Unit 4

Infinite Series

* **Taylor's and maclourin's series**

Taylor's series expansion about point $x=a$ is given by

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \frac{(x-a)^4}{4!}y^{(iv)}(a) + \dots$$

* **maclourins series** expansion given by

$$y(x) = y(0) + x y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(iv)}(0) + \dots$$

problems

① Obtain Taylor's expansion of $\log x$ in powers of $(x-1)$ Hence evaluate $\log_e(1.1)$ correct to 4 ~~dimants places~~ decimal Places.

sol: let $y(x) = \log_e x$ given $(x-1) = (x-a)$

$a=1$.

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \frac{(x-a)^4}{4!}y^{(iv)}(a) + \dots$$

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2}y''(1) + \frac{(x-1)^3}{6}y'''(1) + \frac{(x-1)^4}{24}y^{(iv)}(1) + \dots \rightarrow ①$$

$y(x) = \log_e x$

$y(1) = \log_e 1 = 0.$

$y' = 1/x$

$y'(1) = \frac{1}{1} = 1$

$y'' = -1/x^2$

$y''(1) = -1/1^2 = -1$

$y''' = 2/x^3$

$y'''(1) = 2/1^3 = 2$

$y^{iv} = -6/x^4$

$y^{(iv)}(1) = -6/1^4 = -6.$

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∴ it becomes

$\log_e x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6) + \dots$

$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$

$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \dots$

$\log(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \dots$

$= 0.1 - 0.005 + 0.0003 - 0.000000025$

$\Rightarrow 0.0953$

(2) expaing $\log(\cos x)$ about $x = \pi/3$ up to conditions 3^0

Sol. $y(x) = \log(\cos x)$.

Taylor's series given by.

$y(x) = y(a) + (x-a)\, y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a) + \dots$

$y(x) = y(\pi/3) + (x - \pi/3)\, y'(\pi/3) + \frac{(x-\pi/3)^2}{2} y''(\pi/3) + \frac{(x-\pi/3)^3}{6} \cdot y'''(\pi/3) + \dots \rightarrow (1)$

Now

$y(x) = \log(\cos x)$.

$y(\pi/3) = \log(\cos \pi/3)$

$= \log(1/2) = \log 0.5$.

$y' = \frac{1}{\cos x}(-\sin x)$.

$y' = -\tan x$

$y'(\pi/3) = -\tan(\pi/3)$

$= -\sqrt{3}$

$y'' = -\sec^2 x$

$y''(\pi/3) = -\sec^2(\pi/3) = -2^2$

$= -4$.

$y''' = -(2\sec x \cdot \sec x \tan x)$

$-2\sec^2 x \tan x$

$y'''(\pi/3) = -2\sec^2(\pi/3)\tan(\pi/3)$

$= -2(4)-\sqrt{3}$

$= -8\sqrt{3}$

$$\log(\cos x) = \log(0.5) + (x-\pi/3)(-\sqrt{3}) + \frac{(x-\pi/3)^2}{2}(-4) + \frac{(x-\pi/3)^3}{\cancel{6}\,3}(-8\sqrt{3}) + \cdots$$

$$= \log(0.5) - \sqrt{3}(x-\pi/3) - 2(x-\pi/3)^2 - \frac{4\sqrt{3}}{3}(x-\pi/3)^3 + \cdots$$

(3) Expand $\tan^{-1}x$ in power of $(x-1)$ upto terms contining x^4.

sol Let $\tan^{-1}x$

$$y(x) = y(a) + (x-a)y(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^4}{4!}y^{iv}(a) + \cdots$$

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2}y''(1) + \frac{(x-1)^3}{6}y'''(1) + \frac{(x-1)^4}{24}y^{iv}(1) + \cdots$$

Now, $y = \tan^{-1}(x) = y(1) = \tan^{-1}(1)$

$= 45 = (\pi/4)$

$y' = \frac{1}{1+x^2} = y'(1) = \frac{1}{1+1} = 1/2 = 0.5$

$y_0 = \tan^{-1}x$

$y' = \frac{1}{1+x^2} = y_1$

$y'(1) = \frac{1}{1+1} = \frac{1}{2} = y_1$

$y_1 = \frac{1}{1+x^2}$

$(1+x^2)\,y_1 = 1$

Diff wrt x.

$\cancel{(1+x^2)\,y_2 +}$

$(1+x^2)\,y_2 + y_1(2x) = 0$

$(1+x^2)\,y_2 + 2xy_1 = 0 \rightarrow (1)$

$(1+1)\,y_2 + 2(1)(1/2) = 0$

$2y_2 + 1 = 0$

$2y_2 = -1$

$y_2 = -1/2 = y''$

Diff (1) wrt x

$(1+x^2)y_3 + y_2(2x) + 2(2xy_2 + y_1 \cdot 1) = 0$

$(1+x^2)y_3 + 4xy_2 + 2y_1 = 0 \rightarrow (2)$

$(1+1)y_3 + 4(1)(\frac{-1}{2}) + 2(\frac{1}{2}) = 0$

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$2y_3 - 2 + 1 = 0.$

$2y_3 - 1 = 0$

$2y_3 = 1 \Rightarrow y_3 = 1/2 = y'''$

Diff ② w.r.t x.

$(1+x^2)y_4 + y_3(2x) + 4(xy_3 + y_2) + 2y_2 = 0.$

$(1+x^2)y_4 + 6xy_3 + 6y_2 = 0.$

$(1+1)\,y_4 + 6^3(1)\cdot(1/2) + 6^3\left(\frac{-1}{2}\right) = 0.$

$~~2y + 3 - 3 = 0~~$

$2y_4 + 0 = 0$

$2y_4 = 0 \Rightarrow \boxed{y_4 = 0}$ continue.

Taylor's series expression is given by.

$$y(x) = y(a) + (x-a)\,y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a)$$
$$+ \frac{(x-a)^4}{4!}\cdot y^{\text{iv}}(a) + \cdots$$

$$\tan^{-1}(x) = y(1) + (x-1)\,y'(1) + \frac{(x-1)^2}{2}\,y''(1) + \frac{(x-1)^3}{6}$$
$$y'''(1) + \frac{(x-1)^4}{24}\cdot y^{\text{iv}}(1) \cdots$$

$$= \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + 0 + \cdots$$

Imp Q1.

* Using maclaurin series prove that $\sqrt{1+\sin 2x} = 1+x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots$

Sol: maclaurin series is given by

$y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + \dots$

$y(x) = \sqrt{1+\sin 2x}$

$= \sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x}$

$= \sqrt{\sin^2 x + 2\sin x\cos x + \cos^2 x}$

$= \sqrt{(\sin x + \cos x)^2}$

$y(x) = \sin x + \cos x$

$y = \sin x + \cos x$

$y(0) = \sin 0 + \cos 0$

$= 0 + 1$

$y(0) = 1$

$y' = \cos x - \sin x$

$y'(0) = \cos 0 - \sin 0$

$y' = 1 - 0$

$y' = 1$

$y'' = -\sin 0 - \cos 0$

$= -\sin 0 - \cos 0$

$= 0 - 1$

$y'' = -1$

$y''' = -\cos x + \sin x$

$= -\cos 0 + \sin 0$

$= -\cos 0 + \sin 0$

$= -1 + 0$

$y''' = -1$

$y^{iv} = \sin x + \cos x$

$y^{iv}(0) = \sin 0 + \cos 0$

$= \sin 0 + \cos 0$

$= 0 + 1$

$\boxed{y^{iv} = 1}$

M.S

$\sqrt{1+\sin 2x} = 1 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(-1) + \frac{x^4}{24}(1) + \dots$

$= 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots$

Imp

* obtain the maclorin series function expansion of the function log (1+x)

$\log \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$

M.S.

$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(iv)}(0) + \frac{x^5}{5!} y^{(v)}(0) + \dots$

$\log(1+x)$

$y(x) = y(0) + x\,y'(0)$

$y(0) = \log(1+0)$

$= \log 1$

$y(0) = 0.$

$y' = \frac{1}{1+x}$

$y'(0) = \frac{1}{1+0}$

$\boxed{y'(0) = 1}$

$y'' = \frac{-1}{(1+x)^2}$

$y''(0) = \frac{-1}{(1+0)^2}$

$\boxed{y''(0) = -1}$

$y''' = \frac{2}{(1+x)^3}$

$y'''(0) = \frac{2}{(1+0)^3}$

$\boxed{y'''(0) = 2}$

$y^{iv} = \frac{-6}{(1+x)^4}$

$y^{(iv)}(0) = \frac{-6}{(1+0)^4}$

$\boxed{y^{iv}(0) = -6}$

$\log(1+x) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{\cancel{6}_3}(\cancel{2}) + \frac{x^4}{\cancel{24}_4}(-6) + \dots$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$

put $x = -x$ in ① we get $\log(1-x)$

$\log(1-x) = -x - \frac{(-x)^2}{2} + \frac{x^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \frac{(x)^6}{6} + \dots$

$= \frac{1}{2}\{x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \frac{x^5}{5} - \cancel{\frac{x^6}{6}} + \dots$

$+ x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} + \frac{x^5}{5} + \cancel{\frac{x^6}{6}} + \dots$

$= \frac{1}{2}\{2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots\}$

$= \frac{\cancel{2}}{\cancel{2}}\{x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\}$

$= x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$

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Infinite Series

An expression of form $v_1+v_2+v_3+\dots+v_n+\dots$ containing infinite no' of terms is called as infinite series.

An infinite series denoted by $\sum_{n=1}^{\infty} u_n$ (or) $\sum u_n$

'u_n' is called nth terms of series is called nth partial sum and is denoted by 's_n'

$S_n = u_1+u_2+u_3+\dots+u_n$.

Convergence, divergence, oscillation.

* An infinite series $\sum u_n$ is said to be convergent if $\lim s_n = l$, where 'l' is finite quantity

* An infinity series $\sum u_n$ is said to be divergent if $\lim S_n = \pm\infty$, $n\to\infty$

* An infinite series $\sum u_n$ is said to be oscillatory if s_n tends to more than 1 limit as 'n' tends to infinity (The value may be finite or infinite).

Note:- * theorem statement-

If a series $\sum u_n$ is convergent than $\lim_{n\to\infty} u_n = 0$ but not conversely.

* If $\lim_{n\to\infty} u_n \neq 0$ then given series is divergent

Series of Positive terms

An infinite series $\sum u_n$ whose terms are all positive is called as a series of positive terms.

Cauchy's Root Test

Σu_n is a series of positive terms and
If $\lim_{(n\to\infty)} (u_n)^{1/n} = l$,

Then series Σu_n is convergent if 'l' less 1
Divergent 'l' greater one, and test fails
is equal $1 (l = 1)$.

Imp
$\star \lim_{n\to\infty} n^{1/n} = 1$ $\quad\quad$ $\star \lim_{n\to\infty} (1 + 1/n)^n = e$.

$\star \lim_{(n\to\infty)} \left(1 + \frac{x}{n}\right)^n = e^x$.

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$\star$ Test series for convergence (or) Examine series.

$$\sum_{n=1}^{\infty} \left(1 - 3/n\right)^{n^2}$$

$\Rightarrow$ 1st compar with $\sum_{n=1}^{\infty} u_n$

$\therefore u_n = \left(1 - \frac{3}{n}\right)^{n^2}$ By cauchy's Root test

$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left\{\left(1 - \frac{3}{n}\right)^{n^2}\right\}^{1/n}$

$= \lim_{n\to\infty} (1 - 3/n)^n$.

on comparing $\lim_{n\to\infty} (1 + x/n)^n = e^x$.

$\therefore \lim_{n\to\infty} \left(1 + \left(-3/n\right)\right)^n$

$= e^{-3}$

$= \frac{1}{e^3} = 0.049 < 1$.

$\therefore l < 1$, Σu_n is convergent.

② $\Sigma \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$\Rightarrow$ on comparing $\therefore$ $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

By cauchy's Root test.

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left\{\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right\}^{1/n}$$

$$= \lim_{n\to\infty} \left\{\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right\}^{n^{-1}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2-1}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{1/2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

on comparing with

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e.$$

$$\therefore = \frac{1}{e} = 0.367 < 1.$$

$$l < 1$$

By cauchy's Root test, Σu_n is convergent.

③ $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \ldots$

$\Rightarrow$ To find nth term.

consider $1, 2, 3 \ldots\ldots$ by

$$ap = a+(n-1)d.$$
$$= 1+(n-1)1$$
$$= 1+n-1$$
$$= n,$$

$2, 3, 4 =$ $ap = a+(n-1)d.$

$= 2 + n - 1$
$= n + 1$

$\therefore \quad u_n = \left\{ \frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n} \right\}$

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By cauchy's Root test.

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[\left\{ \frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n} \right\}^{-n} \right]^{1/n}$$

$$= \lim_{n\to\infty} \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right) \right\}^{-1}$$

$$= \lim_{n\to\infty} \left\{ \left(\frac{n+1}{n}\right)^{n} \left(\frac{n+1}{n}\right)^{1} - \left(\frac{n+1}{n}\right) \right\}^{-1}$$

common

$$= \lim_{n\to\infty} \left\{ \left(\frac{n+1}{n}\right) \left[\left(\frac{n+1}{n}\right)^{n} - 1 \right] \right\}^{-1}$$

$$= \lim_{n\to\infty} \left\{ \left(\frac{n(1+1/n)}{n}\right) \left[\left(\frac{\cancel{n}(1+1/n)}{\cancel{n}}\right)^{n} - 1 \right] \right\}^{-1}$$

$$= \lim_{n\to\infty} \left\{ (1+1/n) \left[(1+1/n)^{n} - 1 \right] \right\}^{-1}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)\left[(1+1/n)^{n} - 1\right]}$$

$$= \frac{1}{(1+1/\infty)(e-1)} \qquad \left[\because \lim_{n\to\infty} (1+1/n)^{n} = e \right]$$

$$= \frac{1}{(1+0)(e-1)}$$

$$= \frac{1}{e-1} = 0.58 < 1$$

$l < 1$

$\therefore$ By cauchy's Root test, Σu_n is convergent.

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comparision

If Σu_n are any two series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} \neq 0$ (non zero quantity)

The Σu_n & Σv_n behaves alike

i.e If $\sum u_n$ is convergent $\sum v_n$ also convergent
If $\sum v_n$ is divergent, $\sum u_n$ also divergent

Choose $v_n = \frac{1}{n^{p-q}}$, where 'p' highest power of 'n' in denominator of v_n.

q is highest power of 'n' in numerator u_n.

p - series test:-

A series $\sum \frac{1}{n^p}$ is called p - series.

and is convergent $p > 1$
divergent $p \leq 1$

(1) Test series for convergence or examine series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

=> let $u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$

$u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$

$= \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{n(\sqrt{n+1} + \sqrt{n})} = \frac{(n+1-n)}{n(\sqrt{n+1} + \sqrt{n})}$

$\therefore u_n = \frac{1}{n(\sqrt{n+1} + \sqrt{n})}$

Choose; $v_n = \frac{1}{n^{p-q}}$.

$v_n = \frac{1}{n^{3/2 - 0}} = \frac{1}{n^{3/2}}$

$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{n(\sqrt{n+1} + \sqrt{n})}$

$$\lim_{n\to\infty} \frac{n^{3/2}}{n(\sqrt{n(1+1/n)}+\sqrt{n})}$$

$$\lim_{n\to\infty} \frac{n^{3/2}}{n\sqrt{n}(\sqrt{1+1/n}+1)}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}(\sqrt{1+1/n}+1)}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} \Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt{1+1/\infty}+1}$$

$$= \frac{1}{\sqrt{1}+0+1} = \frac{1}{1+1} = 1/2 \neq 0.$$

$\therefore$ By comparision test Σu_n and Σv_n behaves a like

consider.

$$\Sigma v_n = \Sigma \frac{1}{n^{3/2}} = \Sigma \frac{1}{n^p}.$$

$$\therefore\ v\quad p = 3/2 > 1$$

$\therefore$ p- series test Σv_n is convergent since Σu_n & Σv_n behaves alike alike Σu_n is also convergent

(2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ or $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots$

$$u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

consider, $v_n = \frac{1}{n^{p-q}} = \frac{1}{n^{1/2-0}} = \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{1/n^{1/2}} = \frac{n^{1/2}}{\sqrt{n+1}+\sqrt{n}}.$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n(1+1/n)}+\sqrt{n}} \to \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}(\sqrt{1+1/n}+1)}$$

$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \Rightarrow \frac{1}{\sqrt{1+\frac{1}{\infty}}+1}$

$= \frac{1}{\sqrt{1}+0+1} = \frac{1}{2} \neq 0.$

$\therefore$ By comparision of Σu_n and Σv_n behaves alike.

consider $\Sigma v_n = \Sigma \frac{1}{n^{1/2}} = p = \frac{1}{2} < 1$

③ $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$

$\Rightarrow u_n = \sqrt{n+1} - \sqrt{n}.$

$v_n = \sqrt{n+1} - \sqrt{n} \times \left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right).$

$= \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \Rightarrow u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$v_n = \frac{1}{n^{p-q}} = \frac{1}{n^{1/2-0}} = \frac{1}{n^{1/2}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{1/n^{1/2}}$

$$\lim_{n\to\infty} = \frac{n^{1/2}}{\sqrt{n+1}+\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n(1+1/n)}+\sqrt{n})}$$

$$\Rightarrow \lim_{n\to\infty} = \frac{\sqrt{n}}{\sqrt{n}\,(\sqrt{1+1/n}+1)}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} \Rightarrow \frac{1}{\sqrt{1+0}+1} = 1/2 \neq 0.$$

∴ By comparision Σu_n and v_n behaves alike.

consider $\Sigma v_n = \Sigma \frac{1}{n^{1/2}} = p = 1/2 < 1$

∴ P- series test Σv_n is divergent since Σu_n and Σv_n behave alike

∴ Σu_n is also divergent //.

(11) Test the series for convergence $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots$

Sol: to find the n^{th} term consider the series 1, 2, 3, ...

2, 3, 4 ..... $a+(n-1)d$

$= 2+(n-1)1$

$= 2+n-1$

$= n+1$

$\therefore u_n = \frac{1}{n(n+1)}$

choose $v_n = \frac{1}{n^{p-q}}$

$= \frac{1}{n^{2-0}}$

$= \frac{1}{n^2}$ → by comparison test Σu_n and Σv_n behave alike

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{1/n^2}$

$= \lim_{n\to\infty} \frac{n^2}{n(n+1)}$

$= \lim_{n\to\infty} \frac{n^2}{n^2(1+1/n)}$

$= \frac{1}{1+1/\infty}$

$= \frac{1}{1+0}$

$= 1 \neq 0$

consider

$\Sigma v_n = \Sigma \frac{1}{n^p} = \Sigma \frac{1}{n^2}$

$p = 2 > 1$

the p series test Σu_n convergent. Σu_n is also convergent.

(5) Examine the series $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}$ $+ \dots\dots \infty$.

Sol: ~~1, 2, 3~~

2, 3, 4 ..... $a+(n-1)d$.

$= 2+(n-1)1$

$= 2+n-1$

$= n+1$

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3, 4, 5 .... $a+(n-1)d$

$= 3 + (n-1) 1$

$= n+2$

$u_n = \dfrac{\sqrt{n+1} - 1}{(n+2)^3 - 1}$

$v_n = \dfrac{1}{n^{p-q}}$

$= \dfrac{1}{n^{3 - 1/2}}$

$= \dfrac{1}{n^{5/2}}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{\dfrac{\sqrt{n+1} - 1}{(n+2)^3 - 1}}{1/n^{5/2}}$

$\lim\limits_{n\to\infty} n^{5/2} \left[\dfrac{\sqrt{n(1+1/n)} - 1}{(n(1+2/n))^3 - 1}\right]$

$= \lim\limits_{n\to\infty} n^{5/2} \dfrac{[\sqrt{n}\sqrt{1+1/n} - 1]}{[n^3(1+2/n)^3 - 1]}$

$= \lim\limits_{n\to\infty} n^{5/2} \dfrac{\sqrt{n}(\sqrt{1+1/n} - 1/\sqrt{n})}{n^3[(1+2/n)^3 - 1/n^3]}$

$\lim\limits_{n\to\infty} n^{5/2} \cdot n^{1/2} \dfrac{[\sqrt{1+1/n} - \frac{1}{\sqrt{n}}]}{n^3[(1+2/n)^3 - 1/n^3]}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^3}(\sqrt{1+1/n} - \frac{1}{\sqrt{n}})}{\cancel{n^3}[(1+2/n)^3 - 1/n^3]}$

both Σu_n and Σv_n.

consider $\Sigma v_n = \Sigma 1/n = \Sigma \frac{1}{n^{5/2}}$ $= \frac{5}{2} > 1$

$\therefore$ the p series. that Σv_n is convergent and hence Σu_n is also convergent.

(6). $1 + \cancel{\frac{1}{4^{2/3}}} + \cancel{\frac{1}{9^{2/3}}} + \frac{1}{16^{2/3}} + \cancel{\ldots\ldots \infty}$

(6) $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \dots \infty$

Sol

$1 + \frac{1}{(2^2)^{2/3}} + \frac{1}{(3^2)^{2/3}} + \frac{1}{(4^2)^{2/3}} \dots \infty$

$\frac{1}{1^{4/3}} + \frac{1}{2^{4/3}} + \frac{1}{3^{4/3}} + \frac{1}{4^{4/3}} \dots \infty$

$u_n = \frac{1}{(n)^{4/3}}$

$v_n = \frac{1}{n^{p-q}} = v_n = \frac{1}{n^{4/3 - 0}} = \frac{1}{n^{4/3}}$

$\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{\frac{1}{n^{4/3}}}{\frac{1}{n^{4/3}}} = 1 \neq 0.$

$\Sigma v_n = \frac{1}{n^p} = \frac{1}{n^{4/3}} \quad p = 4/3 > 1$

~~AN~~ =

$a_n = a + (n$

$a = 2$

$a_n = a + (n-1)$

$= \cancel{a} + n -$

$= n$

D' Alemberts Ratio and Raabe's Test

If Σu_n is the series of positive terms and If $\lim_{n \to \infty} \frac{u_{n+1}}{u_n}$ l (finite quantity) then by ratio test

$$\Sigma u_n \text{ is } \begin{cases} \text{convergent if } l<1 \\ \text{divergent if } l>1 \\ \text{test fails if } l=1 \end{cases}$$

when the test fails we apply comparison test on p series are Raabe test.

Raabe test.

$$\lim_{n \to \infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = l \text{ (finite quantity)}$$

$$\text{then } \Sigma u_n \text{ is } \begin{cases} \text{convergent if } l>1 \\ \text{divergent if } l<1 \\ \text{test fail if } l=1 \end{cases}$$

* Examine the series $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \ldots \infty$

Sol:- $\frac{1!}{1^1} + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \ldots \infty$

$u_n = \frac{n!}{n^n}$, $u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

by d alambord Ratio test.

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{n!/n^n}$$

$$= \lim_{n \to \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n \to \infty} \frac{(n+1)\, n!}{(n+1)^n (n+1)} \cdot \frac{n^n}{n!}$$

$$\lim_{n \to \infty} \frac{n^n}{\{n(1+1/n)\}^n}$$

$$\lim_{n \to \infty} \frac{\cancel{n^n}}{\cancel{n^n}(1+1/n)^n}$$

$$\lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$= \frac{1}{e} < 1$ $4! \times 4 \times 5 \times 2 \times 1$

$l < 1$, C $= 4 \times 3!$

$(n+1)! = (n+1) \times n!$

∴ By D Alemberts Ratio test sum. is convergent

(2) $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots \infty$

Sol: $\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots \infty$

$u_n = \frac{n^2}{n!}$ $u_{n+1} \frac{(n+1)^2}{(n+1)!} = (n+1)$

$$\lim_{n\to\infty} = \frac{u_{n+1}}{u_n} = \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}}$$

$$\lim_{n\to\infty} = \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2}$$

$$\lim_{n\to\infty} = \frac{(n+1)^2}{(n+1)n!} \cdot \frac{n!}{n^2}$$

$$\lim_{n\to\infty} = \frac{n(1+1/n)}{n^2}$$

$$\lim_{n\to\infty} = \frac{1+1/n}{n}$$

$$= \frac{1+1/\infty}{\infty}$$

$= 0(1+0)$

$= 0 < 1$

∴ By D.Alemberts Ratio test sum is convergent

(3) Imp. Test the series for convergent.

$$\frac{3}{4} + \frac{3.6}{4.7} + \frac{3.6.9}{4.7.10} + \cdots$$

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$3.6.9 \ldots a+(n-1)d$

$3+(n-1)3$

$\cancel{3}+3n-\cancel{3}$

$3n.$

$4.7.10 \ldots a+(n-1)d$

$4+(n-1)3$

$4+3n-3.$

$3n+1.$

$3.6.9 \ldots 3n$

$4.7.10 \ldots (3n+1).$

$$\therefore u_n = \frac{3.6.9 \ldots 3n}{4.7.10 \ldots (3n+1).}$$

$$\cancel{u_{n+1} = \frac{3.6.9 \ldots 3n\,3(n+1).}{4.7.10 \ldots (3n+1)(3(n+1)+1)}}$$

$$u_{n+1} = \frac{3.6.9 \ldots 3n(3n+3)}{4.7.10 \ldots (3n+1)(3n+4).}$$

~~By D Alembert's Ratio test~~

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\dfrac{3.6.9 \ldots (3n)(3n+3)}{4.7.10 \ldots (3n+1)(3n+4)}}{\dfrac{3.6.9 \ldots 3n}{4.7.10 \ldots (3n+1)}}$$

$$\lim_{n\to\infty} \frac{\cancel{3.6.9 \ldots 3n}(3n+3)}{\cancel{4.7.10 \ldots (3n+1)}(3n+4)} \times \frac{\cancel{4.7.10 \ldots (3n+1)}}{\cancel{3.6.9 \ldots 3n}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+3}{3n+4}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(3+3/n)}{\cancel{n}(3+4/n)}$$

$$\lim_{n\to\infty} \frac{3+3/n}{3+4/n}$$

$$= \frac{3+3/\infty}{3+4/\infty} = \frac{3+0}{3+0} = 3/3 = 1$$

By D.Alemberts ratio test ~~fuerp~~ test

$\therefore$ fails.

Applying Rabbes test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{3n+4}{3n+3}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{3n+4-(3n+3)}{3n+3}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\cancel{3n}+4-\cancel{3n}-3}{3n+3}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{1}{3n+3}\right)$$

$$\lim_{n\to\infty} \cancel{n}\left[\frac{1}{\cancel{n}(3+3/n)}\right]$$

$$\frac{1}{3+3/\infty}$$

$$= \frac{1}{3+0}$$

$$= 1/3 < 1$$

V.D. Imp $\therefore$ by Rabbs test Σu_n is divergent

(4) Examine the series $\sum \frac{4\cdot 7\cdot 10 \cdots (3n+1)}{n!} x^n$

Sol: given

$$\Sigma u_n = \Sigma \frac{4\cdot 7\cdot 10 \cdots (3n+1)}{n!} x^n$$

$$u_{n+1} = \frac{4\cdot 7\cdot 10 \cdots (3n+1)(3(n+1)+1)}{(n+1)!} x^{n+1}$$

$$u_n = \frac{4\cdot 7\cdot 10 \cdots (3n+1)(3n+4)}{(n+1)!} x^{n+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{4\cdot 7\cdot 10 \cdots (3n+1)(3n+4)}{(n+1)!} x^{n+1}$$

$$\times \frac{n!}{4\cdot 7\cdot 10 \cdots (3n+1) x^n}$$

$$= \lim_{n\to\infty} \frac{(3n+4)\cdot \cancel{x^n}\cdot x}{(n+1)\cancel{n!}} \times \frac{n!}{\cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(3+4/n)}{\cancel{n}(1+1/n)} \cdot x$$

$$= \frac{3+4/\infty}{1+1/\infty} \cdot x$$

$$\frac{3+0}{1+0} \cdot x$$

$$3x$$

$$\sum u_n \text{ is } \begin{cases} 3x<1 & C \\ 3x>1 & D \\ 3x=1 & F \end{cases}$$

$\therefore$ by ratio test $\sum u_n$ is convergent $3x<1$ divergent $3x>1$, Test fails $3x=1$,

when $3x=1$ or $x=\frac{1}{3}$ apply Rabbs test.

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty} n\left[\frac{(n+1)}{(3n+4)\frac{1}{3}}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{3(n+1)}{3n+4}-1\right].$$

$$= \lim_{n\to\infty} n\left(\frac{3n+3-(3n+4)}{3n+4}\right)$$

$$= \lim_{n\to\infty} n\left\{\frac{\cancel{3n}+3-\cancel{3n}-4}{3n+4}\right\}$$

$$= \lim_{n\to\infty} n\left(\frac{-1}{3n+4}\right)$$

$$= \lim_{n\to\infty} n\left[\frac{-1}{n(3+4/n)}\right]$$

$$= \frac{-1}{3+4/\infty}$$

$$= \frac{1}{3} < 1$$

$\therefore$ By Raabe's test $\sum u_n$ is Divergent is $3x$

thus sum is $\begin{cases} c, & \text{if } 3x < 1 \\ D, & \text{if } 3x \geq 1 \end{cases}$

(5) examine the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots$

Sol: omiting the first term we get $\frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}}$

$2.4.6 \cdots\cdots 2n \longrightarrow a+(n-1)d$

$2.3.4 \cdots\cdots (n+1)$ $= 2(n-1)2$

$3.4.5. \cdots\cdots (a+(n-1)d.$ $= \cancel{2} + 2n - \cancel{2} = 2n$

$= 3+(n-1)1$

$= n+2$

$u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$ $u_{n+1} = \frac{x^{2(n+1)}}{(n+1+2)\sqrt{n+1+1}}$

$= \frac{x^{2n+2}}{(n+3)\sqrt{n+2}}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{x^{2n+2}}{(n+3)\sqrt{n+2}}}{\frac{x^{2n}}{(n+2)\sqrt{n+1}}}$

$\lim_{n\to\infty} \frac{x^{2n}\cdot x^2}{(n+3)\sqrt{n+2}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}$

$= \lim_{n\to\infty} \frac{(n+2)\sqrt{n+1}}{(n+3)\sqrt{n+2}} x^2$

$= \lim_{n\to\infty} \frac{\cancel{n}(1+2/n)\sqrt{n(1+1/n)}}{\cancel{n}(1+3/n)\sqrt{n(1+2/n)}}$

$= \lim_{n\to\infty} \frac{(1+2/n)\sqrt{n}\sqrt{1+1/n}}{(1+3/n)\sqrt{n}\sqrt{1+2/n}} x^2.$

$= \frac{(1+2/\infty)\sqrt{1+1/\infty}}{(1+3/\infty)\sqrt{1+2/\infty}}$

$= \frac{(1+0)\sqrt{1+0}}{(1+0)\sqrt{1+0}} = x^2$

Σu_n is $\begin{cases} x^2 > 1 & \text{convergent} \\ x^2 < 1 & \text{Divergent} \\ x^2 = 1 & \text{fails.} \end{cases}$

when $x^2 = 1$ Applying P-Series test

$$u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$u_n = \frac{(x^2)^n}{(n+2)\sqrt{n+1}}$$

$$u_n = \frac{1^n}{(n+2)\sqrt{n+1}} = \frac{1}{(n+2)\sqrt{n+1}}$$

~~choose~~ $\cancel{v_n} = \frac{1}{n^{p-q}} = \frac{1}{n^{3/2 - 0}} = \frac{1}{n^{3/2}}$

$$\Sigma v_n = \Sigma \frac{1}{n^p} = \Sigma \frac{1}{n^{3/2}} \left(p = 3/2 > 1\right)$$

By pe-series test

Σu_n is convergent

Hence. Σu_n is $\begin{cases} x^2 < 1, & \text{convergent} \\ x^2 > 1, & \text{divergent.} \end{cases}$