# Ore's theorem

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Ore's theorem is a result in graph theory proved in 1960 by Norwegian mathematician Øystein Ore. It gives a sufficient condition for a graph to be Hamiltonian, essentially stating that a graph with "sufficiently many edges" must contain a Hamilton cycle. Specifically, the theorem considers the sum of the degrees of any two non-adjacent vertices: if this sum is always at least equal to the total number of vertices in the graph, then the graph is Hamiltonian.

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## Formal statement

Let $G$ be a (finite and simple) graph with $n \geq 3$ vertices. We denote by $\deg v$ the degree of a vertex $v$ in $G$, i.e. the number of incident edges in $G$ to $v$. Then, Ore's theorem states that if

$$\deg v + \deg w \geq n \text{ for every pair of non-adjacent vertices } v \text{ and } w \text{ of } G \quad (*)$$

then $G$ is Hamiltonian.

## Proof

It is equivalent to show that every non-Hamiltonian graph $G$ does not obey condition $(*)$. Accordingly, let $G$ be a graph on $n \geq 3$ vertices that is not Hamiltonian, and let $H$ be formed from $G$ by adding edges one at a time that do not create a Hamiltonian cycle, until no more edges can be added. Let $x$ and $y$ be any two non-adjacent vertices in $H$. Then adding edge $xy$ to $H$ would create at least one new Hamiltonian cycle, and the edges other than $xy$ in such a cycle must form a Hamiltonian path $v_1 v_2 \ldots v_n$ in $H$ with $x = v_1$ and $y = v_n$. For each index $i$ in the range $2 \leq i \leq n$, consider the two potential edges in $H$ from $v_1$ to $v_i$ and from $v_{i-1}$ to $v_n$. At most one of these two edges can be present in $H$, for otherwise the cycle $v_1 v_2 \ldots v_{i-1} v_n v_{n-1} \ldots v_i$ would be a Hamiltonian cycle. Thus, the total number of edges incident to either $v_1$ or $v_n$ is at most equal to the number of choices of $i$, which is $n - 1$. Therefore, $H$ does not obey property $(*)$, which requires that this total number of edges $(\deg v_1 + \deg v_n)$ be greater than or equal to $n$. Since the vertex degrees in $G$ are at most equal to the degrees in $H$, it follows that $G$ also does not obey property $(*)$.

# Algorithm

Palmer (1997) describes the following simple algorithm for constructing a Hamiltonian cycle in a graph meeting Ore's condition.

1. Arrange the vertices arbitrarily into a cycle, ignoring adjacencies in the graph.
2. While the cycle contains two consecutive vertices $v_i$ and $v_{i+1}$ that are not adjacent in the graph, perform the following two steps:
   - Search for an index j such that the four vertices $v_i$, $v_{i+1}$, $v_j$, and $v_{j+1}$ are all distinct and such that the graph contains edges from $v_i$ to $v_j$ and from $v_{j+1}$ to $v_{i+1}$
   - Reverse the part of the cycle between $v_{i+1}$ and $v_j$ (inclusive).

Each step increases the number of consecutive pairs in the cycle that are adjacent in the graph, by one or two pairs (depending on whether $v_j$ and $v_{j+1}$ are already adjacent), so the outer loop can only happen at most n times before the algorithm terminates, where n is the number of vertices in the given graph. By an argument similar to the one in the proof of the theorem, the desired index j must exist, or else the nonadjacent vertices $v_i$ and $v_{i+1}$ would have too small a total degree. Finding i and j, and reversing part of the cycle, can all be accomplished in time $O(n)$. Therefore, the total time for the algorithm is $O(n^2)$, matching the number of edges in the input graph.

# Related results

Ore's theorem is a generalization of Dirac's theorem that, when each vertex has degree at least $n/2$, the graph is Hamiltonian. For, if a graph meets Dirac's condition, then clearly each pair of vertices has degrees adding to at least $n$.

In turn Ore's theorem is generalized by the Bondy–Chvátal theorem. One may define a closure operation on a graph in which, whenever two nonadjacent vertices have degrees adding to at least $n$, one adds an edge connecting them; if a graph meets the conditions of Ore's theorem, its closure is a complete graph. The Bondy–Chvátal theorem states that a graph is Hamiltonian if and only if its closure is Hamiltonian; since the complete graph is Hamiltonian, Ore's theorem is an immediate consequence.

Woodall (1972) found a version of Ore's theorem that applies to directed graphs. Suppose a digraph G has the property that, for every two vertices u and v, either there is an edge from u to v or the outdegree of u plus the indegree of v equals or exceeds the number of vertices in G. Then, according to Woodall's theorem, G contains a directed Hamiltonian cycle. Ore's theorem may be obtained from Woodall by replacing every edge in a given undirected graph by a pair of directed edges. A closely related theorem by Meyniel (1973) states that an n-vertex strongly connected digraph with the property that, for every two nonadjacent vertices u and v, the total number of edges incident to u or v is at least $2n - 1$ must be Hamiltonian.

Ore's theorem may also be strengthened to give a stronger condition than Hamiltonicity as a consequence of the degree condition in the theorem. Specifically, every graph satisfying the conditions of Ore's theorem is either a regular complete bipartite graph or is pancyclic (Bondy 1971).

## References


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