MAT 222 Linear Algebra
Week 1
Lecture Notes 2

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Matrices

- Now we define new instruments in order to handle linear systems in a systematic way.

Matrices

A matrix is a rectangular array of numbers. (By "number" we mean real number in this course, unless otherwise stated.)

Examples of matrices

- $A = \begin{bmatrix} 1 & 0 & \pi \\ \frac{\sqrt{2}}{5} & -\frac{2}{7} & 110 \end{bmatrix}$ is a matrix that has 2 rows and 3 columns. Each number inside A is called an entry. So A has 6 entries.
- Normal (round) brackets may also be used: $A = \begin{pmatrix} 1 & 0 & \pi \\ \frac{\sqrt{2}}{5} & -\frac{2}{7} & 110 \end{pmatrix}$.
- $B = \begin{bmatrix} 1 & e \\ 1 & \sin(0.5) \end{bmatrix}$ is a matrix that has 2 rows and 2 columns. B is said to be a square matrix.
- If a matrix has m rows and n columns, it is said to be of size $m \times n$. A is a 2×3 matrix and B is 2×2.

Entries of a Matrix

- Entries of a matrix can be referred to by specifying their position within the matrix by two subscripts. Row position is specified first.
- For $A = \begin{bmatrix} 1 & 0 & \pi \\ \frac{\sqrt{2}}{5} & -\frac{2}{7} & 110 \end{bmatrix}$ we write $a_{1,1} = 1$, $a_{1,2} = 0$, $a_{2,3} = 110$ etc. (Capital letters may also be used: $A_{1,1}, A_{1,2}, \ldots$)
- A matrix may also be referred to by its general entry and size. $A = [a_{i,j}]_{2\times 3}$ or $A = [A_{i,j}]_{2\times 3}$ means "A is a "2×3 matrix."
- Occasionally it is useful to refer to submatrices of a matrix. For instance, the last two columns of A may be denoted by $A_{:,2:3}$. So $A_{:,2:3} = \begin{bmatrix} 0 & \pi \\ -\frac{2}{7} & 110 \end{bmatrix}$. This notation is not standard; it is mainly used in programming languages. This procedure is known as slicing a matrix.
- Slicing may be used to refer to individual columns and rows. For example, $A_{:,2} = \begin{bmatrix} 0 \\ -\frac{2}{7} \end{bmatrix}$ and $A_{1,:} = \begin{bmatrix} 1 & 0 & \pi \end{bmatrix}$.

Origins of Matrices

- A matrix can come up in a wide variety of circumstances.

Example 1: Sales in a store

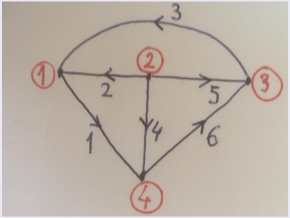
- An electronic device store may organize its weekly sales according to the product type and day of the week as follows:

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Computer	4	2	1	1	0	5	13
Printer	1	2	0	0	1	3	6
Owen	1	3	0	4	1	2	3

- In this table each row corresponds to a certain product type and each column to a certain day. So we have a 3×7 matrix.
- Note that entries in a specific row or column can be added to obtain the number of sales for a specific product or during a certain day. For example, the sum of the second row is 13, which is equal to the number of printer sales during the entire week.

Example 2: Incidence matrix of a directed graph

- A directed graph is a set of vertices (corners) and directed edges that connect a pair of these vertices.
- Consider the below directed graph that has 4 vertices and 6 edges.



- The incidence matrix of the graph is defined as follows:

$$M_{j,k} = \begin{cases} 1, & \text{if edge } k \text{ leaves vertex } j \\ -1, & \text{if edge } k \text{ enters vertex } j \\ 0, & \text{otherwise} \end{cases}$$

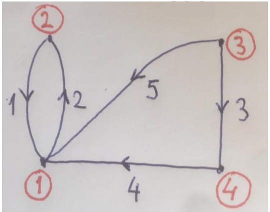
- The incidence matrix uniquely determines the directed graph.
- The incidence matrix of the above directed graph is

$$M = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \text{ which is } 4 \times 6.$$

- **Exercise:** What is the sum of the entries of M? Is it a coincidence?

Origins of Matrices (Example 2 Continued)

- **Exercise:** Find the incidence matrix of the below directed graph.



- **Exercise:** Sketch the directed graph whose incidence matrix is given by

$$M = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

- **Exercise:** Let A be an $m \times n$ matrix, where $m \geq 2$, whose entries are $-1, 0$ or 1 with sum of each column equal to 0. Is A certainly the incidence matrix of a directed graph? In other words, is there certainly a directed graph whose incidence matrix is A?

Matrix Operations

- The most basic operations on matrices are addition and scalar multiplication.

Matrix addition

If $A = [a_{i,j}]_{m\times n}$ and $B = [b_{i,j}]_{m\times n}$, then the matrix $A + B$ is defined as the $m \times n$ matrix whose i, j entry is equal to $a_{i,j} + b_{i,j}$.

- In other words, we add the entries that are on the same position.
- Consider $A = \begin{bmatrix} 4 & 1 \\ -2 & 0 \\ 4 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 0 \\ \frac{3}{2} & -5 \\ -4 & 16 \end{bmatrix}$. Then we have

$$A + B = \begin{bmatrix} 4 & 1 \\ -2 & 0 \\ 4 & 12 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ \frac{3}{2} & -5 \\ -4 & 16 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -\frac{1}{2} & -5 \\ 0 & 28 \end{bmatrix}.$$

- Note that only matrices having the same size can be added. For instance, if $C = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$, then $A + C$ is not defined.

Multiplication of a matrix by a scalar

Let A be an $m \times n$ matrix and c be a real number (called a scalar). Then cA, called the scalar multiple of A by c is the $m \times n$ matrix whose i, j entry is equal to c times $a_{i,j}$.

- For example, if $A = \begin{bmatrix} 4 & 1 \\ -2 & 0 \\ 4 & 12 \end{bmatrix}$, then $-3A = \begin{bmatrix} -12 & -3 \\ 6 & 0 \\ -12 & -36 \end{bmatrix}$.
- A common application of scalar multiplication is unit conversion. For example, if the matrix $D = \begin{bmatrix} 155 & 210 & 342 & 430 \end{bmatrix}$ consists of the distance (in miles) of a certain city to four distinct locations, then these distances can be converted to kilometers by scaling the matrix with 1.609 (approximate value). Distances in km are given by $(1.609)D = \begin{bmatrix} 249.395 & 337.89 & 550.278 & 691.87 \end{bmatrix}$.

- **Exercise:** Define the subtraction of two matrices and give an example.
- **Exercise:** Is scalar multiplication distributive over matrix addition? In other words, if A and B are two matrices of the same size and c is a scalar, is $c(A + B)$ equal to $cA + cB$?

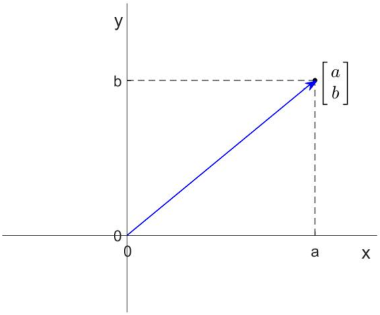
Vectors

A matrix having a single row or a single column is called a vector.

- If a vector has only one row, it is called a row vector.
  $\mathbf{v} = \begin{bmatrix} 4 & \sqrt{17} & -\pi^2 \end{bmatrix}$ is a row vector of size 3.
- If a vector has only one column, it is called a column vector.
  $\mathbf{u} = \begin{bmatrix} -2 \\ 1+\sqrt{5} \end{bmatrix}$ is a column vector of size 2.
- Since vectors are matrices, matrix operations are also defined for them.
- If $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$, then $\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ and
  $5\mathbf{v}_1 = \begin{bmatrix} 20 \\ 10 \\ -5 \end{bmatrix}$.

Geometric interpretation of vectors

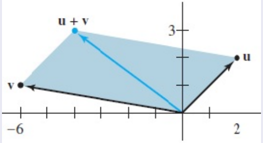
- Vectors having 2 or 3 entries have obvious geometric interpretations. Let us consider vectors in 2 dimensions for this purpose.
- Each column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be associated with the point (a, b) in the 2-dimensional analytic plane $\mathbb{R}^2$. (In physics it is useful to think of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ as the directed line segment that starts at the origin $(0, 0)$ and ends at the point (a, b).) See the figure below.



Geometric interpretation of vectors

- The idea of vectors as geometric objects can be easily extended to addition and scalar multiple of vectors.

Geometric view of vector addition

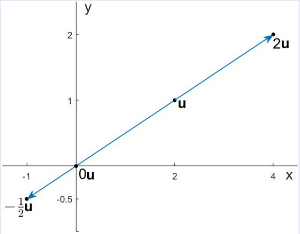


- The sum of two 2-dimensional vectors can be calculated using parallelogram rule: If **u** and **v** are represented by points in the plane, then $\mathbf{u}+\mathbf{v}$ is the fourth vertex of the parallelogram whose three vertices are **u**, **v** and the origin.

- In the above figure[a], you see the addition of the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$. The result is $\mathbf{u}+\mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

[a] The figure is taken from Lay, Lay & Mcdonald, Linear Algebra, 9th Ed., page 26.

Geometric view of scalar multiple of a vector



- If $\mathbf{u}$ is a vector, $c\mathbf{u}$ is a vector pointing to the same direction as $\mathbf{u}$ if $c > 0$ and to the opposite direction as $\mathbf{u}$ if $c < 0$. It is stretched (by a factor of c) version of $\mathbf{u}$ if $|c| > 1$ and a shrinked (by a factor of c) version of $\mathbf{u}$ if $|c| < 1$.

- In the above figure, we have $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The figure shows $2\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $-\frac{1}{2}\mathbf{u} = \begin{bmatrix} -1 \\ -1/2 \end{bmatrix}$. Note that $0\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the origin.

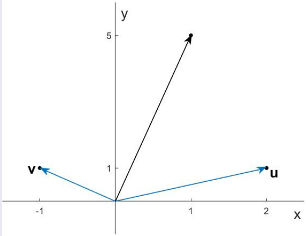
A New Geometric Interpretation of Linear Systems

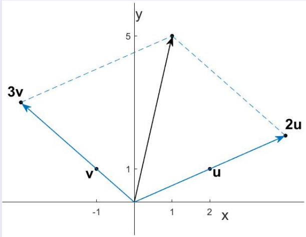
- We have seen that a linear system of 2 unknowns can be seen as relative positions of two planar lines.
- Now we look at linear systems from a new perspective.

"Column picture" of linear systems

- Consider the system $\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$
- We can write it in vector form as $\begin{bmatrix} 2x - y \\ x + y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.
- Alternatively we can write $x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.
- Literally we are trying to answer this question: "Which scalar multiple of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and which scalar multiple of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ add up to give the vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?"

"Column picture" of linear systems





- On the left you see the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in blue. You also see the "desired" result $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ in black.
- We want to "combine" **u** and **v** in such a way as to give the vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.
- Can you see the correct combination?
- It is $x = 2$, $y = 3$, which is the solution of the linear system. See the figure on the left.
- As a result, we have $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Linear Combination of Vectors

Linear combination

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be n vectors of the same size and let $a_1, a_2, \ldots, a_n$ be scalars. Then

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$$

is a vector of the same size, known as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

- Linear combination of vectors may be considered the most important operation in linear algebra.
- For example, if $\mathbf{u} = \begin{bmatrix} 3 & 0 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 & 1 & 5 \end{bmatrix}$, then $3\mathbf{u} - 4\mathbf{v} = \begin{bmatrix} 17 & -4 & -14 \end{bmatrix}$.
- Similarly, if $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\frac{1}{2}\mathbf{v}_1 - 1\mathbf{v}_2 + 4\mathbf{v}_3 = \begin{bmatrix} 6 \\ 5.5 \end{bmatrix}$.
- **Exercise:** Can you express all ordered triples (a, b, c) as a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

Linear Combination of Column Vectors

- We will put emphasis to linear combination of column vectors.
- We can generalize the "column picture"[1] idea to any linear system.
- For example, consider the system $\begin{aligned} 3a + b + 12c &= 35 \\ a + 5b + 4c &= 28. \\ 2a + 6b + 0.5c &= 27 \end{aligned}$
- It can be expressed as $a \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} + c \begin{bmatrix} 12 \\ 4 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 35 \\ 28 \\ 27 \end{bmatrix}$.
- So we are looking for the "correct" combination of $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 12 \\ 4 \\ 0.5 \end{bmatrix}$ that gives the right-hand side $\begin{bmatrix} 35 \\ 28 \\ 27 \end{bmatrix}$.

[1] This expression is due to Gilbert Strang.

Matrix-Vector Product

- Column picture of linear systems gives rise to the following definition.

Matrix-vector product

Let A be a matrix whose columns are $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ and let $\mathbf{v} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ be a column vector of size n. The matrix-vector product of A and $\mathbf{v}$ is defined by

$$A\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n.$$

$A\mathbf{v}$ is the linear combination of columns of A by the entries of $\mathbf{v}$ in the same order.

- Let $A = \begin{bmatrix} 3 & -1 & 2 \\ \frac{1}{2} & 4 & -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$. Then

$$A\mathbf{v} = \begin{bmatrix} 3 & -1 & 2 \\ \frac{1}{2} & 4 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 23 \end{bmatrix}.$$

- Note that the number of columns of A should be equal to the size of the **column vector v**. Otherwise $A\mathbf{v}$ is not defined.

Matrix Representation of Linear Systems

- Matrix-vector multiplication brings us to one of the central ideas in linear algebra: Every linear system can be represented by a matrix.
- For example, our previous system $\begin{aligned} 3x_1 + x_2 + 12x_3 &= 35 \\ x_1 + 5x_2 + 4x_3 &= 28 \\ 2x_1 + 6x_2 + 0.5x_3 &= 27 \end{aligned}$ can be expressed by

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 12 \\ 4 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 12 \\ 1 & 5 & 4 \\ 2 & 6 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 28 \\ 27 \end{bmatrix}.$$

- The matrix $\begin{bmatrix} 3 & 1 & 12 \\ 1 & 5 & 4 \\ 2 & 6 & 0.5 \end{bmatrix}$ is known as the coefficient matrix of the system and $\mathbf{b} = \begin{bmatrix} 35 \\ 28 \\ 27 \end{bmatrix}$ is the vector on the right-hand side. If we define $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to be the vector of unknowns, then we have

$$A\mathbf{x} = \mathbf{b}$$

- **Exercise:** Express enough of today's linear systems in matrix-vector product form until you completely understand the concept.