

THE LANDAU-SELBERG-DELANGE METHOD FOR PRODUCTS OF DIRICHLET L -FUNCTIONS, AND APPLICATIONS, I

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ABSTRACT. The Landau–Selberg–Delange method gives precise asymptotic formulas for the partial sums $\sum_{n \leq x} a_n$ of a Dirichlet series $\sum_n a_n/n^s$ that behaves like a complex power of the Riemann zeta function. However, situations often arise when the Dirichlet series behaves like a product of complex powers of several Dirichlet L –functions to a modulus q . In such situations, one often requires sharp asymptotic formulas for the partial sums $\sum_{n \leq x} a_n$ that apply in much wider ranges of q than permitted by known forms of the Landau–Selberg–Delange method. In this manuscript, we address this problem, giving new estimates on $\sum_{n \leq x} a_n$ in ranges of q that are (in most applications) much wider than attainable from previous results. Our results also weaken certain hypotheses on the size of $\{a_n\}_n$. As applications of our main theorems, we extend Landau’s classical results on the distribution of integers with prime factors restricted to progressions, and improve upon Chang, Martin and Nguyen’s work on the distribution of the least invariant factors and least primary factors of multiplicative groups. We also extend the classical Sathe–Selberg theorem and study the local laws of the functions $\Omega_a(n)$ and $\omega_a(n)$, that count (with and without multiplicity, respectively), the number of prime divisors of n lying in the progression $a \bmod q$.

1. INTRODUCTION

The subject of mean values of multiplicative functions is one of the most active areas of investigation in analytic number theory [14, 40, 15, 6, 41, 42, 11, 12, 21, 39, 8, 9]. The Landau–Selberg–Delange “LSD” method, which saw its beginnings in [18, 33, 7], forms one of the most powerful results in this subject. One of the most general formulations of this was obtained by Tenenbaum [38, Theorem II.5.2], who gives for a sequence $\{a_n\}_n \subset \mathbb{C}$, an asymptotic formula of the shape

$$(1.1) \quad \sum_{n \leq x} a_n = \frac{x}{(\log x)^{1-\alpha}} \sum_{j=0}^N \frac{\kappa_j}{(\log x)^j} + O(\text{Error Terms}),$$

under the hypothesis that the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s = \zeta(s)^{\alpha} G(s)$ for some reasonably well-behaved analytic function $G(s)$.

He also assumed that the coefficients $\{a_n\}_n$ are termwise absolutely bounded by a sequence $\{b_n\}_n$ whose Dirichlet series satisfies similar analytic properties. Owing to its high precision and uniformity, Tenenbaum’s formulation of the LSD method is widely applicable in a variety of problems in analytic number theory. There has also been a lot of work extending this in various directions, by Granville–Koukoulopoulos [10], Tenenbaum–de la Breteche [5], Chang–Martin [2], Cui–Wu [3], and by Phaoibul in his doctoral dissertation [24].

However, situations often arise when the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ behaves like a function of the form $\prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_{\chi}}$, where the product is over all Dirichlet characters χ modulo q , and where

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$\nu > 0$ is a fixed parameter. In such situations, several applications require estimates on the partial sums $\sum_{n \leq x} a_n$, which hold true *uniformly* as the modulus q varies in a wide range, one of the typical ranges being the “Siegel–Walfisz type” range $[1, (\log x)^K]$. One of the first main difficulties here is that there are potentially *two essential singularities* in the zero free region that need to be worried about, one coming from the principal character χ_0 and one from the exceptional character χ_e . One might try to place this in the setting of Tenenbaum’s result (or in the setting of any of its improvements alluded to above) by writing $\prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi} = \zeta(s)^{\alpha_{\chi_0}} \cdot G(s)$, but doing this gives rise to at least three issues:

- (i) The best available bounds on $G(s) = \prod_{\chi \neq \chi_0} L(s\nu, \chi)^{\alpha_\chi}$ often grow way too rapidly with q .
- (ii) The possibility of a Siegel zero modulo q further complicates the analytic behavior of $G(s)$.
(In particular, $G(s)$ itself may have an essential singularity in the zero free region.)
- (iii) The parameter ν (crucial in applications) needs additional care beyond scaling maneuvers.

Out of these, (i) and (ii) are the most major issues, and they cause the “Error Terms” in (1.1) to blow up drastically as the modulus q grows, thereby *severely* impeding uniformity in q . The aforementioned extensions of his work are also unable to deal with any of these issues.

Concrete examples of these phenomena occur in Theorems 3.4 and 3.6, as well as in Proposition 5.4 of [2], where the uniformity in the moduli q is only up to small powers of $\log x$, even with a more explicit version of [38, Theorem II.5.2] that the authors obtain. (In fact, the uniformity in [2, Theorems 3.4 and 3.6] may even drop to a power of $\log \log x$ in some commonly–occurring situations.) More widespread examples arise in residue-class distribution problems of arithmetic functions: For instance, [19] studies the distribution of Euler’s totient $\varphi(n)$ and the sum–of–divisors $\sigma(n)$ in residue classes to varying *prime* moduli p . Here an application of known forms of the LSD method can only give a range $p \leq (\log \log x)^2$. For composite moduli, the ranges obtained are even worse. As such, all previous works [27, 25, 28, 26, 34, 37, 35, 36] studying the equidistribution of arithmetic functions to growing moduli resort to a variety of “ad–hoc” methods.

In this work, we find a new version of the Landau–Selberg–Delange method, which extends Tenenbaum’s formulation in [38] to situations when the Dirichlet series $\sum_n a_n/n^s$ behaves like the product $\prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi}$. In several applications where this used to be unattainable via previous literature, our results give genuine asymptotic formulas in ranges of q that are at least as wide as the “Siegel–Walfisz” range $[1, (\log x)^K]$ (for any fixed $K > 0$), and become even wider as better bounds on the Siegel–zero become available. We also give results (Theorem 1.1 and Corollary 1.2 below) that assume some natural growth conditions on $\{a_n\}_n$ on average, in place of the existence of the sequence $\{b_n\}_n$ that was required for (1.1). As applications of our main results, we can:

- Extend the classical works of Landau [18] on the distribution of integers with prime factors restricted to arithmetic progressions.
- Improve upon the works of Chang, Martin and Nguyen [2, 20], on the distributions of the least invariant factors and least primary factors of unit groups. In these applications, it is also useful that our general results do not require $\{a_n\}_n$ to be a multiplicative function.
- Extend the works of Sathe [30] and Selberg [33], and study the distribution of positive integers with a given number of prime factors from an arithmetic progression.

Before giving the statements of our main results, we summarize some of the ideas we use to deal with issues (i)–(iii) mentioned above. First, we introduce (1.3) a parameter λ_q which controls the correlation of the vector $(\alpha_\chi)_\chi$ with the vectors $(\chi(a))_\chi$ for all coprime residues $a \pmod{q}$; in several applications, λ_q is quite small. Second, inspired by work of Scourfield [31] and other classical ideas, we study the logarithmic derivative of $\mathcal{F}(s\nu) := \prod_\chi L(s\nu, \chi)^{\alpha_\chi}$ by using an “inner contour shift” to give a series representation for $\mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$ in terms of the zeros of $\{L(s, \chi)\}_\chi$. The additional feature here is that the terms of such a series are smoothed by a rapidly decaying function, allowing us to bound $\mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$ via classical zero density estimates. By constructing certain auxiliary functions, we then bound $|\mathcal{F}(s\nu)|$, which in turn allow us to control the residual integrals after performing a second (“outer”) contour shift that is more reminiscent of known forms of the LSD method. Moreover, the contours we consider are modifications of some classical contours that account for the presence of a possible Siegel–zero modulo q as well as for the parameter ν . We deal with the other complications introduced by ν via certain averaging arguments.

1.1. The main results: General statements.

We write complex numbers s as $\sigma + it$, where $\sigma = \operatorname{Re}(s)$ and $t = \operatorname{Im}(s)$. Fix $c_0 \in (0, 1/3)$ such that for any integer $q \geq 3$, the product $\prod_{\chi \pmod{q}} L(s, \chi)$ has no zero in the classical zero–free region $\{\sigma + it : \sigma > 1 - c_0/\log(q(|t|+1))\}$ except at most a simple real zero η_e (the Siegel zero) associated to a real character χ_e (the exceptional character). Fix $\nu > 0$ and let $\Omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a **non-decreasing** function. Given complex numbers $\{a_n\}_{n=1}^\infty$ and $\{\alpha_\chi\}_\chi$ (with the α_χ indexed at the Dirichlet characters $\chi \pmod{q}$), we say that $\{a_n\}_n$ has **property $\mathcal{P}(\nu, \{\alpha_\chi\}_\chi; c_0, \Omega)$** if

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \mathcal{F}(s\nu)G(s) \text{ for all } s \text{ with } \operatorname{Re}(s) > \frac{1}{\nu}, \quad \text{where } \mathcal{F}(s\nu) := \prod_{\chi} L(s\nu, \chi)^{\alpha_\chi},$$

and where $\mathbf{G}(s)$ is a function that analytically continues into the region $\{s = \sigma + it : \sigma > \nu^{-1}(1 - c_0/\log(q(|t\nu|+1)))\}$ and satisfies $|\mathbf{G}(s)| \leq \Omega(|t|)$ in this region. (Note that this region is a dilate of the classical zero–free region by the factor of ν , which will be helpful to work with.)

In what follows, we define

$$(1.3) \quad \lambda_q := 1 + \max_{a \pmod{q}} \left| \sum_{\chi} \alpha_\chi \cdot \chi(a) \right| \quad \text{and} \quad \mu_j := \frac{1}{j!} \cdot \left. \frac{d^j}{ds^j} \right|_{s=1/\nu} \frac{\mathcal{F}(s\nu)G(s)}{s} \left(s - \frac{1}{\nu} \right)^{\alpha_{x_0}},$$

so that μ_j is the j -th coefficient of the power series of the function $s^{-1}\mathcal{F}(s\nu)G(s)(s-1/\nu)^{\alpha_{x_0}}$ around $1/\nu$. (The discussion in subsection § 2.1 shows that this function does analytically continue into some neighborhood of $1/\nu$.) Moreover, writing $\alpha_\chi = \sum_{\psi \pmod{q}} \alpha_\psi \cdot \varphi(q)^{-1} \sum_{a \pmod{q}} \bar{\chi}(a)\psi(a)$ via orthogonality and interchanging sums, we obtain the following important bound

$$(1.4) \quad |\alpha_\chi| \leq \lambda_q \quad \text{and} \quad |\beta_\chi| \leq \lambda_q \quad \text{for all characters } \chi \pmod{q}.$$

Our first main result gives a Landau–Selberg–Delange type estimate on $\sum_{n \leq x} a_n$, assuming an average growth condition on $\{a_n\}_n$ on dyadic intervals $(x, 2x]$.

Theorem 1.1. *Assume that $\{a_n\}_n$ has property $\mathcal{P}(\nu, \{\alpha_\chi\}_\chi; c_0, \Omega)$, and that for all $x > 1$, we have*

$$(1.5) \quad \sum_{x < n \leq 2x} |a_n| \leq \kappa x^{1/\nu}, \quad \text{where } \kappa \geq 2 \text{ is independent of } x.$$

Fix any $K_0 > 0$. The following hold uniformly in all $x \geq q \geq e^{4+5/3\nu}$, $h \in (0, x/2]$, $N \in \mathbb{Z}_{\geq 0}$, and in $\{\alpha_\chi\}_\chi \subset \mathbb{C}$ with $\max\{|\alpha_{x_0}|, |\alpha_{x_e}|\} \leq K_0$; the implied constants depend only on c_0, ν and K_0 .

(1) If the exceptional zero η_e exists and satisfies $1 - c_0/10\lambda_q \log q < \eta_e < 1 - 3\nu/\log x$, then

$$(1.6) \quad \sum_{n \leq x} a_n - \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{j=0}^N \frac{\mu_j (\log x)^{-j}}{\Gamma(\alpha_{\chi_0} - j)} \ll \sum_{x < n \leq x+h} |a_n| + \kappa \cdot \frac{x^{1+1/\nu} \log x}{Th} \\ + (2\lambda_q \log q)^{\lambda_q+2K_0} x^{1/\nu} \left\{ \Omega(T) \frac{(\log(e\nu T))^{1+\lambda_q}}{T} + \Omega(1) \frac{N! (71(1+\nu) \cdot (1-\eta_e)^{-1})^{N+K_0+1}}{(\log x)^{1-|\text{Re}(\alpha_{\chi_0})|} \cdot \min\{x/h, (\log x)^{N+1}\}} \right\}$$

where $T := (q\nu)^{-1/2} \exp\left(0.5\sqrt{\log^2(q\nu) + c_0 \log x / \nu \lambda_q}\right)$.

(2) If η_e does not exist or satisfies $\eta_e \leq 1 - c_0/10\lambda_q \log q$, then for $q < x^{c_0/80\nu\lambda_q}$, we have

$$(1.7) \quad \sum_{n \leq x} a_n - \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{j=0}^N \frac{\mu_j (\log x)^{-j}}{\Gamma(\alpha_{\chi_0} - j)} \ll \sum_{x < n \leq x+h} |a_n| + \kappa \cdot \frac{x^{1+1/\nu} \log x}{Th} \\ + (2\lambda_q \log q)^{\lambda_q} x^{1/\nu} \left\{ \Omega(T) \frac{(\log(e\nu T))^{1+\lambda_q}}{T} + \Omega(1) \frac{N! (2000(1+\nu)c_0^{-1} \cdot \lambda_q \log q)^{N+K_0+1}}{(\log x)^{1-|\text{Re}(\alpha_{\chi_0})|} \cdot \min\{x/h, (\log x)^{N+1}\}} \right\},$$

where $T := (q\nu)^{-1/2} \exp\left(0.5\sqrt{\log^2(q\nu) + c_0 \log x / 20\nu\lambda_q}\right)$. Moreover if η_e does not exist, then

(1.7) holds uniformly for $q < x^{c_0/8\nu\lambda_q}$, we take $T = (q\nu)^{-1/2} \exp\left(0.5\sqrt{\log^2(q\nu) + c_0 \log x / 4\nu\lambda_q}\right)$, we do not need to assume that $|\alpha_{\chi_e}| \leq K$, and 2000ν can be replaced by 200ν .

Theorem 3.1 gives more general versions of this result. Under the Landau–Siegel zeros conjecture, Theorem 1.1 holds uniformly for $q \leq x^{(1-\epsilon)c_0/\nu\lambda_q}$, with $\epsilon \in (0, 1)$ being an absolute constant. With some technical work, we should be able to improve this to allow any fixed $\epsilon \in (0, 1)$. If η_e does exist, then better bounds on it lead to better ranges of uniformity in q (up to the limit $x^{(1-\epsilon)c_0/\nu\lambda_q}$). Also note that in several applications (including all the ones we provide), the parameter $|\lambda_q|$ is absolutely bounded, often just by 1. In such cases, Siegel’s Theorem $1 - \eta_e \gg_\epsilon q^{-\epsilon}$ gives a range of $q \leq (\log x)^K$ (for any fixed $K > 0$) in the above result, while a bound of the form $\eta_e \leq 1 - c(\theta_0)/(\log q)^{\theta_0}$ for some fixed $\theta_0 \in (0, 1]$ and some constant $c(\theta_0) > 0$, such as in the upcoming work of Yitang Zhang [43], gives a range $q \leq \exp((c(\theta_0) \log x / 3\nu)^{1/\theta_0})$. Further, note that the lower bound restriction $q \geq e^{4+5/3\nu}$ is an unserious technicality: For $q < e^{4+5/3\nu} \ll 1$, known forms of the Landau–Selberg–Delange method apply directly anyway.

Our hypothesis $\mathcal{P}(\nu, \{\alpha_\chi\}_\chi; c_0, \Omega)$ generalizes hypothesis “ $\mathcal{P}(z; c_0, \delta, M)$ ” assumed in Tenenbaum [38, Theorem II.5.2], not only by extending it to this setting of Dirichlet L -functions, but also by allowing a reasonably arbitrary growth condition in $G(s)$. Note also that we do not make any assumption on α_χ for $\chi \neq \chi_0, \chi_e$. The additional generality in ν requires additional arguments (compared to the most natural case $\nu = 1$) but has important applications. With these additional arguments, we can also directly generalize [38, Theorem II.5.2] from the case when the Dirichlet series behaves like a (complex) power of $\zeta(s)$ to when it behaves like a power of $\zeta(sv)$.

When $\nu \in \mathbb{N}$, (1.5) follows if $\{a_n\}_n$ is supported on ν -full numbers and bounded on average over intervals of the form $(x, 2x]$ (in particular, if $\nu = 1$, then (1.5) is just saying that $\{a_n\}_n$ is bounded on average on such intervals). In applications, *much stronger* growth conditions than

(1.5) are often available: For instance, one of the most typical kinds intervals where strong growth conditions are easily available are intervals of length $x/(\log x)^A$. In such situations, we have the following useful consequence of Theorem 1.1.

Corollary 1.2. *Assume that $\{a_n\}_n$ has property $\mathcal{P}(\nu, \{\alpha_\chi\}_\chi; c_0, \Omega)$, and that for some $A \geq 1$,*

$$(1.8) \quad \sum_{x < n \leq x + x/(\log x)^A} |a_n| \leq \kappa_A \cdot \frac{x^{1/\nu}}{(\log x)^A} \text{ for all } x > 1, \quad \text{where } \kappa_A \geq 2 \text{ is independent of } x.$$

Then the assertions of Theorem 1.1 hold exactly as stated, but with $\kappa := \kappa_A \cdot 2^{A+1/\nu}$ and $h := x/(\log x)^A$. In particular, $\min\{x/h, (\log x)^{N+1}\} \asymp (\log x)^{\min\{A, N+1\}}$, and we have

$$\sum_{x < n \leq x+h} |a_n| \ll \kappa_A \cdot \frac{x^{1/\nu}}{(\log x)^A} \quad \text{and} \quad \kappa \cdot \frac{x^{1+1/\nu} \log x}{Th} \ll 2^A \kappa_A \cdot \frac{x^{1/\nu} \log x}{T}.$$

Although we just assumed (1.8) for *some* A , in practice, (1.8) is often available for *all* (sufficiently large) A . When this happens, $(\log x)^{1-|\operatorname{Re}(\alpha_{\chi_0})|} \cdot \min\{x/h, (\log x)^{N+1}\} = (\log x)^{N+2-|\operatorname{Re}(\alpha_{\chi_0})|}$, giving clean savings in the power of $\log x$ in (1.6) and (1.7). (Here the uniformity of Corollary 1.2 in A and κ_A , which comes from the uniformity of Theorem 1.1 in the parameters h and κ , is significant.)

In [38, Theorem II.5.2], it is assumed that $\Omega(t) = \mathcal{M}(1+t)^{1-\delta_0}$ for some $\mathcal{M} > 0$, $\delta_0 \in (0, 1]$, and that there is a sequence $\{b_n\}_n$ upper-bounding the $|a_n|$ (termwise), whose Dirichlet series $\sum_n b_n/n^s$ has analytic properties similar to those of $\sum_n a_n/n^s$. In Theorem 1.1, we replaced these hypotheses by (1.5) which is more natural in some applications. However, sometimes the following more direct generalization of [38, Theorem II.5.2] is easier to work with.

Theorem 1.3. *Assume that $\{a_n\}_n \subset \mathbb{C}$ has property $\mathcal{P}(\nu, \{\alpha_\chi\}_\chi; c_0, \Omega)$, and that there exists a sequence $\{b_n\}_n \subset \mathbb{R}_{\geq 0}$ satisfying $|a_n| \leq b_n$ for all n , such that $\{b_n\}_n$ has property $\mathcal{P}(\nu, \{\beta_\chi\}_\chi; c_0, \Omega)$ for some $\{\beta_\chi\}_\chi \subset \mathbb{C}$. Further, suppose $\Omega(t) = \mathcal{M}(1+t)^{1-\delta_0}$ for some $\mathcal{M} > 0$ and $\delta_0 \in (0, 1]$.*

Fix $K_0 > 0$ and define $\Lambda_q := 1 + \max_{a \bmod q} \max \left\{ \left| \sum_\chi \alpha_\chi \cdot \chi(a) \right|, \left| \sum_\chi \beta_\chi \cdot \chi(a) \right| \right\}$. The following estimates hold uniformly in all $x \geq 4$, $N \in \mathbb{Z}_{\geq 0}$, $q \geq e^{4+5/3\nu}$, and in all $\{\alpha_\chi\}_\chi, \{\beta_\chi\}_\chi \subset \mathbb{C}$ with $\max\{|\alpha_{\chi_0}|, |\alpha_{\chi_e}|, |\beta_{\chi_0}|, |\beta_{\chi_e}|\} \leq K_0$; the implied constants depend only on c_0, ν, δ_0 and K_0 .

(1) *If the exceptional zero η_e exists and satisfies $1 - c_0/10\Lambda_q \log q < \eta_e < 1 - 3\nu/\log x$, then*

(1.9)

$$\begin{aligned} \sum_{n \leq x} a_n - \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{j=0}^N \frac{\mu_j (\log x)^{-j}}{\Gamma(\alpha_{\chi_0} - j)} &\ll \frac{(2\Lambda_q \log q)^{\Lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot N! (71\nu)^N \cdot \mathcal{M} x^{1/\nu}}{(1 - \eta_e)^{N+1+|\operatorname{Re}(\alpha_{\chi_e})|} \cdot (\log x)^{N+2-|\operatorname{Re}(\alpha_{\chi_0})|}} \\ &+ \left(\frac{3\Lambda_q \log q}{\delta_0^{1/2}} \right)^{\frac{3\Lambda_q}{2} + 3K_0} \cdot \frac{\mathcal{M} (\log x)^{K_0/2}}{(1 - \eta_e)^{5K_0/2}} \left\{ \frac{x^{1/\nu}}{T^{\delta_0/4}} + x^{(2+\eta_e)/3\nu} T^{\delta_0/4} \right\}, \end{aligned}$$

where $T := (q\nu)^{-1/2} \exp \left(0.5 \sqrt{\log^2(q\nu) + 2c_0 \log x / \nu \delta_0 \Lambda_q} \right)$.

(2) *If η_e does not exist or satisfies $\eta_e \leq 1 - c_0/10\Lambda_q \log q$, then for $q < x^{c_0/80\nu\Lambda_q}$, we have*

(1.10)

$$\begin{aligned} \sum_{n \leq x} a_n - \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{j=0}^N \frac{\mu_j (\log x)^{-j}}{\Gamma(\alpha_{\chi_0} - j)} &\ll \frac{(2\Lambda_q \log q)^{\Lambda_q + |\operatorname{Re}(\alpha_{\chi_0})|} \cdot N! (2000\nu c_0^{-1})^N \cdot \mathcal{M} x^{1/\nu}}{(\Lambda_q \log q)^{-N-1} \cdot (\log x)^{N+2-|\operatorname{Re}(\alpha_{\chi_0})|}} \\ &+ \left(\frac{3\Lambda_q \log q}{\delta_0^{1/2}} \right)^{\frac{3\Lambda_q}{2} + 3K_0} \cdot \mathcal{M} x^{1/\nu} (\log x)^{K_0/2} \exp \left(\frac{\delta_0}{8} \left\{ \log(q\nu) - \sqrt{\frac{c_0 \log x}{10\nu\delta_0\Lambda_q} + \log^2(q\nu)} \right\} \right). \end{aligned}$$

Moreover if η_e does not exist, then (1.10) holds uniformly for $q < x^{c_0/8\nu\Lambda_q}$, we do not need to assume that $|\alpha_{\chi_e}| \leq K$, and 10ν and 2000ν can be replaced by ν and 200ν respectively.

The remarks made after the statement of Theorem 1.1 continue to hold. We now highlight some of the applications of Theorem 1.1, Corollary 1.2 and 1.3, that we had alluded to previously.

1.2. The distribution of integers with prime factors restricted to progressions.

Let $U_q := (\mathbb{Z}/q\mathbb{Z})^\times$ denote the multiplicative group/unit group mod q . Consider an arbitrary subset \mathcal{A} of U_q , and let $\mathcal{N}(x; q, \mathcal{A})$ denote the number of positive integers $n \leq x$ such that any prime dividing n lies in one of the residue classes in \mathcal{A} . Estimating $\mathcal{N}(x; q, \mathcal{A})$ is a classical problem back to the Landau's seminal work [18] on sums of squares, which also forms one of the earliest beginnings of the Landau–Selberg–Delange method.

While Landau's work can deal with the case of *fixed* q and *fixed* \mathcal{A} , it is of interest to ask for extensions of Landau's results when q is allowed to grow with x rapidly, and \mathcal{A} is allowed to vary with q . Such a result has potential applications as well, as we shall see below. However, there seems to be no literature attaining any uniformity in q or \mathcal{A} , – except for work of Chang and Martin [2], – which allows uniformity in $q \leq (\log x)^{1/2-\epsilon}$ and gives a huge error term when \mathcal{A} is reasonably away from its extreme values 1 or $\varphi(q)$. As our first application of Theorem 1.1, we give asymptotics on $\mathcal{N}(x; q, \mathcal{A})$ that allow *complete uniformity* in the modulus q upto *any* fixed power of $\log x$ (the full “Siegel–Walfisz range”) as well as complete uniformity in *all* subsets \mathcal{A} of U_q . Assuming the Landau–Siegel zeros conjecture, we can improve the range of q up to $\exp(c'\sqrt{\log x})$ for some constant c' . (Note that the range of q in [2] is not improvable conditionally either.)

Theorem 1.4. Fix $K > 0$ and $\epsilon_0 \in (0, 1)$. There exists a constant c_1 depending only on ϵ_0 and K such that the following estimate holds uniformly in $x \geq 4$, $N \in \mathbb{Z}_{\geq 0}$, in moduli $q \leq (\log x)^K$ and in $\mathcal{A} \subset U_q$; the implied constants are allowed to depend only on c_0, c_1, K and ϵ_0 .

(1.11)

$$\mathcal{N}(x; q, \mathcal{A}) - \frac{x}{(\log x)^{1-\frac{|\mathcal{A}|}{\varphi(q)}}} \sum_{j=0}^N \frac{k_j (\log x)^{-j}}{\Gamma\left(\frac{|\mathcal{A}|}{\varphi(q)} - j\right)} \ll \frac{N! (142 c_1^{-1})^N \cdot x}{(\log x)^{(N+2)(1-\epsilon_0)-\frac{|\mathcal{A}|}{\varphi(q)}}} + x \exp\left(-\sqrt{\frac{c_0 \log x}{32}}\right).$$

Here the $\{k_j\}_j \subset \mathbb{C}$ depend only on q and \mathcal{A} , and have been made explicit in (5.4) and (5.5). Further, if the Siegel–zero does not exist, then all these assertions hold uniformly in the wider range $q \leq \exp(\sqrt{c_0 \log x}/8)$, and the constants c_1 and ϵ_0 may be omitted.

The last assertion should also hold under a result of the form $\eta_e \gg 1 - c(\theta_0)/(\log q)^{\theta_0}$. The constant c_1 above comes from Siegel's Theorem, and hence is ineffective. Theorems 1.1 and 1.3 can also be used to give more precise error terms, – showing the dependence on the Siegel zero in the above application, – but we have just chosen to highlight the cleanest corollaries.

1.3. Distributions of the least invariant factors of multiplicative groups.

The algebraic structure of multiplicative group $U_n := (\mathbb{Z}/n\mathbb{Z})^\times$ is of remarkable interest to number theorists. For example, its size is the Euler totient function $\varphi(n)$, and U_n is cyclic precisely when n has a primitive root. It turns out that the invariant factor decomposition of U_n also captures a lot of exotic arithmetic information; here by the “invariant factor decomposition”, we mean the unique way of writing $U_n \cong \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_r\mathbb{Z}$, with $d_1, \dots, d_r > 1$ being integers satisfying $d_1 | \cdots | d_r$. (Recall that d_1, \dots, d_r are called the **invariant factors** of U_n .) For instance, the length r of this decomposition is basically the number $\omega(n)$ of distinct primes dividing n , while the *largest* invariant factor d_r is the renowned Carmichael lambda function $\lambda(n)$.

Motivated by this last observation, Chang and Martin [2] study the dual object, namely the *least* invariant factor $\lambda^*(n) := d_1$. Elementary linear algebra over rings shows that $\lambda^*(n)$ must be even for $n > 2$, but in fact they show that $\lambda^*(n)$ must be equal to 2 on a set of asymptotic density 1. Some natural next questions are: How often is $\lambda^*(n)$ equal to a given *even* integer q ? How often is $\lambda^*(n)$ divisible by such a q ? In [2, Proposition 5.4 and Theorem 1.4], Chang and Martin also answer these questions for fixed q or for q varying roughly up to $(\log x)^{1/2-\epsilon}$. (Their range of q is essentially the limit of their methods, even conditional on the Landau–Siegel zeros conjecture.)

As our next application, we qualitatively and quantitatively improve their results to allow q to vary up to *any* fixed power of $\log x$, with this range being further improvable under the Landau–Siegel zeros conjecture. We start by giving an asymptotic on $\#\{n \leq x : q \mid \lambda^*(n)\}$, which will be one of two applications of our general main results in this paper, for a sequence $\{a_n\}_n$ which is *not* defined by a multiplicative function. (The other application will be Theorem 1.7 below.)

Theorem 1.5. *Fix $K > 0$ and $\epsilon_0 \in (0, 1)$. There exists a constant c_1 depending only on ϵ_0 and K such that the following estimate holds uniformly in $x \geq 4$, integers $N \geq 0$, and in **even** $q \leq (\log x)^K$, with the implied constants depending only on c_0, c_1, K and ϵ_0 .*

$$\sum_{\substack{n \leq x \\ q \mid \lambda^*(n)}} 1 = \frac{x}{(\log x)^{1 - \frac{1}{\varphi(q)}}} \sum_{j=0}^N \frac{r_j (\log x)^{-j}}{\Gamma\left(\frac{1}{\varphi(q)} - j\right)} + O\left(\frac{N! (142 c_1^{-1})^N \cdot x}{(\log x)^{(N+2)(1-\epsilon_0) - \frac{1}{\varphi(q)}}} + x \exp\left(-\sqrt{\frac{c_0 \log x}{32}}\right)\right).$$

Here the $\{r_j\}_j \subset \mathbb{C}$ depend only on q and \mathcal{A} , and are defined by (6.1). Moreover,

$$r_0 \asymp \left(\frac{\varphi(q)}{q} \cdot \prod_{\chi \neq \chi_0} L(1, \chi) \right)^{1/\varphi(q)} \ll (\log q) \cdot \left(\frac{\varphi(q)}{q} \right)^{1/\varphi(q)} \text{ uniformly in } q \leq (\log x)^K.$$

If there is no Siegel–zero, then all these assertions hold uniformly for $q \leq \exp(\sqrt{c_0 \log x / 8})$, and c_1 and ϵ_0 do not appear.

This result is closely tied with Theorem 1.4 via Lemma 6.1. Using Theorem 1.5 and with some additional ideas from the anatomy of integers, we can also improve Chang and Martin’s estimate on how often the least invariant factor of the unit group equals a given integer q , giving estimates that become sharper as q grows even slightly rapidly with x . As such, both Theorems 1.5 and 1.6 also highlight a further application of Theorem 1.4.

Theorem 1.6. *Fix $K \geq 1$ and $\epsilon_0 \in (0, 1)$. There exists a constant c_1 depending only on ϵ_0 and K such that the following estimate holds uniformly in $x \geq 4$, integers $N \geq 0$, and in **even** $q \leq (\log x)^K$, with the implied constants depending only on c_0, c_1, K and ϵ_0 .*

$$\begin{aligned} \#\{n \leq x : \lambda^*(n) = q\} &= \frac{x}{(\log x)^{1-\frac{1}{\varphi(q)}}} \sum_{j=0}^N \frac{r_j (\log x)^{-j}}{\Gamma\left(\frac{1}{\varphi(q)} - j\right)} - \frac{\text{li}(x) + \text{li}(x/2)}{\varphi(q)} \\ &\quad + O\left(\frac{N! (142 c_1^{-1})^N \cdot x}{(\log x)^{(N+2)(1-\epsilon_0)-\frac{1}{\varphi(q)}}} + \frac{x (\log \log x \cdot \log \log \log x)^2}{q^2 (\log x)^{1-1/2\varphi(q)}}\right). \end{aligned}$$

Here the r_j are as in Theorem 1.5, and $\text{li}(y) = \int_2^y dt / \log t$ is the logarithmic integral.

With some more work, it should be possible to remove the $(\log \log \log x)^2$ factor.

1.4. Distributions of the least primary factors of multiplicative groups.

Another canonical form of finite abelian groups is their primary decomposition, which for the multiplicative group modulo n , amounts to writing $U_n \cong \mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_k\mathbb{Z}$ for prime powers q_1, \dots, q_k . This decomposition is also unique (up to the order of the q_j), and we may define the **least primary factor** of U_n to be $\lambda'(n) := \min_j q_j$. (For $n = 1, 2$, we define $\lambda'(n) := \infty$.) Martin and Nguyen [20] studied the distribution of $\lambda'(n)$, obtaining asymptotics for how often $\lambda'(n) = q$ for $q = 2$ and for a given **odd** prime power q . (Elementary number theory shows that the least primary factor of U_n can never be 2^k for $k > 1$.)

Their key algebraic idea was to write $\#\{n \leq x : \lambda'(n) = q\}$ as the difference $\mathcal{A}_q(x) - \mathcal{A}_{q^+}(x)$, where $\mathcal{A}_q(x) := \#\{n \leq x : \lambda'(n) \geq q\}$ and q^+ is the next prime power after q . Then they characterized $\mathcal{A}_q(x)$ with a set of congruence restrictions modulo primes less than q (see Lemma 7.1), which brought down the problem of estimating $\mathcal{A}_q(x)$ to the problem of estimating the number $\mathcal{N}(x; Q, \mathcal{B})$ (defined in § 1.2) with Q being an integer having order of magnitude $e^{(1+o(1))q}$, and with \mathcal{B} being a set of coprime residues modulo Q of size really far away from the extreme values 1 or $\varphi(Q)$.

Since this relies on Chang and Martin's estimate [2, Theorem 3.4] on $\mathcal{N}(x; Q, \mathcal{B})$, the results in [20] allow very little uniformity in q , something like $q = o(\log \log x)$ (because of the sizes of both Q and \mathcal{B}), and this range is essentially the limit of their methods (even conditionally). But using the same algebraic idea as in Martin and Nguyen's work, Theorem 1.1 gives an unconditional range of $q \leq K \log \log x$ for any fixed $K > 0$, which can be improved to $q \leq c\sqrt{\log x}$ conditional on the Landau–Siegel zeros conjecture. These limitations again come from Q (but no longer from \mathcal{B}).

Theorem 1.7. Fix $K \geq 1$ and $\epsilon_0 \in (0, 1)$. There exists a constant c_1 depending only on ϵ_0 and K such that the following holds uniformly in $x \geq 4$, integers $N \geq 0$, and in **prime powers** $q \leq K \log \log x$, with the implied constants depending only on c_0, c_1, K and ϵ_0 .

$$\begin{aligned} \sum_{\substack{n \leq x \\ \lambda'(n)=q}} 1 &= \frac{x}{(\log x)^{1-B(q)}} \sum_{j=0}^N \frac{\kappa_j(q) (\log x)^{-j}}{\Gamma(B(q) - j)} - \frac{x}{(\log x)^{1-B(q^+)}} \sum_{j=0}^N \frac{\kappa_j(q^+) (\log x)^{-j}}{\Gamma(B(q^+) - j)} \\ &\quad + O\left(\frac{N! (142 c_1^{-1})^N \cdot x}{(\log x)^{(N+2)(1-\epsilon_0)-B_q}} + x \exp\left(-\sqrt{\frac{c_0 \log x}{32}}\right)\right). \end{aligned}$$

Here, q^+ is the next prime power after q , the $\kappa_j(q)$ are as defined in (7.3), and

$$(1.12) \quad B(q) = \prod_{\ell < q} \left(\frac{\ell - 2}{\ell - 1} + \frac{1}{(\ell - 1)\ell^{\lceil \log q/\log \ell \rceil - 1}} \right).$$

If there is no Siegel–zero, then all these assertions hold uniformly for $q \leq \sqrt{c_0 \log x / 128}$, and c_1 and ϵ_0 do not appear.

1.5. The Sathe–Selberg Theorem in arithmetic progressions.

The functions $\omega(n) = \#\{p \mid n\}$ and $\Omega(n) = \sum_{p \mid n} v_p(n)$ that count the number of primes dividing n without and with multiplicity (respectively), have captured the interest of prominent number theorists for decades. One of the earliest renowned results on this subject, the Hardy–Ramanujan theorem, states that either of these functions concentrates around its mean. A conceptual improvement of this is the celebrated Erdős–Kac theorem, which shows that either of these functions behaves like a Gaussian random variable with mean and variance both equal to $\log \log n$.

The local laws of these functions (namely how often $\omega(n)$ or $\Omega(n)$ takes a given value k) are found in works of L. G. Sathe [30] and A. Selberg [33], which form some of the cornerstones of probabilistic number theory. Their results roughly state that the functions $\omega(n)$ and $\Omega(n)$ locally demonstrate Poisson distribution with parameter $\log \log x$, on the interval $[0, x]$.

Natural generalizations of $\omega(n)$ and $\Omega(n)$ are the functions $\omega_a(n) := \#\{p \mid n : p \equiv a \pmod{q}\}$ and $\Omega_a(n) := \sum_{p \equiv a \pmod{q}} v_p(n)$, which count the number of primes that divide n and lie in the progression $a \pmod{q}$. Heuristic arguments suggest that both of these functions should also satisfy a “Sathe–Selberg type” law, demonstrating local Poisson behavior with parameter $\log \log x / \varphi(q)$. As our final application of Theorem 1.3 in this paper, we establish such a result in a precise quantitative form, – allowing the modulus q to vary within the Siegel–Walfisz range, – and conditionally, – in a still wider range. Our results below extend Theorems II.6.4 and II.6.5 in [38].

Theorem 1.8. Fix $K, \epsilon_0 > 0$. There exists a constant c_1 depending only on ϵ_0 and K such that uniformly in $x \geq 4$, $N \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, in $q \leq (\log x)^K$, and in coprime residues $a \pmod{q}$,

(1.13)

$$\sum_{\substack{n \leq x \\ \omega_a(n)=k}} 1 = x \sum_{j=0}^N \frac{P_{j,k}(\log \log x)}{(\log x)^{j+1/\varphi(q)}} + O \left(\frac{K^{-k} \cdot N! (71 c_1^{-1})^N \cdot x}{(\log x)^{(N+1)(1-\epsilon_0)-\frac{(K+1)}{\varphi(q)}}} + \frac{x}{K^k} \exp \left(-\sqrt{\frac{c_0 \log x}{16K}} \right) \right),$$

where $P_{j,k}$ is a polynomial of degree at most k , defined in (8.8). With $Y := \log \log x$, and with $p_{q,a}$ being the least prime in the progression $a \pmod{q}$, we have uniformly in q, k, a as above

$$(1.14) \quad P_{0,k}(Y) - \left(1 - \frac{1}{p_{q,a}} \right) \frac{(Y/\varphi(q))^k}{k!} - \frac{(Y/\varphi(q))^{k-1}}{p_{q,a} (k-1)!} \ll \begin{cases} \frac{\log q}{\varphi(q)} \cdot \frac{(Y/\varphi(q))^k}{k!}, & \text{if } k \leq \frac{KY}{\varphi(q)} \\ \frac{\log q}{\varphi(q)} \cdot \frac{e^{KY/\varphi(q)}}{K^k}, & \text{for all } k. \end{cases}$$

The implied constants in (1.13) and (1.14) depend only on c_0, c_1, K, ϵ_0 . If there is no Siegel zero, then all these assertions hold uniformly for $q \leq \exp(\sqrt{c_0 \log x / 20K})$, and c_1, ϵ_0 don’t appear.

Theorem 1.8 gives a genuine asymptotic formula for $k \ll \log \log x$, as is also the range in the usual Sathe–Selberg Theorem for the function $\omega(n)$. For $k \leq KY/\varphi(q)$, we will deduce (1.14) from a more precise estimate for $P_{0,k}(Y)$ that we will obtain using a variant of the saddle point method: See Proposition 8.2. Also note that the error term in (1.14) is $\ll \log q / K^k \varphi(q)$ if $\varphi(q) \gg KY$.

A similar result also holds for $\Omega_a(n)$; only this time we need to make sure to stay away from the singularity of the Dirichlet series $\sum_{n=1}^{\infty} z^{\Omega_a(n)}/n^s$ at $z = p_{q,a}$. This is the exact same issue as the one encountered at the prime 2 in the usual Sathe–Selberg theorem for the function $\Omega(n)$.

Theorem 1.9. *Fix $K, \epsilon_0 > 0$. There exists a constant c_1 depending only on ϵ_0 and K such that uniformly in $x \geq 4$, $N \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, in $q \leq (\log x)^K$, and in coprime residues $a \bmod q$,*

(1.15)

$$\sum_{\substack{n \leq x \\ \Omega_a(n)=k}} 1 = x \sum_{j=0}^N \frac{Q_{j,k}(\log \log x)}{(\log x)^{j+1/\varphi(q)}} + O \left(\frac{R^{-k} \cdot N! (71 c_1^{-1})^N \cdot x}{(\log x)^{(N+1)(1-\epsilon_0)-\frac{(K+1)}{\varphi(q)}}} + \frac{x}{R^k} \exp \left(-\sqrt{\frac{c_0 \log x}{16K}} \right) \right),$$

where $R := \min\{K, (1 - \epsilon_0)p_{q,a}\}$ and $Q_{j,k}$ is a polynomial of degree at most k , defined in (8.18). Moreover, the following estimates hold uniformly in q, a as above, with $Y := \log \log x$.

(i) Uniformly in $k \leq RY/\varphi(q)$, we have

$$(1.16) \quad Q_{0,k}(Y) = \frac{(Y/\varphi(q))^k}{k!} \left\{ \left(1 - \frac{1}{p_{q,a}}\right) \left(1 - \frac{k\varphi(q)}{p_{q,a}Y}\right)^{-1} + O \left(\frac{\log q}{\varphi(q)} + \frac{k}{(p_{q,a}Y/\varphi(q))^2} \right) \right\}.$$

(ii) Uniformly in $k \leq (1 - \epsilon_0)p_{q,a}Y/\varphi(q)$, we have

$$(1.17) \quad Q_{0,k}(Y) = \frac{(Y/\varphi(q))^k}{k!} \left(1 - \frac{1}{p_{q,a}}\right) + O \left(\frac{(Y/\varphi(q))^{k-1}}{(k-1)!} + \frac{\log q}{\varphi(q)} \cdot \frac{e^{RY/\varphi(q)}}{R^k} \right).$$

(iii) Uniformly in $k \geq (1 + \epsilon_0)p_{q,a}Y/\varphi(q)$, we have

$$(1.18) \quad Q_{0,k}(Y) = \frac{e^{p_{q,a}Y/\varphi(q)}}{(p_{q,a})^k} \left(1 - \frac{1}{p_{q,a}}\right) + O \left(p_{q,a} \cdot \frac{(Y/\varphi(q))^{k+1}}{(k+1)!} + \frac{\log q}{\varphi(q)} \cdot \frac{e^{RY/\varphi(q)}}{R^k} \right).$$

The implied constants in (1.15), (1.16) and (1.18) depend only on c_0, c_1, K, ϵ_0 . If there is no Siegel zero, then all these assertions hold uniformly for $q \leq \exp(\sqrt{c_0 \log x / 20K})$, and c_1, ϵ_0 don't appear.

In an upcoming sequel, we shall provide more useful variants of Theorems 1.1–1.3, and highlight several more applications.

Notation and Conventions. We do not consider zero function as multiplicative (thus, $f(1) = 1$ for any multiplicative function f). For $y > 0$, any mention of “ $\log y$ ” will always mean the natural logarithm $\ln y$. All other logarithm conventions, branch cuts and analytic continuations have been made explicit in subsection § 2.1. Any mentions of the Siegel zero η_e or the exceptional character χ_e are ignored if η_e doesn't exist, i.e., if $\prod_{\chi} L(s, \chi)$ has no zeros inside the region $\{\sigma + it : \sigma > 1 - c_0 / \log(q(|t| + 1))\}$. Implied constants in \ll and O -notation are allowed to depend on any parameters declared as “fixed”; in particular, they are always allowed to depend on K_0, c_0 and ν . Other dependence will be noted explicitly (for instance, with parentheses or subscripts). We use \log_k to denote the k -th iterate of the natural logarithm.

As is commonplace, we write complex numbers s as $\sigma + it$ (with $\sigma = \operatorname{Re}(s)$ and $t = \operatorname{Im}(s)$). We denote a generic zero of Dirichlet L -functions by $\rho = \beta + i\gamma$, where $\beta = \operatorname{Re}(\rho)$ and $\gamma = \operatorname{Im}(\rho)$; moreover, $\sum_{\rho: L(\rho, \chi)=0}^*$ denotes a sum over all zeros ρ of $L(s, \chi)$ counted with appropriate multiplicity. Other recurring notation has been defined in (1.2), (1.3), (2.2) and (2.5).

2. KEY ANALYTIC INPUTS: LOGARITHMIC DERIVATIVES, AUXILIARY FUNCTIONS, AND THE “INNER CONTOUR SHIFT”

For any $\chi \bmod q$, the function $\text{Log}L(s\nu, \chi) := \sum_{p,r \geq 1} \chi(p^r)/rp^{rs\nu}$ defines an analytic logarithm of $L(s\nu, \chi)$ on the region $\{s : \sigma > 1/\nu\}$. Hence, the function $\mathcal{F}(s\nu)$ is analytic on $\{s : \sigma > 1/\nu\}$, and (2.1)

$$\mathcal{F}(s\nu) = \prod_{\chi} L(s\nu, \chi)^{\alpha_{\chi}} = \exp \left(\sum_{\chi} \alpha_{\chi} \text{Log}L(s\nu, \chi) \right) = \exp \left(\sum_{p,r \geq 1} \frac{1}{rp^{rs\nu}} \sum_{\chi} \alpha_{\chi} \chi(p^r) \right) \quad \text{if } \sigma > 1/\nu.$$

We analytically continue our functions into regions of interest. In what follows, we define

$$(2.2) \quad \mathcal{L}_q(t) := \log(q(|t\nu| + 1)) \quad \text{and} \quad \mathcal{D}(c_0) := \left\{ \sigma + it : \sigma > \frac{1}{\nu} \left(1 - \frac{c_0}{\mathcal{L}_q(t)} \right) \right\}.$$

2.1. Analytic Continuations. Since the functions $L(s\nu, \chi_0)(s - 1/\nu)$, $L(s\nu, \chi_e)(s - \eta_e/\nu)^{-1}$, and $\{L(s\nu, \chi)\}_{\chi \neq \chi_0, \chi_e}$ all continue analytically into nonvanishing functions on $\mathcal{D}(c_0)$, they have (unique) analytic logarithms $\mathcal{T}^*(s, \chi_0)$, $\mathcal{T}^*(s, \chi_e)$, and $\{\mathcal{T}(s, \chi)\}_{\chi \neq \chi_0, \chi_e}$ on $\mathcal{D}(c_0)$ satisfying

$$\mathcal{T}^* \left(\frac{2}{\nu}, \chi_0 \right) = \sum_{p,r \geq 1} \frac{\chi_0(p^r)}{rp^{2r}} + \ln \left(\frac{2}{\nu} - \frac{1}{\nu} \right), \quad \mathcal{T}^* \left(\frac{2}{\nu}, \chi_e \right) = \sum_{p,r \geq 1} \frac{\chi_e(p^r)}{rp^{2r}} - \ln \left(\frac{2}{\nu} - \frac{\eta_e}{\nu} \right),$$

and $\mathcal{T}(2/\nu, \chi) = \sum_{p,r \geq 1} \chi(p^r)/rp^{2r}$ for all other χ . (Thus $\mathcal{T}^*(s, \chi_0)$ is analytic on $\mathcal{D}(c_0)$ and satisfies $e^{\mathcal{T}^*(s, \chi_0)} = L(s\nu, \chi_0)(s - 1/\nu)$ therein, etc.) Comparing derivatives, we see that the functions

$$(2.3) \quad \mathcal{T}(s, \chi_0) := \mathcal{T}^*(s, \chi_0) - \log \left(s - \frac{1}{\nu} \right) \quad \text{and} \quad \mathcal{T}(s, \chi_e) := \mathcal{T}^*(s, \chi_e) + \log \left(s - \frac{\eta_e}{\nu} \right)$$

define unique analytic continuations of the functions $\text{Log}L(s\nu, \chi_0)$ and $\text{Log}L(s\nu, \chi_e)$, into the regions $\mathcal{D}(c_0) \setminus (-\infty, 1/\nu]$ and $\mathcal{D}(c_0) \setminus (-\infty, \eta_e/\nu]$, respectively. (Here $\log z$ is the principal branch of the logarithm, so $\log(s - 1/\nu)$ is analytic on $\mathbb{C} \setminus (-\infty, 1/\nu]$.) From this discussion, we see that the function $\exp(\sum_{\chi} \alpha_{\chi} \mathcal{T}(s, \chi)) = \prod_{\chi} e^{\alpha_{\chi} \mathcal{T}(s, \chi)}$ defines a unique analytic continuation of $\mathcal{F}(s\nu)$ in (2.1) into $\mathcal{D}(c_0) \setminus (-\infty, 1/\nu]$; hence, $\mathcal{F}(s\nu) = \exp(\sum_{\chi} \alpha_{\chi} \mathcal{T}(s, \chi))$ for all s in this region.

Note also that by the first equality in (2.1) and by analytic continuation, we may write

$$(2.4) \quad \frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} = \sum_{\chi} \alpha_{\chi} \frac{L'(s\nu, \chi)}{L(s\nu, \chi)} \quad \text{for all } s \neq 1/\nu \text{ s.t. } s \neq \rho/\nu \text{ for any complex zero } \rho \text{ of } \prod_{\chi} L(s, \chi).$$

This relation is consistent with the analytic continuation of $\mathcal{F}(s\nu)$ in the previous paragraph. We will also need some auxiliary functions. By the above discussion (especially that around (2.3)), we see that if η_e exists, then the function $s^{-1} \exp \left(\alpha_{\chi_0} \mathcal{T}^*(s, \chi_0) + \alpha_{\chi_e} \mathcal{T}^*(s, \chi_e) + \sum_{\chi \neq \chi_0, \chi_e} \alpha_{\chi} \mathcal{T}(s, \chi) \right)$ analytically continues the function $s^{-1} \mathcal{F}(s\nu)(s - 1/\nu)^{\alpha_{\chi_0}} (s - \eta_e/\nu)^{-\alpha_{\chi_e}}$ into the region $\mathcal{D}(c_0)$.

On the other hand, if η_e doesn't exist (i.e. all zeros of $\prod_{\chi} L(s, \chi)$ lie outside $\mathcal{D}(c_0)$), then we can define $\mathcal{T}(s, \chi_e)$ exactly as we defined the functions $\{\mathcal{T}(s, \chi)\}_{\chi \neq \chi_0, \chi_e}$: In this case, $\mathcal{T}(s, \chi_e)$ is analytic on $\mathcal{D}(c_0)$, so that the function $s^{-1} \exp \left(\alpha_{\chi_0} \mathcal{T}^*(s, \chi_0) + \sum_{\chi \neq \chi_0} \alpha_{\chi} \mathcal{T}(s, \chi) \right)$ analytically continues $s^{-1} \mathcal{F}(s\nu)(s - 1/\nu)^{\alpha_{\chi_0}}$ into $\mathcal{D}(c_0)$. Finally if $\eta_e \leq 1 - c_0/10\lambda_q \log q$, then $\mathcal{T}(s, \chi_e)$ is analytic on the smaller region $\mathcal{D}(c_0/10\lambda_q) = \{\sigma + it : \sigma > \nu^{-1}(1 - c_0/10\lambda_q \mathcal{L}_q(t))\}$; as such, the function $s^{-1} \mathcal{F}(s\nu)(s - 1/\nu)^{\alpha_{\chi_0}}$ continues analytically into $\mathcal{D}(c_0/10\lambda_q)$.

The reader may now forget the \mathcal{T} and \mathcal{T}^* . All that we will need from subsection § 2.1 are identities (2.1) and (2.4), that $\mathcal{F}(s\nu)$ *always* continues analytically into $\mathcal{D}(c_0) \setminus (-\infty, 1/\nu]$, and that

$$(2.5) \quad \mathcal{H}(s) := \begin{cases} \frac{\mathcal{F}(s\nu)}{s} \left(s - \frac{1}{\nu}\right)^{\alpha_{x_0}} \left(s - \frac{\eta_e}{\nu}\right)^{-\alpha_{x_e}} \text{ cont. an. into } \mathcal{D}(c_0), & \text{if } \eta_e > 1 - \frac{c_0}{10\lambda_q \log q}. \\ \frac{\mathcal{F}(s\nu)}{s} \left(s - \frac{1}{\nu}\right)^{\alpha_{x_0}} \text{ cont. an. into } \mathcal{D}(c_0), & \text{if } \eta_e \text{ doesn't exist.} \\ \frac{\mathcal{F}(s\nu)}{s} \left(s - \frac{1}{\nu}\right)^{\alpha_{x_0}} \text{ cont. an. into } \mathcal{D}\left(\frac{c_0}{10\lambda_q}\right), & \text{if } \eta_e \leq 1 - \frac{c_0}{10\lambda_q \log q}. \end{cases}$$

Here “cont. an.” abbreviates “continues analytically”. In what follows, we will call the three cases above as **Case 1**, **Case 2** and **Case 3** respectively. (It is *not* necessary to assume that $\eta_e > 1 - c_0/10\lambda_q \log q$ for the function $s^{-1}\mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{x_0}}(s-\eta_e/\nu)^{-\alpha_{x_e}}$ to continue analytically into $\mathcal{D}(c_0)$, however we do so in order to make this case trichotomy convenient for future use.)

2.2. Analysis of Logarithmic Derivatives. To give suitable bounds on $\mathcal{F}(s\nu)$, we will first analyze its logarithmic derivative. To this end, the following known results on Dirichlet L -functions will be useful. Recall that we write $\rho = \beta + i\gamma$ where $\beta = \operatorname{Re}(\rho)$ and $\gamma = \operatorname{Im}(\rho)$. We denote by $\sum_{\rho: L(\rho, \chi)=0}^*$ a sum over all zeros ρ of $L(s, \chi)$ counted with appropriate multiplicity.

Lemma 2.1. *The following hold uniformly in $q \geq 2$ and in all Dirichlet characters $\chi \bmod q$.*

(1) *Uniformly in all real t , we have* $\sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1}}^* \frac{1}{1 + (t - \gamma)^2} \ll \log(q(|t| + 1))$.

(2) *Uniformly in all complex s satisfying $\sigma \in [-1, 2]$, $|t| \geq 2$, and $t \neq \gamma$ for any of the zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$, we have* $\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1, |\gamma-t| \leq 1}}^* \frac{1}{s - \rho} + O(\log(q(|t| + 1)))$.

(3) *We have $L'(s, \chi)/L(s, \chi) \ll \log(q|s|)$, uniformly in all complex s satisfying $\sigma \leq -1$ and lying outside the disks of radius $1/4$ about the trivial zeros of $L(s, \chi)$.*

(4) *Uniformly in real $t \notin (-1, 1)$, we have $\#\{\rho : 0 \leq \beta \leq 1, |\gamma - t| \leq 1, L(\rho, \chi) = 0\} \ll \log(q|t|)$.*

In most standard texts (such as [4, 22, 38]), these results are stated and proved only for primitive characters, however the generality above will be helpful here.

We now give a certain (absolutely convergent) series expansion for the logarithmic derivative of $\mathcal{F}(s\nu)$ in terms of the zeros of the L -functions, with coefficients that are easy to control.

Proposition 2.2. *For any $s \in \mathbb{C}$ satisfying $s \neq 1/\nu$ and $s \neq \rho/\nu$ for any zero ρ of $\prod_{\chi} L(s, \chi)$,*

$$(2.6) \quad \frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} = \sum_{n \leq \xi^2} \frac{\varrho(n)\Lambda(n)}{n^{s\nu}} \tau(n) - \frac{\alpha_{x_0}(\xi^{1-\nu s} - \xi^{2(1-\nu s)})}{(1-\nu s)^2 \log \xi} + \sum_{\chi \bmod q} \sum_{\rho: L(\rho, \chi)=0}^* \frac{\alpha_{\chi}(\xi^{\rho-\nu s} - \xi^{2(\rho-\nu s)})}{(\rho - \nu s)^2 \log \xi},$$

where $\xi := e^{6\mathcal{L}_q(t)}$, $\varrho(n) := \sum_{\chi \bmod q} \alpha_{\chi} \cdot \chi(n)$, and $\tau(n) := \mathbb{1}_{n \leq \xi} + \mathbb{1}_{\xi < n \leq \xi^2} (2 - \log n / \log \xi)$.

Proof. Our starting point is the following identity, which holds for any $b, y > 0$

$$(2.7) \quad \int_{b-i\infty}^{b+i\infty} \frac{y^z}{z^2} dz = \mathbb{1}_{y>1} \cdot 2\pi i \log y$$

To see this, consider any $R \geq 2$, apply the residue theorem to the contour consisting of the vertical segment $[b - iR, b + iR]$ and the *major* arc of the circle centered at the origin passing through $b \pm iR$ if $y > 1$ (resp. *minor* arc for $y \leq 1$), and then let $R \rightarrow \infty$. Now from $L'(s, \chi)/L(s, \chi) = -\sum_n \chi(n)\Lambda(n)/n^s$ and (2.4), we see that $\mathcal{F}'(z\nu)/\mathcal{F}(z\nu)$ has Dirichlet series $\sum_n \varrho(n)\Lambda(n)/n^{z\nu}$ on $\{z \in \mathbb{C} : \operatorname{Re}(z) > 1/\nu\}$. We claim that for all s as in the statement of the proposition,

$$(2.8) \quad \frac{1}{2\pi i} \int_{\frac{2}{\nu}+|s|-i\infty}^{\frac{2}{\nu}+|s|+i\infty} \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)} dz = \nu \sum_{n \leq \xi^2} \frac{\varrho(n)\Lambda(n)}{n^{s\nu}} \tau(n) \log \xi.$$

Indeed by (2.7), the above identity is immediate if $\mathcal{F}'(z\nu)/\mathcal{F}(z\nu)$ were replaced by any finite truncation $\sum_{n \leq Y} \varrho(n)\Lambda(n)/n^{z\nu}$ of its aforementioned Dirichlet series (for any $Y > \xi^6$). Moreover by the same Dirichlet series, the size of the integrand above is at most $2\lambda_q \xi^{4+2\nu|s|} (\sum_n \Lambda(n)/n^2) |z-s|^{-2}$, which is an L^1 -function of z since $\int_{\frac{2}{\nu}+|s|-i\infty}^{\frac{2}{\nu}+|s|+i\infty} |dz|/|z-s|^2 < \infty$ and $\sum_n \Lambda(n)/n^2 \ll 1$. Hence (2.8) follows from the Dominated Convergence Theorem.

We will now shift contours: This is the “inner contour shift” alluded to in the introduction. Note that for any $M \geq 2$, the number of zeros of $\prod_\chi L(s, \chi)$ in the rectangle $[0, 1] \times (M, M+1]$ is $\ll \varphi(q) \log(qM)$ by Lemma 2.1(4). Hence there exists $T_M \in (M, M+1]$ satisfying $|T_M - \gamma| \gg (\varphi(q) \log(qM))^{-1}$ for all zeros $\rho = \beta + i\gamma$ of $\prod_\chi L(s, \chi)$. Since the set of zeros of $\prod_\chi L(s, \chi)$ is closed under complex conjugation, we have

$$(2.9) \quad |T_M \pm \gamma| \gg (\varphi(q) \log(qM))^{-1} \text{ for all zeros } \rho = \beta + i\gamma \text{ of } \prod_\chi L(s, \chi).$$

With the contour ω_M as in Figure 1, we claim that

$$(2.10) \quad \frac{L'(z\nu, \chi)}{L(z\nu, \chi)} \ll \varphi(q) \log^2(qM), \text{ uniformly in } q \geq 3, \chi \bmod q, M \geq 2(1 + \nu + \nu|s|), z \in \omega_M.$$

If $\operatorname{Re}(z) \geq 2/\nu$, this follows from the Dirichlet series of $L'(z\nu, \chi)/L(z\nu, \chi)$. If $\operatorname{Re}(z) \in [-1/\nu, 2/\nu]$, then z must lie on the two horizontal segments in ω_M , so that by (2.9), we have $|z\nu - \rho| \geq |\operatorname{Im}(z)\nu - \gamma| = |T_M \pm \gamma| \gg (\varphi(q) \log(qM))^{-1}$ for any zero $\rho = \beta + i\gamma$ of $\prod_\chi L(s, \chi)$. This gives (2.10) by Lemma 2.1(2) and (4). Lastly if $\operatorname{Re}(z) \leq -1/\nu$, then Lemma 2.1(3) establishes (2.10).

Now for any $M \geq 2\nu|s|$ and any $z \in \omega_M$, we have $|z-s| \geq |z| - |s| \geq |z|/2 \geq M/2\nu$. As such $\int_{\omega_M} |dz|/|z-s|^2 \ll_{\nu, s} \int_{M/2\nu}^\infty dt/t^2 + (M/2\nu)^{-2} \cdot M \ll M^{-1}$, so that (2.4) and (2.10) yield

$$(2.11) \quad \lim_{M \rightarrow \infty} \int_{\omega_M} \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)} dz = 0.$$

Using the residue theorem to shift contours from the vertical line in (2.8) to ω_M , and then letting $M \rightarrow \infty$, we thus find from (2.11), (2.8) and (2.4) that

$$(2.12) \quad \nu \sum_{n \leq \xi^2} \frac{\varrho(n)\Lambda(n)}{n^{z\nu}} \tau(n) \log \xi = \left(\underset{z=s}{\operatorname{Res}} + \underset{z=1/\nu}{\operatorname{Res}} + \sum_{\rho: \prod_\chi L(\rho, \chi)=0} \underset{z=\rho/\nu}{\operatorname{Res}} \right) \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)}.$$

Finally, using (2.4) to compute the above residues, we obtain the proposition. For instance, note that if $\xi^{\rho-\nu s} \neq 1$ for some ρ above, then (2.4) shows that $z = \rho/\nu$ is a simple pole of the function on the right of (2.12) of residue $\nu(\xi^{\rho-\nu s} - \xi^{2(\nu-\rho s)})(\rho - \nu s)^{-2} \sum_{\chi} \alpha_{\chi} \cdot \{\text{multiplicity of } \rho \text{ in } L(s, \chi)\}$. If $\xi^{\rho-\nu s} = 1$, then $z = \rho/\nu$ is a removable singularity, so we can still give the same expression (whose value is zero) for its “residue”. The residue at $z = 1/\nu$ can be computed analogously, and the residue at $z = s$ (which is always necessarily a simple pole) is equal to $-\nu(\log \xi)\mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$. \square

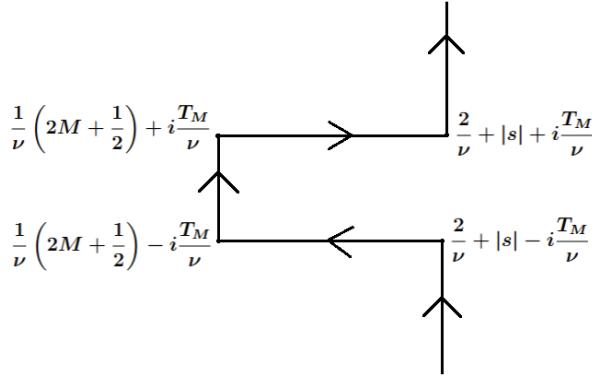


FIGURE 1. The Contour ω_M

We will now use the series representation in Proposition 2.2 to give a suitable bound on $\mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$. A crucial input will be provided the following zero density estimate. In what follows, we define

$$N(\theta, t) := \sum_{\chi} \sum_{\substack{\rho: L(\rho, \chi)=0 \\ \theta \leq \beta \leq 1, |\gamma| \leq t}}^* 1.$$

Lemma 2.3. *We have $N(\theta, t) \ll (qt)^{3(1-\theta)}$, uniformly in $q \geq 3$, $\theta \in [1/2, 1]$, and $t \geq 1$.*

This may be found in works of Heath-Brown [13] and Jutila [17]. (See also the classical text of Iwaniec and Kowalski [16].) We now state the bound alluded to above.

Proposition 2.4. *Uniformly in $q \geq 3$, and in complex numbers s satisfying $\sigma \geq \nu^{-1}(1 - c_0/2\mathcal{L}_q(t))$,*

$$\left| \frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} + \frac{\alpha_{\chi_0}}{s\nu - 1} - \frac{\alpha_{\chi_e}}{s\nu - \eta_e} \right| \ll \lambda_q \mathcal{L}_q(t).$$

The term “ $\alpha_{\chi_e}/(s\nu - \eta_e)$ ” above is omitted if η_e doesn’t exist.

Proof. Most of the argument consists of carefully bounding the different components of the right of (2.6). First, for all $n \leq \xi^2$, we have $|n^{s\nu}| = n^{\sigma\nu} \geq n^{1-c_0/2\mathcal{L}_q(t)} \geq n \exp(-2 \log \xi / 2\mathcal{L}_q(t)) \gg n$, so that the first sum on the right in (2.6) is $\ll \lambda_q \sum_{n \leq \xi^2} \Lambda(n)/n \ll \lambda_q \mathcal{L}_q(t)$ by Mertens’ Theorem.

Next, since the trivial zeros of any $L(s, \chi)$ are simple, the total contribution of all zeros $\{-r/\nu\}_{r \in \mathbb{N}}$ to the right of (2.6) equals $(\log \xi)^{-1} \sum_{r \geq 1} \left(\sum_{\chi: \chi(-1) = (-1)^r} \alpha_{\chi} \right) (\xi^{-(r+\nu s)} - \xi^{-2(r+\nu s)})(r + \nu s)^{-2}$. This expression is $\ll \lambda_q (\log \xi)^{-1} \sum_{r \geq 1} \xi^{-(r+\nu s)} (r + \nu s)^{-2} \ll \lambda_q \mathcal{L}_q(t)^{-1} \sum_{r \geq 1} r^{-2} \ll \lambda_q \mathcal{L}_q(t)^{-1}$, where we noted that $|\sum_{\chi: \chi(-1) = (-1)^r} \alpha_{\chi}| = |\sum_{\chi} \alpha_{\chi} (1 + \chi(-1)(-1)^r)/2| = |\varrho(1) + \varrho(-1)|/2 \leq \lambda_q$.

Now, we observe that $|(\xi^{\theta-\nu s} - \xi^{2(\theta-\nu s)})(\theta - \nu s)^{-2}(\log \xi)^{-1} - (\nu s - \theta)^{-1}| \ll \mathcal{L}_q(t)$ uniformly in $\theta \in (0, 1]$ and s as in the proposition. This follows by a straightforward crude bounding if $|\theta - \nu s| > (\log \xi)^{-1}$, and by the formula $\xi^{\theta-\nu s} = 1 - (\theta - \nu s) \log \xi + O((\theta - \nu s)^2(\log \xi)^2)$ if $|\theta - \nu s| \leq (\log \xi)^{-1}$. Collecting all the observations made so far, we see that this proposition would follow from (2.6), once we show that uniformly in all s with $\sigma \geq \nu^{-1}(1 - c_0/2\mathcal{L}_q(t))$,

$$(2.13) \quad \frac{1}{\mathcal{L}_q(t)} \sum_{\chi} |\alpha_{\chi}| \sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1, \rho \neq \eta_e}}^* \frac{\xi^{\beta-\nu\sigma} + \xi^{2(\beta-\nu\sigma)}}{(\beta - \nu\sigma)^2 + (\gamma - \nu t)^2} \ll \lambda_q \mathcal{L}_q(t).$$

To show this, we start by bounding the entire expression above by $S_1 + S_2 + S_3 + S_4$, where

- S_1 denotes the total contribution of all ρ having $\beta \leq 1/2$, so that

$$S_1 = \frac{1}{\mathcal{L}_q(t)} \sum_{\chi} |\alpha_{\chi}| \sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1/2}}^* \frac{\xi^{\beta-\nu\sigma} + \xi^{2(\beta-\nu\sigma)}}{(\beta - \nu\sigma)^2 + (\gamma - \nu t)^2}.$$

- S_2 denotes the total contribution of all ρ having $\beta \in (1/2, 1]$ and $|\gamma| \leq 2|t\nu| + 1$.
- S_3 denotes the total contribution of all ρ having $\beta \in (1/2, \sigma\nu]$ and $|\gamma| > 2|t\nu| + 1$.
- S_4 denotes the total contribution of all ρ having $\beta \in (\sigma\nu, 1]$.

For any ρ appearing in S_1 , we have $\beta - \nu\sigma \leq 1/2 - (1 - c_0/2\mathcal{L}_q(t)) \leq -1/2 + 1/2 \log q \leq -1/3$, so that $(\beta - \nu\sigma)^2 + (\gamma - \nu t)^2 \geq (1 + (\gamma - \nu t)^2)/9$. Hence (1.4) and Lemma 2.1(1) yield $S_1 \ll \lambda_q \xi^{1/2-\nu\sigma} \mathcal{L}_q(t)^{-1} \sum_{\chi} \sum_{\rho} (1 + (\gamma - \nu t)^2)^{-1} \ll \lambda_q \cdot q \xi^{1/2-\nu\sigma} \ll \lambda_q \cdot q \xi^{-1/2} \cdot \xi^{c_0/2\mathcal{L}_q(t)} \ll \lambda_q$.

For any ρ appearing in S_3 , we have $\beta - \nu\sigma \leq 0$ and $|\gamma - t\nu| \geq |\gamma| - |t\nu| \geq |\gamma|/2$. Thus by (1.4),

$$S_3 \leq \frac{8\lambda_q}{\mathcal{L}_q(t)} \sum_{\chi} \sum_{\substack{\rho \neq \eta_e: L(\rho, \chi)=0 \\ |\gamma| > 2|t\nu| + 1, 1/2 < \beta \leq \min\{\sigma\nu, 1\}}}^* \xi^{\beta-\nu\sigma} \cdot |\gamma|^{-2}.$$

Partitioning the interval $(1/2, \min\{\sigma\nu, 1\}]$ into $R := \lfloor \log \xi/2 \rfloor$ equally spaced intervals, we obtain

$$(2.14) \quad S_3 \leq \frac{8\lambda_q}{\mathcal{L}_q(t)} \sum_{r=1}^R \xi^{1/2+r\mu_0/R-\nu\sigma} \sum_{\chi} \sum_{\substack{\rho \neq \eta_e: L(\rho, \chi)=0, |\gamma| > 2|t\nu| + 1 \\ 1/2+(r-1)\mu_0/R < \beta \leq 1/2+r\mu_0/R}}^* |\gamma|^{-2},$$

where $\mu_0 := \min\{\sigma\nu, 1\} - 1/2$. Now the inner double sum (on χ and ρ) above is at most

$$(2.15) \quad \int_{2|t\nu|+1}^{\infty} \frac{dN\left(\frac{1}{2} + \frac{(r-1)\mu_0}{R}, u\right)}{u^2} \ll \int_{2|t\nu|+1}^{\infty} \frac{N\left(\frac{1}{2} + \frac{(r-1)\mu_0}{R}, u\right)}{u^3} du \ll q^{3(1/2-(r-1)\mu_0/R)}$$

where we have used the Stieltjes integration by parts and Lemma 2.3. The last expression above is $\ll \xi^{1/4-(r-1)\mu_0/2R} \ll \xi^{1/4-r\mu_0/2R}$, as $\mu_0 \leq 1/2$, $\xi \geq q^6$ and $R \geq \log \xi/3$. Inserting these into (2.14), we get $S_3 \ll \lambda_q \mathcal{L}_q(t)^{-1} \xi^{3/4-\nu\sigma} \sum_{r=1}^R \xi^{r\mu_0/2R} \leq \lambda_q \mathcal{L}_q(t)^{-1} \xi^{3/4-\nu\sigma} \cdot R \xi^{\mu_0/2} \ll \lambda_q \xi^{1-\nu\sigma} \ll \lambda_q$.

Next, for any ρ counted in S_2 , we have $|\gamma| \leq 2|t\nu| + 1$ and $\rho \neq \eta_e$, so that $\beta \leq 1 - c_0/\log(q(|\gamma| + 1)) \leq 1 - c_0/\log(2q(|t\nu| + 1))$. Since $\nu\sigma \geq 1 - c_0/2\mathcal{L}_q(t)$, we get

$$\nu\sigma - \beta \geq c_0 \left(\frac{1}{\log(2q(|t\nu| + 1))} - \frac{1}{2\log(q(|t\nu| + 1))} \right) = \frac{c_0}{\mathcal{L}_q(t)} \left(1 - \frac{\log 4}{\log(2q(|t\nu| + 1))} \right) \geq \frac{c_0}{10\mathcal{L}_q(t)}.$$

Hence $(\beta - \nu\sigma)^2 \gg \mathcal{L}_q(t)^{-2}$. Proceeding as in (2.15) (via Lemma 2.3 and integration by parts),

$$\begin{aligned} S_2 &\ll \lambda_q \mathcal{L}_q(t) \sum_{\chi} \sum_{\substack{\rho \neq \eta_e: L(\rho, \chi)=0 \\ 1/2 < \beta \leq 1, |\gamma| \leq 2|t\nu|+1}}^* \xi^{\beta-\nu\sigma} \leq \lambda_q \mathcal{L}_q(t) \left(- \int_{1/2}^1 \xi^{\theta-\nu\sigma} dN(\theta, 2|t\nu|+1) \right) \\ &\leq \lambda_q \mathcal{L}_q(t) \left(\xi^{1/2-\nu\sigma} N(1/2, 2|t\nu|+1) + \log \xi \int_{1/2}^1 \xi^{\theta-\nu\sigma} N(\theta, 2|t\nu|+1) d\theta \right) \\ &\ll \lambda_q \mathcal{L}_q(t) \left(\xi^{3/4-\nu\sigma} + \log \xi \int_{1/2}^1 \xi^{(1+\theta)/2-\nu\sigma} d\theta \right) \ll \lambda_q \mathcal{L}_q(t) \xi^{1-\nu\sigma} \ll \lambda_q \mathcal{L}_q(t). \end{aligned}$$

For any ρ ($\neq \eta_e$) in S_4 , we have $1 - c_0/\log(q(|\gamma|+1)) \geq \beta > \sigma\nu \geq 1 - c_0/2\mathcal{L}_q(t)$, giving $|\gamma| > q(|t\nu|+1)^2 - 1$. Thus also $|\gamma - t\nu| \geq |\gamma| - |t\nu| \geq |\gamma|/2$. Proceeding exactly as we did for S_3 ,

$$S_4 \ll \frac{\lambda_q \xi^{2(1-\nu\sigma)}}{\mathcal{L}_q(t)} \sum_{\chi} \sum_{\substack{\rho: L(\rho, \chi)=0 \\ \sigma\nu < \beta < 1, |\gamma| > q(|t\nu|+1)^2 - 1}}^* |\gamma|^{-2} \ll \frac{\lambda_q}{\mathcal{L}_q(t)} \int_{q(|t\nu|+1)^2 - 1}^{\infty} \frac{dN(\sigma\nu, u)}{u^2} \ll \lambda_q.$$

Collecting all these estimates establishes (2.13), completing the proof of the proposition. \square

As a consequence of Proposition 2.4, we find that the size of the function $\mathcal{H}(s)$ (in (2.5)) remains roughly constant along *short horizontal* segments lying in the region $\sigma \geq \nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t))$.

Corollary 2.5. *In Cases 1 and 2 of (2.5), we have $\mathcal{H}(s) \asymp \mathcal{H}(w)$, uniformly in $s, w \in \mathbb{C}$ having $t = \text{Im}(s) = \text{Im}(w)$ and $\nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t)) \leq \text{Re}(s) \leq \text{Re}(w) \leq \nu^{-1}(1 + 1000/c_0 \lambda_q \mathcal{L}_q(t))$.*

In Case 3, the same assertion holds with $\nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t))$ replaced by $\nu^{-1}(1 - c_0/20\lambda_q \mathcal{L}_q(t))$.

Proof. In Cases 1 and 2 of (2.5), Proposition 2.4 directly gives $|\mathcal{H}'(z)/\mathcal{H}(z)| \ll \lambda_q \mathcal{L}_q(t)$ for all z satisfying $\text{Re}(z) \geq \nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(\text{Im}(z)))$. In Case 3, we just need note that for all z satisfying $\text{Re}(z) \geq \nu^{-1}(1 - c_0/20\lambda_q \mathcal{L}_q(\text{Im}(z)))$, we have $|z\nu - \eta_e| \geq \text{Re}(z)\nu - \eta_e \geq c_0/20\lambda_q \log q$, so that Proposition 2.4 still gives $|\mathcal{H}'(z)/\mathcal{H}(z)| \ll \lambda_q \mathcal{L}_q(t)$ for all such z . The corollary now follows by writing $\log |\mathcal{H}(w)/\mathcal{H}(s)| \leq \int_{\text{Re}(s)}^{\text{Re}(w)} |\mathcal{H}'(u+it)/\mathcal{H}(u+it)| du$ for all s, w as in the statement. \square

2.3. Bounds on $\mathcal{H}(s)$ and $\mathcal{F}(s\nu)$. Corollary 2.5 allows us to study the values of $\mathcal{H}(s)$ for $\text{Re}(s) \leq 1/\nu$ using the values of $\mathcal{H}(w)$ for $\text{Re}(w) > 1/\nu$. The benefit of this is that on this right half plane, $\mathcal{F}(w\nu)$ has an *explicit expression* (2.1) as the exponential of a Dirichlet series. This maneuver allows us to bound the sizes of both $\mathcal{H}(s)$ and $\mathcal{F}(s\nu)$ in suitable regions, setting the stage for the “outer contour shift” part of the “nested contour shift” alluded to in the introduction.

Proposition 2.6. *The following assertions hold in Cases 1 and 2 of (2.5), with all terms involving η_e or χ_e omitted in Case 2 (when η_e doesn’t exist).*

(1) *We have $\mathcal{H}(s) \ll (2\lambda_q \log q)^{\lambda_q + |\text{Re}(\alpha_{x_0})| + |\text{Re}(\alpha_{x_e})|}$, uniformly in moduli $q > 4e^{1/\nu}$ and in complex numbers $s = \sigma + it$ satisfying $\sigma \geq \nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t))$ and $|t| \leq c_0/2\nu\lambda_q \log q$.*

(2) *We have $\mathcal{F}(s\nu) \ll (5\lambda_q \mathcal{L}_q(t)/4)^{\lambda_q}$, uniformly in moduli $q > e^{4+1/\nu}$ and in $s = \sigma + it$ satisfying $\nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t)) \leq \sigma \leq \nu^{-1}(1 + 100/c_0 \lambda_q \mathcal{L}_q(t))$ and $\min\{|s-1/\nu|, |s-\eta_e/\nu|\} \geq c_0/40\nu\lambda_q \log q$.*

In Case 3 of (2.5), these two assertions also hold, with all terms involving η_e or χ_e omitted, and with all occurrences of “ $\nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t))$ ” replaced by “ $\nu^{-1}(1 - c_0/20\lambda_q \mathcal{L}_q(t))$ ”.

Proof. We only give the argument in Case 1, since the arguments in the other two cases are essentially contained in it. Define $\mu(y) := \nu^{-1}(1 + 4/5\lambda_q \mathcal{L}_q(y))$ for any $y \in \mathbb{R}$.¹ By (2.5) and Corollary 2.5, we have, uniformly in all $s = \sigma + it$ satisfying $\sigma \geq \nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t))$,

$$(2.16) \quad \mathcal{H}(s) \ll |\mathcal{H}(\mu(t) + it)| \ll \frac{|\mathcal{F}(\nu(\mu(t) + it))|}{|\mu(t) + it|} \cdot \left| \mu(t) + it - \frac{1}{\nu} \right|^{\operatorname{Re}(\alpha_{x_0})} \cdot \left| \mu(t) + it - \frac{\eta_e}{\nu} \right|^{-\operatorname{Re}(\alpha_{x_e})}.$$

Now since $\operatorname{Re}(\mu(y) + iy) > 1/\nu$, it follows from (2.1) and $|\sum_\chi \alpha_\chi \cdot \chi(p^r)| \leq \lambda_q$ that

$$(2.17) \quad |\mathcal{F}(\nu(\mu(y) + iy))| \leq \exp \left(\lambda_q \sum_{p,r \geq 1} \frac{1}{p^{r\nu\mu(y)}} \right) = \exp \left(\lambda_q \log \zeta(\nu\mu(y)) \right) \ll \left(\frac{5}{4} \lambda_q \mathcal{L}_q(y) \right)^{\lambda_q}.$$

Here the last bound uses that since $\log(\nu\mu(y)) = \log(1 + 4/5\lambda_q \mathcal{L}_q(y)) \leq 4/5\lambda_q \mathcal{L}_q(y) \leq 4/5\lambda_q \log q \leq 1/\lambda_q$, we have $\lambda_q \log \zeta(\nu\mu(y)) \leq \lambda_q \log(\nu\mu(y)) - \lambda_q \log(\nu\mu(y) - 1) \leq 1 + \lambda_q \log(5\lambda_q \mathcal{L}_q(y)/4)$.

Inserting (2.17) into (2.16), we obtain, uniformly in all $s = \sigma + it$ with $\sigma \geq \nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t))$,

$$(2.18) \quad \mathcal{H}(s) \ll \frac{(5\lambda_q \mathcal{L}_q(t)/4)^{\lambda_q}}{|\mu(t) + it|} \cdot \left| \mu(t) + it - \frac{1}{\nu} \right|^{\operatorname{Re}(\alpha_{x_0})} \cdot \left| \mu(t) + it - \frac{\eta_e}{\nu} \right|^{-\operatorname{Re}(\alpha_{x_e})}.$$

Completing the proof of subpart (1). We now observe that $|\mu(t) + it - 1/\nu|$ and $|\mu(t) + it - \eta_e/\nu|$ both lie in the interval $(4/5\nu\lambda_q \mathcal{L}_q(t), 1)$: Indeed, the lower bounds are immediate by definition of $\mu(t)$, while the upper bounds follow from the facts that $\eta_e > 1 - c_0/10\lambda_q \log q$, that $q > e^{1/\nu}$ and $|t| \leq c_0/2\nu\lambda_q \log q$ (by the assumptions in Case 1 in (2.5) and in the statement of subpart (1)).

This observation yields $|\mu(t) + it - 1/\nu|^{\operatorname{Re}(\alpha_{x_0})} \leq |\mu(t) + it - 1/\nu|^{-|\operatorname{Re}(\alpha_{x_0})|} \leq (5\nu\lambda_q \mathcal{L}_q(t)/4)^{|\operatorname{Re}(\alpha_{x_0})|}$, and likewise $|\mu(t) + it - \eta_e/\nu|^{-\operatorname{Re}(\alpha_{x_e})} \leq (5\nu\lambda_q \mathcal{L}_q(t)/4)^{|\operatorname{Re}(\alpha_{x_e})|}$. Inserting these two bounds into (2.18), we get $\mathcal{H}(s) \ll (5\lambda_q \mathcal{L}_q(t)/4)^{\lambda_q + |\operatorname{Re}(\alpha_{x_0})| + |\operatorname{Re}(\alpha_{x_e})|}$. Finally, $\mathcal{L}_q(t) = \log(q(|t\nu| + 1)) < \log(q(c_0/2 + 1)) \leq \log(9q/8) < (8/5) \log q$. (Here we just used $|t| < c_0/2\nu$, $c_0 < 1/4$, and $q > 2$.)

Completing the proof of subpart (2). By (2.5) and (2.18), we have

$$(2.19) \quad \mathcal{F}(s\nu) \ll \left(\frac{5}{4} \lambda_q \mathcal{L}_q(t) \right)^{\lambda_q} \cdot \frac{|s|}{|\mu(t) + it|} \cdot \left| \frac{\mu(t) + it - 1/\nu}{s - 1/\nu} \right|^{\operatorname{Re}(\alpha_{x_0})} \cdot \left| \frac{s - \eta_e/\nu}{\mu(t) + it - \eta_e/\nu} \right|^{\operatorname{Re}(\alpha_{x_e})}$$

uniformly in all s with $\sigma \geq \nu^{-1}(1 - c_0/2\lambda_q \mathcal{L}_q(t))$. Now for s as in subpart (2), we have $|s| \leq \sqrt{4/\nu^2 + t^2} \ll |\mu(t) + it|$, as well as $\mu(t) - \sigma \ll 1/\lambda_q \mathcal{L}_q(t)$, and $\sigma - \mu(t) \leq \sigma - 1/\nu \ll 1/\lambda_q \mathcal{L}_q(t)$, and $\min\{|s - 1/\nu|, |s - \eta_e/\nu|\} \gg 1/\lambda_q \mathcal{L}_q(t)$. These last three inequalities give, for both $\theta \in \{1, \eta_e\}$,

$$\max \left\{ \left| \frac{\mu(t) + it - \theta/\nu}{s - 1/\nu} \right|, \left| \frac{s - \theta/\nu}{\mu(t) + it - 1/\nu} \right| \right\} \leq 1 + \max \left\{ \left| \frac{\mu(t) - \sigma}{s - \theta/\nu} \right|, \left| \frac{\mu(t) - \sigma}{\mu(t) + it - \theta/\nu} \right| \right\} \ll 1,$$

Inserting all these observations into (2.19) completes the proof of the proposition. \square

3. THE LSD METHOD FOR L -FUNCTIONS UNDER AVERAGE GROWTH CONDITIONS: PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

In this section, we will show the following generalizations of Theorem 1.1 and Corollary 1.2.

¹Not to be confused with the Möbius function, which makes no appearance here.

Theorem 3.1. *Under the conditions of Theorem 1.1, the following estimates hold, with the same uniformity as in Theorem 1.1, but also uniformly in $T \geq 1$ satisfying $c_0 \lambda_q \mathcal{L}_q(T) \leq 100 \log x$.*

(1) *If η_e exists and satisfies $1 - c_0/10\lambda_q \log q < \eta_e < 1 - 3\nu/\log x$ then the left side of (1.6) is*

$$(3.1) \quad \ll \frac{x^{1+1/\nu} \log x}{Th} + \left(\frac{5}{4} \lambda_q \mathcal{L}_q(T) \right)^{\lambda_q} \Omega(T) \left\{ \frac{x^{1/\nu}}{T \log x} + x^{1/\nu - c_0/4\nu \lambda_q \mathcal{L}_q(T)} (\log T) \right\} \\ + \sum_{x < n \leq x+h} |a_n| + \frac{\Omega(1) \cdot N! (71(1+\nu))^N \cdot 2^{\lambda_q} (\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot x^{1/\nu}}{(1 - \eta_e)^{N+1+|\operatorname{Re}(\alpha_{\chi_e})|} \cdot (\log x)^{1-|\operatorname{Re}(\alpha_{\chi_0})|} \cdot \min\{x/h, (\log x)^{N+1}\}}.$$

(2) *If η_e does not exist or satisfies $\eta_e \leq 1 - c_0/10\lambda_q \log q$, then for $q < x^{c_0/80\nu\lambda_q}$, the left of (1.7) is bounded by the same expression as (3.1), but only with “ $-c_0/4\nu\lambda_q \mathcal{L}_q(T)$ ” replaced by “ $-c_0/80\nu\lambda_q \mathcal{L}_q(T)$ ”, and with the term involving $\Omega(1)$ replaced by*

$$\frac{\Omega(1) \cdot N! (2000(1+\nu)c_0^{-1})^N \cdot 2^{\lambda_q} (\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot x^{1/\nu}}{(\lambda_q \log q)^{-N} \cdot (\log x)^{1-|\operatorname{Re}(\alpha_{\chi_0})|} \cdot \min\{x/h, (\log x)^{N+1}\}}.$$

The assertions corresponding to the last sentence of Theorem 1.1 also holds, if η_e does not exist.

Finally under (1.8), the above assertions hold exactly as stated, with $h = x/(\log x)^A$.

Theorem 1.1 and Corollary 1.2 follow from the above results by taking T as in the respective statements. (The choice of T in Theorem 1.1(1) comes from writing $1/T \approx x^{-c_0/4\nu\lambda_q \mathcal{L}_q(T)}$ and $\mathcal{L}_q(T) \approx (\log T)^2 + \log(q\nu) \cdot (\log T)$, and then solving the quadratic in $\log T$. There may be better ways of choosing T for specific $\Omega(T)$, which is why we prove Theorem 3.1 in its additional generality.)

3.1. Perron's Formula: Error-term control via averaging. Our proof of Theorem 3.1 begins by relating the partial sum $\sum_{n \leq x} a_n$ with a contour integral via Perron's formula. The following lemma will allow us to control the error terms when we apply this formula.

Lemma 3.2. *Assuming only (1.5), the following hold uniformly in $x > 4e^{1/\nu}$ and $h \in (0, x/2]$.*

(1) *For all $\theta \in (0, 1]$, we have $\sum_{n>1} |a_n|/n^{1/\nu+\lambda} \leq 4\kappa/\theta$.*

(2) *There exists a **half-integer**² $X \in (x, x+h]$ satisfying*

$$\sum_{\frac{3X}{4} < n < \frac{5X}{4}} \frac{|a_n|}{|\log(X/n)|} \ll \kappa \cdot \frac{x^{1+1/\nu} \log x}{h}.$$

Proof. (1) Indeed by (1.5), we have $\sum_{n \geq 2} |a_n|/n^{1/\nu+\theta} \leq \sum_{m \geq 0} 2^{-m/\nu-m\theta} \sum_{2^m < n \leq 2^{m+1}} |a_n| \leq \kappa \sum_{m \geq 0} 2^{-m\theta} = \kappa/(1 - 2^{-\theta})$. Now basic calculus shows that $2^{-\theta} \leq 1 - \theta/4$ for all $\theta \in (0, 1]$.

(2) It suffices to show that uniformly in $h \in (0, x/2]$, we have

$$(3.2) \quad \sum_{\substack{x < X \leq x+h \\ X \in \mathbb{Z}+1/2}} \sum_{\frac{3X}{4} < n < \frac{5X}{4}} \frac{|a_n|}{|\log(X/n)|} \ll \kappa \cdot x^{1+1/\nu} \log x.$$

²i.e. an element of the set $\mathbb{Z} + 1/2 = \{\pm 1/2, \pm 3/2, \dots\}$

Write the total double sum on the left as $S_1 + S_2$, where S_1 denotes the total contribution of all pairs (X, n) for which $n \in (3X/4, X - 1/2]$. Then for any n counted in S_1 , we can write $n = X - r$ for some half-integer $r \in [1/2, X/4) \subset [1/2, (x+h)/4)$. Moreover, $n = X - r \in (x - r, (x - r) + h]$ and $|\log(X/n)| = \log(X/(X-r)) = -\log(1-r/X) \gg r/X \gg r/x$. Combining these observations,

$$(3.3) \quad S_1 \triangleq \sum_{\substack{x < X \leq x+h \\ X \in \mathbb{Z}+1/2}} \sum_{\substack{\frac{3X}{4} < n \leq X - \frac{1}{2}}} \frac{|a_n|}{|\log(X/n)|} \ll x \sum_{\substack{1/2 < r \leq (x+h)/4 \\ r \in \mathbb{Z}+1/2}} \frac{1}{r} \sum_{x-r < n \leq (x-r)+h} |a_n|.$$

Now for any r above, $x - r > x - (x+h)/4 > h$. Hence using (1.5) on each inner sum in (3.3),

$$S_1 \ll \kappa \cdot x \sum_{\substack{1/2 < r \leq (x+h)/4 \\ r \in \mathbb{Z}+1/2}} \frac{1}{r} \cdot (x-r)^{1/\nu} \ll \kappa \cdot x^{1+1/\nu} \sum_{m \leq x} \frac{1}{m} \ll \kappa \cdot x^{1+1/\nu} \log x,$$

proving that S_1 is absorbed in the right of (3.2). The argument for S_2 is entirely analogous, by writing $n = X + r$ for some $r \in [1/2, (x+h)/4)$. This establishes (3.2), and hence the lemma. \square

Remark. We need the growth condition (1.5) *only* in the form of Lemma 3.2.

We will first show Theorem 3.1 with x replaced by the X coming from Lemma 3.2(2); for this we will only need that $\{a_n\}_n$ has property $\mathcal{P}(\nu, \{\alpha_\chi\}_\chi; c_0, \Omega)$ for a *general non-decreasing* $\Omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ for all $m \geq 1$. By Perron's Formula (as stated in [38, Theorem II.2.3]), we have

$$(3.4) \quad \sum_{n \leq X} a_n = \frac{1}{2\pi i} \int_{\frac{1}{\nu}(1+\frac{1}{\log X})-iT}^{\frac{1}{\nu}(1+\frac{1}{\log X})+iT} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds + O\left(\frac{X^{1/\nu}}{T} \sum_{n \geq 1} \frac{|a_n|}{n^{\frac{1}{\nu} + \frac{1}{\nu \log X}} |\log(X/n)|}\right).$$

By Lemma 3.2(2), the total contribution of all $n \in (3X/4, 5X/4)$ to the O -term above is $\ll \kappa \cdot X^{1/\nu} (\log X)/T$. On the other hand, Lemma 3.2(1) shows that the total contribution of all $n \notin (3X/4, 5X/4)$ to the O -term is $\ll X^{1/\nu} T^{-1} (|a_1|/\log X + \kappa \cdot X \log X/h)$. But now letting $s \rightarrow +\infty$ (along the real line) in (1.2) and (2.1), and using that $|G(s)| \leq \Omega(1)$ for all real $s > 1/\nu$, we obtain $|a_1| \leq \Omega(1)$. Inserting all these observations into (3.4), we obtain

$$(3.5) \quad \sum_{n \leq X} a_n = \frac{1}{2\pi i} \int_{\frac{1}{\nu}(1+\frac{1}{\log X})-iT}^{\frac{1}{\nu}(1+\frac{1}{\log X})+iT} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds + O\left(\frac{\Omega(1) X^{1/\nu}}{T \log X} + \kappa \cdot \frac{X^{1+1/\nu} \log X}{Th}\right).$$

The rest of the argument breaks up into the two cases, in the two subparts of Theorem 3.1.

3.2. When $1 - c_0/10\lambda_q \log q < \eta_e < 1 - 3\nu/\log x$: Proof of Theorem 3.1(1).

In this subsection, we define

$$(3.6) \quad \sigma_\nu(t) := \frac{1}{\nu} \left(1 - \frac{c_0}{4\nu\lambda_q \mathcal{L}_q(t)} \right), \quad r_e = \frac{1 - r_e}{6\nu}, \quad r_1 = \frac{1}{2 \max\{1, \nu\} \log X}.$$

We also define Γ_0 to be the contour consisting of the following components. (See Figure 2.)

- Γ_2 , the horizontal segment traversed from $\sigma_\nu(T) + iT$ to $\nu^{-1}(1 + 1/\log x) + iT$.
- Γ_3 , the part of the curve $\sigma_\nu(T) + it$ traversed upwards from $t = 0$ to $t = T$.
- Γ_4 , the horizontal segment traversed from $\eta_e/\nu - r_e$ to $\sigma_\nu(0)$ **above** the branch cut.
- Γ_5 , the anticlockwise semicircle in the upper half plane with center η_e/ν , radius r_e .
- Γ_6 , the horizontal segment traversed from $(2 + \eta_e)/3\nu$ to $\eta_e/\nu + r_e$ **above** the branch cut.

- Γ_7 , the horizontal segment traversed from $1/\nu - r_1$ to $(2 + \eta_e)/3\nu$ **above** the branch cut.
- Γ_8 , the circle with center at $1/\nu$, radius r_1 , traversed anticlockwise as shown in Figure 2.
- $\bar{\Gamma}_j$ (for $2 \leq j \leq 7$), the reflection of Γ_j about the real line, directed as in Figure 2.

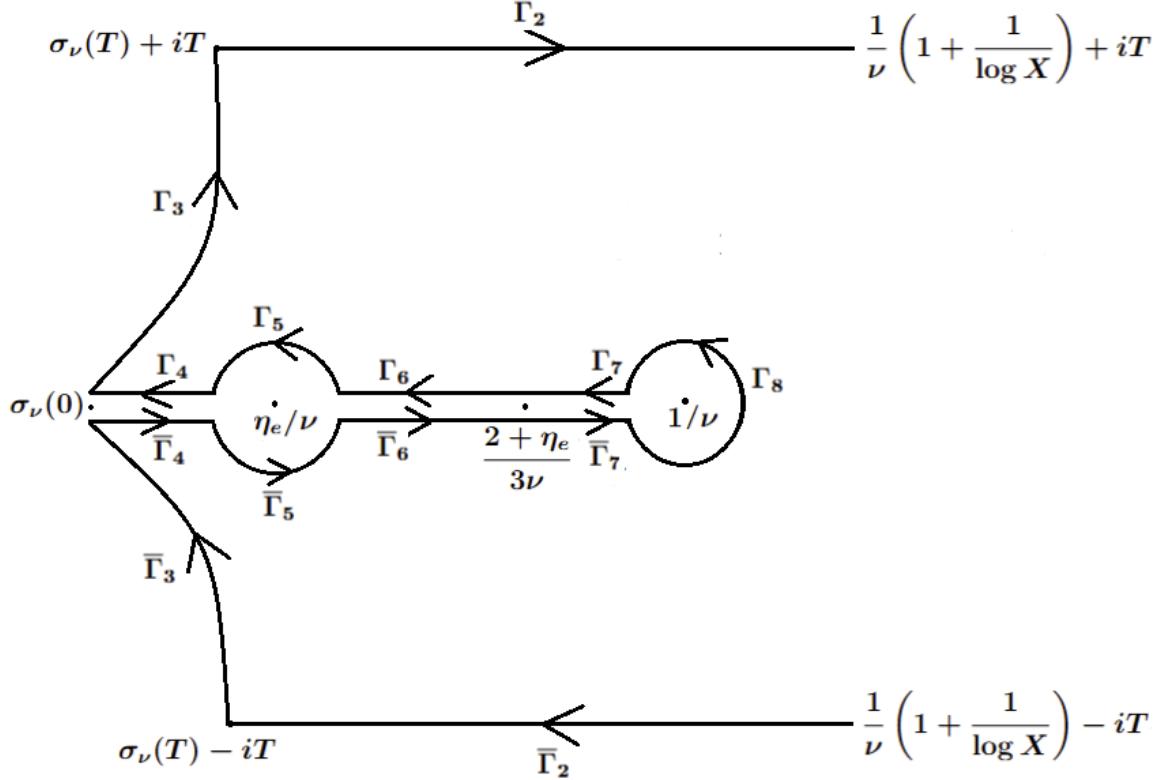


FIGURE 2. Contour Γ_0 for Theorem 3.1(1), i.e. when $1 - \frac{c_0}{10\lambda_q \log q} < \eta_e < 1 - \frac{3\nu}{\log x}$.

Since we are in the first case of (2.5), it follows that the function $\mathcal{F}(s\nu)G(s)X^s/s$ is analytic in a region containing the one enclosed by Γ_0 and the vertical segment $[\nu^{-1}(1 + 1/\log X) - iT, \nu^{-1}(1 + 1/\log X) + iT]$. As such, Cauchy's integral theorem yields, from (3.5),

$$(3.7) \quad \sum_{n \leq X} a_n = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds + O\left(\frac{\Omega(1)X^{1/\nu}}{T \log X} + \kappa \cdot \frac{X^{1+1/\nu} \log X}{Th}\right).$$

We will now show that the contribution of all parts of Γ_0 except $\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8$ are negligible. We will be repeatedly using the hypotheses $q \geq e^{4+5/3\nu}$, $\eta_e \geq 1 - c_0/10\lambda_q \log q$, and $\max\{|\alpha_{\chi_0}|, |\alpha_{\chi_e}|\} \leq K_0$. Our implied constants are allowed to depend on c_0, ν and K_0 .

Contribution of $\sum_{j=2}^3 (\Gamma_j + \bar{\Gamma}_j)$. Any s on these four contours satisfies the conditions of Proposition 2.6(2): The condition on σ follows from the definition of σ_ν and the hypothesis “ $c_0\lambda_q \mathcal{L}_q(T) \leq \log x$ ” in Theorem 3.1(1). The other condition on s in Proposition 2.6(2) is clear if $|t| \geq 1/\nu$, whereas if $|t| < 1/\nu$, then $s \in \Gamma_3 + \bar{\Gamma}_3$, so $|s - 1/\nu| \geq 1/\nu - \sigma_\nu(t) = c_0/4\nu\lambda_q \mathcal{L}_q(t)$ and

$$\left|s - \frac{\eta_e}{\nu}\right| \geq \frac{\eta_e}{\nu} - \sigma_\nu(t) \geq \frac{c_0}{4\nu\lambda_q \log(2q)} - \frac{c_0}{10\nu\lambda_q \log q} \geq \frac{c_0}{8\nu\lambda_q \log q} - \frac{c_0}{10\nu\lambda_q \log q} \geq \frac{c_0}{40\nu\lambda_q \log q}.$$

Hence Proposition 2.6(2) yields $\mathcal{F}(s\nu) \ll (5\lambda_q \mathcal{L}_q(T)/4)^{\lambda_q}$ uniformly for $s \in \Gamma_2 + \bar{\Gamma}_2 + \Gamma_3 + \bar{\Gamma}_3$. Moreover, $|G(s)| \leq \Omega(|t|) \leq \Omega(T)$ for all such s . Lastly, $\int_{\Gamma_2 + \bar{\Gamma}_2} |X^s/s| ds \leq T^{-1} \int_{\sigma_\nu(T)}^{\frac{1}{\nu} + \frac{1}{\nu \log X}} X^\sigma d\sigma \ll X^{1/\nu}/T \log X$, and $\int_{\Gamma_3 + \bar{\Gamma}_3} |X^s/s| ds \leq \int_0^T X^{\sigma_\nu(t)} dt/(t+1) \leq X^{\sigma_\nu(T)}(\log T)$, where we have noted that $|s| \gg |t| + 1$ for all $s \in \Gamma_3 + \bar{\Gamma}_3$. Combining all the observations in this paragraph, we get

$$(3.8) \quad \sum_{j \in \{2,3\}} \left| \int_{\Gamma_j + \bar{\Gamma}_j} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds \right| \ll \left(\frac{5}{4} \lambda_q \mathcal{L}_q(T) \right)^{\lambda_q} \Omega(T) \left\{ \frac{X^{1/\nu}}{T \log X} + X^{1/\nu - c_0/4\nu\lambda_q \mathcal{L}_q(T)} (\log T) \right\}.$$

Contribution of $\sum_{j=4}^6 (\Gamma_j + \bar{\Gamma}_j)$. Since $r_e = (1 - \eta_e)/6\nu \leq c_0/60\nu\lambda_q \log q$, any s on these six contours satisfies the conditions of Proposition 2.6(1), as well as $|G(s)| \leq \Omega(1)$. Any such s also satisfies $\sigma \leq (2 + \eta_e)/3\nu$ and $|s| \geq \sigma_\nu(0) \gg 1$. Hence by definition of $\mathcal{H}(s)$ and Proposition 2.6(1),

$$(3.9) \quad \sum_{j=4}^6 \left| \int_{\Gamma_j + \bar{\Gamma}_j} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds \right| \ll (2\lambda_q \log_q)^{\lambda_q + |\operatorname{Re}(\alpha_{x_0})| + |\operatorname{Re}(\alpha_{x_e})|} \Omega(1) X^{(2+\eta_e)/3\nu} \sum_{j=4}^6 \int_{\Gamma_j + \bar{\Gamma}_j} \left| s - \frac{1}{\nu} \right|^{-\operatorname{Re}(\alpha_{x_0})} \cdot \left| s - \frac{1}{\nu} \right|^{\operatorname{Re}(\alpha_{x_e})} |ds|.$$

As Figure 2 shows, any s above satisfies $(1 - \eta_e)/3\nu = 1/\nu - (2 + \eta_e)/3\nu \leq |s - 1/\nu| \leq 1/\nu - \sigma_\nu(0) = c_0/4\nu\lambda_q \log q < 1$ and $(1 - \eta_e)/6\nu = r_e \leq |s - \eta_e/\nu| \leq 1/\nu - \sigma_\nu(0) < 1$. In particular, both $|s - 1/\nu|$ and $|s - \eta_e/\nu|$ lie between $(1 - \eta_e)/6\nu$ and 1, so that $|s - 1/\nu|^{-\operatorname{Re}(\alpha_{x_0})} \cdot |s - \eta_e/\nu|^{\operatorname{Re}(\alpha_{x_e})} \leq |s - 1/\nu|^{-|\operatorname{Re}(\alpha_{x_0})|} \cdot |s - \eta_e/\nu|^{-|\operatorname{Re}(\alpha_{x_e})|} \ll_{\nu, K_0} (1 - \eta_e)^{-|\operatorname{Re}(\alpha_{x_0})| - |\operatorname{Re}(\alpha_{x_e})|}$. Hence (3.9) yields

$$(3.10) \quad \sum_{j=4}^6 \left| \int_{\Gamma_j + \bar{\Gamma}_j} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds \right| \ll \Omega(1) \cdot \frac{(2\lambda_q \log_q)^{\lambda_q + |\operatorname{Re}(\alpha_{x_0})| + |\operatorname{Re}(\alpha_{x_e})|} \cdot X^{(2+\eta_e)/3\nu}}{(1 - \eta_e)^{|\operatorname{Re}(\alpha_{x_0})| + |\operatorname{Re}(\alpha_{x_e})|}}.$$

Now we extract the main term from the contribution of $\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8$. Our method for this is partly inspired from works of Scourfield [32] and Tenenbaum [38, Chapter II.5]. **We first claim that the disk $\{s : |s - 1/\nu| \leq 2(1 - \eta_e)/3\nu\}$ is contained in the region**

$$\left\{ s = \sigma + it : \sigma > \frac{1}{\nu} \left(1 - \frac{c_0}{4\nu\lambda_q \mathcal{L}_q(t)} \right), |t| \leq \frac{c_0}{15\nu\lambda_q \log q}, s \notin \left(-\infty, \frac{\eta_e}{\nu} \right] \right\}.$$

Indeed for any s in this disk, we have $\sigma - \eta_e/\nu \geq (1/\nu - 2(1 - \eta_e)/3\nu) - \eta_e/\nu > 0$, and $|t| \leq 2(1 - \eta_e)/3\nu < c_0/15\nu\lambda_q \log q < 1/\nu$. (Recall $1 - \eta_e < c_0/10\lambda_q \log q$.) The claim now follows from

$$\sigma \geq \frac{1}{\nu} - \frac{2(1 - \eta_e)}{3\nu} > \frac{1}{\nu} - \frac{c_0}{15\nu\lambda_q \log q} > \frac{1}{\nu} \left(1 - \frac{c_0}{4\lambda_q \log(2q)} \right) > \frac{1}{\nu} \left(1 - \frac{c_0}{4\lambda_q \log(q(|t\nu| + 1))} \right),$$

Two consequences of this claim are that any s in the above disk satisfies the hypotheses of Proposition 2.6(1), and that the function $\mathcal{H}(s)G(s)(s - \eta_e/\nu)^{\alpha_{x_e}}$ is analytic on this disk. Hence

$$(3.11) \quad \mathcal{H}(s)G(s) \left(s - \frac{\eta_e}{\nu} \right)^{\alpha_{x_e}} = \sum_{j=0}^{\infty} \mu_j \left(s - \frac{1}{\nu} \right)^j \text{ for all } s \text{ satisfying } \left| s - \frac{1}{\nu} \right| \leq \frac{2(1 - \eta_e)}{3\nu}.$$

where by Cauchy's integral formula and Proposition 2.6(1), we have

$$(3.12) \quad |\mu_j| = \frac{1}{2\pi i} \int_{|z-\frac{1}{\nu}|=\frac{2(1-\eta_e)}{3\nu}} \frac{\mathcal{H}(z)G(z)(z-\eta_e/\nu)^{\alpha_{\chi_e}}}{(z-1/\nu)^{j+1}} dz \ll \Omega(1) \cdot \frac{(2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|}}{(1-\eta_e)^{|\operatorname{Re}(\alpha_{\chi_e})|} \cdot (2(1-\eta_e)/3\nu)^j}$$

uniformly in all $j \geq 0$. (Here, we also observed that if $|z - 1/\nu| = 2(1 - \eta_e)/3\nu$, then $|z - \eta_e/\nu|$ lies in the interval $[(1 - \eta_e)/3\nu, 4(1 - \eta_e)/3\nu]$ by the triangle inequality.)

Now for $s \in \Gamma_7 + \bar{\Gamma}_7 + \Gamma_8$, we have $|s - 1/\nu| \leq 1/\nu - (2 + \eta_e)/3\nu = (1 - \eta_e)/3\nu$. Hence (3.12) yields

$$(3.13) \quad \sum_{j \geq N+1} \left| \mu_j \left(s - \frac{1}{\nu} \right)^j \right| \ll \Omega(1) \cdot \frac{(2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|}}{(1-\eta_e)^{|\operatorname{Re}(\alpha_{\chi_e})|}} \cdot \left(\frac{|s - 1/\nu|}{2(1-\eta_e)/3\nu} \right)^{N+1}$$

uniformly in all such s and in $N \geq 0$; here we noted that $\sum_{j \geq N+1} \left(\frac{|s - 1/\nu|}{2(1-\eta_e)/3\nu} \right)^{j-(N+1)} \leq \sum_{j \geq N+1} (1/2)^{j-(N+1)} \leq 2$. Hence by definition of $\mathcal{H}(s)$, along with (3.11) and (3.13), we obtain

$$(3.14) \quad \frac{1}{2\pi i} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds = \sum_{j=0}^N \frac{\mu_j}{2\pi i} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} X^s \left(s - \frac{1}{\nu} \right)^{j-\alpha_{\chi_0}} ds \\ + O \left(\Omega(1) \cdot \frac{(2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot (3\nu/2)^N}{(1-\eta_e)^{N+1+|\operatorname{Re}(\alpha_{\chi_e})|}} \cdot \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} X^\sigma \left| s - \frac{1}{\nu} \right|^{N+1-\operatorname{Re}(\alpha_{\chi_0})} |ds| \right).$$

Now setting $w := (s - 1/\nu) \log X$, we see that the total main term on the right hand of (3.14) is

$$(3.15) \quad \sum_{j=0}^N \frac{\mu_j}{2\pi i} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} X^s \left(s - \frac{1}{\nu} \right)^{j-\alpha_{\chi_0}} ds = \sum_{j=0}^N \frac{\mu_j X^{1/\nu}}{(\log X)^{j+1-\alpha_{\chi_0}}} \cdot \frac{1}{2\pi i} \int_{\mathcal{W}} e^w w^{j-\alpha_{\chi_0}} dw \\ = \frac{X^{1/\nu}}{(\log X)^{1-\alpha_{\chi_0}}} \sum_{j=0}^N \frac{\mu_j (\log X)^{-j}}{\Gamma(\alpha_{\chi_0} - j)} + O \left(\frac{X^{1/\nu + (\eta_e - 1)/6\nu}}{(\log X)^{1-\operatorname{Re}(\alpha_{\chi_0})}} \sum_{j=0}^N |\mu_j| \Gamma(j+1+|\alpha_{\chi_0}|) \left(\frac{47}{\log X} \right)^j \right),$$

where \mathcal{W} is the truncated Hankel contour in Figure 3, and we have used [38, Corollary II.0.18]. (The hypothesis $\eta_e < 1 - 3\nu/\log x$ in Theorem 3.1(1) guarantees that [38, Corollary II.0.18] applies.) By (3.12), the entire O -term in (3.15) is

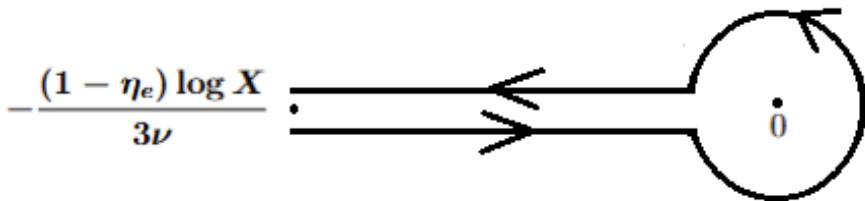


FIGURE 3. The truncated Hankel contour \mathcal{W} after the substitution $w = \left(s - \frac{1}{\nu} \right) \log X$.

$$\ll \frac{(2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|} \Omega(1) X^{1/\nu + (\eta_e - 1)/6\nu}}{(1 - \eta_e)^{|\operatorname{Re}(\alpha_{\chi_e})|} \cdot (\log X)^{1 - \operatorname{Re}(\alpha_{\chi_0})}} \sum_{j=0}^N \frac{\Gamma(j + 1 + |\alpha_{\chi_0}|)}{(2(1 - \eta_e) \log X / 141\nu)^j}$$

Since $\Gamma(N + 1 + |\alpha_{\chi_0}|) = \Gamma(j + 1 + |\alpha_{\chi_0}|) \cdot \prod_{i=j+1}^N (i + |\alpha_{\chi_0}|) \geq \Gamma(j + 1 + |\alpha_{\chi_0}|) \cdot (N - j)!$, we get

$$\sum_{j=0}^N \frac{\Gamma(j + 1 + |\alpha_{\chi_0}|)}{(2(1 - \eta_e) \log X / 141\nu)^j} = \frac{(141\nu/2)^N \Gamma(N + 1 + |\alpha_{\chi_0}|)}{(1 - \eta_e)^N \cdot (\log x)^N} \sum_{j=0}^N \frac{(2(1 - \eta_e) \log X / 141\nu)^{N-j}}{(N - j)!},$$

and the last sum above is at most $X^{2(1 - \eta_e)/141\nu}$. Inserting this into (3.15), we obtain

$$(3.16) \quad \begin{aligned} & \sum_{j=0}^N \frac{\mu_j}{2\pi i} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} X^s \left(s - \frac{1}{\nu}\right)^{j - \alpha_{\chi_0}} ds = \frac{X^{1/\nu}}{(\log X)^{1 - \alpha_{\chi_0}}} \sum_{j=0}^N \frac{\mu_j (\log X)^{-j}}{\Gamma(\alpha_{\chi_0} - j)} \\ & + O\left(\Omega(1) (2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot \frac{(141\nu/2)^N \Gamma(N + 1 + K_0) \cdot X^{1/\nu + 43(\eta_e - 1)/282\nu}}{(1 - \eta_e)^{N + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot (\log X)^{N + 1 - \operatorname{Re}(\alpha_{\chi_0})}}\right), \end{aligned}$$

Next, we bound the integral in the O -term of (3.14). Note that for all $s \in \Gamma_7 + \bar{\Gamma}_7 + \Gamma_8$, we have $r_1 \leq |1/\nu - s| < 1/\nu - (2 + \eta_e)/3\nu = (1 - \eta_e)/3\nu < c_0/30\lambda_q \log q < 1$. As such $|1/\nu - s|^{-\operatorname{Re}(\alpha_{\chi_0})} \leq |1/\nu - s|^{-|\operatorname{Re}(\alpha_{\chi_0})|} < (r_1)^{-|\operatorname{Re}(\alpha_{\chi_0})|} \ll (\log x)^{|\operatorname{Re}(\alpha_{\chi_0})|}$ via (3.6). Parametrizing the circle Γ_8 as $s = 1/\nu + r_1 e^{i\theta}$, $-\pi < \theta \leq \pi$, we thus find that the integral in the O -term of (3.14) is

$$\ll (\log x)^{|\operatorname{Re}(\alpha_{\chi_0})|} \left\{ \int_{\frac{2+\eta_e}{3\nu}}^{\frac{1}{\nu}-r_1} X^\sigma \left(\frac{1}{\nu} - \sigma\right)^{N+1} d\sigma + X^{1/\nu+r_1} r_1^{N+2} \right\} \ll \frac{(N+1)! X^{1/\nu}}{(\log X)^{N+2-|\operatorname{Re}(\alpha_{\chi_0})|}}.$$

To get the last bound, we have again used (3.6), and we have made the change of variable $\sigma = 1/\nu - u/\log X$ in the last integral above to see that its value is at most $\Gamma(N+2)X^{N+1}/(\log X)^{N+2}$. Inserting the above bound into (3.14), we find that the total O -term in (3.14) is

$$\ll \Omega(1) \cdot \frac{(2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot (3\nu/2)^N (N+1)! X^{1/\nu}}{(1 - \eta_e)^{N+1+|\operatorname{Re}(\alpha_{\chi_e})|} \cdot (\log X)^{N+2-|\operatorname{Re}(\alpha_{\chi_0})|}}.$$

Inserting this observation along with (3.16) into (3.14), we now obtain

$$(3.17) \quad \begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} \frac{\mathcal{F}(s\nu)G(s)X^s}{s} ds - \frac{X^{1/\nu}}{(\log X)^{1 - \alpha_{\chi_0}}} \sum_{j=0}^N \frac{\mu_j (\log X)^{-j}}{\Gamma(\alpha_{\chi_0} - j)} \\ & \ll \Omega(1) \cdot \frac{(2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{\chi_0})| + |\operatorname{Re}(\alpha_{\chi_e})|} \cdot N! (71\nu)^N X^{1/\nu}}{(1 - \eta_e)^{N+1+|\operatorname{Re}(\alpha_{\chi_e})|} \cdot (\log X)^{N+2-|\operatorname{Re}(\alpha_{\chi_0})|}}; \end{aligned}$$

to absorb the O -term in (3.16) into the right side above, we noted that $\Gamma(N+1+K_0)/(N+1)! \leq \Gamma(1+K_0) \cdot \prod_{i=1}^{N+1} (1+K_0/i) \ll_{K_0} \exp(K_0 \sum_{i=1}^{N+1} 1/i) \ll (N+1)^{K_0} \ll (N+1)^{-1} \cdot (142/141)^N$.

Combining (3.8), (3.10) and (3.17), we deduce that Theorem 3.1 holds with $X \in (x, x+h]$ in place of x . Now $|\sum_{n \leq X} a_n - \sum_{n \leq x} a_n| \leq \sum_{x < n \leq x+h} |a_n|$ is absorbed in the right of (3.1). Hence, to complete the proof of Theorem 3.1(1), it only remains to show that the difference between $\sum_{j=0}^N \mu_j X^{1/\nu} (\log X)^{-j-1+\alpha_{\chi_0}} / \Gamma(\alpha_{\chi_0} - j)$ and its “ x -analogue” is absorbed in the right of (3.1).

To this end, note that since $x < X \leq x(1+h/x)$ and $h < x/2$, we have $\mathbf{X}^{1/\nu} = \mathbf{x}^{1/\nu}(\mathbf{1} + \mathbf{O}(\mathbf{h}/\mathbf{x}))$ and $\log x < \log X \leq \log x + h/x$, so that $|\log \mathbf{X} - \log \mathbf{x}| \leq \mathbf{h}/\mathbf{x}$. This in turn shows that the difference $|\log X|^{-j-1+\alpha_{\chi_0}} - |\log x|^{-j-1+\alpha_{\chi_0}}$ is at most $|j+1 - \alpha_{\chi_0}| \cdot \int_{\log x}^{\log X} t^{-j-2+\operatorname{Re}(\alpha_{\chi_0})} dt \ll$

$(jh/x) \cdot (\log x)^{-j-2+\operatorname{Re}(\alpha_{x_0})}$, uniformly in all $j \geq 0$. (Here the last “ \ll ” bound is tautological for $j+2 > \operatorname{Re}(\alpha_{x_0})$, while for $j+2 \leq \operatorname{Re}(\alpha_{x_0}) \leq K_0$, it follows from the fact that $\log X = (\log x)(1+h/x \log x)$.) As such, $(\log \mathbf{X})^{-j-1+\alpha_{x_0}} = (\log x)^{-j-1+\alpha_{x_0}} \cdot (1 + O(jh/x \log x))$.

We will also need to upper bound $1/|\Gamma(\alpha_{x_0} - j)|$ uniformly in all $j \in \{0, \dots, N\}$. If $j \leq 2K_0 + 1$, then the fact that $1/\Gamma$ is entire yields $1/|\Gamma(\alpha_{x_0} - j)| \ll_{K_0} 1$. On the other hand, if $j > 2K_0 + 1$, then from $|\Gamma(\lceil K_0 \rceil + 1 - \alpha_{x_0})| \ll_{K_0} 1$ and the reflection identity, we find that

$$\begin{aligned} \frac{1}{|\Gamma(\alpha_{x_0} - j)|} &\ll |\sin(\pi\alpha_{x_0})| \cdot |\Gamma(j+1-\alpha_{x_0})| \ll_{K_0} |\Gamma(\lceil K_0 \rceil + 1 - \alpha_{x_0})| \cdot \prod_{i=\lceil K_0 \rceil + 1}^j |i - \alpha_{x_0}| \\ &\ll_{K_0} \prod_{i=1}^j (i + K_0) \leq N! \cdot \prod_{i=1}^{N+1} \left(1 + \frac{K_0}{i}\right) \leq N! \cdot \exp\left(K_0 \sum_{i=1}^{N+1} \frac{1}{i}\right) \ll N! \cdot (N+1)^{K_0} \end{aligned}$$

Hence, $1/|\Gamma(\alpha_{x_0} - j)| \ll N! (N+1)^{K_0}$ uniformly in all $j \in \{0, \dots, N\}$. Collecting all the observations in bold in this paragraph and the last, and using (3.12), we find that

$$\begin{aligned} &\left| \sum_{j=0}^N \frac{\mu_j}{\Gamma(\alpha_{x_0} - j)} \left\{ \frac{X^{1/\nu}}{(\log X)^{j+1-\operatorname{Re}(\alpha_{x_0})}} - \frac{x^{1/\nu}}{(\log x)^{j+1-\operatorname{Re}(\alpha_{x_0})}} \right\} \right| \\ &\ll \frac{h}{x} \cdot \Omega(1) \cdot \frac{(2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{x_0})| + |\operatorname{Re}(\alpha_{x_e})|} \cdot N! (N+1)^{K_0} \cdot x^{1/\nu}}{(1 - \eta_e)^{|\operatorname{Re}(\alpha_{x_e})|} \cdot (\log x)^{1-\operatorname{Re}(\alpha_{x_0})}} \sum_{j=0}^N j \left(\frac{2(1 - \eta_e) \log x}{3\nu} \right)^{-j}, \end{aligned}$$

which is absorbed in the right of (3.1), since by the hypothesis $(1 - \eta_e) \log x > 3\nu$, the sum on j above is at most $\sum_{j \geq 0} j/2^j \ll 1$. This concludes the proof of subpart (1) of Theorem 3.1. \square

3.3. When η_e does not exist or $\eta_e \leq 1 - c_0/10\lambda_q \log q$: Proof of Theorem 3.1(2).

We are in cases 2 and 3 of (2.5). We just mention the main changes from the above arguments. First, we redefine $\sigma_\nu(t) := \nu^{-1}(1 - c_0/8\lambda_q \mathcal{L}_q(t))$ for case 2, and $\sigma_\nu(t) := \nu^{-1}(1 - c_0/80\lambda_q \mathcal{L}_q(t))$ for case 3. Second, we redefine Γ_0 by replacing all the contours $\sum_{j=4}^7 (\Gamma_j + \bar{\Gamma}_j)$ by $\Gamma'_4 + \bar{\Gamma}'_4$, where Γ'_4 is the horizontal segment traversed from $1/\nu - r_1$ to $\sigma_\nu(0)$ above the branch cut. Here $\Gamma_3 + \bar{\Gamma}_3$ have been automatically redefined using the respective σ_ν 's, and r_1 is still as defined in (3.6).

In the rest of the subsection, we continue only with case 3 of (2.5) (namely, that $\eta_e \leq 1 - c_0/10\lambda_q \mathcal{L}_q(t)$), since case 2 will be entirely analogous and simpler. Note that by the “case 3” assertion of Proposition 2.6, the analogue of (3.8) continues to hold, only with the “ $X^{-c_0/4\nu\lambda_q \mathcal{L}_q(T)}$ ” term replaced “ $X^{-c_0/80\nu\lambda_q \mathcal{L}_q(T)}$ ”. Our main terms come from $\Gamma'_4 + \bar{\Gamma}'_4 + \Gamma_8$. The analogue of (3.11) is that $\mathcal{H}(s)G(s) = \sum_{j=0}^\infty \mu_j (s-1/\nu)^j$ for all s satisfying $|s-1/\nu| \leq c_0/40\nu\lambda_q \log q$. As such, the respective analogues of (3.12) and (3.13) continue to hold, with all instances of “ η_e ” and “ α_{x_e} ” removed, and with all instances of “ $2(1 - \eta_e)/3\nu$ ” replaced by “ $c_0/40\nu\lambda_q \log q$ ”. Finally, the analogues of (3.14), (3.16) and (3.17) hold with “ $\Gamma'_4 + \bar{\Gamma}'_4$ ” playing the role of “ $\Gamma_7 + \bar{\Gamma}_7$ ”; the error term in the analogue of (3.17) is $\ll \Omega(1) \cdot (2\lambda_q \log q)^{\lambda_q + |\operatorname{Re}(\alpha_{x_0})|} N! (2000\nu c_0^{-1} \cdot \lambda_q \log q)^{N+1} X^{1/\nu}/(\log X)^{N+2-|\operatorname{Re}(\alpha_{x_0})|}$. \square

3.4. When the stronger growth condition (1.8) is available.

To complete the proof of Theorem 3.1, it thus only remains to show its very last assertion (which would then also establish Corollary 1.2). To this end, we just need to show that condition (1.8), (assumed for some $A \geq 1$ and $\kappa_A \geq 2$) implies condition (1.5) with $\kappa := \kappa_A \cdot 2^{A+1/\nu}$.

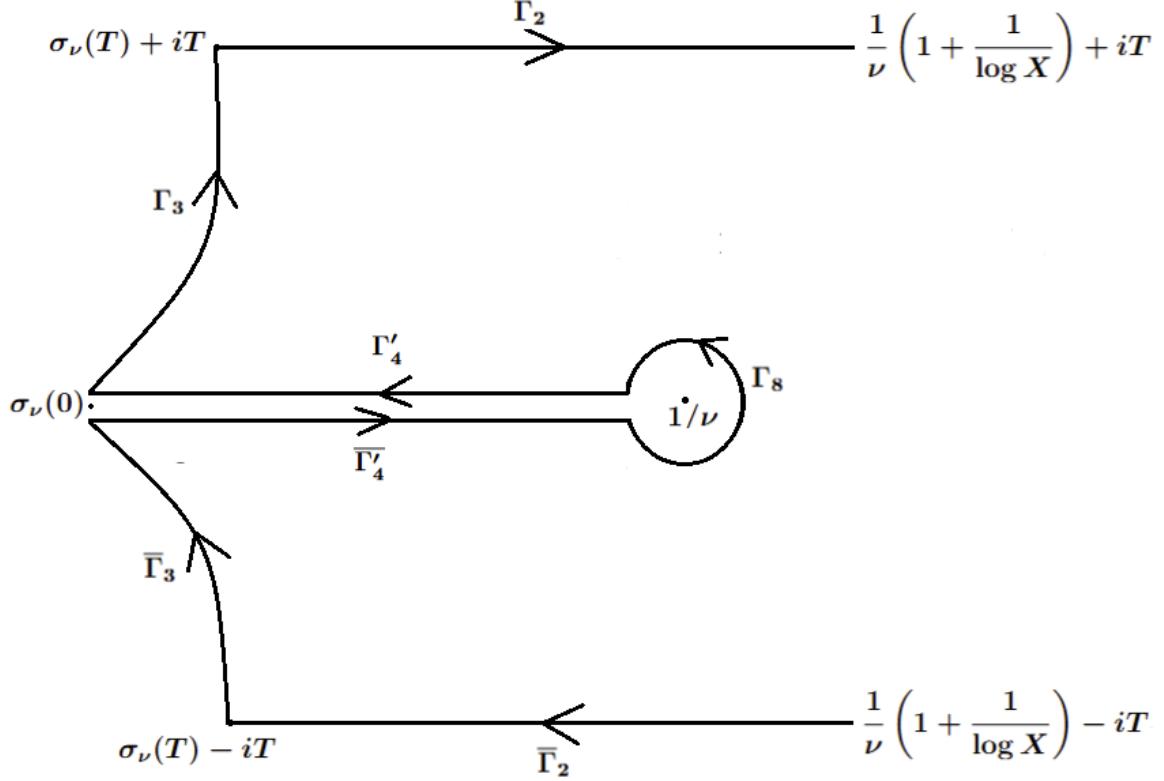


FIGURE 4. Contour Γ_0 when $\eta_e \leq 1 - \frac{c_0}{10\lambda_q \log q}$. Here $\sigma_\nu(t) = \frac{1}{\nu} \left(1 - \frac{c_0}{80\lambda_q \mathcal{L}_q(t)} \right)$.

Indeed, given any $x \geq 2$, consider the sequence $\{y_m\}_{m=0}^\infty$ defined by $y_0 := x$ and $y_{m+1} := y_m + y_m/(\log y_m)^A$. Let $M := M(x)$ be the unique index satisfying $y_M \leq x < y_{M+1}$. Since $\{y_m\}_m$ is clearly increasing, we have $x < y_m \leq 2x$ for all $m \in \{0, \dots, M\}$, so that by (1.8) yields

$$\sum_{x < n \leq 2x} |a_n| \leq \sum_{0 \leq m \leq M} \sum_{y_m < x \leq y_{m+1}} |a_n| \leq \kappa_A \sum_{0 \leq m \leq M} \frac{y_m^{1/\nu}}{(\log y_m)^A} \leq \kappa_A \cdot \frac{M(2x)^{1/\nu}}{(\log x)^A}.$$

But again the fact that $y_m \in (x, 2x]$ and the recurrence defining y_m yield $2x \geq y_M \geq y_{M-1} + x/(\log(2x))^A \geq y_{M-2} + 2x/(\log(2x))^A \geq \dots \geq x + Mx/(\log(2x))^A$, leading to $M \leq (\log(2x))^A$. Inserting this into the above display yields (1.5) with the desired value of κ . \square

4. THE LSD METHOD FOR L -FUNCTIONS VIA A SIMILARLY-BEHAVING BOUNDING SEQUENCE: PROOF OF THEOREM 1.3

In this section, we are in the setting of Theorem 1.3, so we only assume that $\{a_n\}_n$ has property $\mathcal{P}(\nu, \{\alpha_\chi\}_\chi; c_0, \Omega)$ with $\Omega(t) = \mathcal{M}(1+t)^{1-\delta_0}$, and that there exist $\{b_n\}_n$ satisfying $|a_n| \leq b_n$ for all n , such that $\{b_n\}_n$ has property $\mathcal{P}(\nu, \{\beta_\chi\}_\chi; c_0, \Omega)$ for some $\{\beta_\chi\}_\chi \subset \mathbb{C}$. In the entire section, $\max\{|\alpha_{\chi_0}|, |\alpha_{\chi_e}|, |\beta_{\chi_0}|, |\beta_{\chi_e}|\} \leq K_0$, and the implied constants depend only on c_0, ν, δ_0 and K_0 . The parameter $\Lambda_q := 1 + \max_{a \pmod{q}} \max \left\{ \left| \sum_\chi \alpha_\chi \cdot \chi(a) \right|, \left| \sum_\chi \beta_\chi \cdot \chi(a) \right| \right\}$ plays the role of “ λ_q ” from section 2: As such, the analogue of Proposition 2.6 (which we will use without further reference) holds, with λ_q replaced by Λ_q , and with “cases 1-3” from (2.5) redefined accordingly.

We will only highlight the main changes required from the arguments in the previous section. The first main change is that we use a different version of Perron's formula [38, Theorem II.2.5] to write

$$(4.1) \quad \int_0^x A(t) dt = \frac{1}{2\pi i} \int_{\frac{1}{\nu}(1+\frac{1}{\log x})-i\infty}^{\frac{1}{\nu}(1+\frac{1}{\log x})+i\infty} \frac{\mathcal{F}(s\nu)G(s)x^{s+1}}{s(s+1)} ds,$$

where $\mathbf{A}(\mathbf{x}) := \sum_{n \leq x} \mathbf{a}_n$. Note that there is no error term here, and “ x ” itself will play the role of the “ X ” in the previous section. Next, all our contours Γ_0 from the previous section are redefined, with the following additional specifications (everything else being as before).

- $T \geq 1$ is a parameter satisfying $c_0 \Lambda_q \mathcal{L}_q(T) \leq 100 \log(x/2)$.
- $\sigma_\nu(t)$ is as in section 3, defined according to the (analogues of the) three cases of (2.5).
- “ X ” and “ λ_q ” are replaced by “ x ” and “ Λ_q ”, respectively (in all three cases).
- We add the infinite vertical line $\Gamma_1 := [\nu^{-1}(1 + \log x) + iT, \nu^{-1}(1 + \log x) + i\infty)$ and its reflection about the real axis, both of these traversed upwards (in all three cases).
- r_1 , the radius of the circle Γ_8 , is just taken to be any positive parameter satisfying $r_1 \leq (4 \max\{1, \nu\} \log x)^{-1}$. We do not choose r_1 until the very end (to avoid technical issues).

In all cases, our integrand on the right of (4.1) is holomorphic on the region enclosed by (the respective) Γ_0 and the infinite vertical line $(\nu^{-1}(1 + \log x) + iT, \nu^{-1}(1 + \log x) + i\infty)$, so that

$$(4.2) \quad \int_0^x A(t) dt = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\mathcal{F}(s\nu)G(s)x^{s+1}}{s(s+1)} ds.$$

The third nontrivial change from the previous section is that we will use that

$$(4.3) \quad \mathcal{L}_q(t)^{\Lambda_q} \leq (2\delta_0^{-1} \Lambda_q \log q)^{\Lambda_q} (1 + |t\nu|)^{\delta_0/2} \ll_{\nu, \delta_0} (2\delta_0^{-1} \Lambda_q \log q)^{\Lambda_q} (1 + |t|)^{\delta_0/2}$$

uniformly in **all real t** . Here the first inequality can be seen by calculus: If $\Lambda_q \leq \delta_0(\log q)/2$, the function $\mathcal{L}_q(t)^{\Lambda_q}/(1 + t\nu)^{\delta_0/2}$ is strictly decreasing on $(0, \infty)$; in the other case, the function attains its global maximum (over all positive reals) at the unique $t_{\max} > 0$ satisfying $\mathcal{L}_q(t_{\max}) = 2\Lambda_q/\delta_0$.

We henceforth restrict to (the redefined) case 1 of (2.5) and establish subpart (1) of Theorem 1.3; the changes required for the other cases are entirely analogous to those described in subsection § 3.3. Hence, we are assuming that $1 - c_0/10\Lambda_q \log q < \eta_e < 1 - 3\nu/\log x$, and we have $\sigma_\nu(t) = \nu^{-1}(1 - c_0/4\Lambda_q \mathcal{L}_q(t))$. The analogue of Proposition 2.6(2), combined with (4.3), yields $\mathcal{F}(s\nu)G(s) \ll \mathcal{M} (5\Lambda_q^2 \log q/2\delta_0)^{\Lambda_q} \cdot (1 + t)^{1-\delta_0/2}$ uniformly in $s \in \sum_{j=1}^3 (\Gamma_j + \bar{\Gamma}_j)$. Hence the analogue of (3.8) is

$$(4.4) \quad \sum_{j=1}^3 \left| \int_{\Gamma_j + \bar{\Gamma}_j} \frac{\mathcal{F}(s\nu)G(s)x^{s+1}}{s(s+1)} ds \right| \ll \left(\frac{5\Lambda_q^2 \log q}{2\delta_0} \right)^{\Lambda_q} \mathcal{M} x \left\{ \frac{x^{1/\nu}}{T^{\delta_0/2}} + x^{\sigma_\nu(T)} \right\}.$$

Likewise, following the method leading to (3.10), we find that its analogue here is

$$(4.5) \quad \sum_{j=4}^6 \left| \int_{\Gamma_j + \bar{\Gamma}_j} \frac{\mathcal{F}(s\nu)G(s)x^{s+1}}{s(s+1)} ds \right| \ll (2\Lambda_q \log q)^{\Lambda_q+2K_0} (1 - \eta_e)^{-2K_0} \mathcal{M} x^{1+(2+\eta_e)/3\nu}.$$

From this point on, we closely follow part of the argument given for [38, Theorem II.5.2]. Inserting (4.4) and (4.5) into (4.2), we obtain $\int_0^x A(t) dt = \Phi(x) + O(\mathcal{E})$, where

$$\Phi(x) := \frac{1}{2\pi i} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} \frac{\mathcal{F}(s\nu)G(s)x^{s+1}}{s(s+1)} ds,$$

and where \mathcal{E} is the sum of the two expressions on the right hand sides of (4.4) and (4.5). With $u \in [-x/2, x/2]$ a parameter to be specified later, it thus follows that

$$(4.6) \quad \int_x^{x+u} A(t) dt = \Phi(x+u) - \Phi(x) + O(\mathcal{E}) = u\Phi'(x) + u^2 \int_0^1 (1-t)\Phi''(x+tu) + O(\mathcal{E}).$$

Differentiating under the integral sign, we have $\Phi'(x) = (2\pi i)^{-1} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} \mathcal{F}(s\nu)G(s)x^s/s ds$ and $\Phi'(x) = (2\pi i)^{-1} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} \mathcal{F}(s\nu)G(s)x^{s-1} ds$. Now uniformly in $y \in [x/2, 2x]$, we see that

$$\begin{aligned} |\Phi''(y)| &\ll \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} |\mathcal{H}(s)G(s)| y^{\sigma-1} \left| s - \frac{1}{\nu} \right|^{-\operatorname{Re}(\alpha_{\chi_0})} \cdot \left| s - \frac{\eta_e}{\nu} \right|^{\operatorname{Re}(\alpha_{\chi_e})} |ds| \\ &\ll (2\Lambda_q \log q)^{\Lambda_q+2K_0} (r_1(1-\eta_e))^{-K_0} \cdot \mathcal{M}x^{1/\nu-1}, \end{aligned}$$

where we have used the definition of $\mathcal{H}(s)$ and the analogue of Proposition 2.6(1), along with the facts that $r_1 \leq |s - 1/\nu| < 1/\nu - \sigma_\nu(0) < 1$, that $2(1-\eta_e)/3\nu \leq |s - \eta_e/\nu| \leq 7(1-\eta_e)/6\nu$, that $\max\{|\operatorname{Re}(\alpha_{\chi_0})|, |\operatorname{Re}(\alpha_{\chi_e})|\} \leq K_0$, and that $y^{\sigma-1} \asymp x^{\sigma-1} \leq x^{1/\nu+r_1-1} \ll x^{1/\nu-1}$ as $r_1 < (4 \max\{1, \nu\} \log x)^{-1}$. Inserting the bound in the above display into (4.6), we obtain

$$(4.7) \quad \frac{1}{u} \int_x^{x+u} A(t) dt = \Phi'(x) + O\left(\frac{\mathcal{E}}{u} + u \cdot (2\Lambda_q \log q)^{\Lambda_q+2K_0} (r_1(1-\eta_e))^{-K_0} \cdot \mathcal{M}x^{1/\nu-1}\right).$$

Now we use the properties of $\{b_n\}_n$. Since $|a_n| \leq b_n$ for all n , we find that for all $u > 0$,

$$\begin{aligned} \left| \frac{1}{u} \int_x^{x+u} A(t) dt - A(x) \right| &\leq \frac{1}{u} \int_x^{x+u} \left| \sum_{x < n \leq t} a_n \right| dt \leq \frac{1}{u} \int_x^{x+u} \left(\sum_{x < n \leq t} b_n \right) dt \\ (4.8) \quad &= \frac{1}{u} \int_x^{x+u} (B(t) - B(x)) dt \leq \left| \frac{1}{u} \int_x^{x+u} B(t) dt - \frac{1}{u} \int_{x-u}^x B(t) dt \right|, \end{aligned}$$

where we have noted that $B(t) := \sum_{n \leq t} b_n$ is an increasing function of t . Now since $\{b_n\}_n$ has property $\mathcal{P}(\nu, \{\beta_\chi\}_\chi; c_0, \Omega)$ and $\max\{|\beta_{\chi_0}|, |\beta_{\chi_e}|\} \leq K_0$, we see that all the arguments given for (4.7) go through for $\{b_n\}_n$, so that the analogue of (4.7) holds for $\{b_n\}_n$ as well. Hence uniformly in $u \in (0, x/2]$, the last difference in (4.8) is absorbed into the O -term in (4.7). Thus from (4.7),

$$A(x) = \frac{1}{2\pi i} \int_{\Gamma_7 + \bar{\Gamma}_7 + \Gamma_8} \frac{\mathcal{F}(s\nu)G(s)x^s}{s} ds + O\left(\frac{\mathcal{E}}{u} + u \cdot (2\Lambda_q \log q)^{\Lambda_q+2K_0} (r_1(1-\eta_e))^{-K_0} \mathcal{M}x^{1/\nu-1}\right),$$

where we have also used the expression for $\Phi'(x)$ given after (4.6). Finally, the analogue of (3.17) (with “ X ” and “ Λ_q ” replaced by “ x ” and “ λ_q ” respectively) estimates the last integral above, showing that the left hand side of (1.10) is bounded by the expression

$$\begin{aligned} &\frac{(2\Lambda_q \log q)^{\Lambda_q+|\operatorname{Re}(\alpha_{\chi_0})|+|\operatorname{Re}(\alpha_{\chi_e})|} N! (71\nu)^N \mathcal{M} x^{1/\nu}}{(1-\eta_e)^{N+1+|\operatorname{Re}(\alpha_{\chi_e})|} \cdot (\log x)^{N+2-|\operatorname{Re}(\alpha_{\chi_0})|}} + u (2\Lambda_q \log q)^{\Lambda_q+2K_0} (r_1(1-\eta_e))^{-K_0} \mathcal{M}x^{1/\nu-1} \\ &+ u^{-1} \left(\frac{5\Lambda_q^2 \log q}{2\delta_0} \right)^{\Lambda_q} \mathcal{M}x \left\{ \frac{x^{1/\nu}}{T^{\delta_0/2}} + x^{\sigma_\nu(T)} \right\} + u^{-1} (2\Lambda_q \log q)^{\Lambda_q+2K_0} (1-\eta_e)^{-2K_0} \mathcal{M}x^{1+(2+\eta_e)/3\nu}, \end{aligned}$$

uniformly in all $u \in (0, x/2]$, and in $T \geq 1$ satisfying $c_0 \lambda_q \mathcal{L}_q(T) \leq 100 \log(x/2)$. Theorem 1.3 subpart (1) follows by taking $r_1 := (4 \max\{1, \nu\} \log x)^{-1}$, choosing T as in its statement (which ensures that $x^{1/\nu} T^{-\delta_0/2} \asymp x^{\sigma_\nu(T)}$), and lastly, by choosing u so as to equate the first two terms in the above display that contain u (namely, the term with “ u ” and the first term with “ u^{-1} ”). \square

5. DISTRIBUTION OF INTEGERS WITH PRIME FACTORS RESTRICTED TO PROGRESSIONS WITH VARYING MODULI: PROOF OF THEOREM 1.4

We intend to apply Theorem 1.1 with $\nu = 1$ and with a_n being the indicator function of the property that all the prime factors of n lie in the residue classes in \mathcal{A} . Note that (1.5) is tautological. Moreover, since $\{a_n\}_n$ is a multiplicative sequence, we can use the Euler product to write

$$(5.1) \quad \sum_n \frac{a_n}{n^s} = \prod_{p \bmod q \in \mathcal{A}} \left(1 + \sum_{r \geq 1} \frac{1}{p^{rs}} \right) = \prod_{p \bmod q \in \mathcal{A}} \left(1 - \frac{1}{p^s} \right)^{-1}$$

for all complex s having $\operatorname{Re}(s) > 1$. Now by the orthogonality of Dirichlet characters, we see that for any $b \in U_q$, we have $\sum_{p \equiv b \pmod{q}} 1/p^s = \varphi(q)^{-1} \sum_{\chi} \bar{\chi}(b) \sum_p \chi(p)/p^s$.

Moreover, the Euler product of $L(s, \chi)$ shows that $\log L(s, \chi) = \sum_p \chi(p)/p^s + \sum_{p,r \geq 2} \chi(p^r)/rp^{rs}$. Eliminating $\sum_p \chi(p)/p^s$ from these two identities, and once again invoking orthogonality, we obtain

$$(5.2) \quad \sum_{p \equiv b \pmod{q}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(b) \log L(s, \chi) - \sum_{\substack{p,r \geq 2 \\ p^r \equiv b \pmod{q}}} \frac{1}{rp^{rs}}$$

for all $b \in U_q$ and all complex numbers s having $\operatorname{Re}(s) > 1$. Now writing

$$- \sum_{p \bmod q \in \mathcal{A}} \log \left(1 - \frac{1}{p^s} \right) = - \sum_{b \in \mathcal{A}} \sum_{p \equiv b \pmod{q}} \log \left(1 - \frac{1}{p^s} \right) = \sum_{b \in \mathcal{A}} \sum_{\substack{p,r \geq 1 \\ p^r \equiv b \pmod{q}}} \frac{1}{rp^{rs}},$$

and using (5.2) replace the contribution of $r = 1$, we find that

$$(5.3) \quad - \sum_{p \bmod q \in \mathcal{A}} \log \left(1 - \frac{1}{p^s} \right) = \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(b) \log L(s, \chi) + \sum_{\substack{p,r \geq 2 \\ p \bmod q \in \mathcal{A}}} \frac{1}{rp^{rs}} - \sum_{\substack{p,r \geq 2 \\ p^r \bmod q \in \mathcal{A}}} \frac{1}{rp^{rs}}.$$

Inserting (5.3) into (5.1), we find that $\sum_{n \geq 1} a_n/n^s = \mathcal{F}(s)G(s)$, where $\mathcal{F}(s) := \prod_{\chi} L(s, \chi)^{\alpha_{\chi}}$, where $\alpha_{\chi} = \varphi(q)^{-1} \sum_{b \in \mathcal{A}} \bar{\chi}(b)$, and where $G(s) := \exp \left(\sum_{\substack{p,r \geq 2 \\ p \bmod q \in \mathcal{A}}} 1/rp^{rs} - \sum_{\substack{p,r \geq 2 \\ p^r \bmod q \in \mathcal{A}}} 1/rp^{rs} \right)$. Note that $\sum_{p,r \geq 2} |1/rp^{rs}| \leq \sum_{p,r \geq 2} p^{-3r/4} \ll \sum_p p^{-3/2} \ll 1$ uniformly in s with $\operatorname{Re}(s) \geq 3/4$. Hence $G(s)$ is analytic on the half-plane $\operatorname{Re}(s) \geq 3/4$ and satisfies $G(s) \ll 1$ uniformly therein (with the last implied constant being absolute). This shows that the sequence $\{a_n\}_n$ satisfies property $\mathcal{P}(1, \{\alpha_{\chi}\}_{\chi}; c_0, \Omega)$, with α_{χ} as defined above and with $\Omega(t) \ll 1$.

We now observe one of the first concrete applications showing the benefit of introducing the parameter λ_q in our main result Theorem 1.1: Note that for any coprime residue class $a \bmod q$,

$$\sum_{\chi} \alpha_{\chi} \cdot \chi(a) = \frac{1}{\varphi(q)} \sum_{b \in \mathcal{A}} \sum_{\chi} \bar{\chi}(b) \chi(a) = \sum_{b \in \mathcal{A}} \mathbb{1}_{b \equiv a \pmod{q}} = \mathbb{1}_{a \in \mathcal{A}}.$$

This shows that $|\lambda_q| \leq 2$. The asymptotic formula (1.11) now follows from Theorem 1.1(1) in the case $q \in [e^6, (\log x)^{K_0}]$, upon using Siegel's Theorem to note that $1 - \eta_e \geq c_1 \cdot q^{-\epsilon_0/K_0}$ for some constant $c_1 > 0$ depending only on ϵ_0 and K_0 . Likewise, (1.11) follows from the last assertion of Theorem 1.1(2) in the case when the Siegel zero does not exist.

Finally, we use (1.3) to see that the coefficients k_j in (5.1) are given by

$$(5.4) \quad k_j = \frac{1}{j!} \cdot \left. \frac{d^j}{ds^j} \right|_{s=1} \frac{\mathcal{F}(s)G(s)}{s} (s-1)^{\alpha_{\chi_0}},$$

with $\mathcal{F}(s)$, $G(s)$ and α_χ as defined after (5.3). In particular, it is worth noting that

$$k_0 = \lim_{s \rightarrow 1+} \frac{\mathcal{F}(s)G(s)}{s} (s-1)^{\alpha_{\chi_0}} = G(1) \cdot \left(\prod_{\chi \neq \chi_0} L(1, \chi)^{\alpha_\chi} \right) \cdot \lim_{s \rightarrow 1+} (L(s, \chi_0)(s-1))^{\alpha_{\chi_0}},$$

Now since $L(s, \chi_0) = \zeta(s) \cdot \prod_{\ell \bmod q} (1 - 1/\ell^s)$, and since $\lim_{s \rightarrow 1+} \zeta(s)(s-1) = 1$, we obtain the following explicit formula for the first coefficient k_0 of the asymptotic series (1.11).

$$(5.5) \quad k_0 = \left(\frac{\varphi(q)}{q} \right)^{\frac{|\mathcal{A}|}{\varphi(q)}} \left(\prod_{\chi \neq \chi_0} L(1, \chi)^{\alpha_\chi} \right) \cdot \exp \left(\sum_{\substack{p, r \geq 2 \\ p \bmod q \in \mathcal{A}}} \frac{1}{rp^r} - \sum_{\substack{p, r \geq 2 \\ p^r \bmod q \in \mathcal{A}}} \frac{1}{rp^r} \right).$$

6. DISTRIBUTION OF THE LEAST INVARIANT FACTOR OF THE MULTIPLICATIVE GROUP: PROOF OF THEOREMS 1.5 AND 1.6

The following algebraic result characterizes when an even integer $q > 2$ divides the least invariant factor $\lambda^*(n)$ of the multiplicative group modulo n . This is a restatement of [2, Proposition 5.3]. In this section, $\mathbf{P}(q)$ denotes the **largest** prime divisor of q and $e_p = v_{\mathbf{P}(q)}(q)$ is the exponent (highest power) of $P(q)$ in the prime decomposition of q .

Lemma 6.1. *Let $q \geq 4$ be an even integer.*

(i) *If $q \nmid \varphi(P(q)^{e_p+1})$, then the positive integers n for which $q \mid \lambda^*(n)$ are precisely those of the form $n = 2^{e_2} \cdot m$, where $e_2 \in \{0, 1\}$ and m is only divisible by primes $\equiv 1 \pmod{q}$.*

(ii) *If $P(q) > 2$ and $q \mid \varphi(P(q)^{e_p+1})$, then the positive integers n for which $q \mid \lambda^*(n)$ are precisely those of the form $n = 2^{e_2} \cdot P(q)^v \cdot m$, where $e_2 \in \{0, 1\}$, where m is only divisible by primes $\equiv 1 \pmod{q}$, and where v is either 0 or at least $e_p + 1$.*

To see that the above statement is indeed equivalent to [2, Proposition 5.3], we just need to note that if $q \mid \ell^{v_\ell(q)}(\ell - 1)$ for some prime $\ell \mid q$, then we have $P(q) \mid \ell^{v_\ell(q)}(\ell - 1)$, forcing $\ell = P(q)$. In order to establish Theorem 1.5, we use the above characterization to see that for all complex s with $\sigma = \operatorname{Re}(s) > 1$, we may write

$$\begin{aligned} \sum_n \frac{\mathbb{1}_{q \mid \lambda^*(n)}}{n^s} &= \sum_{e_2 \in \{0, 1\}} \frac{1}{2^{e_2 s}} \sum_{m: p \mid m \implies p \equiv 1 \pmod{q}} \frac{1}{m^s} \\ &\quad + \mathbb{1}_{q \mid \varphi(P(q)^{e_p+1})} \sum_{e_2 \in \{0, 1\}} \frac{1}{2^{e_2 s}} \sum_{r \geq e_p+1} \frac{1}{P(q)^{rs}} \sum_{m: p \mid m \implies p \equiv 1 \pmod{q}} \frac{1}{m^s} \end{aligned}$$

$$= \left(1 + \frac{1}{2^s}\right) \cdot \left(1 + \mathbb{1}_{q|\varphi(P(q)^{ep+1})} \sum_{r \geq e_p+1} \frac{1}{P(q)^{rs}}\right) \cdot \prod_{p \equiv 1 \pmod{q}} \left(1 + \sum_{r \geq 1} \frac{1}{p^{rs}}\right)$$

Invoking (5.3) with $\mathcal{A} := \{1\} \subset U_q$, we get $\sum_n \mathbb{1}_{q|\lambda^*(n)}/n^s = \left(\prod_\chi L(s, \chi)\right)^{1/\varphi(q)} \cdot \mathcal{G}(s)$, where

$$\mathcal{G}(s) := \left(1 + \frac{1}{2^s}\right) \cdot \left(1 + \frac{P(q)^{-se_p}}{P(q)^s - 1}\right)^{\mathbb{1}_{q|\varphi(P(q)^{ep+1})}} \cdot \exp\left(\sum_{\substack{p,r \geq 2 \\ p \equiv 1 \pmod{q}}} \frac{1}{rp^{rs}} - \sum_{\substack{p,r \geq 2 \\ p^r \equiv 1 \pmod{q}}} \frac{1}{rp^{rs}}\right).$$

Once again, since $\sum_{p,r \geq 2} |1/rp^{rs}| \ll 1$ uniformly in the half plane $\text{Re}(s) \geq 3/4$, we find that $\mathcal{G}(s)$ is analytic and of size $O(1)$ uniformly on this half place. Hence, our sequence $\{\mathbb{1}_{q|\lambda^*(n)}\}_n$ has property $\mathcal{P}(1, \{\alpha_\chi\}_\chi; c_0, \Omega)$, with $\Omega(t) \ll 1$ and with all $\alpha_\chi = 1/\varphi(q)$. As such, $\sum_\chi \alpha_\chi \cdot \chi(a) = \mathbb{1}_{a \equiv 1 \pmod{q}}$ for any coprime residue class $a \pmod{q}$, showing that once again $|\lambda_q| \leq 2$ in Theorem 1.1. The rest of the argument for Theorem 1.5 goes through exactly as in the previous section.

The coefficients r_j (and in particular r_0) have analogous explicit formulas to (5.4) and (5.5) respectively, and they are proven analogously as well. We spell them out below, for completeness.

$$(6.1) \quad r_j = \frac{1}{j!} \cdot \left. \frac{d^j}{ds^j} \right|_{s=1} \frac{\mathcal{G}(s)}{s} \cdot \left((s-1) \cdot \prod_\chi L(s, \chi) \right)^{1/\varphi(q)}.$$

In particular, since $L(1, \chi) \ll \log q$ for all nontrivial characters $\chi \pmod{q}$, we see that

$$r_0 = \mathcal{G}(1) \cdot \left(\frac{\varphi(q)}{q} \prod_{\chi \neq \chi_0} L(1, \chi) \right)^{1/\varphi(q)} \asymp \left(\frac{\varphi(q)}{q} \prod_{\chi \neq \chi_0} |L(1, \chi)| \right)^{1/\varphi(q)} \ll \log q.$$

This completes the proof of Theorem 1.5. \square

Remark. This approach of directly applying our Theorem 1.1 on the non-multiplicative sequence $a_n = \mathbb{1}_{q|\lambda^*(n)}$ also simplifies Chang and Martin's approach in [2, Proposition 5.4] of first estimating the count $\mathcal{N}(x; q, 1)$, and then passing to the count $\sum_{n \leq x} \mathbb{1}_{q|\lambda^*(n)}$ via an additional combinatorial argument.

We now come to Theorem 1.6; **in the rest of this section, we assume that $q \leq (\log x)^K$ with $K > 0$ fixed**. We will refine some of the ideas of Chang and Martin with an additional input from the anatomy of integers, where we split off the largest prime factor of n and carefully study the case when this factor grows somewhat rapidly with x . To this end, we define $z := x^{1/\log_2 x}$, and let $\mathcal{D}_q(x)$ denote the number of positive integers $n \leq x$ satisfying all the following properties

- $q \mid \lambda^*(n)$.
- $P(n) > z$, $P(n)^2 \nmid n$.
- $\#\{p \mid n : p \equiv 1 \pmod{q}\} \geq 2$.

We define $\mathcal{E}_q(x)$ the same way, only with the first (divisibility) condition replaced by the equality $q = \lambda^*(n)$. We start by giving the following upper bounds on $\mathcal{D}_q(x)$.

Lemma 6.2. *Uniformly in $x \geq 16$, we have*

$$(6.2) \quad \mathcal{D}_Q(x) \begin{cases} \ll \frac{x}{(\log x)^{1-1/\varphi(Q)}} \cdot \left(\frac{\log_2 x}{\varphi(Q)} \right)^2, & \text{uniformly in even } Q \leq (\log x)^{100K}. \\ \leq x \left(\frac{\log_2 x}{\varphi(Q)} + O\left(\frac{\log Q}{Q}\right) \right)^2, & \text{uniformly in even } Q \in [2, x]. \end{cases}$$

Proof of Lemma 6.2. To see the first bound, we consider any $Q \leq (\log x)^K$, and write $\mathcal{D}_Q(x) = \mathcal{D}_{\text{odd}}(x) + \mathcal{D}_{\text{even}}(x)$, where $\mathcal{D}_{\text{odd}}(x)$ and $\mathcal{D}_{\text{even}}(x)$ denote the contribution of all odd and even positive integers counted in $\mathcal{D}_Q(x)$, respectively. By Lemma 6.1 and the definition of $\mathcal{D}_Q(x)$, any positive integer n counted in $\mathcal{D}_{\text{odd}}(x)$ can be written as mpP , where $P = P(n) > \max\{z, P(mp)\}$, where $P \equiv p \equiv 1 \pmod{Q}$, where any prime ℓ dividing m must either be equal to $P(Q)$ or must be 1 mod ℓ . (Here we have noted that $P = P(n) > z > q \geq P(Q)$.)

Now given m and p , the number of primes $P \in (z, x/mp]$ satisfying $P \equiv 1 \pmod{Q}$ is $\ll x/\varphi(Q)mp \log z$ by the Brun–Titchmarsh inequality. Moreover, by Brun–Titchmarsh and partial summation, we have $\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{Q}}} 1/p \ll \log_2 x/\varphi(Q)$. This bound also shows that

$$\begin{aligned} \sum_{\substack{m \leq x \\ \ell|m \implies \ell=P(Q) \text{ or } \ell \equiv 1 \pmod{Q}}} \frac{1}{m} &\leq \left(1 + \sum_{r \geq 1} \frac{1}{P(Q)^r} \right) \cdot \prod_{\substack{\ell \leq x \\ \ell \equiv 1 \pmod{Q}}} \left(1 + \sum_{r \geq 1} \frac{1}{\ell^r} \right) \\ &\leq \left(1 + \sum_{r \geq 1} \frac{1}{2^r} \right) \cdot \exp \left(\sum_{\substack{\ell \leq x \\ \ell \equiv 1 \pmod{Q}}} \frac{1}{\ell} + O\left(\sum_{\ell, r \geq 2} \frac{1}{\ell^r}\right) \right) \ll (\log x)^{1/\varphi(Q)}. \end{aligned}$$

Collecting all these observations, we find that $\mathcal{D}_{\text{odd}}(x) \ll x(\log x)^{\varphi(Q)-1} \cdot (\log_2 x/\varphi(Q))^2$. An entirely analogous argument shows that $\mathcal{D}_{\text{even}}(x)$ satisfies the same bound: We just start by writing all the n counted in $\mathcal{D}_{\text{even}}(x)$ as $2mpP$, with m, p, P satisfying similar conditions as above. (Here we recalled that $v_2(n) \in \{0, 1\}$ by Lemma 6.1.) This shows the first bound in (6.2).

For the second bound, we simply note that by the last condition in the definition of $\mathcal{D}_Q(x)$,

$$\mathcal{D}_Q(x) \leq \sum_{\substack{p_1, p_2 \leq x \\ p_1 \equiv p_2 \equiv 1 \pmod{Q}}} \sum_{\substack{n \leq x \\ p_1 p_2 | n}} 1 \leq \sum_{\substack{p_1, p_2 \leq x \\ p_1 \equiv p_2 \equiv 1 \pmod{Q}}} \frac{x}{p_1 p_2} \leq x \left(\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{Q}}} \frac{1}{p} \right)^2.$$

The second bound in (6.2) now follows by using an estimate due independently to Pomerance (see Remark 1 of [29]) and Norton (see the Lemma on p. 699 of [23]). \square

Using Lemma 6.2, we will now provide an estimate on $\mathcal{E}_q(x)$ (also defined before the lemma) uniformly for $q \leq (\log x)^K$. To do this, we adapt the argument given for Lemma 6.2 in [2]. Note that by definition of $\mathcal{D}_q(x)$ and $\mathcal{E}_q(x)$, we have $\mathcal{D}_q(x) = \sum_{m \geq 1} \mathcal{E}_{mq}(x)$, so that by a version of the Möbius Inversion Formula, we have

$$(6.3) \quad \mathcal{E}_q(x) = \sum_{m \geq 1} \mu(m) \mathcal{D}_{mq}(x).$$

(Here both the seemingly infinite sums are actually finite, and only go up to $m \leq x/q$.) Now from the first bound in (6.2) and the well-known estimate $\varphi(mq) \gg mq/\log_2(mq)$, we see that

$$\begin{aligned} \sum_{2 \leq m \leq (\log x)^{3K+3}} \mathcal{D}_{mq}(x) &\ll \sum_{2 \leq m \leq (\log x)^{3K+3}} \frac{x}{(\log x)^{1-1/\varphi(mq)}} \cdot \left(\frac{\log_2 x \cdot \log_2(mq)}{mq} \right)^2 \\ (6.4) \quad &\ll \frac{x (\log_2 x \cdot \log_3 x)^2}{q^2 (\log x)^{1-1/2\varphi(q)}} \sum_{m \geq 2} \frac{1}{m^2} \ll \frac{x (\log_2 x \cdot \log_3 x)^2}{q^2 (\log x)^{1-1/2\varphi(q)}}, \end{aligned}$$

where in the second line, we have used that $mq \leq (\log x)^{4K+3}$, and that $\varphi(mq) \geq 2\varphi(q)$ for any $m \geq 2$ as q is even. On the other hand, using the second bound in (6.2), we find that

$$\begin{aligned} \sum_{(\log x)^{3K+3} < m \leq x/q} \mathcal{D}_{mq}(x) &\ll \sum_{(\log x)^{3K+3} < m \leq x/q} x \cdot \left(\frac{\log_2 x \cdot \log_2(mq)}{mq} + \frac{\log(mq)}{mq} \right)^2 \\ (6.5) \quad &\ll x \sum_{m > (\log x)^{3K+3}} \left(\frac{\log x}{mq} \right)^2 \ll \frac{x}{q^2 (\log x)^{3K}}. \end{aligned}$$

Inserting (6.4) and (6.5) into (6.3), we obtain

$$(6.6) \quad \mathcal{E}_q(x) = \mathcal{D}_q(x) + O\left(\frac{x (\log_2 x \cdot \log_3 x)^2}{q^2 (\log x)^{1-1/2\varphi(q)}}\right).$$

Now by known estimates on smooth numbers (see [1, p. 15] or [38, Theorem 5.13 and Corollary 5.19, Chapter III.5]), the number of $n \leq x$ having $P(n) \leq z = x^{1/\log_2 x}$ is $\ll x/(\log x)^{100K}$. Hence

$$\begin{aligned} \sum_{\substack{n \leq x \\ q \mid \lambda^*(n)}} 1 &= \mathcal{D}_q(x) + \sum_{\substack{n \leq x: q \mid \lambda^*(n) \\ P(n) > z, P(n)^2 \nmid n \\ \#\{p|n: p \equiv 1 \pmod{q}\} \in \{0,1\}}} 1 + O\left(\frac{x}{(\log x)^{100K}}\right). \\ \sum_{\substack{n \leq x \\ q = \lambda^*(n)}} 1 &= \mathcal{E}_q(x) + \sum_{\substack{n \leq x: q = \lambda^*(n) \\ P(n) > z, P(n)^2 \nmid n \\ \#\{p|n: p \equiv 1 \pmod{q}\} \in \{0,1\}}} 1 + O\left(\frac{x}{(\log x)^{100K}}\right). \end{aligned}$$

Subtracting the second estimate from the first, and using (6.6), we obtain

$$(6.7) \quad \sum_{\substack{n \leq x \\ q = \lambda^*(n)}} 1 = \sum_{\substack{n \leq x \\ q \mid \lambda^*(n)}} 1 - \sum_{\substack{n \leq x: q \mid \lambda^*(n), \lambda^*(n) > q \\ P(n) > z, P(n)^2 \nmid n \\ \#\{p|n: p \equiv 1 \pmod{q}\} \in \{0,1\}}} 1 + O\left(\frac{x (\log_2 x \cdot \log_3 x)^2}{q^2 (\log x)^{1-1/2\varphi(q)}}\right).$$

We now estimate the second sum on the right hand above, which we call \mathcal{D}_{rem} .

Case 1. If $q \nmid \varphi(P(q)^{e_p+1})$, then Lemma 6.1(i) shows that any n counted in the sum \mathcal{D}_{rem} must necessarily be of the form $n = P$ or $n = 2P$ where $P \in (z, x)$ satisfies $P \equiv 1 \pmod{q}$. This condition is also sufficient since from the isomorphisms $U_P \cong \mathbb{Z}/(P-1)\mathbb{Z}$ and $U_{2P} \cong U_2 \times U_P \cong \mathbb{Z}/(P-1)\mathbb{Z}$, we have $\lambda^*(2P) = \lambda^*(P) = P-1 \equiv 0 \pmod{q}$. Hence by the Siegel–Walfisz Theorem,

$$(6.8) \quad \sum_{\substack{n \leq x: q \mid \lambda^*(n), \lambda^*(n) > q \\ P(n) > z, P(n)^2 \nmid n \\ \#\{p|n: p \equiv 1 \pmod{q}\} \in \{0,1\}}} 1 = \sum_{\substack{z < P \leq x \\ P \equiv 1 \pmod{q}}} 1 + \sum_{\substack{z < P \leq x/2 \\ P \equiv 1 \pmod{q}}} 1 = \frac{\text{li}(x) + \text{li}(x/2)}{\varphi(q)} + O(x \exp(-\sqrt{\log x})).$$

Case 2. Now assume that $q \mid \varphi(P(q)^{e_p+1})$. By Lemma 6.1(ii), any $n \leq x$ satisfying $q \mid \lambda^*(n)$ but having no prime factor $\equiv 1 \pmod{q}$, must be of the form $n = 2^{e_2} P(q)^v$. Hence the number of such n is $\ll \sum_{0 \leq z \leq \log x} 1 \ll \log x$. On the other hand, if any n counted in \mathcal{D}_{rem} has exactly one prime factor $\equiv 1 \pmod{q}$, then Lemma 6.1(ii) shows that n must be of the form $2^{e_2} P(q)^v P$ with $e_2 \in \{0, 1\}$, with $P \in (z, x]$ satisfying $P \equiv 1 \pmod{q}$, and with $v \in \{0, e_p + 1, e_p + 2, \dots\}$. Hence the total number of n counted in \mathcal{D}_{rem} exactly one prime factor in the residue class 1 mod q is

$$\sum_{\substack{z < P \leq x \\ P \equiv 1 \pmod{q}}} 1 + \sum_{\substack{z < P \leq x/2 \\ P \equiv 1 \pmod{q}}} 1 + O\left(\sum_{e_2 \in \{0, 1\}} \sum_{v \geq e_p+1} \sum_{\substack{z < P \leq x/2^{e_2} P(q)^v \\ P \equiv 1 \pmod{q}}} 1\right),$$

where the first two sums come from the case $v = 0$ (and then correspond to whether $e_2 = 0$ or $e_2 = 1$). Now the first two sums above can again be estimated by the Siegel–Walfisz Theorem, whereas by the Brun–Titchmarsh theorem, the total O -term in the above display is

$$\ll \frac{x}{\varphi(q) \log z} \sum_{e_2 \in \{0, 1\}} \frac{1}{2^{e_2}} \sum_{v \geq e_p+1} \frac{1}{P(q)^v} \ll \frac{x}{P(q)^{e_p+1} \varphi(q) \log z} \ll \frac{x \log_2 x \cdot \log_3 x}{q^2 \log x}$$

where we have noted that $q \mid \varphi(P(q)^{e_p+1})$ forces $q \leq P(q)^{e_p+1}$. By the discussion under Case 2,

$$(6.9) \quad \sum_{\substack{n \leq x: q \mid \lambda^*(n), \lambda^*(n) > q \\ P(n) > z, P(n)^2 \nmid n \\ \#\{p|n: p \equiv 1 \pmod{q}\} \in \{0, 1\}}} 1 = \frac{\text{li}(x) + \text{li}(x/2)}{\varphi(q)} + O\left(\frac{x \log_2 x \cdot \log_3 x}{q^2 \log x}\right).$$

Finally, inserting (6.8) and (6.9) into (6.7), we find that

$$\sum_{\substack{n \leq x \\ q \mid \lambda^*(n)}} 1 = \sum_{\substack{n \leq x \\ q \nmid \lambda^*(n)}} 1 - \frac{\text{li}(x) + \text{li}(x/2)}{\varphi(q)} + O\left(\frac{x (\log_2 x \cdot \log_3 x)^2}{q^2 (\log x)^{1-1/2\varphi(q)}}\right),$$

whereupon Theorem 1.6 follows from the very first assertion in Theorem 1.5. \square

7. DISTRIBUTION OF THE LEAST PRIMARY FACTOR OF THE MULTIPLICATIVE GROUP: PROOF OF THEOREM 1.7

As alluded to in subsection § 1.4, our algebraic input will be the same as in Martin and Nguyen’s work [20], namely we will start by writing $\#\{n \leq x : \lambda'(n) = q\} = \mathcal{A}_q(x) - \mathcal{A}_{q+}(x)$, and then estimating $\mathcal{A}_q(x)$. The following characterization of $\mathcal{A}_q(x)$ will be really useful for this purpose: This is a restatement of [20, Proposition 3.6], but we also summarize their argument.

Lemma 7.1. *Given a prime power $q \geq 3$, the integers $n \geq 3$ having $\lambda'(n) \geq q$ are precisely those of the form $2^{e_2} m$, with $e_2 \in \{0, 1\}$, and with the odd integer m satisfying the following property:*

$$(7.1) \quad \text{For all primes } p \mid m \text{ and all primes } \ell < q, \text{ either } \ell \nmid (p-1) \text{ or } p \equiv 1 \pmod{\ell^{\lceil \log q / \log \ell \rceil}}.$$

In particular, any such n cannot be divisible by any odd prime at most q .

Proof of Lemma 7.1. Since $U_n \cong \prod_{p^k \parallel n} U_{p^k}$, we have $\lambda'(n) = \min_{p^k \parallel n} \lambda'(p^k)$, so that the condition $\lambda'(n) \geq q$ amounts to $\lambda'(p^k) \geq q$ for all prime powers $p^k \parallel n$. Now if $p = 2$ but $k \geq 2$, then from $U_{2^k} \cong U_2 \times U_{2^{k-2}}$, we see that $\lambda'(2^k) = 2 < q$. This forces $4 \nmid n$. (Conversely, $4 \nmid n$ also guarantees that $\lambda'(2^{v_2(n)}) = \infty$ by our convention.) On the other hand, if $p > 2$, then from $U_{p^k} \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^{k-1}\mathbb{Z}$, we see that $\lambda'(p^k)$ is the smallest prime power $\ell^j \parallel (p-1)$, so that

the condition $\lambda'(p^k) \geq q$ amounts to having $\ell^j \geq q$ for all $\ell^j \parallel (p-1)$, which in turn is equivalent to having $v_\ell(p-1) \geq \log q / \log \ell$, i.e., to $p \equiv 1 \pmod{\ell^{\lceil \log q / \log \ell \rceil}}$.

This proves the equivalence in the first assertion of the lemma. For the final assertion, note that if an odd prime $p \leq q$ divides n , then any prime divisor ℓ of $p-1$ violates (7.1). \square

In this section, we define $Q := \prod_{\ell < q} \ell^{\lceil \log q / \log \ell \rceil}$. By the prime number theorem, we have $e^{q/2} \leq Q \leq e^{4q}$ for all sufficiently large q . Note that condition (7.1) amounts to throwing each odd prime factor of n into $\prod_{\ell < q} (1 + (\ell-2) \cdot \ell^{\lceil \log q / \log \ell \rceil - 1})$ many coprime residue classes modulo Q . We use \mathcal{B}_Q to denote this set of residues mod Q , so that the quantity $B(q)$ in (1.12) is exactly $\#\mathcal{B}_Q / \varphi(Q)$. (Note that the residue classes in \mathcal{B}_Q are precisely those that for each prime $\ell \in (2, q)$, either leave a remainder of 1 modulo $\ell^{\lceil \log q / \log \ell \rceil}$, or are obtained by lifting the residues $\{2, \dots, \ell-1\}$ from modulus ℓ to modulus $\ell^{\lceil \log q / \log \ell \rceil}$.)

Hereafter proceeding exactly as we did in section 5 with the integer Q playing the role of “ q ” in that section, we find that with $\mathbb{1}_{(7.1)}$ being the indicator function of condition (7.1),

$$\sum_n \frac{\mathbb{1}_{\lambda'(n) \geq q}}{n^s} = \sum_{2 \nmid n} \frac{\mathbb{1}_{(7.1)}}{n^s} + \frac{1}{2^s} \sum_{2 \nmid n} \frac{\mathbb{1}_{(7.1)}}{n^s} = \left(1 + \frac{1}{2^s}\right) \cdot \prod_{p \text{ mod } q \in \mathcal{B}_Q} \left(1 - \frac{1}{p^s}\right)^{-1}$$

can be written in the form $\left(\prod_{\chi \text{ mod } Q} L(s, \chi)^{\alpha_\chi}\right) \cdot G_Q(s)$, with $\alpha_\chi := \varphi(Q)^{-1} \sum_{b \in \mathcal{B}_Q} \bar{\chi}(b)$ (so that $\alpha_{\chi_0} = B(q)$), and with the function

$$G_Q(s) := \left(1 + \frac{1}{2^s}\right) \left(1 - \frac{1}{2^s}\right)^{\mathbb{1}_{2 \in \mathcal{B}_Q}} \cdot \exp\left(\sum_{\substack{p, r \geq 2 \\ p \text{ mod } q \in \mathcal{B}_Q}} \frac{1}{rp^{rs}} - \sum_{\substack{p, r \geq 2 \\ pr \text{ mod } q \in \mathcal{B}_Q}} \frac{1}{rp^{rs}}\right)$$

being analytic of size $O(1)$ uniformly in the half plane $\sigma \geq 3/4$. This shows that the sequence $\{\mathbb{1}_{S(n) \geq q}\}_n$ has property $\mathcal{P}(1, \{\alpha_\chi\}_\chi; c_0, \Omega)$ with $\Omega(t) \ll 1$. Theorem 1.1 thus gives (7.2)

$$\mathcal{A}_q(x) = \frac{x}{(\log x)^{1-B(q)}} \sum_{j=0}^N \frac{\kappa_j(q) (\log x)^{-j}}{\Gamma(B(q)-j)} + O\left(\frac{N! (142 c_1^{-1})^N \cdot x}{(\log x)^{(N+2)(1-\epsilon_0)-\frac{1}{\varphi(q)}}} + x \exp\left(-\sqrt{\frac{c_0 \log x}{32}}\right)\right),$$

uniformly in $Q \leq e^{4K} \leq (\log x)^{4K}$ (and uniformly in $Q \leq \exp(\sqrt{c_0 \log x / 8})$ if the Siegel zero does not exist), with the coefficients κ_j being defined by

$$(7.3) \quad \kappa_j = \frac{1}{j!} \cdot \frac{d^j}{ds^j} \Bigg|_{s=1} \frac{G_Q(s)}{s} \cdot \left(\prod_{\chi \text{ mod } Q} L(s, \chi)^{\alpha_\chi} \right) \cdot (s-1)^{B(q)}.$$

Note that the exponents $\alpha_\chi = \varphi(Q)^{-1} \sum_{b \in \mathcal{B}_Q} \bar{\chi}(b)$ have been computed explicitly in [20, Section 6]. Finally, Theorem 1.7 follows from (7.2) by writing $\#\{n \leq x : \lambda'(n)\} = \mathcal{A}_q(x) - \mathcal{A}_{q^+}(x)$, and then noting that $B(q^+) \leq B(q)$ by (1.12) as well as $q^+ \leq 2q$ by Bertrand’s Postulate. \square

8. THE SATHE–SELBERG THEOREM IN ARITHMETIC PROGRESSIONS TO VARYING MODULI: PROOFS OF THEOREMS 1.8 AND 1.9

We first prove Theorem 1.8 and then mention the changes required for Theorem 1.9. In this entire section, we may assume that $q > \max\{e^6, 25(K+2)^2\}$, for otherwise all assertions can be

deduced by following our arguments below, and by replacing our upcoming uses of Theorem 1.3 by the standard form of the Landau–Selberg–Delange method as in [38, Theorem II.5.2].

Inspired by Selberg’s idea in [33], we will identify the number $\#\{n \leq x : \omega_a(n) = k\}$ as the coefficient of $z^{\omega_a(n)}$ in the polynomial $\sum_{n \leq x} z^{\omega_a(n)}$. We thus start by giving an estimate on $\sum_{n \leq x} z^{\omega_a(n)}$ for complex numbers z having $|z| \leq K + 1$. To this end, note that since $\omega_a(n)$ is an additive function, the function $n \mapsto z^{\omega_a(n)}$ is multiplicative. Hence for all s with $\sigma = \operatorname{Re}(s) > 1$, we have

$$(8.1) \quad \sum_n \frac{z^{\omega_a(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\omega_a(p^r)}}{p^{rs}} \right) = \prod_{p \equiv a \pmod{q}} \left(1 + z \sum_{r \geq 1} \frac{1}{p^{rs}} \right) \cdot \prod_{p \not\equiv a \pmod{q}} \left(1 + \sum_{r \geq 1} \frac{1}{p^{rs}} \right)$$

$$(8.2) \quad = \zeta(s) \cdot \prod_{p \equiv a \pmod{q}} \left(1 + \frac{z}{p^s} \cdot \left(1 - \frac{1}{p^s} \right)^{-1} \right) \left(1 - \frac{1}{p^s} \right) = \zeta(s) \cdot \prod_{p \equiv a \pmod{q}} \left(1 + \frac{z-1}{p^s} \right)$$

where we have used the Euler product of $\zeta(s)$ and the fact that $\omega_a(p^r) = \mathbb{1}_{p \equiv a \pmod{q}}$. Continuing,

$$\sum_n \frac{z^{\omega_a(n)}}{n^s} = \zeta(s) \cdot \exp \left(\sum_{p \equiv a \pmod{q}} \frac{z-1}{p^s} \right) \cdot \left(\prod_{p \equiv a \pmod{q}} \left(1 + \frac{z-1}{p^s} \right) e^{(1-z)/p^s} \right).$$

Invoking (5.2) for the sum on the right hand side above and using the fact that $e^{(1-z)/p^s} = 1 - (z-1)/p^s + \sum_{r \geq 2} (1-z)^r / r! p^{rs}$ and that $\zeta(s) = L(s, \chi_0) \prod_{\ell \mid q} (1 - 1/\ell^s)^{-1}$, we thus obtain

$$(8.3) \quad \sum_n \frac{z^{\omega_a(n)}}{n^s} = \left(\prod_{\chi} L(s, \chi)^{\mathbb{1}_{\chi=\chi_0} + (z-1)\bar{\chi}(a)/\varphi(q)} \right) \cdot \mathcal{G}_z(s),$$

where the function $\mathcal{G}_z(s)$ is holomorphic on the half plane $\sigma \geq 3/4$, and has the formula

$$(8.4) \quad \mathcal{G}_z(s) = \exp \left((1-z) \sum_{\substack{p, r \geq 2 \\ p^r \equiv a \pmod{q}}} \frac{1}{rp^{rs}} \right) \cdot \left(\prod_{\ell \mid q} \left(1 - \frac{1}{\ell^s} \right)^{-1} \right) \cdot \left(\prod_{p \equiv a \pmod{q}} \left(1 + \frac{z-1}{p^s} \right) e^{(1-z)/p^s} \right).$$

We now bound $|\mathcal{G}_z(s)|$. For $\sigma \geq 3/4$, we have $\sum_{p, r \geq 2} 1/|rp^{rs}| \leq \sum_p \sum_{r \geq 2} 1/p^{3r/4} \leq \sum_p 1/p^{3/2} \ll 1$. Thus the exponential factor in (8.4) has size $O_K(1)$ for all $|z| \leq K + 1$.

Moreover, for $\sigma \geq 3/4$ and $|z| \leq K + 1$, the last infinite product in (8.4) is

$$(8.5) \quad \prod_{p \equiv a \pmod{q}} \left(1 + \frac{z-1}{p^s} \right) \left(1 + \frac{1-z}{p^s} + O \left(\frac{|z-1|^2}{p^{2\sigma}} \right) \right) = \prod_p \left(1 + O_K \left(\frac{1}{p^{3/2}} \right) \right) \ll_K 1,$$

Lastly, for all s with $\sigma \geq 1 - c_0/\log q$, we have $\ell^\sigma \geq \ell^{1-c_0/\log q} \geq e^{-c_0} \cdot \ell \geq 2\ell/3$, so that

$$\sum_{\ell \mid q} \left| \log \left(1 - \frac{1}{\ell^s} \right) \right| \leq \sum_{\ell \mid q} \frac{1}{\ell^\sigma} + O \left(\sum_{\ell \geq 2} \frac{1}{\ell^{3/2}} \right) \leq \frac{3}{2} \sum_{\ell \mid q} \frac{1}{\ell} + O(1) \leq \frac{3}{2} \sum_{\ell \leq \omega(q)} \frac{1}{\ell} + O(1),$$

and by Mertens’ Theorem, this last sum is $\leq (3/2) \log_3 q + O(1)$. Inserting all these observations into (8.4), we get $\mathcal{G}_z(s) \ll (\log_2 q)^{3/2}$, uniformly for all s with $\sigma \geq 1 - c_0/\log q$.

Hence by (8.3), we find that the sequence $\{z^{\omega_a(n)}\}_n$ has property $\mathcal{P}(1, \{\alpha_\chi\}_\chi; c_0, \Omega)$, where $\alpha_\chi = \mathbb{1}_{\chi=\chi_0} + (z-1)\bar{\chi}(a)/\varphi(q)$ and $\Omega(t) \ll (\log_2 q)^{3/2}$. An entirely analogous argument shows that the sequence $\{|z|^{\omega_a(n)}\}_n$ has property $\mathcal{P}(1, \{\beta_\chi\}_\chi; c_0, \Omega)$, with $\beta_\chi = \mathbb{1}_{\chi=\chi_0} + (|z|-1)\bar{\chi}(a)/\varphi(q)$. This neatly places us in the setting of Theorem 1.3, with $\nu = 1$, $\mathcal{M} \ll (\log_2 q)^{3/2}$, $\delta_0 = 1$, and with $\max\{|\alpha_{\chi_0}|, |\alpha_{\chi_e}|, |\beta_{\chi_0}|, |\beta_{\chi_e}|\} \leq 1 + (K+1)/\varphi(q) \leq 2$. (Note that this is one of the situations where the hypotheses of Theorem 1.3 are easier to verify than those of Theorem 1.1.) Since $\sum_\chi \alpha_\chi \cdot \chi(b) = 1 + (z-1) \sum_\chi \bar{\chi}(b) \chi(b)/\varphi(q) = z$, and likewise $\sum_\chi \beta_\chi \cdot \chi(b) = |z|$, for any b coprime to q , it follows that $\Lambda_q = |z| \leq K$. Theorem 1.3(1) yields, uniformly in $q \leq (\log x)^K$ and $|z| \leq K$,

$$(8.6) \quad \sum_{n \leq x} z^{\omega_a(n)} - \frac{x}{(\log x)^{(1-z)/\varphi(q)}} \sum_{j=0}^N \frac{C_j(z)}{(\log x)^j} \ll \frac{N! (71 c_1^{-1})^N \cdot x}{(\log x)^{(N+1)(1-\epsilon_0)-\frac{K+1}{\varphi(q)}}} + x \exp\left(-\sqrt{\frac{c_0 \log x}{16K}}\right).$$

Here we have used $1 - \eta_e \geq c_1 q^{-\epsilon_0/(K+1)}$ to see that $(1 - \eta_e)^{-|\text{Re}(\alpha_{\chi_e})|} \ll q^{\epsilon_0/\varphi(q)} \ll 1$. Moreover, $C_j(z)$ is a holomorphic function (of z) on the disk $|z| \leq K$, defined by

$$(8.7) \quad C_j(z) = \frac{1/j!}{\Gamma\left(1-j+\frac{z-1}{\varphi(q)}\right)} \cdot \frac{d^j}{ds^j} \Bigg|_{s=1} \frac{\mathcal{G}_z(s)}{s} \cdot \left(\prod_\chi L(s, \chi)^{\mathbb{1}_{\chi=\chi_0} + \frac{(z-1)\bar{\chi}(a)}{\varphi(q)}} \right) \cdot (s-1)^{1+\frac{z-1}{\varphi(q)}}.$$

Now since $\#\{n \leq x : \omega_a(n) = k\}$ is the coefficient of z^k in the polynomial $\sum_{n \leq x} z^{\omega_a(n)}$, estimate (1.13) follows from (8.6) by applying Cauchy's integral formula on the circle of radius K centered at the origin. The polynomials $P_{j,k}(T)$ appearing in (1.13) are given by the following identities.

(8.8)

$$P_{j,k}(T) := \frac{1}{2\pi i} \int_{|z|=K} \frac{C_j(z) e^{zT/\varphi(q)}}{z^{k+1}} dz = \frac{1}{k!} \cdot \frac{d^k}{dz^k} \Bigg|_{z=0} C_j(z) e^{zT/\varphi(q)} = \sum_{r=0}^k \frac{C_j^{(r)}(0)}{r!} \cdot \frac{(T/\varphi(q))^{k-r}}{(k-r)!},$$

The last equality above uses the generalized product formula, with $C_j^{(r)}$ being the r -th derivative of C_j . To show (1.14), we will need the following lemma, which also seems to be of general interest.

Lemma 8.1. *Uniformly in all moduli $q \geq 1$ and in coprime residues $a \pmod{q}$, we have*

$$\lim_{s \rightarrow 1} \sum_{p \equiv a \pmod{q}} \frac{1}{p^s} - \frac{1}{\varphi(q)} \log\left(\frac{1}{s-1}\right) = \frac{1}{p_{q,a}} + O\left(\frac{\log(2q)}{\varphi(q)}\right).$$

Proof of Lemma 8.1. It is well known that the limit exists, hence it suffices to show the lemma with $s \rightarrow 1+$ along the real numbers. Write $s = 1 + 1/\log X$ with $X \rightarrow \infty$. For all $p \leq X$, we can write $p^{-1-1/\log X} = p^{-1} \exp(-\log p/\log X) = 1/p + O(\log p/p \log X)$. Now by [23, Lemma (6.3), p. 699] or [29, Remark 1], we have $\sum_{p \leq X: p \equiv a \pmod{q}} 1/p = \log_2 X/\varphi(q) + 1/p_{q,a} + O(\log(2q)/\varphi(q))$ for all $X \geq 3q$. Moreover by partial summation, the last estimate also yields $\sum_{p \leq X: p \equiv a \pmod{q}} 1/p = \log(p_{q,a})/p_{q,a} + O(\log(2q) \cdot \log X/\varphi(q))$ for all such X . Combining these three observations, we get

$$(8.9) \quad \sum_{\substack{p \leq X \\ p \equiv a \pmod{q}}} \frac{1}{p^{1+1/\log X}} = \frac{\log_2 X}{\varphi(q)} + \frac{1}{p_{q,a}} + O\left(\frac{\log(2q)}{\varphi(q)} + \frac{\log(p_{q,a})/p_{q,a}}{\log X}\right),$$

uniformly in all $X \geq 2q$. Next, by partial summation and the Brun–Titchmarsh theorem,

$$\sum_{\substack{p > X \\ p \equiv a \pmod{q}}} \frac{1}{p^{1+1/\log X}} \ll \int_X^\infty \left(\sum_{\substack{X < p \leq t \\ p \equiv a \pmod{q}}} 1 \right) \frac{dt}{t^{2+1/\log X}} \ll \frac{1}{\varphi(q) \log X} \int_X^\infty \frac{dt}{t^{1+1/\log X}} \ll \frac{1}{\varphi(q)},$$

uniformly in all $X \geq q^2$. Letting $X \rightarrow \infty$ in this estimate and in (8.9), we deduce that

$$\lim_{s \rightarrow 1+} \sum_{p \equiv a \pmod{q}} \frac{1}{p^s} - \frac{1}{\varphi(q)} \log \left(\frac{1}{s-1} \right) = \lim_{X \rightarrow \infty} \sum_{p \equiv a \pmod{q}} \frac{1}{p^{1+1/\log X}} - \frac{\log_2 X}{\varphi(q)}$$

is equal to $1/p_{q,a} + O(\log(2q)/\varphi(q))$. This completes the proof of Lemma 8.1. \square

We now proceed to estimate $C_0(z)$ uniformly in all complex z with $|z| \leq K+1$. Since $L(s, \chi_0) = \zeta(s) \prod_{\ell|q} (1 - 1/\ell^s)$, we see from (8.4) and (5.2) that for all s with $\sigma > 1$, we have

$$\begin{aligned} & \mathcal{G}_z(s) \cdot \left(\prod_{\chi} L(s, \chi)^{\mathbb{1}_{\chi=\chi_0} + (z-1)\bar{\chi}(a)/\varphi(q)} \right) \cdot (s-1)^{1+(z-1)/\varphi(q)} \\ &= \zeta(s)(s-1) \cdot \exp \left((z-1) \left\{ \sum_{p \equiv a \pmod{q}} \frac{1}{p^s} + \frac{\log(s-1)}{\varphi(q)} \right\} \right) \cdot \left(\prod_{p \equiv a \pmod{q}} \left(1 + \frac{z-1}{p^s} \right) e^{(1-z)/p^s} \right). \end{aligned}$$

Letting $s \rightarrow 1+$ above and using $\lim_{s \rightarrow 1+} \zeta(s)(s-1) = 1$, we obtain from (8.7) and Lemma 8.1,

$$(8.10) \quad C_0(z) = e^{(z-1)/p_{q,a}} \left(1 + O \left(\frac{\log q}{\varphi(q)} \right) \right) \cdot \left(\prod_{p \equiv a \pmod{q}} \left(1 + \frac{z-1}{p} \right) e^{(1-z)/p} \right),$$

uniformly in $|z| \leq K+1$. Here we also used the fact that $1/\Gamma(1+(z-1)/\varphi(q)) = 1 + O_K(1/\varphi(q))$, which holds true as $(K+2)/\varphi(q) \leq 5(K+2)/2q^{1/2} \leq 1/2$.

Now since $|1-z|/p \leq (K+2)/q \leq 1/(K+2)$ for all $p > q$, we have

$$\sum_{\substack{p > q \\ p \equiv a \pmod{q}}} \left\{ \log \left(1 + \frac{z-1}{p} \right) + \frac{1-z}{p} \right\} \ll_K \sum_{p > q} \frac{1}{p^2} \ll \frac{1}{q \log q},$$

which shows that the contribution of all primes $p > q$ to the last infinite product in (8.10) is $1 + O(1/q \log q)$. Hence from (8.10), we deduce that uniformly in $|z| \leq K+1$, we have

$$(8.11) \quad C_0(z) = e^{(z-1)/p_{q,a}} \left(1 + O \left(\frac{\log q}{\varphi(q)} \right) \right) \cdot \left(1 + \frac{z-1}{p_{q,a}} \right) e^{(1-z)/p_{q,a}} = \left(1 - \frac{1}{p_{q,a}} \right) + \frac{z}{p_{q,a}} + O \left(\frac{\log q}{\varphi(q)} \right).$$

(Note that if $p_{q,a} > q$, then $1/p_{q,a}$ gets absorbed in the error term in all these computations.) As a consequence, we also obtain uniformly in all positive integers r ,

$$(8.12) \quad C_0^{(r)}(0) = \frac{r!}{2\pi i} \int_{|z|=K} \frac{C_0(z)}{z^{r+1}} dz = \frac{\mathbb{1}_{r=1}}{p_{q,a}} + O_K \left(\frac{r!}{K^r} \cdot \frac{\log q}{\varphi(q)} \right).$$

Inserting (8.11) and (8.12) into the last equality in (8.8), we obtain with $Y := \log_2 x$,

$$(8.13) \quad P_{0,k}(Y) = \left(1 - \frac{1}{p_{q,a}} \right) \frac{(Y/\varphi(q))^k}{k!} + \frac{(Y/\varphi(q))^{k-1}}{p_{q,a} (k-1)!} + O \left(\frac{\log q}{K^k \varphi(q)} \sum_{r=0}^k \frac{(KY/\varphi(q))^{k-r}}{(k-r)!} \right).$$

Since the sum in the O -term above is at most $e^{KY/\varphi(q)}$, we obtain the second bound of (1.14). We will now show that the (stronger) first bound holds uniformly in the range $k \leq KY/\varphi(q)$.

It is worth noting that in the more restricted range $k \leq (K - \epsilon_0)Y/\varphi(q)$, estimate (8.13) already yields the first bound in (1.14): This is because for all such k , the sum in the O -term of (8.13) is at most $(KY/\varphi(q))^k \cdot (k!)^{-1} \sum_{r=0}^k (k\varphi(q)/KY)^r \leq (KY/\varphi(q))^k \cdot (k!)^{-1} \sum_{r \geq 0} (1 - \epsilon_0/K)^r$. However in the *full range* $k \leq KY/\varphi(q)$, the first bound in (1.14) follows immediately from (8.11) and

Proposition 8.2. *With $Y := \log_2 x$, we have uniformly in all $k \leq KY/\varphi(q)$,*

$$(8.14) \quad P_{0,k}(Y) = \frac{(Y/\varphi(q))^k}{k!} \left\{ C_0 \left(\frac{k\varphi(q)}{Y} \right) + O \left(\frac{k\varphi(q) \log q}{Y^2} \right) \right\}.$$

Proof of Proposition 8.2. We use a variant of the saddle point method, adapting the argument in [38, Theorem II.6.3]. We start by using the first equality in (8.8) to write

$$(8.15) \quad P_{0,k}(Y) = \frac{C_0(\tau)}{2\pi i} \int_{|z|=\tau} \frac{e^{zY/\varphi(q)}}{z^{k+1}} dz + \frac{1}{2\pi i} \int_{|z|=\tau} \{C_0(z) - C_0(\tau) - (z - \tau)C'_0(\tau)\} \cdot \frac{e^{zY/\varphi(q)}}{z^{k+1}} dz,$$

where we have chosen $\tau := k\varphi(q)/Y \leq K$, so as to ensure that

$$\int_{|z|=\tau} (z - \tau) \frac{e^{zY/\varphi(q)}}{z^{k+1}} dz = \int_{|z|=\tau} \frac{e^{zY/\varphi(q)}}{z^k} - \tau \int_{|z|=\tau} \frac{e^{zY/\varphi(q)}}{z^{k+1}} dz = 0$$

via Cauchy's integral formula. As such, it follows from (8.15) that

$$(8.16) \quad P_{0,k}(Y) = C_0(\tau) \frac{(Y/\varphi(q))^k}{k!} + \frac{1}{2\pi i} \int_{|z|=\tau} (z - \tau)^2 \left(\int_0^1 (1 - u) C''_0(uz + (1 - u)\tau) du \right) \frac{e^{zY/\varphi(q)}}{z^{k+1}} dz;$$

here the identity $C_0(z) - C_0(\tau) - (z - \tau)C'_0(\tau) = (z - \tau)^2 \int_0^1 (1 - u) C''_0(\tau + u(z - \tau)) du$ can be verified by integrating by parts twice. To complete the proof of Proposition 8.2, it only remains to show that the entire double integral above is absorbed in the error term of (8.14).

Now we observe that $C''_0(w) \ll \log q/\varphi(q)$ uniformly in all complex w with $|w| \leq K$: This follows by using (8.11) in Cauchy's integral formula on the disk of radius 1 centered at w . By this observation, we have $C''_0(uz + (1 - u)\tau) \ll \log q/\varphi(q)$ uniformly in all u and z appearing on the right of (8.16). Writing $z = \tau e^{i\theta}$, we thus find that the entire double integral in (8.16) has size

$$\ll \frac{\log q}{\tau^{k-2} \varphi(q)} \int_0^{2\pi} |e^{i\theta} - 1|^2 \cdot e^{\tau Y \cos \theta / \varphi(q)} d\theta = \frac{\log q}{\varphi(q)} \cdot \left(\frac{Y}{k\varphi(q)} \right)^{k-2} \int_0^{2\pi} (1 - \cos \theta) e^{k \cos \theta} d\theta$$

The integral above is at most $2\pi + 2 \int_0^{\pi/2} (1 - \cos \theta) e^{k \cos \theta} d\theta \leq 2\pi + 2 \int_0^1 e^{ku} \sqrt{1-u} du \leq 2\pi + 2e^k k^{-3/2} \int_0^k e^{-v} v^{1/2} dv \ll e^k k^{-3/2}$. (In this chain of inequalities, we first took $u := \cos \theta$, noted that $\sqrt{1+u} \geq 1$, and then let $v := k(1-u)$, noting that $\int_0^\infty e^{-v} v^{1/2} dv = \Gamma(3/2)$.) Collecting these observations, we find that the double integral in (8.16) is $\ll (\log q/\varphi(q)) \cdot (Y/\varphi(q))^{k-2} \cdot k e^k / k^{k+1/2}$. By Stirling's formula, this expression is absorbed in the error term of (8.14), as desired. \square

This establishes all the assertions of Theorem 1.8, except the very last one. But the last assertion (corresponding to the case when the Siegel zero does not exist) can be dealt with by all the same arguments, but only by replacing the use of subpart (1) of Theorem 1.3 by subpart (2). \square

8.1. Proof of Theorem 1.9. We just mention the main changes from the proof of Theorem 1.9. This time we start by estimating $\sum_{n \leq x} z^{\Omega_a(n)}$ for $|z| \leq \min\{K + 2, p_{q,a}(1 - \epsilon_0/100)\}$. The analogue of (8.3) is $\sum_n z^{\Omega_a(n)}/n^s = \left(\prod_\chi L(s, \chi)^{\mathbb{1}_{x=x_0} + (z-1)\bar{\chi}(a)/\varphi(q)} \right) \cdot \tilde{\mathcal{G}}_z(s)$, where $\tilde{\mathcal{G}}_z(s)$ is defined the same as $\mathcal{G}_z(s)$ in (8.4), but only with the last infinite product changed to

$$(8.17) \quad \prod_{p \equiv a \pmod{q}} \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right) e^{(1-z)/p^s}.$$

Now since K and ϵ_0 are fixed throughout, we may choose our constant $c_0 := c_0(K, \epsilon_0)$ (defined at the start of subsection § 1.1) to be small enough that $e^{c_0} < \min\{K, (1 - \epsilon_0/100)^{-1}\}$. For all primes $p > q$ and for all complex numbers s with $\sigma = \operatorname{Re}(s) \geq 1 - c_0/\log q$, we have $|z/p^s| \leq (K+1)/q^{1-c_0/\log q} \leq e^{c_0}/(K+2) < 1$, where we have recalled that $q > (K+2)^2$. Moreover, if $p_{q,a} \leq q$, then for all such s , we also have $|z/(p_{q,a})^s| \leq (|z|/p_{q,a}) \cdot \exp(c_0 \log(p_{q,a})/\log q) \leq e^{c_0}(1 - \epsilon_0/100) < 1$. These two observations show that the product (8.17) defines an analytic function of size $O(1)$ on the half plane $\sigma \geq 1 - c_0/\log q$. (Here we wrote $(1 - z/p^s)^{-1} = 1 + z/p^s + O(1/p^{2\sigma})$.)

We can now proceed as we did for Theorem 1.8 to see that once again Theorem 1.3 applies with the **same** $\alpha_\chi, \beta_\chi, \nu, \Omega, \mathcal{M}$ and δ_0 as before. Hence, the analogue of (8.6) for $\sum_{n \leq x} z^{\Omega_a(n)}$ holds true, uniformly in $q \leq (\log x)^K$ and in $|z| \leq \min\{K+2, p_{q,a}(1 - \epsilon_0/100)\}$: The only difference is that $C_j(z)$ is replaced by the function $\tilde{C}_j(z)$ which is defined exactly as in (8.7), with the obvious replacement of $\mathcal{G}_z(s)$ by $\tilde{\mathcal{G}}_z(s)$. This proves the first assertion (1.15), with

$$(8.18) \quad Q_{j,k}(T) := \frac{1}{2\pi i} \int_{|z|=R} \frac{\tilde{C}_j(z) e^{zT/\varphi(q)}}{z^{k+1}} dz = \frac{1}{k!} \cdot \frac{d^k}{dz^k} \Big|_{z=0} \tilde{C}_j(z) e^{zT/\varphi(q)} = \sum_{r=0}^k \frac{\tilde{C}_j^{(r)}(0)}{r!} \cdot \frac{(T/\varphi(q))^{k-r}}{(k-r)!},$$

and with $R := \min\{K, p_{q,a}(1 - \epsilon_0)\}$ as defined in the statement of Theorem 1.9.

Now proceeding exactly as we did for (8.11) and (8.12), but replacing all instances of “ $|z| \leq K+1$ ” by “ $|z| \leq \min\{K+2, p_{q,a}(1 - \epsilon_0/100)\}$ ”, we see that the analogues of these two estimates are

$$(8.19) \quad \tilde{C}_0(z) = \left(1 - \frac{z}{p_{q,a}}\right)^{-1} \left(1 - \frac{1}{p_{q,a}}\right) + O\left(\frac{\log q}{\varphi(q)}\right),$$

uniformly in all complex z with $|z| \leq \min\{K+2, p_{q,a}(1 - \epsilon_0/100)\}$, and

$$(8.20) \quad \tilde{C}_0^{(r)}(0) = \frac{r!}{(p_{q,a})^r} \left(1 - \frac{1}{p_{q,a}}\right) + O\left(\frac{r!}{R^r} \cdot \frac{\log q}{\varphi(q)}\right), \text{ uniformly in } r \in \mathbb{N} \cup \{0\}.$$

Inserting these two estimates into the last expression for $Q_{0,k}$ in (8.18), we find that

$$(8.21) \quad \begin{aligned} Q_{0,k}(Y) &= \frac{1}{(p_{q,a})^k} \cdot \left(1 - \frac{1}{p_{q,a}}\right) \sum_{r=0}^k \frac{(p_{q,a}Y/\varphi(q))^{k-r}}{(k-r)!} + O\left(\frac{\log q}{R^k \varphi(q)} \sum_{r=0}^k \frac{(RY/\varphi(q))^{k-r}}{(k-r)!}\right) \\ &= \frac{1}{(p_{q,a})^k} \cdot \left(1 - \frac{1}{p_{q,a}}\right) \sum_{r=0}^k \frac{(p_{q,a}Y/\varphi(q))^r}{r!} + O\left(\frac{\log q}{\varphi(q)} \cdot \frac{e^{RY/\varphi(q)}}{R^k}\right) \end{aligned}$$

with $Y := \log_2 x$. Now for $k \leq (1 - \epsilon_0)p_{q,a}Y/\varphi(q)$, we see that

$$\sum_{r=0}^{k-1} \frac{(p_{q,a}Y/\varphi(q))^r}{r!} \leq \frac{(p_{q,a}Y/\varphi(q))^{k-1}}{(k-1)!} \sum_{r=0}^{k-1} \left(\frac{k\varphi(q)}{p_{q,a}Y}\right)^{k-1-r} \leq \frac{(p_{q,a}Y/\varphi(q))^{k-1}}{(k-1)!} \sum_{m \geq 0} (1 - \epsilon_0)^m,$$

which is $\ll (p_{q,a}Y/\varphi(q))^{k-1}/(k-1)!$. Inserting this into (8.21), we get (1.17). On the other hand, for $k \geq (1 + \epsilon_0)p_{q,a}Y/\varphi(q)$, the difference between $e^{p_{q,a}Y/\varphi(q)}$ and the last sum in (8.21) is

$$\sum_{r \geq k+1} \frac{(p_{q,a}Y/\varphi(q))^r}{r!} \leq \frac{(p_{q,a}Y/\varphi(q))^{k+1}}{(k+1)!} \cdot \sum_{r \geq k+1} \left(\frac{p_{q,a}Y}{k\varphi(q)} \right)^{r-k-1} \ll_{\epsilon_0} \frac{(p_{q,a}Y/\varphi(q))^{k+1}}{(k+1)!},$$

which also establishes estimate (1.18) in Theorem 1.9. Finally (1.16) follows from (8.19) and the following analogue of Proposition 8.2.

Proposition 8.3. *With $Y = \log_2 x$ and $R = \min\{K, p_{q,a}(1 - \epsilon_0)\}$ as above, we have uniformly in all $k \leq RY/\varphi(q)$,*

$$(8.22) \quad Q_{0,k}(Y) = \frac{(Y/\varphi(q))^k}{k!} \left\{ \tilde{C}_0 \left(\frac{k\varphi(q)}{Y} \right) + O \left(\frac{k}{(p_{q,a}Y/\varphi(q))^2} + \frac{k\varphi(q) \log q}{Y^2} \right) \right\}.$$

The proof of this result is entirely analogous to that of Proposition 8.2: The only main difference is that this time we use the bound $\tilde{C}_0''(w) \ll 1/(p_{q,a})^2 + \log q/\varphi(q)$, which holds uniformly in $|w| \leq R$, and is obtained by an application of (8.19) in conjunction with Cauchy's integral formula on the disk of radius ϵ_0 centered at w . This concludes the proof of Theorem 1.9. \square

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