Derivation of Nonlocal I-eqns vs. VAR(p)

Consider normalized SIR model with external infection influ

$$dS/dt = -\beta_1 S I - \beta_2 S I_e \quad (1)$$

$$dI/dt = \beta_1 S I + \beta_2 S I_e - \gamma I \quad (2)$$

$$dR/dt = \gamma I \quad (3)$$

Clearly

$$S + I + R = 1 = 100\% \text{ population of a region}$$

$I_e := I_e(t)$. external influx of infectious people.

From (3): $R(t) = R(t_0) + \gamma \int_{t_0}^{t} I(\tau)\,d\tau$

So: $$S(t) = 1 - I(t) - R(t_0) - \gamma \int_{t_0}^{t} I(\tau)\,d\tau \quad (4)$$

Substituting (4) in (2):

$$dI/dt = (\beta_1 I + \beta_2 I_e)\left(1 - I(t) - R(t_0) - \gamma \int_{t_0}^{t} I(\tau)\,d\tau\right) - \gamma I \quad (5)$$

the continuous I-eqn.

Forward Euler + Riemann Sum approximation of (5) give:

$$I_{t+1} = I_t + \eta(\beta_1 I_t + \beta_2 I_{e,t})\left(1 - I_t - R(t_0) - \gamma \frac{t-t_0}{p+1} \sum_{j=0}^{p} I_{t-j}\right) - \gamma I_t$$

or

$$I_{t+1} = \left(1 + \eta\beta_1(1 - R(t_0)) + \eta\beta_2 I_{e,t} \cdot \left(-1 - \gamma \frac{t-t_0}{p+1}\right) - \gamma\right) I_t$$

$$- \eta\beta_2 I_{e,t} \left(\gamma \cdot \frac{t-t_0}{p+1}\right) \cdot \sum_{j=1}^{p} I_{t-j}$$

linear like VAR(p)

$$- \eta\beta_1 I_t \left(I_t + \gamma \frac{t-t_0}{p+1} \sum_{j=0}^{p} I_{t-j}\right)$$

nonlinear in I_t ...

$$+ \beta_2 I_{e,t}(1 - R(t_0))$$

external forcing

Remark: $\eta, \gamma, \beta_1, \beta_2$

4 parameters to estimate