

Unit AInfinite Series

## \* Taylor's and MacLaurin's series

Taylor's series expansion about point  $x=a$

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a)$$

$$+ \frac{(x-a)^4}{4!} y^{(IV)}(a) + \dots$$

## \* MacLaurin's series expansion given by

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0)$$

$$+ \frac{x^4}{4!} y^{(IV)}(0) + \dots$$

## problems.

- (1) Obtain Taylor's expansion of  $\log x$  in part of  $(x-1)$ . Hence evaluate  $\log e$  (1.1) correct to 4 decimal places.

Sol. Let  $y(x) = \log x$  given  $f(x) = (x-1) = 0$ .

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2!} y''(1) + \frac{(x-1)^3}{3!} y'''(1)$$

$$+ \frac{(x-1)^4}{4!} y^{(IV)}(1) + \dots$$

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2!} y''(1) + \frac{(x-1)^3}{3!} y'''(1) +$$

$$\frac{(x-1)^4}{4!} y^{(IV)}(1) + \dots \Rightarrow ①$$

$$y(x) = \log x$$

$$y(1) = \log e^1 = 0.$$

$$y = \frac{1}{x}$$

$$y'(1) = \frac{1}{1} = 1$$

$$y'' = -\frac{1}{x^2}$$

$$y''(1) = -\frac{1}{1^2} = -1$$

$$y''' = \frac{2}{x^3}$$

$$y'''(1) = \frac{2}{1^3} = 2$$

$$y^{(IV)} = -\frac{6}{x^4}$$

$$y^{(IV)}(1) = -\frac{6}{1^4} = -6.$$

$$\log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(1) + \frac{(x-1)^3}{3}(-\frac{1}{2}) + \frac{(x-1)^4}{4}(0)$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \dots$$

$$\log(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4},$$

$$= 0.1 - 0.005 + 0.0003 - 0.0000025$$

$$0.1 - 0.005 + 0.0003 - 0.0000025 \Rightarrow 0.0953$$

(2) expanding  $\log(\cos x)$  about  $x = \frac{\pi}{3}$  up to 4 terms

conditions  $x > 0$ ,  $x < \frac{\pi}{2}$

$$\text{sol. } y(x) = \log(\cos x)$$

Taylor's series given by

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2}y''(a) +$$

$$+ \frac{(x-a)^3}{3!}y'''(a) + \dots$$

$$y(\frac{\pi}{3}) = y(\frac{\pi}{3}) + (x-\frac{\pi}{3})y'(\frac{\pi}{3}) + \frac{(x-\frac{\pi}{3})^2}{2}y''(\frac{\pi}{3})$$

$$+ \frac{(x-\frac{\pi}{3})^3}{6}y'''(\frac{\pi}{3}) + \dots \quad \text{①}$$

now

$$y(x) = \log(\cos x)$$

$$y(\frac{\pi}{3}) = \log(\cos \frac{\pi}{3})$$

$$= \log(\frac{1}{2}) = \log 0.5.$$

$$y' = \frac{1}{\cos x} (-\sin x).$$

$$y'(\frac{\pi}{3}) = -\tan x$$

$$y'(\frac{\pi}{3}) = -\tan(\frac{\pi}{3})$$

$$= -\sqrt{3}$$

$$y'' = -\sec^2 x$$

$$y''(\frac{\pi}{3}) = -\sec^2(\frac{\pi}{3}) = -2$$

$$= -4.$$

$$y''' = -(2\sec x \cdot \sec x \tan x - 2\sec^2 x \tan x)$$

$$y'''(\frac{\pi}{3}) = -2\sec^2(\frac{\pi}{3})$$

$$\begin{aligned} &= \tan(\frac{\pi}{3}) \\ &= -2(\sqrt{3}) - \sqrt{3} \\ &= -8\sqrt{3} \end{aligned}$$

$$\log(\cos x) = \log(0.5) + (x - \pi/3)(-1/3) + (x - \pi/3)^2/2$$

$$= \log(0.5) - \frac{1}{3}(x - \pi/3) - \frac{1}{2}(x - \pi/3)^2 - \frac{4}{3}(x - \pi/3)^3$$

③ Expand  $\tan^{-1}x$  in powers of  $(x-1)$  up to terms containing  $x^4$ .

Let  $y = \tan^{-1}x$

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^4}{4!}$$

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2} y''(1) + \frac{(x-1)^4}{24} y''''(1)$$

$$\text{Now, } y = \tan^{-1}(x) = y(1) = \tan^{-1}(1).$$

$$y' = \frac{1}{1+x^2} = y'(1) = \frac{1}{1+1} = \frac{1}{2} = 0.5,$$

$$y = \tan^{-1}x$$

$$y' = \frac{1}{1+x^2} = y_1, \quad (1+x^2)y_2 + y_1(2x) = 0.$$

$$(1+x^2)y_2 + 2xy_1 = 0 \rightarrow ①.$$

$$(1+x^2)y_2 + 2y_1 = 0 \rightarrow ②.$$

$$2y_2 + 1 = 0.$$

$$2y_2 = -1 \Rightarrow y_2 = -\frac{1}{2} = y_1$$

$$y_1 = -\frac{1}{2} = y_1$$

$$\text{Solve } 0 \text{ from } ①$$

$$(1+x^2)y_3 + y_2(2x) + 2(2xy_2 + y_1 \cdot 1) = 0$$

$$(1+x^2)y_3 + 4xy_2 + 2y_1 = 0 \rightarrow ③$$

$$(1+x^2)y_3 + x^2(-\frac{1}{2}) + x(\frac{1}{2}) = 0$$

$$2y_3 - 2 + 1 = 0.$$

$$2y_3 - 1 = 0$$

$$2y_3 = 1 \Rightarrow y_3 = \frac{1}{2} = 0.5$$

Putting  $y_3 = 0.5$  in  $y_2$ :

$$(1+x^2)y_3(2x) + 4(xy_3 + y_2) + 2y_2 = 0.$$

$$(1+x^2)y_4 + (6xy_3 + 6y_2) = 0.$$

$$(1+1)y_4 + 6^3(1)(\frac{1}{2}) + \frac{1}{6}(\frac{1}{2}) = 0.$$

$$2y_4 + 3 - 3 = 0$$

$$2y_4 + 0 = 0$$

$$2y_4 = 0 \Rightarrow \underline{\underline{y_4 = 0}}. \quad \text{continued}$$

~~1~~ Taylor's series expression is given by.

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \frac{(x-a)^4}{4!}y^{(4)}(a) + \dots$$

$$\tan^{-1}(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2}y''(1) + \frac{(x-1)^3}{3!}$$

$$y''(1) + \frac{(x-1)^4}{24}y^{(4)}(1) + \dots$$

$$= \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{3!} + 0 + \dots$$

$$(\frac{\pi}{4} + 6y^{(4)}(1))$$

$$(1 - \frac{1}{16})$$

$$(\frac{15}{16} - \frac{1}{16})$$

$$(\frac{14}{16} - \frac{1}{16})$$

$$(\frac{13}{16} - \frac{1}{16})$$

$$(\frac{12}{16} - \frac{1}{16})$$

$$(\frac{11}{16} - \frac{1}{16})$$

$$(\frac{10}{16} - \frac{1}{16})$$

$$(\frac{9}{16} - \frac{1}{16})$$

$$(\frac{8}{16} - \frac{1}{16})$$

$$(\frac{7}{16} - \frac{1}{16})$$

$$(\frac{6}{16} - \frac{1}{16})$$

$$(\frac{5}{16} - \frac{1}{16})$$

$$(\frac{4}{16} - \frac{1}{16})$$

$$(\frac{3}{16} - \frac{1}{16})$$

$$(\frac{2}{16} - \frac{1}{16})$$

$$(\frac{1}{16} - \frac{1}{16})$$

$$(\frac{0}{16} - \frac{1}{16})$$

$$(\frac{-1}{16} - \frac{1}{16})$$

$$(\frac{-2}{16} - \frac{1}{16})$$

$$(\frac{-3}{16} - \frac{1}{16})$$

$$(\frac{-4}{16} - \frac{1}{16})$$

$$(\frac{-5}{16} - \frac{1}{16})$$

$$(\frac{-6}{16} - \frac{1}{16})$$

$$(\frac{-7}{16} - \frac{1}{16})$$

$$(\frac{-8}{16} - \frac{1}{16})$$

$$(\frac{-9}{16} - \frac{1}{16})$$

$$(\frac{-10}{16} - \frac{1}{16})$$

$$(\frac{-11}{16} - \frac{1}{16})$$

$$(\frac{-12}{16} - \frac{1}{16})$$

$$(\frac{-13}{16} - \frac{1}{16})$$

$$(\frac{-14}{16} - \frac{1}{16})$$

$$(\frac{-15}{16} - \frac{1}{16})$$

$$(\frac{-16}{16} - \frac{1}{16})$$

$$(\frac{-17}{16} - \frac{1}{16})$$

$$(\frac{-18}{16} - \frac{1}{16})$$

$$(\frac{-19}{16} - \frac{1}{16})$$

$$(\frac{-20}{16} - \frac{1}{16})$$

$$(\frac{-21}{16} - \frac{1}{16})$$

$$(\frac{-22}{16} - \frac{1}{16})$$

$$(\frac{-23}{16} - \frac{1}{16})$$

$$(\frac{-24}{16} - \frac{1}{16})$$

$$(\frac{-25}{16} - \frac{1}{16})$$

$$(\frac{-26}{16} - \frac{1}{16})$$

$$(\frac{-27}{16} - \frac{1}{16})$$

$$(\frac{-28}{16} - \frac{1}{16})$$

$$(\frac{-29}{16} - \frac{1}{16})$$

$$(\frac{-30}{16} - \frac{1}{16})$$

$$(\frac{-31}{16} - \frac{1}{16})$$

$$(\frac{-32}{16} - \frac{1}{16})$$

$$(\frac{-33}{16} - \frac{1}{16})$$

$$(\frac{-34}{16} - \frac{1}{16})$$

$$(\frac{-35}{16} - \frac{1}{16})$$

$$(\frac{-36}{16} - \frac{1}{16})$$

$$(\frac{-37}{16} - \frac{1}{16})$$

$$(\frac{-38}{16} - \frac{1}{16})$$

$$(\frac{-39}{16} - \frac{1}{16})$$

$$(\frac{-40}{16} - \frac{1}{16})$$

$$(\frac{-41}{16} - \frac{1}{16})$$

$$(\frac{-42}{16} - \frac{1}{16})$$

$$(\frac{-43}{16} - \frac{1}{16})$$

$$(\frac{-44}{16} - \frac{1}{16})$$

$$(\frac{-45}{16} - \frac{1}{16})$$

$$(\frac{-46}{16} - \frac{1}{16})$$

$$(\frac{-47}{16} - \frac{1}{16})$$

$$(\frac{-48}{16} - \frac{1}{16})$$

$$(\frac{-49}{16} - \frac{1}{16})$$

$$(\frac{-50}{16} - \frac{1}{16})$$

$$(\frac{-51}{16} - \frac{1}{16})$$

$$(\frac{-52}{16} - \frac{1}{16})$$

$$(\frac{-53}{16} - \frac{1}{16})$$

$$(\frac{-54}{16} - \frac{1}{16})$$

$$(\frac{-55}{16} - \frac{1}{16})$$

$$(\frac{-56}{16} - \frac{1}{16})$$

$$(\frac{-57}{16} - \frac{1}{16})$$

$$(\frac{-58}{16} - \frac{1}{16})$$

$$(\frac{-59}{16} - \frac{1}{16})$$

$$(\frac{-60}{16} - \frac{1}{16})$$

$$(\frac{-61}{16} - \frac{1}{16})$$

$$(\frac{-62}{16} - \frac{1}{16})$$

$$(\frac{-63}{16} - \frac{1}{16})$$

$$(\frac{-64}{16} - \frac{1}{16})$$

$$(\frac{-65}{16} - \frac{1}{16})$$

$$(\frac{-66}{16} - \frac{1}{16})$$

$$(\frac{-67}{16} - \frac{1}{16})$$

$$(\frac{-68}{16} - \frac{1}{16})$$

$$(\frac{-69}{16} - \frac{1}{16})$$

$$(\frac{-70}{16} - \frac{1}{16})$$

$$(\frac{-71}{16} - \frac{1}{16})$$

$$(\frac{-72}{16} - \frac{1}{16})$$

$$(\frac{-73}{16} - \frac{1}{16})$$

$$(\frac{-74}{16} - \frac{1}{16})$$

$$(\frac{-75}{16} - \frac{1}{16})$$

$$(\frac{-76}{16} - \frac{1}{16})$$

$$(\frac{-77}{16} - \frac{1}{16})$$

$$(\frac{-78}{16} - \frac{1}{16})$$

$$(\frac{-79}{16} - \frac{1}{16})$$

$$(\frac{-80}{16} - \frac{1}{16})$$

$$(\frac{-81}{16} - \frac{1}{16})$$

$$(\frac{-82}{16} - \frac{1}{16})$$

$$(\frac{-83}{16} - \frac{1}{16})$$

$$(\frac{-84}{16} - \frac{1}{16})$$

$$(\frac{-85}{16} - \frac{1}{16})$$

$$(\frac{-86}{16} - \frac{1}{16})$$

$$(\frac{-87}{16} - \frac{1}{16})$$

$$(\frac{-88}{16} - \frac{1}{16})$$

$$(\frac{-89}{16} - \frac{1}{16})$$

$$(\frac{-90}{16} - \frac{1}{16})$$

$$(\frac{-91}{16} - \frac{1}{16})$$

$$(\frac{-92}{16} - \frac{1}{16})$$

$$(\frac{-93}{16} - \frac{1}{16})$$

$$(\frac{-94}{16} - \frac{1}{16})$$

$$(\frac{-95}{16} - \frac{1}{16})$$

$$(\frac{-96}{16} - \frac{1}{16})$$

$$(\frac{-97}{16} - \frac{1}{16})$$

$$(\frac{-98}{16} - \frac{1}{16})$$

$$(\frac{-99}{16} - \frac{1}{16})$$

$$(\frac{-100}{16} - \frac{1}{16})$$

$$(\frac{-101}{16} - \frac{1}{16})$$

$$(\frac{-102}{16} - \frac{1}{16})$$

$$(\frac{-103}{16} - \frac{1}{16})$$

$$(\frac{-104}{16} - \frac{1}{16})$$

$$(\frac{-105}{16} - \frac{1}{16})$$

$$(\frac{-106}{16} - \frac{1}{16})$$

$$(\frac{-107}{16} - \frac{1}{16})$$

$$(\frac{-108}{16} - \frac{1}{16})$$

$$(\frac{-109}{16} - \frac{1}{16})$$

$$(\frac{-110}{16} - \frac{1}{16})$$

$$(\frac{-111}{16} - \frac{1}{16})$$

$$(\frac{-112}{16} - \frac{1}{16})$$

$$(\frac{-113}{16} - \frac{1}{16})$$

$$(\frac{-114}{16} - \frac{1}{16})$$

$$(\frac{-115}{16} - \frac{1}{16})$$

$$(\frac{-116}{16} - \frac{1}{16})$$

$$(\frac{-117}{16} - \frac{1}{16})$$

$$(\frac{-118}{16} - \frac{1}{16})$$

$$(\frac{-119}{16} - \frac{1}{16})$$

$$(\frac{-120}{16} - \frac{1}{16})$$

$$(\frac{-121}{16} - \frac{1}{16})$$

$$(\frac{-122}{16} - \frac{1}{16})$$

$$(\frac{-123}{16} - \frac{1}{16})$$

$$(\frac{-124}{16} - \frac{1}{16})$$

$$(\frac{-125}{16} - \frac{1}{16})$$

$$(\frac{-126}{16} - \frac{1}{16})$$

$$(\frac{-127}{16} - \frac{1}{16})$$

$$(\frac{-128}{16} - \frac{1}{16})$$

$$(\frac{-129}{16} - \frac{1}{16})$$

$$(\frac{-130}{16} - \frac{1}{16})$$

$$(\frac{-131}{16} - \frac{1}{16})$$

$$(\frac{-132}{16} - \frac{1}{16})$$

$$(\frac{-133}{16} - \frac{1}{16})$$

$$(\frac{-134}{16} - \frac{1}{16})$$

$$(\frac{-135}{16} - \frac{1}{16})$$

$$(\frac{-136}{16} - \frac{1}{16})$$

$$(\frac{-137}{16} - \frac{1}{16})$$

$$(\frac{-138}{16} - \frac{1}{16})$$

$$(\frac{-139}{16} - \frac{1}{16})$$

$$(\frac{-140}{16} - \frac{1}{16})$$

$$(\frac{-141}{16} - \frac{1}{16})$$

$$(\frac{-142}{16} - \frac{1}{16})$$

$$(\frac{-143}{16} - \frac{1}{16})$$

$$(\frac{-144}{16} - \frac{1}{16})$$

$$(\frac{-145}{16} - \frac{1}{16})$$

$$(\frac{-146}{16} - \frac{1}{16})$$

$$(\frac{-147}{16} - \frac{1}{16})$$

Ques 01.

\* using maclaurin series prove that

$$1 + \sin 2x = 1 + x - \frac{x^2}{2} - \frac{x^3}{8} + \frac{x^4}{24} + \dots$$

sol

mclaurin series is given by

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \dots$$

$$y(x) = \sqrt{1 + \sin 2x}$$

$$= \sqrt{\cos^2 x + \sin^2 x + 2 \sin x \cos x}$$

$$= \sqrt{\sin^2 x + 2 \sin x \cos x + \cos^2 x}$$

$$= \sqrt{(\sin x + \cos x)^2}$$

$$y(x) = \sin x + \cos x$$

$$y = \sin x + \cos x$$

$$y(0) = \sin 0 + \cos 0$$

$$= 0 + 1$$

$$y'(0) = 1 \quad y''(0) = -\sin 0 - \cos 0$$

$$y'(x) = \cos x \Rightarrow \sin x + (\cos x) + (-\sin x) = -\sin x + \cos x$$

$$y'(0) = \cos 0 - \sin 0 \Rightarrow 1 - 0 = 1$$

$$y' = 1 - 0 \quad y'' = -1$$

$$y''(x) = -\sin x + \cos x \quad y''(0) = -\sin 0 + \cos 0$$

$$= -\cos x + \sin x$$

$$= -\cos 0 + \sin 0$$

$$= -1 + 0$$

$$y'''(0) = -1$$

$$y''(0) = \sin x + \cos x$$

$$= \sin 0 + \cos 0$$

$$= 0 + 1$$

$$(y''(0)) = 1$$

m.s

$$\sqrt{1 + \sin 2x} = 1 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(1) + \dots$$

$$= 1 + x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \dots$$

Ques

\* obtain the mclaurin series function expand

from the function  $w(x) = \sqrt{1+x}$

$$\log \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

M.S.

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!}$$

$$y(0) + \frac{x^5}{5!} y^{(5)}(0) + \dots$$

$y(x) = y(0) + xy'(0)$	$y' = \frac{1}{1+x}$	$y'' = -\frac{1}{(1+x)^2}$
$y(0) = \log(1+0)$	$y'(0) = \frac{1}{1+0}$	$y'''(0) = -\frac{1}{(1+0)^2}$
$y(0) = 0$	$y''(0) = 1$	$y''''(0) = -1$

$$y'''(0) = \frac{2}{(1+0)^3}$$

$$y'''(0) = \frac{2}{(1+0)^3}$$

$$(y'''(0) = 2)$$

$$y''''(0) = -6$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{3}(-2) + \frac{x^4}{4}(-6) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$$

put  $x = -6$ . In ① we get  $\log(1-x)$

$$\log(1-x) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \frac{(-x)^6}{6} + \dots$$

$$= \frac{1}{2} \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots \right\}$$

$$+ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \dots$$

$$= \frac{1}{2} \left\{ 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots \right\}$$

$$\text{Therefore } \frac{1}{2} \left\{ 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots \right\}$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

so next mark and give in below in writing. Now

11/12/22

## Infinite series

An expression of form  $v_1 + v_2 + v_3 + \dots + v_n$  containing infinite no' of terms is called by infinite series.

An infinite series denoted by  $\sum_{n=1}^{\infty} u_n$  (62),  $\Sigma u_n$ .

' $u_n$ ' is called nth terms of series is called partial sum and is denoted by  $s_n$   
 $s_n = v_1 + v_2 + v_3 + \dots + v_n$

convergence, divergence, oscillation.

\* An infinite series  $\sum u_n$  is said to be convergent if  $\lim s_n = l$ , where  $l$  is finite quantity.

\* An infinity series  $\sum u_n$  is said to be diverge if  $\lim s_n = \pm \infty$ ,  $n \rightarrow \infty$

\* An infinite series  $\sum u_n$  is said to be oscillate if  $s_n$  tends to more than 1 limit as 'n' tends to infinity (the value may be finite or infinite).

Note:- \* theorem statement -

If a series  $\sum u_n$  is convergent then  $\lim u_n = 0$  but not conversely.

\* If  $\lim_{n \rightarrow \infty} u_n \neq 0$  then given series is divergent

series of positive terms

An infinite series  $\sum u_n$  whose terms are all positive is called as a series of positive terms.

cauchy's Root Test.

Even if a series of positive terms and  
 $\lim_{n \rightarrow \infty} (u_n)^{1/n} = l$ ,

then series even is convergent if  $l < 1$   
 Divergent if greater, one and rest tails  
 is equal to  $l$  ( $l = 1$ ).

$$\star \lim_{n \rightarrow \infty} n^{1/n} = 1 \quad \star \lim_{n \rightarrow \infty} (1 + 1/n)^n = e.$$

$$\star \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2.$$

15/12/23

\* test series for convergence (69) examine  
 series

$$\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^{n^2}$$

$\Rightarrow$  test compare with  $\sum_{n=1}^{\infty} e^{-3n}$

$$\therefore u_n = \left(1 - \frac{3}{n}\right)^{n^2} \text{ by Cauchy's Root Test}$$

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left\{ \left(1 - \frac{3}{n}\right)^{n^2} \right\}^{1/n}$$

$$= \lim_{n \rightarrow \infty} \left(1 - \frac{3}{n}\right)^n$$

on comparing  $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2$ .

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \left(-\frac{3}{n}\right)\right)^n$$

$$= e^{-3}$$

$$= \frac{e^{-3}}{(1-0.3)^{-3}} = 0.049 \approx 1.$$

$\therefore l < 1$ ,  $\therefore$  series is convergent.

$$10/12/23$$

$$10/12/23 = 90 \approx 4.8.6$$

$$\textcircled{2} \quad \sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$\Rightarrow$  on comparing  $\therefore a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

By cauchy's root test.

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_n)^{1/n} &= \lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}} \right\}^{1/n} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2} \cdot \frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2-1}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{1/2}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{n^{1/2}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} \end{aligned}$$

on comparing with  $e^n$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = e$$

$$\therefore = \frac{1}{e} < 1$$

By cauchy's root test,  $\sum a_n$  is convergent.

$$\textcircled{3} \quad \left( \frac{2^2}{1^2} - \frac{2}{1} \right)^{-1} + \left( \frac{3^3}{2^3} - \frac{3}{2} \right)^{-2} + \left( \frac{4^4}{3^4} - \frac{4}{3} \right)^{-3}$$

$\Rightarrow$  To find  $n$ th term.

Consider  $1, 2, 3, \dots$  by.

$$a_p = a + (n-1)d$$

$$= 1 + (n-1)1$$

$$= n/1$$

$$2, 3, 4 = a_p = a + (n-1)d$$

$$= 2 + (n-1)$$

$$= 2 + n - 1 \\ = n + 1$$

$$\therefore u_n = \frac{2(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n}$$

JBW

Date \_\_\_\_\_  
Page \_\_\_\_\_

By Cauchy's Root Test.

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left\{ \frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n} \right\}^{1/n}$$

$$= \lim_{n \rightarrow \infty} \left\{ \frac{(n+1)^{n+1}}{n^n} \cdot \left( \frac{n+1}{n} \right)^{-1} \right\}^{1/n}$$

$$= \lim_{n \rightarrow \infty} \left\{ \left( \frac{n+1}{n} \right) \left( \left( \frac{n+1}{n} \right)^n - 1 \right) \right\}^{-1/n}$$

$$= \frac{\lim}{n \rightarrow \infty} \left\{ \left( \frac{n(1+1/n)}{n} \right) \left[ \left( \frac{(1+1/n)^n}{n} - 1 \right) \right]^{-1/n} \right\}$$

$$= \lim_{n \rightarrow \infty} \left\{ (1+1/n) \left[ (1+1/n)^n - 1 \right]^{-1/n} \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{(1+1/n)^n}{(1+1/n)^n - 1}$$

$$= \frac{(1+0)^e}{(1+0)^e - 1} \quad \left[ \lim_{n \rightarrow \infty} (1+1/n)^n = e \right]$$

$$(1+0)^{e-1}$$

$$= \frac{1}{e-1} = 0.58 < 1$$

$$l < 1$$

By Cauchy's Root Test,  $\sum u_n$  is convergent by comparison.

If  $\sum u_n$  and  $\sum v_n$  are any two series of positive terms such that  $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} \neq 0$  (non-zero quantity),

The  $\sum u_n$  &  $\sum v_n$  behave alike.

$$\frac{(1+1/n)^n}{n} \approx n^{\frac{1}{n}}$$

i.e If  $\sum a_n$  is convergent then also converges  
if  $\sum a_n$  is divergent;  $\sum b_n$  also diverges.

choose  $v_n = \frac{1}{n^{p-\alpha}}$ , where 'P' highest power of 'n' in denominator.  
 $a_n$  is highest power of 'n' in numerator.

p - series test:-

A series  $\sum \frac{1}{n^p}$  is called p-series.

and is convergent  $\cdot P > 1$ .

divergent  $\cdot P \leq 1$

(1) Test series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{J_{n+1} - J_n}{(1+\ln n)^2}$$

$$\Rightarrow \text{let } u_n = \frac{J_{n+1} - J_n}{(1+\ln n)^2}$$

$$u_n = \frac{J_{n+1} - J_n}{\frac{(J_{n+1} + J_n)}{2}} \times \frac{J_{n+1} + J_n}{J_{n+1} + J_n}$$

$$= \frac{(J_{n+1})^2 - (J_n)^2}{n(J_{n+1} + J_n)} = \frac{(n+1-n)}{n(J_{n+1} + J_n)}$$

$$\frac{n(J_{n+1} + J_n)}{n(J_{n+1} + J_n)}$$

choose,  $v_n = \frac{1}{n^{p-\alpha}}$

$$v_n = \frac{1}{n^{p-\alpha}} \cdot \frac{1}{n^{3/2-\alpha}} = \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{n(J_{n+1} + J_n)}$$

$$\lim_{n \rightarrow \infty} \frac{n^{3/2}}{n(\sqrt{n(1+\frac{1}{n})+2n})}$$

$$\lim_{n \rightarrow \infty} \frac{n^{3/2}}{n\sqrt{n}(\sqrt{1+\frac{1}{n}}+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{3/2}}{n^{3/2}(\sqrt{1+\frac{1}{n}}+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{1}{\sqrt{1+0+1}} = \frac{1}{1+1} = \frac{1}{2} \neq 0.$$

∴ by comparison test  $\sum u_n$  and  $\sum v_n$  behaves alike.

(consider):

$$\sum v_n = \sum \frac{1}{n^{3/2}} = \sum \frac{1}{n^p}$$

$$\therefore p = \frac{3}{2} > 1$$

∴  $p$ -series test  $\sum v_n$  is convergent since  $\sum u_n$  and  $\sum v_n$  behaves alike. Since  $\sum v_n$  is also convergent

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1+n}} \text{ or } \frac{1}{\sqrt{1+\frac{1}{n}}+1} + \frac{1}{\sqrt{2+\frac{1}{n}}+1} + \frac{1}{\sqrt{3+\frac{1}{n}}+1} + \dots$$

$$u_n = \frac{1}{\sqrt{n+1+n}}$$

$$\text{consider: } v_n = \frac{1}{n^{p-a}} = \frac{1}{n^{1/2-0}} = \frac{1}{n^{1/2}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} \xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1+n}} = \frac{n^{1/2}}{\sqrt{n+1+n}}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}(1+\frac{1}{n})+1} \Rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}(1+\frac{1}{n}+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \Rightarrow \frac{1}{\sqrt{1+\frac{1}{\infty}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2+0}.$$

$\therefore$  By comparison of  $\sum v_n$  and  $\sum u_n$  behavior alike.

$$\text{consider } \sum v_n = \sum \frac{1}{n^{1/2}} = p - \frac{1}{2} < 1$$

$$(3) \quad \sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$

$$\Rightarrow u_n = \sqrt{n+1} - \sqrt{n}$$

$$v_n = \sqrt{n+1} - \sqrt{n} \times \left( \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right)$$

$$(\sqrt{n+1})^2 - (\sqrt{n})^2$$

$$= \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} \Rightarrow v_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\sqrt{n} = \frac{1}{n^{p-1/2}} = n^{1/2-1} = n^{-1/2}$$

$$\lim_{n \rightarrow \infty} \frac{v_n}{u_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \cdot \frac{1}{n^{1/2}}$$

$$\lim_{n \rightarrow \infty} \frac{J_n}{J_{n+1}} = \lim_{n \rightarrow \infty} \frac{n^{1/2}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \frac{J_n}{\sqrt{n(n+1) + n}} =$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}(\sqrt{n+1} + 1)} =$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}}} = \frac{1}{\sqrt{1+0+1}} = \frac{1}{2} \neq 0.$$

$\therefore$  By comparison test  $\sum v_n$  and  $v_n$  behave alike.

$$\text{consider } v_n = \epsilon^{\frac{1}{n^{1/p}}} = p = \frac{1}{2} < 1$$

$\therefore$  P-series test  $v_n$  is divergent since  $v_n$  and  $v_n$  behave alike

$\therefore v_n$  is also divergent.

also it is known that  $\sum v_n$  is also divergent

$$\frac{1}{1-\epsilon^p} + \frac{1}{1-\epsilon^p} + \dots + \frac{1}{1-\epsilon^p} \quad (3)$$

$$p(1-\alpha) + \beta =$$

$$(1-\alpha) + \beta =$$

(11) Test the series for convergence,  $\sum_{n=1}^{\infty} \frac{1}{n^{p+q}}$

Q: Total test of comparison consider the series  $\sum_{n=1}^{\infty} \frac{1}{n^{p+q}}$

$$D_n = n^{p+q} - (n+1)^{p+q}$$

$$= q \cdot n^{p+q-1}$$

$$= q + n^{p+q-1}$$

$$= n^{p+q-1}$$

$$\therefore u_n = \frac{1}{n^{p+q}}$$

$$\text{choose } v_n = \frac{1}{n^{p+q-1}}$$

$$= \frac{1}{n^{p+q-1}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{n^{p+q}} \cdot \frac{n^{p+q-1}}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^{p+q-1}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{n^{p+q-1}} = \lim_{n \rightarrow \infty} n^{2-p-q}$$

$$= \frac{1}{1+\frac{1}{n^{p+q-1}}}$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n^{p+q-1}}} \text{ and type } \frac{1}{1+0} = 1$$

$$= 1 \neq 0$$

$$\text{consider } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$p = 2 > 1$$

the p series test sum convergent.  $v_n$  is also convergent.

(12)

Examining the Series  $\sum_{n=1}^{\infty} \frac{\sqrt{n}-1}{n^3-1} + \frac{\sqrt{n}-1}{n^3-1}$

$\leftarrow \dots \infty$

Q: ~~Ex. 2.3.4~~

$$2, 3, 4, \dots, a+(n-1)d$$

$$= 2 + (n-1)d$$

$$= 2 + n - 1$$

$$= n + 1$$

$$3, 4, 5, \dots, a + (n-1)d$$

$$= 3 + (n-1)1$$

$$= n+2$$

$$u_n = \frac{\sqrt{n+1} - 1}{(n+2)^{3/2}}$$

$$v_n = \frac{1}{n^{5/2}}$$

$$n^3 - \frac{1}{2}$$

$$\frac{1}{n^{5/2}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n+1} - 1}{(n+2)^{3/2}}$$

$$= \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n+1/2 + 1/2}} = \frac{1}{\sqrt{n+1/2}}$$

$$\lim_{n \rightarrow \infty} n^{5/2} \left[ \frac{\sqrt{n(1 + \frac{1}{n})} - 1}{(n+1)^{3/2}} \right]$$

$$= \lim_{n \rightarrow \infty} n^{5/2} \left[ \sqrt{n} \left( \sqrt{1 + \frac{1}{n}} - 1 \right) \right]$$

$$= \lim_{n \rightarrow \infty} n^{5/2} \left[ \sqrt{n} \left( 1 + \frac{1}{2n} - 1 \right) \right]$$

$$= \lim_{n \rightarrow \infty} n^{5/2} \frac{\sqrt{n} \left( \sqrt{1 + \frac{1}{n}} - \frac{1}{\sqrt{n}} \right)}{n^3 \left( \left( 1 + \frac{1}{2n} \right)^3 - \frac{1}{n^3} \right)}$$

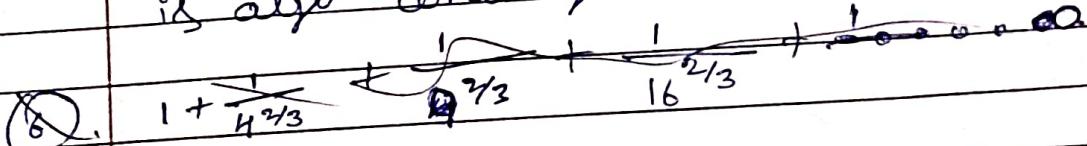
$$= \lim_{n \rightarrow \infty} \frac{n^{5/2} \cdot n^{1/2} \left( \sqrt{1 + \frac{1}{n}} - \frac{1}{\sqrt{n}} \right)}{n^3 \left( \left( 1 + \frac{1}{2n} \right)^3 - \frac{1}{n^3} \right)}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{5/2} \left( \sqrt{1 + \frac{1}{n}} - \frac{1}{\sqrt{n}} \right)}{n^3 \left( \left( 1 + \frac{1}{2n} \right)^3 - \frac{1}{n^3} \right)}$$

both sum and sum.

$$\text{consider } \sum v_n = \sum \frac{1}{n^{5/2}}$$

$\therefore$  the p-series that sum is convergent and hence sum is also convergent.



$$⑥ 1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \dots = \infty$$

$$\text{Sv} 1 + \frac{1}{(2^2)^{2/3}} + \frac{1}{(3^2)^{2/3}} + \frac{1}{(4^2)^{2/3}} + \dots = \infty$$

$$1 + \frac{1}{1^{4/3}} + \frac{1}{2^{4/3}} + \frac{1}{3^{4/3}} + \frac{1}{4^{4/3}} + \dots = \infty$$

$$v_n = \frac{1}{(n^2)^{4/3}}$$

$$v_n = \frac{1}{n^{p-q}} = v_n = \frac{1}{n^{4/3}-0} = \frac{1}{n^{4/3}}$$

$$\lim_{n \rightarrow \infty} \frac{v_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{n^{4/3}}$$

$$\lim_{n \rightarrow \infty} \frac{v_n}{v_n} = \frac{1}{\infty^{4/3}} = 0$$

$$= 1 \neq 0$$

$$\sum v_n = \frac{1}{n^p} = \frac{1}{n^{4/3}} \quad p = \frac{4}{3} > 1$$

# ① Alernbord's Ratio and Raabe's Test

Date \_\_\_\_\_  
Page \_\_\_\_\_

If sum is the series is positive infinity and if  
 $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} > 1$  (finite quantity) then by ratio test

sum is   
 ) convergent if  $l < 1$   
 ) divergent if  $l > 1$   
 ( test fails if  $l = 1$

when the test fails we apply comparison test  
 on p-series are grabbed kept.

Raabe's test.

$$\lim_{n \rightarrow \infty} n \left[ \frac{u_n}{u_{n+1}} - 1 \right] = l \quad (\text{finite quantity})$$

then sum is   
 ) convergent if  $l > 1$   
 ) divergent if  $l < 1$   
 ( test fails if  $l = 1$

\* Examining the series  $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots \infty$

$$\text{Sofir: } \frac{1!}{1^1} + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots \infty$$

$$u_n = \frac{n!}{n^n}, u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

by d'Alambert ratio test.

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1) n!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \quad (n+1)^{n+1} = \infty$$

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^{n+1}} \quad 1 > 0 =$$

∴ sum is not finite therefore it is  $\infty$ .

$$\lim_{n \rightarrow \infty} \frac{n}{n^n (1 + 1/n)^n} \text{ follows}$$

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$$

$$= 111 \times 4^{(23)(2)} = 111 \times 4^{46}$$

∴ By D'Alembert's Ratio Test sum is convergent

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n!} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{n^2 + 2n + 1} = 1$$

$$\text{Let } \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n!} = \frac{(n+1)^2}{(n+1)!} = \frac{(n+1)^2}{n^2 + 2n + 1}$$

$$u_n = \frac{n^2}{n!} \quad u_{n+1} = \frac{(n+1)^2}{(n+1)!} = \frac{(n+1)^2}{n^2 + 2n + 1}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2 + 2n + 1} = 1$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+1)!} = \frac{(n+1)^2}{n^2 + 2n + 1} = \frac{1}{n+1} \cdot \frac{(n+1)^2}{n^2} = \frac{1}{n+1} \cdot n^2 = \frac{n^2}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+1)!} = \frac{(n+1)^2}{(n+1)!} = \frac{n^2}{n+1} = \frac{n^2}{n(n+1)} = \frac{n}{n+1} = \frac{n}{n(1+\frac{1}{n})} = \frac{1}{1+\frac{1}{n}}$$

$$\lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} = 1 \quad \text{because as } n \rightarrow \infty, \frac{1}{n} \rightarrow 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

$$= 1 + \frac{1}{\infty} \cdot \frac{\frac{1}{n}}{1 + \frac{1}{n}(1+\frac{1}{n})} = 1 + 0 \cdot \frac{1}{1+0} = 1$$

$$= 0(1+0) \cdot \frac{1}{1+0} = 0 \cdot \frac{1}{1+0} = 0$$

$$= 0 < 1$$

∴ By D'Alembert's Ratio Test sum is convergent

~~Time Test for series convergence~~

$$\textcircled{3} \quad \frac{3}{4} + \frac{3 \cdot 6}{4 \cdot 7} + \frac{3 \cdot 6 \cdot 9}{4 \cdot 7 \cdot 10} + \dots$$

JBW

Date \_\_\_\_\_  
Page \_\_\_\_\_

$$3 \cdot 6 \cdot 9 \dots a + (n-1)d$$

$$3 + (n-1)3$$

$$3 + 3n - 3$$

$$3n$$

$$4 \cdot 7 \cdot 10 \dots a + (n-1)d$$

$$4 + (n-1)3$$

$$4 + 3n - 3$$

$$3 \cdot 6 \cdot 9 \dots 3n \quad 3(3n+1)$$

$$4 \cdot 7 \cdot 10 \dots (3n+1)$$

$$\therefore u_n = \frac{3 \cdot 6 \cdot 9 \dots 3n}{4 \cdot 7 \cdot 10 \dots (3n+1)}$$

$$u_{n+1} = \frac{3 \cdot 6 \cdot 9 \dots 3n \cdot 3(n+1)}{4 \cdot 7 \cdot 10 \dots (3n+1)(3(n+1)+1)}$$

$$u_{n+1} = \frac{3 \cdot 6 \cdot 9 \dots 3n(3n+3)}{4 \cdot 7 \cdot 10 \dots (3n+1)(3n+4)}$$

~~Ratio Test~~ Alternating Series Test

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{3 \cdot 6 \cdot 9 \dots (3n)(3n+3)}{4 \cdot 7 \cdot 10 \dots (3n+1)(3n+4)}$$

$$3 \cdot 6 \cdot 9 \dots 3n$$

$$(1+n^2) \dots 01 \cdot 4 \cdot 7 \cdot 10 \dots (3n+1)$$

$$\lim_{n \rightarrow \infty} \frac{3 \cdot 6 \cdot 9 \dots 3n(3n+3)}{4 \cdot 7 \cdot 10 \dots (3n+1)(3n+4)} \times \frac{4 \cdot 7 \cdot 10 \dots (3n+1)}{3 \cdot 6 \cdot 9 \dots 3n}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{3n+3}{3n+4}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + 4/n}$$

$$\lim_{n \rightarrow \infty} \frac{3 + 3/n}{1 + 4/n} = \frac{3+0}{1+0} = 3/3 = 1$$

$$= \frac{3+3/0}{1+4/0}$$

$$= \frac{3+0}{3+0} = 3/3 = 1$$

By P. Alembaris ratio test & root test  
tails.

APPLY RABBE'S TEST

$$\lim_{n \rightarrow \infty} n \sqrt[n]{\frac{v_n}{v_{n+1}} - 1} = \lim_{n \rightarrow \infty} n \sqrt[n]{\frac{3n+4}{3n+3} - 1}$$

$$= \lim_{n \rightarrow \infty} n \left( \frac{3n+4 - (3n+3)}{3n+3} \right)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} n \left( \frac{1}{3n+3} \right)$$

$$= \lim_{n \rightarrow \infty} n \left( \frac{1}{3n+3} \right)$$

$$\lim_{n \rightarrow \infty} n \left[ \frac{1}{(3n+3)^{\frac{1}{n}}} \right]$$

$$(1 + \frac{1}{n})^n \rightarrow e$$

$$= \frac{1}{e} < 1$$

V.D. ∴ by Rabbe's Test sum is divergent

True

(4) ~~Examine the series~~  $\sum 4 \cdot 7 \cdot 10 \dots (3n+1)$

$$\text{Sof. given } \sum u_n = 01 \sum 4 \cdot 7 \cdot 10 \dots (3n+1) \frac{n!}{n!}$$

$$u_n = \frac{(3n+1)!}{(3n+1)(3n+2)\dots(3n+1+1)}$$

$$\frac{(n+1)!}{(n+1)!}$$

$$u_n = \frac{4 \cdot 7 \cdot 10 \dots (3n+1) (3n+4)}{(n+1)!} \frac{n+1}{(n+1)!}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{4 \cdot 7 \cdot 10 \dots (3n+1) (3n+4)}{(n+1)!} \frac{n+1}{(n+1)!}$$

$$n! \cdot 01 \cdot 8 \dots$$

$$= \infty = \frac{0 \cdot 4 \cdot 7 \cdot 10 \dots (3n+1) x^n}{0+x} = \infty$$

$$= \lim_{n \rightarrow \infty} \frac{(3n+4)^{1/n} \cdot x^n}{(n+1)^{1/n}} \times \frac{n!}{x^n}$$

JBW  
Date \_\_\_\_\_  
Page \_\_\_\_\_

$$= \lim_{n \rightarrow \infty} \frac{x^{(3+4/n)}}{x^{(1+1/n)}}$$

$$= \frac{3+4/x}{1+1/x}$$

$$= \frac{3+0}{1+0}$$

$$\text{so } 3x$$

$$\begin{cases} \text{converges} & 3x < 1 \\ & 3x > 1 \\ & 3x = 1 \end{cases}$$

$\therefore$  by ratio test sum is convergent  $3x < 1$   
divergent  $3x > 1$ , test fails  $3x = 1$

when  $3x = 1 \Rightarrow x = \frac{1}{3}$ , apply Raab's test.

$$\lim_{n \rightarrow \infty} n \left[ \frac{u_n}{u_{n+1}} - 1 \right] = \lim_{n \rightarrow \infty} n \left[ \frac{(n+1)}{(3n+4)^{1/2}} - 1 \right]$$

$$= \lim_{n \rightarrow \infty} n \left[ \frac{3(n+1)}{3n+4} - 1 \right].$$

$$= \lim_{n \rightarrow \infty} n \left( \frac{3n+3 - (3n+4)}{3n+4} \right)$$

$$= \lim_{n \rightarrow \infty} n \left\{ \frac{-1}{3n+4} \right\}$$

$$= \lim_{n \rightarrow \infty} n \left( \frac{-1}{n(3+4/n)} \right)$$

$$= \frac{-1}{3+4/8}$$

$$= \frac{1}{3} < 1$$

$\therefore$  By Raab's test sum is divergent if  $x > 1$ .

thus sum is  $\begin{cases} C, \text{ if } 3x < 1 \\ D, \text{ if } 3x \geq 1 \end{cases}$

$$(5) \text{ examine the terms } \frac{(n+1)^{2n}}{2J_1} + \frac{x^2}{3J_2} + \frac{x^4}{4J_3} + \dots$$

sol: omitting the first term we get  $\frac{x^2}{3J_2} + \frac{x^4}{4J_3} + \dots$

$$2 \cdot 4 \cdot 6 \cdots 2n \rightarrow a + (n-1)d$$

$$2 \cdot 3 \cdot 4 \cdots (n+1) = 2(n-1)2$$

$$3 \cdot 4 \cdot 5 \cdots (a + (n-1)d) = 7 + 2n - 2 = 2n$$

$$= 3 + (n-1)1$$

$$\therefore = 2 + 2$$

$$u_n = \frac{x^{2n}}{(n+2)J_{n+1}} \quad u_{n+1} = \frac{x^{2(n+1)}}{(n+1+2)J_{n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{x^{2n+2}}{(n+3)J_{n+2}}$$

$$\lim_{n \rightarrow \infty} \frac{x^{2n+2}}{(n+3)J_{n+2}} \cdot \frac{x^2}{(n+2)J_{n+1}}$$

$$= \lim_{n \rightarrow \infty} \frac{x^{2n+2}}{(n+3)J_{n+2}} \cdot \frac{(n+2)J_{n+1}}{x^{2n+2}}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+2)J_{n+1}}{4(n+3)J_{n+2}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}(1+3/n)J_{\sqrt{n}(1+1/n)}}$$

$$= \lim_{n \rightarrow \infty} \frac{(1+3/n)\sqrt{n}J_{1+1/n}}{(1+3/n)\sqrt{1+1/n}}$$

$$= \frac{(1+3/\infty)\sqrt{1+1/\infty}}{(1+3/\infty)\sqrt{1+1/\infty}}$$

$$= (1+0)\sqrt{1+0}$$

$$= \frac{(1+0)\sqrt{1+0}}{(1+0)\sqrt{1+0}}$$

$\sum n^x$  is  $\begin{cases} \text{convergent} & x^2 > 1 \\ \text{divergent} & x^2 < 1 \end{cases}$

when  $x^2 = 1$  fails.

when  $x^2 = 1$  applying P-series test

$$u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$u_n = \frac{(x^2)^n}{(n+2)\sqrt{n+1}}$$

$$\text{Now } u_n = \frac{1^n}{(n+2)\sqrt{n+1}} = \frac{1}{(n+2)\sqrt{n+1}}$$

$$\text{choose } v_n = \frac{1}{n^{p-2}} = \frac{1}{n^{3/2-2}} = \frac{1}{n^{1/2}}$$

$$\sum u_n = \sum \frac{1}{n^p} = \sum \frac{1}{n^{3/2}} \quad (p = 3/2 > 1)$$

By P-series test

$\sum u_n$  is convergent

Hence,  $\sum n^x$  is  $\begin{cases} \text{convergent} & x^2 < 1 \\ \text{divergent} & x^2 > 1 \end{cases}$

for  $x^2 = 1$  fails.

for  $x^2 = 1$  fails.