January 5, 2018, Friday, 2pm | Final Exam | Duration: 90 minutes

Name: Correct Answers / Solutions   Student No: ______   Grade: ______

P1 [15 points]

(a) Suppose that there is a knowledge contest, and high schools are allowed to send **at most two** groups containing **three students each**. Assume that a high school has 100 students. In how many ways can this school attend(or may not attend) this contest? (Number of possible team configurations?)

$$\underbrace{\binom{100}{3}\binom{97}{3}/2}_{\text{Attending with 2 teams}} + \underbrace{\binom{100}{3}}_{\text{1 team}} + \underbrace{1}_{\text{no teams.}}$$

(b) A football team has 3 goalkeepers, 10 defenders, 10 midfielders, and 6 forwards. In how many ways can a manager select the starting eleven players if he wants a 4-4-2 (meaning 4 defenders, 4 midfielders, 4 forwards, and of course a goalkeeper) or 3-5-2 (meaning 3 defenders, 5 midfielders, 2 forwards, and a goalkeeper) tactics? (Think simply, forget about right-left-center distinction).

$$\binom{3}{1}\binom{10}{4}\binom{10}{4}\binom{6}{2}+\binom{3}{1}\binom{10}{3}\binom{10}{5}\binom{6}{2} = \binom{3}{1}\binom{6}{2}\left(\binom{10}{4}^2+\binom{10}{3}\binom{10}{5}\right)$$

(c) For which positive integer n will the equations

$$x_1 + x_2 + x_3 + \cdots + x_{19} = n, \text{ and}$$
$$y_1 + y_2 + y_3 + \cdots + y_{64} = n$$

have the same number of **positive** integer solutions?

Substitute $x_i = a_i + 1$ or $a_i = x_i - 1$ for $1 \le i \le 19$ $\Rightarrow \sum_{i=1}^{19} a_i = n-19 \Rightarrow \#\text{sol} = \binom{n-19+18}{18}$

" $y_i = b_i + 1$ or $b_i = y_i - 1$ for $1 \le i \le 64$ $\Rightarrow \sum_{i=1}^{64} b_i = n-64 \Rightarrow \#\text{sol} = \binom{n-64+63}{63}$

$\binom{n-1}{18} = \binom{n-1}{63} \Rightarrow n-1 = 18+63 = 81$

$n = 82$ //

P2 [10 points] Questions on integers:

(a) Do there exist integers x, y, z so that $5x + 9y + 15z = 105$ and $x \cdot y = 144$? Explain why/why not.

No. Because since all $5x$, $15z$ and 105 are divisible by 5, so is $9y$. Since $5 \nmid 9$, 5 has to divide y. But then xy must also be divisible by 5 but 144 is not! QED.

(b) Prove that for any positive integer $n \in \mathbb{Z}^+$, $\gcd(5n+3, 7n+4) = 1$. Hint: Consider odd even cases of n and try to find out the gcd in both cases.

Let n be even $\Rightarrow n = 2k \Rightarrow \gcd(5n+3, 7n+4) = \gcd(10k+3, 14k+4)$
$= \gcd(10k+3, 4k+1) = \gcd(6k+2, 4k+1)$
$= \gcd(2k+1, 4k+1) = \gcd(2k+1, 2k) = \gcd(1, 2k)$
$= 1.$

Let n be odd. $\Rightarrow n = 2k+1. \Rightarrow \gcd(5n+3, 7n+4) = \gcd(10k+8, 14k+11)$
$= \gcd(10k+8, 4k+3) = \gcd(2k+2, 4k+3)$
$= \gcd(2k+2, 2k+1) = 1$

So, no matter n is even or odd, it turns out that the gcd is 1. QED.

P3 [12 points] Let $|A| = 5$.

(a) What is $|A \times A|$? Answer: 25

(b) How many functions $f : A \times A \to A$ are there? Answer: 5^{25}

(c) Can there be a injective function $f : A \times A \to A$? Answer: Y · N

(d) Can there be a surjective function $f : A \times A \to A$? Answer: Y · N

P4 [20 points] (Clearly state pigeons and pigeonholes)

(a) Let $S = \{2, 16, 128, 1024, 8192, 65536\}$. If four numbers are selected from S, prove that two of them must have the product 131072.

Let the subsets $\{2, 65536\}$, $\{16, 8192\}$, and $\{128, 1024\}$ be our pigeonholes. Let the four numbers to be selected be our pigeons. By P.H.P., if we choose 4 numbers at least two of them will be in the same subset. Then, their product is 131072.

QED.

(b) If $\{x_1, x_2, \cdots, x_7\} \subseteq \mathbb{Z}^+$, show that for some $i \neq j$, either $x_i + x_j$ or $x_i - x_j$ is divisible by 10.

If for some i, j: $x_i = x_j \mod 10$, we are done. Assume otherwise: all x_i's are different mode 10. Now, let our pigeonholes be the sets $\{0\}, \{1,9\}, \{2,8\}, \{3,7\}$ $\{4,6\}$ and $\{5\}$. By P.H.P. if we select 7 integers x_i, at least two will be in the same set, since we have only 6 sets. And for these x_i and x_j, $x_i + x_j$ is div. by 10.

QED.

P5 [15 points] What will this Java program print on the screen?

```
int count = 0;
for(int a=2; a<7; a++)
  for(int b=2; b<7; b++)
    for(int c=2; c<7; c++)
      for(int d=2; d<7; d++)
        if(a+b+c+d==18)
          count++;
System.out.println(count);
```

a b c d

This can be translated into mathematics as follows:

$x_1 + x_2 + x_3 + x_4 = 18 \qquad 2 \leq x_i < 7 \quad \text{for all } 1 \leq i \leq 4.$

Substitute $y_i = x_i - 2 \Rightarrow y_1 + y_2 + y_3 + y_4 = 10$

$y_i < 5.$

We'll use inclusion-exclusion principle:

Let our conditions be c_i: ~~...~~ $y_i \geq 5$. Now, we need $\bar{N} = N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4)$

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 = \binom{13}{3} + 4\binom{8}{3} + 6.$$

$S_0 = \binom{10+3}{3} = \binom{13}{3}$

$S_1 = \binom{4}{1}\binom{5+3}{3} = 4\binom{8}{3}$

$S_2 = \binom{4}{2}\binom{3}{3} = 6$

$S_3 = 0$

$S_4 = 0$

P6 [20 points] Solve each of the questions below using generating functions. Write down the generating function that represents this problem clearly [5 points each]. Then, find the needed coefficient. [5 points each]

a) In how many ways can a farmer distribute 20 apples to 3 children and 5 adults so that everyone gets at least one apple but no child gets more than four?

$\boxed{(x+x^2+x^3+x^4)^3\,(x+x^2+x^3+\dots)^5}$ → We need the coeff. of $\boxed{x^{20}}$

⇒ $(1+x+x^2+x^3)^3\,(1+x+x^2+\dots)^5$ → we need the coeff. of x^{12}.

$(1-x^4)^3(1-x)^{-3}\;(1-x)^{-5} = \boxed{(1-x^4)^3(1-x)^{-8}}$ Needed Term

$\binom{3}{0}\cdot 1 \to [x^{12}] = \binom{-8}{12}(-x)^{12} = \binom{19}{12}(-1)^{12}(-1)^{12}x^{12}$

$\binom{3}{1}\cdot -x^4 \to [x^{8}] = \binom{-8}{8}(-x)^{8} = \binom{15}{8}(-1)^{8}(-1)^{8}x^{8}$

$\binom{3}{2}\cdot x^8 \to [x^{4}] = \binom{-8}{4}(-x)^{4} = \binom{11}{4}(-1)^{4}(-1)^{4}x^{4}$

$\binom{3}{3}\cdot -x^{12} \to [1] = \binom{-8}{0}(-x)^{0} = \binom{7}{0}(-1)^{0}(-1)^{0}x^{0}$

So, the coeff is:

$\boxed{\binom{19}{12}-3\binom{15}{8}+3\binom{11}{4}-1}$

The term we need, $1\cdot\binom{19}{12}x^{12}+3(-x^4)\binom{15}{8}x^8+3x^8\binom{11}{4}x^4+1(-x^{12})\binom{7}{0}x^0$

b) You play a game with Bill Gates. He tells you that he'll give you 1 million dollars if you win the game. Rules are simple: you'll choose an integer n first. Then you'll roll a fair die 10 times and if the sum is exactly n, you win. 1. What n will you choose to maximize your chance? (This is very easy.) 2. With this n, what is your chance (probability) of winning?

1. When you roll a die you get 1, 2, 3, 4, 5, 6 with the same probability. The average is 3.5 So, after 10 rolls 35 is the most probable sum. $n=35$.

2. We'll find in how many ways we can get 35 and divide it by 6^{10} to get the probability

$\boxed{(x+x^2+\dots+x^6)^{10}}$ ⇒ coeff of $\boxed{x^{35}=?}$ This is the same as:

$(1+x+\dots+x^5)^{10}$ ⇒ " " $x^{25}=?$

$\boxed{(1-x^6)^{10}\,(1-x)^{-10}}$ Needed term from right

$\binom{10}{0}\;1 \to [x^{25}] = \binom{-10}{25}(-x)^{25} = \binom{34}{25}(-1)^{25}(-1)^{25}x^{25} = \binom{34}{25}x^{25}$

$\binom{10}{1}\cdot -x^6 \to [x^{19}] = \binom{-10}{19}(-x)^{19} = \binom{28}{19}(-1)^{19}(-1)^{19}x^{19} = \binom{28}{19}x^{19}$

$\binom{10}{2}\cdot x^{12} \to [x^{13}] = \binom{-10}{13}(-x)^{13} = \binom{22}{13}(-1)^{13}(-1)^{13}x^{13} = \binom{22}{13}x^{13}$

$\binom{10}{3}\cdot -x^{18} \to [x^{7}] = \binom{-10}{7}(-x)^{7} = \binom{16}{7}(-1)^{7}(-1)^{7}x^{7} = \binom{16}{7}x^{7}$

$\binom{10}{4}\cdot x^{24} \to [x] = \binom{-10}{1}(-x)^{1} = \binom{10}{1}(-1)^{1}(-1)^{1}x^{1} = \binom{10}{1}x$

Overall needed term: $\binom{34}{25}x^{25}-10x^6\binom{28}{19}x^{19}+\binom{10}{2}x^{12}\binom{22}{13}x^{13}-\binom{10}{3}x^{18}\binom{16}{7}x^7+\binom{10}{4}x^{24}\binom{10}{1}x$.

⇒ needed coeff $=\boxed{\binom{34}{25}-10\binom{28}{19}+\binom{10}{2}\binom{22}{13}-\binom{10}{3}\binom{16}{7}+\binom{10}{4}\binom{10}{1}}$

P7 [15 points] Remember Lucas Numbers are defined as $L_0 = 2$; $L_1 = 1$; and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 2$. Consider the summation:

$$S_n = {L_1}^2 + {L_2}^2 + {L_3}^2 + \cdots + {L_n}^2$$

(a) Simply fill the table to see the first few terms of the Lucas Series, and the sum series S_n.

n	L_n	L_n^2	S_n	
0	2	4	-	
1	1	1	1	$=3.1-2$
2	3	9	10	$=4.3-2$
3	4	16	26	$=7.4-2$
4	7	49	75	$=11.7-2$
5	11	121	196	

(b) By looking at the first few numbers in the series, conjecture (hypothesize) a formula for S_n. (So, you need to guess a nice and simple formula for S_n) [Hint: look for a formula having L_n and L_{n+1}]

$S_n = L_n \cdot L_{n+1} - 2$

(c) Prove your conjecture (your guess) by using mathematical induction.

- Base step — It works for $n=1$. $1.3-2=1$.
- Inductive hypothesis: Assume for k: $S_k = L_k L_{k+1} - 2$.
- Show for $k+1$: $S_{k+1} = S_k + L_{k+1}^2 = L_k L_{k+1} - 2 + L_{k+1}^2$

$$= L_{k+1}(L_k + L_{k+1}) - 2$$

$$= L_{k+1} L_{k+2} - 2$$

$$= L_n L_{n+1} - 2 \Big|_{n=k+1}.$$

QED.

Table 1: Some generating functions that can be useful. For all $m, n \in \mathbb{Z}^+$, $a \in \mathbb{R}$

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$
2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$
3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$
4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + x^3 + \cdots + x^n$
5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$
6) $1/(1-ax) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$
7) $1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots = 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \cdots$
8) $1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots = 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \cdots$