n) > q$, if q is also squarefree.

The smaller the value of R in (1.2), the larger our set wherein equidistribution occurs, and the better our result. In section 6, we show that the third value of R is optimal, while for any K, the second value is at most "one step away" from being optimal (in that it cannot be reduced to $2K - 1$). Note that even the $K = 1$ special case of Theorem 1.3 improves over [29, Theorem 1.4].

One important theme in our arguments is a *"mixing"* phenomenon originally observed in [29]: Intuitively, given $G \in \mathbb{Z}[T]$ and q supported on large primes, if we choose uniformly at random, units $u_1, u_2, \dots$ mod q such that each $G(u_j)$ is a unit mod q, then in the sequence of partial products $\{G(u_1)\dots G(u_J)\}_{J=1}^{\infty}$, *every* unit mod q appears equally often. Here we detect and generalize this "mixing" via more refined arguments from sieve theory and from the anatomy of integers.

However, we also need several new ideas. First, we require two inputs from classical analytic number theory: One from Halász's Theorem (which requires us to carefully estimate certain "pretentious distances"[5]), and another from *modifying* the Landau–Selberg–Delange "LSD" method. (Known forms of the LSD method or other mean value results are *not* sufficient to get the desired uniformity in q.) Further, we need machinery from character sums (extensions of Weil bounds), linear algebra over rings (via Smith normal forms) as well as algebraic and arithmetic geometry (by constructing regular sequences and counting rational points on varieties over finite fields).

1.2. **Beyond the polynomially–defined regime.** The aforementioned "mixing" (suitably extended) can be used to study the joint weak equidistribution of multiplicative functions $h_1, \dots, h_K : \mathbb{Z}^+ \to \mathbb{Z}$ assuming only a natural control on the *average* behavior of the h_i at the primes. Owing to this generality, we expect the following results to have applications beyond the "polynomially–defined" regime, such as for Ramanujan's τ-function and Fourier coefficients of modular forms.

We assume that for some (large) *threshold parameter* $\boldsymbol{y}$ and for all Dirichlet characters $(\chi_1, \dots, \chi_K)$ mod q, we have an estimate of the form $\sum_{y<p\le Y} \prod_i \chi_i(h_i(p)) \;\approx\; \alpha_{(\chi_1,\dots,\chi_K)} \cdot \sum_{y<p\le Y} \prod_i \chi_0(h_i(p))$ for all $Y \ge y$, where $\alpha_{(\chi_1,\dots,\chi_K)} \in \mathbb{C}$ depends only on $(\chi_1, \dots, \chi_K)$, and where $\boldsymbol{\chi_0}$ is the trivial character mod q. Assuming this with an essentially arbitrary error term, we bound the extent to which $(h_1, \dots, h_K)$ fails to be weakly equidistributed mod q. Our first result shows that for any $J \in \mathbb{Z}^+$ for which the J-th moment $\sum_{\chi_1,\dots,\chi_K} |\alpha_{(\chi_1,\dots,\chi_K)}|^J$ is small enough, $(h_1, \dots, h_K)$ are WUD mod q, **upon** restricting to inputs n with J many large prime factors (exceeding the *threshold* y).

In what follows, we let $\boldsymbol{\widehat{\chi} := (\chi_1, \dots, \chi_K)}$ and let $\boldsymbol{\mathfrak{f}(\widehat{\chi})}$ denote the lcm of the conductors of χ_i. We write $\boldsymbol{\widehat{h} := (h_1, \dots, h_K)}$, $\boldsymbol{h := h_1 \cdots h_K}$, and $\boldsymbol{\widehat{\chi}(\widehat{h}(n)) := \chi_1(h_1(n)) \cdots \chi_K(h_K(n))}$.

Theorem 1.4. *Let $h_1, \dots, h_K : \mathbb{Z}^+ \to \mathbb{Z}$ be multiplicative functions. Consider any $q \in \mathbb{Z}^+$, $M \ge 1$, $y \ge q$, and a decreasing function $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $\widehat{\chi}$ mod q and all $Y \ge y$,*

$$\sum_{y<p\le Y} \widehat{\chi}(\widehat{h}(p)) \;=\; \alpha_{\widehat{\chi}} \sum_{y<p\le Y} \chi_0(h(p)) + O(MY\mathcal{E}(y)), \tag{1.3}$$

with $\alpha_{\widehat{\chi}} \in \mathbb{C}$ in the unit disk. Then for all d dividing q, all $x \ge y$ and all $J \in \mathbb{Z}^+$, we have

$$\sum_{\substack{n\le x:\ P_J(n)>y\\ (\forall i)\ h_i(n)\equiv a_i \pmod q}} 1 \;-\; \left(\frac{\varphi(d)}{\varphi(q)}\right)^K \sum_{\substack{n\le x:\ P_J(n)>y,\ (h(n),q)=1\\ (\forall i)\ h_i(n)\equiv a_i \pmod d}} 1 \tag{1.4}$$
$$\ll \frac{1}{\varphi(q)^K} \sum_{\substack{\widehat{\chi} \bmod q\\ \mathfrak{f}(\widehat{\chi}) \nmid d}} |\alpha_{\widehat{\chi}}|^J \sum_{\substack{n\le x\\ (h(n),q)=1}} 1 \;+\; \frac{Jx}{y} \;+\; Mx\mathcal{E}(y)\log x.$$

The implied constant here depends only on the implied constant in (1.3).

Note that (1.3) is in general weaker than a hypothesis of the form "$\sum_{y<p\le Y} \widehat{\chi}(\widehat{h}(p)) = \beta_{\widehat{\chi}} \sum_{y<p\le Y} 1 + O(MY\mathcal{E}(y))$". As the proof will reveal, (1.4) just uses (1.3) for all $\widehat{\chi}$ mod q with $\mathfrak{f}(\widehat{\chi}) \nmid d$. Taking $J = d = 1$ directly bounds the failure of weak equidistribution. However, to make the right side

[5] in the terminology of Granville and Soundararajan [14]

of (1.4) negligible in applications (including in the polynomially–defined cases), we often need to select a parameter $J \to \infty$. We thus see what happens if we include the inputs n having $P_J(n) \le y$.

Our estimate will be in terms of a parameter "$\xi_1(q;z)$" that controls the (suitably normalized) count of large prime solutions to the congruences $h_i(p) \equiv a_i \pmod q$. Given any $h_i : \mathbb{Z}^+ \to \mathbb{Z}$, and $q, r \in \mathbb{Z}^+$ and $z > 0$, **let $\boldsymbol{\xi_r(q;z)}$** $:= \xi_r(q; z; \{h_i\}_{i=1}^K) > 0$ be *any* parameter satisfying

(K_r)
$$\#\{P_1 \cdots P_r \le x : P_i \text{ primes},\ P_1 > z,\ q < P_r < \cdots < P_1,\ (\forall i)\ h_i(P_1)\cdots h_i(P_r) \equiv a_i \pmod q\} \le\ \xi_r(q;z) \cdot x(\log\log x)^r \Big/ \log(z/q)$$

for all $x \ge z$ and all units $a_1, \dots, a_K \bmod q$. Thus, $\xi_r(q;z)$ controls the number of squarefree integers $A \le x$ with exactly r prime factors (all suitably large), that satisfy $h_i(A) \equiv a_i \pmod q$ for all i. In particular, $\xi_1(q;z)$ controls the number of primes $P \in (z, x)$ satisfying $h_i(P) \equiv a_i \pmod q$ for all i. Note that this is genuinely a *definition*, not a *hypothesis*, for the choice $\xi_r(q;z) = 1$ always satisfies (K_r) by Chebyshev's estimates, for all possible $\{h_i\}_i, q, r$ and z.

Let $\Psi(x,z)$ be the number of z-smooth numbers up to x, a well–studied quantity.

Proposition 1.5. *Fix $K \in \mathbb{Z}^+$. Uniformly in **all** multiplicative functions $h_1, \dots, h_K : \mathbb{Z}^+ \to \mathbb{Z}$, in all real numbers $x \ge z > 0$, and in all positive integers J, we have*
$$\sum_{\substack{n \le x:\, P_J(n) \le y \\ (\forall i)\ h_i(n) \equiv a_i \pmod q}} 1 \ \ll\ \xi_1(q;z) \frac{x(2\log\log x)^{R+J}}{\log(z/q)} \exp\Bigg(\sum_{\substack{p \le y \\ (h(p), q) = 1}} \frac{1}{p}\Bigg) \ +\ \Psi(x,z)\ +\ \frac{x}{y}.$$
The implied constant depends only on K. More generally, for any $R \le J$, if we also impose the condition $P_R(n) > q$ on the left, then the same bound holds, but only with $\xi_1(q;z)$ replaced by $\xi_R(q;z) + R^K \sum_{\substack{1 \le r < R \\ 1 \le s < K}} \xi_r(q;z)/q^{\max\{s, R-r-s\}}$, and with the implied constant depending only on K.

Combining Theorem 1.4 with Proposition 1.5, we get the following flexible result, which removes the restriction $P_J(n) > y$ in Theorem 1.4 *while* giving us the freedom to choose J and z.

Theorem 1.6. *Assume the hypotheses of Theorem 1.4. For all d dividing q, all $x, z > 0$, and all $J, R \in \mathbb{Z}^+$ satisfying $y \le z \le x$ and $J \ge R$, we have*

$$\sum_{\substack{n \le x:\ P_R(n) > q \\ (\forall i)\ h_i(n) \equiv a_i \pmod q}} 1 \ -\ \left(\frac{\varphi(d)}{\varphi(q)}\right)^K \sum_{\substack{n \le x:\ (h(n), q) = 1 \\ (\forall i)\ h_i(n) \equiv a_i \pmod d}} 1 \tag{1.5}$$
$$\ll\ \frac{1}{\varphi(q)^K} \sum_{\substack{\widehat{\chi} \bmod q \\ \mathfrak{f}(\widehat{\chi})\, \nmid\, d}} |\alpha_{\widehat{\chi}}|^J \sum_{\substack{n \le x \\ (h(n), q) = 1}} 1\ +\ \Psi(x,z)\ +\ \frac{Jx}{y}\ +\ Mx\mathcal{E}(y)\log x$$
$$+\ \left\{\left(\frac{\varphi(d)}{\varphi(q)}\right)^K + \xi_R(q;z) + R^K \sum_{\substack{1 \le r < R \\ 1 \le s < K}} \frac{\xi_r(q;z)}{q^{\max\{s, R-r-s\}}}\right\} \frac{x(2\log\log x)^{R+J}}{\log(z/q)} \exp\Bigg(\sum_{\substack{p \le y \\ (h(p), q) = 1}} \frac{1}{p}\Bigg).$$

The implied constant here depends only on K and on the implied constant in (1.3).

The flexibility in choosing the parameters d, y, z, J, R (especially the fact that z and J now appear only on the right of (1.5)) are *all* useful in applications: For instance, by taking $d = R = 1$ and a large J, we can make the right side negligible while including *all* inputs n on the left. The strength of Theorem 1.6 basically depends on the bounds available on $\sum_{\mathfrak{f}(\widehat{\chi}) \nmid d} |\alpha_{\widehat{\chi}}|^J$, and on $\xi_r(q; z)$ for $r \in [R]$. As discussed in Remark 2, Theorems 1.4 and 1.6 can be extended by replacing the condition "$\mathfrak{f}(\widehat{\chi}) \nmid d$" by the more general "$\widehat{\chi} \notin \mathcal{S}$", where $\mathcal{S}$ is an arbitrary subset of the set of K-tuples of characters mod q; the corresponding variant of (1.5) would be uniform in $\mathcal{S}$.

Finally, we mention the following estimate, which forms a key input in the proofs of Theorems 1.2 and 1.3, and may also be of independent interest or have applications to other problems.

Proposition 1.7. *Under the hypotheses of Theorem 1.4, we have for any $\widehat{\chi}$ mod q,*

$$\sum_{n \le x} \widehat{\chi}(\widehat{h}(n)) \ll \frac{x}{\log x} \exp\left(\sum_{\substack{p \le y \\ (h(p), q)=1}} \frac{1}{p} + |\alpha_{\widehat{\chi}}| \sum_{\substack{y < p \le x \\ (h(p), q)=1}} \frac{1}{p} + O(M (\log x)^3 \mathcal{E}(y)) \right).$$

The implied constant depends only on the implied constant in (1.3).

To conclude the introduction, we remark that Narkiewicz's general weak equidistribution criteria in [21, 23] are *much* more general than Theorem 1.1: They allow the possibility that the behavior of our f_i at the ν-th powers of primes is *most significant* (in a certain sense), for some fixed $\nu \in \mathbb{Z}^+$; this also allows the possibility that our input sets are sparse. However, all the results and methods from all previous work on varying moduli are unable to handle any $\nu > 1$. Adapting our methods for Theorems 1.2 and 1.3 with further additional arguments, we obtain (in section 7) optimal uniform results in this greater generality, along with several new applications. We also give extensions of Theorems 1.4 and 1.6 that apply for certain sparse input sets.

In sections 2 to 5, we will first prove the results in subsection § 1.2, and then deduce those in § 1.1 with additional arguments. In a future paper, we shall study the joint weak equidistribution of a family of multiplicative functions $(f_1, \dots, f_K)$ to moduli $(q_1, \dots, q_K)$, defined appropriately.

Notation and Conventions. We do not consider the zero function as multiplicative (thus, $f(1) = 1$ for any multiplicative function f). By $P(n)$ and $P^-(n)$, we mean the largest and least prime factors of n respectively (with $P(1) \coloneqq 1$, $P^-(1) \coloneqq \infty$). Further, $P_R(n)$ denotes the R-th largest prime factor of n listed with multiplicity (with the convention $P_R(n) \coloneqq 1$ if $\Omega(n) < R$).

Given $z > 0$, we say that $n \in \mathbb{Z}^+$ is z-smooth (respectively, z-rough) if $P(n) \le z$ (resp. $P^-(n) > z$). By the z-smooth part (resp. z-rough part) of n, we shall mean the largest z-smooth (resp. z-rough) positive integer dividing n. When there is no danger of confusion, we shall write $(a_1, \dots, a_K)$ in place of $\gcd(a_1, \dots, a_K)$. Throughout, we use p and ℓ to denote primes.

Implied constants in $\ll$ and O-notation, and implicit constants in qualifiers like "sufficiently large", may depend on any parameters declared as "fixed". Other dependence will be noted explicitly (with, say, parentheses or subscripts). $\log_k$ denotes the k-th iterate of the natural logarithm.

2. The general mixing phenomenon: Proof of Theorems 1.4–1.6

Proof of Theorem 1.4. Recall that $\boldsymbol{\widehat{\chi} \coloneqq (\chi_1, \dots, \chi_K)}$, $\mathfrak{f}(\widehat{\chi})$ is the lcm of the conductors of χ_i, $\boldsymbol{\widehat{h} \coloneqq (h_1, \dots, h_K)}$, $\boldsymbol{h \coloneqq h_1 \cdots h_K}$, and $\boldsymbol{\widehat{\chi}(\widehat{h}(n)) \coloneqq \chi_1(h_1(n)) \cdots \chi_K(h_K(n))}$. We say that

$n \le x$ is "convenient" (abbreviated as "conv") if the J largest prime divisors of n exceed y and are distinct. Thus any convenient n can be uniquely written as $mP_J \cdots P_1$, where

$$L_m := \max\{y, P(m)\} < P_J < \cdots < P_1. \tag{2.1}$$

Now, the number of $n \le x$ having a repeated prime factor exceeding y is at most $\sum_{p>y} \sum_{m \le x/p^2} 1 \ll x/y$, which is absorbed in the right of (1.5). Hence, to complete the proof of Theorem 1.6, it suffices to show (1.4) with the condition "$P_J(n) > y$" replaced by the condition that n is convenient.

By the orthogonality of Dirichlet characters, we detect the congruences $h_i(n) \equiv a_i \pmod q$.

$$\sum_{\substack{n \le x \text{ conv} \\ (\forall i)\ h_i(n) \equiv a_i \ (\mathrm{mod}\ q)}} 1 \;=\; \frac{1}{\varphi(q)^K} \sum_{\widehat{\chi} \bmod q} \overline{\chi}_1(a_1) \cdots \overline{\chi}_K(a_K) \sum_{n \le x \text{ conv}} \widehat{\chi}(\widehat{h}(n)). \tag{2.2}$$

For any $\widehat{\chi}$ mod q for which $\mathfrak{f}(\widehat{\chi}) \mid d$, there exists a unique tuple of characters $\widehat{\psi} := (\psi_1, \dots, \psi_K)$ mod d such that $\chi_i = \chi_0 \psi_i$ for all i. Thus the total of contribution of all such $\widehat{\chi}$ in (2.2) equals

$$\frac{1}{\varphi(q)^K} \sum_{\widehat{\psi} \bmod d} \overline{\psi}_1(a_1) \cdots \overline{\psi}_K(a_K) \sum_{\substack{n \le x \text{ conv} \\ (h(n), q)=1}} \widehat{\psi}(\widehat{h}(n)) = \left(\frac{\varphi(d)}{\varphi(q)}\right)^K \sum_{\substack{n \le x \text{ conv: } (h(n), q)=1 \\ (\forall i)\ h_i(n) \equiv a_i \ (\mathrm{mod}\ d)}} 1. \tag{2.3}$$

Now consider any $\widehat{\chi}$ mod q. Splitting any convenient n uniquely as $mP_J \cdots P_1$ above, we obtain

$$\sum_{n \le x \text{ conv}} \widehat{\chi}(\widehat{h}(n)) \;=\; \sum_{m \le x} \widehat{\chi}(\widehat{h}(m)) \cdot \frac{1}{J!} \sum_{\substack{P_1, \dots, P_J > L_m \\ P_1, \dots, P_J \text{ distinct} \\ P_1 \cdots P_J \le x/m}} \widehat{\chi}(\widehat{h}(P_1)) \cdots \widehat{\chi}(\widehat{h}(P_J)), \tag{2.4}$$

where we have have replaced the ordering on $P_1, \dots, P_J$ by the weaker condition "$P_1, \dots, P_J$ distinct" at the cost of $J!$. For each $i \in [J]$, we see that

$$\begin{aligned} &\sum_{\substack{P_i \le x/mP_1 \cdots P_{i-1} P_{i+1} \cdots P_J \\ P_i > L_m,\ r \ne i \implies P_r \ne P_i}} \widehat{\chi}(\widehat{h}(P_i)) \\ &\qquad = \alpha_{\widehat{\chi}} \sum_{\substack{P_i \le x/mP_1 \cdots P_{i-1} P_{i+1} \cdots P_J \\ P_i > L_m,\ r \ne i \implies P_r \ne P_i}} \chi_0(h(P_i)) \;+\; O\left(J + \frac{Mx\mathcal{E}(y)}{mP_1 \cdots P_{i-1} P_{i+1} \cdots P_J}\right), \end{aligned} \tag{2.5}$$

where we have used (1.3), and removed and restored the "$\ne$" condition with error $O(J)$. The total O-term in (2.5) summed over all m and all the other P_r in (2.4) is $\ll x/y + J^{-1} Mx\mathcal{E}(y) \sum_{n \le x/y} 1/n \ll x/y + J^{-1} Mx\mathcal{E}(y) \log x$. (Here we used the $(J-1)!$ from (2.4) to restore the ordering on $P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_J$, and then noted that $n = mP_1 \cdots P_{i-1} P_{i+1} \cdots P_J \le x/y$.) Using this observation along with (2.5) for each $i = 1, \dots, J$ successively, we get from (2.4),

$$\sum_{n \le x \text{ conv}} \widehat{\chi}(\widehat{h}(n)) \;=\; (\alpha_{\widehat{\chi}})^J \sum_{m \le x} \widehat{\chi}(\widehat{h}(m)) \sum_{\substack{P_1, \dots, P_J \\ L_m < P_J < \cdots < P_1 \\ P_1 \cdots P_J \le x/m}} \chi_0(h(P_1 \cdots P_J)) \;+\; O\left(\frac{Jx}{y} + Mx\mathcal{E}(y) \log x\right). \tag{2.6}$$

The first sum on the right side is $\ll |\alpha_{\widehat{\chi}}|^J \#\{n \le x \text{ conv} : (h(n), q) = 1\}$. Using this for all character tuples $\widehat{\chi}$ mod q for which $\mathfrak{f}(\widehat{\chi}) \nmid d$, we obtain Theorem 1.4. $\square$

Proof of Proposition 1.5. Now consider **any** multiplicative functions $h_1, \dots, h_K : \mathbb{Z}^+ \to \mathbb{Z}$. The first assertion of Proposition 1.5 is a special case of the second, hence it suffices to show the claimed bound on the number of $n \le x$ satisfying $P_R(n) > q$, $P_J(n) \le y$, and $h_i(n) \equiv a_i \pmod q$ for all i. Proceeding as in the sentence following (2.1), we see that up to an error of $O(\Psi(x, z) + x/y)$, we may also assume that $P(n) > z$ and that n has no repeated prime factor exceeding y.

Hence, let $\sum^*_{n\le x}$ denote a sum over $n \le x$ with no repeated prime factor exceeding y, that satisfy $P_R(n) > q$, $P_J(n) \le y$, $P(n) > z$, and $h_i(n) \equiv a_i \pmod q$ for all i. Let $E_{r,s}$ be the number of n counted in $\sum^*_{n\le x} 1$ with $\#\{p > q : p \parallel n\} = r$ and $\#\{p > q : p^2 \mid n\} = s$. Then

$$\sum\nolimits^*_{n\le x} 1 \;\le\; \sideset{}{^*}\sum_{\substack{n\le x\\ \#\{p>q:\ p\parallel n\}\ge R}} 1 \;+\; \sideset{}{^*}\sum_{\substack{n\le x\\ \#\{p>q:\ p^2\mid n\}\ge K}} 1 \;+\; \sum_{\substack{1\le r<R\\ 1\le s<K}} E_{r,s}\,. \tag{2.7}$$

To bound $E_{r,s}$, note that any n counted in it can be decomposed as $mp_1^{c_1}\cdots p_s^{c_s}A$, where $P(m) \le q$, all $p_j > q$ are primes, all $c_j \ge 2$, where A is squarefree and q-rough with $\Omega(A) = r$, and where the factors are all pairwise coprime. Then $(h(m), q) = 1$, $h_i(A) \equiv a_i h_i(mp_1^{c_1}\cdots p_s^{c_s})^{-1} \bmod q$ for all i, and $c_1 + \cdots + c_s \ge R - r$ (as $P_R(n) > q$). By (K_r), the number of possible A is $\ll \xi_r(q; z)x(\log_2 x)^r/mp_1^{c_1}\cdots p_s^{c_s} \log(z/q)$. Next, $\sum_{\substack{m:\ P(m)\le q\\ (h(m),q)=1}} 1/m \ll \exp(\sum_{p\le q} \mathbb{1}_{(h(p),\,q)=1}/p)$, and $\sum_{p_1,\dots,p_s>q} p_1^{-c_1}\cdots p_s^{-c_s} \ll q^{-(c_1+\cdots+c_s-s)}$. The sum of $q^{-(c_1+\cdots+c_s-s)}$ over all possible $(c_1, \dots, c_s)$ can be bounded in two ways: On the one hand, it is $\ll q^s \prod_{j=1}^s (\sum_{c_j\ge 2} q^{-c_j}) \ll q^{-s}$, and on the other (writing $t := c_1 + \cdots + c_s$), it is $\ll \sum_{t\ge R-r} t^s/q^{t-s} \ll R^K/q^{R-r-s}$. Collecting everything, we get

$$E_{r,s} \ll R^K \frac{\xi_r(q; z)}{q^{\max\{s,R-r-s\}}} \cdot \frac{x(\log_2 x)^R}{\log(z/q)} \;\exp\left(\sum_{p\le q} \frac{\mathbb{1}_{(h(p),\,q)=1}}{p}\right),$$

The other two sums on the right of (2.7) are bounded analogously: For example, to bound the second sum, we start by writing $n = mp_1^{c_1}\cdots p_s^{c_s}P$ (with $P_J(m) \le y$, $P > z$, and all $p_j > q$, $c_j \ge 2$) and bounding the count of P by Chebyshev's estimates. This establishes Proposition 1.5. ☐

Proof of Theorem 1.6. Since $R \le J$ and $y \ge q$, the condition $P_J(n) > y$ appearing in (1.4) is stronger than the condition $P_R(n) > q$ appearing in (1.5). Hence to deduce Theorem 1.6 from Theorem 1.4 and Proposition 1.5, it only remains to show that

$$\sum_{\substack{n\le x:\ P_J(n)\le y\\ (h(n),\,q)=1}} 1 \;\ll\; \frac{x(2\log_2 x)^J}{\log z}\;\exp\left(\sum_{p\le y} \frac{\mathbb{1}_{(h(p),\,q)=1}}{p}\right) \;+\; \Psi(x, z) \;+\; \frac{x}{y}. \tag{2.8}$$

Up to an error of $O(\Psi(x, z)+x/z)$, we may also assume that $P(n) > z$ and that n has no repeated prime factor exceeding z. Write any such n as $n = BAP$, where $P(B) \le y < P^-(A)$, and $P := P(n) \in (z, x/AB]$. Then B, A, P are pairwise coprime and $\Omega(A) \le J$ (since $P_J(n) \le y$). By Chebyshev's estimates, the number of P is $\ll x/AB \log z$. Moreover, $\sum_{A\le x:\ \Omega(A)\le J} 1/A \le (1 + \sum_{p\le x} 1/p)^J \le (2\log_2 x)^J$, while $\sum_{\substack{B:\ P(B)\le y\\ (h(B),\,q)=1}} 1/B \le \prod_{p\le y} \left(1 + \sum_{r\ge 1} \mathbb{1}_{(h(p^r),\,q)=1}/p^r\right) \ll \exp(\sum_{p\le y} \mathbb{1}_{(h(p),\,q)=1}/p)$. Collecting all the estimates in this paragraph yields (2.8). ☐

Remark. Given an arbitrary subset $\mathcal{S}$ of the set of K-tuples of characters mod q, applying (2.6) and its ensuing observation for $\widehat{\chi} \notin \mathcal{S}$, we see that the more general version of Theorem 1.6 holds with the condition "$\mathfrak{f}(\widehat{\chi}) \nmid d$" on the right of (1.5) replaced by "$\widehat{\chi} \notin \mathcal{S}$", and with the second term on the left replaced by $\varphi(q)^{-K}\sum_{\widehat{\chi}\in\mathcal{S}}\overline{\chi}_1(a_1)\cdots\overline{\chi}_K(a_K)\sum_{n\le x}\widehat{\chi}(\widehat{h}(n))$. This more general estimate doesn't involve d and is uniform in all $\mathcal{S}$. (The analogous comment holds for Theorem 1.4 as well.)

3. A General Character Sum Bound: Proof of Proposition 1.7

Consider any $\widehat{\chi}$ mod q. By Halász's Theorem (as stated in [38, Corollary III.4.12]), we have

$$\sum_{n\le x}\widehat{\chi}(\widehat{h}(n)) \ll \frac{x}{L} + x\exp\left(-\min_{-L\le t\le L}\mathcal{D}(x,t)\right), \tag{3.1}$$

uniformly in $x, L \ge 2$, where by Mertens' Theorem,

$$\mathcal{D}(x,t) \coloneqq \sum_{p\le x}\frac{1-\mathrm{Re}(\widehat{\chi}(\widehat{h}(p))p^{-it})}{p} \ge \log_2 x - \sum_{\substack{p\le y\\(h(p),q)=1}}\frac{1}{p} - \left|\sum_{y<p\le x}\frac{\widehat{\chi}(\widehat{h}(p))}{p^{1+it}}\right| + O(1). \tag{3.2}$$

Set $\boldsymbol{L \coloneqq \log x}$, and cover the interval $(y,x]$ with disjoint "multiplicatively narrow" subintervals of the form $\boldsymbol{I_\eta \coloneqq (\eta, \eta(1+1/\log^2 x)]}$, such that the rightmost of these juts out slightly past x but not past $x(1+1/\log^2 x)$. Note that $\sum_{x<p\le x(1+1/\log^2 x)} 1/p \ll 1/\log^3 x$, and that for any $p\in I_\eta$, we have $p=\eta(1+O(1/\log^2 x))$ so that $p^{-(1+it)}=\eta^{-(1+it)}(1+O(1/\log x))$ uniformly in $t\in[-L,L]$. Combining these and noting that $\sum_\eta\sum_{p\in I_\eta}1/\eta \ll \sum_{p\le 2x}1/p \ll \log_2 x$, we get

$$\sum_{y<p\le x}\frac{\widehat{\chi}(\widehat{h}(p))}{p^{1+it}} = \sum_\eta\frac{1}{\eta^{1+it}}\sum_{p\in I_\eta}\widehat{\chi}(\widehat{h}(p)) + O\left(\frac{\log_2 x}{\log x}\right). \tag{3.3}$$

Using (1.3) to estimate $\sum_{p\in I_\eta}\widehat{\chi}(\widehat{h}(p))$ and noting that there are $O(\log^3 x)$ many I_η's, we obtain

$$\sum_{y<p\le x}\frac{\widehat{\chi}(\widehat{h}(p))}{p^{1+it}} = \sum_\eta\frac{\alpha_{\widehat{\chi}}}{\eta^{1+it}}\sum_{p\in I_\eta}\chi_0(h(p)) + O\left(\frac{\log_2 x}{\log x} + M\mathcal{E}(y)(\log x)^3\right).$$

By the two observations before (3.3), we replace the entire main term by $\alpha_{\widehat{\chi}}\sum_{y<p\le x}\chi_0(h(p))/p^{1+it}$ (upto an error $\ll \log_2 x/\log x$). Inserting the resulting estimate into (3.2), we obtain

$$\mathcal{D}(x,t) \ge \log_2 x - \sum_{\substack{p\le y\\(h(p),q)=1}}\frac{1}{p} - |\alpha_{\widehat{\chi}}|\sum_{\substack{y<p\le x\\(h(p),q)=1}}\frac{1}{p} + O(1+M\mathcal{E}(y)(\log x)^3). \tag{3.4}$$

Finally, inserting this bound into (3.1) completes the proof of Proposition 1.7. ☐

4. Setting the stage for theorems 1.2 and 1.3: Character sums, module theory and a little arithmetic geometry

We remind the reader of the set-up in Theorems 1.2 and 1.3, which we will be assuming throughout until section 6: $f_1,\dots,f_K:\mathbb{Z}^+\to\mathbb{Z}$ are multiplicative functions and $F_1,\dots,F_K\in\mathbb{Z}[T]$ satisfy $f_i(p)=F_i(p)$ for all primes p and all i. We define $\boldsymbol{f \coloneqq \prod_{i=1}^K f_i}$ and $\boldsymbol{F \coloneqq \prod_{i=1}^K F_i}$. We assume that $(F_i)_{i=1}^K$ are **multiplicatively independent**, hence the matrix $\boldsymbol{E_0} = \left((\mu_{i,j})_{\substack{1\le i\le K\\1\le j\le M}}\right)^{\mathsf{T}}$ has rank K $(\le M)$. Its Smith normal form is $\mathrm{diag}(\beta_1,\dots,\beta_K)$ where $\beta_i\in\mathbb{Z}^+$ and $\beta_i \mid \beta_{i+1}$ for all i. Recall that $\boldsymbol{\beta = \beta(F_1, \dots, F_K) = \beta_K}$. Further, we are assuming that $\boldsymbol{\alpha(q)} = \frac{1}{\varphi(q)}\#\{u \in U_q : F(u) \in U_q\} \neq 0$. By the Chinese Remainder Theorem and Mertens' Theorem, it is easy to see that

$$\boldsymbol{\alpha(q) \gg (\log_2(3q))^{-D}} \text{ uniformly in } q \geq 1, \text{ where } \boldsymbol{D = \deg F = \sum_{i=1}^{K} \deg F_i}. \tag{4.1}$$

In what follows, for an odd prime power ℓ^e, let ψ_{ℓ^e} denote a character mod ℓ^e which generates the character group of U_{ℓ^e}. We will need two bounds on character sums. The first is a version of the Weil bounds, a special case of [39, Corollary 2.3] (see also [7], [40] and [32] for older results).

Proposition 4.1. *Let ℓ be a prime. Consider $H \in \mathbb{Z}[T]$ not of the form $c \cdot G^{\ell-1}$ in $\mathbb{F}_\ell[T]$. Then $|\sum_{u \bmod \ell} \psi_\ell(H(u))| \leq (r-1)\sqrt{\ell}$ where r is the degree of the largest squarefree divisor of H.*

The second is an extension of this to higher prime powers, which comes from Theorems 1.2 and 7.1 and eqn. (1.15) in work of Cochrane [5] (see [6] for related results).

Proposition 4.2. *Let ℓ be a prime, $H \in \mathbb{Z}[T]$ nonconstant with $\mathrm{ord}_\ell(H) = 0$. Define $t := \mathrm{ord}_\ell(H')$, $\mathcal{C}_H := \ell^{-t}H'$ and $M := \max\{\mathrm{mult}_\theta(\mathcal{C}_H) : \theta \in \mathbb{F}_\ell, H(\theta) \neq 0\}$. Let $e \geq t+2$ be an integer.*

(i) *If $\ell > 2$, then* $\left|\sum_{u \bmod \ell^e} \psi_{\ell^e}(H(u))\right| \leq \left(\sum_{\substack{\theta \in \mathbb{F}_\ell \\ H(\theta) \neq 0}} \mathrm{mult}_\theta(\mathcal{C}_H)\right) \ell^{t/(M+1)}\ \ell^{e(1-1/(M+1))}$.

(ii) *Assume that $e \geq t+3$ and let ψ be the character mod 2^e defined by $\psi(5) := \exp(2\pi i/2^{e-2})$ and $\psi(-1) := 1$. Then* $\left|\sum_{u \bmod 2^e} \psi(H(u))\right| \leq (12.5)2^{t/(M+1)}\ 2^{e(1-1/(M+1))}$.

The condition "$\mathrm{ord}_\ell(H) = 0$" guarantees that t is finite. Further, $\mathcal{C}_H$ is being viewed as a nonzero element of $\mathbb{F}_\ell[T]$, and M is the maximum multiplicity of a zero of $\mathcal{C}_H$ (in $\mathbb{F}_\ell$) that is not a zero of H. To use these bounds, we need the following crucial inputs from linear algebra over rings.

Lemma 4.3. *There exists a constant $C := C(F_1, \dots, F_K) > 0$ satisfying all the following:*

(1) *For any prime $\ell > C$ satisfying $(\ell-1, \beta) = 1$, the only tuple $(A_1, \dots, A_K) \in [\ell-1]^K$ for which $\prod_{i=1}^K F_i^{A_i}$ is of the form $c \cdot G^{\ell-1}$ in $\mathbb{F}_\ell[T]$ is $(A_1, \dots, A_K) = (\ell-1, \dots, \ell-1)$.*

(2) *Consider any integer $r \geq 2$, prime ℓ, and $(A_1, \dots, A_K) \in (\mathbb{Z}^+)^K$ with $\ell \nmid (A_1, \dots, A_K)$. Let $\tau_\ell := \mathrm{ord}_\ell\Big((T^{\varphi(\ell^r)} \prod_{i=1}^K F_i(T)^{A_i})'\Big)$ and $\widetilde{F} := \sum_{i=1}^K A_i F_i' \prod_{j \neq i} F_j$. Then*

$$\tau_\ell \begin{cases} = \mathrm{ord}_\ell(\widetilde{F}) = 0, & \text{if } \ell > C, r \geq 2 \\ = \mathrm{ord}_\ell(\widetilde{F}) \leq C, & \text{if } \ell \leq C,\ \mathrm{ord}_\ell(F) = 0, \text{ and } r \geq C+2. \end{cases}$$

In both cases above, for any $\theta \in \mathbb{F}_\ell$ for which $\theta F(\theta) \neq 0$, the multiplicity of θ in the polynomials $\ell^{-\tau_\ell}\widetilde{F}$ and $\ell^{-\tau_\ell}(T^{\varphi(\ell^r)} \prod_{i=1}^K F_i(T)^{A_i})'$ are equal.

Proof. Writing $F_i' \prod_{j \neq i} F_j = \sum_{r=0}^{D-1} c_{i,r} T^r$ in $\mathbb{Z}[T]$ for each $i \in [K]$, let $\widetilde{\boldsymbol{\beta}} \in \mathbb{Z}^+$ denote the largest invariant factor of the matrix $\boldsymbol{M_0} := \Big((c_{i,r})_{\substack{1 \leq i \leq K \\ 0 \leq r < D}}\Big)^\intercal$. Fix $\boldsymbol{C}$ **exceeding 2,** $\widetilde{\boldsymbol{\beta}}$, the product

$\prod_{j=1}^{M} |\mathrm{disc}(G_j)| \cdot \prod_{1\le j\ne j'\le M} |\mathrm{Res}(G_j, G_{j'})|$ (>0), and (the sizes of) the leading coefficients of all F_i and G_j. (Recall that $\{G_j\}_{j=1}^M$ are pairwise coprime primitive irreducibles and $F_i = \prod_{j=1}^M G_j^{\mu_{ij}}$, as in § 1.1.) We show that any such C satisfies all the assertions in this proposition. Note that for all $\ell > C$, the polynomial $\prod_{j=1}^M G_j$ is separable over $\mathbb{F}_\ell$, i.e. squarefree in $\overline{\mathbb{F}}_\ell[T]$.

Proof of (1). Consider any prime $\ell > C$ and any $(A_1, \dots, A_K) \in [\ell-1]^K$ satisfying $(\ell-1, \beta) = 1$ and $\prod_{i=1}^K F_i^{A_i} = c \cdot G^{\ell-1}$ for some $c \in U_\ell$ and $G \in \mathbb{F}_\ell[T]$. Since $F_i = r_i \prod_{j=1}^M G_j^{\mu_{i,j}}$, we have $c \cdot G^{\ell-1} = \rho \prod_{j=1}^M G_j^{\sum_{i=1}^K \mu_{i,j} A_i}$ for some $\rho \in U_\ell$. Factoring out all leading coefficients and comparing exponents, we get $\sum_{i=1}^K \mu_{i,j} A_i \equiv 0 \pmod{\ell-1}$ for all j. (Here we used the last sentence of the previous paragraph.) This gives the matrix congruence $E_0(A_1 \dots A_K)^\intercal \equiv 0 \pmod{\ell-1}$.

Now considering invertible integer matrices P_0, R_0 for which $P_0 E_0 R_0$ is the Smith normal form $\mathrm{diag}(\beta_1, \dots, \beta_K)$, the last congruence above yields $\mathrm{diag}(\beta_1, \dots, \beta_K) R_0^{-1} (A_1 \dots A_K)^\intercal \equiv 0 \pmod{\ell-1}$. Since $(\ell-1, \beta_K) = 1$, we have $(\ell-1, \beta_1 \cdots \beta_K) = 1$, so that the last congruence forces $R_0^{-1}(A_1 \dots A_K)^\intercal \equiv 0 \pmod{\ell-1} \implies (A_1 \dots A_K)^\intercal \equiv 0 \pmod{\ell-1}$. Hence each $A_i = \ell-1$.

Proof of (2). Since $(\prod_{i=1}^K F_i^{b_i})' = \sum_{i=1}^K b_i F_i' \prod_{j\ne i} F_j$ for any $(b_i)_{i=1}^K \in \mathbb{Z}^K$, the multiplicative independence of the $(F_i)_{i=1}^K$ forces the polynomials $\left(F_i' \prod_{j\ne i} F_j\right)_{i=1}^K$ to be $\mathbb{Q}$–linearly independent. Hence the $D \times K$ matrix $M_0 = \left((c_{i,r})_{\substack{1\le i\le K\\ 0\le r<D}}\right)^\intercal$ defined above has rank K, its Smith Normal Form has K positive diagonal entries, and its largest invariant factor $\widetilde{\beta}$ is the last diagonal entry.

Now with $d := \mathrm{ord}_\ell(\widetilde{F})$, since ℓ^d divides all the coefficients of $\widetilde{F}(T) = \sum_{i=1}^K A_i F_i'(T) \prod_{j\ne i} F_j(T) = \sum_{r=0}^{D-1} (\sum_{i=1}^K c_{i,r} A_i) T^r$, it follows that $M_0(A_1 \dots A_K)^\intercal \equiv 0 \pmod{\ell^d}$. An argument analogous to that given for subpart (1) shows that if $\ell^d \nmid \widetilde{\beta}$, then $\ell \mid (A_1, \dots, A_K)$, a contradiction. Thus $\ell^d \mid \widetilde{\beta}$, forcing $\mathrm{ord}_\ell(\widetilde{F}) = d \le v_\ell(\widetilde{\beta}) \le \mathbb{1}_{\ell\le C} \cdot C$. All assertions in subpart (2) now follow easily from the fact that $\left(T^{\varphi(\ell^r)} \prod_{i=1}^K F_i(T)^{A_i}\right)' = \varphi(\ell^r) T^{\varphi(\ell^r)-1} \prod_{i=1}^K F_i(T)^{A_i} + T^{\varphi(\ell^r)} \left(\prod_{i=1}^K F_i(T)^{A_i-1}\right) \widetilde{F}(T)$. $\square$

Let $\boldsymbol{\alpha_{\widehat{\chi}} := (\alpha(q)\varphi(q))^{-1} \sum_{u\in U_q} \widehat{\chi}(\widehat{F}(u)) = (\alpha(q)\varphi(q))^{-1} \sum_{u \bmod q} \chi_0(u) \prod_{i=1}^K \chi_i(F_i(u))}$. With B_0 and $D = \sum_{i=1}^K \deg F_i$ in Theorems 1.2 and 1.3, and with C as in the proof of Lemma 4.3,

$$\textbf{Fix } \boldsymbol{C_0 > \max\{B_0, C, (32D)^{2D+2}\}}, \text{ and fix an } \textbf{integer } \boldsymbol{\kappa > 100D(DC_0^{2C_0})^{4C_0}}. \tag{4.2}$$

Let $\boldsymbol{Q_0}$ be the largest C_0-smooth $(\kappa+1)$-free divisor of q, i.e. $\boldsymbol{Q_0 = \prod_{\ell\le C_0} \ell^{\min\{v_\ell(q),\kappa\}} \ll 1}$.

Proposition 4.4. *Fix any $\varepsilon > 0$.*
(1) *We have* $\sum_{\substack{\widehat{\chi} \bmod q\\ \mathfrak{f}(\widehat{\chi}) \nmid Q_0}} |\alpha_{\widehat{\chi}}|^N = o(1)$ *as $N \to \infty$, uniformly in all q. Moreover, uniformly in all q,*

$$\sum_{\widehat{\chi} \bmod q} |\alpha_{\widehat{\chi}}|^N \ll_N \begin{cases} \exp(O((\log q)^{1-1/D})), & \textit{for each fixed } N \ge KD+1. \\ q^{K-N/D+\varepsilon}, & \textit{for each fixed } N \le KD. \end{cases} \tag{4.3}$$

(2) *The bound* (K_r) *holds, where uniformly in all* $q \in \mathbb{Z}^+$ *and* $z \in (q, x]$, *we have*

$$\xi_r(q;z) \ll \begin{cases} \varphi(q)^{-K}\exp(O((\log q)^{1-1/D})), & \text{for each fixed } r \ge KD+1. \\ q^{-r/D+\varepsilon}, & \text{for each fixed } r \le KD. \\ q^{-1/D_{\min}} \cdot \log_2(3q), & \text{if } r = 1, \text{ where } D_{\min} = \min_i \deg(F_i). \end{cases} \tag{4.4}$$

(3) *The bound* (K_r) *holds, where uniformly in all* ***squarefree*** q *and* $z \in (q, x]$, *we have*

$$\xi_r(q;z) \ll \begin{cases} \varphi(q)^{-K}\exp(O(\sqrt{\log q})), & \text{for each fixed } r \ge 2K+1. \\ q^{-r/2+\varepsilon}, & \text{for each fixed } r \le 2K. \\ D_{\min}^{\omega(q)} \cdot \varphi(q)^{-1}, & \text{if } r = 1. \\ \varphi(q)^{-1}\exp(O(\sqrt{\log q})), & \text{if } r = 2, \ K = 1, \text{ and } F_1 \text{ is not squarefull.} \end{cases} \tag{4.5}$$

Proof. We give the argument for $D > 1$; the case $D = 1$ (i.e. when $K = 1$ and F_1 is linear) is much simpler. Define $\boldsymbol{\omega_{>C_0}(m)} := \#\{\ell \mid m : \ell > C_0\}$ and $\boldsymbol{\widetilde{\omega}(m)} := \#\{\ell \mid m : \ \ell \le C_0, \ v_\ell(m) \ge \kappa + 1\}$. We claim that for any tuple $\widehat{\chi} = (\chi_1, \dots, \chi_K)$ of Dirichlet characters mod q, we have

$$|\alpha_{\widehat{\chi}}| \ \le \ (DC_0^{C_0+1})^{\widetilde{\omega}(\mathfrak{f}(\widehat{\chi}))} \cdot (4D)^{\omega_{>C_0}(\mathfrak{f}(\widehat{\chi}))} \cdot \mathfrak{f}(\widehat{\chi})^{-1/D}. \tag{4.6}$$

To show this, we start by writing each $\chi_i = \prod_{\ell^e \| q} \boldsymbol{\chi_{i,\ell}}$ for some character $\chi_{i,\ell}$ mod ℓ^e. Then with $\boldsymbol{\alpha_{\widehat{\chi},\ell} := (\alpha(\ell)\varphi(\ell^e))^{-1} \sum_{u \bmod \ell^e} \ \chi_{0,\ell}(u) \prod_{i=1}^K \chi_{i,\ell}(F_i(u))}$, we have $\alpha_{\widehat{\chi}} = \prod_{\ell^e \| q} \ \alpha_{\widehat{\chi},\ell}$ by the Chinese Remainder Theorem. (Here $\chi_{0,\ell}$ is the trivial character mod ℓ.) To bound each $\alpha_{\widehat{\chi},\ell}$, let $\boldsymbol{r_\ell := v_\ell(\mathfrak{f}(\widehat{\chi}))}$ so that $\boldsymbol{\ell^{r_\ell}} = \mathrm{lcm}[\boldsymbol{\mathfrak{f}(\chi_{1,\ell}), \dots, \mathfrak{f}(\chi_{K,\ell})}]$. Let $\boldsymbol{\psi_{i,\ell}}$ denote the character mod ℓ^{r_ℓ} inducing $\chi_{i,\ell}$, so that **at least one of** $\boldsymbol{(\psi_{1,\ell}, \dots, \psi_{K,\ell})}$ **is a primitive character mod** ℓ^{r_ℓ}. It is easy to see that $\alpha_{\widehat{\chi},\ell} = (\alpha(\ell)\varphi(\ell^{r_\ell}))^{-1} \sum_{u \bmod \ell^{r_\ell}} \ \chi_{0,\ell}(u) \prod_{i=1}^K \psi_{i,\ell}(F_i(u))$.

We first consider the case $\boldsymbol{\ell > 2}$. With ψ_ℓ being a character generating the character group mod ℓ^{r_ℓ}, note that each $\boldsymbol{\psi_{i,\ell} = \psi_\ell^{A_i}}$ for some $\boldsymbol{A_i} \in [\boldsymbol{\varphi(\ell^{r_\ell})}]$. Since some $\psi_{i,\ell}$ is primitive mod ℓ^{r_ℓ},

$$(A_1, \dots, A_K) \not\equiv \begin{cases} 0 \pmod{\ell - 1}, & \text{if } r_\ell = 1 \\ 0 \pmod{\ell}, & \text{if } r_\ell > 1. \end{cases}$$

Writing $\alpha_{\widehat{\chi},\ell} = (\alpha(\ell)\varphi(\ell^{r_\ell}))^{-1} \ \sum_{u \bmod \ell^{r_\ell}} \ \psi_\ell\left(u^{\varphi(\ell^{r_\ell})} \prod_{i=1}^K F_i(u)^{A_i}\right)$, we claim that

$$|\alpha_{\widehat{\chi},\ell}| \le \begin{cases} (4D) \cdot \ell^{-v_\ell(\mathfrak{f}(\widehat{\chi}))/D} & \text{if } \ell > C_0 \\ (DC_0^{C_0+1}) \cdot \ell^{-v_\ell(\mathfrak{f}(\widehat{\chi}))/D}, & \text{if } 2 < \ell \le C_0, \ v_\ell(\mathfrak{f}(\widehat{\chi})) \ge \kappa + 1. \end{cases} \tag{4.7}$$

When $\ell > C_0$ and $r_\ell = 1$, we apply Proposition 4.1 and Lemma 4.3(1), recalling that the prime divisors $\ell > C_0 > B_0$ satisfy $\gcd(\ell - 1, \beta) = 1$ by hypothesis $IFH(F_1, \dots, F_K; B_0)$. In all other cases, we apply Proposition 4.2(i) on the polynomial $H(T) := T^{\varphi(\ell^{r_\ell})} \prod_{i=1}^K F_i(T)^{A_i}$, noting that Lemma 4.3(2) gives both $\mathrm{ord}_\ell(H') \le r_\ell - 2$ and $\sum_{\theta \in \mathbb{F}_\ell : \ H(\theta) \ne 0} \mathrm{mult}_\theta(\mathcal{C}_H) \ \le D - 1$. Now (4.7) follows upon noting that since $\alpha(\ell) \ne 0$, we have $\alpha(\ell) = 1 - \frac{1}{\ell - 1}\#\{u \in U_\ell : F_1(u) \cdots F_K(u) \equiv 0 \pmod{\ell}\} \ge 1 - \frac{\min\{\ell-2, D\}}{\ell - 1}$, which gives $\alpha(\ell) \ge 1/2$ for $\ell > C_0$, and $\alpha(\ell) \ge 1/C_0$ for $\ell \le C_0$.

Now consider the case $\boldsymbol{\ell = 2}$. Hence we are assuming that $2 \mid q$ (which forces $\alpha(2) = 1$), and that $r_2 = v_2(\mathfrak{f}(\widehat{\chi})) \ge \kappa + 1$. The character group mod 2^{r_2} is generated by the characters ψ, η mod 2^{r_2}

defined by $\psi(5) = \exp(2\pi i/2^{r_2-2})$, $\psi(-1) = 1$, $\eta(5) = 1$ and $\eta(-1) = -1$. Since at least one of the characters $(\psi_{i,2})_{i=1}^K$ is primitive mod 2^{r_2}, we can write each $\psi_{i,2} = \psi^{A_i}\eta^{B_i}$, where $A_i \in [2^{r_2-2}]$, $B_i \in [2]$, and $2 \nmid (A_1, \dots, A_K)$. Then $\alpha_{\widehat{\chi},2} = \varphi(2^{r_2})^{-1} \sum_{u \bmod 2^{r_2}} \psi\left(\prod_{i=1}^K F_i(u)^{A_i}\right) \eta\left(u^2 \prod_{i=1}^K F_i(u)^{B_i}\right)$. Defining $g_\lambda(T) = \prod_{i=1}^K F_i(4T+\lambda)^{A_i}$ for $\lambda = \pm 1$, and noting that η has conductor 4, we write

$$\alpha_{\widehat{\chi},2} = \frac{1}{2^{r_2-1}} \sum_{\lambda=\pm 1} \eta\left(\prod_{i=1}^K F_i(\lambda)^{B_i}\right) \cdot \frac{1}{4} \sum_{v \bmod 2^{r_2}} \psi(g_\lambda(v)).$$

If $\eta\left(\prod_{i=1}^K F_i(\lambda)^{B_i}\right) \neq 0$, then $\prod_{i=1}^K F_i(4T+\lambda) \equiv \prod_{i=1}^K F_i(\lambda) \equiv 1 \pmod 2$, which means that

$$t_{2,\lambda} := \mathrm{ord}_2(g'_\lambda) = 2 + \mathrm{ord}_2(\widetilde{F}(4T+\lambda)) \leq 2D + \mathrm{ord}_2(\widetilde{F}) \leq 2D + C_0 \leq \kappa - 3 \leq r_2 - 3. \tag{4.8}$$

Here the second inequality uses Lemma 4.3(2) (with $\widetilde{F} = \sum_{i=1}^K A_i F_i' \prod_{j \neq i} F_j$ as defined there), and the first inequality uses the general fact that $\mathrm{ord}_2(W(4T+\lambda)) \leq \mathrm{ord}_2(W) + 2\deg(W)$ for any $W \in \mathbb{Z}[T]$. (This follows from a straightforward divisibility argument.) The identity $g'_\lambda(T) = 4\widetilde{F}(4T+\lambda) \prod_{i=1}^K F_i(4T+\lambda)^{A_i-1}$ gives $\mathrm{mult}_\theta(2^{-t_{2,\lambda}} g'_\lambda) = \mathrm{mult}_\theta(2^{-(t_{2,\lambda}-2)} \widetilde{F}(4T+\lambda)) \leq D-1$ for any $\theta \in \mathbb{F}_2$ for which $\theta F(\theta) \neq 0$. An application of Proposition 4.2(ii) and (4.8) now yields

$$|\alpha_{\widehat{\chi},2}| \leq (DC_0^{C_0+1}) \cdot 2^{-v_2(\mathfrak{f}(\widehat{\chi}))/D} \quad \text{if} \quad 2 \mid q \quad \text{and} \quad v_2(\mathfrak{f}(\widehat{\chi})) \geq \kappa + 1.$$

Combining this with (4.7) in the factorization $|\alpha_{\widehat{\chi}}| = \prod_{\ell^e \| q} |\alpha_{\widehat{\chi},\ell}|$ establishes our claim (4.6).

Proof of (1). Since $\#\{\widehat{\chi} \bmod q : \mathfrak{f}(\widehat{\chi}) \mid m\} \leq m^K$, (4.6) yields for any fixed N,

$$\sum_{\widehat{\chi} \bmod q} |\alpha_{\widehat{\chi}}|^N \;\leq\; (DC_0^{C_0+1})^{NC_0} \sum_{m \mid q} \frac{(4D)^{N\omega_{>C_0}(m)}}{m^{N/D-K}} \;\ll_N\; \prod_{\ell^e \| q} \left(1 + (4D)^{N\mathbb{1}_{\ell>C_0}} \sum_{1 \leq v \leq e} \frac{1}{\ell^{v(N/D-K)}}\right),$$

where we have noted that $\omega_{>C_0}$ is an additive function. If $N \geq KD+1$, the product above is $\ll \exp(O(\sum_{\ell \mid q} \ell^{-1/D})) \leq \exp(O(\sum_{\ell \leq \omega(q)} \ell^{-1/D})) \leq \exp(O((\log q)^{1-1/D}))$. If $N \leq KD$, the same product is $\ll \prod_{\ell^e \| q} \left(2(4D)^N e^{\mathbb{1}_{N=KD}} \ell^{e(K-N/D)}\right) \leq \left(\prod_{\ell^e \| q} e\right)^{\mathbb{1}_{N=KD}} \cdot q^{K-N/D} \exp(O(\omega(q))) \ll_\varepsilon q^{K-N/D+\varepsilon}$.

To finish subpart (1), it thus only remains to show its very first assertion. As $N \to \infty$, (4.6) gives

$$\sum_{\substack{\widehat{\chi} \bmod q \\ \mathfrak{f}(\widehat{\chi}) \nmid Q_0}} |\alpha_{\widehat{\chi}}|^N \;\leq\; \sum_{\substack{m \mid q \\ m \nmid Q_0}} \frac{(DC_0^{C_0+1})^{N\widetilde{\omega}(m)} \cdot (4D)^{N\omega_{>C_0}(m)}}{m^{N/D-K}} \leq \sum_{\substack{m \mid q \\ m \nmid Q_0}} \frac{(DC_0^{C_0+1})^{N\widetilde{\omega}(m)} \cdot (4D)^{N\omega_{>C_0}(m)}}{m^{N/2D}}. \tag{4.9}$$

Since Q_0 is defined to be the largest C_0-smooth $(\kappa+1)$-free divisor of q, any m counted above either: (i) has $P(m) > C_0$ or (ii) has $P(m) \leq C_0$ but is not $(\kappa+1)$-free. Let E_1 denote the contribution of all m of type (i) to the rightmost sum in (4.9), and E_2 be that of type (ii). Then

$$E_1 \leq \sum_{\ell > C_0} \frac{(4D)^N}{\ell^{N/2D}} \sum_{M \mid q} \frac{(DC_0^{C_0+1})^{N\widetilde{\omega}(M)} \cdot (4D)^{N\omega_{>C_0}(M)}}{M^{N/2D}}. \tag{4.10}$$

Now $\sum_{\ell > C_0} (4D)^N \ell^{-N/2D} = (4D)^N \sum_{\ell > C_0} \ell^{-N/4D} \cdot \ell^{-N/4D} \leq (4DC_0^{-1/4D})^N \sum_{\ell > C_0} \ell^{-5} = o(1)$ as $N \to \infty$. Since $\widetilde{\omega}$ and $\omega_{>C_0}$ are additive functions, the second sum in (4.10) is at most

$$\prod_{\ell \mid q \,:\, \ell \leq C_0} \left(1 + \sum_{1 \leq v \leq \kappa} \ell^{-vN/2D} + (DC_0^{C_0+1})^N \sum_{v > \kappa} \ell^{-vN/2D}\right) \cdot \prod_{\ell \mid q \,:\, \ell > C_0} \left(1 + (4D)^N \sum_{v \geq 1} \ell^{-vN/2D}\right)$$

$$\leq \exp\left(\sum_{\ell \leq C_0}\sum_{v \leq \kappa} \ell^{-5v} + (DC_0^{C_0+1} 2^{-\kappa/4D})^N \sum_{\ell \leq C_0}\sum_{v \geq 0} \ell^{-5v}\right) \cdot \exp\left((4DC_0^{-1/4D})^N \sum_{\ell > C_0}\sum_{v \geq 1} \ell^{-5v}\right) \ll 1,$$

where we have recalled (4.2). Collecting all these estimates gives $E_1 = o(1)$ as $N \to \infty$.

On the other hand any m counted in E_2 is C_0-smooth but not $(\kappa+1)$-free. Any such m can be written as $m = Am'$ where $A > 1$ is $(\kappa+1)$-full and m' is $(\kappa+1)$-free. Then $m' \ll_{C_0,\kappa} 1$, giving

$$E_2 \ll (DC_0^{C_0+1})^{NC_0} \sum_{\substack{A \mid q \,:\, P(A) \leq C_0 \\ A > 1 \text{ is } (\kappa+1)\text{-full}}} \frac{1}{A^{N/2D}} \leq (DC_0^{C_0+1})^{NC_0} \left\{ \prod_{\substack{\ell^e \| q \\ \ell \leq C_0}} \left(1 + \sum_{\kappa+1 \leq v \leq e} \frac{1}{\ell^{vN/2D}}\right) - 1 \right\}$$

As $N \to \infty$, this is $\ll (DC_0^{C_0+1})^{NC_0}\{\exp(O(2^{-\kappa N/2D})) - 1\} \ll \left((DC_0^{C_0+1})^{C_0} 2^{-\kappa/2D}\right)^N \ll 50^{-N} = o(1)$. Inserting this and the conclusion on E_1 into (4.9), we get the first assertion of subpart (1).

Proof of (2). Fix any $r \in \mathbb{Z}^+$ and consider any $z \in (q, x]$. We may write

$$\sum_{\substack{P_1, \dots, P_r:\ P_1 \cdots P_r \leq x \\ P_1 > z,\ q < P_r < \cdots < P_1 \\ (\forall i)\ f_i(P_1) \cdots f_i(P_r) \equiv a_i \pmod q}} 1 \leq \sum_{(v_1, \dots, v_r) \in \mathcal{V}_{r,K}(q; \widehat{a})} \sum_{\substack{q < P_r < \cdots < P_2 \leq x \\ (\forall j \geq 2)\ P_j \equiv v_j \pmod q}} \sum_{\substack{q < P_1 \leq x/P_2 \cdots P_r \\ P_1 \equiv v_1 \pmod q}} 1,$$

with $\mathcal{V}_{r,K}(q; \widehat{a}) \coloneqq \{(v_1, \dots, v_r) \in U_q^r :\ (\forall i)\ \prod_{j=1}^r F_i(v_j) \equiv a_i \pmod q\}$. Brun–Titchmarsh shows that the innermost sum above is $\ll x/\varphi(q) P_2 \cdots P_r \log(z/q)$; combined with partial summation, it also shows that $\sum_{\substack{q < P_j \leq x \\ P_j \equiv v_j \pmod q}} 1/P_j \ll \log_2 x/\varphi(q)$ for each $j \in \{2, \dots, r\}$. Consequently

$$\sum_{\substack{P_1, \dots, P_r:\ P_1 \cdots P_r \leq x \\ P_1 > z,\ q < P_r < \cdots < P_1 \\ (\forall i)\ f_i(P_1) \cdots f_i(P_r) \equiv a_i \pmod q}} 1 \ll \frac{\#\mathcal{V}_{r,K}(q; \widehat{a})}{\varphi(q)^r} \cdot \frac{x (\log_2 x)^{r-1}}{\log(z/q)}. \tag{4.11}$$

Hereafter, the first two bounds in (4.4) follow immediately by detecting the congruences $\prod_{j=1}^r F_i(v_j) \equiv a_i \pmod q$ via orthogonality to get

$$\frac{\#\mathcal{V}_{r,K}(q; \widehat{a})}{\varphi(q)^r} = \frac{\alpha(q)^r}{\varphi(q)^K} \sum_{\widehat{\chi} \bmod q} \overline{\chi}_1(a_1) \cdots \overline{\chi}_K(a_K)\ (\alpha_{\widehat{\chi}})^r, \tag{4.12}$$

and then using (4.3). The last bound in (4.4) follows from (4.11), upon noting that $\#\mathcal{V}_{1,K}(q; \widehat{a}) \ll q^{1-1/D_{\min}} \ll q^{-1/D_{\min}}\ \varphi(q)\ \log_2(3q)$ by a result of Konyagin [16, 17].

Proof of (3). A simpler version of the argument for (4.6) yields $|\alpha_{\widehat{\chi}}| \ll (4D)^{\omega_{>C_0}(\mathfrak{f}(\widehat{\chi}))} \cdot \mathfrak{f}(\widehat{\chi})^{-1/2}$ for any $\widehat{\chi}$ modulo squarefree q. Proceeding analogously to the proof of (4.3) to give suitable bounds on $\sum_{\widehat{\chi} \bmod q} |\alpha_{\widehat{\chi}}|^r$ (for any fixed r), and then invoking (4.12) and (4.11), we obtain the first two bounds in (4.5). The third bound in (4.5) follows from the fact that $\#\mathcal{V}_{1,K}(q; \widehat{a}) = \prod_{\ell \mid q} \#\mathcal{V}_{1,K}(\ell; \widehat{a}) \leq (D_{\min})^{\omega(q)}$. (Here we use the Chinese Remainder Theorem for the first equality.)

It only remains to show the fourth and final bound. In the rest of the proof, we thus assume that $r = 2$, $K = 1$ and that $F_1 \in \mathbb{Z}[T]$ is not squarefull. Since $\#\mathcal{V}_{2,1}(q; a) = \prod_{\ell \mid q} \#\mathcal{V}_{2,1}(\ell; a)$ for any $a \in U_q$, the fourth bound would follow once we show that $\#\mathcal{V}_{2,1}(\ell; a) \leq \varphi(\ell)(1 + O(\ell^{-1/2}))$ uniformly in all primes $\ell \gg 1$ and all $a \in U_\ell$. Since F_1 is not squarefull, proceeding as in the first paragraph of the proof of Lemma 4.3, we are guaranteed that F_1 has at least one simple root in $\overline{\mathbb{F}}_\ell$ for all $\ell > C_1$, for some constant $C_1 = C_1(F_1) > 0$.

We claim that $F_1(X)F_1(Y) - w$ is irreducible in $\overline{\mathbb{F}}_\ell[X, Y]$ for all $\ell > C_1$ and all $w \in U_\ell$. Indeed, if $F_1(X)F_1(Y) - w = U(X, Y)V(X, Y)$ with $U, V \in \overline{\mathbb{F}}_\ell[X, Y]$, then for any root $\theta \in \overline{\mathbb{F}}_\ell$ of F_1, we have $-w = U(X, \theta)V(X, \theta)$, forcing $U(X, \theta)$ and $V(X, \theta)$ to be constant. Writing $U(X, Y) = \sum_{0\le j\le r} u_j(Y)X^j$ and $V(X, Y) = \sum_{0\le j\le s} v_j(Y)X^j$, we get $u_j(\theta) = v_j(\theta) = 0$ for all $j > 0$, so that $\prod_{\theta\in\overline{\mathbb{F}}_\ell\,:\,F_1(\theta)=0} (Y - \theta) \mid (\gcd_{j>0} u_j(Y), \gcd_{j>0} v_j(Y))$. Now if both $r, s > 0$, then comparing the leading (highest degree) terms in X on both sides of $F_1(X)F_1(Y) - w = U(X, Y)V(X, Y)$ yields $\prod_{\theta\in\overline{\mathbb{F}}_\ell\,:\,F_1(\theta)=0} (Y - \theta)^2 \mid F_1(Y)$ in $\overline{\mathbb{F}}_\ell[Y]$, contradicting that F_1 has a simple root in $\overline{\mathbb{F}}_\ell$. Hence, either $r = 0$ or $s = 0$. Assuming the latter (wlog), we have $V(X, Y) = v_0(Y)$ with $F_1(X)F_1(Y) - w = U(X, Y)v_0(Y)$. Substituting a root of F for X forces $V(X, Y) = v_0(Y)$ to be constant, proving our claim that $F_1(X)F_1(Y) - w$ is irreducible in $\overline{\mathbb{F}}_\ell[X, Y]$.

Finally by the version of the Hasse–Weil bound in [20, Corollary 2b], we get $\#\mathcal{V}_{2,1}(\ell; a) \le \varphi(\ell)(1 + O(\ell^{-1/2}))$ uniformly in $\ell > C_1$ and $a \in U_\ell$, as desired. □

4.1. **Setting the stage for Theorems 1.2 and 1.3.** With C_0 and κ as in (4.2), define the constant $\boldsymbol{\delta_1 := \delta_1(C_0, \kappa)} < 1$ to be the maximum of $(\alpha(M)\varphi(M))^{-1}|\sum_{u\in U_M} \widehat{\eta}(\widehat{F}(u))|$, taken over all integers $M \le C_0^{C_0\kappa}$ and over all tuples of characters $\widehat{\eta} := (\eta_1, \dots, \eta_K)$ mod M for which $\widehat{\eta}(\widehat{F}(u)) = \prod_{i=1}^K \eta_i(F_i(u))$ is not constant on its unit support $\{u \in U_M : F(u) \in U_M\}$. Note that $\delta_1 < 1$.

Let $\boldsymbol{y := \exp((\log x)^{\min\{\epsilon/2, (1-\delta_1)/2\}})}$, where **$\boldsymbol\epsilon$ is as in the statement of Theorem 1.2** (with **$\boldsymbol{\epsilon := 1}$ for Theorem 1.3**). For any $\widehat{\chi}$ mod q and $Y \ge y$, writing $\sum_{y<p\le Y} \widehat{\chi}(\widehat{f}(p)) = \sum_{u\in U_q} \widehat{\chi}(\widehat{F}(u)) \sum_{\substack{y<p\le Y\\ p\equiv u \pmod q}} 1$ and using Siegel–Walfisz to estimate the inner sum shows that (1.3) holds uniformly in $q \le (\log x)^{K_0}$, with $h_i = f_i$, $\boldsymbol{M := \varphi(q)}$, $\boldsymbol{\alpha_{\widehat{\chi}} = (\alpha(q)\varphi(q))^{-1} \sum_{u\in U_q} \widehat{\chi}(\widehat{F}(u))}$ as above, and $\boldsymbol{\mathcal{E}(y) := \exp(-c\sqrt{\log y})}$. ($c > 0$ being a constant coming from Siegel–Walfisz.)

We thus apply Theorem 1.6 with $\boldsymbol{d := Q_0}$ (here Q_0 was defined after (4.2)), $\boldsymbol{z := x^{1/\log_2 x}}$ and $\boldsymbol{J := \lfloor \log_3 x \rfloor}$. Take **$\boldsymbol R$ as in Theorem 1.3, and for Theorem 1.2, we take $\boldsymbol{R := 1}$**, noting that the condition "$P_1(n) > q$" (on the left of (1.5)) can be ignored with an error at most $\Psi(x, q) \le \Psi(x, z)$. To bound the right of (1.5), note that $\sum_{\substack{\widehat{\chi} \bmod q\\ \mathfrak{f}(\widehat{\chi}) \nmid Q_0}} |\alpha_{\widehat{\chi}}|^J = o(1)$ by Proposition 4.4(1), and $\Psi(x, z) \le x/(\log x)^{(1+o(1))\log_3 x}$ by known estimates on smooth numbers (see [4, p. 15] or [38, Theorem 5.13 and Corollary 5.19, Chapter III.5]). We use the relevant subparts of Proposition 4.4 to bound the $\xi_r(q; z)$. Finally, note that by [29, Proposition 2.1 and Lemma 2.4], we have

(4.13)
$$\sum_{\substack{n\le X\\ (f(n),q)=1}} 1 = \frac{X}{(\log X)^{1-\alpha(q)}} \exp(O((\log_2(3q))^{O(1)})), \qquad \sum_{\substack{p\le X\\ (f(p),q)=1}} \frac{1}{p} = \alpha(q) \log_2 X + O((\log_2(3q))^{O(1)}),$$

while $\alpha(q) \gg (\log_2(3q))^{-D}$ by (4.1). Combining all these observations, we deduce that (in the settings of both Theorems 1.2 and 1.3), the right of (1.5) is $o(\varphi(q)^{-K}\#\{n \le x : (f(n), q) = 1\})$. Further, the second asymptotic in (1.2) follows from [27, Lemma 2.3] and the first estimate in (4.13). Theorems 1.2 and 1.3 would be thus be proven once we show that

$$\#\{n \le x :\ (f(n), q) = 1,\ (\forall i)\ f_i(n) \equiv a_i \pmod{Q_0}\} \sim \varphi(Q_0)^{-K}\#\{n \le x : (f(n), q) = 1\}.$$

Now if the radical (product of distinct prime divisors) of q is of size $O(1)$, then this is immediate by applying Theorem 1.1 to the integer $Q^* \coloneqq \mathrm{lcm}[Q_0, \prod_{\ell | q} \ell] \ll 1$. Hence we may assume that q has sufficiently large radical. By orthogonality, (4.1) and the first estimate in (4.13), Theorems 1.2 and 1.3 would follow once we show that: There exists a constant $\delta_0 \coloneqq \delta_0(C_0, \kappa) < 1$ such that

$$(4.14)\quad \sum_{n \le x} \mathbb{1}_{(f(n),\, q)=1}\ \widehat{\psi}(\widehat{f}(n)) \ \ll\ \frac{x}{(\log x)^{1-\delta_0 \alpha(q)}} \quad \text{for all nontrivial character tuples } \widehat{\psi} \bmod Q_0,[6]$$

uniformly in $\boldsymbol{q \le (\log x)^{K_0}}$ lying in $\boldsymbol{\mathcal{Q}(f_1, \dots, f_K)}$, **having $\boldsymbol{Q_0 > 1}$ and having sufficiently large radical**. (Here for a given q, its "Q_0" is defined as after (4.2).)

Now if $\widehat{\psi}(\widehat{F}(u))$ is **not** constant on $\{u \in U_{Q_0} : F(u) \in U_{Q_0}\}$, then (4.14) follows by applying Proposition 1.7 to the character tuple $\widehat{\chi}$ mod q induced by $\widehat{\psi}$, for which we have $|\alpha_{\widehat{\chi}}| = (\alpha(Q_0)\varphi(Q_0))^{-1} |\sum_{u \in U_{Q_0}} \widehat{\psi}(\widehat{F}(u))| \le \delta_1$ by definition of δ_1. It thus only remains to show (4.14) if $\boldsymbol{\widehat{\psi}(\widehat{F}(u))}$ **is constant on** $\boldsymbol{\{u \in U_{Q_0} : F(u) \in U_{Q_0}\}}$. We do this in the next section.

5. Completing the proof of Theorems 1.2 and 1.3: Modifying the Landau–Selberg–Delange method

5.1. **Perron's formula and analytic continuations.** Recall we are assuming that $\boldsymbol{Q_0 > 1}$, that $\widehat{\psi}(\widehat{F}(u))$ takes a constant value $\boldsymbol{c_{\widehat{\psi}}}$ on its unit support $\{u \in U_{Q_0} : F(u) \in U_{Q_0}\}$, and that the radical $\boldsymbol{Q = \prod_{\ell | q} \ell}$ **is sufficiently large**. It suffices to show (4.14) for $x \in \mathbb{Z}^+$, hence we assume this throughout. We will modify the Landau–Selberg–Delange method, borrowing ideas from work of Scourfield [33]. We start by considering the Dirichlet series

$$\mathcal{F}(s) \ =\ \sum_{n \ge 1} \frac{\mathbb{1}_{(f(n),\, q)=1}\ \widehat{\psi}(\widehat{f}(n))}{n^s} \ =\ \sum_{n \ge 1} \frac{\mathbb{1}_{(f(n),\, Q)=1}\ \widehat{\psi}(\widehat{f}(n))}{n^s},$$

and applying Perron's Formula (as stated in [38, Theorem II.2.3]) to write

(5.1)
$$\sum_{n \le x} \mathbb{1}_{(f(n),\, q)=1}\ \widehat{\psi}(\widehat{f}(n)) \ =\ \frac{1}{2\pi i} \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \frac{\mathcal{F}(s) x^s}{s}\, \mathrm{d}s \ +\ O\left(x \sum_{n \ge 1} \frac{\mathbb{1}_{(f(n),\, q)=1}}{n^{1+\frac{1}{\log x}} (1 + T |\log(x/n)|)} \right).$$

uniformly in $T \ge 1$. The error term here is estimated as in the prime number theorem: First, note that the total contribution of all $n \le 3x/4$ and $n \ge 5x/4$ to the O-term above is

(5.2)
$$\ll \frac{x}{T} \sum_{n \ge 1} \frac{1}{n^{1+1/\log x}} \ll \frac{x}{T} \exp\left(\sum_p \frac{1}{p^{1+1/\log x}} \right) \ll \frac{x}{T} \exp\left(\sum_{p \le x} \frac{1}{p} \ +\ \sum_{j \ge 0} \exp(-2^j) \sum_{x^{2^j} < p \le x^{2^{j+1}}} \frac{1}{p} \right),$$

which is $\ll x(\log x)/T$. Second, note that for any $n \in (3x/4, x-1]$, we can write $n = x - v$ for some integer $v \in [1, x/4)$, so that $|\log(x/n)| = -\log(1 - v/x) \gg v/x$. As such, the contribution of all such n to the O-term in (5.1) is $\ll T^{-1} \sum_{1 \le v < x/4} x/v \ll x(\log x)/T$, and likewise, so is the contribution of all $n \in [x+1, 5x/4)$. Collecting all estimates, we obtain

$$(5.3)\qquad \sum_{n \le x} \mathbb{1}_{(f(n),\, q)=1}\ \widehat{\psi}(\widehat{f}(n)) \ =\ \frac{1}{2\pi i} \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \frac{\mathcal{F}(s) x^s}{s}\, \mathrm{d}s \ +\ O\left(\frac{x \log x}{T} \right).$$

[6] i.e., for all tuples of characters $\widehat{\psi} = (\psi_1, \dots, \psi_K)$ mod Q_0, except for the tuple of trivial characters mod Q_0

In what follows, we write a complex number s as $\sigma + it$, where $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$. Set $\boldsymbol{\mathcal{L}_Q(t) := \log(Q(|t|+1))}$. There exists an absolute constant $\boldsymbol{c_1} > 0$ such that $\prod_{\eta \bmod Q} L(s,\eta)$ has at most one zero $\boldsymbol{\beta_e}$ (the "Siegel zero") in the region $\sigma > 1 - c_1/\mathcal{L}_Q(t)$; moreover, β_e is real and simple and is associated to a real character $\boldsymbol{\eta_e}$ mod Q (the "exceptional character").

We make branch cuts of the complex plane: If $(\alpha(Q), c_{\widehat{\psi}}) \neq (1,1)$, we cut the complex plane along the real line up to $\sigma \leq 1$. If $(\alpha(Q), c_{\widehat{\psi}}) = (1,1)$ and β_e exists then we cut the complex plane along the real line up to $\sigma \leq \beta_e$. In the remaining case, we make no branch cut. The necessary analytic properties of our Dirichlet series $\mathcal{F}(s)$ are given below.

Lemma 5.1. *For $\sigma > 1$, we have $\mathcal{F}(s) = F_{\widehat{\psi}}(s) \cdot G_{\widehat{\psi}}(s)$, where*
(i) $F_{\widehat{\psi}}(s) := \left(\prod_{\eta \bmod Q} L(s,\eta)^{\gamma(\eta)}\right)^{\alpha(Q)c_{\widehat{\psi}}}$, $\gamma(\eta) := (\alpha(Q)\varphi(Q))^{-1}\sum_{u \in U_Q:\ F(u) \in U_Q} \overline{\eta}(u)$,
(ii) $G_{\widehat{\psi}}(s)$ *is analytic on $\sigma > 1 - 1/\log Q$ and satisfies $G_{\widehat{\psi}}(s) \ll (\log_2 Q)^3$ uniformly therein.*
As such, with the aforementioned branch cut conventions, $\mathcal{F}(s)$ analytically continues into the region $\{\sigma + it :\ \sigma > 1 - c_1/\mathcal{L}_Q(t)\} - [1-c_1, 1]$. (Here the real segment $[1-c_1,1]$ has been omitted.)

Proof. We start by using the Euler product of the Dirichlet series $\mathcal{F}(s)$ to write

(5.4)
$$\mathcal{F}(s) = \left(\prod_{p:\ pF(p) \in U_Q} \left(1 - \frac{c_{\widehat{\psi}}}{p^s}\right)^{-1}\right) \left(\prod_p \left(1 + \sum_{v \geq 1} \frac{\mathbb{1}_{(f(p^v),q)=1}\ \widehat{\psi}(\widehat{f}(p^v))}{p^{vs}}\right)\left(1 - \frac{c_{\widehat{\psi}}\ \mathbb{1}_{pF(p) \in U_Q}}{p^s}\right)\right).$$

Expand the logarithm of the first product into Taylor series and split the sum $\sum_{p:\ pF(p) \in U_Q} p^{-s}$ as $\sum_{b \in U_Q:\ F(b) \in U_Q} \sum_{p \equiv b \pmod Q} p^{-s}$. By orthogonality and $\log L(s,\eta) = \sum_{p,v \geq 1} \eta(p^v)/vp^{vs}$, we have

$$\sum_{p \equiv b \pmod Q} \frac{1}{p^s} = \frac{1}{\varphi(Q)} \sum_{\eta \bmod Q} \overline{\eta}(b) \log L(s,\eta) - \sum_{\substack{p,v \geq 2 \\ p^v \equiv b \pmod Q}} \frac{1}{vp^{vs}}.$$

Hence, the logarithm of the first product in (5.4) equals

$$\alpha(Q)c_{\widehat{\psi}} \sum_{\eta \bmod Q} \gamma(\eta) \log L(s,\eta) + \left(\sum_{\substack{p,v \geq 2 \\ pF(p) \in U_Q}} \frac{(c_{\widehat{\psi}})^v}{vp^{vs}} - \sum_{\substack{p,v \geq 2 \\ pF(p^v) \in U_Q}} \frac{c_{\widehat{\psi}}}{vp^{vs}}\right). \tag{5.5}$$

The product formula for $\mathcal{F}(s)$ now follows immediately from (5.4).

Next, note that for $\sigma > 2/3$, we have $\sum_{p,v \geq 2} |vp^{vs}|^{-1} \leq \sum_{p,v \geq 2} p^{-2v/3} \ll 1$, which also shows that the expression in parentheses in (5.5) defines an analytic function on the region $\sigma > 2/3$. Moreover, the total contribution of all primes $p \nmid Q$ to the second infinite product on the right of (5.4) is equal to $\prod_{p \nmid Q}\left(1 + \sum_{v \geq 2} p^{-vs} \left\{\mathbb{1}_{(f(p^v),Q)=1} \cdot \widehat{\psi}(\widehat{f}(p^v)) - c_{\widehat{\psi}} \cdot \mathbb{1}_{(F(p)f(p^{v-1}),Q)=1} \cdot \widehat{\psi}(\widehat{f}(p^{v-1}))\right\}\right)$, which defines an analytic function of size $O(1)$ for $\sigma > 2/3$. Finally for $\sigma > 1 - 1/\log Q$, the total contribution of all primes $p \mid Q$ to the second infinite product on the right of (5.4) has absolute value at most $\exp(\sum_{p|Q}\sum_{v \geq 1} p^{-v\sigma}) \ll \exp(\sum_{p|Q} p^{-1}\exp(\log p/\log Q)) \ll \exp(3\sum_{p \leq \omega(Q)} p^{-1}) \ll (\log_2 Q)^3$. Collecting all the observations in this paragraph proves the assertions in (ii). $\square$

5.2. **The contour shift.** We assume that $\boldsymbol{\beta_e}$ **exists and that** $\boldsymbol{(\alpha(Q), c_{\widehat{\psi}}) \neq (1,1)}$, otherwise much simpler versions of the argument below suffice. (We summarize the modifications for the other possibilities at the end of this section.) We may thus also assume that

$$\beta_e > 1 - \frac{5}{24} \cdot \frac{c_1}{\log Q}, \tag{5.6}$$

otherwise scaling down c_1 by a constant factor, we are in the simpler case when β_e doesn't exist. By scaling down c_1, we may also assume that the **conductor of** η_e (which is squarefree) **is large enough that it is divisible by a prime exceeding** $D + 2 = \sum_{i=1}^{K} \deg F_i + 2$**.**

Define

$$\boldsymbol{T = \exp(\sqrt{\log x}),\ \beta^* = \frac{2}{3} + \frac{\beta_e}{3}} \text{ and } \boldsymbol{\vartheta(t) = 1 - \frac{c_1}{4\mathcal{L}_Q(t)}}.$$

Let $\boldsymbol{\delta_e, \delta > 0}$ be any parameters satisfying $\vartheta(0) < \beta_e - 2\delta_e$ and $\beta_e + 2\delta_e < \beta^* < 1 - 2\delta$. (These constraints are only imposed to ensure that Figure 1 below makes sense. We will eventually let $\delta, \delta_e \to 0+$.) We consider the contour $\boldsymbol{\Gamma_1}$ consisting of the following components. (See Figure 1.)

- $\boldsymbol{\Gamma_2}$, the horizontal segment traversed from $\vartheta(T) + iT$ to $(1 + 1/\log x) + iT$.
- $\boldsymbol{\Gamma_3}$, the part of the curve $\vartheta(t) + it$ traversed upwards from $t = 0$ to $t = T$.
- $\boldsymbol{\Gamma_4}$, the horizontal segment traversed from $\beta_e - \delta_e$ to $\vartheta(0)$ **above** the branch cut.
- $\boldsymbol{\Gamma_5}$, the semicircle in the upper half plane with center β_e, radius δ_e, traversed anticlockwise.
- $\boldsymbol{\Gamma_6}$, the horizontal segment traversed from β^* to $\beta_e + \delta_e$ **above** the branch cut.
- $\boldsymbol{\Gamma_7}$, the horizontal segment traversed from $1 - \delta$ to β^* **above** the branch cut.
- $\boldsymbol{\Gamma_8}$, the circle with center at 1, radius δ, traversed anticlockwise as shown in Figure 1.
- $\boldsymbol{\overline{\Gamma}_j}$ **(for** $\boldsymbol{2 \leq j \leq 7}$**)**, the reflection of Γ_j about the real line, directed as in Figure 1.

By Cauchy's integral formula and the final assertion of Lemma 5.1, we may thus rewrite (5.3) as

$$\sum_{n \leq x} \mathbb{1}_{(f(n),\, q)=1}\ \widehat{\psi}(\widehat{f}(n)) \ = \ \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\mathcal{F}(s) x^s}{s}\, \mathrm{d}s \ + \ O\left(\frac{x \log x}{T}\right). \tag{5.7}$$

We define the function

$$\boldsymbol{H_{\widehat{\psi}}(s) \coloneqq F_{\widehat{\psi}}(s)\ (s-1)^{\alpha(Q) c_{\widehat{\psi}}}\ (s - \beta_e)^{-\alpha(Q)\gamma(\eta_e) c_{\widehat{\psi}}}}, \tag{5.8}$$

which analytically continues into the region $\sigma > 1 - c_1/\mathcal{L}_Q(t)$.

We now proceed to show that the contributions of all parts of Γ_1, except for Γ_7 and $\overline{\Gamma}_7$, are negligible. The following bounds will be useful for this.

Proposition 5.2.
(1) *We have* $|\mathcal{F}(s)| \ll \log x$ *uniformly in* $s \in \Gamma_2 \cup \Gamma_3 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$.
(2) *For any fixed* $\epsilon > 0$*, we have* $|H_{\widehat{\psi}}(s)| \ll (\log x)^{\epsilon\alpha(Q)/5}$ *uniformly in real* $s \in [1 - c_1/4\log Q, 1]$.

Proof. We start with the following useful observation:

(5.9) We have $H_{\widehat{\psi}}(s) \asymp H_{\widehat{\psi}}(w)$, uniformly in $s, w \in \mathbb{C}$ having $\mathrm{Im}(s) = \mathrm{Im}(w) =: t$ and $1 - c_1/2\mathcal{L}_Q(t) \leq \mathrm{Re}(s) \leq \mathrm{Re}(w) \leq 1 + 100/\mathcal{L}_Q(t)$.

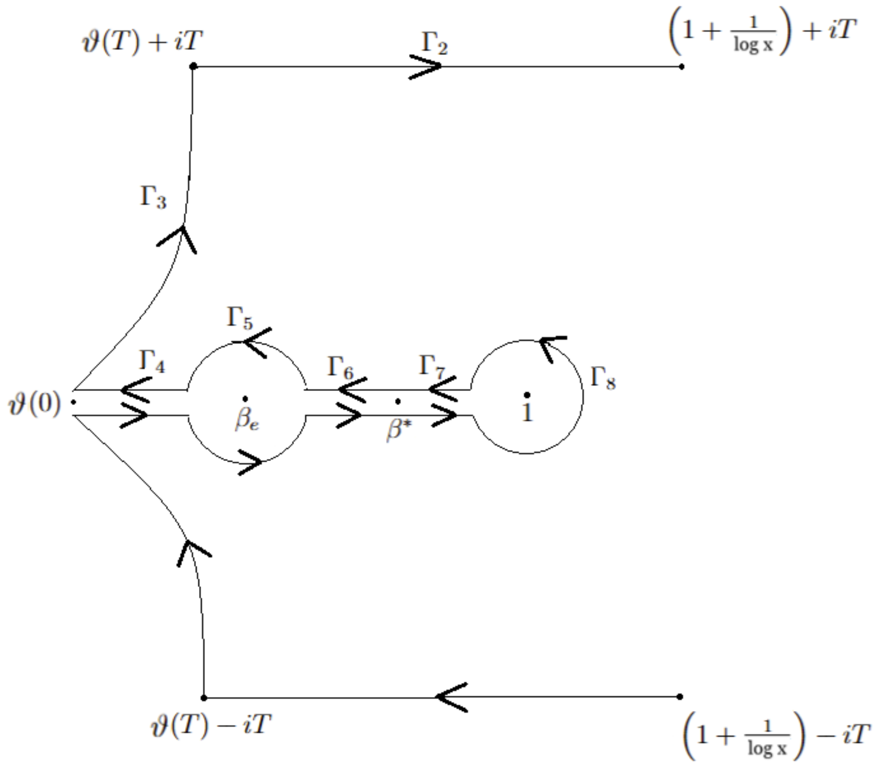

FIGURE 1. The Contour Γ_1 in the case when the Siegel–zero exists and $(\alpha(Q), c_{\widehat{\psi}}) \neq (1,1)$

Indeed by Lemmas 15(i) and 9(ii) in [33],[7] we have $|H'_{\widehat{\psi}}(z)/H_{\widehat{\psi}}(z)| = |F'_{\widehat{\psi}}(z)/F_{\widehat{\psi}}(z)+\alpha(Q)c_{\widehat{\psi}}/(z-1) -\alpha(Q)\gamma(\eta_e)c_{\widehat{\psi}}/(z-\beta_e)| \ll \mathcal{L}_Q(\mathrm{Im}(z))$ uniformly in $z \in \mathbb{C}$ having $\mathrm{Re}(z) > 1 - c_1/2\mathcal{L}_Q(\mathrm{Im}(z))$. (To be in the setting of the two lemmas, we take "ξ" there to be $\exp(6\mathcal{L}_Q(t))$, and use the facts that $L(z,\eta) = L(z,\eta^*)\prod_{\ell|Q}(1-\eta^*(\ell)/\ell^z)$ and $\gamma(\eta) = \gamma(\eta^*)$ when $\eta \bmod Q$ is induced by the primitive character η^*.) Now (5.9) follows from $\log|H_{\widehat{\psi}}(w)/H_{\widehat{\psi}}(s)| \le \int_{\mathrm{Re}(s)}^{\mathrm{Re}(w)} |H'_{\widehat{\psi}}(u+it)/H_{\widehat{\psi}}(u+it)|\,\mathrm{d}u$.

Proof of (1). Let $\boldsymbol{\mu(t) := 1 + c_1/7\mathcal{L}_Q(t)}$. Consider any $s \in \Gamma_2 \cup \Gamma_3 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$. By Lemma 5.1(ii), $\mathcal{F}(s) \ll (\log_2 Q)^3 |F_{\widehat{\psi}}(s)|$. By (5.9), $|H_{\widehat{\psi}}(s)| \ll |H_{\widehat{\psi}}(\mu(t)+it)|$. Using these with (5.8),

$$\mathcal{F}(s) \ll (\log_2 Q)^3 \cdot |H_{\widehat{\psi}}(\mu(t)+it)| \cdot |s-1|^{-\alpha(Q)\mathrm{Re}(c_{\widehat{\psi}})}\ |s-\beta_e|^{\alpha(Q)\mathrm{Re}(\gamma(\eta_e)c_{\widehat{\psi}})}. \tag{5.10}$$

Since $\mathrm{Re}(\mu(t)+it) > 1$, we can use the Dirichlet series of $\log L(\mu(t)+it, \eta)$ and interchange sums to write $|F_{\widehat{\psi}}(\mu(t)+it)| \le \exp(\sum_{p, r\ge 1}\ p^{-r\mu(t)} |\sum_{\eta \bmod Q}\ \alpha(Q)\gamma(\eta)\eta(p^r)|)$. By definition of $\gamma(\eta)$,

$$\sum_{\eta \bmod Q} \alpha(Q)\gamma(\eta)\eta(p^r) \ = \ \frac{1}{\varphi(Q)} \sum_{\substack{u \in U_Q \\ F(u) \in U_Q}} \sum_{\eta \bmod Q} \overline{\eta}(u)\eta(p^r) \ = \ \sum_{\substack{u \in U_Q \\ F(u) \in U_Q}} \mathbb{1}_{p^r \equiv u \bmod Q} \ = \ \mathbb{1}_{p^r F(p^r) \in U_Q}.$$

[7]These two lemmas form an extremely crucial input in our argument, and it is to apply them that we require the full strength of our hypothesis that $\widehat{\psi}(\widehat{F}(u))$ is constant on its unit support.

Hence $|F_{\widehat{\psi}}(\mu(t)+it)| \le \exp(\sum_{p,r\ge 1} p^{-r\mu(t)}) \ll \exp(\sum_p p^{-\mu(t)}) \ll \zeta(\mu(t)) = 1/(\mu(t)-1)+O(1) \ll \mathcal{L}_Q(t)$. By (5.8), this gives the following bound, which holds uniformly in all $t \in \mathbb{R}$.

$$|H_{\widehat{\psi}}(\mu(t)+it)| \ll \mathcal{L}_Q(t)\ |\mu(t)+it-1|^{\alpha(Q)\mathrm{Re}(c_{\widehat{\psi}})}\ |\mu(t)+it-\beta_e|^{-\alpha(Q)\mathrm{Re}(\gamma(\eta_e)c_{\widehat{\psi}})}. \tag{5.11}$$

Finally, we observe that for both $\theta \in \{1, \beta_e\}$, we have $|s-\theta| \ge c_1/48\mathcal{L}_Q(t)$ for all $s = \sigma+it$ lying on $\Gamma_2 \cup \Gamma_3 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$. (This is immediate for $|t| > 1/100$, and also if $\theta = 1$ and $|t| \le 1/100$. If $\theta = \beta_e$ and $|t| \le 1/100$, this follows from (5.6).) This observation easily yields $|s-\theta| \asymp |\mu(t)+it-\theta|$. Inserting this estimate and (5.11) into (5.10), we get $\mathcal{F}(s) \ll (\log_2 Q)^3 \mathcal{L}_Q(t) \ll (\log_3 x)^3 \log(QT) \ll \log x$.

Proof of (2). For any real $s \in [1 - c_1/4\log Q, 1]$, (5.9) and (5.11) yield $|H_{\widehat{\psi}}(s)| \ll |H_{\widehat{\psi}}(\mu(0))| \ll (\log Q)^3 (1-\beta_e)^{-\alpha(Q)}$. To complete the proof, note that $1 - \beta_e \gg_\epsilon Q^{-\epsilon/20K_0} \gg (\log x)^{-\epsilon/20}$ for any fixed $\epsilon > 0$ (by Siegel's Theorem), and that $\alpha(Q) \gg (\log_3 x)^{-D}$ by (4.1). ☐

Proposition 5.3. *Set* $I_j := \int_{\Gamma_j} \mathcal{F}(s) x^s/s\, \mathrm{d}s$ *and* $\overline{I}_j := \int_{\overline{\Gamma}_j} \mathcal{F}(s) x^s/s\, \mathrm{d}s$.
(1) $|I_2| + |I_3| + |\overline{I}_2| + |\overline{I}_3| \ \ll\ x \exp(-(\log x)^{1/3})$.
(2) $|I_4 + \overline{I}_4| + |I_6 + \overline{I}_6| \ \ll\ x \exp(-(\log x)^{99/100})$.
(3) $\lim_{\delta_e \to 0+} I_5 = \lim_{\delta_e \to 0+} \overline{I}_5 = 0$. *Moreover,* $\lim_{\delta \to 0+} I_8 = 0$.

Proof. **(1)** By Proposition 5.2, we have $|I_2| \ll T^{-1}(\log x) \int_{\vartheta(T)}^{1+1/\log x} x^u\, \mathrm{d}u \ll x/T$. For $s \in \Gamma_3$, note that $|s| \gg t+1$, which gives $|I_3| \ll x^{\vartheta(T)}(\log x) \int_0^T \mathrm{d}t/(t+1) \ll x(\log x)^{3/2} \exp(-c_1 \log x/4\mathcal{L}_Q(T)) \ll x\exp(-(\log x)^{1/3})$. The same arguments go through for $\overline{I}_2$ and $\overline{I}_3$.

(2) We only show the bound for $I_4 + \overline{I_4}$; the argument for $I_6 + \overline{I_6}$ is analogous. Note that for $s \in \Gamma_4$, we have $(s-1)^{-\alpha(Q)c_{\widehat{\psi}}} \cdot (s-\beta_e)^{\alpha(Q)\gamma(\eta_e)c_{\widehat{\psi}}} = |1-s|^{-\alpha(Q)c_{\widehat{\psi}}} \cdot |\beta_e - s|^{\alpha(Q)\gamma(\eta_e)c_{\widehat{\psi}}} \cdot e^{-i\pi\alpha(Q)c_{\widehat{\psi}} + i\pi\alpha(Q)\gamma(\eta_e)c_{\widehat{\psi}}}$, while for $s \in \overline{\Gamma}_4$, the same equality holds but with the opposite sign of the power of "e". As such, by (5.8), Lemma 5.1 and Proposition 5.2(2), we obtain

$$|I_4 + \overline{I}_4| \ll \int_{\vartheta(0)}^{\beta_e - \delta_e} \frac{|H_{\widehat{\psi}}(u) G_{\widehat{\psi}}(u)| x^u}{u}\ (1-u)^{-\alpha(Q)\mathrm{Re}(c_{\widehat{\psi}})} \cdot (\beta_e - u)^{\alpha(Q)\mathrm{Re}(\gamma(\eta_e)c_{\widehat{\psi}})}\, \mathrm{d}u$$

$$\ll\ x^{\beta_e}\ (\log x)^{\alpha(Q)/200} \cdot (\log_2 Q)^3 \cdot (1-\beta_e)^{-\alpha(Q)} \int_{\vartheta(0)}^{\beta_e - \delta_e} (\beta_e - u)^{\alpha(Q)\mathrm{Re}(\gamma(\eta_e)c_{\widehat{\psi}})}\, \mathrm{d}u. \tag{5.12}$$

Now by our hypothesis in the paragraph following (5.6), the conductor $\mathfrak{f}(\eta_e)$ has a prime factor $\ell_0 > D+2$. As such, factoring $\eta_e = \prod_{\ell \mid Q} \eta_{e,\ell}$ with $\eta_{e,\ell}$ a character mod ℓ, we observe that

$$|\alpha(Q)\gamma(\eta_e)| \ =\ \prod_{\ell \mid Q} \frac{1}{\varphi(\ell)} \left| \sum_{\substack{u \in U_\ell \\ F(u) \in U_\ell}} \eta_{e,\ell}(u) \right| \ \le\ \frac{1}{\ell_0 - 1} \left| - \sum_{\substack{u \in U_{\ell_0} \\ F(u) \equiv 0 \pmod{\ell_0}}} \eta_{e,\ell_0}(u) \right| \ \le\ \frac{D}{D+1}. \tag{5.13}$$

Hence the integral in (5.12) is at most $(\beta_e - \vartheta(0))^{1+\alpha(Q)\mathrm{Re}(\gamma(\eta_e)c_{\widehat{\psi}})}/(1+\alpha(Q)\mathrm{Re}(\gamma(\eta_e)c_{\widehat{\psi}})) \ll 1$. The bound on $|I_4 + \overline{I}_4|$ now follows from $1 - \beta_e \gg (\log x)^{-1/200}$ (Siegel's Theorem) and (4.1).

(3) Let $M_{\widehat{\psi}}$ be the maximum of $|H_{\widehat{\psi}}(s)|$ on a fixed small closed disk centered at β_e that is contained in the region $\sigma > 1 - c_1/4\mathcal{L}_Q(t)$. Note that $M_{\widehat{\psi}}$ is finite as $H_{\widehat{\psi}}(s)$ is holomorphic on the region $\sigma > 1 - c_1/\mathcal{L}_Q(t)$. Parametrize the points on the semicircle Γ_5 as $s = \beta_e + \delta_e \cdot e^{i\theta}$ for $-\pi \le \theta \le \pi$. Invoking Lemma 5.1 and (5.8), and arguing as above, we find that

$$|I_5| \ll M_{\widehat{\psi}}\ (\log_2 Q)^3\ x^{\beta_e + \delta_e} \cdot (1 - \beta_e - \delta_e)^{-\alpha(Q)} \cdot \delta_e^{1+\alpha(Q)\mathrm{Re}(\gamma(\eta_e)c_{\widehat{\psi}})}$$

By (5.13), the power of δ_e above is at least $1/(D+1)$, hence the above display shows that $\lim_{\delta_e \to 0+} I_5 = 0$. The arguments for $\overline{I}_5$ and I_8 are analogous; for I_8, we only need the fact that $\lim_{\delta \to 0+} \delta^{1-\alpha(Q)\mathrm{Re}(c_{\widehat{\psi}})} = 0$ since $(\alpha(Q), c_{\widehat{\psi}}) \neq (1,1)$ as assumed in the paragraph before (5.6). $\square$

Letting $\delta, \delta_e \to 0+$ and invoking Proposition 5.3, we thus obtain from (5.7),

$$\sum_{n \le x} \mathbb{1}_{(f(n),\, q)=1}\ \widehat{\psi}(\widehat{f}(n)) \ = \ \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\mathcal{F}(s) x^s}{s}\, \mathrm{d}s \ + \ O\left(x \exp(-(\log x)^{1/3})\right), \tag{5.14}$$

with Γ_0 being the contour consisting of the two horizontal lines joining the points 1 and β^* above and below the branch cut, directed like Γ_7. (It is easily seen that the integral in (5.14) converges.)

5.3. **Bounding the remaining integral in** (5.14)**.** We start as in the proof of Proposition 5.3(2), by using Lemma 5.1 and (5.8), and then rewriting as a real integral. We get, for any fixed $\epsilon > 0$,

$$\begin{aligned} \left| \int_{\Gamma_0} \frac{\mathcal{F}(s) x^s}{s}\, \mathrm{d}s \right| &\ll \int_{\beta^*}^{1} \frac{|H_{\widehat{\psi}}(u) G_{\widehat{\psi}}(u)| x^u}{u} (1-u)^{-\alpha(Q)\mathrm{Re}(c_{\widehat{\psi}})} \cdot (u-\beta_e)^{\alpha(Q)\mathrm{Re}(\gamma(\eta_e) c_{\widehat{\psi}})}\, \mathrm{d}u \\ &\ll \ (\log x)^{\epsilon\alpha(Q)/4} \int_{\beta^*}^{1} |G_{\widehat{\psi}}(u)| \cdot x^u (1-u)^{-\alpha(Q)\mathrm{Re}(c_{\widehat{\psi}})}\, \mathrm{d}u; \end{aligned} \tag{5.15}$$

the last line uses Proposition 5.2(2) and $(u-\beta_e)^{\alpha(Q)\mathrm{Re}(\gamma(\eta_e) c_{\widehat{\psi}})} \ll (1-\beta_e)^{-\alpha(Q)} \ll_\epsilon (\log x)^{\epsilon\alpha(Q)/20}$.

Set $\boldsymbol{\epsilon_0 := \epsilon_0(C_0, \kappa) := \max\{\cos(2\pi/r) : 1 < r \le C_0^{C_0\kappa}\}} \in [-1,1)$ and note that since $1 < Q_0 \le C_0^{C_0\kappa}$ and $c_{\widehat{\psi}}$ is a $\varphi(Q_0)$-th root of unity, we must either have $\mathrm{Re}(c_{\widehat{\psi}}) \le \epsilon_0$ or $c_{\widehat{\psi}} = 1$. In the former case, using Lemma 5.1(ii) and writing $u =: 1 - v/\log x$ yields, from (5.15),

$$\left| \int_{\Gamma_0} \frac{\mathcal{F}(s) x^s}{s}\, \mathrm{d}s \right| \ \ll \ \frac{x (\log_3 x)^3}{(\log x)^{1-\alpha(Q)(\epsilon_0 + \epsilon/4)}} \cdot \Gamma(1 - \alpha(Q)\mathrm{Re}(c_{\widehat{\psi}})) \ll \frac{x}{(\log x)^{1-\alpha(Q)(\epsilon_0+\epsilon/3)}},$$

where Γ is the Gamma-function, and we have noted that Γ is of size $O(1)$ on the interval $[1-\epsilon_0, 1]$. Inserting the above bound into (5.14) proves (4.14) when $c_{\widehat{\psi}} \neq 1$ (by taking, say, $\delta_0 \le 1 - \epsilon_0$.)

Now assume that $c_{\widehat{\psi}} = 1$, so that $\widehat{\psi}(\widehat{F}(u)) = 1$ for all $u \in U_{Q_0}$ satisfying $F(u) \in U_{Q_0}$. Since $q \in \mathcal{Q}(f_1, \dots, f_K)$, applying the definition of $\mathcal{Q}(f_1, \dots, f_K)$ to the characters mod q induced by $\widehat{\psi} = (\psi_1, \dots, \psi_K)$, we find that $\mathbb{1}_{(f(2^j),q)=1}\ \widehat{\psi}(\widehat{f}(2^j)) = -1$ for all $j \ge 1$. Isolating the contribution of $p = 2$ from the second infinite product in (5.4), we can thus write $G_{\widehat{\psi}}(s) = G_{\widehat{\psi},1}(s) G_{\widehat{\psi},2}(s)$, where $G_{\widehat{\psi},2}(s) := (1 + \sum_{j\ge1} \mathbb{1}_{(f(2^j),q)=1}\ \widehat{\psi}(\widehat{f}(2^j))/2^{js})\ (1 - c_{\widehat{\psi}}\ \mathbb{1}_{(2F(2),Q)=1}/2^s)$ is analytic on $\sigma > 0$ and satisfies $G_{\widehat{\psi},2}(1) = 0$, and where the function $G_{\widehat{\psi},1}(s)$ is also analytic on the region $\sigma > 1 - 1/\log Q$ and satisfies $|G_{\widehat{\psi},1}(s)| \ll (\log_2 Q)^3$ uniformly therein. (The assertions on $G_{\widehat{\psi},1}$ can be shown by following the arguments in Lemma 5.1.) Hence for all $u \in [\beta^*, 1]$, we have

$$|G_{\widehat{\psi}}(u)| \ \ll \ (\log_2 Q)^3 \cdot |G_{\widehat{\psi},2}(1) - G_{\widehat{\psi},2}(u)| \ \ll \ (\log_3 x)^3 \int_u^1 |G'_{\widehat{\psi},2}(w)|\, \mathrm{d}w \ \ll \ (\log_3 x)^3 (1-u). \tag{5.16}$$

Inserting this bound into (5.15) and making the substitution $u =: 1 - v/\log x$, we obtain

$$\left| \int_{\Gamma_0} \frac{\mathcal{F}(s) x^s}{s}\, \mathrm{d}s \right| \ \ll \ \frac{x (\log_3 x)^3}{(\log x)^{2-\alpha(Q)(1+\epsilon/4)}} \cdot \Gamma(2 - \alpha(Q)) \ll \frac{x}{(\log x)^{1-\alpha(Q)\epsilon/3}}, \tag{5.17}$$

establishing (4.14) in the remaining case $c_{\widehat{\psi}} = 1$.

To conclude, we summarize the changes if either β_e doesn't exist or if $(\alpha(Q), c_{\widehat{\psi}}) = (1,1)$. If β_e doesn't exist and $(\alpha(Q), c_{\widehat{\psi}}) \neq (1,1)$, then we replace the semicircles Γ_5 and $\overline{\Gamma}_5$ by straight segments above and below the branch cut (there is no δ_e). If $(\alpha(Q), c_{\widehat{\psi}}) = (1,1)$ and β_e exists, then (as mentioned in the paragraph preceding Lemma 5.1), we only make a branch cut up to $\sigma \leq \beta_e$, so we can replace all of Γ_j and $\overline{\Gamma}_j$ (for $j \geq 5$) by a circle of radius δ_e centered at β_e, going from below to above the branch cut; we eventually let $\delta_e \to 0+$. If $(\alpha(Q), c_{\widehat{\psi}}) = (1,1)$ and β_e doesn't exist either, then there is no branch cut, and our contour Γ_1 just consists of Γ_2, Γ_3, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$. By the discussion preceding (4.14), this completes the proofs of Theorems 1.2 and 1.3. $\square$

6. Optimalities in Theorems 1.2 and 1.3

Lemma 6.1. *Let $f_1, \ldots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ be multiplicative functions and $\{F_i\}_i \subset \mathbb{Z}[T]$ be nonconstant such that $\prod_{i=1}^K F_i$ is squarefree, and $f_i(p) = F_i(p)$ for all i and all primes p. There exists a constant $C_{\widehat{F}} > 0$ depending only on $\{F_i\}_i$ such that any $q \in \mathbb{Z}^+$ with $P^-(q) > C_{\widehat{F}}$ lies in $\mathcal{Q}(f_1, \ldots, f_K)$.*

Proof. If $P^-(q) > D+1$, then $\alpha(q) = \prod_{\ell | q}(1 - (\ell-1)^{-1}\#\{u \in U_\ell : \prod_{i=1}^K F_i(u) \equiv 0 \pmod{\ell}\}) \geq \prod_{\ell | q}(1 - D/(\ell-1)) > 0$. Moreover, if $\mathcal{T}(q) := \{(F_1(u), \ldots, F_K(u)) : u \prod_{i=1}^K F_i(u) \in U_q\}$ generates U_q^K, then $q \in \mathcal{Q}(f_1, \ldots, f_K)$ vacuously, i.e. there is no tuple of characters $\widehat{\chi} \neq (\chi_0, \ldots, \chi_0) \bmod q$ for which $\prod_{i=1}^K \chi_i(F_i(u)) = 1$ on $\{u \in U_q : \prod_{i=1}^K F_i(u) \in U_q\}$. Hence, it suffices to show that there exists a constant $C_{\widehat{F}}$ such that $\mathcal{T}(q)$ generates U_q^K for all q with $P^-(q) > C_{\widehat{F}}$.

Now under the isomorphism $U_q^K \cong \prod_{\ell^e \| q} U_{\ell^e}^K$, the set $\mathcal{T}(q)$ maps to $\prod_{\ell^e \| q} \mathcal{T}(\ell^e)$. Thus if $\mathcal{T}(q)$ doesn't generate U_q^K, then by [25, Lemma 5.13], there is some $\ell^e \| q$ and characters $(\psi_1, \ldots, \psi_K)$ mod ℓ^e, not all trivial, for which $\prod_{i=1}^K \psi_i(F_i(u))$ is constant on $\{u \in U_{\ell^e} : \prod_{i=1}^K F_i(u) \in U_{\ell^e}\}$. The lemma now follows from [23, Lemma 5]. $\square$

We show that the ranges of q in Theorem 1.2 are essentially optimal, in that "$1-\epsilon$" cannot be replaced by "$1+\epsilon$" in either subpart. In all our examples below, $\{F_i\}_{i=1}^K$ will be nonconstant and $\prod_{i=1}^K F_i$ will be separable, – guaranteeing that $\{F_i\}_{i=1}^K$ are multiplicatively independent, and that any $q \in \mathbb{Z}^+$ satisfies $IFH(F_1, \ldots, F_K; B_0)$ for any $B_0 > 0$. Fix $r \in \mathbb{Z}^+$. Our K_0 will be fixed large enough in terms of $F_1, \ldots, F_K$, and we will always have $q \leq (\log x)^{K_0}$.

Optimality in Theorem 1.2(i). Let $F_i(T) := (T-1)^r + i$, and q be a perfect r-th power with $P^-(q) > \max\{C_{\widehat{F}}, 2K\}$ where $C_{\widehat{F}}$ is as in Lemma 6.1. Then $q \in \mathcal{Q}(f_1, \ldots, f_K)$ and any prime $P \equiv 1 \pmod{q^{1/r}}$ satisfies $f_i(P) = F_i(P) \equiv i \pmod{q}$. By Siegel–Walfisz, $\#\{n \leq x : (\forall i)\ f_i(n) \equiv i \pmod{q}\} \gg x/(q^{1/r}\log x)$ uniformly in $q \leq (\log x)^{K_0}$. The last expression grows faster than the expected main term $\varphi(q)^{-K}\#\{n \leq x : (f(n), q) = 1\}$, once $q > (\log x)^{(1+\epsilon)\alpha(q)(K-1/r)^{-1}} = (\log x)^{(1+\epsilon)\alpha(q)(K-1/D_{\min})^{-1}}$ for any fixed ϵ. (To see this, use (4.1) and the first estimate in (4.13).)

Stronger optimality in Theorem 1.2(i) for $\boldsymbol{K > 1, D_{\min} = 1}$ (i.e., at least one of the $\boldsymbol{F_i}$ is linear). We show that in this case, the range of q in Theorem 1.2(i) is optimal in a much stronger sense: It is optimal even if **all** the F_i are **any** pairwise coprime linear polynomials. Indeed, if $F_i(T) = c_iT + b_i$, then fixing any $b \in \mathbb{Z} \setminus \{0\}$ with $\prod_{i=1}^K F_i(b) \neq 0$, and taking any q with $P^-(q) > \max\{C_{\widehat{F}}, 1 + |b\prod_{i=1}^K F_i(b)|\}$, we note that any prime $P \equiv b \pmod{q}$ satisfies $f_i(P) \equiv F_i(b) \pmod{q}$. By Siegel–Walfisz, $\#\{n \leq x : (\forall i)\ f_i(n) \equiv F_i(b) \pmod{q}\} \gg x/\varphi(q)\log x$. This expression grows faster than the expected main term as soon as $q > (\log x)^{(1+\epsilon)\alpha(Q)/(K-1)}$.

Optimality in Theorem 1.2(ii). Let $F_i(T) := \prod_{j=1}^r (T-2j) + 2(2i-1)$. Each F_i is irreducible by Eisenstein's criterion at 2, hence $\prod_{i=1}^K F_i$ is separable. Now let $q \le (\log x)^{K_0}$ be a **squarefree** integer having $P^-(q) > \max\{C_{\widehat{F}}, 4Kr\}$. By the Chinese Remainder Theorem, the congruence $\prod_{j=1}^r (v-2j) \equiv 0 \pmod q$ has exactly $r^{\omega(q)}$ distinct solutions $v \in U_q$. Hence by Siegel–Walfisz,

$$\sum_{\substack{n \le x \\ (\forall i)\ f_i(n) \equiv 2(2i-1) \pmod q}} 1 \ \ge \sum_{\substack{n \le x:\ P(n) > q \\ (\forall i)\ f_i(n) \equiv 2(2i-1) \pmod q}} 1 \ \ge \sum_{\substack{q < P \le x \\ \prod_{j=1}^r (P-2j) \equiv 0 \pmod q}} 1 \ \gg \frac{r^{\omega(q)} x}{\varphi(q) \log x}. \tag{6.1}$$

By (4.13) and (4.1), the rightmost expression grows faster than $\varphi(q)^{-K} \#\{n \le x : (f(n), q) = 1\}$ as soon as $q^{K-1} D_{\min}^{\omega(q)} = q^{K-1} r^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha(q)}$ for any fixed $\epsilon > 0$.

For completeness, we construct arbitrarily large squarefree q satisfying the last inequality; in fact, we ensure that $r^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha(q)}$. Indeed, let $q := \prod_{\max\{C_{\widehat{F}}, 4Kr\} < \ell \le Y} \ell$, with $Y \le (K_0/2) \log_2 x$ a parameter to be chosen later. Then $q \le (\log x)^{K_0}$, $\omega(q) \ge Y/2 \log Y$, and by the Chinese Remainder Theorem and the Prime Ideal Theorem, we have $\alpha(Q) \le c/\log Y$ for some constant $c > 0$ depending only on $F_1, \dots, F_K$. The inequality $r^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha(q)}$ is then ensured once we choose any $Y \in (4c \log_2 x / \log r, (K_0/2) \log_2 x)$. (We can fix $K_0 > 16c$ at the start.)

Optimality in Theorem 1.3. Note that by the $K = 1$ case of (6.1), the third value of R in Theorem 1.3 is optimal, in that it cannot be reduced to 1. We now show that the second value of R is "nearly optimal", in that it cannot be reduced to $2K-1$. With $F_i(T) = \prod_{j=1}^r (T-2j) + 2(2i-1)$ as above, we consider multiplicative functions $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ satisfying $f_i(p) = F_i(p)$ and $f_i(p^2) = 1$ for all i and all primes p. Consider any squarefree q as in the previous paragraph, and let $n = (p_1 \cdots p_{K-1})^2 P$ where p_i, P are primes satisfying $q < p_{K-1} < \cdots < p_1 < x^{1/4K} < x^{1/3} < P$ and $\prod_{j=1}^r (P-2j) \equiv 0 \pmod q$. Then $f_i(n) \equiv 2(2i-1) \pmod q$, so that by Siegel–Walfisz,

$$\sum_{\substack{n \le x:\ P_{2K-1}(n) > q \\ (\forall i)\ f_i(n) \equiv 2(2i-1) \pmod q}} 1 \ \gg \frac{r^{\omega(q)} x}{\varphi(q) \log x} \sum_{\substack{p_1, \dots, p_{K-1} \in (q, x^{1/4K}) \\ p_1, \dots, p_{K-1} \text{ distinct}}} \frac{1}{(p_1 \cdots p_{K-1})^2}, \tag{6.2}$$

where we have replaced the ordering condition on $p_1, \dots, p_{K-1}$ by a distinctness condition at the cost of $(K-1)!$. Note that $\sum_{p_1, \dots, p_{K-1} \in (q, x^{1/4K})} (p_1 \cdots p_{K-1})^{-2} \gg (\sum_{q < p \le x^{1/4K}} p^{-2})^{K-1} \gg (q \log q)^{-(K-1)}$. On the other hand, the sum of $1/p_1 \cdots p_{K-1}$ over all primes $p_1, \dots, p_{K-1} > q$ when $p_i = p_j$ for some $i \ne j$, is $\ll (\sum_{p > q} p^{-4}) \cdot (\sum_{p > q} p^{-2})^{K-3} \ll q^{-K}$. Combining, we get

$$\sum_{\substack{n \le x:\ P_{2K-1}(n) > q \\ (\forall i)\ f_i(n) \equiv 2(2i-1) \pmod q}} 1 \ \gg \frac{r^{\omega(q)}}{\varphi(q)(q \log q)^{K-1}} \cdot \frac{x}{\log x} \ \gg \frac{r^{\omega(q)}}{\varphi(q)^K (\log_2 x)^K} \cdot \frac{x}{\log x}, \tag{6.3}$$

where we used $q \ll \varphi(q) \log_2 q$. The last expression above grows faster than $\varphi(q)^{-K} \#\{n \le x : (f(n), q) = 1\}$ once $r^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha(q)}$ for any fixed $\epsilon > 0$, which is already satisfied by our q.

7. The general case

The hypothesis "$\alpha(q) \ne 0$" played a key role in Theorem 1.1 and its uniform analogues Theorems 1.2 and 1.3: Not only did it prevent the input sets from becoming too sparse (via (4.13)), but also guaranteed that the "polynomial–type" control on the f_i at the *primes* was most significant in a certain sense. It turns out that if $\alpha(q) = 0$, then the behavior of the f_i at a *higher prime power* becomes more significant. Narkiewicz's general criterion in [23] allows for this flexibility, and we obtain best possible uniform analogues of his general criterion as well.

7.1. **Extending Theorems 1.4 and 1.6 for certain sparse input sets.** In this entire subsection, we fix $\boldsymbol{\nu} \in \mathbb{Z}^+$ and $\boldsymbol{C > 0}$. Given functions $h_1, \dots, h_K : \mathbb{Z}^+ \to \mathbb{Z}$ and $q \in \mathbb{Z}^+$, we say that $(h_1, \dots, h_K)$ are **ν-supported (with respect to q)** if the ν-free part of any n satisfying $(h_1(n) \cdots h_K(n), q) = 1$ is at most C. Equivalently, any such n can be written as $n_1 n_2$, where $n_1 \le C$ and n_2 is ν-free. In what follows, we will just write "$(h_1, \dots, h_K)$ are ν-supported" to mean that they are ν-supported with respect to some q which will always be clear from the context.

Note that if $(h_1, \dots, h_K)$ are ν-supported, then they are also $(\nu-1)$-supported. By convention, any functions $(h_1, \dots, h_K)$ are 1-supported. The general paradigm in this entire section is that the control of the h_i at the ν-th powers of primes is most significant: All hypotheses, results and arguments are modified accordingly. In particular, taking $\nu = 1$ here recovers everything before.

In what follows, let $\boldsymbol{\Psi_\nu(x, z)}$ denote the **number of $\boldsymbol{\nu}$-full $\boldsymbol{z}$-smooth numbers up to $\boldsymbol{x}$**; we give a bound for this in Lemma 7.2. Given any ν-supported integer-valued multiplicative functions $(h_1, \dots, h_K)$, and any $q, r \in \mathbb{Z}^+$ and $z > 0$, we **redefine $\boldsymbol{\xi_r(q; z)}$** $:= \xi_r(q; z; \{h_i\}_{i=1}^K) > 0$ be *any* parameter satisfying, for all $x \ge z$ and all coprime residues $a_1, \dots, a_K$ mod q, the bound

$$\#\{P_1 \cdots P_r \le x : P_i \text{ primes},\ P_1 > z,\ q < P_r < \cdots < P_1,\ (\forall i)\ h_i(\boldsymbol{P_1^\nu}) \cdots h_i(\boldsymbol{P_r^\nu}) \equiv a_i \pmod q\} \\ \le\ \xi_r(q; z) \cdot x (\log\log x)^r \Big/ \log(z/q). \tag{7.1}$$

The following theorem generalizes Theorems 1.4 and 1.6; the differences have been highlighted. We still use the notation preceding the statement of Theorem 1.4.

Theorem 7.1. *Consider any $q \in \mathbb{Z}^+$ with $\boldsymbol{q > C}$. Consider any **$\boldsymbol{\nu}$-supported** multiplicative functions $(h_1, \dots, h_K)$, any $M \ge 1$, $y \ge q$, and a decreasing function $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+$, such that*

$$\sum_{y \le p \le Y} \widehat{\chi}(\widehat{h}(\boldsymbol{p^\nu})) \ =\ \alpha_{\widehat{\chi}} \sum_{y \le p \le Y} \chi_0(h(\boldsymbol{p^\nu})) + O(MY\mathcal{E}(y)) \tag{7.2}$$

for all $\widehat{\chi}$ mod q and all $Y \ge y$, where $\alpha_{\widehat{\chi}} \in \mathbb{C}$ lies in the unit disk.

Then for all d dividing q, all $x, z > 0$, and all $J, R \in \mathbb{Z}^+$ satisfying $y \le z \le x$ and $J \ge R$, we have

$$\sum_{\substack{n \le x:\ P_R(n) > q \\ (\forall i)\ h_i(n) \equiv a_i \ (\mathrm{mod}\ q)}} 1 \ -\ \left(\frac{\varphi(d)}{\varphi(q)}\right)^K \sum_{\substack{n \le x:\ (h(n), q) = 1 \\ (\forall i)\ h_i(n) \equiv a_i \ (\mathrm{mod}\ d)}} 1 \tag{7.3}$$

$$\ll\ \frac{1}{\varphi(q)^K} \sum_{\substack{\widehat{\chi} \bmod q \\ \mathfrak{f}(\widehat{\chi}) \nmid d}} |\alpha_{\widehat{\chi}}|^J \sum_{\substack{n \le x \\ (h(n), q) = 1}} 1 \ +\ \boldsymbol{\Psi_\nu(x, z)} \ +\ J\left(\frac{\boldsymbol{x}}{\boldsymbol{y}}\right)^{\boldsymbol{1/\nu}} \ +\ M\boldsymbol{x^{1/\nu}}\mathcal{E}(y)\log x$$

$$+ \left\{ \left(\frac{\varphi(d)}{\varphi(q)}\right)^K + \boldsymbol{\xi_{\lceil R/\nu \rceil}(q; z)} + R^K \sum_{\substack{\boldsymbol{1 \le r < R/\nu} \\ \boldsymbol{1 \le s < K\nu}}} \frac{\xi_r(q; z)}{q^{\boldsymbol{\max\{\frac{s}{\nu}, \frac{R}{\nu} - r - s\}}}} \right\} \frac{\boldsymbol{x^{1/\nu}}(2\log_2 x)^{R+J}}{\log(z/q)} \exp\left(\sum_{\substack{p \le y \\ \boldsymbol{(h(p^\nu), q) = 1}}} \frac{1}{p} \right).$$

The analogue of (1.4) *also holds, with the relevant replacements on the right hand side, and with the condition "$P_J(n) > y$" replaced by "$\boldsymbol{P_{J\nu}(n) > y}$". The implied constants in both estimates depend only on $\boldsymbol{\nu, K, C}$ and the implied constant in* (7.2).

Proof. The proofs are essentially analogous, so we mention the main changes. First, we define n to be "convenient" if the J largest **distinct** prime factors of n all exceed y and each appear exactly to the ν-th power in n. Thus, any convenient n can be uniquely written as $m(P_J \cdots P_1)^\nu$ where (2.1) holds. Once again, to prove the analogue of (1.4), it suffices to show the corresponding estimate with the condition "$P_{J\nu}(n) > y$" replaced by "n convenient": This is because any inconvenient n having $P_{J\nu}(n) > y$ must be divisible by the $(\nu+1)$-th power of a prime exceeding y, and because (7.4)

$$\sum_{\substack{n \le x:\ (h(n),q)=1 \\ \exists\ p>y \text{ s.t. } p^{\nu+1}\mid n}} 1 \le \sum_{\substack{p>y \\ r\ge\nu+1}} \sum_{\substack{m\le x/p^r \\ (h(m),q)=1}} 1 \le \sum_{\substack{p>y \\ r\ge\nu+1}} \sum_{m_1\le C} \sum_{\substack{m_2\le x/p^r \\ m_2 \text{ is } \nu\text{-full}}} 1 \ll x^{1/\nu} \sum_{\substack{p>y \\ r\ge\nu+1}} \frac{1}{p^{r/\nu}} \ll \left(\frac{x}{y}\right)^{1/\nu},$$

where we've used the Erdős–Szekeres estimate [12] on ν-full numbers. Now (2.2) and (2.3) hold as stated. The analogues of (2.4) and (2.5) hold with all x/m and $\widehat{\chi}(\widehat{h}(P_i))$ replaced by $x^{1/\nu}/m^{1/\nu}$ and $\widehat{\chi}(\widehat{h}(P_i^\nu))$; the O-term in (2.5) is replaced by $O\left(J + Mx^{1/\nu}\mathcal{E}(y)\big/m^{1/\nu}P_1\cdots P_{i-1}P_{i+1}\cdots P_J\right)$. Reversing the splitting of convenient n, we see that this O-term summed over m and all other P_i is $\ll \sum_{n\le x/y^\nu:\ (h(n),q)=1} 1 + J^{-1}Mx^{1/\nu}\sum_{n\le x/y^\nu:\ (h(n),q)=1} n^{-1/\nu}$. The first of the two sums is $\ll \sum_{n_1\le C}\sum_{\substack{n_2\le x/y^\nu \\ n_2 \text{ is } \nu\text{-full}}} 1 \ll x^{1/\nu}/y$. The second of the two sums is at most

$$\sum_{\substack{n\le x \\ (h(n),q)=1}} \frac{1}{n^{1/\nu}} \le \sum_{n_1\le C} \frac{1}{n_1^{1/\nu}} \sum_{\substack{n_2\le x \\ n_2 \text{ is } \nu\text{-full}}} \frac{1}{n_2^{1/\nu}} \ll \prod_{p\le x}\left(1+\sum_{r\ge\nu}\frac{1}{p^{r/\nu}}\right) \ll \exp\left(\sum_{p\le x}\frac{1}{p}\right) \ll \log x. \tag{7.5}$$

We thus find that the analogue of (2.6) holds with all products $P_1\cdots P_J$ replaced by $(P_1\cdots P_J)^\nu$, and with all instances of x in the O-term replaced by $x^{1/\nu}$. This proves the analogue of (1.4).

Next to obtain (7.3), we show the analogue of (2.8), which is

$$\sum_{\substack{n\le x:\ P_{J\nu}(n)\le y \\ (h(n),q)=1}} 1 \ll \frac{x^{1/\nu}(2\log_2 x)^J}{\log z} \exp\left(\sum_{p\le y} \frac{\mathbb{1}_{(h(p^\nu),q)=1}}{p}\right) + \Psi_\nu(x,z) + \left(\frac{x}{y}\right)^{1/\nu}. \tag{7.6}$$

By (7.4), we may assume that the y-rough part of n is $(\nu+1)$-free, so that (by definition of ν-supported and $y \ge q > C$) the y-rough part must be a perfect ν-th power. We may also assume that $P(n) > z$ since $\#\{n \le x : P(n) \le z,\ (h(n),q) = 1\} \le \sum_{n_1\le C}\sum_{\substack{n_2\le C:\ P(n_2)\le z \\ n_2 \text{ is } \nu \text{ full}}} 1 \ll \Psi_\nu(x,z)$. Under these two assumptions, $n = BA^\nu P^\nu$, with $P = P(n) > z$, $P(B) \le y < P^-(A)$, $\Omega(A) \le J$ and $(h(B),q) = 1$. Proceeding as in the proof of (2.8), and noting that $\sum_{B:\ P(B)\le y} \mathbb{1}_{(h(B),q)=1}/B^{1/\nu} \ll \exp(\sum_{p\le y} \mathbb{1}_{(h(p),q)=1}/p)$ by the method used in (7.5), we get (7.6).

Finally, we bound the n satisfying $P_R(n) > q$, $P_{J\nu}(n) \le y$ and $h_i(n) \equiv a_i \pmod q$. Again, it suffices to bound the number $\sum_{n\le x}^* 1$ of all $n \le x$ with a $(\nu+1)$-free y-rough part, satisfying $P_R(n) > q$, $P_{J\nu}(n) \le y$, $P(n) > z$, and $h_i(n) \equiv a_i \pmod q$. Let $E_{r,s}$ be the number of n counted in $\sum_{n\le x}^* 1$ with $\#\{p > q : p^\nu \parallel n\} = r$ and $\#\{p > q : p^{\nu+1} \mid n\} = s$. The analogue of (2.7) holds with "R", "K", "$p \parallel n$", "$p^2 \mid n$" replaced by "R/ν", "$K\nu$", "$p^\nu \parallel n$", "$p^{\nu+1} \mid n$" respectively. (Here we noted that the q-rough part of n is ν-full, as $q > C$ and $h_1, \dots, h_K$ are ν-supported.)

The rest of the proof goes through by adapting the argument for Proposition 1.5: For instance, any n counted in $E_{r,s}$ is of the form $mp_1^{c_1}\cdots p_s^{c_s}\boldsymbol{A^\nu}$, where $P(m) \le q$, $p_j > q$, $c_j \ge \nu+1$, $c_1+\cdots+c_s \ge R-\nu r$, and A is squarefree with $P^-(A) > q$, $P(A) > z$, $\Omega(A) = r$. Given $m, p_1, \dots, p_s$

and $c_1, \dots, c_s$, the number of A is at most $\xi_r(q; z) \cdot x^{1/\nu} (\log_2 x)^r / \big(m^{1/\nu} p_1^{c_1/\nu} \cdots p_s^{c_s/\nu} \log(z/q)\big)$. The sum of $p_1^{-c_1/\nu} \cdots p_s^{-c_s/\nu}$ over all possible $p_j > q$ and over all $c_j \geq \nu+1$ satisfying $c_1+\dots+c_s \geq R-\nu r$, is $\ll R^K q^{-\max\{s/\nu, R/\nu - r - s\}}$ by the same argument as given in Proposition 1.5. On the other hand, since $h_1, \dots, h_K$ are ν-supported, we see that the sum of $1/m^{1/\nu}$ over all $P(m) \leq q$ satisfying $(f(m), q) = 1$ is at most $\big(\sum_{m_1 \leq C} 1/m\big) \cdot \big(\sum_{\substack{m_2 \leq x:\, P(m_2) \leq q \\ m_2 \text{ is } \nu\text{-full}}} 1/m_2\big) \ll \exp(\sum_{p \leq q} 1/p)$. ☐

The following lemma gives a general uniform bound on $\Psi_\nu(x, z)$, the number of ν-full z-smooth numbers up to x. This generalizes some known bounds on smooth numbers, such as in [4, p. 15].

Lemma 7.2. *Fix $\nu \in \mathbb{Z}^+$. As $x, z \to \infty$, we have*

$$\Psi_\nu(x, z) \ \ll\ x^{1/\nu} (\log z) \exp(-u\nu^{-1} \log u + O(u \log_2(3u))),$$

uniformly for $(\log x)^{\max\{3, 2\nu\}} \leq z \leq x^{1/2}$. Here $u \coloneqq \log x / \log z$.

Proof. This is a classic application of Rankin's trick. Let $\eta \coloneqq (\log u)/(\log z) \leq \min\{1/3, 1/2\nu\}$. Noting that $\sum_p \sum_{r \geq \nu+1} p^{-r(1-\eta)/\nu} \ll_\nu \sum_p p^{-(1-\eta)(1+1/\nu)} \ll_\nu 1$, we have

$$\Psi_\nu(x, z) \leq \sum_{\substack{n:\ P(n) \leq z \\ n \text{ is } \nu\text{-full}}} \left(\frac{x}{n}\right)^{(1-\eta)/\nu} \leq\ x^{(1-\eta)/\nu} \prod_{p \leq z} \left(1 + \sum_{r \geq \nu} \frac{1}{p^{r(1-\eta)/\nu}}\right) \ll x^{(1-\eta)/\nu}\ \exp\left(\sum_{p \leq z} \frac{1}{p^{1-\eta}}\right).$$

Now $\sum_{p \leq z} p^{-(1-\eta)} = \log_2 z + \sum_{p \leq z} (\exp(\eta \log p) - 1)/p + O(1)$. In the last sum, the contribution of all $p \leq 2^{1/\eta}$ is $\ll\ \eta \sum_{p \leq 2^{1/\eta}} \log p/p \ll 1$, while that of $p \in (2^{1/\eta}, z]$ is at most $(\exp(\eta \log z) - 1) \sum_{2^{1/\eta} < p \leq z} 1/p \leq u(\log_2(3u) + O(1))$. Collecting all these bounds completes the proof. ☐

We also mention the following generalization of Proposition 1.7 which will be useful to us soon.

Proposition 7.3. *In the setting of Theorem 7.1, we have for any $\widehat{\chi}$ mod q,*

$$\sum_{n \leq x} \widehat{\chi}(\widehat{h}(n)) \ \ll\ \frac{x^{1/\nu}}{\log x}\ \exp\Bigg(\sum_{\substack{p \leq y \\ (h(p^\nu), q) = 1}} \frac{1}{p} \ +\ |\alpha_{\widehat{\chi}}| \sum_{\substack{y < p \leq x \\ (h(p^\nu), q) = 1}} \frac{1}{p} \ +\ O(M (\log x)^3 \mathcal{E}(y)) \Bigg) \\ +\ \left(\frac{x}{y}\right)^{1/\nu} \ +\ x^{1/\nu} (\log x)^2 \exp\left(-u(2\nu)^{-1} \log u + O(u \log_2(3u))\right). \tag{7.7}$$

where $u = \log x / \log y$. The implied constant depends only on the implied constant in (1.3).

Proof. By ν-supportedness (and the fact that $y > C$), the y-rough part of any n on the left of (7.7) is ν-full. By (7.4), it also suffices to restrict the sum on the left of (7.7) to the n's whose y-rough part is $(\nu+1)$-free. These restricted n are thus of the form BMA^ν where $P(BM) \leq Y < P^-(A)$, where A is squarefree, M is ν-full, B is ν-free with $B \ll 1$, and $(h(B)h(M)h(A^\nu), q) = 1$.

If $M > x^{1/2}$, then $A \leq (x/BM)^{1/\nu} \leq x^{1/2\nu}/B^{1/\nu}$. Given B and A, the number of M is $\ll (x^{1/\nu}/B^{1/\nu}A) \cdot (\log y) \exp(-u(2\nu)^{-1} \log u + O(u \log_2(3u)))$ by Lemma 7.2, where $u = \log x / \log y$. Since $\sum 1/B \ll 1$ and $\sum 1/A \ll \log x$, the total contribution of n with $M > x^{1/2}$ is absorbed in the right of (7.7). On the other hand, the total contribution of all n having $M \le x^{1/2}$ to the left hand side of (7.7) equals

$$\sum_{\substack{B\le x:\ P(B)\le y\\ B \text{ is } \nu\text{-free}}} \widehat{\chi}(\widehat{h}(B)) \sum_{\substack{M\le x^{1/2}:\ P(M)\le y\\ M \text{ is } \nu\text{-full}}} \widehat{\chi}(\widehat{h}(M)) \sum_{A\le (x/BM)^{1/\nu}} \mu(A)^2\ \mathbb{1}_{P^-(A)>y}\ \widehat{\chi}(\widehat{h}(A^\nu)). \tag{7.8}$$

Now we observe that uniformly in all B and M in (7.8), we have

$$\sum_{A\le (x/BM)^{1/\nu}} \mu(A)^2\ \mathbb{1}_{P^-(A)>y}\ \widehat{\chi}(\widehat{h}(A^\nu)) \ll \frac{(x/BM)^{1/\nu}}{\log x} \exp\left(|\alpha_{\widehat{\chi}}| \sum_{\substack{y<p\le x\\ (h(p^\nu),\,q)=1}} \frac{1}{p} + O(M(\log x)^3 \mathcal{E}(y))\right). \tag{7.9}$$

To show this, we proceed as in section 3: Writing $X := (x/BM)^{1/\nu}$, the analogue of (3.4) is

$$\mathcal{D}(X,t) = \sum_{p\le X} \frac{1-\mathrm{Re}(\mathbb{1}_{p>y}\widehat{\chi}(\widehat{h}(p^\nu))p^{-it})}{p} \ge \log_2 X - |\alpha_{\widehat{\chi}}| \sum_{y<p\le X} \frac{\mathbb{1}_{(h(p^\nu),\,q)=1}}{p} + O(1{+}M\mathcal{E}(y)(\log x)^3).$$

Inserting (7.9) into (7.8), recalling that $B \ll 1$, and observing that $\sum_{\substack{M:\ P(M)\le y\\ M \text{ is } \nu\text{-full}}} \mathbb{1}_{(h(M),\,q)=1}/M^{1/\nu} \le \prod_{p\le y}\left(1+\sum_{r\ge\nu} \mathbb{1}_{(h(p^r),\,q)=1}/p^{r/\nu}\right) \ll \exp(\sum_{p\le y} \mathbb{1}_{(h(p^\nu),\,q)=1}/p)$ completes the proof. $\square$

7.2. **Narkiewicz's general criterion and its uniform analogues.** As alluded to above, Narkiewicz's criterion allows for the possibility that the behavior of the f_i at the ν-th powers of primes is most significant, for any (fixed) $\nu \in \mathbb{Z}^+$. We thus start with multiplicative functions $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ for which there exist polynomials $\{F_{i,j}\}_{\substack{1\le i\le K\\ 1\le j\le V}} \subset \mathbb{Z}[T]$ satisfying $f_i(p^j) = F_{i,j}(p)$ for all $i \in [K]$, $j \in [V]$, and all primes p. (Thus, $F_{i,j}$ controls the behavior of f_i at the j-th powers of primes.) For each $j \in [V]$, define $\alpha_{\boldsymbol{j}}(q) := \varphi(q)^{-1}\#\{u \in U_q : \prod_{i=1}^K F_{i,\boldsymbol{j}}(u) \in U_q\}$. For a fixed $\nu \in [V]$, we say that q is ***$\boldsymbol{\nu}$*-admissible** if $\alpha_\nu(q) \ne 0$ but $\alpha_j(q) = 0$ for all $j < \nu$. (In other words, ν is the minimal index j for which $\prod_{i=1}^K F_{i,j}$ carries some unit mod q to some unit mod q.) Finally, let $\mathcal{Q}(\nu; f_1, \cdots, f_K)$ be the set of all ***$\boldsymbol{\nu}$*-admissible moduli *$\boldsymbol{q}$*** satisfying the following property

$$\text{For all } \widehat{\chi} \ne (\chi_0,\dots,\chi_0) \bmod q \text{ satisfying } \prod_{i=1}^K \chi_i(F_{i,\nu}(u)) = 1 \text{ on its unit support} \tag{7.10}$$

$$\{u \in U_q : \prod_{i=1}^K F_{i,\nu}(u) \in U_q\},\ \text{there exists a prime } p \text{ satisfying } \sum_{m\ge 0} \widehat{\chi}(\widehat{f}(p^m))p^{-m/\nu} = 0.$$

By the triangle inequality, such a prime $p \le 2^\nu$; thus $\mathcal{Q}(1; f_1, \dots, f_K)$ is precisely the set $\mathcal{Q}(f_1, \dots, f_K)$ that we had been working with before. We now state Narkiewicz's general criterion.

Theorem 7.4. [23] *Fix ν-admissible q. Then $(f_1, \dots, f_K)$ are WUD mod q iff $q \in \mathcal{Q}(\nu; f_1, \cdots, f_K)$.*

As the reader may expect by now, the polynomials $(F_{1,\nu}, \dots, F_{K,\nu})$ play the role of $(F_1, \dots, F_K)$ from the previous sections. Our first complete uniform analogue of Theorem 7.4 is thus the following generalization of Theorem 1.1, giving uniformity without any restrictions on inputs. All implied constants in what follows depend only on the polynomials $\{F_{i,j}\}_{\substack{1\le i\le K\\ \boldsymbol{1\le j\le \nu}}}$.

Theorem 7.5. *Fix $K_0 > 0$ and $\epsilon \in (0,1)$, and assume that $F_{1,\nu}, \dots, F_{K,\nu}$ are multiplicatively independent. Then $(f_1, \dots, f_K)$ are WUD mod $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(\nu; f_1, \cdots, f_K)$ and satisfying $IFH(F_{1,\nu}, \dots, F_{K,\nu}; B_0)$, provided at least one of the following two conditions holds:*

(i) *$q \le (\log x)^{(1-\epsilon)\alpha_\nu(q)(K-1/D_{\min})^{-1}}$, where $D_{\min} := \min\{\deg F_{i,\nu} : 1 \le i \le K\}$,* ***or***

(ii) *q is squarefree and $q^{K-1} D_{\min}^{\omega(q)} \le (\log x)^{(1-\epsilon)\alpha_\nu(q)}$.*

Again, both these ranges are essentially optimal. If $K > 1$ and $D_{\min} = 1$, then optimality holds in the stronger sense, i.e. even if $F_{1,\nu}, \dots, F_{K,\nu}$ are **any** pairwise coprime linear polynomials. The examples are analogous to those constructed in section 6; we discuss the changes in subsection § 7.5. As in Theorem 1.3, we need to restrict our inputs to restore full "Siegel–Walfisz uniformity".

Theorem 7.6. *Fix $K_0 > 0$ and assume that $F_{1,\nu}, \dots, F_{K,\nu}$ are multiplicatively independent. Then (1.2) holds uniformly in moduli $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(\nu; f_1, \cdots, f_K)$ and satisfying $IFH(F_{1,\nu}, \dots, F_{K,\nu};\ B_0)$, and uniformly in $a_i \in U_q$. Here for a general q, we can take*

$$R = \begin{cases} \nu(KD+1), & \text{if } \nu < D := \sum_{i=1}^K \deg F_{i,\nu}. \\ \text{the least integer exceeding } \nu(1 + (\nu+1)(K - 1/D)), & \text{if } \nu \ge D. \end{cases} \tag{7.11}$$

If q is squarefree, we can take $R = \nu(K\nu + K - \nu + 1) + 1$, and we can also improve this to

$$R = \begin{cases} 2, & \text{if } K = \nu = 1 \text{ and } F_{1,\nu} \text{ is not squarefull.} \\ \nu(K\nu + K - \nu) + 1, & \text{if } \nu > 1 \text{ and at least one of } \{F_{i,\nu}\}_{i=1}^K \text{ is not squarefull.} \end{cases} \tag{7.12}$$

Both values of R in (7.12) are optimal in general; we discuss this in subsection § 7.5. The *second* value of R in (7.11) is also optimal, as shown by the example of $\sigma(n)$ in the discussion below. Note that the second values of R in (7.11) and (7.12) had no analogues in Theorem 1.3.

We mention concrete examples which can only be addressed with this greater generality. Recall the definition of $\mathcal{Q}(\nu; f_1, \cdots, f_K)$ given in (7.10).

- Śliwa [37] showed that $\sigma(n)$ is WUD modulo *fixed* q iff $6 \nmid q$; in fact, $\mathcal{Q}(1;\sigma) = \{q : 2 \nmid q\}$ and $\mathcal{Q}(2;\sigma) = \{q : 2 \mid q, 3 \nmid q\}$. While Theorem 1.2 only dealt with odd q, Theorem 7.5 shows that $\sigma(n)$ is WUD modulo all $q \le (\log x)^{(2-\delta)\widetilde{\alpha}(q)}$ in $\mathcal{Q}(2;\sigma)$, as well as modulo all squarefree $q \le (\log x)^{K_0}$ in $\mathcal{Q}(2;\sigma)$ satisfying $2^{\omega(q)} \le (\log x)^{(1-\delta)\widetilde{\alpha}(q)}$, where $\widetilde{\alpha}(q) = \prod_{\ell \mid q:\ \ell \equiv 1 \pmod 3} (1 - 2/(\ell - 1))$. (Here $\nu = D = 2$ and IFH is trivial as $T^2 + 1$ is separable.) The restriction on squarefree q is optimal by the example in [34, subsection 7.2].
- Moreover, Theorem 7.6 shows that $\sigma(n)$ is WUD modulo all $q \le (\log x)^{K_0}$ in $\mathcal{Q}(2;\sigma)$ if we restrict to inputs n with $P_5(n) > q$ (respectively, $P_3(n) > q$ for squarefree q). By the examples constructed in [34, subsections 6.1 and 7.1], both these restrictions are optimal. (Note that our inputs n must be of the form m^2 or $2m^2$ since $\sigma(n)$ is odd, q being even.)
- Narkiewicz [24] studied the weak equidistribution of $\sigma_3(n) = \sum_{d \mid n} d^3$. He showed that $\mathcal{Q}(1;\sigma_3) = \{q : (q,14) = 1\}$, $Q(2;\sigma_3) = \{q : (q,6) = 2\}$, and $Q(\nu;\sigma_3) = \emptyset$ for all $\nu > 2$. Our results extend this to show that $\sigma(n)$ is WUD modulo all $q \le (\log x)^{(3/2-\delta)\alpha(q)}$ in $\mathcal{Q}(1;\sigma_3)$, as well as modulo all squarefree $q \le (\log x)^{K_0}$ in $\mathcal{Q}(1;\sigma_3)$ that satisfy $3^{\omega(q)} \le (\log x)^{(1-\delta)\widetilde{\alpha}(q)}$, where $\widetilde{\alpha}(q) = \prod_{\substack{\ell \mid q \\ \ell \equiv 1 \pmod 3}} (1 - 3/(\ell - 1)) \cdot \prod_{\substack{\ell \mid q \\ \ell \not\equiv 1 \pmod 3}} (1 - 1/(\ell - 1))$. Uniformity is restored

modulo all $q \le (\log x)^{K_0}$ in $\mathcal{Q}(1;\sigma_3)$ by restricting to inputs n with $P_4(n) > q$ (resp. $P_2(n) > q$ for squarefree $q \le (\log x)^{K_0}$ in $\mathcal{Q}(1;\sigma_3)$). Analogous results can be given for varying q in $\mathcal{Q}(2;\sigma_3)$; this genuinely requires the additional generality in this section.

Our results apply to any $\sigma_r(n)$ (for which all involved polynomials are separable), as well as to families like $(n, \varphi(n), \sigma(n))$ $(\sigma, \sigma_2, \sigma_3)$, $(\varphi, \sigma, \sigma_2, \sigma_3)$, etc. In general, once we have an explicit description of the sets $\{\mathcal{Q}(\nu; f_1, \dots, f_K)\}_{\nu=1}^{\infty}$ (which is a "fixed-modulus" problem for each fixed ν), we have explicit analogues of Siegel–Walfisz for $(f_1, \dots, f_K)$ with optimal arithmetic restrictions. However, this problem of giving explicit descriptions of $\{\mathcal{Q}(\nu; f_1, \dots, f_K)\}_{\nu=1}^{\infty}$ is not solved in general. In fact, even just for the single function σ_r (with $r > 1$ fixed), the sets $\{\mathcal{Q}(\nu;\sigma_r)\}_{\nu=1}^{\infty}$ have only been computed upto $r \le 200$ in works of Narkiewicz and Rayner [24, 30, 31]. Narkiewicz [22, 25] shows that for $(f_1, \dots, f_K)$ satisfying some natural additional constraints, we have $\mathcal{Q}(\nu; f_1, \dots, f_K) = \emptyset$ for all $\nu \gg 1$; he also gives algorithms to compute the nonempty $\mathcal{Q}(\nu; f_1, \dots, f_K)$.

7.3. **Additional ingredients for Theorems 7.5 and 7.6.** From now on, we are in the setting of Theorem 7.4 (as described before the statement of the theorem). Our first key observation:

$$\text{If } q \text{ is } \nu\text{-admissible, then } \boldsymbol{(f_1, \dots, f_K)} \textbf{ are } \boldsymbol{\nu}\textbf{-supported.}^{8} \tag{7.13}$$

To see this, let $\mathcal{A}$ be the set of primes ℓ which satisfy $\alpha_j(\ell) = 0$ for some $j < \nu$. Note that since $\alpha_j(\ell) \ge 1 - (\ell-1)^{-1}\sum_{i=1}^{K} \deg F_{i,j}$, the primes in the set $\mathcal{A}$ are no more than $1 + \max_{j\in[\nu]} \sum_{i=1}^{K} \deg F_{i,j} \ll 1$. To show (7.13), it thus suffices to show that for any n satisfying $(f(n), q) = 1$, the primes dividing the ν-free part of n must lie in $\mathcal{A}$. So assume by way of contradiction that $p^j \parallel n$ for some $p \notin \mathcal{A}$ and $j < \nu$. Then $\prod_{i=1}^{K} F_{i,j}(p) = f(p^j)$ divides $f(n)$. Since $j < \nu$ and q is ν-admissible, we have $\alpha_j(q) = 0$, so that by the Chinese Remainder Theorem, $\alpha_j(\ell_0) = 0$ for some prime $\ell_0 \mid q$. Then $\ell_0 \in \mathcal{A}$, so that $p \in U_{\ell_0}$. This forces $\ell_0 \mid \prod_{i=1}^{K} F_{i,j}(p) = f(p^j)$ as $\alpha_j(\ell_0) = 0$. Hence $\ell_0 \mid f(n)$, violating the fact that $(f(n), q) = 1$. This establishes (7.13).

We intend to apply Theorem 7.1. To set the stage, we need two additional ingredients. The first is an estimate on the input sets that generalizes and plays the role of the first estimate in (4.13).

Proposition 7.7. *As $x \to \infty$, we have uniformly in ν-admissible $q \le (\log x)^{K_0}$,*

$$\#\{n \le x : \ (f(n), q) = 1\} = \frac{x^{1/\nu}}{(\log x)^{1-\alpha_\nu(q)}} \exp(O((\log_2(3q))^{O(1)})). \tag{7.14}$$

Proof. The lower bound follows just by looking at the n of the form m^ν and applying the first estimate in (4.13) to the multiplicative function $m \mapsto f(m^\nu)$. To show the upper bound implied in (7.14), define $Y := \exp(\sqrt{\log x})$, and note that by the arguments leading to (7.8), it suffices to count the n of the form BMA^ν where $P(BM) \le Y < P^-(A)$, where A is squarefree, M is ν-full and $M \le x^{1/2}$, where B is ν-free with $B \ll 1$, and where $(f(B)f(M)f(A^\nu), q) = 1$. By [15, Theorem 01, p. 2], given B and M, the total number of A is

$$\sum_{A \le (x/BM)^{1/\nu}} \mu(A)^2 \mathbb{1}_{P^-(A) > Y} \mathbb{1}_{(f(A^\nu), q)=1} \ll \frac{(x/BM)^{1/\nu}}{\log x} \exp\left(\sum_{Y < p \le x} \frac{\mathbb{1}_{(f(p^\nu),\, q)=1}}{p}\right).$$

[8] where we can take the "C" in the definition of ν-supported to be some constant depending only on $\{F_{i,j}\}_{\substack{1\le i\le K\\ 1\le j\le \nu}}$.

Bounding $\sum 1/M^{1/\nu}$ as at the end of the proof of Proposition 7.3, we see that the number of such n is $\ll (x^{1/\nu}/\log x)\exp(\sum_{p\le x}\mathbb{1}_{(f(p^\nu),\,q)=1}/p)$. This is absorbed in the right of (7.14) because

$$\sum_{\substack{p\le X\\ (f(p^\nu),\,q)=1}}\frac{1}{p} \;=\; \sum_{\substack{p\le X\\ (F_{1,\nu}(p)\cdots F_{K,\nu}(p),q)=1}}\frac{1}{p} \;=\; \alpha_\nu(q)\log_2 X + O((\log_2(3q))^{O(1)}) \text{ uniformly in } X\ge 3q, \tag{7.15}$$

by [29, Lemma 2.4]. This establishes the upper bound implied in (7.14). □

The second ingredient is a partial improvement of (4.5) in the case $r=3$.

Lemma 7.8. *Fix $\varepsilon>0$. If at least one of $\{F_{i,\nu}\}_{i=1}^K$ is not squarefull (in $\mathbb{C}$), then*

$$\xi_3(q;z)\ll q^{-2+\varepsilon}, \text{ uniformly in squarefree } q \text{ and in } z\in(q,x]. \tag{7.16}$$

Proof. The analogue of (4.11) holds with $f_i(P_j)$ replaced by $f_i(P_j^\nu)$ (and with $\mathcal{V}_{r,K}(q,\widehat{a})$ redefined accordingly), so we need only show that $\#\mathcal{V}_{3,K}(q;\widehat{a})\ll\varphi(q)^{1+\epsilon}$. As $\#\mathcal{V}_{3,K}(q;\widehat{a})=\prod_{\ell\mid q}\#\mathcal{V}_{3,K}(\ell;\widehat{a})$ and $\omega(q)\ll\log q/\log_2 q$, it suffices to show (to complete the proof of the lemma) that

$$\#\mathcal{V}_{3,K}(\ell;\widehat{a})\ll\varphi(\ell) \text{ uniformly in primes } \ell\gg 1 \text{ and } \widehat{a}=(a_1,a_2,a_3)\in U_\ell^3. \tag{7.17}$$

Assume w.l.o.g. that $F_{1,\nu}$ is not squarefull (in $\mathbb{C}$). As in the last bound in (4.5), our idea will be to embed $\mathcal{V}_{3,K}(\ell;\widehat{a})$ into the set $V_\ell(\mathbb{F}_\ell)$ of all $\mathbb{F}_\ell$-rational points of the variety

$$V_\ell := \{(X,Y,Z)\in\overline{\mathbb{F}}_\ell^3:\ F_{1,\nu}(X)F_{1,\nu}(Y)F_{1,\nu}(Z)-a_1=F_{2,\nu}(X)F_{2,\nu}(Y)F_{2,\nu}(Z)-a_2=0\}.$$

Now, any sufficiently large prime ℓ satisfies all the following properties:

(i) ℓ doesn't divide the leading coefficients of $F_{1,\nu}$ or $F_{2,\nu}$;

(ii) $F_{1,\nu}$ is not squarefull in $\overline{\mathbb{F}}_\ell$;

(iii) If $F_{1,\nu}^{c_1}\cdot F_{2,\nu}^{c_2}$ is constant in $\overline{\mathbb{F}}_\ell(T)$ for some $c_1,c_2\in\mathbb{Z}$, then $\ell\mid(c_1,c_2)$.

We ensure (ii) by arguing as in the first paragraph of the proof of Lemma 4.3. To ensure (iii), we proceed as in the proof of Lemma 4.3(2) to see that the polynomials $(F_{1,\nu}'F_{2,\nu},F_{1,\nu}F_{2,\nu}')$ must be $\mathbb{F}_\ell$-linearly independent for all $\ell\gg 1$. (Basically, ℓ shouldn't divide the invariant factors of the 2-column matrix listing the coefficients of $F_{1,\nu}'F_{2,\nu}$ and $F_{1,\nu}F_{2,\nu}'$.) Now if $F_{1,\nu}^{c_1}\cdot F_{2,\nu}^{c_2}$ is constant in $\overline{\mathbb{F}}_\ell(T)$ for any such ℓ, then from its derivative, we get $c_1F_{1,\nu}'F_{2,\nu}+c_2F_{1,\nu}F_{2,\nu}'=0$, forcing $\ell\mid(c_1,c_2)$.

Our next important observation is that if ℓ is large enough to satisfy (i)-(iii) above, then

$$\begin{aligned}F_1^*(X,Y,Z) &:= F_{1,\nu}(X)F_{1,\nu}(Y)F_{1,\nu}(Z)-a_1 \text{ is irreducible over } \overline{\mathbb{F}}_\ell\\ &\text{and does \textbf{not} divide } F_2^*(X,Y,Z):=F_{2,\nu}(X)F_{2,\nu}(Y)F_{2,\nu}(Z)-a_2 \text{ in } \overline{\mathbb{F}}_\ell[X,Y,Z].\end{aligned} \tag{7.18}$$

The first assertion can be shown by simply replicating the arguments given for the irreducibility of "$F_1(X)F_1(Y)-w$" in the proof of Proposition 4.4(3). Next, assume for the sake of contradiction $F_2^*=F_1^*H_1$ for some $H_1\in\overline{\mathbb{F}}_\ell[T]$. Comparing the coefficients of the monomial $Y^{r_1}Z^{r_2}$ of maximum total degree r_1+r_2 on both sides of the last identity, we obtain $F_{1,\nu}(X)\mid F_{2,\nu}(X)$. Hence $F_{2,\nu}=F_{1,\nu}^m\cdot H$ for some $m\in\mathbb{Z}^+$ and $H\in\overline{\mathbb{F}}_\ell[X]$ such that $F\nmid H$. We will show that H must be constant.

To see this, note that by an easy induction argument, we have for all $t \in \{0, 1, \dots, m\}$,

$$F_1^* \text{ divides } G_t(X,Y,Z) := F_{1,\nu}(X)^{m-t}F_{1,\nu}(Y)^{m-t}F_{1,\nu}(Z)^{m-t}H(X)H(Y)H(Z) - a_1^{-t}a_2 \text{ in } \overline{\mathbb{F}}_\ell[X,Y,Z]. \tag{7.19}$$

Indeed, since $G_0 = F_2^*$, this is tautological for $t = 0$. If (7.19) holds for some $t \le m-1$, then writing $G_t = F_1^* Q_t$ shows that $F_{1,\nu}(X)F_{1,\nu}(Y)F_{1,\nu}(Z) \mid (Q_t(X,Y,Z) - a_1^{-(t+1)}a_2)$. Defining Q_{t+1} by the relation $Q_t(X,Y,Z) - a_1^{-(t+1)}a_2 = F_{1,\nu}(X)F_{1,\nu}(Y)F_{1,\nu}(Z)Q_{t+1}(X,Y,Z)$, the identity $G_t = F_1^* Q_t$ leads to $G_{t+1} = F_1^* Q_{t+1}$, completing the induction and proving (7.19). Applying (7.19) for $t = m$, we get $H(X)H(Y)H(Z) - a_1^{-m}a_2 = F_1^*(X,Y,Z)Q(X,Y,Z)$ for some $Q \in \overline{\mathbb{F}}_\ell[X,Y,Z]$. Now if H were not constant, then again comparing the coefficient of the monomial $Y^{r_1}Z^{r_2}$ with maximal $r_1 + r_2$ on both sides of the last identity would show that $F \mid H$, a contradiction. Hence $F_{1,\nu}^{-m} \cdot F_{2,\nu} = H$ must be constant in $\overline{\mathbb{F}}_\ell(T)$, violating property (iii) from before. This proves (7.18).

The final stretch of the argument involves some key inputs from commutative algebra and algebraic geometry. Note that by (7.18), (F_2^*, F_1^*) form an $\overline{\mathbb{F}}_\ell[X,Y,Z]$-regular sequence[9]: This is because if F_1^*Q lies in the ideal (F_2^*), then (7.18) forces $Q \in (F_2^*)$. By [3, Proposition 1.2.14], it thus follows that the ideal (F_2^*, F_1^*) has height at least 2, so that the variety V_ℓ has Krull dimension at most 1 (since the ambient ring $\overline{\mathbb{F}}_\ell[X,Y,Z]$ has Krull dimension 3). Finally, the version of the Lang–Weil bound in [11, Claim 7.2] yields $\#V_\ell(\mathbb{F}_\ell) \ll \ell$ for all $\ell \gg 1$, yielding the desired (7.17). ☐

We are now ready to apply Theorem 7.1 in the context of Theorems 7.5 and 7.6. **Defining $\boldsymbol{\alpha}_{\widehat{\chi}}$ and $\boldsymbol{y}$ as in subsection § 4.1** (but with $F_{i,\nu}$ playing the role of "F_i" there), we see that (7.2) holds with same M and $\mathcal{E}(y)$ as in § 4.1. We thus apply Theorem 7.1 with the same J, z as in § 4.1, and with $d := Q_0$ (with Q_0 defined as after (4.2).) Proposition 4.4 (which just required multiplicative independence) continues to hold as stated, with $F_{i,\nu}$ playing the role of F_i, and with the $\xi_r(q,z)$ (re)defined in (7.1). This proposition and Lemma 7.8 bound all the relevant quantities on the right of (7.3). Lemma 7.2 and (7.15) estimate $\Psi_\nu(x,z)$ and $\sum_{p \le y} \mathbb{1}_{(f(p^\nu),\,q)=1}/p$ respectively.

Collecting all these observations shows that the right of (7.3) is negligible compared to the main term $\varphi(q)^{-K}\#\{n \le x : (f(n),\, q) = 1\}$. (For instance, to show that the values of R in (7.11) work, we just need to check that $\max\{s/\nu + r/D, R/\nu - r + r/D - s\} > K$. If $s/\nu + r/D \le K$, then $R/\nu - r + r/D - s \ge R/\nu - K\nu + ((\nu+1)/D - 1)r$, and this last quantity attains its minimum over all $r \in [1, R/\nu]$, either at $r = 1$ or $r = R/\nu$, depending on whether $\nu \ge D$ or $\nu < D$. The corresponding check for squarefree q is just more mechanically tedious; one needs only use the best available bound out of (7.16) and the analogues of (4.5) for each $r \in [1, R/\nu)$.)

Hence, proving Theorems 7.5 and 7.6 once again comes down to showing the following analogue of (4.14): There exists a constant $\delta_0 := \delta_0(C_0, \kappa) < 1$ such that

$$\sum_{n \le x} \mathbb{1}_{(f(n),\,q)=1}\ \widehat{\psi}(\widehat{f}(n)) \ \ll\ \frac{x^{1/\nu}}{(\log x)^{1-\delta_0\alpha(q)}} \quad \text{for all nontrivial tuples } \widehat{\psi} \bmod Q_0, \tag{7.20}$$

uniformly in $\boldsymbol{q \le (\log x)^{K_0}}$ lying in $\boldsymbol{\mathcal{Q}(f_1, \dots, f_K)}$, **having $\boldsymbol{Q_0 > 1}$ and having sufficiently large radical**. Once again, Proposition 7.3 yields (7.20) when $\prod_{i=1}^K \psi_i(F_{i,\nu}(u))$ is not constant on its unit support $\{u \in U_{Q_0} : \prod_{i=1}^K F_{i,\nu}(u) \in U_{Q_0}\}$. Hence, we need only show (7.20) when

[9]Following [3, Chapter 1], given a module M over a ring R, we say that $x_1, \dots, x_n \in R$ form an M-regular sequence if $M(x_1, \dots, x_n) \neq M$, and if for each $i \ge 1$, the multiplication-by-x_i map is injective in the quotient module $M/(x_1, \dots, x_{i-1})M$. (For $i = 1$, this means that the map $M \to M : \lambda \mapsto x_i\lambda$ is injective.)

$\prod_{i=1}^K \psi_i(F_{i,\nu}(u))$ takes a constant value $c_{\widehat{\psi}}$ on its unit support. To do this, we just need to modify the arguments in section 5 as follows.

7.4. **The additional analytic ingredients.** Our first (and most nontrivial) modification is that in order to get the analogue of (5.3), we need the following observation.

Lemma 7.9. *For any $x \geq 2$, there exists an integer $X \in [x, x + x/\log^2 x)$ such that*

$$\sum_{\substack{3X/4<n<5X/4 \\ n \neq X}} \frac{\mathbb{1}_{(f(n),\,q)=1}}{|\log(X/n)|} \ll X^{1/\nu} \log X.$$

Proof. The lemma would follow once we show that with $h \coloneqq x/\log^2 x$, we have

$$\sum_{X \,\in\, \mathbb{Z}\cap[x,x+h)} \sum_{\substack{3X/4<n<5X/4 \\ n \neq X}} \frac{\mathbb{1}_{(f(n),\,q)=1}}{|\log(X/n)|} \ll x^{1/\nu} h \log x. \tag{7.21}$$

We write the total double sum on the left as $S_1 + S_2$, where S_1 denotes the contribution of $n \in (3X/4, X-1]$. Then for any n in S_1, we can write $n = X-r$, where $r \in [1, X/4) \subset [1, (x+h)/4]$. By (7.13), we have $n = n_1 n_2$ where $n_1 \ll 1$ and n_2 is ν-full. Hence, $n_2 = (X-r)/n_1 \in [(x-r)/n_1, (x+h-r)/n_1)$. Also $|\log(X/n)| = -\log(1-v/X) \gg v/X \gg v/x$. Combining these,

$$S_1 \leq x \sum_{1 \leq r \leq \frac{x+h}{4}} \frac{1}{r} \sum_{n_1 \ll 1} \sum_{\substack{\frac{x-r}{n_1} \leq n_2 < \frac{x+h-r}{n_1} \\ n_2 \text{ is } \nu\text{-full}}} 1 \ll x \log x \left(x^{1/\nu} \cdot \frac{h}{x} + x^{1/(\nu+1)} \right) \ll x^{1/\nu} h \log x,$$

where we have noted that the sum on n_2 above is $\ll x^{1/\nu}\{(1+O(h/x))^{1/\nu} - 1\} + x^{1/(\nu+1)}$ by the Erdős-Szekeres estimate [12]. Likewise, we have $S_2 \ll x^{1/\nu} h \log x$, establishing (7.21). ☐

Note that $\sum_{x<n\leq x+x/\log^2 x} \mathbb{1}_{(f(n),\,q)=1} \ll x^{1/\nu}/\log^2 x$ (write $n = n_1 n_2$ and proceed as above). Hence, it suffices to show (7.20) with "x" replaced by the "X" in Lemma 7.9. Now, our Dirichlet series $\mathcal{F}(s) = \sum_{n\geq 1} \mathbb{1}_{(f(n),\,q)=1}\, \widehat{\psi}(\widehat{f}(n))/n^s$ absolutely converges on $\mathrm{Re}(s) > 1/\nu$, since in the Euler product $\mathcal{F}(s) = \prod_p \left(1 + \sum_{r\geq 1} \mathbb{1}_{(f(p^r),\,q)=1}/p^{rs}\right)$, all but finitely many of the factors are of the form $1 + \sum_{r \geq \nu} \mathbb{1}_{(f(p^r),\,q)=1}/p^{rs}$ by ν-admissibility. Thus by Perron's formula [38, Theorem II.2.3],

$$\sum_{n\leq X} \mathbb{1}_{(f(n),\,q)=1}\, \widehat{\psi}(\widehat{f}(n)) = \frac{1}{2\pi i} \int_{\frac{1}{\nu}(1+\frac{1}{\log X})-iT}^{\frac{1}{\nu}(1+\frac{1}{\log X})+iT} \frac{\mathcal{F}(s) X^s}{s}\, \mathrm{d}s + O\left(\frac{X^{1/\nu}\log X}{T}\right).$$

To bound the error term from Perron, we used Lemma 7.9, and the technique in (5.2) which gave

$$\sum_{\substack{n\geq 1 \\ n\notin(3X/4,5X/4)}} \frac{\mathbb{1}_{(f(n),\,q)=1}}{n^{\frac{1}{\nu}(1+\frac{1}{\log X})}|\log(X/n)|} \ll \sum_{\substack{n_2\geq 1 \\ n_2 \text{ is } \nu\text{-full}}} \frac{1}{n_2^{\frac{1}{\nu}(1+\frac{1}{\log X})}} \ll \prod_p \left(1 + \sum_{r\geq\nu} \frac{1}{p^{\frac{r}{\nu}(1+\frac{1}{\log X})}}\right) \ll \log X.$$

The rest of the argument in section 5 goes through only by scaling things by ν appropriately. Branch cut conventions are analogous to those before the statement of Lemma 5.1: Branch cuts up to $\sigma \le 1$ (respectively, $\sigma \le \beta_e$) there are replaced by those up to $\sigma \le 1/\nu$ (resp. $\sigma \le \beta_e/\nu$). Redefining $\boldsymbol{\mathcal{L}_Q(t) = \log(Q(|t\nu|+1))}$, Lemma 5.1 holds with $F_{\widehat{\psi}}(s)$ replaced by

$$F_{\widehat{\psi}}(s\nu) := \left(\prod_{\eta \bmod Q} L(s\nu,\eta)^{\gamma(\eta)}\right)^{\boldsymbol{\alpha_\nu(Q)} c_{\widehat{\psi}}}, \quad \text{with } \gamma(\eta) := \frac{1}{\alpha_\nu(Q)\varphi(Q)} \sum_{\substack{u\in U_Q \\ \prod_{i=1}^K F_{i,\nu}(u)\in U_Q}} \overline{\eta}(u),$$

so that $G_{\widehat{\psi}}(s)$ (redefined accordingly) is analytic on $\sigma > \nu^{-1}(1-1/\log Q)$ and satisfies $G_{\widehat{\psi}}(s) \ll (\log_2 Q)^3$ uniformly therein. (The analogue of (5.4) holds with $1 - c_{\widehat{\psi}} \cdot \mathbb{1}_{p \prod_{i=1}^K F_{i,\nu}(p)\in U_Q}/p^{s\nu}$ replacing $1 - c_{\widehat{\psi}} \cdot \mathbb{1}_{pF(p)\in U_Q}/p^s$ there.) We redefine our contour Γ_1 (Figure 1) by replacing $1+1/\log x$, 1, β^*, β_e, by $\nu^{-1}(1+1/\log X)$, $1/\nu$, β^*/ν, β_e/ν, and by taking $\vartheta(t) := \nu^{-1}(1 - c_1/4\mathcal{L}_Q(t))$, $T = \exp(\sqrt{\log x})$.

In place of (5.8), we work with $H_{\widehat{\psi}}(s) := F_{\widehat{\psi}}(s\nu)\ (s-1/\nu)^{\alpha_\nu(q)c_{\widehat{\psi}}}\ (s-\beta_e/\nu)^{-\alpha_\nu(q)\gamma(\eta_e)c_{\widehat{\psi}}}$ which analytically continues into $\sigma > \nu^{-1}(1-c_1/\mathcal{L}_Q(t))$. The analogue of Proposition 5.2 holds with s in subpart (2) lying in $[\nu^{-1}(1-c_1/4\log Q),\ \nu^{-1}]$. Its proof is analogous, the main changes being:

- (5.9) holds with "$1 - c_1/2\mathcal{L}_Q(t) \le \mathrm{Re}(s)$" replaced by "$\nu^{-1}(1-c_1/2\mathcal{L}_Q(t)) \le \mathrm{Re}(s)$".
- In the proof of subpart (1), we redefine $\mu(t)$ to $\nu^{-1}(1+c_1/7\mathcal{L}_Q(t))$. Hence the mention of $\zeta(\mu(t))$ is replaced by $\zeta(\nu\mu(t))$.

Proposition 5.3 and everything thereafter goes through with analogous arguments; the analogue of Γ_0 in (5.14) is the union of segments joining β^*/ν and $1/\nu$ above and below the branch cut. At the very end (in the case when $c_{\widehat{\psi}} = 1$), we can write $G_{\widehat{\psi}}(s) = G_{\widehat{\psi},1}(s)G_{\widehat{\psi},2}(s)$, where $G_{\widehat{\psi},2}(s) := \prod_{p\le 2^\nu}\left(1+\sum_{r\ge1}\mathbb{1}_{(f(p^r),q)=1}/p^{rs}\right)$, and where $G_{\widehat{\psi},1}(s)$ is analytic on $\sigma > \nu^{-1}(1-1/\log Q)$ and satisfies $G_{\widehat{\psi},1}(s) \ll (\log_2 Q)^3$ uniformly therein. By definition of $\mathcal{Q}(\nu; f_1,\cdots,f_K)$ (i.e., (7.10) and its ensuing observation), we have $G_{\widehat{\psi},2}(1/\nu) = 0$, allowing us to adapt (5.16) and (5.17). This establishes the desired (7.20), completing the proofs of Theorems 7.5 and 7.6. □

7.5. **Optimality in Theorems 7.5 and 7.6.** The arguments for Lemma 6.1 show that given $\widehat{F} := (F_{1,\nu},\ldots,F_{K,\nu})$ with $\prod_{i=1}^K F_{i,\nu}$ squarefree, there exists a constant $C_{\widehat{F}} > 0$ such that any q with $P^-(q) > C_{\widehat{F}}$ has $\alpha_\nu(q) \ne 0$ and satisfies (7.10) vacuously (there is **no** tuple of characters $\widehat{\chi} \ne (\chi_0,\ldots,\chi_0) \bmod q$ for which $\prod_{i=1}^K \chi_i(F_{i,\nu}(u))$ is constant on $\{u \in U_q : \prod_{i=1}^K F_{i,\nu}(u) \in U_q\}$). Henceforth, we fix $\nu \in \mathbb{Z}^+$, a sufficiently large prime $\ell_0 > C_{\widehat{F}}$, and $\{F_{i,j}\}_{\substack{1\le i\le K\\1\le j\le \nu-1}} \subset \mathbb{Z}[T]$ with **all** coefficients divisible by ℓ_0. Our $\{F_{i,\nu}\}_{i=1}^K$ play the role of the $\{F_i\}_{i=1}^K$ in section 6. Our moduli $q \le (\log x)^{K_0}$ will have $P^-(q) = \ell_0$, so that by this discussion, $q \in \mathcal{Q}(\nu; f_1,\cdots,f_K)$. [10]

The ranges (i) and (ii) in Theorem 7.5 are both optimal, and if $D_{\min} = 1$, the range in (i) is optimal for $\{F_{i,\nu}\}_{i=1}^K$ being **any** pairwise coprime linear polynomials. All this can be shown by using the set-up in the previous paragraph, and having the prime powers P^ν play the role of the prime inputs "P" in section 6. (Example: To prove optimality of Theorem 7.5(i), take $F_{i,\nu}(T) := (T-1)^r + i$, and q a perfect r-th power with $P^-(q) = \ell_0 > \max\{C_{\widehat{F}}, 2K\}$. Then any prime $P \equiv 1 \pmod{q^{1/r}}$ satisfies $f_i(P^\nu) \equiv i \pmod q$, showing that $\#\{n \le x : (\forall i)\ f_i(n) \equiv i \pmod q\} \gg x^{1/\nu}/(q^{1/r}\log x)$.)

[10] The construction of $\{F_{i,j}\}_{\substack{1\le i\le K\\1\le j\le \nu-1}}$ guarantees that $\alpha_j(q) = 0$ for all $j < \nu$, and the definition of $C_{\widehat{F}}$ guarantees both that $\alpha_\nu(q) \ne 0$ and that q satisfies (7.10).

Finally, we show the optimality of the values of R in (7.12); hereafter q is **squarefree**. The value $R = 2$ is optimal by the constructions in section 6. To show that $R = \nu(K\nu + K - \nu) + 1$ is optimal in the setting of (7.12), we adapt the last construction in section 6 as follows:

Take $F_{i,\nu}(T) := \prod_{j=1}^{r}(T-2j)+2(2i-1)$, and continue with the set-up in the first paragraph of this subsection (with $P^-(q) = \ell_0 > \max\{C_{\widehat{F}}, 4Kr\}$). Let $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ be any multiplicative functions satisfying $f_i(p^j) = F_{i,j}(p)$ for all $i \in [K]$ and all $j \in [\nu]$, as well as $f_i(p^{\nu+1}) := 1$ for all i. Let $n = (p_1 \dots p_{(K-1)\nu})^{\nu+1} \cdot P^\nu$, with $p_1, \dots, p_{(K-1)\nu}, P$ being distinct primes satisfying $q < p_{K-1} < \cdots < p_1 < x^{1/8K\nu^2} < x^{1/3\nu} < P \le \left(x/(p_1 \dots p_{(K-1)\nu})^{\nu+1}\right)^{1/\nu}$ and $\prod_{j=1}^{r}(P-2j) \equiv 0 \pmod{q}$. (Recall that there are exactly $r^{\omega(q)}$ many possible coprime residue classes mod q that P could lie in.) Then $n \le x$, and $f_i(n) = F_{i,\nu}(P) \equiv 2(2i-1) \pmod{q}$ for all i, so that a computation entirely analogous to that done in (6.2) and (6.3) shows that

$$\sum_{\substack{n \le x:\ P_{\nu(K\nu+K-\nu)}(n) > q \\ (\forall i)\ f_i(n) \equiv 2(2i-1) \pmod{q}}} 1 \gg \frac{r^{\omega(q)}}{\varphi(q)^K (\log_2 x)^{\nu(K-1)+1}} \cdot \frac{x^{1/\nu}}{\log x}$$

As in section 6, the right can be made to grow much faster than our expected main term.

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