Honours Project-3
Multipartite Entanglement Detection

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Abstract- In this project we are now moved on from bipartite to multipartite entanglement detection. We tried to detect entanglement using Transposition-based GME criteria. We applied Transposition based GME criteria on both the W-state and GHZ state. At last we observed that by using naive transposition, one can detect negative eigen values for W-state, but not for GHZ state and to make the GME criteria more robust to detect entanglement for GHZ state we need to do some modifications to the criteria i.e direct lifting of maps.

I. Introduction

This project is an exploration of multipartite entanglement. Here we discuss the structure of entanglement when more than two parties are involved. We first discuss different notions of entanglement and separability for the case of three qubits. Then, we discuss the case of general multipartite systems.

A. Entanglement of three qubits

1. Pure states:

Let us first consider pure three-qubit states. There are two different types of separability: the **fully separable states** that can be written as

$$|\Phi^{fs}\rangle_{A|B|C} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C \tag{1}$$

and the **Bi-separable states** that can be written as a product state in the bipartite system.

$$|\Phi^{bs}\rangle_{A|BC} = |\alpha\rangle_A \otimes |\delta\rangle_{BC}$$

$$|\Phi^{bs}\rangle_{B|AC} = |\alpha\rangle_B \otimes |\delta\rangle_{AC}$$

$$|\Phi^{bs}\rangle_{C|AB} = |\alpha\rangle_C \otimes |\delta\rangle_{AB} \tag{2}$$

Here, $|\delta\rangle$ denotes a two-party state that might be entangled. Therefore a pure state is called **Genuine Tripartite Entangled** if it is neither fully separable nor bi-separable.

For example: **GHZ state** and **W** state,

$$|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{3}$$

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{4}$$

2. Mixed states:

We define a mixed state ϱ^{fs} as fully separable if ϱ^{fs} can be written as a convex combination of fully separable pure states, i.e., if there are convex weights p_i and fully separable states $|\phi^{fs}\rangle such that we can write$

$$\varrho^{fs} = \sum_i p_i |\phi_i^{fs}\rangle \langle\phi_i^{fs}| \tag{5}$$

A state ϱ^{bs} that is not fully separable is called **bi-separable** if it can be written as a convex combination of bi-separable pure states:

$$\varrho^{bs} = \sum_i p_i |\phi_i^{bs}\rangle \langle\phi_i^{bs}| \tag{6}$$

Note that the bi-separable states $|\phi_i^{bs}\rangle$ might be bi-separable with respect to different partitions **[1]**.

B. Full Separability:

The strongest notion of separability is the complete absence of any type of entanglement, i.e. full separability. An n-partite finite dimensional quantum state $\rho_{sep} \in P(H1 \otimes ... \otimes Hn)$ is called fully-separable, and will be denoted by ρ_{sep}, iff it can be decomposed as

$$\rho_{sep} = \sum_i p_i \rho_1^i \otimes ... \otimes \rho_n^i \tag{7}$$

where p_i is a probability distribution. If a quantum state ρ cannot be decomposed into the form (9) there must be some entanglement.

C. Bi-seperability:

It is the weakest notion of separability usually referred to as bi-separability. An n-partite finite dimensional quantum state $\rho_{2-sep} \in P(H1 \otimes ... \otimes Hn)$ is called fully-separable, and will be denoted by ρ_{sep} and $A \subset 1, ..., n$ denote a proper subset of the parties. A state ρ_{2-sep} is bi-separable iff it can be decomposed as

$$\rho_{sep} = \sum_A \sum_i p_A^i \rho_A^i \otimes \rho_{\bar{A}}^i \tag{8}$$

where $p_A^i \geq 0$, $\sum_A \sum_i p_A^i = 1$ ρ_A denotes a quantum state for the subsystem defined by the subset A and Σ_A stands for the sum over all bi-partitions $A|\bar{A}$.

D. Genuine N-partite entangled:

An *n*-partite state which cannot be decomposed as (10) is called **genuine *n*-partite entangled** or genuine multipartite entangled (**GME**).

II. GME Criteria Based On Positive Maps

Our goal is to explore a direct lifting method and seek for maps of the form,

$$\Phi_{GME} := \sum_A \wedge_A \otimes I_{\bar{A}} \circ \mathcal{U}^{(A)} + M, \tag{9}$$

where M is a positive map, $\mathcal{U}^{(A)}[\rho] := \sigma_i p_i^{(A)} U_i^{(A)} \rho (U_i^{(A)})^\dagger$ is a family of convex combinations of local unitaries, and $\rho_{GME}[\rho_{2-sep}] \geq 0 \forall \rho_{2-sep}$.

III. Transposition-based GME criteria

In this section, we tried to apply the transposition-based GME criteria to check whether it can detect entanglement in W-state and GHZ state. Here we set $M = c\mathbb{1}.Tr$, $\mathcal{U}^{(A)} = I$ and $\wedge = T$, the transposition, in a tripartite setting, and find the value of c for which Φ_{GME} defined by (11) is a GME-map **[2]**.

$$\Phi_T[.] = (T_A \otimes I_B \otimes I_C + I_A \otimes T_B \otimes I_C + I_A \otimes I_B \otimes T_C + c\mathbb{1}.Tr)[.] \tag{10}$$

We got to know that for **c = 1** it holds that for all tripartite bi-separable state ρ_{2-sep}, $\Phi_T[\rho_{2-sep}] \geq 0$ and this value of c is optimal.

Now, lets apply this map to a system or a state where it directly works. Firstly we will be applying it on **W-state**.

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{11}$$

To apply the map on the W state, first and main step is we need to get the matrix form of the transposition map.

Let us consider, $A = |0\rangle\langle 0|$, $B = |0\rangle\langle 1|$, $C = |1\rangle\langle 0|$, $D = |1\rangle\langle 1|$, $\rho = |W\rangle\langle W|$. Following the criteria and form of the map we create the matrix form for tripartite setting is given below,

$$\begin{aligned} I \otimes \Phi_T[\rho] = & \, I \otimes I \otimes A[\rho] I \otimes I \otimes A + I \otimes I \otimes B[\rho] I \otimes I \otimes B + I \otimes I \otimes C[\rho] I \otimes I \otimes C + I \otimes I \otimes D[\rho] I \otimes I \otimes D \\ & + I \otimes A \otimes I[\rho] I \otimes A \otimes I + I \otimes B \otimes I[\rho] I \otimes B \otimes I + I \otimes C \otimes I[\rho] I \otimes C \otimes I + I \otimes D \otimes I[\rho] I \otimes D \otimes I \\ & + A \otimes I \otimes I[\rho] A \otimes I \otimes I + B \otimes I \otimes I[\rho] B \otimes I \otimes I + C \otimes I \otimes I[\rho] C \otimes I \otimes I + D \otimes I \otimes I[\rho] D \otimes I \otimes I \end{aligned} \tag{12}$$

After applying the transposition map we find that $\Phi_T[|W\rangle\langle W|] \ngeq 0$ with negative eigenvalue ***- 0.1547***.

Given below is the **Matlab Code** for applying the transposition map based GME criteria on a multipartite setting (i.e. tripartite case) on **W-state**.

Matlab Code

```
% pure states

a = [1; 0]
b = [0; 1]

%-------------------------------
%-------------------------------

```

```
%Computation for W-state

ak = kron(a,a);
a1k = kron(ak,b);

bk = kron(a,b);
b1k = kron(bk,a);

ck = kron(b,a);
c1k= kron(ck, a);

%------------------------------

% Resulting W-state
Wstate = a1k + b1k + c1k

Wstate_t = transpose(Wstate)

% Final 8x8 matrix of |W >< W|-state
final = kron(Wstate, Wstate_t)

%------------------------------
%------------------------------

% Creating Matrix Form of Transposition Map

A = [1 0; 0 0] %|0><0|
B = [0 1; 0 0] %|0><1|
C = [0 0; 1 0] %|1><0|
D = [0 0; 0 1] %|1><1|

I = [1 0; 0 1]

% I x I x A
t = kron(I,I)
IxIxA = kron(t,A)

% I x I x B
IxIxB = kron(t,B)

% I x I x C
IxIxC = kron(t,C)

% I x I x D
IxIxD = kron(t,D)

%------------------------------

% I x A x I
t = kron(I,A);
IxAxI = kron(t,I)

% I x B x I
t = kron(I,B);
```

```
IxBxI = kron(t,I)

% I x C x I
t = kron(I,C);
IxCxI = kron(t,I)

% I x D x I
t = kron(I,D);
IxDxI = kron(t,I)

%------------------------------

% A x I x I
t = kron(A,I);
AxIxI = kron(t,I)

% B x I x I
t = kron(B,I);
BxIxI = kron(t,I)

% C x I x I
t = kron(C,I);
CxIxI = kron(t,I)

% D x I x I
t = kron(D,I);
DxIxI = kron(t,I)

%------------------------------
%------------------------------

id = kron(I,I);
id1 = kron(id, I) * trace(final) % added the trace, c = 1

sum1 = IxIxA * final * IxIxA + IxIxB * final * IxIxB + IxIxC * final * IxIxC +
    IxIxD * final * IxIxD + id1;

sum2 = IxAxI * final * IxAxI + IxBxI * final * IxBxI + IxCxI * final * IxCxI +
    IxDxI * final * IxDxI;

sum3 = AxIxI * final * AxIxI + BxIxI * final * BxIxI + CxIxI * final * CxIxI +
    DxIxI * final * DxIxI;

%resultant 8x8 matrix
totalsum = sum3 + sum2 + sum1

% Eigen values of the reultant matrix after applying the map.
resut = eig(totalsum)/3
```

OUTPUT

```
%-------- OUTPUT --------%
result = 8x1
```

```
   -0.1547 % <--- Negative  Eigenvalue
   1.0000
   1.0000
   1.0000
   1.6667
   1.6667
   2.1547
   2.6667
%-------- OUTPUT --------%
```

Similarly we did for GHZ state (i.e. $|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + \langle 111|)$) and tried to find out whether this simple transposition map based GME criteria can detect negative eigenvalue (or entanglement) **But this map was not successful in getting negative eigenvalues**.

IV. Conclusion

After observing the results we can conclude that this transposition map based GME criteria can **only detect negative eigenvalues in W-state** and not for GHZ state.

V. Future Goals Improvements

As our criteria was not able to detect negative eigenvalues in GHZ state, to make it more robust we need to do modifications to this transposition map based GME criteria. By operating **Unitary Pauli operator** σ_x we can obtain negative eigenvalue. Further we may also create a **Anti-transposition based GME criteria** to detect negative eigenvalues in multipartite setting.

References

[1] Guhne, O., and TothK, G., *Entanglement detection*, Cornell University, 2009. https://doi.org/10.1016/j.physrep.2009.02.004.

[2] Fabien Clivaz, L. L. G. M., Marcus Huber, *Genuine-multipartite entanglement criteria based on positive mapss*, 10.1016/j.physrep.2009.02.004, 2017. https://doi.org/10.1063/1.4998433.