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Some New Fuzzy Entropy Formulae

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Abstract

Fuzzy entropy plays a significant role in characterization and quantification of fuzzy uncertainty. This study introduces some new formulae of fuzzy entropy. Unlike most of the existing geometrical measures of fuzziness, the measures proposed here are based on the distance between a fuzzy set and the centre of a fuzzy unit hypercube. Their structure is particularly simple and intuitive.

Keywords: Fuzzy entropy, Distance Measure, Fuzzy hypercube

1 Introduction

Since its inception [11], the theory of fuzzy sets has continued to play a leading role in modeling non-statistical uncertainty-more specifically, its contribution to characterization and quantification of fuzzy uncertainty is undeniable [12, 9]. Fuzzy sets provide a mathematical framework within which vague concepts can be quantitatively defined and analyzed more naturally without much loss of information.

Fuzziness is measured using fuzzy entropy [3]. Motivated by the classical Shannom entropy, Deluca and Termin introduced the now widely adopted definition of fuzzy entroy [3]. Since then, numerous researchers have successfully formulated new fuzzy entropy formulae satisfying this definition ([7],[8],[6],[10]). This study proposes some new distance based fuzzy entropy formulae. The structure of the formulae is based the distance between a fuzzy set and the centre of the a fuzzy unit hypercube.

The following section provides definitions and notations. The third section reviews some existing measures of fuzzy entropy. Then some new formulae of fuzzy entropy are given in the last section.

2 Preliminaries

Let $\zeta = \{x_1, x_2, \ldots, x_n\}$ be a universal set. A set of all crisp and fuzzy set over ζ is denoted by 2^ζ and $F\left(2^\zeta\right)$ respectively. A fuzzy subset $A = \{(x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n)\}$ is characterized by its membership function m_A, a mapping

$$m_A : \zeta \to [0, 1] \tag{1}$$

such that $m_A(x_i) = a_i \in [0, 1]$ is a membership (or fit) value of x_i in A. Fuzzy set A represented by a fit vector $(a_1, a_2, \ldots, a_n)$ is regarded as a point in an n-dimensional fuzzy unit hypercube $[0, 1] \times [0, 1] \times \cdots \times [0, 1] = I^n$ [8]. A constant $A_{[a]}$ fuzzy set on ζ is such that $m_A(x_i) = a \forall x_i \in \zeta$. The complement $\overline{A}$ of A is fuzzy set with membership function $m_{\overline{A}}(x) = 1 - m_A(x)$. The cardinality of A is a non-negative real scalar $c(A) = \sum_{i=1}^{n} m_A(x_i)$. A fuzzy set A^* is said to be a sharpened form of A if $m_A(x) \geq \frac{1}{2}$ then $m_{A^*}(x) \geq m_A(x)$ and if $m_A(x) \leq \frac{1}{2}$ then $m_{A^*}(x) \leq m_A(x)$.

For any $A \in F\left(2^\zeta\right)$, denote respectively by A_{near} and A_{far} the nearest and farthest crisp sets with membership functions

$$A_{near} = \begin{cases} 1, & \text{if} \quad m_A(x) \geq \frac{1}{2} \\ 0, & \text{if} \quad m_A(x) < \frac{1}{2} \end{cases} \tag{2}$$

and

$$A_{far} = \begin{cases} 0, & \text{if} \quad m_A(x) \geq \frac{1}{2} \\ 1, & \text{if} \quad m_A(x) < \frac{1}{2} \end{cases} \tag{3}$$

A general class of distance measure, namely the Minkowski distance between fuzzy sets A and B is defined as,

$$\gamma_p(A,B) = \left(\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|^p \right)^{\frac{1}{p}}, \quad p \geq 1 \tag{4}$$

3 Fuzzy Entropy

Fuzzy entropy is a set to point mapping [3]

$$E : F\left(2^{\zeta}\right) \rightarrow [0,1] \tag{5}$$

given by

$$E(A) = -k\left\{m_A(x_i)\log m_A(x_i) + (1 - m_A(x_i))\log(1 - m_A(x_i))\right\} \tag{6}$$

with $k \geq 0$, satisfying the following properties.

$(A1)$ $E(A) = 0 \forall A \in 2^{\zeta}$

$(A2)$ $E(A)$ attains its maximum at $A_{[\frac{1}{2}]}$

$(A3)$ if A^* is a sharpened version of any $A \in F\left(2^{\zeta}\right)$ then $E(A^*) \leq E(A)$

$(A4)$ For any $A \in F\left(2^{\zeta}\right)$, $E(A) = E\left(\overline{A}\right)$

Based on this axiomatic definition, Kaufman [6] proposed a measure of fuzziness based the distance between fuzzy set A and the nearest crisp set A_{near} as

$$E_{ka}(A) = \left(\sum_{i=1}^{n} |m_A(x_i) - m_{A_{near}}(x_i)|^p \right)^{\frac{1}{p}} \tag{7}$$

For $p = 1$ and $p = 2$ this measure collapses to special cases namely,

$$E_{ka}(A) = \sum_{i=1}^{n} |m_A(x_i) - m_{A_{near}}(x_i)| \tag{8}$$

and

$$E_{ka}(A) = \left(\sum_{i=1}^{n} |m_A(x_i) - m_{A_{near}}(x_i)|^2 \right)^{\frac{1}{2}} \tag{9}$$

the Hamming and Euclidean distances respectively. Viewing fuzziness as indistinguishability between fuzzy set A and its opposite $\overline{A}$, Yager suggested a mesure of fuzziness given by the formula[10],

$$E_y(A) = 1 - \frac{\gamma_p\left(A, \overline{A}\right)}{n^{\frac{1}{p}}} \tag{10}$$

where

$$\gamma_p\left(A,\overline{A}\right)=\left(\sum_{i=1}^{n}|m_A\left(x_i\right)-m_{\overline{A}}\left(x_i\right)|^p\right)^{\frac{1}{p}} \tag{11}$$

For $p=1$, this measure is of particular simplicity,

$$E_y\left(A\right)=1-\widehat{\gamma}_1\left(A,\overline{A}\right) \tag{12}$$

where $\widehat{\gamma}_1\left(A,\overline{A}\right)$ is the normalized Hamming distance.

Kosko's fuzzy entropy formula[8],

$$E_{k1}\left(A\right)=\frac{\gamma_p\left(A,A_{near}\right)}{\gamma_p\left(A,A_{far}\right)} \tag{13}$$

was largely motivated by the geometrical representation of fuzzy sets. Kosko formulated another fuzzy entropy measure based on the fuzzy operators of the union and intersection as[8]

$$E_{k2}\left(A\right)=\frac{c\left(A\cap\overline{A}\right)}{c\left(A\cup\overline{A}\right)} \tag{14}$$

The formula given in is sometimes known as the fuzzy entropy theorem [8].

Distance induced fuzzy entropy is also introduced in [5]. Bhandari and Pal [1] have suggested fuzzy entropy of a fuzzy set with respect to some fixed fuzzy set.

4 Some new fuzzy entropy formulae

For any $A\in F\left(2^{\varsigma}\right)$ it is directly that,

$$\begin{aligned}\gamma_p\left(A,A_{\left[\frac{1}{2}\right]}\right)=\gamma_p\left(\overline{A},A_{\left[\frac{1}{2}\right]}\right)=\left(\sum_{i=n}^{n}|m_A\left(x_i\right)-\frac{1}{2}|^p\right)^{\frac{1}{p}}\\ \leq\frac{n^{\frac{1}{p}}}{2}\end{aligned} \tag{15}$$

The metric $\gamma_p\left(A,A_{\left[\frac{1}{2}\right]}\right)$ inversely expresses the degree of fuzziness of A. Thus,

$$E_1\left(A\right)=1-\frac{n^{\frac{1}{p}}}{2}\gamma_p\left(A,A_{\left[\frac{1}{2}\right]}\right) \tag{16}$$

gives a measure of fuzzy degree of set A. We can also intuitively introduce a measure of entropy of fuzzy set A as,

$$E_2\left(A\right)=\frac{n^{\frac{1}{p}}-2\gamma_p\left(A,A_{\left[\frac{1}{2}\right]}\right)}{n^{\frac{1}{p}}+2\gamma_p\left(A,A_{\left[\frac{1}{2}\right]}\right)} \tag{17}$$

is a fuzzy entropy measure.

Recognize that this measure can be expressed as,

$$E_2(A) = \frac{n^{\frac{1}{p}} - 2\left(\sum_{i=1}^{n} |m_A(x_i) - \frac{1}{2}|^p\right)^{\frac{1}{p}}}{n^{\frac{1}{p}} + 2\left(\sum_{i=1}^{n} |m_A(x_i) - \frac{1}{2}|^p\right)^{\frac{1}{p}}} \tag{18}$$

Similarly $(A1), (A2)$ and $(A4)$ are easy to check. The proof for $(A3)$ follows from the fact that fuzziness of A is inversely related to the distance between A and the centre of the hypercube.

5 Conclusion

We have intuitively introduced new fuzzy entropy measures based on the distance between a fuzzy set and the centre of the fuzzy unit hypercube. These measures offer more choices from which we can choose in modeling uncertainty due to fuzziness

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