Unit: 4. Markov chain and queuing theory

1] Define probability vector
2] Define stochastic matrix
3] Define regular stochastic matrix
4] unique fixed probability vector

i] **Probability vector** :- A vector $v = \{v_1, v_2, --- v_n\}$ is said to be probability vector if

i] $v_i \geq 0$

ii] $\Sigma v_i = 1$

Ex :- i] $v = \left[\frac{1}{2}, \frac{1}{2}\right]$

ii] $v = [0, 1, 0]$

iii] $v = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$

iv] $v = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right]$

Note:-

i] If v is not probability vector but each one of v_i are non-negative then λv is probability vector where $\lambda = \frac{1}{\Sigma v_i}$

Ex :-

$v = [1, 2, 3]$

$$\lambda = \frac{1}{\Sigma v_i} = \frac{1}{1+2+3} = \frac{1}{6}$$

$$\lambda v = \frac{1}{6}[1, 2, 3]$$

$\lambda v = [\frac{1}{6}, \frac{2}{6}, \frac{3}{6}]$

$\lambda v = 1$

λv is a probability vector.

ii] Stochastic Matrix :- A square matrix $P = [P_{ij}]$ with each row being a probability vector is called stochastic matrix

Ex :- i] $P = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

2] $A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

iii] Regular stochastic matrix :- A stochastic matrix P is said to be regular stochastic if all the entries of P^n are positive

Ex :- i] $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$A^2 = A.A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 + 1/2 & 0 + 1/2 \\ 0 + 1/4 & \frac{1}{2} + \frac{1}{4} \end{bmatrix}$

$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

Q] A stochastic matrix which is not a regular stochastic

$$A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 3/4 + 1/8 & 3/8 + 1/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ & 1/2 \end{bmatrix}$$

iv] Unique fixed probability vector :-

A regular stochastic matrix p is said to be unique fixed probability vector $v = (x, y, z)$ if

i] $x + y + z = 1$

ii] $vp = v$

Problem

1] Show that the matrix $A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a regular stochastic matrix and hence find the unique fixed vector.

soln given stochastic matrix

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 + \frac{1}{2} & 0 + \frac{1}{2} \\ 0 + \frac{1}{4} & \frac{1}{2} + \frac{1}{4} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

All the enteries of A^2 are positive.

$\therefore$ A is the regular stochastic matrix of order $n=2$

unique fixed probability vector :-

Let $v = (x, y)$ be the unique fixed probability vector

such that i] $x + y = 1$ — (1)

ii] $VA = V$ — (2)

$$[x, y] \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [x, y]$$

$$\left[0x + \tfrac{1}{2}y, \; x + \tfrac{1}{2}y\right] = [x, y]$$

$0x + \frac{1}{2}y = x$	$x + \frac{1}{2}y = y$
$-x + \frac{1}{2}y = 0$ (3)	$x + (\frac{1}{2} - 1)y = 0$

$x = \frac{1}{2} y$ — (2)

eqn (1) $\Rightarrow$

$x + y = 1$

$\frac{1}{2} y + y = 1$

$\frac{3}{2} y = 1$

$3y = 2$

$$\boxed{y = \frac{2}{3}}$$

put $y = 2/3$ in eqn (2)

$x = \frac{1}{\cancel{2}} \left(\frac{\cancel{2}}{3} \right)$

$x = \frac{1}{3}$

$$\boxed{x = 1/3}$$

$V = [x, y] = [\frac{1}{3} \cdot \frac{2}{3}] = 1$

2 imp) If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is a stochastic matrix and $v = (v_1, v_2)$ is a probability vector. show that vA is also probability vector.

Soln By data $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ be the stochastic matrix

and $v = (v_1, v_2)$ be the probability vector

$$vA = [v_1 \ v_2] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$$vA = [v_1 a_1 + v_2 b_1 , v_1 a_2 + v_2 b_2]$$

We have prove that $(v_1 a_1 + v_2 b_1) + (v_1 a_2 + v_2 b_2) = 1$

$LHS = v_1 a_1 + v_2 b_1 + v_1 a_2 + v_2 b_2$

$= v_1 (a_1 + a_2) + v_2 (b_1 + b_2)$ [$\because$ By the defn of S.M)

$= v_1 (1) + v_2 (1)$ (By the defn of pr)

$= 1 = RHS$

imp

3 If A is a square matrix of order 'n' whose rows are each the same vector $a = \{a_1, a_2, a_3 ---- a_n\}$ and $v = \{v_1 \ v_2 --- v_n\}$ is a probability vector prove that $vA = a$

Soln By data

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & & \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

and $v = \{v_1\ v_2 \cdots v_n\}$ be the probability ...

$$vA = [v_1, v_2 \cdots v_n] \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & & \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$= [v_1 a_1 + v_2 a_1 \cdots + v_n a_1,\ v_1 a_2 + v_2 a_2 + \cdots + v_n a_2,$$

$$v_1 a_n + v_2 a_n \cdots v_n a_n]$$

$$= [a_1(v_1+v_2+\cdots+v_n),\ a_2(v_1+v_2+\cdots+v_n),\ \cdots a_n(v_1+v_2 \cdots + v_n)]$$

$$= [a_1(1),\ a_2(1) \cdots a_n(1)] = a$$

4] Which of the following matrix are stochastic matrix

i] $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ → If we add sum should be 1 & it should be square matrix.

→ It is not a square matrix. So it is not a stochastic matrix

ii] $\begin{bmatrix} 15/16 & 1/16 \\ 2/3 & 2/3 \end{bmatrix}$

→ This is not a stochastic matrix because $\frac{2}{3} + \frac{2}{3} \neq 1$

iii] $\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$ → It is a stochastic matrix, because it is square, $1, 0, 1/2, 1/2 \geq 0$ &

$1+0=1$, $1/2+1/2=1$

iv] $\begin{bmatrix} 1/2 & -1/2 \\ 1/4 & 3/4 \end{bmatrix}$

→ It is a not stochastic matrix

$-1/2$ is a negative value

$\frac{1}{2} + (-\frac{1}{2}) \neq 1$

Find the unique fixed probability vector of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

Soln Let $v = [x, y, z]$ be the unique fixed probability vector.

$x+y+z = 1$ — (1)

and $vP = v$

$$[x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$[0x + 1/6\,y + 0z, \quad 1x + 1/2\,y + 2/3\,z, \quad 0x + 1/3\,y + 1/3\,z] = [x, y, z]$$

$0x + \frac{1}{6}y + 0z = x$

$\boxed{\frac{1}{6}y = x}$ — (2)

$y = 6x$

$1x + \underbrace{1/2\,y + 2/3\,z}_{\text{taking LCM}} = y$

$\frac{6x+3y+4z}{6} = y$

$6x+3y+4z = 6y$

$\boxed{6x-3y+4z=0}$

$0x + 1/3\,y + 1/3\,z = z$

$\frac{y+z}{3} = z$

$y+z = 3z$

$y-2z=0$

Substituting eqn (2) and eqn (4) in eqn (1)

$x+y+z=1$

$\frac{y}{6}+y+\frac{y}{2}=1$

$\frac{y+6y+3y}{6}=1$

$10y=6$

$\boxed{y=\frac{6}{10}}=\frac{3}{5} \quad -(5)$

(2) $\Rightarrow x = y\frac{1}{6} = \frac{1}{\cancel{6}_2}\left(\frac{\cancel{3}^1}{5}\right) = \frac{1}{10}$

$z=\frac{y}{2} = \frac{1}{2}\left(\frac{3}{5}\right) = \frac{3}{10}$

$v=(x, y, z) = \left(\frac{3}{5}, \frac{1}{10}, \frac{3}{10}\right)$

i] $\frac{3}{5}, \frac{1}{10}, \frac{3}{10} \geq 0$

ii] $\frac{3}{5}+\frac{1}{10}+\frac{3}{10}=1$

6] $P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Soln Let $v=[x, y, z]$ be the unique fixed probability vector

$x+y+z=1 \quad -(1)$

and $vP=v$

$$[x \;\; y \;\; z]\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [x \;\; y \;\; z]$$

$$[0x + 1/3y + 0z,\quad 1/2x + 2/3y + 1z,\quad 1/2x + 0y + 0z] = (x, y, z)$$

$0x + 1/3y + 0z = x$ $\qquad$ $1/2x + 2/3\,y + 1z = y$ $\qquad$ $1/2x + 0y + 0z = z$

$\boxed{\frac{1}{3}y = x}$ — (2) $\qquad$ $\frac{3x + 4y + 6z}{6} = y$ $\qquad$ $\boxed{\frac{1}{2}x = z}$ → (4)

$\boxed{3x + 4y + 6z = 6y}$ → (3)

Substituting eqn (2) & eqn (4) in eqn (1)

$x + y + z = 1$

$\frac{1}{3}y + y + \frac{1}{2}x = 1$

$\frac{1}{3}y + y + \frac{1}{2}\left(\frac{1}{3}y\right) = 1$

$\frac{1}{3}y + y + \frac{1}{6}y = 1$

$\frac{2y + 6y + 1y}{6} = 1$

$\frac{9y}{6} = 1$

$9y = 6$

$y = \frac{\cancel{6}^{\,2}}{\cancel{9}_{\,3}}$ $\qquad$ $\boxed{y = \frac{2}{3}}$

Substitute $y = \frac{2}{3}$ in eqn (2)

$\frac{1}{3}y = x \implies \frac{1}{3}\left(\frac{2}{3}\right) = x$ $\qquad$ $\boxed{x = \frac{2}{9}}$

eqn ④ $\Rightarrow$

$$\frac{1}{2}x = z$$

$$\frac{1}{2}\left(\frac{2}{9}\right) = z$$

$$z = \frac{2}{18}$$

$$\boxed{z = 1/9}$$

$$v = (x, y, z) = \left(\frac{2}{9}, \frac{2}{3}, 1/9\right)$$

i] $\frac{2}{9}, \frac{2}{3}, \frac{1}{9} \geq 0$

ii] $\frac{2}{9} + \frac{2}{3} + \frac{1}{9} = 1$

7] $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$

Soln Let $v = [x, y, z]$ be the unique fixed probability vector

$$x + y + z = 1 \quad \text{—} ①$$

and $vp = v$

$$[x, y, z]\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} = [x, y, z]$$

$[0x + \frac{1}{2}y + \frac{1}{2}z,\ 1x + 0y + \frac{1}{4}z,\ 0x + \frac{1}{2}y + \frac{1}{4}z] = [x, y, z]$

$0x + \frac{1}{2}y + \frac{1}{2}z = x$

$\boxed{y + z = x}$ → 2

$1x + 0y + \frac{1}{4}z = y$

$\frac{4x + 1z}{4} = y$

$\boxed{\frac{4x + 1z}{4} = y}$ → 3

$0x + \frac{1}{2}y + \frac{1}{4}z = z$

$\frac{2y + 1z}{4} = z$

$2y + 1z = 4z$

$2y = 4z - 1z$

$2y = 3z$

$\boxed{\frac{2y}{3} = z}$ → 4

substituting eqn 2 and eqn 4 in eqn 1

$x + y + z = 1$

$y + z + y + \frac{2}{3}y = 1$

$y + \frac{2}{3}y + y + \frac{2}{3}y = 1$

$2y + \frac{4}{3}y = 1$

$\frac{6y + 4y}{3} = 1$

$\frac{10y}{3} = 1$

$10y = 3$

$\boxed{y = \frac{3}{10}}$

eqn 4 $\Rightarrow$ $z = \frac{2}{3}y$

$z = \frac{2}{3}\left(\frac{3}{10}\right)$

$z = \frac{\cancel{6}^{2}}{\cancel{30}_{10}}$

$\boxed{z = \frac{2}{10}}$

eqn 2

$y + z = x$

$\frac{3}{10} + \frac{2}{10} = x$

$\frac{\cancel{5}^{1}}{\cancel{10}_{2}} = x$

$\boxed{x = \frac{1}{2}}$

$v = (x; y, z) = \left(\frac{1}{2}, \frac{3}{10}, \frac{2}{10}\right)$

i] $\frac{1}{2}, \frac{3}{10}, \frac{2}{10} \geq 0$

ii] $\frac{1}{2} + \frac{3}{10} + \frac{2}{10} = 1$

8] $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Soln. Let $v = [x, y]$ be the unique fixed probability vector

$x + y = 1$ — 1 and

$vP = v$

$$[x, y] \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x, y]$$

$$\left[\frac{3}{4}x + \frac{1}{2}y,\ \frac{1}{4}x + \frac{1}{2}y\right] = [x, y]$$

$\frac{3}{4}x + \frac{1}{2}y = x$

$\frac{3x + 2y}{4} = x$

$3x + 2y = 4x$

$2y = 4x - 3x$

$\boxed{2y = x}$ — 2

$\frac{1}{4}x + \frac{1}{2}y = y$

$\frac{1x + 2y}{4} = y$

$\boxed{x + 2y = 4y}$ — 3

eq^n (2) ~~(3)~~ in (1)

$x + y = 1$

$2y + y = 1$

$3y = 1$

$\boxed{y = \frac{1}{3}}$

eq^n (2) $\Rightarrow$ $2y = x$

$2(\frac{1}{3}) = x$

$\boxed{x = \frac{2}{3}}$

$v = (x, y) = (\frac{2}{3}, \frac{1}{3})$

i] $\frac{2}{3}, \frac{1}{3} \geq 0$

ii] $\frac{2}{3} + \frac{1}{3} = 1$

Q] which of the following is a probability vector and given reason

i] $v_1 = (\frac{1}{3}, 0, -\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$ $\rightarrow$ It is not a probability vector because $-\frac{1}{6}$ is not greater than or equal to zero

$v_i \ngeq 0$

(i.e., $-\frac{1}{6} \ngeq 0$)

ii] $v_2 = (\frac{1}{3}, 0, \frac{1}{6}, \frac{1}{2}, \frac{1}{3})$

$\Rightarrow$ $v_i \geq 0$

$\Sigma v_i = \frac{1}{3} + 0 + \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1.3 \neq 1$

$\therefore$ It is not a probability vector

iii] $v_3 = (\frac{1}{3}, 0, 0, \frac{1}{6}, \frac{1}{2})$

$v_i \geq 0$

$\Sigma v_i = \frac{1}{3} + 0 + 0 + \frac{1}{6} + \frac{1}{2} = 1$ $\therefore$ v_3 is a probability vector

10] Show that the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$ is a regular stochastic matrix and hence find the unique fixed probability vector.

Soln

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+1+0 \\ 0+0+1/2 & 0+0+1/2 & 0+0+0 \\ 0+0+0 & \frac{1}{2}+0+0 & 0+\frac{1}{2}+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \cdots \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \cdots \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0+0+\frac{1}{2} & 0+0+\frac{1}{2} & 0+0+0 \\ 0+0+0 & \frac{1}{2}+0+0 & 0+\frac{1}{2}+0 \\ 0+0+\frac{1}{4} & 0+0+\frac{1}{4} & 0+\frac{1}{2}+0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$A^4 = A^3 . A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0+0+0 & \frac{1}{2}+0+0 & 0+\frac{1}{2}+0 \\ 0+0+\frac{1}{4} & 0+0+\frac{1}{4} & 0+\frac{1}{2}+0 \\ 0+0+\frac{1}{4} & \frac{1}{4}+0+\frac{1}{4} & 0+\frac{1}{4}+0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$A^5 = A^4 . A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 0+0+\frac{1}{4} & 0+0+\frac{1}{4} & 0+\frac{1}{2}+0 \\ 0+0+\frac{1}{4} & \frac{1}{4}+0+\frac{1}{4} & 0+\frac{1}{4}+0 \\ 0+0+\frac{1}{8} & \frac{1}{4}+0+\frac{1}{8} & 0+\frac{1}{2}+0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

All the entries of A^5 are +ve

A is a regular stochastic matrix of the order $n=5$

unique probability vector

Let $v = [x, y, z]$ be the unique probability vector

$x + y + z = 1$ — ①

and $vP = v$

$$[x, y, z]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x, y, z]$$

$$\left[0x + 0y + \tfrac{1}{2}z, \quad 1x + 0y + \tfrac{1}{2}z, \quad 0x + 1y + 0z\right] = [x, y, z]$$

$0x + 0y + \frac{1}{2}z = x$

$\frac{1}{2}z = x$ — ②

$1x + 0y + \frac{1}{2}z = y$

$\frac{2x + 1z}{2} = y$

$2x + 1z = 2y$ — ③

$0x + 1y + 0z = z$

$1y = z$ — ④

Substitute ② & ④ in ①

$x + y + z = 1$

$\frac{1}{2}z + y + y = 1$

$\frac{1}{2}(y) + y + y = 1$

$\frac{1}{2}y + 2y = 1$

$\frac{1y + 4y}{2} = 1$

$\underline{5y} = 1$

$5y = 2$

$$\boxed{y = \frac{2}{5}}$$

eq^n (4) $\Rightarrow$ $y = z$

$$\boxed{\frac{2}{5} = z}$$

eq^n (2) $\Rightarrow$ $\frac{1}{2} z = x$

$$\frac{1}{2}\left(\frac{2}{5}\right) = x$$

$$x = \frac{2}{10}$$

$$\boxed{x = \frac{1}{5}}$$

$\pi = (x, y, z) = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$

i] $\frac{1}{5}, \frac{2}{5}, \frac{2}{5} \geq 0$

ii] $\frac{1}{5} + \frac{2}{5} + \frac{2}{5} = 1$

~~Find the unique~~

Markov chain:-

stochastic process:- It is a set of random variable $\{X(t) \mid t \in T\}$ defined on S with parameter d

Markov chain:- A stochastic process which is such that the generation of probability distribution depend only on the present state is called a markov process.

If this state space is discrete is called markov chain.

Transition probability matrix :- The P_{ij}, which are non-negative real numbers form a square matrix of order m is called the transition probability matrix and it satisfies

(i) $0 \leq P_{ij} \leq 1$

(ii) $\sum_{j=1}^{m} P_{ij} = 1$

$i = 1, 2, 3, \text{----} m$

$$P = [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & & & \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix}$$

With the help TPM we predict the moment of the system from one state to other state.

Ex :- Three boy A, B, C throwing the ball to each other. A always throws ball to B, B always throws ball to C and C just as likely to throw ball to B as to A.

Form the TPM

Soln

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Higher transition probability matrix

The probability that the system changes from state a_i to the state a_j in exactly n steps is defined by

$$P^n = [P_{ij}^{(n)}] = \begin{bmatrix} P_{11}^{(n)} & P_{12}^{(n)} & \cdots & P_{1m}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & \cdots & P_{2m}^{(n)} \\ \vdots & & & \\ P_{m1}^{(n)} & P_{m2}^{(n)} & \cdots & P_{mm}^{(n)} \end{bmatrix}$$

Let $p^{(0)} = [p_1^{(0)} \; p_2^{(0)} \cdots p_n^{(0)}]$ denote initial probability distribution.

Let $p^{(n)} = [p_1^{(n)}, p_2^{(0)} \cdots p_n^{(n)}]$ denotes the n^{th} step probability distribution.

$$p^{(1)} = p^{(0)} P$$

$$p^{(2)} = p^{(0)} P^2$$

$$p^{(3)} = p^{(0)} P^3$$

$$\vdots$$

$$p^{(n)} = p^{(0)} P^n$$

Irreducible :- The markov chain is said to irreducible if the associated TPM is regular.

Absorbing state :- A state i is called an absorbing state if the transition probability P_{ij} are such that $P_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases}$

$$P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

a_2 state is the absorbing state

$$P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

a_1 and a_3 are the absorbing state.

Transient state :- A state i is said to be transient state if the system is in this state at same step and their is a chance it will not return to that state

State transition diagram :-

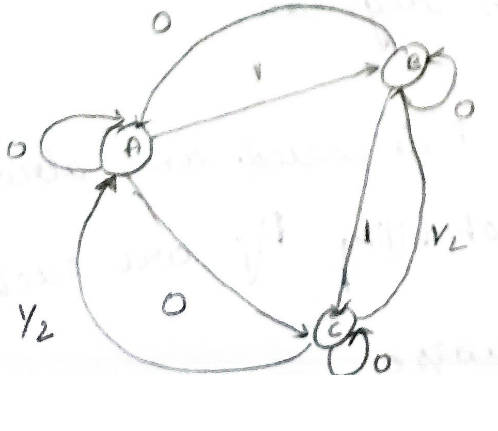
A markov chain is usually shown by a state transition diagram.

The pictorial representation of transition probabilites from state a_i to a_j is called state transition diagram.

Ex :-

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

There are three 3 states A, B and C



Problems

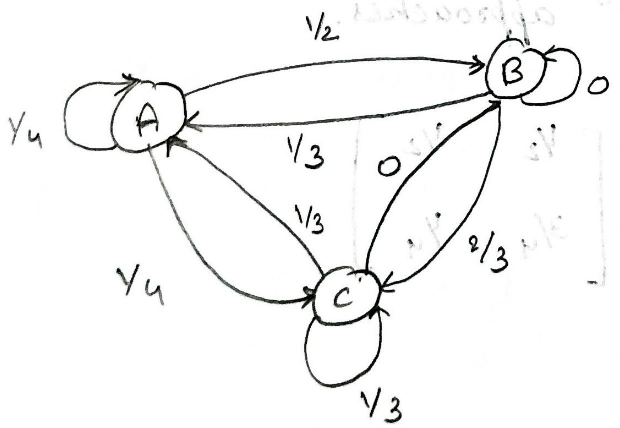
1) write the state transition diagram of the following T.P.M P. Also verify that P is irreducible

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Soln state transition diagram

we have three state space

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/3 \end{bmatrix} \end{array}$$



$$P^2 = P \cdot P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{16} + \frac{1}{6} + \frac{1}{8} & \frac{1}{8} + 0 + 0 & \frac{1}{16} + \frac{1}{3} + \frac{1}{8} \\ \frac{1}{12} + 0 + \frac{1}{3} & \frac{1}{6} + 0 + 0 & \frac{1}{12} + 0 + \frac{1}{3} \\ \frac{2}{8} + 0 + \frac{1}{4} & \frac{1}{4} + 0 + 0 & \frac{1}{8} + 0 + \frac{1}{4} \end{bmatrix}$$

Problems

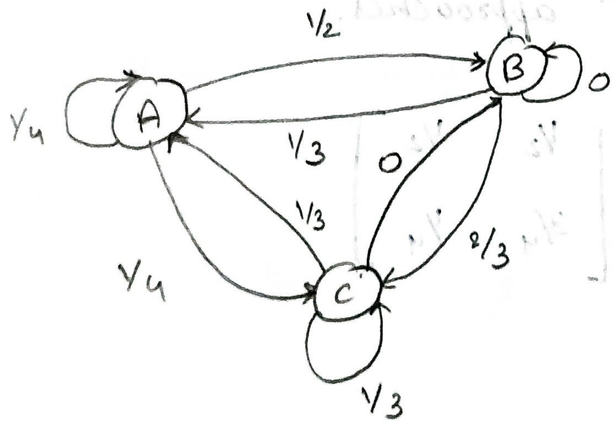
1] write the state transition diagram of the following T.P.M P Also verify that P is irreducible

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Soln state transition diagram

we have three state space

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/3 \end{bmatrix} \end{array}$$



$$P^2 = P.P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{16} + \frac{1}{6} + \frac{1}{8} & \frac{1}{8} + 0 + 0 & \frac{1}{16} + \frac{1}{3} + \frac{1}{8} \\ \frac{1}{12} + 0 + \frac{1}{3} & \frac{1}{6} + 0 + 0 & \frac{1}{12} + 0 + \frac{1}{3} \\ \frac{1}{8} + 0 + \frac{1}{4} & \frac{1}{4} + 0 + 0 & \frac{1}{8} + 0 + \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 17/48 & 1/8 & 25/48 \\ \frac{5}{12} & \frac{1}{6} & \frac{5}{12} \\ 3/8 & \frac{1}{4} & 3/8 \end{bmatrix}$$

All the entries are +ve

$\therefore$ P is irreducible for $n=2$

imp

2] The transition probability matrix of markov chain is given by $P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with initial probability = Distribution $\left(\frac{1}{4}, \frac{3}{4}\right)$

Find i] $P_{21}^{(2)}$, $P_{12}^{(2)}$, $P_1^{(2)}$, $P_2^{(2)}$

ii] write the state transition diagram.

iii] The vector $p^{(0)} p^n$ approaches.

Soln :- given tPm $P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$

i] $p^{(0)} = \left[1/4 , 3/4 \right]$

$$P^2 = P.P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = P^2 = \begin{bmatrix} \frac{1}{4} + \frac{3}{8} & \frac{1}{4} + \frac{1}{8} \\ 3/8 + 3/16 & 3/8 + 1/16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{8} & 3/8 \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{bmatrix}$$

Comparing $P_{21}^{(2)} = \frac{9}{16}$ $\quad P_{12}^{(2)} = 3/8$

WKT $\boxed{p(n) = p^{(0)} p^n}$

$p^{(2)} = p^{(0)} p^2$

$p^2 = \left[\frac{1}{4}, \frac{3}{4}\right] \begin{bmatrix} \frac{5}{8} & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$

$= \left[\frac{5}{32} + \frac{27}{64}, \quad \frac{3}{32} + \frac{21}{64}\right] = p^{(2)} = \left[\frac{37}{64}, \frac{27}{64}\right]$

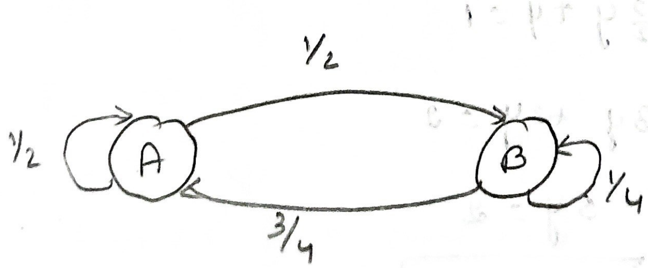
$= \left[P_1^{(2)}, P_2^{(2)}\right]$

Comparing :-

$P_1^{(2)} = \frac{37}{64}$ $\quad P_2^{(2)} = \frac{27}{64}$

ii] state transition diagram.

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} \end{array}$$



iii] The vector $p^{(0)} p^n$ approaches to the unique fixed probability vector.

Let $v = [x, y]$ be the unique fixed probability vector.

such that $x + y = 1$ — ①

and $vP = v$ — ②

$$[x, y] \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = [x, y]$$

$$\left[\frac{1}{2}x + \frac{3}{4}y, \quad \frac{1}{2}x + 1/4\, y\right] = [x, y]$$

$$\frac{1}{2}x + \frac{3}{4}y = x \quad \text{— ③} \qquad \frac{1}{2}x + \frac{1}{4}y = y \quad \text{— ④}$$

$$\cancel{\frac{1}{2}\left(\frac{3}{2}y\right) + \frac{1}{4}y = y}$$

$$\frac{2x + 3y}{4} = x$$

$$2x + 3y = 4x$$

$$3y = 4x - 2x$$

$$3y = 2x.$$

$$\boxed{x = 3/2\, y} \quad \text{— ⑤}$$

eqn ③ in ①.

x value in ①

$$x + y = 1$$

$$\frac{3}{2}y + y = 1$$

$$3y + 2y = 2$$

$$5y = 2$$

$$\boxed{y = \frac{2}{5}} \quad \text{— ⑥}$$

$$x = \frac{3}{2}\left(\frac{2}{5}\right)$$

$$\boxed{x = \frac{6}{10}}$$

③ The TPM of a markov chain is given by

$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$ and the initial probability distribution is $p^{(0)} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ find $p_{13}^{(2)}$, $p_{23}^{(2)}$ and $p_1^{(2)}$

Soln given = $\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$

$$P^2 = P \cdot P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}+0+\frac{1}{8} & 0+0+\frac{1}{4} & \frac{1}{4}+0+\frac{1}{8} \\ \frac{1}{2}+0+0 & 0+0+0 & \frac{1}{2}+0+0 \\ \frac{1}{8}+\frac{1}{2}+\frac{1}{16} & 0+0+\frac{1}{8} & \frac{1}{8}+0+\frac{1}{16} \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix} = P = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} & p_{13}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} & p_{23}^{(2)} \\ p_{31}^{(2)} & p_{32}^{(2)} & p_{33}^{(2)} \end{bmatrix}$$

comparing $p_{13}^{(2)} = 3/8$ $\quad p_{23}^{(2)} = 1/2$

WKT : $p^{(n)} = p^{(0)} P^n$

$= p^2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)(P^2)$

$$p^2 = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$= \left[\frac{3}{16} + \frac{1}{4} + 0 \,,\; \frac{1}{8} + 0 + 0 \,,\; \frac{3}{16} + \frac{1}{4} + 0 \right]$$

$$= \left[\frac{7}{16} \,,\; \frac{1}{8} \,,\; \frac{7}{16} \right] = \left[p_1^{(2)}, \; p_2^{(2)}, \; p_3^{(3)} \right].$$

Comparing $p_1^{(2)} = \frac{7}{16}$

4] Three boys A, B, C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball. Find the probabilities that after three throws.

i] A has the ball

ii] B has the ball

iii] C has the ball.

Soln The state space {A, B, C} the associated TPM P is as follows.

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Since ... person to throw the ball, the associated initial probability vector is given by

$$p^{(0)} = (0, 0, 1)$$

Since the probabilities are desired after three throws we have to find

$$p^{(3)} = p^{(0)} P^3 \quad — (1)$$

$$P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\text{eq}^n (1) \longrightarrow p^{(3)} = p^{(0)} P^3$$

$$= [0, 0, 1] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$p^{(3)} = [1/4, 1/4, 1/2] = [P_A, P_B, P_C]$$

5] A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night on the other hand if he does not study one night, he is 60% sure not to study the next night. In the long run how often does he study?

Soln The state space of the system is $\{A, B\}$

where A : studying

B : not studying.

The associated tpm is

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

In order to find the happening in the long run we have to find the unique fixed probability vector $v(P)$ of P $v(P)$

Let $v = (x, y)$ be the unique fixed probability vector

such that $x + y = 1$ — ① and

$$vP = v$$

$$[x, y]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x, y]$$

$$[0.3x + 0.4y \quad , 0.7x + 0.6y] = [x . y]$$

$0.3x + 0.4y = x$ ① $0.7x + 0.6y = y$ - ②

$0.4y = 0.7x$

$\boxed{\frac{0.4}{0.7} y = x}$ (or) $\boxed{y = \frac{0.7}{0.4} x}$ — ④ $\Rightarrow$ $\boxed{y = \frac{7}{4} x}$

eq^n (ii) in eq^n (i)

$$x + y = 1$$

$$x + \frac{0.7}{0.4}x = 1$$

$$\frac{0.4x + 0.7x}{0.4} = 1$$

$$\frac{0.11x}{0.4} = 1$$

$$\frac{11}{4}x = 1$$

$$11x = 4$$

$$\boxed{x = \frac{4}{11}}$$

eq^n (ii)

$$y = \frac{7}{4}x$$

$$y = \frac{7}{4} \times \frac{4}{11}$$

$$\boxed{y = \frac{28}{44}} = \frac{7}{11}$$

$$v = \left(\frac{4}{11}, \frac{7}{11}\right)$$

$$(P_A, P_B)$$

In long run
He will study $\frac{4}{11} = 0.36$ or 36%.

queuing theory

A line of people or vehicle awaiting for their turn.

queuing theory :- It is the mathematical study of waiting line / queue. This technique provide basis of decision making about the resource needed to provide service.

arrival departure

System	costomer	service
Hospital	patient	Doctor or Nurse.

Queue discipline.

1) First come first serve
2) Last come first serve
3) Service in random order
4) Emergency

Queuing model

Deterministic queuing model

λ = mean no of arrivals per unit time

μ = mean no of services per unit time

imp.

Q) A self service store employees has 1 cashier at its counter. 9 customers arrived on an average of every 5 mins while the cashier can serve 10 customers in 5 mins. Assuming Poisson's distribution for arrival rate and exponential distribution for service rate. Find.

i) Average number of customers in the system

ii) Average queue length

iii) Average time a customer spends in the system

iv) Average time a customer waits before being served.

5

Soln λ = Arrival rate of a customer = 9 per 5 minutes.

μ = Service rate of customer = 10 per 5 minutes.

i] Average no of customer in system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{9}{10-9} = 9$$

ii] Average queue length.

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{9^2}{10(10-9)} = \frac{81}{10} = 8.1 \text{ customers per 5 minutes}$$

iii] Average time a customer spends in system

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{10-9} = 1 \text{ unit.}$$

iv] Average time a customer waits before being served

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{9}{10(10-9)} = \frac{9}{10} = 0.9 \text{ unit} \times 5\text{min}$$

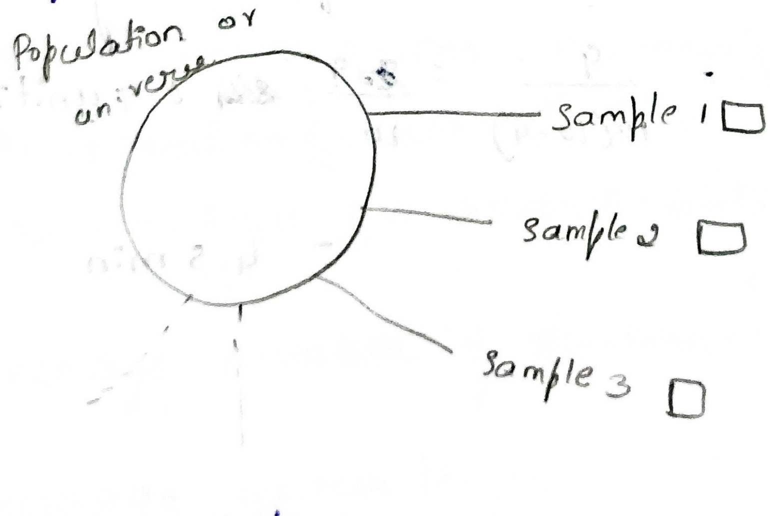
$$= 4.5 \text{ min}$$

new delhi Q imp Q

Unit-5 · Sampling and Statistical Inference

A large collection of individuals or attributes or numerical data can be understood as a population or universe.

* A finite subset of the population is called a sample
* The process of selecting a sample from the population is called a sampling
* The number of individuals in a population is called the size of the population. And is denoted by N.
* The number of individuals in a sample is called is called as sample size and is denoted by 'n'.
* The selection of an individual or item from the population in such a way that each has same chance of being selected is called as random sample.



imp
Sampling distribution:-

A probability distribution of a statics obtained from a large number of samples drawn from a specific

* Sampling ... members of population may be selected more than once is called as sampling with replacement.

* On other hand if a member of population cannot be chosen more than once is called as sampling without replacement

Random sampling with replacement.

* Here the items are drawn one by one and are put back to the population before the next draw.

* If N is the size of the finite population and n is the size of the sample then we have N^n samples.

* The mean $\mu_{\bar{x}}$ of the frequency distribution of the sample means will be equal to the mean of the population

* The variance $\sigma^2_{\bar{x}}$ of the frequency distribution of the sample means will be equal to the variance of the population (σ^2)

i.e $\mu_{\bar{x}} = \mu$

and $\sigma^2_{\bar{x}} = \sigma^2$

where $\sigma_{\bar{x}}$ is called standard error of the means.

Random sampling without replacement

* Here the items are drawn one by one and are not put back to the population before the next draw.

(2)

* Sampling where a member of population may be selected more than once is called as sampling with replacement.

* On other hand if a member of population cannot be choosen more than once is called as sampling without replacement.

Random sampling with replacement.

* Here the items are drawn one by one and are put back to the population before the next draw.

* If N is the size of the finite population and n is the size of the sample then we have N^n samples.

* The mean $\mu_{\bar{x}}$ of the frequency distribution of the sample means will be equal to the mean of the population.

* The variance $\sigma_{\bar{x}}^2$ of the frequency distribution of the sample means will be equal to the variance of the population (σ^2)

i.e $\mu_{\bar{x}} = \mu$

and $\sigma_{\bar{x}}^2 = \sigma^2$

where $\sigma_{\bar{x}}$ is called standard error of the means.

Random sampling without replacement

* Here the items are drawn one by one and are not put back to the population before the next draw.

* If N is the size of finite population & n is the size of sample then we have $^{N}C_{n}$ samples.

* In this case we have the following results.

i) $\mu_{\bar{x}} = \mu$

ii) $\sigma^2_{\bar{x}} = \left[\frac{N-n}{N-1}\right]\frac{\sigma^2}{n}$