Honours Project-2

Anti-Transposition Map

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***Abstract-** In this paper we study applications of the Anti-transposition map in quantum information theory.We are interested in particular in the problem of detecting entanglement.Our main goal is to check whether a given quantum state is entangled in four dimension when anti-transposition map is applied. We show, by using the theory of positive maps which are not completely positive and Choi-Jamiolkowski isomorphism, that in four dimension Anti-transposition map gives satisfactory results i.e detects entanglement in four dimension.*

I. Introduction

This project is further implementation of Anti-Transposition map on 4x4 dimension and detect entanglement.

To achieve it, let's first understand some definitions, basic maps and how they are benificial for us to achieve desired result.

A. Tensor Product:

They are used to describe system consisting of multiple subsystems. Each subsystem is described by a vector in a vector space(Hilbert Space),

$$v \otimes w \tag{1}$$

where $v \in V$, $w \in W$. The tensor product $v \otimes w$, is thus defined to be the complex vector space of states of the two particle system.

B. Entanglement:

It is a physical phenomenon that occurs when a pair or group of particles is generated, interact, or share spatial proximity in a way such that the quantum state of each particle of the pair or group cannot be described independently of the state of the others, including when the particles are separated by a large distance.

For a bi-partite system, $|\Psi\rangle_{AB}$ is entangled or non-seperable if it's schmidt number **[1]** is greater than 1, otherwise it is seperable (or unentangled). Thus a seperable bipartite pure state is a direct product of pure states in H_A and H_B.

$$|\Psi\rangle_{AB} = |\Psi\rangle_A \otimes |\Psi\rangle_B \tag{2}$$

then the reduced density matrices

$$\rho_A = |\Psi\rangle_A \otimes \langle\Psi|_A \ \ \& \ \rho_B = |\Psi\rangle_B \otimes \langle\Psi|_B \tag{3}$$

Any state that cannot be expressed as such as direct product is entangled, then ρ_A & ρ_B are mixed states.

C. Positive map:

Let **A** and **B** be **C*** -algebras, then a linear map $\phi : A \longrightarrow$ B is said to be positive if $\phi(A_+) \subseteq B_+$. Here, A_+ is denoted the positive part of A . In mathematics, a positive map is a map between C*-algebras that sends positive elements to positive elements.

D. Completely positive map

A completely positive map is one which satisfies a stronger, more robust condition i.e. positive in all the dimensions or we can say K-positive. Complete positivity of the map ensures that the positivity condition is satisfied by the output state of any combined system **[2]**.

E. Positive, but not completely posive map

Let us understand this with the help of an example, consider the transposition map T $(\rho) = \rho^T$. Since for any density matrix $\rho^T \geq 0$, T is a positive map. However if we consider the action of I $\otimes$ T on the maximally entangled 2-qubit state $\frac{1}{\sqrt{2}}$ $(|00\rangle + |11\rangle)$, the density matrix for the final state of the joint system is,

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

which has a negative eigenvalue of $-\frac{1}{2}$ and so is not a physical density matrix for the joint system. Thus a map getting negative eigenvalues for any of it's principal minors is not completely positive map.

F. Choi-Jamiolkowski isomorphism:

To study a quantum channel $\mathcal{E}$ from system S to S', which is a trace-preserving complete positive map from operator spaces $\mathcal{L}(\mathcal{H}_S)$ to $\mathcal{L}(\mathcal{H}_{S'})$, we introduce an auxiliary system A with the same dimension as system S. Consider the maximally entangled state,

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle \tag{5}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}}(|0\rangle \otimes |0\rangle + \cdots + |d-1\rangle \otimes |d-1\rangle)|\Phi^+\rangle \tag{6}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle \tag{7}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}}(|0\rangle \otimes |0\rangle + \cdots + |d-1\rangle \otimes |d-1\rangle) \tag{8}$$

in the space of $\mathcal{H}_A \otimes \mathcal{H}_S$, since $\mathcal{E}$ is complete positive, $\mathrm{I} \otimes \mathcal{E}(|\Phi^+\rangle\langle\Phi^+|)$ is a non-negative operator. Conversely, for any non-negative operator on $\mathcal{H}_A \otimes \mathrm{H}_{S'}$, we can associate a complete positive map from $\mathcal{L}(\mathcal{H}_S)$ to $\mathcal{L}(\mathcal{H}_{S'})$, this kind of correspondence is called **Choi-Jamiołkowski isomorphism [3]**.

II. Anti-Transposition Map

Let us consider a map $\wedge$ and let ρ be a quantum state in 4-dimesion such that,

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \tag{9}$$

where,

$$\wedge(\rho) = \wedge \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \longrightarrow \begin{pmatrix} \rho_{44} & \rho_{34} & \rho_{24} & \rho_{14} \\ \rho_{43} & \rho_{33} & \rho_{23} & \rho_{13} \\ \rho_{42} & \rho_{32} & \rho_{22} & \rho_{12} \\ \rho_{41} & \rho_{31} & \rho_{21} & \rho_{11} \end{pmatrix} \tag{10}$$

$$\therefore \wedge(\rho) = \begin{pmatrix} \rho_{44} & \rho_{34} & \rho_{24} & \rho_{14} \\ \rho_{43} & \rho_{33} & \rho_{23} & \rho_{13} \\ \rho_{42} & \rho_{32} & \rho_{22} & \rho_{12} \\ \rho_{41} & \rho_{31} & \rho_{21} & \rho_{11} \end{pmatrix} \tag{11}$$

here the pure states will be,

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

A map $\wedge$ is positive, if $\wedge(\rho)$ is positive semi-definite i.e all the principal minors of $\rho \geq 0$. For checking the positivity of $\wedge(\rho)$ we have to check whether it's principal submatrices are ≥ 0. Also proving it for higher dimensions is not so easy, so checking on arbitrary pure states would be sufficient.

III. Proof for Positivity of $\wedge$

To prove the positivity of our anti-transposition map it is sufficient for us to perform operations on arbitrary pure states.

Let $|\phi\rangle$ be a pure state, i.e.

$$|\phi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \langle\phi| = \begin{pmatrix} \alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* \end{pmatrix} \tag{13}$$

The outer product of the pure state $|\phi\rangle$ can be written in the form,

$$|\phi\rangle\langle\phi| = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \begin{pmatrix} \alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* \end{pmatrix} \tag{14}$$

After doing the simple matrix multiplication we get,

$$|\phi\rangle\langle\phi| = \begin{vmatrix} |\alpha_1|^2 & \alpha_1\alpha_2^* & \alpha_1\alpha_3^* & \alpha_1\alpha_4^* \\ \alpha_2\alpha_1^* & |\alpha_2|^2 & \alpha_2\alpha_3^* & \alpha_2\alpha_4^* \\ \alpha_3\alpha_1^* & \alpha_3\alpha_2^* & |\alpha_3|^2 & \alpha_3\alpha_4^* \\ \alpha_4\alpha_1^* & \alpha_4\alpha_2^* & \alpha_4\alpha_3^* & |\alpha_4|^2 \end{vmatrix} \tag{15}$$

where $|\alpha_1|^2 = \alpha_1\alpha_1^*$, $|\alpha_2|^2 = \alpha_2, \alpha_2^*|\alpha_3|^2 = \alpha_3\alpha_3^*$, $|\alpha_4|^2 = \alpha_4\alpha_4^*$ is nothing but property of complex conjugate**[4]**.

Now applying the anti-transposition map $\wedge$ we get,

$$\wedge(|\phi\rangle\langle\phi|) = \begin{vmatrix} |\alpha_4|^2 & \alpha_3\alpha_4^* & \alpha_2\alpha_4^* & \alpha_1\alpha_4^* \\ \alpha_4\alpha_3^* & |\alpha_3|^2 & \alpha_2\alpha_3^* & \alpha_1\alpha_3^* \\ \alpha_4\alpha_2^* & \alpha_3\alpha_2^* & |\alpha_2|^2 & \alpha_1\alpha_2^* \\ \alpha_4\alpha_1^* & \alpha_3\alpha_1^* & \alpha_2\alpha_1^* & |\alpha_1|^2 \end{vmatrix} \tag{16}$$

The Principal Minors of $\wedge(|\phi\rangle\langle\phi|)$, **1st** order PM's, i.e.

$$|\alpha_1|^2 \geq 0, |\alpha_2|^2 \geq 0, |\alpha_3|^2 \geq 0, |\alpha_4|^2 \geq 0.\textit{(trivial)} \tag{17}$$

The **2nd** order Principal Minor's are ρ_{11}, ρ_{22}, ρ_{33}, ρ_{44}, here ρ_{11} means checking the value of output matrix after removing first row and first column, similarly for ρ_{22}, ρ_{33}, ρ_{44},

$$\rho_{11} = \begin{vmatrix} |\alpha_3|^2 & \alpha_2\alpha_3^* & \alpha_1\alpha_3^* \\ \alpha_3\alpha_2^* & |\alpha_2|^2 & \alpha_1\alpha_2^* \\ \alpha_3\alpha_1^* & \alpha_2\alpha_1^* & |\alpha_1|^2 \end{vmatrix} = -(|\alpha_2|^2|\alpha_3|^2|\alpha_1|^2 - |\alpha_1|^2|\alpha_2|^2|\alpha_3|^2) = 0 \tag{18}$$

Similarly after simplifying the minors we get,

$$\rho_{22} = \begin{vmatrix} |\alpha_4|^2 & \alpha_2\alpha_4^* & \alpha_1\alpha_4^* \\ \alpha_4\alpha_2^* & |\alpha_2|^2 & \alpha_1\alpha_2^* \\ \alpha_4\alpha_1^* & \alpha_2\alpha_1^* & |\alpha_1|^2 \end{vmatrix} = 0, \tag{19}$$

$$\rho_{33} = \begin{vmatrix} |\alpha_4|^2 & \alpha_3\alpha_4^* & \alpha_1\alpha_4^* \\ \alpha_4\alpha_3^* & |\alpha_3|^2 & \alpha_1\alpha_3^* \\ \alpha_4\alpha_1^* & \alpha_3\alpha_1^* & |\alpha_1|^2 \end{vmatrix} = 0, \tag{20}$$

$$\rho_{44} = \begin{vmatrix} |\alpha_4|^2 & \alpha_3\alpha_4^* & \alpha_2\alpha_4^* \\ \alpha_4\alpha_3^* & |\alpha_3|^2 & \alpha_2\alpha_3^* \\ \alpha_4\alpha_2^* & \alpha_3\alpha_2^* & |\alpha_2|^2 \end{vmatrix} = 0 \tag{21}$$

The **3rd** order Principal Minor's are $\rho_{12,1'2'}$, $\rho_{34,3'4'}$, $\rho_{23,2'3'}$, $\rho_{14,1'4'}$, $\rho_{24,2'4'}$, $\rho_{13,1'3'}$ (where let's say in $\rho_{12,1'2'}$ 12 - is first and second row, 1'2' - first column and second column).

$$\rho_{12,1'2'} = \begin{vmatrix} |\alpha_2|^2 & \alpha_1\alpha_2^* \\ \alpha_2\alpha_1^* & |\alpha_1|^2 \end{vmatrix} = 0 \tag{22}$$

$$\rho_{13,1'3'} = \begin{vmatrix} |\alpha_3|^2 & \alpha_1\alpha_3^* \\ \alpha_3\alpha_1^* & |\alpha_1|^2 \end{vmatrix} = 0 \tag{23}$$

$$\rho_{14,1'4'} = \begin{vmatrix} |\alpha_3|^2 & \alpha_2\alpha_3^* \\ \alpha_3\alpha_2^* & |\alpha_2|^2 \end{vmatrix} = 0, \tag{24}$$

$$\rho_{23,2'3'} = \begin{vmatrix} |\alpha_4|^2 & \alpha_1\alpha_4^* \\ \alpha_4\alpha_1^* & |\alpha_1|^2 \end{vmatrix} = 0 \tag{25}$$

$$\rho_{24,2'4'} = \begin{vmatrix} |\alpha_4|^2 & \alpha_2\alpha_4^* \\ \alpha_4\alpha_2^* & |\alpha_2|^2 \end{vmatrix} = 0, \tag{26}$$

$$\rho_{34,3'4'} = \begin{vmatrix} |\alpha_4|^2 & \alpha_3\alpha_4^* \\ \alpha_4\alpha_3^* & |\alpha_3|^2 \end{vmatrix} = 0 \tag{27}$$

The **4th** order Principal Minor's are $\rho_{123,1'2'3'}, \rho_{124,1'2'4'}, \rho_{134,1'3'4'} \& \rho_{234,2'3'4'}$ (where 123 - are 1st,2nd & 3rd row, 1'2'3' - are 1st,2nd & 3rd column) i.e finding the value by removing $\rho_{123,1'2'3'} \& \rho_{234,2'3'4'}$,

$$\rho_{123,1'2'3'} = |\alpha_1^2| \geq 0 \tag{28}$$

$$\rho_{124,1'2'4'} = |\alpha_2^2| \geq 0 \tag{29}$$

$$\rho_{134,1'3'4'} = |\alpha_3^2| \geq 0 \tag{30}$$

$$\rho_{234,2'3'4'} = |\alpha_4^2| \geq 0. \tag{31}$$

The **5th** order Principal Minor or determinant of ρ i.e. $\Delta(\rho)$,

$$\Delta(\rho) = \begin{vmatrix} |\alpha_4|^2 & \alpha_3\alpha_4^* & \alpha_2\alpha_4^* & \alpha_1\alpha_4^* \\ \alpha_4\alpha_3^* & |\alpha_3|^2 & \alpha_2\alpha_3^* & \alpha_1\alpha_3^* \\ \alpha_4\alpha_2^* & \alpha_3\alpha_2^* & |\alpha_2|^2 & \alpha_1\alpha_2^* \\ \alpha_4\alpha_1^* & \alpha_3\alpha_1^* & \alpha_2\alpha_1^* & |\alpha_1|^2 \end{vmatrix} \tag{32}$$

$$= |\alpha_4|^2 \begin{vmatrix} |\alpha_3|^2 & \alpha_2\alpha_3^* & \alpha_1\alpha_3^* \\ \alpha_3\alpha_2^* & |\alpha_2|^2 & \alpha_1\alpha_2^* \\ \alpha_3\alpha_1^* & \alpha_2\alpha_1^* & |\alpha_1|^2 \end{vmatrix} - (\alpha_3\alpha_4^*) \begin{vmatrix} \alpha_4\alpha_3^* & \alpha_2\alpha_3^* & \alpha_1\alpha_3^* \\ \alpha_4\alpha_2^* & |\alpha_2|^2 & \alpha_1\alpha_2^* \\ \alpha_4\alpha_1^* & \alpha_2\alpha_1^* & |\alpha_1|^2 \end{vmatrix} \ldots$$

$$\ldots + (\alpha_2\alpha_4^*) \begin{vmatrix} \alpha_4\alpha_3^* & |\alpha_3|^2 & \alpha_1\alpha_3^* \\ \alpha_4\alpha_2^* & \alpha_3\alpha_2^* & \alpha_1\alpha_2^* \\ \alpha_4\alpha_1^* & \alpha_3\alpha_1^* & |\alpha_1|^2 \end{vmatrix} - (\alpha_1\alpha_4^*) \begin{vmatrix} \alpha_4\alpha_3^* & |\alpha_3|^2 & \alpha_2\alpha_3^* \\ \alpha_4\alpha_2^* & \alpha_3\alpha_2^* & |\alpha_2|^2 \\ \alpha_4\alpha_1^* & \alpha_3\alpha_1^* & \alpha_2\alpha_1^* \end{vmatrix} \tag{33}$$

$$\tag{34}$$

$$\Delta(\rho) = |\alpha_4|^2(0) - (\alpha_3\alpha_4^*) * (0) + (\alpha_2\alpha_4^*) * (0) + (\alpha_1\alpha_4^*) * (0) = 0.$$

$$\Delta(\rho) = 0. \tag{35}$$

Thus we can conclude that the anti-transposition map "$\wedge$" is **Positive map** as for arbitrary pure state $|\phi\rangle$ we get all order principal minors **non-negative**.

IV. Proof for Complete positivity of $\wedge$

In this section we will prove that our anti-transposition map is completely positive. As the map is applied on state in four dimension, then maximally entangled state we get is,

$$\Phi = \frac{1}{\sqrt{4}}(|00\rangle + |11\rangle + |22\rangle + |33\rangle). \tag{36}$$

The pure states are,

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{37}$$

Using Choi-Jamiolkowski isomorphism

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle \tag{38}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}}(|0\rangle \otimes |0\rangle + \cdots + |d-1\rangle \otimes |d-1\rangle) \tag{39}$$

In our case d = 4, therefore

$$\Phi = \frac{1}{\sqrt{4}}(|00\rangle + |11\rangle + |22\rangle + |33\rangle) \tag{40}$$

As we have to see whether our map is completely positive we have to consider every possible combination of pure states in four dimensions,

$$\begin{aligned}\Phi = {} & |00\rangle\langle 00| + |00\rangle\langle 11| + |00\rangle\langle 22| + |00\rangle\langle 33| \\ & \ldots + |11\rangle\langle 00| + |11\rangle\langle 11| + |11\rangle\langle 22| + |11\rangle\langle 33| + |22\rangle\langle 00| \\ & \ldots + |22\rangle\langle 11| + |22\rangle\langle 22| + |22\rangle\langle 33| + |33\rangle\langle 00| + |33\rangle\langle 11| + |33\rangle\langle 22| + |33\rangle\langle 33|\end{aligned} \tag{41}$$

Before solving let's explore the internal subsystem of the ket and bra we get,

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\\0\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\\0\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\\1\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}, |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\0\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\1\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\0\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix}$$

$$|22\rangle = |2\rangle \otimes |2\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\\1\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\\0\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\\0\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, |33\rangle = |3\rangle \otimes |3\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \tag{42}$$

Similarly we can find the subsystems for their corresponding bra's,

$$
\begin{aligned}
\langle 00| &= \langle 0| \otimes \langle 0| \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} & 1\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\langle 11| &= \langle 1| \otimes \langle 1| \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
\langle 22| &= \langle 2| \otimes \langle 2| \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\langle 33| &= \langle 3| \otimes \langle 3| \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{46}
$$

Now we can formulate the sum of outer-products,

$$
|00\rangle \langle 00| = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$|00\rangle\langle 11| = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$|00\rangle\langle 22| = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$
|00\rangle\langle 33| = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
|11\rangle\langle 00| = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$|11\rangle\langle 11| = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$$|11\rangle\langle 22| = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$$|11\rangle\langle 33| = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|22\rangle\langle 00| = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
|22\rangle\langle 11| = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$
|22\rangle\langle 22| = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$
|22\rangle\langle 33| = \begin{pmatrix} 0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
|33\rangle\langle 00| = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
|33\rangle\langle 11| = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

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|33\rangle\langle 22| = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$|33\rangle\langle 33| = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\end{pmatrix}$$

$$\Phi = (16,16) = \begin{pmatrix}1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\end{pmatrix} \tag{47}$$

After applying our anti-transposition map on Φ i.e. $\wedge(\Phi)$ we get,

$$
\wedge(\Phi) \quad = \quad \begin{pmatrix}
\wedge\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \\
\wedge\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix} \\
\wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \\
\wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix} & \wedge\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}
\end{pmatrix}
$$

$$
\wedge(\Phi) \quad = \quad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0\\
0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0\\
0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0\\
0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0\\
0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0\\
0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0\\
\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

by removing $\Phi_{2-15,2-15}$ (removing the rows from row-2 to row-15 and col-2 to col-15) i.e. one of it's principal minor we get the following matrix,

$$
\Phi_{2-15,2-15} = \begin{pmatrix}0&1\\1&0\end{pmatrix}
$$

the determinant of the following principal minor is $\Delta(\Phi_{2-15,2-15}) = 0 - 1 = -1 \leq 0$.

As the the value of the principal minor is negative i.e. -1 $\leq$ 0 also if we find the eigen values of Φ we get several negative eigen values, Thus we can conclude that the anti-transposition map is not completely positive map.

V. Conclusion

At the end we would like to conclude that the Anti-Transposition map satisfies all the conditions to be a positive map, but does not satisfies the conditions for completely positive map. So Anti-Transposition map $\wedge$ is positive, but not completely positive map. Therefore, we can conclude that $\wedge$ can detect entanglement in three and four dimensions.

VI. Future Goals

Looking forward to implement this map for n-dimensions. And try to make it a general map for entanglement detection.

References

[1]: Schmidt decomposition

[2]: Completely Positive map, (page no. 16,17)

[3]: Choi-JamiolKowski Isomorphism

[4]: Complex Conjuate Properties