

THE LANDAU-SELBERG-DELANGE METHOD FOR PRODUCTS OF DIRICHLET L -FUNCTIONS

AKASH SINGHA ROY

1. INTRODUCTION

We write complex numbers s as $\sigma + it$, where $\sigma = \operatorname{Re}(s)$ and $t = \operatorname{Im}(s)$. Fix $\mathbf{c}_0 \in (\mathbf{0}, \mathbf{1}/3)$ such that for any integer $q \geq 3$, the product $\prod_{\chi \bmod q} L(s, \chi)$ has no zero in the region $\{\sigma + it : \sigma > 1 - c_0/\log(q(|t| + 1))\}$ except at most a simple real zero $\boldsymbol{\eta}_e$ (the ‘‘Siegel zero’’) associated to a real character χ_e (the ‘‘exceptional character’’). We also fix any $\boldsymbol{\nu} > \mathbf{0}$ and $\boldsymbol{\delta}_0 \in (\mathbf{0}, \mathbf{1}]$, and define $\mathcal{L}_q(t) = \log(q(|t\boldsymbol{\nu}| + 1))$, $\mathcal{D}(\mathbf{c}_0) = \{\boldsymbol{\sigma} + i\mathbf{t} : \sigma > 1 - \mathbf{c}_0/\mathcal{L}_q(t)\}$, and

$$\lambda_q := 1 + \max_{a \bmod q} \max \left\{ \left| \sum_{\chi} \alpha_{\chi} \chi(a) \right|, \left| \sum_{\chi} \beta_{\chi} \chi(a) \right| \right\}.$$

Writing $\alpha_{\chi} = \sum_{\psi \bmod q} \alpha_{\psi} \cdot \mathbb{1}_{\psi=\chi} = \sum_{\psi \bmod q} \alpha_{\psi} \cdot \varphi(q)^{-1} \sum_{a \bmod q} \bar{\chi}(a) \psi(a)$ and interchanging sums, we obtain the following important bound

$$(1.1) \quad |\alpha_{\chi}| \leq \lambda_q \quad \text{and} \quad |\beta_{\chi}| \leq \lambda_q \quad \text{for all characters } \chi \bmod q.$$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, $\{\alpha_{\chi}\}_{\chi \bmod q}$ be a set of complex numbers (indexed at the Dirichlet characters $\chi \bmod q$), and $\Omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a non-decreasing function. We say that $\{a_n\}_{n=1}^{\infty}$ has **property** $\mathcal{P}(\{\alpha_{\chi}\}_{\chi}, c_0, \Omega)$ if the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ is of the form $\mathcal{F}(s\boldsymbol{\nu})G(s)$ for all complex numbers s having $\sigma > 1/\boldsymbol{\nu}$, where $\mathcal{F}(s\boldsymbol{\nu}) := \prod_{\chi \bmod q} L(s\boldsymbol{\nu}, \chi)^{\alpha_{\chi}}$, and where $G(s)$ is a function that analytically continues into the region $\mathcal{D}(c_0)$ and satisfies $|G(s)| \leq \Omega(t)$ therein. We shall also say that a positive integer N is **good** (with respect to $\{a_n\}_{n=1}^{\infty}$) if for any constant $c > 0$, there exists a constant $\kappa_c(N) > 0$ depending only on c and N such that

$$\sum_{x \leq n \leq x+cx/(\log x)^N} |a_n| \leq \kappa(c, N) \cdot \frac{x^{1/\boldsymbol{\nu}}}{(\log x)^N} \quad \text{for all } x \geq 2.$$

Our first main result is the following:

Theorem 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We say that such that the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ has property $\mathcal{P}(\{\alpha_{\chi}\}_{\chi}, c_0, \delta, M)$. Then uniformly in $x \geq 4$, in good $N \geq 0$, and in moduli $q \geq 4$ satisfying $(1 - \eta_e) \log x > 3\boldsymbol{\nu}$, we have*

$$\begin{aligned} & \sum_{n \leq x} a_n - \frac{x^{1/\boldsymbol{\nu}}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{0 \leq j \leq N} \frac{1}{\Gamma(\alpha_{\chi_0} - j)} \cdot \frac{\kappa_j}{(\log x)^j} \\ & \ll (4\lambda_q \log q)^{\lambda_q + 2K} \cdot \kappa_c(N) x^{1/\boldsymbol{\nu}} \left\{ \frac{\Omega_{\text{gr}}(T)(\log T)^{1+\lambda_q}}{T} + \frac{\Omega_{\text{gr}}(1/\boldsymbol{\nu})(1 - \eta_e)^{-2K} \Gamma(N + 2 + |\alpha_{\chi_0}|)}{(2(1 - \eta_e) \log x / 141\boldsymbol{\nu})^{N+2-\operatorname{Re}(\alpha_{\chi_0})}} \right\}. \end{aligned}$$

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2. KEY ANALYTIC INPUTS: LOGARITHMIC DERIVATIVES AND AUXILIARY FUNCTIONS

For any $\chi \bmod q$, the function $\text{Log}L(s\nu, \chi) := \sum_{p,r \geq 1} \chi(p^r)/rp^{rs\nu}$ defines an analytic logarithm of $L(s\nu, \chi)$ on the region $\{s : \sigma > 1/\nu\}$. Hence, the function $\mathcal{F}(s\nu)$ is analytic on $\{s : \sigma > 1/\nu\}$, and

$$\mathcal{F}(s\nu) = \prod_{\chi} L(s\nu, \chi)^{\alpha_{\chi}} = \exp \left(\sum_{\chi} \alpha_{\chi} \text{Log}L(s\nu, \chi) \right) = \exp \left(\sum_{p,r \geq 1} \frac{1}{rp^{rs\nu}} \sum_{\chi} \alpha_{\chi} \chi(p^r) \right) \quad \text{if } \sigma > 1/\nu. \quad (2.1)$$

We now make it clear how our functions can be analytically continued into regions of interest. In what follows, anything involving the Siegel zero η_e is to be ignored if η_e doesn't exist.

2.1. Analytic Continuations. Since the functions $L(s\nu, \chi_0)(s-1/\nu)$, $L(s\nu, \chi_e)(s-\eta_e/\nu)^{-1}$, and $\{L(s\nu, \chi)\}_{\chi \neq \chi_0, \chi_e \bmod q}$ all continue analytically into nonvanishing functions on $\mathcal{D}(c_0)$, they have (unique) analytic logarithms $\mathcal{T}^*(s, \chi_0)$, $\mathcal{T}^*(s, \chi_e)$, and $\{\mathcal{T}(s, \chi)\}_{\chi \neq \chi_0, \chi_e \bmod q}$ on $\mathcal{D}(c_0)$ satisfying

$$\mathcal{T}^* \left(\frac{2}{\nu}, \chi_0 \right) = \sum_{p,r \geq 1} \frac{\chi_0(p^r)}{rp^{2r}} + \ln \left(\frac{2}{\nu} - \frac{1}{\nu} \right), \quad \mathcal{T}^* \left(\frac{2}{\nu}, \chi_e \right) = \sum_{p,r \geq 1} \frac{\chi_e(p^r)}{rp^{2r}} - \ln \left(\frac{2}{\nu} - \frac{\eta_e}{\nu} \right),$$

and $\mathcal{T}(2/\nu, \chi) = \sum_{p,r \geq 1} \chi(p^r)/rp^{2r}$ for all other χ . (Thus $\mathcal{T}^*(s, \chi_0)$ is analytic on $\mathcal{D}(c_0)$ and satisfies $e^{\mathcal{T}^*(s, \chi_0)} = L(s\nu, \chi_0)(s-1/\nu)$ therein, etc.) Comparing derivatives, we see that the functions

$$(2.2) \quad \mathcal{T}(s, \chi_0) := \mathcal{T}^*(s, \chi_0) - \log \left(s - \frac{1}{\nu} \right) \quad \text{and} \quad \mathcal{T}(s, \chi_e) := \mathcal{T}^*(s, \chi_e) + \log \left(s - \frac{\eta_e}{\nu} \right)$$

define unique analytic continuations of the functions $\text{Log}L(s, \chi_0)$ and $\text{Log}L(s, \chi_e)$, into the regions $\mathcal{D}(c_0) \setminus (-\infty, 1/\nu]$ and $\mathcal{D}(c_0) \setminus (-\infty, \eta_e/\nu]$, respectively. (Here $\log z$ is the principal branch of the logarithm, so $\log(s-1/\nu)$ is analytic on $\mathbb{C} \setminus (-\infty, 1/\nu]$.) From this discussion, we see that the function $\exp(\sum_{\chi} \alpha_{\chi} \mathcal{T}(s, \chi)) = \prod_{\chi} e^{\alpha_{\chi} \mathcal{T}(s, \chi)}$ defines a unique analytic continuation of $\mathcal{F}(s\nu)$ in (2.1) into $\mathcal{D}(c_0) \setminus (-\infty, 1/\nu]$; hence, $\mathcal{F}(s\nu) = \exp(\sum_{\chi} \alpha_{\chi} \mathcal{T}(s, \chi))$ for all s in this region.

Note also that by the first equality in (2.1) and by analytic continuation, we may write

$$(2.3) \quad \frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} = \sum_{\chi} \alpha_{\chi} \frac{L'(s\nu, \chi)}{L(s\nu, \chi)} \quad \text{for all } s \neq 1/\nu \text{ s.t. } s \neq \rho/\nu \text{ for any complex zero } \rho \text{ of } \prod_{\chi} L(s, \chi).$$

This relation is consistent with the analytic continuation of $\mathcal{F}(s\nu)$ in the previous paragraph.

We will also need the following **two auxiliary functions**: By the above discussion (especially (2.2)),

- The function $\tilde{\mathcal{H}}(s) := \exp \left(\alpha_{\chi_0} \mathcal{T}^*(s, \chi_0) + \alpha_{\chi_e} \mathcal{T}^*(s, \chi_e) + \sum_{\chi \neq \chi_0, \chi_e} \alpha_{\chi} \mathcal{T}(s, \chi) \right)$ analytically continue the function $\mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}}(s-\eta_e/\nu)^{-\alpha_{\chi_e}}$ into the region $\mathcal{D}(c_0)$.
- The function $\mathcal{H}(s) := s^{-1} \exp \left(\alpha_{\chi_0} \mathcal{T}^*(s, \chi_0) + \sum_{\chi \neq \chi_0} \alpha_{\chi} \mathcal{T}(s, \chi) \right)$ analytically continues the function $s^{-1} \mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}}$ into the region $\mathcal{D}(c_0) \setminus (-\infty, \eta_e/\nu]$.

The reader may now forget all the \mathcal{T} and \mathcal{T}^* . All that needs to be remembered from this subsection are (2.1) and (2.3), that $\mathcal{F}(s\nu)$ continues analytically into $\mathcal{D}(c_0) \setminus (-\infty, 1/\nu]$, and that

$$(2.4) \quad \tilde{\mathcal{H}}(s) = \mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}}(s-\eta_e/\nu)^{-\alpha_{\chi_e}} \quad \text{for all } s \in \mathcal{D}(c_0),$$

$$(2.5) \quad \mathcal{H}(s) = s^{-1} \mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}} \quad \text{for all } s \in \mathcal{D}(c_0) \setminus (-\infty, \eta_e/\nu],$$

with $\tilde{\mathcal{H}}(s)$ and $\mathcal{H}(s)$ being analytic on $\mathcal{D}(c_0)$ and $\mathcal{D}(c_0) \setminus (-\infty, \eta_e/\nu]$, respectively.

2.2. Analysis of Logarithmic Derivatives. To give suitable bounds on $\mathcal{F}(s\nu)$, we will first analyze its logarithmic derivative. To this end, the following known results on Dirichlet L -functions will be useful. In what follows, we write $\rho = \beta + i\gamma$ where $\beta = \mathbf{Re}(\rho)$ and $\gamma = \mathbf{Im}(\rho)$. We denote by $\sum_{\rho: L(\rho, \chi)=0}^*$ a sum over all zeros ρ of $L(s, \chi)$ counted with appropriate multiplicity.

Lemma 2.1. *The following hold uniformly in $q \geq 2$ and in **all** Dirichlet characters $\chi \bmod q$.*

- (1) *Uniformly in all real t , we have $\sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1}}^* \frac{1}{1 + (t - \gamma)^2} \ll \log(q(|t| + 1))$.*
- (2) *Uniformly in all complex s satisfying $\sigma \in [-1, 2]$, $|t| \geq 2$, and $t \neq \gamma$ for any of the zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$, we have $\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1, |\gamma - t| \leq 1}}^* \frac{1}{s - \rho} + O(\log(q(|t| + 1)))$.*
- (3) *We have $L'(s, \chi)/L(s, \chi) \ll \log(q|s|)$, uniformly in all complex s satisfying $\sigma \leq -1$ and lying outside the disks of radius $1/4$ about the trivial zeros of $L(s, \chi)$.*
- (4) *Uniformly in real $t \notin (-1, 1)$, we have $\#\{\rho : 0 \leq \beta \leq 1, |\gamma - t| \leq 1, L(\rho, \chi) = 0\} \ll \log(q|t|)$.*

In most standard texts, these results are stated and proved only for primitive characters, however the generality above will be helpful here. (Section 8 discusses this lemma for general $\chi \bmod q$.)

We now give a certain (absolutely convergent) series expansion for the logarithmic derivative of $\mathcal{F}(s\nu)$ in terms of the zeros of the L -functions, with coefficients that are easy to control.

Proposition 2.2. *For any $s \in \mathbb{C}$ satisfying $s \neq 1/\nu$ and $s \neq \rho/\nu$ for any zero ρ of $\prod_{\chi} L(s, \chi)$,*

$$(2.6) \quad \frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} = \sum_{n \leq \xi^2} \frac{\varrho(n)\Lambda(n)}{n^{s\nu}} \tau(n) - \frac{\alpha_{\chi_0}(\xi^{1-\nu s} - \xi^{2(1-\nu s)})}{(1 - \nu s)^2 \log \xi} + \sum_{\chi \bmod q} \sum_{\rho: L(\rho, \chi)=0}^* \frac{\alpha_{\chi}(\xi^{\rho-\nu s} - \xi^{2(\rho-\nu s)})}{(\rho - \nu s)^2 \log \xi},$$

where $\xi := e^{6\mathcal{L}_q(t)}$, $\varrho(n) := \sum_{\chi \bmod q} \alpha_{\chi} \chi(n)$, and $\tau(n) := \mathbb{1}_{n \leq \xi} + \mathbb{1}_{\xi < n \leq \xi^2} (2 - \log n / \log \xi)$.

Proof. Our starting point is the identity $\int_{b-i\infty}^{b+i\infty} y^z/z^2 dz = \mathbb{1}_{y>1} \cdot 2\pi i \log y$ which holds for any $b, y > 0$. To see this, consider any $R \geq 2$, apply the residue theorem to the contour consisting of the vertical segment $[b - iR, b + iR]$ and the major arc (respectively, minor arc) of the circle centered at the origin passing through $b \pm iR$ if $y > 1$ (resp. $y \leq 1$), and then let $R \rightarrow \infty$.

The Dirichlet series of $L'(s, \chi)/L(s, \chi)$ and (2.3) give $\mathcal{F}'(z\nu)/\mathcal{F}(z\nu) = \sum_{n \geq 1} \varrho(n)\Lambda(n)/n^{z\nu}$ for all z with $\mathbf{Re}(z) > 1/\nu$. We now claim that for all s as in the statement of the proposition,

$$(2.7) \quad \frac{1}{2\pi i} \int_{\frac{2}{\nu} + |s| - i\infty}^{\frac{2}{\nu} + |s| + i\infty} \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)} dz = \nu \sum_{n \leq \xi^2} \frac{\varrho(n)\Lambda(n)}{n^{z\nu}} \tau(n) \log \xi.$$

Indeed by the identity in the first paragraph of the proof, (2.7) is immediate if $\mathcal{F}'(z\nu)/\mathcal{F}(z\nu)$ were replaced by any finite truncation $\sum_{n \leq Y} \varrho(n)\Lambda(n)/n^{z\nu}$ of its aforementioned Dirichlet series (for any $Y > \xi^6$). Moreover by the same Dirichlet series, the absolute value of the integrand above is at most $2\lambda_q \xi^{4+2\nu|s|} (\sum_n \Lambda(n)/n^2) |z-s|^{-2}$, which is an L^1 -function since $\int_{\frac{2}{\nu} + |s| - i\infty}^{\frac{2}{\nu} + |s| + i\infty} |dz|/|z-s|^2 < \infty$ and $\sum_n \Lambda(n)/n^2 \ll 1$. Thus (2.7) follows by the Dominated Convergence Theorem.

We will now shift contours. For this, note that for any $M \geq 2$, the number of zeros of $\prod_{\chi} L(s, \chi)$ in the rectangle $[0, 1] \times (M, M + 1]$ is $\ll \varphi(q) \log(qM)$ by Lemma 2.1(4). Hence there exists $T_M \in (M, M + 1]$ satisfying $|T_M - \gamma| \gg (\varphi(q) \log(qM))^{-1}$ for **all** zeros $\rho = \beta + i\gamma$ of $\prod_{\chi} L(s, \chi)$. Since the set of zeros of $\prod_{\chi} L(s, \chi)$ is closed under complex conjugation, this also means that

$$(2.8) \quad |T_M \pm \gamma| \gg (\varphi(q) \log(qM))^{-1} \text{ for all zeros } \rho = \beta + i\gamma \text{ of } \prod_{\chi} L(s, \chi).$$

With the contour ω_M as in Figure 1, we claim that

$$(2.9) \quad \frac{L'(z\nu, \chi)}{L(z\nu, \chi)} \ll \varphi(q) \log^2(qM), \text{ uniformly in } q \geq 3, \chi \bmod q, M \geq 2(1 + \nu + \nu|s|), z \in \omega_M.$$

If $\operatorname{Re}(z) \geq 2/\nu$, this follows from the Dirichlet series of $L'(z\nu, \chi)/L(z\nu, \chi)$. If $\operatorname{Re}(z) \in [-1/\nu, 2/\nu]$, then z must lie on the two horizontal segments in ω_M , so that by (2.8), we have $|z\nu - \rho| \geq |\operatorname{Im}(z)\nu - \gamma| = |T_M \pm \gamma| \gg (\varphi(q) \log(qM))^{-1}$ for any zero $\rho = \beta + i\gamma$ of $\prod_{\chi} L(s, \chi)$. This gives (2.9) by Lemma 2.1(2) and (4). Lastly if $\operatorname{Re}(z) \leq -1/\nu$, then Lemma 2.1(3) establishes (2.9).

Now for any $M \geq 2\nu|s|$ and any $z \in \omega_M$, we have $|z - s| \geq |z| - |s| \geq |z|/2 \geq M/2\nu$. As such $\int_{\omega_M} |dz|/|z - s|^2 \ll_{\nu, s} \int_{M/2\nu}^{\infty} dt/t^2 + (M/2\nu)^{-2} \cdot M \ll M^{-1}$, so that (2.3) and (2.9) yield

$$(2.10) \quad \lim_{M \rightarrow \infty} \int_{\omega_M} \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)} dz = 0.$$

Using the residue theorem to shift contours from the vertical line in (2.7) to ω_M , and then letting $M \rightarrow \infty$, we thus find from (2.10), (2.7) and (2.3) that

$$(2.11) \quad \nu \sum_{n \leq \xi^2} \frac{\varrho(n) \Lambda(n)}{n^{z\nu}} \tau(n) \log \xi = \left(\operatorname{Res}_{z=s} + \operatorname{Res}_{z=1/\nu} + \sum_{\rho: \prod_{\chi} L(\rho, \chi)=0} \operatorname{Res}_{z=\rho/\nu} \right) \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)}.$$

Finally, using (2.3) to compute the above residues, we obtain the proposition. For instance, note that if $\xi^{\rho-\nu s} \neq 1$ for some ρ above, then (2.3) shows that $z = \rho/\nu$ is a simple pole of the function on the right of (2.11) of residue $\nu(\xi^{\rho-\nu s} - \xi^{2(\nu-\rho s)})(\rho - \nu s)^{-2} \sum_{\chi} \alpha_{\chi} \cdot \{\text{multiplicity of } \rho \text{ in } L(s, \chi)\}$. If $\xi^{\rho-\nu s} = 1$, then $z = \rho/\nu$ is a removable singularity, so we can still give the same expression (whose value is zero) for its “residue”. The residue at $z = 1/\nu$ can be computed analogously, and the residue at $z = s$ (which is always necessarily a simple pole) is equal to $-\nu(\log \xi) \mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$. \square

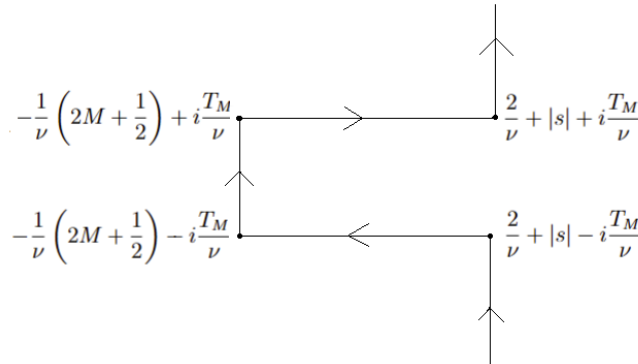


FIGURE 1. The Contour ω_M

We will now use the series representation in Proposition 2.2 to give a suitable bound on $\mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$. A crucial input will be provided the following zero density estimate. In what follows, we define

$$N(\theta, t) := \sum_{\chi \bmod q} \sum_{\substack{\rho: L(\rho, \chi)=0 \\ \theta \leq \beta \leq 1, |\gamma| \leq t}}^* 1.$$

Lemma 2.3. *We have $N(\theta, t) \ll (qt)^{3(1-\theta)}$, uniformly in $q \geq 3$, $\theta \in [1/2, 1]$, and $t \geq 1$.*

This may be found in work of Heath–Brown and Jutila. We now state the bound alluded to above.

Proposition 2.4. *Uniformly in $q \geq 3$, and in complex numbers s satisfying $\sigma \geq \nu^{-1}(1 - c_0/2\mathcal{L}_q(t))$,*

$$\left| \frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} + \frac{\alpha_{\chi_0}}{s\nu - 1} - \frac{\alpha_{\chi_e}}{s\nu - \eta_e} \right| \ll \lambda_q \mathcal{L}_q(t).$$

Proof. Most of the argument consists of carefully bounding the different components of the right of (2.6). First, for all $n \leq \xi^2$, we have $|n^{s\nu}| = n^{\sigma\nu} \geq n^{1-c_0/2\mathcal{L}_q(t)} \geq n \exp(-2 \log \xi / 2\mathcal{L}_q(t)) \gg n$, so that the first sum on the right in (2.6) is $\ll \lambda_q \sum_{n \leq \xi^2} \Lambda(n)/n \ll \lambda_q \mathcal{L}_q(t)$ by Mertens' Theorem.

Next, since the trivial zeros of any $L(s, \chi)$ are simple, the total contribution of all zeros $\{-r/\nu\}_{r \in \mathbb{N}}$ to the right of (2.6) equals $(\log \xi)^{-1} \sum_{r \geq 1} \left(\sum_{\chi: \chi(-1)=(-1)^r} \alpha_\chi \right) (\xi^{-(r+\nu s)} - \xi^{-2(r+\nu s)})(r + \nu s)^{-2}$. Since the sum on χ is $\sum_\chi \alpha_\chi (1 + \chi(-1)(-1)^r)/2 = (\varrho(1) + \varrho(-1))/2$, it follows that the last expression has size $\ll \lambda_q (\log \xi)^{-1} \sum_{r \geq 1} \xi^{-(r+\nu\sigma)} (r + \nu\sigma)^{-2} \ll \lambda_q \mathcal{L}_q(t)^{-1} \sum_{r \geq 1} r^{-2} \ll \lambda_q \mathcal{L}_q(t)^{-1}$.

Now, we observe that $|(\xi^{\theta-\nu s} - \xi^{2(\theta-\nu s)})(\theta - \nu s)^{-2}(\log \xi)^{-1} - (\nu s - \theta)^{-1}| \ll \mathcal{L}_q(t)$ uniformly in $\theta \in (0, 1]$ and s as in the proposition. This follows by a straightforward crude bounding if $|\theta - \nu s| > (\log \xi)^{-1}$, and by the formula $\xi^{\theta-\nu s} = 1 - (\theta - \nu s) \log \xi + O((\theta - \nu s)^2 (\log \xi)^2)$ if $|\theta - \nu s| \leq (\log \xi)^{-1}$. Collecting all the observations made so far, we see that this proposition would follow from (2.6), once we show that uniformly in all s with $\sigma \geq \nu^{-1}(1 - c_0/2\mathcal{L}_q(t))$,

$$(2.12) \quad \frac{1}{\mathcal{L}_q(t)} \sum_\chi |\alpha_\chi| \sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1, \rho \neq \eta_e}}^* \frac{\xi^{\beta-\nu\sigma} + \xi^{2(\beta-\nu\sigma)}}{(\beta - \nu\sigma)^2 + (\gamma - \nu t)^2} \ll \lambda_q \mathcal{L}_q(t).$$

To show this, we start by bounding the entire expression above by $S_1 + S_2 + S_3 + S_4$, where

- S_1 denotes the total contribution of all ρ having $\beta \leq 1/2$, so that

$$S_1 = \frac{1}{\mathcal{L}_q(t)} \sum_\chi |\alpha_\chi| \sum_{\substack{\rho: L(\rho, \chi)=0 \\ 0 \leq \beta \leq 1/2}}^* \frac{\xi^{\beta-\nu\sigma} + \xi^{2(\beta-\nu\sigma)}}{(\beta - \nu\sigma)^2 + (\gamma - \nu t)^2}.$$

- S_2 denotes the total contribution of all ρ having $\beta \in (1/2, 1]$ and $|\gamma| \leq 2|t\nu| + 1$.
- S_3 denotes the total contribution of all ρ having $\beta \in (1/2, \sigma\nu]$ and $|\gamma| > 2|t\nu| + 1$.
- S_4 denotes the total contribution of all ρ having $\beta \in (\sigma\nu, 1]$.

For any ρ appearing in S_1 , we have $\beta - \nu\sigma \leq 1/2 - (1 - c_0/2\mathcal{L}_q(t)) \leq -1/2 + 1/2 \log q \leq -1/3$, so that $(\beta - \nu\sigma)^2 + (\gamma - \nu t)^2 \geq (1 + (\gamma - \nu t)^2)/9$. Hence (1.1) and Lemma 2.1(1) yield $S_1 \ll \lambda_q \xi^{1/2-\nu\sigma} \mathcal{L}_q(t)^{-1} \sum_\chi \sum_\rho (1 + (\gamma - \nu t)^2)^{-1} \ll \lambda_q \cdot q \xi^{1/2-\nu\sigma} \ll \lambda_q \cdot q \xi^{-1/2} \cdot \xi^{c_0/2\mathcal{L}_q(t)} \ll \lambda_q$.

For any ρ appearing in S_3 , we have $\beta - \nu\sigma \leq 0$ and $|\gamma - t\nu| \geq |\gamma| - |t\nu| \geq |\gamma|/2$. Thus by (1.1),

$$S_3 \leq \frac{8\lambda_q}{\mathcal{L}_q(t)} \sum_{\chi} \sum_{\substack{\rho \neq \eta_e: L(\rho, \chi)=0 \\ |\gamma| > 2|t\nu|+1, 1/2 < \beta \leq \min\{\sigma\nu, 1\}}}^* \xi^{\beta-\nu\sigma} \cdot |\gamma|^{-2}.$$

Partitioning the interval $(1/2, \min\{\sigma\nu, 1\}]$ into $R := \lfloor \log \xi/2 \rfloor$ equally spaced intervals, we obtain

$$(2.13) \quad S_3 \leq \frac{8\lambda_q}{\mathcal{L}_q(t)} \sum_{r=1}^R \xi^{1/2+r\mu_0/R-\nu\sigma} \sum_{\chi} \sum_{\substack{\rho \neq \eta_e: L(\rho, \chi)=0, |\gamma| > 2|t\nu|+1 \\ 1/2+(r-1)\mu_0/R < \beta \leq 1/2+r\mu_0/R}}^* |\gamma|^{-2},$$

where $\mu_0 := \min\{\sigma\nu, 1\} - 1/2$. Now the inner double sum (on χ and ρ) above is at most

$$(2.14) \quad \int_{2|t\nu|+1}^{\infty} \frac{dN\left(\frac{1}{2} + \frac{(r-1)\mu_0}{R}, u\right)}{u^2} \ll \int_{2|t\nu|+1}^{\infty} \frac{N\left(\frac{1}{2} + \frac{(r-1)\mu_0}{R}, u\right)}{u^3} du \ll q^{3(1/2-(r-1)\mu_0/R)}$$

where we have used the Stieltjes integration by parts and Lemma 2.3. The last expression above is $\ll \xi^{1/4-(r-1)\mu_0/2R} \ll \xi^{1/4-r\mu_0/2R}$, as $\mu_0 \leq 1/2$, $\xi \geq q^6$ and $R \geq \log \xi/3$. Inserting these into (2.13), we get $S_3 \ll \lambda_q \mathcal{L}_q(t)^{-1} \xi^{3/4-\nu\sigma} \sum_{r=1}^R \xi^{r\mu_0/2R} \leq \lambda_q \mathcal{L}_q(t)^{-1} \xi^{3/4-\nu\sigma} \cdot R \xi^{\mu_0/2} \ll \lambda_q \xi^{1-\nu\sigma} \ll \lambda_q$.

Next, for any ρ counted in S_2 , we have $|\gamma| \leq 2|t\nu|+1$ and $\rho \neq \eta_e$, so that $\beta \leq 1 - c_0/\log(q(|\gamma|+1)) \leq 1 - c_0/\log(2q(|t\nu|+1))$. Since $\nu\sigma \geq 1 - c_0/2\mathcal{L}_q(t)$, we get

$$\nu\sigma - \beta \geq c_0 \left(\frac{1}{\log(2q(|t\nu|+1))} - \frac{1}{2\log(q(|t\nu|+1))} \right) = \frac{c_0}{\mathcal{L}_q(t)} \left(1 - \frac{\log 4}{\log(2q(|t\nu|+1))} \right) \geq \frac{c_0}{10\mathcal{L}_q(t)}.$$

Hence $(\beta - \nu\sigma)^2 \gg \mathcal{L}_q(t)^{-2}$. Proceeding as in (2.14) (via Lemma 2.3 and integration by parts),

$$\begin{aligned} S_2 &\ll \lambda_q \mathcal{L}_q(t) \sum_{\chi} \sum_{\substack{\rho \neq \eta_e: L(\rho, \chi)=0 \\ 1/2 < \beta \leq 1, |\gamma| \leq 2|t\nu|+1}}^* \xi^{\beta-\nu\sigma} \leq \lambda_q \mathcal{L}_q(t) \left(- \int_{1/2}^1 \xi^{\theta-\nu\sigma} dN(\theta, 2|t\nu|+1) \right) \\ &\leq \lambda_q \mathcal{L}_q(t) \left(\xi^{1/2-\nu\sigma} N(1/2, 2|t\nu|+1) + \log \xi \int_{1/2}^1 \xi^{\theta-\nu\sigma} N(\theta, 2|t\nu|+1) d\theta \right) \\ &\ll \lambda_q \mathcal{L}_q(t) \left(\xi^{3/4-\nu\sigma} + \log \xi \int_{1/2}^1 \xi^{(1+\theta)/2-\nu\sigma} d\theta \right) \ll \lambda_q \mathcal{L}_q(t) \xi^{1-\nu\sigma} \ll \lambda_q \mathcal{L}_q(t). \end{aligned}$$

For any $\rho (\neq \eta_e)$ in S_4 , we have $1 - c_0/\log(q(|\gamma|+1)) \geq \beta > \sigma\nu \geq 1 - c_0/2\mathcal{L}_q(t)$, giving $|\gamma| > q(|t\nu|+1)^2 - 1$. Thus also $|\gamma - t\nu| \geq |\gamma| - |t\nu| \geq |\gamma|/2$. Proceeding exactly as we did for S_3 ,

$$S_4 \ll \frac{\lambda_q \xi^{2(1-\nu\sigma)}}{\mathcal{L}_q(t)} \sum_{\chi} \sum_{\substack{\rho: L(\rho, \chi)=0 \\ \sigma\nu < \beta < 1, |\gamma| > q(|t\nu|+s1)^2-1}}^* |\gamma|^{-2} \ll \frac{\lambda_q}{\mathcal{L}_q(t)} \int_{q(|t\nu|+1)^2-1}^{\infty} \frac{dN(\sigma\nu, u)}{u^2} \ll \lambda_q.$$

Collecting all these estimates establishes (2.12), completing the proof of the proposition. \square

The following is an important consequence of (2.4) and Proposition 2.4.

Corollary 2.5. *We have $\tilde{\mathcal{H}}'(s)/\tilde{\mathcal{H}}(s) \ll \lambda_q \mathcal{L}_q(t)$ uniformly in complex s having $\sigma \geq 1 - c_0/2\mathcal{L}_q(t)$.*

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GA 30602

Email address: `akash01s.roy@gmail.com`