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Unit - 3

Partial Differential Equations.

$P = \frac{\partial z}{\partial x} = 2x$

$q = \frac{\partial z}{\partial y} = 2y$

$R = \frac{\partial^2 z}{\partial x^2} = 2xx$

$S = \frac{\partial^2}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2} = 2yy$

(1) form the PDE by eliminating the arbitrary constants for the following equations

$\Rightarrow P = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, t = \frac{\partial z^2}{\partial y^2}, s = \frac{\partial^2 z}{\partial x \partial y}$

$z = (x-a)(y-b)$ constant

Sol:- Given $z = (x-a)(y-b)$ — (1).

Diff w.r.t x as partially - P → (2)

$\frac{\partial z}{\partial x} = (1)(y-b) = P \Rightarrow$ (2)

Diff w.r.t y as partially

$\frac{\partial z}{\partial y} = (x-a)(1) = q \Rightarrow$ (3)

using eqn (2) & (3) in (1)

$$z = (x-a)(y-b)$$

$$z = qp$$

$$\boxed{z = pq}$$

This required soln.

(2) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow$

Sol Given $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow (1)$

Now Diff wrto 'x' as partially

$$\frac{\cancel{2}\partial z}{\partial x} = \frac{\cancel{2}x}{a^2} + 0.$$

$$\frac{\partial z}{\partial x} = \frac{x}{a^2} = 0$$

$$\Rightarrow p = \frac{x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{p}{x} \rightarrow (2)$$

Next Diff wrto y as partially.

$$\frac{\cancel{2}\partial z}{\partial y} = \frac{\cancel{2}y}{b^2}$$

$$\frac{\partial z}{\partial y} = \frac{y}{b^2} = q.$$

$$q = \frac{y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{q}{y} - (3)$$

using eqn (2) & (3) in (1)

$$2z = x^2 \times \frac{1}{a^2} + y^2 \times \frac{1}{b^2}$$

$$2z = x^{\cancel{2}} \times \frac{p}{\cancel{x}} + y^{\cancel{2}} \times \frac{q}{\cancel{y}}$$

$$\boxed{2z = px + qy}$$

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(3) $z = xy + y\sqrt{x^2-a^2}$

Diff w.r.to 'x' partially

$$\frac{\partial z}{\partial x} = y + y\frac{1}{\cancel{2}\sqrt{x^2-a^2}}\cancel{2}x = p$$

$$p = \frac{\partial z}{\partial x} = y + \frac{xy}{\sqrt{x^2-a^2}} \rightarrow (1)$$

Now diff w.r.to 'y' partially.

$$\frac{\partial z}{\partial y} = x + \sqrt{x^2-a^2} = q$$

$$q = x\sqrt{x^2-a^2} =$$

$$q - x = \sqrt{x^2-a^2} - (2)$$

Using eqn (2) in (1)

$$p = y + \frac{xy}{q-x}$$

$$(p-y) = \frac{xy}{q-x}$$

$$(p-y)(q-x) = xy.$$

$$pq - px - qy + xy = xy.$$

$$pq = \cancel{xy} + px + qy - \cancel{xy}$$

$$\boxed{pq = px + qy} - (8)$$

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A $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where, a, b, c are constants

$\Rightarrow$ Diff wrt x

$$\frac{2x}{a^2} + 0 + 2z\frac{d^2}{2} = 0$$

$$\frac{2x}{d^2} = \frac{d}{2} \quad \frac{-2zdz}{dx/c^2}$$

$$\frac{x}{a^2} = \frac{-2p}{x^2}$$

$$\frac{c^2}{-a^2} = \frac{-zp}{x}$$

$$-\frac{c^2}{a^2} = \frac{zp}{x}$$

Diff wrt x

$$zp\left(\frac{1}{x}\right) = \frac{-c^2}{a^2}$$

$$\frac{d}{dx}(uvw) = uvw' + uv'w + u'vw$$

$$\frac{zdz}{dx}\left(\frac{-1}{x^2}\right) + \frac{zd^2z}{dx^2}\frac{1}{x} + \frac{\partial z}{dx}\frac{\partial z}{\partial x}\frac{1}{x} = 0$$

multiply by x^2

$$-\frac{1}{x^2} \times x^2 - z\frac{\partial z}{dx} + \frac{z\,\partial^2 z}{dx^2}\frac{1}{x} \times x^2$$

$$+\left(\frac{\partial z}{\partial x}\right)^2 \frac{1}{x} \times x^2 = 0$$

$\Rightarrow -zp + zrx + p^2x = 0$

★ Form PDE by eliminating arbitrary function following eq:-

★ $z = f(x^2 - y^2)$

$z = f(x^2 - y^2) \rightarrow$ Diff w.r.t x partially

$\frac{\partial z}{\partial x} = f'(x^2-y^2) \cdot 2x \rightarrow (1)$

Diff w.r.t y

$\frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y) \rightarrow (2)$

$(1) \div (2)$

$p = f'(x^2-y^2)\, 2x$

$q = f'(x^2-y^2) \cdot (-2y)$

$\frac{p}{q} = \frac{x}{-y}$

$-py = qx$

$qx + pq = 0$

★ $z = x f_1(x+t) + f_2(x+t)$

Diff w.r.t x

$\frac{\partial z}{\partial x} = x \cdot f_1'(x+t) + f_1(x+t) + f_2'(x+t)$

$\frac{\partial^2 z}{\partial x^2} = x \cdot f_1''(x+1) + f_1(x+t) + (x+t)$
$+ f_2''(x+t)$

Diff w.r.t 't' partially.

$\frac{\partial z}{\partial t} = x f_1(x+t) + f_2'(x+t)$

$\frac{\partial^2 z}{\partial t^2} = x f_1''(x+t) + f_2''(x+t) \rightarrow (2)$

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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(x f_1'(x+t) + f_2'(x+t)\right)$$

$$\frac{\partial^2 z}{\partial x \partial t} = x f_1''(x+t)\, f_1(x+t) + f_2''(x+t)$$

$- (4)$

(3) in (2) & (4)

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2} + 2 f_1'(x+t) - (5)$$

$$\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial^2 z}{\partial t^2} + f_1'(x+t) - (6)$$

multiply eq $\times$ (2)

$$\frac{2\partial^2 z}{\partial x dt} = \frac{2\partial^2 z}{\partial t 2} + 2 f_1(x+t) - (7)$$

eq (7) $-$ (5)

$$2\frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial^2} = \frac{2\partial^2 z}{\partial t^2} + 2 f_1'\cancel{(x+t)}$$

$$- \frac{\partial^2 z}{\partial t^2} - 2 f_1'\cancel{(x+t)}$$

$$\frac{2\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$$

$$\frac{2\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$$

$z = \psi(x+ay) + \phi(x-ay)$

* $z = \psi(x+ay) + \phi(x-ay)$

Sol. consider $z = \psi(x+ay) + \phi(x-ay)$

Diff w.r.to 'x' partially.

$$\frac{\partial z}{\partial x} = \psi'(x+ay)(1) + \phi'(x-ay)(1)$$

Again Diff w.r.to 'x' partially.

Now Diff w.r.to 'y' partially.

$$\frac{\partial z}{\partial y} = \psi'(x+ay)(a) + \phi(x-ay)(-a)$$

Again diff w.r.to 'y' partially.

$$\frac{\partial^2 z}{\partial y^2} = \psi''(x+ay)a^2 + \phi'(x-ay)a^2$$

$$\frac{\partial^2 z}{\partial y^2} = a^2(\psi''(x+ay) \mp \phi''(x-ay))$$

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 r}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}$$

So.

$$t = a^2 \frac{\partial z}{\partial r^2}$$

$$\boxed{t = a^2 r}$$

Thus. req^ed sol^n.

* $\phi(xy+z^2, x+y+z) = 0$.

Sol. consider $\phi(xy+z^2, x+y+z) = 0$

It is in the form of $\phi(u, v) = 0$.

$u = xy + z^2$, $v = x+y+z$

$$\frac{\partial u}{\partial x} = y + 2z \cdot \frac{\partial z}{\partial x} = y + 2zp$$

$$\frac{\partial u}{\partial y} = x + 2z\frac{\partial z}{\partial y} = x + 2zq$$

$v = x+y+z$

$$\frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial y} = 1+q$$

$p = \frac{\partial z}{\partial x}$. $q = \frac{\partial z}{\partial y}$. $r = \frac{\partial^2 z}{\partial x^2}$. $s = \frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$

$t = a^2 \frac{\partial^2 z}{\partial x^2}$

$t = a^2 r$

Thus Reqd soln.

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

$$\frac{y+2zp}{x+2zq} = \frac{1+p}{1+q}$$

$(1+p)(x+2zq) = (1+q)(y+2zp)$.

$x + 2zq + px + 2zpq = y + 2zp + qy + 2zpq$.

$px - 2zp + 2zp - qy + x - y = 0$

$p(x - 2z) + q(2z - y) + x - y = 0$

Q. $f(x^2 + 2yz, y^2 + 2xz) = 0$.

~~$\partial u = \ldots$~~

consider $f(x^2 + 2yz \cdot y^2 + 2xz) = 0$

It is in the form $f(u, v) = 0$

$u = x^2 + 2yz$ $v = y^2 + 2xz) = 0$.

$\frac{\partial u}{\partial x} = 2x + 2y\frac{\partial z}{\partial x} = 2(x + yp)$

$\frac{\partial u}{\partial y} = 2\left(y\frac{\partial z}{\partial y} + z\right) = 2(yq + z)$

$v = y^2 + 2xz$

$\frac{\partial v}{\partial x} = 2\left(x\frac{\partial z}{\partial x} + z\right) = 2(xp + z)$

$\frac{\partial v}{\partial y} = 2y + 2x\frac{\partial z}{\partial y} = 2(y + xq)$.

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$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y} \quad r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$

$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial u/\partial x}{\partial y/\partial y}$

$= \frac{2(x+yp)}{2(yq+z)} = \frac{2(xp+z)}{2(y+xq)}$

$(x+yp)(y+xq) = (xp+z)(yq+z)$

$(xy + x^2 q + y^2 p + xypq = xypq + zxp + zyq + z^2$

$(py^2 - zxp) + (qx^2 - zyq) + (xy - z^2) = 0$

$p(y^2 - zx) + q(x^2 - zy) + xy - z^2 = 0$

★ Solve $\frac{\partial^2 u}{\partial x^2} = x+y$ by direct Integration method.

Ans:- Integrating w.r.t. x

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = x+y$

$\int \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \int x\,dx \int y\,dx$

$\frac{\partial u}{\partial x} = \frac{x^2}{2} + yx + f(y)$

Again Integrating x

$\int \frac{\partial u}{\partial x} = \frac{1}{2}\int x^2 + y\int x + f(y)$

$u = \frac{x^3}{6} + y\frac{x^2}{2} + f(y).x + g(y)$

$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y} \quad r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y}$

$u = \frac{x^3}{6} + y\frac{x^2}{2} + x f(y) + g(y)$

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Solve $\frac{\partial^3 y}{\partial x^2 \cdot \partial y} = \cos(2x+3y)$ By direct Integration method.

considering - $\frac{\partial^3 u}{\partial x^2 \partial y} = \cos(2x+3y)$

$\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\right) = \cos(2x+3y)$

Integrationg w.r.to 'x'

$\int \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \int \cos(2x+3y)$

$\int\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\right) = \int \frac{\sin(2x+3y)}{2} + f(y)$

Again Integration w.r.to 'x'

$\int \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{1}{2}\int \sin(2x+3y) + \int f(y)$

$\frac{\partial u}{\partial y} = \frac{1}{2} - \frac{\cos(2x+3y)}{2} + f(y)x + g(y)$

Integrating w.r.t 'y'

$\int \frac{\partial u}{\partial y} = -\frac{1}{4}\int \cos(2x+3y) + \int x f(y) + \int g(y)$

$u = \frac{-1}{4} \cdot \frac{\sin(2x+3y)}{3} + x f(y) + C_1(y) + h(x)$

$u = -\frac{\sin(2x+3y)}{12} + x f(y) + G(y) + h(x)$

★ Solve $\frac{\partial^2 y}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ and $z=0$ if y is add multiply of $\frac{\pi}{2}$ by direct Integration method

Sol: $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$$

Integration w.r.to 'x'

$$\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) dx = \int \sin x \sin y \, dx$$

$$\frac{\partial z}{\partial y} = \sin y\,(-\cos x) + f(y)$$

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad -(1)$$

Integration w.r.to 'y'

$$\int\left(\frac{\partial z}{\partial y}\right) dy = -\int \cos x \sin y \, dy + \int f(y)\, dy$$

$$z = -\cos x\,(-\cos y) + f(y) + g(x)$$

$$z = \cos x \cos y + F(y) + g(x) \quad -(2)$$

using when $x=0$. $\frac{\partial z}{\partial y} = -2\sin y$ in eqn (1)

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$$

$$-2\sin y = -\cos(0)\sin y + f(y).$$

$$-2\sin y = -\sin y + f(y). \qquad \cos 0 = 1$$

$$f(y) = -2\sin y + \sin y$$

$$f(y) = -\sin y$$

Next: $F(y) = \int f(y)\, dy$

$$= \int -\sin y \, dy.$$

$$F(y) = \cos y \quad -(3)$$

using eqn (3) in eqn (2)

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$z = \cos x \cos y + f(y) + g(x)$

$z = \cos x \cos y + \cos y + g(x)$

when $z = 0$, $y = (2n+1)\frac{\pi}{2}$

using this condn in eqn (4)

$z = \cos x \cos y + \cos y + g(x)$

$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x)$

$\Rightarrow \cos(2n+1)\frac{\pi}{2} = 0$

$0 = 0 + 0 + g(x) \Rightarrow g(x) = 0$

$z = \cos x \cos y + \cos y + 0$

$z = \cos y(\cos x + 1)$ //

* use the method of separation of variables to solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$

Sol let $u = XY$ be the soln. PDE eqn → (1)

where $X = X(x)$, $Y = Y(y)$

now, Diff u w.r.to x and y partially.

$\frac{\partial u}{\partial x} = X'Y$ | $\frac{\partial u}{\partial y} = XY'$ → (2)

consider $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ → (3)

usinging eqn (1) & (2) in eqn (3).

$X'Y + XY' = 2(x+y)XY$

÷ by XY

$\frac{X'Y}{XY} + \frac{XY'}{XY} = \frac{2(x+y)XY}{XY}$

$\frac{X'}{X} + \frac{Y'}{Y} = 2(x+y)$

where $X' = \frac{dX}{dx}$, $Y' = \frac{dY}{dy}$

$\frac{1}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 2x + 2y$

$\frac{dX}{X} = 2x\,dx$ $\qquad$ $\frac{dY}{Y} = 2y\,dy$

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$\frac{dx}{x} - 2x\,dx = k \qquad \frac{dy}{y} - 2y\,dy = k$

$\frac{dx}{x} = (k + 2x)dx \qquad -\frac{dy}{y} + 2y\,dy = -k$

$\int \frac{dx}{x} = \int (2x + k)dx \qquad \frac{dy}{y} = (-k - 2y)dy$

$\log x = x^2 + kx + c_1 \qquad \int \frac{dy}{y} = (-ky - y^2) + c_2$

$x = e^{x^2 + kx + c_1} \qquad \log_e y = (-ky - y^2) + c_2$

$\Rightarrow a^n \cdot a^n = a^{m+n}. \qquad y = e^{-ky - y^2 + c_2}$

$\therefore u = xy$

$u = e^{x^2 + kx + c_1 - ky - y^2 + c_2} \qquad e^{c_1} \cdot e^{c_2} = c$

$u = ce^{x^2 + kx\ y - y^2}$

$= ce^{x^2 - y^2 + k(x-y)}$

$= ce^{x^2 - y^2 + k(x-y)}$

$\frac{x'}{x} - 2x = k \qquad \frac{-y'}{y} + 2y = k$

$\frac{1}{x}$ where $x' = \frac{dx}{dx} \quad y' = \frac{dy}{dy}$

$\frac{1}{x}\frac{dx}{dx} - 2x = k \quad . \quad -\frac{1}{y}\frac{dy}{dy} + 2y = k$

$\frac{1}{x}\frac{dx}{dx} = (2x + k) \quad . \quad -\frac{1}{y}\frac{dy}{dy} = (k - 2y)$

$\frac{1}{x}\frac{dx}{dx} = (2x + k)$

$\frac{dx}{x} = \int (2x + k)dx$

$\log_e x = x^2 + kx + c_1$

$x = e^{x^2 + kx + c_1}$

eq^n ① becomes.

$u = xy$

u

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$u = e^{x^2} + kx + c_1 \cdot e^{y^2} + ky + c_2$

$= ce^{x^2 + y^2} + kx - ky$

$= ce^{x^2 + y^2} + k(x-y)$

$= ce^{x^2 + y^2} + k(x-y)$

$\frac{1}{y}\frac{dy}{dy} = (2y - k)$

$\frac{dY}{Y} = (2y-k)dy$

$\int \frac{dy}{y} = \int (2y-k)dy$

$\log y = y^2 - ky + c_2$

$Y = e^{y^2 - ky + c_2}$

* use the method of separation of variable to solve $\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + z$ where $z(x,0) = 6e^{-3x}$

let $z = xy$ —① be soln.

diff z w.r.to x and y partially

$\frac{\partial z}{\partial x} = x'y \quad \frac{\partial z}{\partial y} = xy' \rightarrow$ ②

consider $\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + z$

$x'y = 2(xy') + xy$

$x'y = x(2y' + y)$

$\frac{x'}{x} = \frac{2y' + y}{y} = k$

$\frac{x'}{x} = k, \quad \frac{2y' + y}{y} = k$

$2y' + y = ky$

$2y' = ky - y$

$2y' = y(k-1)$

$\frac{y'}{y} = \left(\frac{k-1}{2}\right)$

where $x' = \frac{dx}{dx} \quad y' = \frac{dy}{dy}$

$\frac{1}{x}\frac{dx}{dx} = k, \quad \frac{1}{y}\frac{dy}{dy} = \left(\frac{k-1}{2}\right)$

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$$\frac{dx}{x} = k\,dx \qquad\qquad \int \frac{dy}{y} = \int \frac{(k-1)}{2} dy$$

$$\int \frac{dx}{x} = \int k\,dx \qquad\qquad \int \frac{dy}{y} = \int \left(\frac{k-1}{2}\right) dy$$

$$\log_e X = kx + c_1 \qquad\qquad \log_e Y = \left(\frac{k-1}{2}\right) y + c_2$$

$$X = e^{kx+c_1}$$

eqn ① becomes.

$$z = XY \qquad\qquad Y = e^{\left(\frac{k-1}{2}\right) y + c_2}$$

$$z = e^{kx+c_1} \cdot e^{\left(\frac{k-1}{2}\right) y + c_2}$$

$$z = C e^{kx} \cdot e^{\left(\frac{k-1}{2}\right) y} \quad - ②$$

$$z(x,0) = 6e^{-3x}$$

$$z(x,0) = C e^{kx} \cdot e^{\frac{(k-1)}{2} 0}$$

$$z(x,0) = C e^{kx}$$

$$6e^{-3x} = C e^{kx}$$

on comparing

$$C = 6 \qquad k = -3$$

eqn ② becomes

$$z = 6e^{-3x} e^{-2y}$$

$$= 6e^{-(3x+2y)}$$

A) use the method of separation of variables to solve $4\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3z$ where $z(0,y) = 2e^{5y}$

let $z = XY$ — ① be soln.

Diff z w.r.t x and y partially

$$\frac{\partial z}{\partial x} = X'Y \qquad \frac{\partial z}{\partial y} = XY' \quad - ③$$

$$\frac{4\,\partial z}{\partial x} = X'Y \qquad \frac{\partial z}{\partial y} = XY' \Rightarrow ①$$

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$$\frac{4\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3z$$

$$40x'y + x.y' = 3(xy)$$

$$4x'y + xy' = 3xy$$

~~Divided by xy~~

$$\frac{4x'y}{xy} + \frac{xy'}{xy} = \frac{3xy}{xy}$$

$$\frac{4x'}{x} + \frac{y'}{y} = 3$$

$$\frac{4x'}{x} = K \quad \frac{y'}{y} = K$$

$$4x'y = 3xy - xy' \Rightarrow 4\frac{x'}{y} = \frac{3y-y'}{y}$$

$$\cancel{4x'y + xy' = 3xy}$$

$$4x'y = x(3y - y')$$

$$4\frac{x'}{x} = \frac{3y-y'}{y} = K$$

$$= K$$

$$\frac{4xx'}{x} = K \qquad \frac{3y-y'}{y} = K$$

$$\frac{x'}{x} = \frac{K}{4} \qquad 3y - y' = Ky$$

$$3y - Ky = y'$$

$$y(3-K) = y'$$

$$(3-K) = \frac{y'}{y}$$

$$\frac{y'}{y} = 3-K$$

$$x' = \frac{dx}{dx} \qquad y' = \frac{dy}{dy}$$

$$\frac{1}{x}\frac{dx}{dx} = \frac{K}{4} \qquad \frac{1}{y}\frac{dy}{dy} = 3-K$$

Integrate

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$\log_e x = \frac{1}{4} k\,dx \qquad \log_e y = 3y - Y$

$X = e^{1/4 kx + c_1} \qquad Y = e^{3y - kY + c_2}$

$z = XY$

$\Rightarrow e^{1/4 kx + c_1} \; 3y - ky + c_2$

$\Rightarrow c e^{1/4 kx} \; e^{3y - ky}$

$\Rightarrow c e^{1/4 kx} \; e^{y(3-k)}$

$\Rightarrow e^{1/4 kx} + y(3-k)$

$z(0, y) = 2e^{5y}$

$z(0, y) = (0) + e^{y(3-k)}$

$= ce^{y(3-k)}$

$2e^{5y} = ce^{3y - ky}$

$c = 2$

$e^{5y} = e^{(3-k)y}$

$5y = (3-k)y$

$5 = 3 - k$

$\boxed{k = -2}$

$z = 2e^{-2/4 k} + (3-(-2))y$

$\boxed{= 2e^{(-x/2 + 5y)}}$ //.

(7) obtain various possible solution one dimensional wave equations $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ by method of

Separation variable.

$\Rightarrow u = XT \rightarrow (1)$

$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial^2 u}{\partial x^2} = X''T$

$\frac{\partial u}{\partial t} = XT' \qquad \frac{\partial^2 u}{\partial t^2} = XT''$

Then, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

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$XT'' = c^2 X'' T$

$\frac{T''}{c^2 T} = \frac{X''}{X} = k$

$\frac{X''}{X} = k \qquad \frac{T''}{c^2 T} = k$

$X' = \frac{dx}{dx} \qquad X'' = \frac{\partial^2 X}{\partial x^2} \qquad T'' = \frac{\partial^2 T}{\partial t^2}$

$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = k, \qquad \frac{1}{c^2} \frac{d^2 T}{dt^2} = k$

$\frac{\partial^2 X}{\partial x^2} = kx \qquad \frac{d^2 T}{dt^2} = c^2 T k$

$D^2 X = kX \qquad D^2 T = c^2 T k$

$D^2 X - kX = 0 \qquad D^2 T - c^2 T k = 0$

$(D^2 - k) X = 0 \qquad (D^2 - c^2 k) T = 0$

$D^2 - k = 0 \qquad D^2 - c^2 k = 0$

Case 1) $k = 0 \qquad D^2 - c^2 (0) = 0$

$D^2 - D = 0 \qquad D^2 = 0$

$D^2 = 0$

By Auxillary $\qquad m^2 - c^2 (0) = 0$

$m^2 = 0$

$m_3 = 0 \quad m_4 = 0$

$m^2 = 0$

$m_1 = 0 \quad m_2 = 0 \quad m_3 = 0 . \quad m_4 = 0.$

$X = (c_1 + c_2 x) e^{0x} \qquad T = (c_3 + c_4 t) e^{0x}$

$X = (c_1 + c_2 x) \qquad T = (c_3 + c_4 t)$

$u = (c_1 + c_2 x)(c_3 + c_4 t).$

case 2 $\quad k$ be positive $\quad k = +p^2$

$D^2 - k = 0 \qquad D^2 - c^2 k = 0$

$D^2 - p^2 = 0 \qquad D^2 - cp^2 = 0$

$m^2 - p^2 = 0 \qquad m^2 - c^2 p^2 = 0$

$m^2 = p^2 \qquad m = \pm \sqrt{c^2 p^2}$

$m = \pm \sqrt{p^2} \qquad m = \pm cp.$

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$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$ $T = c_3 e^{cpt} + c_4 e^{-cpt}$

$u = (c_1 e^{px} + c_2 e^{-px}) \cdot (c_3 e^{cpt} + c_4 e^{-cpt})$

case 3)

K is negative $K = -p^2$

$D^2 - K = 0$ $D^2 - c^2 K = 0$

$D^2 - p^2 = 0$ $D^2 - c^2 p^2 = 0$

$m^2 - p^2 = 0$ $D^2 = -c^2 p^2$

$m^2 = -p^2$ $m^2 = -c^2 p^2$

$m = \pm \sqrt{-p^2}$ $m = \pm \sqrt{-c^2 p^2}$

$m = \pm ip$ $m = \pm icp$

$y = c_1 \cos px + c_2 \sin px$ $T = c_3 \cos c p t +$
$c_4 \sin c p t$

$u = (c_1 \cos px + c_2 \sin px)(c_3 \cos c p t)$

(8) Obtain various possible solution one dimensional Heat equations $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ by method separation variable

Ans $u = XT$

$\frac{\partial u}{\partial x} = X'T$

$\frac{\partial^2 u}{\partial x^2} = X''T$

$\frac{\partial u}{\partial t} = XT'$

Then $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$XT' = c^2 X''T$

$\frac{T'}{c^2 T} = \frac{X''}{X} = K$

$\frac{T'}{c^2 T} = K$ $\frac{X''}{X} = K$

$T' = \frac{\partial T}{\partial t}$ $X'' = \partial^2 X$

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$$\frac{1}{c^2 T}\frac{\partial T}{\partial t} = K \qquad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = K$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = K$$

$$D^2 X = KX$$

$$D^2 X - KX = 0$$

$$(D^2 - K)X = 0$$

$$D^2 - K = 0$$

$$m^2 - K = 0$$

$$\frac{1}{c^2 T}\frac{\partial T}{dt} = K$$

$$DT = Kc^2 T$$

$$DT - c^2 T K = 0$$

$$(D - c^2 K)T = 0$$

$$D - c^2 K = 0$$

$$m - c^2 K = 0$$

case 1) $K = 0$

$$m^2 - 0 = 0$$

$$m^2 = 0$$

$$m_1 = 0 \quad m_2 = 0$$

$$X = c_1 + c_2 x$$

$$m - c^2(0) = 0$$

$$m - 0 = 0$$

$$m = 0$$

$$T = c_3 e^{0x}$$

$$T = c_3$$

case 2) $K = +p^2$

$$m^2 - p^2 = 0$$

$$m^2 = p^2$$

$$m = \pm\sqrt{p^2}$$

$$m = \pm p$$

$$m = +p \quad m = -p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$m - c^2 p^2 = 0$$

$$m = c^2 p^2$$

$$T = c_1 e^{c^2 p^2 t}$$

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{c^2 p^2 t})$$

case (3) $K = -p^2$

$$m^2 + p^2 = 0$$

$$m^2 = -p^2$$

$$m = \pm\sqrt{-p^2}$$

$$m = \pm ip$$

$$X = c_1 \cos px + c_2 \sin px$$

$$m + c^2 p^2$$

$$m = -c^2 p^2$$

$$T = c_1 e^{-c^2 p^2 t}$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{-c^2 p^2 t})$$

★ Langrage's linear partial differential eqⁿ

P.D eqⁿ form $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$

$Pp + Qq = R$ this known as langrage's linear P.D eqⁿ. where (P, Q, R are expression

$q = \frac{\partial z}{\partial y}$.

solⁿ of the langrage's linear PDE eqⁿs in x, y, z

$$\int \frac{dx}{P} = \int \frac{dy}{Q}$$

$$Pp + Qp = R.$$

(1) A eqⁿ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{P} = \frac{dy}{Q}$ or $\frac{dy}{Q} = \frac{dz}{R}$ or $\frac{dx}{P} = \frac{dz}{R}$

$\frac{dx}{P} = \frac{dy}{Q}$ + $\frac{dy}{Q} = \frac{dz}{R}$

$$\int \frac{dy}{Q} = \int \frac{dz}{R}.$$

$$V = C_2$$

complete solⁿ. $f(u,v) = 0$ OR $\phi(u,v) = 0$.

★ 1) solve $p\cot x + q\cot y = \cot z$

consider $p\cot x + q\cot y = \cot z$

$Pp + Qq = R$

where $P = \cot x$ $Q = \cot y$ $R = \cot z$

A eqⁿ is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

consider $\frac{dx}{\cot x} = \frac{dy}{\cot y}$

$$\int \frac{dx}{\cot x} = \int \frac{dy}{\cot y}.$$

$$\int \tan x\, dx = \int \tan y\, dy.$$

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$\log(\sec x) = \log(\sec y) + \log c_1$

$\log(\sec x) = \log(\sec y) + \log c_1$

~~$\log(\sec x) - \log(\sec y) = c$~~

$\cancel{\log}\left(\frac{\sec x}{\sec y}\right) = \cancel{\log} c_1$

$\frac{\sec x}{\sec y} = c_1 \Rightarrow u = \frac{\sec x}{\sec y}$

$\frac{dy}{\cot y} = \frac{dz}{\cot z}$

$\int \frac{dy}{\cot y} = \int \frac{dz}{\cot z}$

$\int \tan y\, dy = \int \tan z\, dz$

$\log(\sec y) = \log(\sec z) = \log c_2$

$\cancel{\log}\left(\frac{\sec y}{\sec z}\right) \cancel{\log} c_2$

$\frac{\sec y}{\sec z} = c_2 \Rightarrow v = \frac{\sec y}{\sec z}$

complete sol $\phi(u, v) = 0$

$\phi\left(\frac{\sec x}{\sec y}, \frac{\sec y}{\sec z}\right) = 0$

② $x^2 p + y^2 p = z^2$

$P = x^2 \quad Q = y^2 \quad R = z^2$

$P \quad Q \quad R$

$\frac{dx}{x^2} \quad \frac{dy}{y^2} \quad \frac{dz}{z^2}$

$\frac{dx}{x^2} = \frac{dy}{dy^2}$

Integrate

$\frac{1}{y} - \frac{1}{x} = c_1 \Rightarrow \frac{1}{y} - \frac{1}{x}$

$\frac{-1}{x} = \frac{y^{-1}}{-1} + c_1 \quad x + \frac{1}{y} = c_1$

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$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

$$\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$$

$$-\frac{1}{y} = -\frac{1}{z} + c_2$$

$$\boxed{\frac{1}{z} - \frac{1}{y} = c_2} \Rightarrow \boxed{\frac{1}{z} - \frac{1}{y}}$$

complete solⁿ $\phi(u, v) = 0$

$$\boxed{\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0.}$$

* solve = $y^2 zp = x^2 (zp + y)$

Sol $y^2 zp = x^2 zp + x^2 y.$

$P = y^2 zp \quad Q = -x^2 zp \quad R = x^2 y$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2 \cancel{zp}} = \frac{dy}{-x^2 \cancel{zp}} = \frac{dz}{x^2 y.}$$

$$-x^2 dx = y^2 dy = 0$$

$$y^2 dy + x^2 dx = 0$$

$$\frac{y^3}{3} + \frac{x^3}{3} = c_1 \Rightarrow y^3 + x^3 = 3c_1$$

or $x^3 + y^3 = c_1$

$$\frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$$

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$$\cancel{x^2} y\,dy = -\cancel{x^2} z\,dz$$

$$\frac{y^2}{2} = -\frac{z^2}{2}$$

$$\frac{y^2}{2} + \frac{z^2}{2} = c_2 \quad (\text{or}) \Rightarrow y^2 + z^2 = 2c_2$$

$$y^2 + z^2 = c_2$$

$$\phi(u,v) = \phi\left(\frac{y^3}{3} + \frac{x^3}{3}, \frac{y^2}{2} + \frac{z^2}{2}\right) = 0$$

$$(\text{or}) \quad \phi(u,v) \Rightarrow \phi\left(x^3+y^3, y^2+z^2\right) = 0 //.$$

Solve :- $(mz - ny)p + (nx - lz)q = (ly - mx)$

consider : $Pp + Qq = R$

$P = (mz - ny)$ $Q = (nx - lz)$ $R = (ly - mx)$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow$$

$$\frac{dx}{(mz-ny)} \quad \frac{dy}{(nx-lz)} \quad \frac{dz}{(ly-mx)}$$

⇒ using multiplication x, y, z

$$\frac{dx + dy + dz}{(mz-ny) + (nx-lz) + (ly-mx)} = k.$$

$$\frac{x\,dx + y\,dy + z\,dz}{(mxz - nxy) + (nyx - lyz)(lzy - mzx)} = k$$

$$\frac{x\,dx + y\,dy + z\,dz}{0} = k.$$

$$= \int x\,dx + \int y\,dy + \int z\,dz$$

$$= \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = 2c_1$$

$$= x^2 + y^2 + z^2 = 2c_1 \quad (\text{or}) \Rightarrow \boxed{x^2+y^2+z^2 = c_1}$$

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By ~~mud~~ multiple - $\underline{LMN}$

$l\,dx + m\,dy + n\,dz$

$(m\,dz - n\,dy) + (n\,dx - L\,dz) + (L\,dy - m\,dx) = k$

$= 0$

$\int L\,dx + \int m\,dy + \int N\,dz = k$

$Lx + my + Nz = c_2$

$\phi(u, v) = \phi(x^2 + y^2 + z^2 = c_1,\ Lx + my + nz = c_2)$

★ —— ★ —— ★

MP Tour