UNIT - 3

Partial Derivatives.

let u we function of two independent variable x, y that is u = f(x, y). we defined partial derivatives of u with respect x and y as follows.

The derivative of u with r to x, treating y as a constant is called partial derivatives of u w.r.to x and is derivative as.

$$u_x = \frac{\partial x}{\partial y} = \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

The derivative of y wrt 'y' treating as const is called P.D. of 'y' & is defined as.

$$u_y = \frac{\partial y}{\partial y} = \frac{f(x, y+\delta y) - f(x,y)}{\partial y}$$

thus $\frac{\partial u}{\partial x}$ $\frac{\partial u}{\partial y}$ are called 1st ord P.D

* 2nd order P.D

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial u^2}{\partial x^2} = u_{xx}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial u^2}{\partial y^2} = u_{yy}.$$

* mixed P.D

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = u_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial x} = u_{xy}.$$

*

It is Imp to note that for all most all functions $\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}}$

Problems

(1) if $u = x^2 + y^2 + xy$ then find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$
$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$.

Ans $u = x^2 + y^2 + xy \rightarrow (1)$

Diff wrt 'x' partially.

$\frac{\partial u}{\partial x} = 2x + 0 + 1 \cdot y.$

Diff wrt 'x' partially.

$\frac{\partial^2 u}{\partial x^2} = 2 + 0$

$= 2$

Diff (1) wrt 'y' partially.

$\frac{\partial u}{\partial y} = 0 + 2y + x(1)$

$= 2y + x$

Diff wrt 'y' partially

$\frac{\partial^2 u}{\partial y^2} = 2 + 0$

$= 2$

(2) If $u = 3x^2y + 5xy + 7$ then find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$
$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$.

Ans Diff wrt 'x' partially.

$\frac{\partial u}{\partial x} = 3(2x)y + 5(1)y + 0$

$= 6xy + 5y.$

Diff wrt 'x' partially.

$\frac{\partial^2 u}{\partial x^2} = 6(1)y + 0 = 6y.$

Diff w.r.t 'y' partially

$$\frac{\partial u}{\partial y} = 3x^2(1) + 5x(1) + 0$$
$$3x^2 + 5x$$

Diff w.r.t 'y' partially.

$$\frac{\partial u}{\partial y} = 0 + 0 = 0$$

(3) $z = x^3 + y^3 - 3axy$. then find $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$. $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$.

A) Diff w.r.t 'x' partially.

$$\frac{\partial z}{\partial x} = \cancel{x^3 + y} \; 3x^2 + 3ay.$$

Diff w.r.t 'x' partially.

$$\frac{\partial^2 z}{\partial x^2} = 3(2x) - 0$$
$$= 6x$$

diff w.r to y partially

$$\frac{\partial z}{\partial y} = 3y^2 - 3ax$$

diff w.r to y partially

$$\frac{\partial^2 z}{\partial y^2} = 6y$$

Note:- symmetric function of $f(x,y)$ is said to be symmetric If $f(y,x)$
ex = $u = x^2 + y^2$

A function $f(x, y, z)$ is said to be symmetric If $f(x, y, z) = f(y, z, x) = f(z, x, y)$.

$u = \log(\tan x + \tan y + \tan z)$

PAGE NO:
DATE:

* If $u = (x^2+y^2+z^2)^{-1/2}$ then prove that $u_{xx} + u_{yy} + u_{zz} = 0$.

Sol :- $u = (x^2+y^2+z^2)^{-1/2} \to (1)$

The given function is symmetric function.

Diff w.r.t 'x' partially.

$\frac{\partial u}{\partial x} = u_x = \left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-1/2-1}(2x+0+0)$.

$\frac{\partial u}{\partial x} = \left(\frac{-1}{\cancel{2}}\right)(\cancel{2}x)(x^2+y^2+z^2)^{-3/2}$

$= -x(x^2+y^2+z^2)^{-3/2}$

Diff w.r.t 'x' partially.

$\frac{\partial^2 u}{\partial x^2} = u_{xx} = -\left[x\cdot\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-3/2-1}\, 2x+0+0\right.$

$+$

$u_{xx} = \frac{3}{\cancel{2}}(\cancel{2}x^2)(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

$u_{xx} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{3/2}$

similar

$u_{yy} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

$u_{zz} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

Consider.

$LHS = u_{xx} + u_{yy} + u_{zz}$

$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

$+ 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

$+ 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

PAGE NO
DATE

$LHS = [x^2+y^2+z^2]^{-5/2}(3x^2+3y^2+3z^2)$
$-3(x^2+y^2+z^2)^{-3/2}$

$= 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2} - 3(x^2+y^2+z^2)^{-3/2}$

$= 3(x^2+y^2+z^2)^{1-5/2} - 3(x^2+y^2+z^2)^{-3/2}$

$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$

$= 0$

$\Rightarrow$ LHS = RHS.

* $u = \log(\tan x + \tan y + \tan z) \rightarrow ①$

$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

~~the given function is symmetric function~~

Diff ① w.r.t 'x' partially.

$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x)$

iiiy

$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$

$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$

$LHS = \sin 2x \frac{\partial u}{\partial x} + \sin y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$

$x\left[\frac{\sec^2 x}{\tan x + \tan y + \tan z}\right] + \sin 2y \left[\frac{\sec^2 y}{\tan x + \tan y + \tan z}\right]$

$+ \sin 2z \left(\frac{\sec^2 z}{\tan x + \tan y + \tan z}\right)$

$= \frac{1}{\tan x + \tan y + \tan z}\left[\sin 2x \cdot \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z\right].$

$$= \frac{1}{\tan x + \tan y + \tan z}\left[2\sin x\cos x \cdot \frac{1}{\cos x} + 2\sin y \cos y \cdot \frac{1}{\cos y}\right.$$

$$2\sin z\cos z \cdot \frac{1}{\cos z}$$

$$\Rightarrow \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)}$$

$= 2$

$=$ RHS

$=$ LHS $=$ RHS.

If $u = \log(x^3+y^3+z^3-3xyz)$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ and hence show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

solu.

let $u = \log(x^3+y^3+z^3-3xyz) \rightarrow ①$.

Diff ① wrt 'x' partially.

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$= \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

III^{ly}.

$$\frac{\partial u}{\partial y} = \frac{3.(y^2-zx)}{x^3+y^3+z^3-3xyz} \qquad \frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$\text{LHS} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz} + \frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz} + \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

PAGE NO.
DATE:

$$= \frac{3}{x^3+y^3+z^3-3xyz}(x^2+y^2+z^2-xy-yz-zx)$$

$$= \frac{3}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}\cancel{(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z} = RHS$$

$\therefore$ LHS = RHS.

continue

~~$u = \log(x^3+y^3$~~

$$LHS = \frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$\frac{-3}{(x+y+z)^2} \quad \frac{-3}{(x+y+z)^2} \quad \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)} =$$

RHS

LHS = RHS

(4) $z = e^{ax+by} f(ax-by) \rightarrow (1).$

PAGE NO:
DATE:

$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Diff (1) w.r.t 'x' partially.

$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)(a).$
$+ e^{ax+by} . a f(ax-by)$

$b\frac{\partial z}{\partial x} = abe^{ax+by} f'(ax-by).$
$+ abe^{ax+by} f(ax-by)$

$b\frac{\partial z}{\partial x} = abe^{ax+by} f'(ax-by) + abz$

Diff (1) w.r.t 'y' partially.

$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b)$
$+ e^{ax+by} . b f(ax-by)$

$a\frac{\partial z}{\partial y} = -abe^{ax+by} f'(ax-by)$
$+ abe^{ax+by} f(ax-by).$

$= abe^{ax+by} f'(ax-by).$
$+ abz$

$LHS = b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$

$= abe^{ax+by} f'(ax-by) + abz$
$- abe^{ax+by} f'(ax-by) + abz$

$= 2abz = RHS$

$\therefore LHS = RHS$

(5) $z(x+y) = x^2+y^2$ then prove that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

PAGE NO

DATE

let: $z = \frac{x^2+y^2}{x+y} \rightarrow (1)$ the given function is symmetric function

Diff. w.r.t 'x' partially.

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$|||^{ly}$

$$\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

LHS $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$

$$= \left\{\frac{x^2+2xy-y^2}{(x+y)^2} - \left[\frac{y^2+2xy-x^2}{(x+y)^2}\right]\right\}^2$$

$$= \left\{\frac{x^2+2xy-y^2-y^2-2xy+x^2}{(x+y)^2}\right\}^2$$

$$= \left(\frac{2x^2-2y^2}{(x+y)^2}\right)^2$$

$$= \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2$$

$$= \frac{4(x-y)^2}{(x+y)^2} \rightarrow (2)$$

$$\text{RHS} = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= 4\left\{1 - \left(\frac{x^2+2xy-y^2}{(x+y)^2}\right) - \left(\frac{y^2+2xy-x^2}{(x+y)^2}\right)\right\}$$

$$= 4\left\{\frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right.$$

$$= 4\left\{\frac{x^2+y^2+2xy-4xy}{(x+y)^2}\right\}$$

$$= 4\left\{\frac{x^2-2xy+y^2}{(x+y)^2}\right\}$$

$$= 4\cdot\frac{(x-y)^2}{(x+y)^2} \rightarrow (3)$$

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right).$$

comparing eq^n (2) & (3) we have

$$\left[\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]^2 = 4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right].$$

★ (6) If $x = r\cos\theta$, $y = r\sin\theta$.

(i) $\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2 = 1$

(ii) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\}$

consider $x = r\cos\theta$; $y = r\sin\theta$.

squaring we get

$x^2 = r^2\cos^2\theta$. $y^2 = r^2\sin^2\theta$.

$= r^2(1)$.

$r^2 = x^2+y^2$ (or) $r = \sqrt{x^2+y^2}$

$r^2 = x^2+y^2$.

The function is symmetric function.

Diff w.r.t 'x' partially.

$$2r\frac{\partial r}{\partial x} = 2x.$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

lll^{ly} $\frac{\partial r}{\partial y} = \frac{y}{r}$

consider,

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Diff w.r.t x Partially

$$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x \cdot \frac{x}{r}}{r^2}$$

$$= \frac{\frac{r^2 - x^2}{r}}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r^2 - x^2}{r^3}$$

$$\text{III}^{ly} \cdot \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3}$$

(i) LHS $= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$

$$= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$$

$$= \frac{x^2}{r^2} + \frac{y^2}{r^2}$$

$$= \frac{1}{r^2}(x^2 + y^2)$$

$$= \frac{r^2}{r^2} \quad \{\because \text{from } (1)\}$$

$= 1 =$ RHS.

(ii) LHS $= \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}$.

$$= \left(\frac{r^2 - x^2}{r^3}\right) + \left(\frac{r^2 - y^2}{r^3}\right)$$

$$= \frac{r^2 - x^2 + r^2 - y^2}{r^3}.$$

PAGE NO:
DATE:

$$= \frac{2r^2 - r^2}{r^3} \quad [\because x^2+y^2=r^2)$$

$$= \frac{r^2}{r^3}$$

$$= \frac{1}{r} \rightarrow (2).$$

$$RHS = \frac{1}{r}\left\{\left(\frac{Dr}{Dx}\right)^2 + \left(\frac{Dr}{Dy}\right)^2\right\}$$

$$= \frac{1}{r}\left\{\frac{x^2}{r^2} + \frac{y^2}{r^2}\right\}.$$

$$= \frac{1}{r}\left\{\frac{x^2+y^2}{r^2}\right\}$$

$$= \frac{1}{r}\left\{\frac{\cancel{r^2}}{\cancel{r^2}}\right\} \quad (\because x^2+y^2=r^2)$$

$$= \frac{1}{r} \rightarrow (3).$$

$\therefore$ LHS = RHS, comparing eq (1) & (2).

Total Derivative of composite function

(1) **Total defferention :-** If $u = f(x,y)$ then total defferention or exxtecte differention of u is defined as $du = \frac{\partial u}{\partial x} . dx + \frac{\partial u}{\partial y} dy$.

total derivative rule

* If $u = f(x,y)$ where $x = x(t)$; $y = y(t)$. then u is the composite function of single variable 't'

$$u \cancel{t} \to (x,y) \longrightarrow t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt}$$

This is called as total derivative of u.

* If $u = f(x,y)$ where $x = x(r,s)$ $y = y(r,s)$. then u is composite function of two independent variable of r and s.

$$u \longrightarrow (x,y) \longrightarrow (r,s) \begin{cases} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{cases}$$

~~This~~. $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{dx}{dr} + \frac{\partial u}{\partial y} \frac{dy}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial s}$$

this is called as chain Rule.

Problems:

PAGE NO.
DATE:

(1) if $u = x^2 + y^2$, $x = e^t \cos t$, $y = e^t \sin t$ find $\frac{du}{dt}$ verify the result of direction by direct substitution.

Ans

$$u \to (x, y) \to t$$

u is the composite function as 't'

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \to (1)$$

$$\frac{\partial u}{\partial x} = 2x \cdot \quad \frac{\partial u}{\partial y} = 2y.$$

$$\frac{dx}{dt} = -e^t \sin t + e^t \cos t$$

$$\frac{dy}{dt} = e^t \cos t + e^t \sin t$$

$$\frac{du}{dt} = 2x(-e^t \sin t + e^t \cos t) + 2y (e^t \cos t + e^t \sin t)$$

$$= 2\{(e^t \cos t)(-e^t \sin t + e^t \cos t) + e^t \sin t (e^t \cos t + e^t \sin t).$$

$$= 2\{-e^t \cdot e^t \sin t \cdot \cos t + e^{2t} \cdot e^t \cos^2 t + e^t \cdot e^t \sin t \cos t + e^t \cdot e^t \sin^2 t\}.$$

$$= 2\{-e^{2t} \sin t \cos t + e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{2t} \sin^2 t\}$$

$$= 2e^{2t} \{\cos^2 t + \sin^2 t).$$

$$= 2e^{2t}(1) \qquad = 2e^{2t} \to (2)$$

$$u = x^2 + y^2$$

$$= (e^t \cos t)^2 + (e^2 \sin t)$$

$$= e^{2t} \cos^2 t + e^{2t} \sin^2 t$$

$$= e^{2t} (\cos^2 t + \sin^2 t)$$

$$= e^{2t} (1)$$

$$u = e^{2t}$$

$$\frac{du}{dt} = e^{2t} \cdot 2$$

$$= 2e^{2t} \to (3)$$

Q. If $u = xy^2 + x^2y$ where $x = at^2$ & $y = 2at$ find $\frac{du}{dt}$ and verify the result.

Ans $u \rightarrow (x, y) \longrightarrow t$

u is the composite function of single variable of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \rightarrow (1)$$

$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$.

$= \{4a^2t^2 + 2(at^2)(2at)\}(2at) + \{2(at^2)(2at) + a^2t^4\}(2a)$

$= (4a^2t^2 + 4a^2t^3)(2at) + (4a^2t^3 + a^2t^4)(2a)$

$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$

$= 16a^3t^3 + 10a^3t^4 \longrightarrow (1)$.

$u = xy^2 + x^2y$.

$u = (at^2)(4a^2t^2) + (a^2t^4)(2at)$.

$= 4a^3t^4 + 2a^3t^5$

$\frac{du}{dt} = 16a^3t^3 + 10a^3t^4 \rightarrow (2)$

∴ comparing eq (1) & (2)

Imp.

(3) If $u = f(x-y, y-z, z-x)$. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Ans let $p = x-y$, $q = y-z$, $r = z-x$

$u = f(p, q, r)$.

$$u \rightarrow (p, q, r) \longrightarrow (x, y, z) \begin{cases} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{cases}$$

by chain Rule we have.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r},$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} \rightarrow (3)$$

(1) $\Rightarrow$

$$\frac{\partial x}{\partial x} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial r}(0) \cdot \frac{\partial u}{\partial r}(-1) \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}$$

(2) $\Rightarrow$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0) \Rightarrow \frac{\partial u}{\partial y} =$$

$$+ \frac{\partial u}{\partial q}$$

(3) $\Rightarrow$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1) \Rightarrow \frac{\partial u}{\partial z} = \frac{-\partial u}{\partial z}$$

$$-\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$$

$$LHS = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial p} - \cancel{\frac{\partial u}{\partial r}} - \frac{\partial u}{\partial p} + \cancel{\frac{\partial u}{\partial q}} - \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial x}}$$

$= 0$

$= RHS \quad \therefore LHS = RHS.$

---

* If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0.$$

Ans let $p = x/y$, $q = y/z$, $r = z/x$.

$\therefore u = f(p, q, r)$.

$$u \rightarrow (p, q, r) \rightarrow (x, y, z).$$

u is the composite function of x y & z

PAGE NO:
DATE:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial u}{\partial x} \rightarrow ①$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial u}{\partial y} \rightarrow ②$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial u}{\partial z} \rightarrow ③$$

$$① \Rightarrow \frac{\partial u}{\partial x} = \left(\frac{1}{y}\right)\frac{\partial u}{\partial p} + (0)\frac{\partial u}{\partial q} + \left(\frac{-2}{x^2}\right)\frac{\partial u}{\partial r}$$

$$= \left(\frac{1}{y}\right)\frac{\partial u}{\partial p}\left(-\frac{2}{x^2}\right)\frac{\partial u}{\partial r}$$

$$② \Rightarrow \frac{\partial u}{\partial y} = \left(\frac{-x}{y^2}\right)\frac{\partial u}{\partial p} + \left(\frac{1}{z}\right)\frac{\partial u}{\partial q} + (0)\frac{\partial u}{\partial r} = \left(\frac{-x}{y^2}\right)$$

$$\frac{\partial u}{\partial p} + \left(\frac{1}{z}\right)\left(\frac{\partial u}{\partial q}\right)$$

$$③ \Rightarrow \frac{\partial u}{\partial z} = (0)\frac{\partial u}{\partial p} + \left(\frac{-y}{z^2}\right)\frac{\partial u}{\partial q} + \left(\frac{1}{x}\right)\frac{\partial u}{\partial x} = \left(\frac{-y}{z^2}\right)$$

$$\frac{\partial u}{\partial q} + \left(\frac{1}{x}\right)\frac{\partial u}{r}$$

$$LHS = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= \left(\frac{x}{y}\right)\frac{\partial u}{\partial p} - x\left(\frac{2}{x}\right)\frac{\partial u}{\partial r} + y\left(\frac{-x}{y^2}\right)\frac{\partial u}{\partial p} + \left(\frac{y}{z}\right)$$

$$\frac{\partial u}{\partial q} + z\left(\frac{-y}{\partial q} + \left(\frac{2}{x}\right)\frac{\partial u}{\partial r}\right.$$

$$= \cancel{\left(\frac{x}{y}\right)\frac{\partial u}{\partial p}} \cancel{\left(\frac{2}{x}\right)\frac{\partial u}{\partial r}} - \cancel{\left(\frac{x}{y}\right)\frac{\partial u}{\partial p}} + \cancel{\frac{y}{z}\left(\frac{\partial u}{\partial q}\right.}$$

$$- \cancel{\frac{y}{z}\left(\frac{\partial u}{\partial q}\right)} + \cancel{\left(\frac{2}{x}\left(\frac{\partial u}{\partial r}\right.\right.}$$

$= 0 = RHS$. $\therefore LHS = RHS$

If $u = f(xz, y/z)$

$$x = \frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0.$$

let $p = xz \qquad q = y/z.$

$\therefore u = f(pq).$

$$u \to (pq) \to (xyz)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}.$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z}.$$

(1) $\Rightarrow \frac{\partial u}{\partial x} = z\left(\frac{\partial u}{\partial p}\right) + (0)\frac{\partial u}{\partial q} \Rightarrow z\frac{\partial u}{\partial p}$

(2) $= \frac{\partial u}{\partial y} = (0)\frac{\partial u}{\partial p} + \frac{1}{z}\left(\frac{\partial u}{\partial q}\right) \Rightarrow \frac{1}{z}\frac{\partial u}{\partial q}$

(3) $\frac{\partial u}{\partial z} = x\left(\frac{\partial u}{\partial p}\right) + \frac{-y}{z^2}\left(\frac{\partial u}{\partial q}\right)$

$\Rightarrow x\frac{\partial u}{\partial p} - \frac{y}{z^2}\frac{\partial u}{\partial q}.$

LHS

$$x = \left[z\frac{\partial u}{\partial p}\right] - y\left(\frac{1}{z}\frac{\partial u}{\partial q}\right) - z\left(x\frac{\partial u}{\partial p} - \frac{y}{z^2}\frac{\partial u}{\partial q}\right)$$

$$xz\left(\frac{\partial u}{\partial p}\right) - \frac{y}{z}\cancel{\frac{\partial u}{\partial q}} - xz\cancel{\frac{\partial u}{\partial p}} + \frac{y}{z}\cancel{\frac{\partial u}{\partial q}}$$

$= 0$ RHS

LHS = RHS

Jacobians

(*) let (u, v) function of 2 independent variable (x, y) the Jacobians u, v w.r.t x, y is denoted by.

$$J\left(\frac{u,v}{x,y}\right) \text{ or } \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

similarly (u, v, w) is function three independent variable (x y z) then

$$J\left(\frac{u,v,w}{x,y,z}\right) \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

(1) provetis.

If $(u,v) \to (x,y) \to (r,s)$ then :

$$J\left(\frac{u,v}{r,s}\right) = J\left(\frac{u,v}{x,y}\right) \cdot J\left(\frac{x,y}{r,s}\right)$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

(2) property

If $(u,v) \to (x,y)$ and $(x,y) \to (u,v)$ then.

$\boxed{J \times J' = 1}$, $\boxed{J J' = 1}$ where J' is the Inverse of J.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix}$$

① x find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ $u = x+y+z$, $v = y+z$, $w = z$

sol:- $\frac{\partial(u\,v\,w)}{\partial(x,y,z)}$ $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 1(0-0) + 1(0-0)$$

$$= 1 + 0 + 0 = 1$$

② If $u = \frac{xy}{z}$, $v = \frac{yz}{x}$, $w = \frac{zx}{y}$ $\frac{\partial(u\,v\,w)}{\partial(x,y,z)}$

sol: $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \\ \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \end{vmatrix}$

$= \frac{y}{z}\left\{\frac{xz}{xy} + \frac{xyz}{xyz}\right\} - \frac{x}{z}\left\{\frac{xyz}{x^2y} - \frac{yz}{xy}\right\} + \left(\frac{-xy}{z^2}\right)\left\{\frac{xyz^2}{x^2y^2} - \frac{z^2}{xy}\right\}.$

$= \frac{y}{z}\left\{\frac{z}{y} + \frac{z}{y}\right\} \frac{-x}{z}\left\{\frac{-z}{x} - \frac{z}{x}\right\} - \frac{xy}{z^2}\left\{\frac{z^2}{xy} - \frac{z^2}{xy}\right\}$

$= \frac{y}{\cancel{z}}\left(\frac{2z}{y}\right) - \frac{x}{\cancel{z}}\left\{\frac{-2z}{\cancel{x}}\right\} - 0 =$

$2 + 2 = 4$ //.

2/12/23

Imp

(3) $x = r\cos\theta,\ y = r\sin\theta$ ~~JJ=1~~ then prove that $JJ' = 1$

Sol: let $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta - (-r\sin^2\theta).$

$= r[\cos^2\theta + \sin^2\theta].$

$= r(1)$

$\therefore \boxed{J = r}$

$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix}$

$x = r\cos\theta \rightarrow (1).$ $\quad y = r\sin\theta \rightarrow (2)$

squaring on both side

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2(1)$

$\therefore r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2+y^2}$

dividing eq 2/1 we get $\cdot\ y/x = \frac{r\sin\theta}{r\cos\theta}$

$\tan\theta = y/x$

$\theta \cdot$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{r}.$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} 2y = y/r.$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$= \frac{-y}{x^2\left(1+\frac{y^2}{x^2}\right)}$$

$$= \frac{-y}{x^2\left(\frac{x^2+y^2}{x^2}\right)}$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}.$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}(1/x) = \frac{1}{x\left(\frac{x^2+y^2}{x^2}\right)} = \frac{x}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

$$\therefore J = r;\ J' = 1/r \qquad JJ' = r(1/r) = 1 //.$$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} \quad \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix}$$

$$\frac{x^2}{r^3} - \left(\frac{-y^2}{r^3}\right)$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3}$$

$$= \frac{1}{r^3}(x^2+y^2) = \frac{1}{r^3}(r^2) = 1/r$$

$$\therefore J = r \quad J' = 1/r$$

$$JJ' = \cancel{r}\left(\frac{1}{\cancel{r}}\right) = 1$$

$$= 1 //$$

4) If $x = u(1-v)$, $y = uv$ P.T $JJ' = 1$ Date: 2/12/23

Sol:- $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$

$= u(1-v) - (-uv)$

$= u - uv + uv$

$\boxed{J = u}$

consider

$x = u - uv \rightarrow (1)$ $y = uv \rightarrow (2)$.

$x = u - y$ $v = \frac{y}{u}$.

$\therefore x + y = u$

$\boxed{u = x + y}$ $\boxed{v = \frac{y}{x+y}}$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial x} = y\left(\frac{-1}{(x+y)^2}\right)$

$\frac{\partial u}{\partial y} = 1$ $= \frac{-y}{(x+y)^2}$

$= \frac{-y}{u^2}$

$\frac{\partial v}{\partial y} = \frac{(x+y)\cdot 1 - y(1)}{(x+y)^2}$

$= \frac{x + \cancel{y} - \cancel{y}}{(x+y)^2}$

$= \frac{x}{(x+y)^2}$

$= \frac{x}{u^2}$

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ -y/u^2 & x/u^2 \end{vmatrix}$

$= \frac{x}{u^2} - \left(-\frac{y}{u^2}\right)$

$= \left(\frac{x+y}{u^2}\right) = \left(\frac{u}{u^2}\right)$

$\boxed{J' = \frac{1}{u}}$

$JJ' = u\left(\frac{1}{u}\right)$

$\boxed{= 1}$ //.

Source : Internet

Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(A - \lambda I)x = 0$

consider a characteristic eq. $= (A - \lambda I)x = 0$

$A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ $\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= (-1-\lambda)(4-\lambda) - (-6) = 0$

$= -4 - 4\lambda + \lambda + \lambda^2 + 6 = 0$

$= -4 - 3\lambda + \lambda^2 + 6$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda^2 - 2\lambda - 1\lambda + 2 = 0$

$\lambda(\lambda - 2)\ (\lambda - 2)$

$\lambda = 2$ $= \boxed{\lambda - 1}$

Case :- $\boxed{\lambda = 2}$

$A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \lambda \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{matrix} -1-2 & 3-0 \\ -2-0 & 4-2 \end{matrix}$

$$\begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$-3x + 3y = 0 \rightarrow (1) \quad \Rightarrow -x + y = 0$$

$$-2x + 2y = 0 \rightarrow (2). \quad -x + y = 0.$$

$$\frac{x}{1} = \frac{y}{1} \qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\lambda = 1}$$

$$A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} -1-1 & 3-0 \\ -2-0 & 4-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= -2x + 3y = 0. \rightarrow (3).$$

$$-2x + 3y = 0 \rightarrow (4).$$

$$-2x + 3y$$

$$+2x + 3y$$

$$\frac{x}{3} = \frac{y}{2} \qquad x_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$x = 3, \; y = 2$$

$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$(A - \lambda I) x = 0.$

consider of charecterstic of equation

$(A - \lambda I) = 0.$

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & 4-0 \\ 1-0 & 2-\lambda \end{bmatrix}$$

$(5-\lambda)(2-\lambda) - 4 = 0.$

$\cancel{10 - 2\lambda + 5\lambda + \lambda^2 - 4 = 0.}$

$\cancel{6 + 3\lambda + \lambda^2}$

$\cancel{\lambda^2 + 3 \quad 6 = 0.}$

$\cancel{\lambda^2 + 3\lambda + 6 = 0.}$

$10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0.$

$6 - 7\lambda + \lambda^2$

$\lambda^2 - 6\lambda - 1\lambda + 6$

$\lambda(\lambda - 6) - 1(\lambda - 6)$.

$(\lambda - 6)(\lambda - 1)$

$\boxed{\lambda = 6}$ $\boxed{\lambda = 1}$

case (i).

$\lambda = 6.$

$(A = \lambda I)\, x = 0.$

$\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}.$

$\begin{bmatrix} 5-6 & 4-0 \\ 1-0 & 2-6 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}.$

$-1x + 4y = 0 \rightarrow 1.$

$1x - 4y = 0 \rightarrow 2.$

$x - 4y \rightarrow 1 \quad x = 4y \quad \frac{x}{4} = \frac{y}{1}$

$\cancel{x + y = 0}.$

$\cancel{\frac{x}{1} = \frac{y}{1}} \quad \cancel{x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$x - 4y \rightarrow 1$

$x = 4y$

$\frac{x}{4} = \frac{y}{1} \qquad x_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$

$\lambda = 1$

Date ____
Page ____

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5-1 & 4-0 \\ 0-0 & 2-1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}$$

$4x + 4y = 0 \rightarrow 3$

$x1 + y1 = 0 \rightarrow 4$

$4(x+y) = 0$

$(x+y) = \frac{0}{4} = 0$

$x + y = 0.$

$x + y = 0.$

$\frac{x}{1} = \frac{-y}{1}$

$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x = 1 \quad y = -1$

$x_2 \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

Assignement 02.

(5) $J\left(\frac{x, y, z}{u, v, w}\right)$, $x = u(1-v)$, $y = uv(1-uv)$.
$z = uvw$.

$x = u - uv$, $y = uv - u^2v^2$.

Sol:-

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -v & 0 \\ v-2uv^2 & u-2vu^2 & 0 \\ vw & uw & uv \end{vmatrix}$$

$$\Rightarrow (1-v)\left[(u-2uv^2)(uv)\right] + v\left[(v-2uv^2)(uv)\right]$$

$$\Rightarrow (1-v)(u^2v - 2v^2u^3) + v(uv^2 - 2u^2v^3)$$

$$\Rightarrow u^2v - 2v^2u^3 - u^2v^2 - 2v^3u^3 + u^2v^2 - 2u^3\ldots$$

$$= \boxed{u^2v - 2v^2u^3}$$

(6) $u = x + y + z$, $v = y + z$, $z = uvw$.

$$J = \frac{(x, y, z)}{(u, v, w)}$$

Sol

$u = x + v$ $\quad v = y + uvw$ $\quad z = uvw$.

$x = u - v$ $\quad y = v - uvw$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$\Rightarrow 1\left[(1-uw)(uv) + (uw)(uv)\right] + 1(-vw)(uv) + (uw)(v\ldots$$

$$= uv - \cancel{u^2vw} + \cancel{u^2vw} - \cancel{v^2uw} + \cancel{uwv^2}$$

$$= \boxed{uv}$$

Extreme Values

Date
Page

Extreme Values of function of two variabl

consider a function $f(x,y)$ then 1st order partial derivatives are $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$ and second order prartial derivatives are

$$r = \frac{\partial^2 f}{\partial x^2}, \quad q = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}$$

* The nessary for maximum or/minimum value of (x,y) is $p=0, q=0$ $rt-s^2>0$ the suffient condition for maximam is $r<0$ and for minima $r>0$.

* If $rt-s^2<0$ then the point is called saddle point

* If $rt-s^2=0$ the further investigation is ~~nessa~~ necessary.

* Both maximum and mininnum value are called by a single name exteeme value or critical point (or) stationary points.

(1) Show that $f(x,y) = xy(a-x-y)$ $a>0$ is maximum at the point $[a/3, a/3]$.

sol) let $f = axy - x^2y - xy^2 \rightarrow (1)$.

Diff w.r.t x partially.

$\frac{\partial f}{\partial x} = p = ay - 2xy - y^2$.

Diff w.r.t 'x' paratially.

$\frac{\partial^2 f}{\partial x^2} = r = 0 - 2y - 0 = -2y$.

Diff ① w.r.t 'y' partially.

$q = \frac{\partial f}{\partial y} = ax - x^2 - 2xy.$

Diff w.r.t 'y' partially.

$\frac{\partial^2 f}{\partial y^2} = t = 0 - 0 - 2x = -2x$

Diff ① w.r.t 'x' partially.

$\frac{\partial^2 f}{\partial x \partial y} = s = a - 2x - 2y.$

At $(a/3, a/3)$

$p = a(a/3) - 2(a/3)(a/3) - (a/3)^2$

$= a^2/3 - 2a^2/9 - a^2/9.$

$= \frac{3a^2 - 2a^2 - a^2}{9}$

$p = 0/9 = 0$

$q = a(a/3) - (a/3)^2 - 2(a/3)(a/3)$

$= a^2/3 - a^2/9 - 2a^2/9.$

$= \frac{3a^2 - a^2 - 2a^2}{9}$

$\boxed{q = 0}$

$r = -2(a/3) = -2a/3 < 0.$

$s = a - 2(a/3) - 2(a/3) = a - 2a/3 - 2a/3 =$

$\frac{3a - 2a - 2a}{3}$

$\boxed{s = -a/3}$

$t = -2(a/3) = -(+a/3)^2$

Date
Page

$rt - s^2 (-2a/3)(-2a/3) - (a/3)^2$

$= 4a^2/9 - a^2/9 = 3a^2/9 = a^2/3 > 0.$

Since $p=0; q=0, Art - s^2 > 0$ & $r<0$, the given function is maximum at $a/3, a/3$

$\therefore$ maximum value is $f(x,y) = axy - x^2y - xy^2$

$f(a/3, a/3) = \cancel{a(a/3)} = a(a/3)(a/3) - (a/3)^2(a/3)$

$- (a/3)(a/3)^2.$

$= \left(a^3/9\right) - \left(a^2/9\right)(a/3) - (a/3)\left(a^2/9\right).$

$= \frac{a^3}{9} - \frac{a^3}{27} - \frac{a}{27}$

$= \frac{3a^3 - a^3 - a^3}{27}$

$\boxed{= a^3/27}$

* Show that $f(x,y) = x^2 + 2xy + 2y^2 + 2x + y$ is minimum at $(-3/2, 1/2)$

Sol. let $f = x^2 + 2xy + 2y^2 + 2x + y \rightarrow (1)$

Diff wrt 'x' Partially.

$\frac{\partial f}{\partial x} = p = 2x + 2y + 2 \Rightarrow \frac{d^2 f}{dx^2} = r = 2$

Diff (1) wrt 'y' Partially.

$\boxed{\frac{\partial^2 f}{\partial x^2} = r = 2}$

$q = \frac{\partial f}{\partial y} = 2x + 4y + 1 \rightarrow (2)$

Diff wrt 'y' partially.

$\frac{\partial^2 f}{\partial y^2} = t = 4$

Diff (2) wrt x partially.

$\frac{\partial^2 f}{\partial x \partial y} = s = 2$

At $(-3/2, 1/2)$

$p = 2(-3/2) + 2(1/2) + 2$

$= -3 + 1 + 2 = 0$

$\boxed{p = 0}$

$q = 2(-3/2) + 4(1/2) + 1$

$= -3 + 2 + 1 = 0$

$r = 2 > 0$

$t = 4$

$s = 2$

$rt - s^2 = 2(4) - 2^2 = 8 - 4 = 4 > 0$.

$\therefore$ $p = 0$ $q = 0$ $rt - s^2 > 0$ & $r < 0$ the given function is maximum at $(-3/2, 1/2)$.

$\therefore$ maximum value is.

$f(x, y) = x^2 + 2xy + 2y^2 + 2x + y$.

$f(-3/2, 1/2) = (-3/2)^2 + 2(-3/2)(1/2) + 2(1/2)^2 +$
$2(-3/2) + 1/2$

$= 9/4 - 3/2 + 2(1/4) - 3 + 1/2$

$= \dfrac{9 - 6 + 2 - 12 + 2}{4}$

$\boxed{= -5/4}$

* Show that $z(x, y) = x^3 + y^3 - 3xy + 1$ is minimum at the point $(1, 1)$.

Sol

$z = x^3 + y^3 - 3xy + 1 \rightarrow (1)$.

Diff wrt 'x' partially.

$\dfrac{\partial z}{\partial x} = p = 3x^2 - 3y$.

Diff wrt 'x' partially.

$\dfrac{\partial^2 z}{\partial x^2} = r = 6x$

Diff ① w.r.t 'y' partially.

$$q = \frac{\partial z}{\partial y} = 3y^2 - 3x \rightarrow ②$$

Diff. w.r.t 'y' partially.

$$\frac{\partial^2 z}{\partial y^2} = t = 6y.$$

Diff ② w.r.t x partially.

$$\frac{\partial^2 z}{\partial x \partial y} = s = -3$$

At $(1,1)$ $6>0$

$p = 3(1)^2 - 3(1) = 0$

$q = 3(1)^2 - 3(1) = 0$

$r = 6(1) = 6 > 0.$

$s = -3$

$t = 6(1) = 6$

$rt - s^2 = 6(6) - (-3)^2 = 36 - 9 = 27 > 0.$

since $p=0$, $q=0$ & $rt - s^2 > 0$ & $r > 0$ the given function minimum at $(1,1)$.

$z(x,y) = x^3 + y^3 - 3xy + 1$

$z(1,1) = 1 + 1 - 3(1)(1) + 1$

$= 2 - 3 + 1$

$= 0$