MAT 222 Linear Algebra
Week 1
Lecture Notes 1

Murat Karaçayır

Akdeniz University
Department of Mathematics

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Course Outline

- Vectors, Matrices and Systems of Linear Equations (Weeks 1-5)
- Vector Spaces and Linear Transformations (Weeks 6-7)
- Eigenvalues and Eigenvectors (Weeks 8-9)
- Inner Product and Orthogonal Matrices (Weeks 10-11)
- Least Squares Problems and Optimization (Weeks 12-13)

Suggested Sources

There are lots of online or printed sources on linear algebra. A few are given below.

- D.C. Lay, S.R. Lay & J.J. Mcdonald, Linear Algebra and Its Applications, 5th Edition, Pearson, 2016.
- G. Strang, Introduction to Linear Algebra, 6th Edition, Cambridge University Press, 2023.
- J. Hefferon, Linear Algebra, 4th Edition, Orthogonal Publishing L3C, 2020.
- This web page
- Linear Algebra series by 3Blue1Brown
- ...and many more

Grading

- Midterm (% 25-30)
- 2 or more assignments (% 20-30)
- Final (% 40-45)
- Exercises will be posted in class and also on Teams group every 3-4 weeks.
- Assignments will be announced on Teams.
- Office hours: Thursday 10.30-12.20 (in person), anytime I can respond (on Teams)
- **Doing the exercises is essential to understand the course content.**

Linear Equations

Linear equation

A linear equation is an equation which is linear in the unknowns.

Examples

- $\sqrt{2}x - 3y + \pi^2 z = 15 \quad \longrightarrow$ Linear in 3 unknowns
- $a^2 - 5b = 0 \quad \longrightarrow$ Nonlinear in 2 unknowns
- $x_1 x_2 + 2x_3 = -\dfrac{1}{2} \quad \longrightarrow$ Nonlinear in 3 unknowns
- $\sin x = 0.4 \quad \longrightarrow$ Nonlinear in 1 unknown
- $2a = 5b - \sqrt{75} \quad \longrightarrow$ Linear in 2 unknowns
- $\sqrt{x} + y = 2 \quad \longrightarrow$ Nonlinear in 2 unknowns

Solutions of a Linear Equation

Solution of an equation

A solution of an (linear) equation is a certain collection of values that satisfies the equation.

Examples

(i) $2x + 5 = 4 \quad \longrightarrow x = -1/2$ is a solution. (It is the only solution)

(ii) $3a - 4b = 14 \quad \longrightarrow a = 6, b = 1$ is a solution. If the order of unknowns is fixed as (a, b) then it can be written as $(6, 1)$. (It is **not** the only solution. For example $(\frac{22}{3}, 2)$ is another solution. The equation actually has infinitely many solutions.)

(iii) $3x + 5y - \frac{1}{4}z = 0 \quad \longrightarrow x = -1, y = \frac{1}{2}, z = -2$ is a solution. It can be written as $(-1, \frac{1}{2}, -2)$. (It has infinitely many solutions.)

- **Question:** Which has "more" solutions: Equation (ii) or (iii)? (Thinking exercise)

System of linear equations

Often we are interested in solving two or more linear equations simultaneously. In such cases we have a system of linear equations.

Examples

(1) $$\begin{aligned} 2a - b &= 4 \\ -a + 3b &= 3 \end{aligned}$$

A system of 2 equations in 2 unknowns

(2) $$\begin{aligned} 2x_1 - x_2 - x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= -5 \end{aligned}$$

A system of 2 equations in 3 unknowns. It is an underdetermined system.

(3) $$\begin{aligned} 2x + y &= 5 \\ -x + 4y &= \sqrt{3} \\ 2x - 3y &= 0 \end{aligned}$$

A system of 3 equations in 2 unknowns. It is an overdetermined system.

Origins of Linear Systems

Example 1: Setting up a diet

Rabbits in a lab are given a strict diet consisting of three food types labeled Mix A, Mix B, Mix C. Total micro contents of these mixtures are given below.

	Carbohydrate (g)	Protein (g)	Fat (g)
Mix A	3	1	2
Mix B	1	5	6
Mix C	12	4	0.5

If each rabbit is required to take 35 grams of carbohydrates, 28 grams of proteins and 27 grams of fats daily, set up the system of equations required to calculate how many units of each food type should be given to the rabbits daily to meet the mentioned dietary need.

Suppose each rabbit takes $\begin{cases} a \text{ units of Mix A} \\ b \text{ units of Mix B} \\ c \text{ units of Mix C} \end{cases}.$

Origins of Linear Systems (Example 1 continued)

Therefore each rabbit takes

$$\begin{cases} 3a + 1b + 12c \text{ grams of carbohydrates} \\ 1a + 5b + 4c \text{ grams of proteins} \\ 2a + 6b + 0.5c \text{ grams of fats} \end{cases}.$$

The dietary need for each macro can be expressed as a linear equation.

35 grams of carbohydrates $\longrightarrow \quad 3a + 1b + 12c = 35$
28 grams of proteins $\longrightarrow \quad 1a + 5b + 4c = 28$
27 grams of fats $\longrightarrow \quad 2a + 6b + 0.5c = 27$

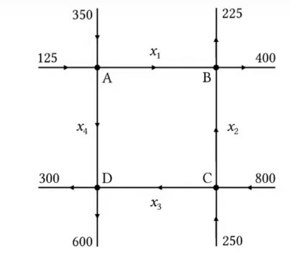
Thus, we have obtained the following linear system of 3 equations:

$$\begin{aligned} 3a + b + 12c &= 35 \\ a + 5b + 4c &= 28 \\ 2a + 6b + 0.5c &= 27 \end{aligned}$$

- **Question:** Is it possible to satisfy every possible dietary need using only these three food types? (Thinking exercise)

Example 2: Balancing a traffic network

Below is the diagram of a traffic network with four junctions. The indicated numbers are the count of cars entering or leaving each junction during a given day.



According to the diagram, set up the system of equations to calculate the number of cars that passed through adjacent pair of junctions during the day.

Exercise!

Solutions of a System of Linear Equations

Solution of a system

A solution of a (linear) system is a collection of values that satisfies every equation in the system.

An example system

- Consider the following system: $$\begin{aligned} 2x - y &= 4 \\ -x + 3y &= 3 \end{aligned}$$
- Note that $(5, 6)$ is a solution of the first equation. But it is not a solution of the system since it does not satisfy the second equation.
- Note also that $(1, \frac{4}{3})$ is a solution of the second equation. But it is not a solution of the system since it does not satisfy the first equation.
- $(3, 2)$ satisfies both equations (Check this by substituting $x = 3, y = 2$ in both equations), so it is a solution of the system. Note that it is the only solution. (Can you see why?)

Solutions of a System of Linear Equations

An example system with infinitely many solutions

- Consider the following system: $$\begin{aligned} 2x - y &= 4 \\ -4x + 2y &= -8 \end{aligned}$$
- Note that $(1, -2)$ is a solution of the system. $(8, 12)$ is also a solution of the system. Indeed every pair of the form $(a, 2a - 4)$, where $a \in \mathbb{R}$, is a solution of the system. Hence the system has infinitely many solutions.

An example system with no solutions

- Consider the following system: $$\begin{aligned} 2x - y &= 4 \\ 6x - 3y &= 9 \end{aligned}$$
- Try to find a pair of values for x and y that satisfies both equations. You won't be able to find it since the system does not have any solutions. (Can you see why?)

Solutions of An Underdetermined System

- An underdetermined system usually has infinitely many solutions.
- Consider the system $\begin{aligned} x + y + z &= 2 \\ x - 2y + z &= 5 \end{aligned}$
- Every triple of the form $(a, -1, 3 - a)$, where $a \in \mathbb{R}$ satisfies the system, so the system has infinitely many solutions.
- On the other hand, an underdetermined system may have no solutions.
- For instance, the system $\begin{aligned} 2x - y + 3z &= 5 \\ -4x + 2y - 6z &= 8 \end{aligned}$

  does not have any solutions.

- **Question:** Is it possible that an underdetermined system has a unique solution? (Exercise)

Solutions of An Overdetermined System

- An overdetermined system usually has no solutions.
- Consider the system
$$\begin{aligned} 2x + y &= 2 \\ x - 2y &= 5 \\ -x + 7y &= -4 \end{aligned}$$
- This system does not have any solutions. (Why?)
- On the other hand, sometimes an overdetermined system may have a solution.
- For instance, you can check that the system
$$\begin{aligned} 2x + y &= 2 \\ x - 2y &= 5 \\ -x + 7y &= -13 \end{aligned}$$
has the unique solution $x = \frac{9}{5}, y = -\frac{8}{5}$.
- **Question:** Is it possible that an overdetermined system has infinitely many solutions? (Exercise)

Number of Solutions of a Linear System

- Given a linear system, there are 3 possible cases for the number of its solutions:

(i) The system may have a unique solution.

(ii) The system may have infinitely many solutions.

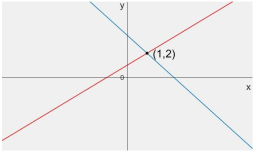
(iii) The system may have no solution.

- If the system has at least one solution (cases (i) and (ii)), it is called a consistent system.
- If the system does not have any solutions (case (iii)), it is called an inconsistent system.

- **Exercise:** Give an explanation as to why a 4th case (2 or more finitely many solutions) is not possible. (Hint: Even if you cannot find an explanation for the general case, try to find such an explanation in the case of 2 equations in 2 unknowns.)

- Each equation of the form $ax + by = c$, where a, b, c are known constants, corresponds to a line in the analytic plane.
- This makes it possible to analyze systems of 2 equations in 2 unknowns from a geometric point of view.

3x+2y = 7
x-y=-1



3 x+2y=7
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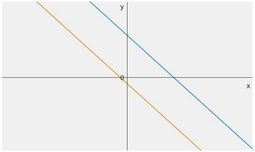
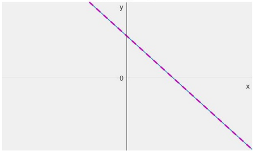


3x+2y=7
6x+4y=14

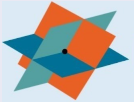


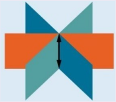
- Two lines intersect. A unique solution.
- Two lines are parallel. No solution.
- Two lines are the same. Infinitely many solutions.

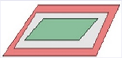
Geometric Interpretation of Systems of 3 Unknowns

- A similar analysis can be made in case of 3 unknowns; this time an equation of the form $ax + by + cz = d$ corresponds to a plane in space.
- The geometric subcases are more diverse this time.

Case 1: Consistent Systems







(a) The "normal" case: Three planes intersect at a single point. System has a unique solution.

(b) Three planes intersect at a line. System has infinitely many solutions.

(c) Three planes are coincident (the same). System has infinitely many solutions.

- **Exercise:** There is a fourth possibility for a consistent system of three unknowns. What is it?
- **Exercise:** Consider subcases (b) and (c), which both have infinitely many solutions. Which one has a "bigger" solution set? Think about it.

Geometric Interpretation of Systems of 3 Unknowns

Case 2: Inconsistent Systems

(a) Three planes are parallel. System has no solutions.

(b) Two planes are parallel. System has no solutions.

(c) Each plane intersects the other two planes at a line.

- **Exercise:** There is a fourth possibility for an inconsistent system of three unknowns. What is it?
- **Exercise:** Construct a system of 3 linear equations for each of the $6 + 2 = 8$ cases in this and the previous slides.