SIMPLE HARMONIC MOTION
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1.Which of the following represent simple harmonic motion?

(i) $x = A \sin \omega t + B \cos \omega t$

(ii) $x = A \sin \omega t + B \cos 2\omega t$

(iii) $x = A e^{i\omega t}$

(iv) $x = A \ln \omega t$

Solution

(i) $x = A \sin \omega t + B \cos \omega t$

$$\frac{dx}{dt} = A\,\omega \cos \omega t - B\,\omega \sin \omega t$$

$$\frac{d^2x}{dt^2} = -\,\omega^2 (A \sin \omega t + B \cos \omega t)$$

$$\frac{d^2x}{dt^2} = -\,\omega^2 x$$

2. Consider a particle undergoing simple harmonic motion. The velocity of the particle at position x_1 is v_1 and velocity of the particle at position x_2 is v_2. Show that the ratio of time period and amplitude is

$$\frac{T}{A} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 x_2^2 - v_2^2 x_1^2}}$$

Solution

Using equation (10.8)

$v = \omega\sqrt{A^2 - x^2} \Rightarrow v^2 = \omega^2 (A^2 - x^2)$

Therefore, at position x_1,

$$v_1^2 = \omega^2 (A^2 - x_1^2) \qquad (1)$$

Similarly, at position x_2,

$$v_2^2 = \omega^2 (A^2 - x_2^2) \qquad (2)$$

Subtrating (2) from (1), we get

$$v_1^2 - v_2^2 = \omega^2 (A^2 - x_1^2) - \omega^2 (A^2 - x_2^2)$$

$$= \omega^2 (x_2^2 - x_1^2)$$

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} \Rightarrow T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}} \qquad (3)$$

Dividing (1) and (2), we get

$$\frac{v_1^2}{v_2^2} = \frac{\omega^2 \left(A^2 - x_1^2\right)}{\omega^2 \left(A^2 - x_2^2\right)} \Rightarrow A = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}} \qquad (4)$$

Dividing equation (3) and equation (4), we have

$$\frac{T}{A} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 x_2^2 - v_2^2 x_1^2}}$$

3. A nurse measured the average heart beats of a patient and reported to the doctor in terms of time period as 0.8*s*. Express the heart beat of the patient in terms of number of beats measured per minute.

Solution

Let the number of heart beats measured be *f*. Since the time period is inversely proportional to the heart beat, then

$$f = \frac{1}{T} = \frac{1}{0.8} = 1.25\,s^{-1}$$

One minute is 60 second,

(1 second = $\frac{1}{60}$ minute $\Rightarrow$ 1 s^{-1} = 60 min^{-1})

f=1.25 s^{-1} $\Rightarrow$ f = 1.25 × 60 min^{-1} = 75 beats per minute

4. Calculate the amplitude, angular frequency, frequency, time period and initial phase for the simple harmonic oscillation given below

a. $y = 0.3 \sin(40\pi t + 1.1)$

b. $y = 2 \cos(\pi t)$

c. $y = 3 \sin(2\pi t - 1.5)$

Solution

Simple harmonic oscillation equation is $y = A \sin(\omega t + \varphi_0)$ or $y = A \cos(\omega t + \varphi_0)$

a. For the wave, $y = 0.3 \sin(40\pi t + 1.1)$

Amplitude is $A = 0.3$ unit

Angular frequency $\omega = 40\pi$ rad s^{-1}

Frequency $f = \frac{\omega}{2\pi} = \frac{40\pi}{2\pi} = 20\,Hz$

Time period $T = \frac{1}{f} = \frac{1}{20} = 0.05\,s$

Initial phase is $\varphi_0 = 1.1$ rad

b. For the wave, $y = 2\cos(\pi t)$

Amplitude is $A = 2$ unit

Angular frequency $\omega = \pi$ rad s^{-1}

Frequency $f = \frac{\omega}{2\pi} = \frac{\pi}{2\pi} = 0.5 Hz$

Time period $T = \frac{1}{f} = \frac{1}{0.5} = 2s$

Initial phase is $\varphi_0 = 0$ rad

c. For the wave, $y = 3\sin(2\pi t + 1.5)$

Amplitude is $A = 3$ unit

Angular frequency $\omega = 2\pi$ rad s^{-1}

Frequency $f = \frac{\omega}{2\pi} = \frac{2\pi}{2\pi} = 1 Hz$

Time period $T = \frac{1}{f} = \frac{1}{1} = 1s$

Initial phase is $\varphi_0 = 1.5$ rad

5. Show that for a simple harmonic motion, the phase difference between

a. displacement and velocity is $\pi/2$ radian or 90°.

b. velocity and acceleration is $\pi/2$ radian or 90°.

c. displacement and acceleration is π radian or 180°.

Solution

a. The displacement of the particle executing simple harmonic motion

$y = A \sin\omega t$

Velocity of the particle is

$v = A\omega\cos\omega t = A\omega\sin(\omega t + \pi/2)$

The phase difference between displacement and velocity is $\pi/2$.

b. The velocity of the particle is

$v = A\,\omega\cos\omega t$

Acceleration of the particle is

$a = A\omega^2 \sin\omega t = A\omega^2 \cos(\omega t + \pi/2)$

The phase difference between velocity and acceleration is $\pi/2$.

c. The displacement of the particle is $y = A \sin\omega t$

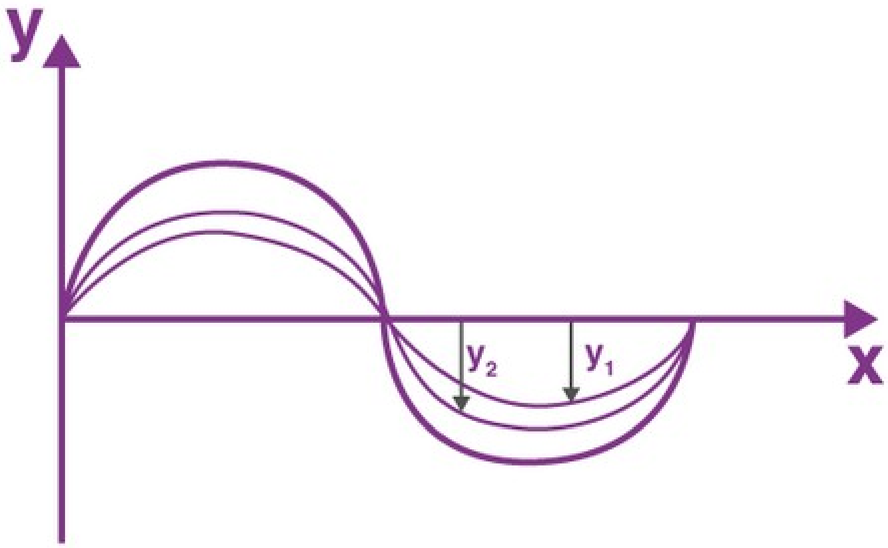
Acceleration of the particle is

$a = -A\,\omega^2 \sin\omega t = A\,\omega^2 \sin(\omega t + \pi)$

The phase difference between displacement and acceleration is π.

Superposition Principal

According to the **principle of superposition**, the resultant displacement of a number of waves in a medium at a particular point is the vector sum of the individual displacements produced by each of the waves at that point.



Let us say two waves are travelling alone, and the displacements of any element of these two waves can be represented by $y_1(x, t)$ and $y_2(x, t)$. When these two waves overlap, the resultant displacement can be given as $y(x,t)$.
Mathematically, $y(x, t) = y_1(x, t) + y_2(x, t)$.
As per the principle of superposition, we can add the overlapped waves algebraically to produce a resultant wave. Let us say the wave functions of the moving waves are

$y_1 = f_1(x–vt)$,

$y_2 = f_2(x–vt)$

……….

$y_n = f_n(x–vt)$

Then, the wave function describing the disturbance in the medium can be described as,

$y = f_1(x - vt) + f_2(x - vt) + \ldots + f_n(x - vt)$

or,$y = \sum_{i=1 \text{ to } n} = f_i\,(x - vt)$

Let us consider a wave travelling along a stretched string given by, $y_1(x, t) = A \sin(kx - \omega t)$ and another wave, shifted from the first by a phase φ, given as $y_2(x, t) = A \sin(kx - \omega t + \varphi)$
From the equations, we can see that both the waves have the same angular frequency, the same angular wave number k, and hence the same wavelength and the same amplitude A.

Now, applying the superposition principle, the resultant wave is the algebraic sum of the two constituent waves and has displacement $y(x, t) = A \sin(kx - \omega t) + A \sin(kx - \omega t + \varphi)$.
The above equation can be written as,

$y(x, t) = 2A \cos(\phi/2).\ \text{Sin}\,(kx - \omega t + \phi/2)$

The resultant wave is a sinusoidal wave, travelling in the positive X direction, where the phase angle is half of the phase difference of the individual waves and the amplitude as [2cos ϕ/2] times the amplitudes of the original waves.

Let two waves of vertical displacements, y_1 and y_2, superimpose at a point *p* in space, as shown in the figure, then the resultant displacement is given by
$y = y1 + y2$

Waves meet at some point *p*, and at the same time, the only difference occurs in their phases. Displacements of individual waves are given by

$y_1 = a \sin \omega t$

$y_2 = b \sin(\omega t + \varphi)$

Where, a and b are their respective amplitudes, and Φ is the constant phase difference between the two waves.

Applying the superposition principle as stated above, we get

$y = a \sin \omega t + b \sin(\omega t + \theta)$ . . . . (1)

The resultant having an amplitude A and a phase angle with respect to wave —1

$y = A \sin(\omega t + \theta)$

$A \sin(\omega t + \theta) = a \sin \omega t + b \sin(\omega t + \theta)$

Using $\left[\sin(A+B) = SinA \cos B + CosA\, SinB\right]$

$A\left[Sinwt \cos\theta + CoswtSin\theta\right] = a \sin wt + b\left[\sin wt \cos\theta + Coswt \sin\theta\right]$

Equating 'sinwt' and 'coswt'terms on both sides

$ACos\theta = a + b\,Cos\theta$ ------------------------2

$ASin\theta = b \sin\theta$ -----------------------------3

Squaring and adding equation 2 and equation 3

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\theta \text{ ------------------6}$$

For constructive interference, intensity should be maximum , $I = I_{max}$ only when

$Cos\theta = \pm 1$

When $\cos\theta = \pm 1$

$\theta = 0, 2\pi, 4\pi, ...(2n\pi)$

If Δx is the path difference b/w the waves at point p.

$$\Delta x = \frac{\lambda}{2\pi}(\theta)$$

$\Delta x = \frac{\lambda}{2\pi}(2n\pi) \quad n = 1, 2, 3.....$

$\Delta x = n\lambda$

Condition for constructive interference:

Phase difference = $0 \; or \; 2n\pi \; where \; n = 1, 2, 3 \; ....$

Path difference = $n\lambda \; where \; n = 1, 2, 3.......$

$$I = I_1 + I_2 + 2\left(\sqrt{k}a\right)\left(\sqrt{k}\,b\right)\cos\theta$$

For During constructive interference $I = I_{max}$, then

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

$$= \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 |$$

$$= I_{max} = K(a+b)^2$$

destructive interference, intensity should be minimum I = I_{min}, which happens only when cos φ = -1

When cos φ = -1

φ = π , 3π, 5π, ...

⇒ φ = (2n – 1)π, when n = 1, 2, 3, ...

Of Δx is the path difference between the waves at point *p*.

$$\Delta x = \frac{\lambda}{2\pi}\phi$$
$$\Delta x = \frac{\lambda}{2\pi}(2n-1)\pi$$

Therefore,

$$\Delta x = \frac{(2n-1)}{2}\lambda$$

Condition for Destructive Interference

Phase difference = $(2n - 1)\pi$

Path difference = $(2n - 1)\lambda/2$

$I = I_{min}$

$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2} \\ = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$$

Virtual Lab: https://phet.colorado.edu/en/simulations/wave-interference